Bachelor of Science in Computer Science
University of Colombo School of Computing

SCS 1211 – Mathematical Methods I (Linear Algebra)

Topic -1: Matrices

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Matrix

Definition: A matrix is a two dimensional array of numbers or expressions arranged in rows and columns. An $m \times n$ matrix A has m rows and n columns and is written

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

where the element a_{ij} located in the i^{th} row and the j^{th} column. The rows of A are the horizontal line of elements. The columns of A are the vertical line of elements. A can also be denoted by $A = \left[a_{ij}\right]_{m\times n}$.

Notations: $M_{(m\times n)}(\mathbb{R})$ – the set of all $m \times n$ matrices with real numbers as its elements.

Special Matrices

Square Matrix: If the number of rows and the number of columns of a matrix are equal, then it is called a square matrix.

Thus $A = \left[a_{ij}\right]_{m\times n}$ is a square matrix if $m = n$.

Example: $\mathrm{A} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ and $\mathrm{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are square matrices of order 2, and 3 respectively.

Zero Matrix: A zero matrix or null matrix is a matrix with all elements equal to zero.

Example: $0_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $0_{3\times 4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ are zero matrices.

Special Matrices

Diagonal Matrix: A square matrix is said to be diagonal if each one of the non diagonal entry is zero.

Thus $A = [a_{ij}]_{n\times n}$ is a diagonal matrix if $a_{ij} = 0$, for all $i \neq j$.

Example: $A = \begin{bmatrix} 7 & 0 \\ 0 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 11 \end{bmatrix}$ are diagonal matrices of order 2, and 3 respectively.

Scalar Matrix: A diagonal matrix in which all the diagonal entries are equal is said to be a scalar matrix.

Example: $A = \begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix}$ and $B = \begin{bmatrix} -7 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -7 \end{bmatrix}$ are scalar matrices of order 2, and 3 respectively.

Special Matrices

Column Vector or Column Matrix: A matrix with only one column is called a column vector or column matrix.
For example,

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}_{n\times 1} \quad \text{is an } n \text{ diamensional column vector.}$$

Example: $\mathrm{u} = \begin{bmatrix} 7 \\ -1 \\ 4 \end{bmatrix}$ is a 3 dimensional column vector. u can also be written with respect to the standard basis as $u = 7\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Special Matrices

Row Vector or Row Matrix: A matrix with only one row is called a row vector or row matrix.
For example,

$$v = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}_{1\times n} \text{ is an } n \text{ diamensional row vector.}$$

Example: $\mathrm{u} = \begin{bmatrix} 2 & -1 & 4 & 11 \end{bmatrix}$ is a 4 dimensional row vector.
u can also be written as $u = (2, -1, 4, 11)$.

Special Matrices

Upper Triangular Matrix: A square matrix $A = \left[a_{ij}\right]_{n\times n}$ is said to be an upper triangular matrix if $a_{ij} = 0 \ \ for \ \ i > j$ and usually denoted by U.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}_{n\times n}$$

is an upper triangular matrix.

Strictly Upper Triangular Matrix: If the entries on the main diagonal of an upper triangular matrix are all zero, the matrix is said to be strictly upper triangular. ($\begin{bmatrix} 0 & 2 & 7 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ is a strictly upper triangular matrix.)

Special Matrices

Lower Triangular Matrix: A square matrix $A = [a_{ij}]_{n\times n}$ is said to be a lower triangular matrix if $a_{ij} = 0 \;\; for \;\; i < j$ and usually denoted by L.

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & 0 & \dots & 0 \\ a_{31} & a_{32} & a_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}_{n\times n}$$

is a lower triangular matrix.

Strictly Lower Triangular Matrix: If the entries on the main diagonal of a lower triangular matrix are all zero, the matrix is said to be strictly lower triangular.

Special Matrices

Identity Matrix: A square matrix in which all the main diagonal elements are 1's and all the remaining elements are 0's is called an identity matrix. Identity matrix is also called as unit matrix.

For example $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an identity matrix of order 3.

Equality of Matrices

Definition: Two matrices are equal if they have the same size and the same corresponding entries. More precisely, if $A = [a_{ij}]$ and $B = [b_{ij}]$ are two $m \times n$-matrices, then $A = B$ means that $a_{ij} = b_{ij}$ for all $i = 1, \dots, m$ and $j = 1, \dots, n$.

Example:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$ because they are different sizes.

$$\begin{bmatrix} 4 & 3 & 2 \\ 5 & 4 & 3 \\ 6 & 5 & 4 \end{bmatrix} \neq \begin{bmatrix} 4 & 3 & 2 \\ 5 & 4 & 2 \\ 6 & 5 & 4 \end{bmatrix}$$ because $a_{23} = 3 \neq 2 = b_{23}$.

Operations on Matrices - Addition

Let $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$ be two real matrices ($M_{(m\times n)}(\mathbb{R})$) and let $c \in \mathbb{R}$.

- **Addition of Matrices:** The addition of A and B is denoted by $A + B$, and is defined as $A + B = [a_{ij} + b_{ij}]_{m\times n}$.

- **Subtraction of Matrices:** The subtraction of A and B is denoted by $A - B$, and is defined as $A - B = [a_{ij} - b_{ij}]_{m\times n}$.

- Scalar Multiplication: The scalar multiplication of A with a real number c is denoted by cA and is defined as $cA = [c\ a_{ij}]_{m\times n}$.

- Note that $-A = (-1)A = [-\ a_{ij}]_{m\times n}$.

Examples:

Let $A = \begin{bmatrix} 2 & -3 & 4 \\ 1 & 5 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} -1 & 1 & 3 \\ 4 & 7 & 2 \end{bmatrix}$. Find $A + B$, $A - B$, and $3A$.

$$A + B = \begin{bmatrix} 2 + (-1) & -3 + 1 & 4 + 3 \\ 1 + 4 & 5 + 7 & 0 + 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 7 \\ 5 & 12 & 6 \end{bmatrix}.$$

$$A - B = \begin{bmatrix} 2 - (-1) & -3 - 1 & 4 - 3 \\ 1 - 4 & 5 - 7 & 0 - 2 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 1 \\ -3 & -2 & -2 \end{bmatrix}.$$

$$3A = 3\begin{bmatrix} 2 & -3 & 4 \\ 1 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 3 \times 2 & 3 \times (-3) & 3 \times 4 \\ 3 \times 1 & 3 \times 5 & 3 \times 0 \end{bmatrix} = \begin{bmatrix} 6 & -9 & 12 \\ 3 & 15 & 0 \end{bmatrix}.$$

Theorem 2.1

Let $M_{(m\times n)}(\mathbb{R})$ be the collection of all m × n matrices over field of real numbers. If $A, B, C \in M_{(m\times n)}(\mathbb{R})$ and $\alpha, \beta \in \mathbb{R}$, then

1. $A + B = B + A$
2. $A + (B + C) = (A + B) + C$
3. $A + \mathbf{0} = A$
4. $A + (-A) = \mathbf{0}$
5. $(\alpha + \beta)A = \alpha A + \beta A$
6. $\alpha(A + B) = \alpha A + \alpha B$
7. $\alpha(\beta A) = (\alpha\beta)A$
8. $1 \cdot A = A$,

where $\mathbf{0}$ is a zero matrix of order $m \times n$.

Matrix Multiplication

Let $A = [a_{ij}]$ be an $m \times n$ real matrix, and $B = [b_{ij}]$ be an $n \times p$ real matrix. The product of A and B is a matrix $C = [C_{ij}]$ of order $m \times p$, with

$$AB = C = [C_{ij}] = \left[\sum_{k=1}^{n} a_{ik}\, b_{kj}\right] = [a_{i1}\, b_{1j} + a_{i2}\, b_{2j} + \cdots + a_{in}\, b_{nj}].$$

Example: Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}_{2\times 2}$, and $B = \begin{bmatrix} 2 & 0 & -2 \\ 3 & 1 & 4 \end{bmatrix}_{2\times 3}$. Then

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -2 \\ 3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1\times 2 + 3\times 3 & 1\times 0 + 3\times 1 & 1\times(-2) + 3\times 4 \\ 2\times 2 + 1\times 3 & 2\times 0 + 1\times 1 & 2\times(-2) + 1\times 4 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 3 & 10 \\ 7 & 1 & 0 \end{bmatrix}_{2\times 3}.$$

The product of a matrix and a vector, by columns

The product of an $m \times n$ −matrix A and an n −dimensional column vector x is an m −dimensional column vector, defined as a linear combination of the columns of A as follows:

$$Ax = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}.$$

In other words, we can think of the vector x as encoding instructions for how to take a linear combination of the columns of A. The product Ax is computed by taking x_1 times the first column of A, plus x_2 times the second column of A, and so on.

Example: Compute the product

$$\begin{bmatrix} 2 & 3 & 1 \\ 5 & 4 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

The product of a matrix and a vector, by rows

The product of an $m \times n$ −matrix A and an n −dimensional column vector, can also be written like this:

$$A\,x = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}\,x_1 + a_{12}\,x_2 + \cdots + a_{1n}\,x_n \\ a_{21}\,x_1 + a_{22}\,x_2 + \cdots + a_{2n}\,x_n \\ \vdots \\ a_{m1}\,x_1 + a_{m2}\,x_2 + \cdots + a_{mn}\,x_n \end{bmatrix}$$

Example: Compute the product

$$\begin{bmatrix} 2 & 3 & 1 \\ 5 & 4 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

Matrix Multiplication – Column Method

Let $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{bmatrix}$ be two real matrices with orders $m \times n$ and $n \times p$ respectively. Suppose that the columns of B are $\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_p$. Then the columns of AB are

$$A\boldsymbol{b}_1, A\boldsymbol{b}_2, \cdots, A\boldsymbol{b}_p.$$

In other words, the k^{th} column of the matrix product AB is equal to A times the k^{th} column of B.

Matrix Multiplication – Column Method

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}_{2\times 2}$, and $B = \begin{bmatrix} 2 & 0 & -2 \\ 3 & 1 & 4 \end{bmatrix}_{2\times 3}$.

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & -2 \\ 3 & 1 & 4 \end{bmatrix} = \left[\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 4 \end{bmatrix}\right]$$

$$= \left[2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad 0\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad (-2)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 4\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right]$$

$$= \begin{bmatrix} 11 & 3 & 10 \\ 7 & 1 & 0 \end{bmatrix}_{2\times 3}.$$

Note that columns of AB are combinations of columns of A.

Matrix Multiplication – Row Method

Let $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{bmatrix}$ be two real matrices with orders $m \times n$ and $n \times p$ respectively. Suppose that the rows of A are $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_m$. Then the rows of AB are

$$\boldsymbol{a}_1 B, \boldsymbol{a}_2 B, \cdots, \boldsymbol{a}_m B.$$

In other words, the i^{th} row of the matrix product AB is equal to the i^{th} row of A times B.

Matrix Multiplication – Row Method

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}_{2\times 2}$, and $B = \begin{bmatrix} 2 & 0 & -2 \\ 3 & 1 & 4 \end{bmatrix}_{2\times 3}$.

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & -2 \\ 3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} [1 \quad 3]\begin{bmatrix} 2 & 0 & -2 \\ 3 & 1 & 4 \end{bmatrix} \\ [2 \quad 1]\begin{bmatrix} 2 & 0 & -2 \\ 3 & 1 & 4 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1\,[2 \quad 0 \quad -2] + 3\,[3 \quad 1 \quad 4] \\ 2\,[2 \quad 0 \quad -2] + 1\,[3 \quad 1 \quad 4] \end{bmatrix} = \begin{bmatrix} 11 & 3 & 10 \\ 7 & 1 & 0 \end{bmatrix}_{2\times 3}.$$

Note that rows of AB are combinations of rows of B.

Matrix Multiplication – 4th way

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}_{2\times 2}$, and $B = \begin{bmatrix} 2 & 0 & -2 \\ 3 & 1 & 4 \end{bmatrix}_{2\times 3}$.

$AB =$ Sum of [(Columns of A) X (Rows of B)].

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -2 \\ 3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & -2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & -2 \\ 4 & 0 & -4 \end{bmatrix}_{2\times 3} + \begin{bmatrix} 9 & 3 & 12 \\ 3 & 1 & 4 \end{bmatrix}_{2\times 3}$$

$$= \begin{bmatrix} 11 & 3 & 10 \\ 7 & 1 & 0 \end{bmatrix}_{2\times 3}.$$

Theorem 2.2

Let A, B, C be three matrices with appropriate orders over field of real numbers and $\alpha \in \mathbb{R}$ such that the operations on the following identities are well defined. Then the following identities hold:

1. $(AB)C = A(BC)$
2. $A(B + C) = AB + AC$
3. $(A + B)C = AC + BC$
4. $\alpha(AB) = (\alpha A)B = A(\alpha B)$
5. $IA = AI = A$, where I is the identity matrix.

Proof of $A(B + C) = AB + AC$

Let $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{n\times p}$, and $C = [c_{ij}]_{n\times p}$.

$$
\begin{aligned}
ij^{th} \text{ element of } A(B+C) &= \sum_{k=1}^{n} a_{ik}(kj^{th}\ element\ of\ (B+C)) \\
&= \sum_{k=1}^{n} a_{ik}(b_{kj} + c_{kj}) \\
&= \sum_{k=1}^{n} (a_{ik}b_{kj} + a_{ik}c_{kj}) \\
&= \sum_{k=1}^{n} a_{ik}b_{kj} + \sum_{k=1}^{n} a_{ik}c_{kj} \\
&= ij^{th} \text{ element of } AB + ij^{th} \text{ element of } AC \\
&= ij^{th} \text{ element of } (AB + AC).
\end{aligned}
$$

Matrix Multiplication is not Commutative

In general matrix multiplication is not commutative. That is $AB \neq BC$.

For example, let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = BA.$$

Transpose of a Matrix

Definition: The transpose of an $m \times n$ matrix $A = [a_{ij}]$ is defined as the $n \times m$ matrix $B = [b_{ij}]$, with $b_{ij} = a_{ji}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. The transpose of A is denoted by A^T.

That is, by the transpose of an $m \times n$ matrix A, we mean a matrix of order $n \times m$ having the rows of A as its columns and the columns of A as its rows.

For example, if A $= \begin{bmatrix} 2 & 0 & -2 \\ 3 & 1 & 4 \end{bmatrix}_{2\times 3}$, then $A^T = \begin{bmatrix} 2 & 3 \\ 0 & 1 \\ -2 & 4 \end{bmatrix}_{3\times 2}$.

Theorem 2.3

Let A, B be two matrices with appropriate orders over field of real numbers and $\alpha \in \mathbb{R}$ such that the operations on each of the following identities are well defined. Then the following identities hold:

1. $(A + B)^T = A^T + B^T$
2. $(A^T)^T = A$.
3. $(\alpha A)^T = \alpha\, A^T$
4. $(AB)^T = B^T A^T$.

Proof of $(AB)^T = B^T A^T$

Let $A = [a_{ij}]_{m\times n}$, and $B = [b_{ij}]_{n\times p}$.

ij^{th} element of $AB = a_{i1}b_1 j + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$.

Hence, ji^{th} element of $(AB)^T = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$.

Now, j^{th} row of $B^T = (b_{1j} \quad b_{2j} \quad \cdots \quad b_{nj})$, i^{th} column of $A^T = (a_{i1} \quad a_{i2} \quad \cdots \quad a_{in})^T$.

Then ji^{th} element of $B^T A^T = b_{1j}a_{i1} + b_{2j}a_{i2} + \cdots + b_{nj}a_{in}$

$$= \sum_{k=1}^{n} b_{kj}a_{ik} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

$$= ji^{th} \text{ element of } (AB)^T.$$

Thus $(AB)^T = B^T A^T$.

Trace of a Matrix

Definition: The Trace of a square matrix $A = [a_{ij}]_{n\times n}$ is defined as the sum of its diagonal elements and is denoted by $\mathrm{tr}(\mathrm{A})$.

$$tr(A) = \sum_{i=1}^{n} a_{ii}$$

Example: Let $\mathrm{A} = \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 2 \\ 1 & 1 & -2 \end{bmatrix}$. Then

$tr(A) = 3 + 5 + (-2) = 6.$

Theorem 2.4

Let A, B be two square matrices of order n and $\alpha \in \mathbb{R}$. Then

1. $tr(\alpha A) = \alpha\, tr(A)$.
2. $tr(A + B) = tr(A) + tr(B)$.
3. $tr(AB) = tr(BA)$.

Proof of $\mathrm{tr(AB) = tr(BA)}$

Let $A = \left[a_{ij}\right]_{n\times n}$, and $B = \left[b_{ij}\right]_{n\times n}$ be two square matrices of order n.

Let $AB = C = \left[c_{ij}\right]_{n\times n}$, and $BA = D = \left[d_{ij}\right]_{n\times n}$.

$$tr(AB) = \sum_{i=1}^{n} c_{ii} = \sum_{i=1}^{n}\left(\sum_{k=1}^{n} a_{ik}b_{ki}\right)$$

$$= \sum_{k=1}^{n}\left(\sum_{i=1}^{n} b_{ki}a_{ik}\right)$$

$$= \sum_{k=1}^{n} d_{kk}$$

$$= tr(BA).$$

Symmetric & Skew-Symmetric Matrices

Definition: A square matrix A is said to be **symmetric** if $A = A^T$.

Example: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$. Then $A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$.

Since $A = A^T$, A is symmetric.

Definition: A square matrix A is said to be **skew-symmetric** if $A^T = -A$.

Example: Let $A = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & -4 \\ -3 & 4 & 0 \end{bmatrix}$. Then $A^T = \begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & 4 \\ 3 & -4 & 0 \end{bmatrix} = -A$.

Since $A^T = -A$, A is skew-symmetric.

Orthogonal Matrices

Definition: A square matrix A of order n is said to be **orthogonal** if $A^T A = \mathrm{I}_n$.

Example: Let $A = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Then $A^T A = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & 0 & 0 \\ 0 & \sin^2\alpha + \cos^2\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathrm{I}_3.$$

Since $A^T A = \mathrm{I}_3$, A is orthogoanal.

Inverse of a Square Matrix

Definition: Let A be a square matrix of order n. If there exists a square matrix B of order n such that $AB = BA = I$, then B is called the inverse of A and is denoted by A^{-1}, where I is the identity matrix of order n.

That is; $AA^{-1} = A^{-1}A = I$.

In this case A is said to be **invertible** or **nonsingular**.

If a square matrix A has no inverse, then it is said to be **singular**.

Exercise

Prove that if a square matrix A has an inverse then it is unique.

Solution: Suppose that A has two inverses B and C. Then
$AB = BA = I$ and $AC = CA = I$.

$$
\begin{aligned}
B &= BI \\
&= B(AC) \qquad (\because AC = I) \\
&= (BA)C \\
&= IC \qquad (\because BA = I) \\
&= C.
\end{aligned}
$$

Exercise

Let A, B be two invertible square matrices of order n. Show that $(AB)^{-1} = B^{-1}A^{-1}$ and $(A^T)^{-1} = (A^{-1})^T$.

$$(AB)(B^{-1}A^{-1}) = \mathrm{A}(\mathrm{B}\,B^{-1})A^{-1} \quad (\because \text{matrix multiplication is associative})$$

$$= A\,IA^{-1}$$

$$= AA^{-1}$$

$$= I.$$

Similarly, we can show that $(B^{-1}A^{-1})(AB) = I$.

Hence, $(AB)^{-1} = B^{-1}A^{-1}$.

$$AA^{-1} = \mathrm{I} \implies (AA^{-1})^T = I^T = I$$

$$\implies (A^{-1})^T A^T = I \quad (\because (AB)^T = B^T A^T)$$

$$\implies (A^T)^{-1} = (A^{-1})^T.$$

Right and Left Inverses

Definition: Let A be an $m \times n$ real matrix and B an $n \times m$ real matrix.

- We say that B is a left inverse of A if $BA = I$, where I is the identity matrix of order n.
- We say that B is a right inverse of A if $AB = I$, where I is the identity matrix of order m.

If A has a left inverse, we also say that A is left invertible. Similarly, if A has a right inverse, we say that A is right invertible.

Example: Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Then B is a right inverse of A, since $AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$.

Note that B is not a left inverse, of A.