# Assignment 1 - Image Deblurring

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## 1 Finite difference approximation of the objective function $E$

The deblurring problem can be written as the following optimization

$$\hat{u} = \arg \underbrace{min}_{u} E[u] \tag{1}$$

$$E[u] = |g - u * k|^2 + \lambda R[u] \tag{2}$$

where the term $|g - u * k|^2$ equals to:

$$|g - u * k|^2 = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \left| g[i,j] - \sum_{p=0}^{1} \sum_{q=0}^{1} k[p,q] u[i-p+1, j-q+1] \right|_2^2 \tag{3}$$

where $g$ is the blurred image, $k$ is the kernel and $u$ should be our result, the deblurred image. The kernel used for this assignment is:

$$k_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix} \tag{4}$$

### 1.1 Derive the corresponding discretization for the anisotropic prior

We want to discretize the anisotropic prior:

$$R[u] = |\nabla u|_1 = \sum_{i=0}^{m} \sum_{j=0}^{n} |\nabla u[i,j]|_1 \tag{5}$$

the discretized version is similar but takes the absolute value.

$$R[u] = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} |u[i+1,j] - u[i,j]| + |u[i,j+1] - u[i,j]| + \sum_{j=0}^{n-1} |u[m,j+1] - u[m,j]| + \sum_{i=0}^{m-1} |u[i+1,n] - u[i,n]| \tag{6}$$

## 2 Calculation of the exact gradient of the discretized $E$

### 2.1 $\lambda = 0$

With $\lambda = 0$ the problem reduces to:

$$E[u] = |g - u * k|^2$$
$$E[u] = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \Big| g[i,j] - \sum_{p=0}^{1} \sum_{q=0}^{1} k[p,q] u[i-p+1, j-q+1] \Big|_2^2$$

As the kernel has only four entries, we get rid of the summation:

$$k = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix}$$

$$\begin{aligned} \sum_{p=0}^{1} \sum_{q=0}^{1} k[p,q] u[i-p+1, j-q+1] &= \frac{1}{2} u[i-0+1, j-0+1] + 0u[i-0+1, j-1+1] + \frac{1}{2} u[i-1+1, j-0+1] + 0u[i-1+1, j-1+1] \\ &= \frac{1}{2} u[i-0+1, j-0+1] + \frac{1}{2} u[i-1+1, j-0+1] \\ &= \underline{\frac{1}{2} u[i+1, j+1] + \frac{1}{2} u[i, j+1]} \end{aligned}$$

$$\nabla E[u] = \nabla \Big( \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \Big| g[i,j] - \frac{1}{2} u[i+1, j+1] - \frac{1}{2} u[i, j+1] \Big|_2^2 \Big)$$
$$\nabla E[u] = \nabla \Big( \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \Big| g[i,j] - \frac{1}{2} u[i+1, j+1] - \frac{1}{2} u[i, j+1] \Big|_2^2 \Big)$$
$$\nabla E[u] = \nabla \Big( \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} (g[i,j] - \frac{1}{2} u[i+1, j+1] - \frac{1}{2} u[i, j+1])^2 \Big)$$
$$\nabla E[u] = \underline{\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} 2g[i,j] - u[i+1, j+1] - u[i, j+1]}$$

we finish here with the basis form where then we go further.

Pixels influence each other, we have to look from all perspectives for $\nabla E[i,j]$: center pixel (i, j), northern pixel (i, j - 1), southern pixel (i, j + 1), western pixel (i - 1, j), eastern pixel (i + 1, j), north-western pixel (i - 1, j - 1), north-eastern pixel (i + 1, j - 1), south-western pixel (i - 1, j + 1), south-eastern pixel (i + 1, j + 1).

$$\begin{array}{ccccccc} 2g[i-1,j-1] - u[i,j] - u[i-1,j] & + & 2g[i,j-1] - u[i+1,j] - u[i,j] & + & 2g[i+1,j-1] - u[i+2,j] - u[i+1,j] & + \\ 2g[i-1,j] - u[i,j+1] - u[i-1,j+1] & + & 2g[i,j] - u[i+1,j+1] - u[i,j+1] & + & 2g[i+1,j] - u[i+2,j+1] - u[i+1,j+1] & + \\ 2g[i-1,j+1] - u[i,j+2] - u[i-1,j+2] & + & 2g[i,j+1] - u[i+1,j+2] - u[i,j+2] & + & 2g[i+1,j+1] - u[i+2,j+2] - u[i+1,j+2] \end{array}$$

As we only try to find the derivative for $\nabla E[i,j]$, most value indexes that don't have $i$ or $j$ will be zero. This leaves us with this term:

$$\begin{aligned}\nabla E[i,j] =& -u[i,j] + 2g[i,j-1] - u[i+1,j] - u[i,j] - u[i+2,j] \\ & - u[i+1,j] + 2g[i-1,j] - u[i,j+1] + 2g[i,j] - u[i,j+1] \\ & + 2g[i+1,j] - u[i,j+2] + 2g[i,j+1] - u[i,j+2] \\ \nabla E[i,j] =& \underline{\underline{-2u[i,j] + 2g[i,j-1] - 2u[i+1,j] - u[i+2,j]}} \\ & \underline{\underline{+2g[i-1,j] - 2u[i,j+1] + 2g[i,j]}} \\ & \underline{\underline{+2g[i+1,j] - 2u[i,j+2] + 2g[i,j+1]}}\end{aligned}$$

Lets define the behavior at the boundaries: We use the Dirichlet approach, where I set the function to a constant, in this case 0. We define our range as $\forall i \in [1, M-2], \forall j \in [1, N-2]$

$$\begin{array}{cc} [i,0] & \rightarrow & 0 \\ [0,j] & \rightarrow & 0 \\ [i,N] & \rightarrow & 0 \\ [M,J] & \rightarrow & 0 \\ [i,N-1] & \rightarrow & 0 \\ [M-1,J] & \rightarrow & 0 \end{array}$$

Our loop will go from 1 to $M-2$ or $N-2$.

## 2.2 Gaussian prior

The derivative of the data term stays the same, so we only have to look at the Gaussian prior which is:

$$R[u] = |\nabla u|_2^2 = \sum_{i=0}^{m}\sum_{j=0}^{n} |\nabla u[i,j]|_2^2$$

$$R[u] = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} (u[i+1,j] - u[i,j])^2 + (u[i,j+1] - u[i,j])^2 + \sum_{j=0}^{n-1} (u[m,j+1] - u[m,j])^2 + \sum_{i=0}^{m-1} (u[i+1,n] - u[i,n])^2$$

The derivate then would be:

$$\nabla R[u] = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} -2(u[i+1,j] - u[i,j]) - 2(u[i,j+1] - u[i,j]) + \sum_{j=0}^{n-1} -2(u[m,j+1] - u[m,j]) + \sum_{i=0}^{m-1} -2(u[i+1,n] - u[i,n])$$

For the iterative version this would result into this derivative:

$$\begin{aligned} u_{\nabla} R[i,j] &= \nabla\Big(-2(u[i+1,j] - u[i,j]) - 2(u[i,j+1] - u[i,j]) - 2(u[m,j+1] - u[m,j]) - 2(u[i+1,n] - u[i,n])\Big) \\ u_{\nabla} R[i,j] &= -2(u[i+1,j] - u[i,j]) - 2(u[i,j+1] - u[i,j]) - 2(0 - u[m,j]) - 2(0 - u[i,n]) \\ u_{\nabla} R[i,j] &= -2(u[i+1,j] - u[i,j]) - 2(u[i,j+1] - u[i,j]) + 2u[m,j] + 2u[i,n]\end{aligned}$$

Complete Equation:

$$\begin{aligned}\nabla E[i,j] =& \underline{\underline{-2u[i,j] + 2g[i,j-1] - 2u[i+1,j] - u[i+2,j]}} \\ & \underline{\underline{+2g[i-1,j] - 2u[i,j+1] + 2g[i,j]}} \\ & \underline{\underline{+2g[i+1,j] - 2u[i,j+2] + 2g[i,j+1]}} \\ & \underline{\underline{+\lambda(-2(u[i+1,j] - u[i,j]) - 2(u[i,j+1] - u[i,j]) + 2u[m,j] + 2u[i,n])}}\end{aligned}$$

hint: $\forall i \in [1, M-2], \forall j \in [1, N-2]$
The boundaries behave the same as before.

## 2.3 Anisotropic Total Variation

The derivative of the data term stays the same, so we only have to look at the anisotropic Gaussian prior which is:

$$R[u] = |\nabla u|_1 = \sum_{i=0}^{m} \sum_{j=0}^{n} |\nabla u[i,j]|_1$$

$$R[u] = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} |u[i+1,j] - u[i,j]| + |u[i,j+1] - u[i,j]| + \sum_{j=0}^{n-1} |u[m,j+1] - u[m,j]| + \sum_{i=0}^{m-1} |u[i+1,n] - u[i,n]|$$

$$\nabla R[u] = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} -\frac{u[i+1,j] - u[i,j]}{|u[i+1,j] - u[i,j]|} - \frac{u[i,j+1] - u[i,j]}{|u[i,j+1] - u[i,j]|} + \sum_{j=0}^{n-1} -\frac{u[m,j+1] - u[m,j]}{|u[m,j+1] - u[m,j]|} + \sum_{i=0}^{m-1} -\frac{u[i+1,n] - u[i,n]}{|u[i+1,n] - u[i,n]|}$$

For the iterative version this would result into this derivative:

$$\nabla R[i,j] = \nabla\left( -\frac{u[i+1,j] - u[i,j]}{|u[i+1,j] - u[i,j]|} - \frac{u[i,j+1] - u[i,j]}{|u[i,j+1] - u[i,j]|} - \frac{u[m,j+1] - u[m,j]}{|u[m,j+1] - u[m,j]|} - \frac{u[i+1,n] - u[i,n]}{|u[i+1,n] - u[i,n]|} \right)$$

$$\nabla R[i,j] = -\frac{u[i+1,j] - u[i,j]}{|u[i+1,j] - u[i,j]|} - \frac{u[i,j+1] - u[i,j]}{|u[i,j+1] - u[i,j]|} - \frac{u[i+1,n] - u[i,n]}{|u[i+1,n] - u[i,n]|}$$

Complete Equation:

$$\begin{aligned}\nabla E[i,j] = & \underline{\underline{-2u[i,j] + 2g[i,j-1] - 2u[i+1,j] - u[i+2,j]}} \\ & \underline{\underline{+2g[i-1,j] - 2u[i,j+1] + 2g[i,j]}} \\ & \underline{\underline{+2g[i+1,j] - 2u[i,j+2] + 2g[i,j+1]}} \\ & \underline{\underline{+\lambda\left( -\frac{u[i+1,j] - u[i,j]}{|u[i+1,j] - u[i,j]|} - \frac{u[i,j+1] - u[i,j]}{|u[i,j+1] - u[i,j]|} - \frac{u[i+1,n] - u[i,n]}{|u[i+1,n] - u[i,n]|} \right)}}\end{aligned}$$

hint: $\forall i \in [1, M-2], \forall j \in [1, N-2]$
The boundaries behave the same as before.