Assignment 1 - Image Deblurring

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1 Finite difference approximation of the objective function E

The deblurring problem can be written as the following optimization

$$\hat{u} = \arg \underbrace{min}_{u} E[u] \tag{1}$$

$$E[u] = |g - u * k|^2 + \lambda R[u] \tag{2}$$

where the term $|g - u * k|^2$ equals to:

$$|g - u * k|^2 = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \left| g[i,j] - \sum_{p=0}^{1} \sum_{q=0}^{1} k[p,q] u[i-p+1, j-q+1] \right|_2^2 \tag{3}$$

where g is the blurred image, k is the kernel and u should be our result, the deblurred image. The kernel used for this assignment is:

$$k_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix} \tag{4}$$

1.1 Derive the corresponding discretization for the anisotropic prior

We want to discretize the anisotropic prior:

$$R[u] = |\nabla u|_1 = \sum_{i=0}^{m} \sum_{j=0}^{n} |\nabla u[i,j]|_1 \tag{5}$$

the discretized version is similar but takes the absolute value.

$$R[u] = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} |u[i+1,j] - u[i,j]| + |u[i,j+1] - u[i,j]| + \sum_{j=0}^{n-1} |u[m,j+1] - u[m,j]| + \sum_{i=0}^{m-1} |u[i+1,n] - u[i,n]| \tag{6}$$

2 Calculation of the exact gradient of the discretized E

2.1 $\lambda = 0$

With $\lambda = 0$ the problem reduces to:

$$E[u] = |g - u * k|^2$$
$$E[u] = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\left|g[i,j] - \sum_{p=0}^{1}\sum_{q=0}^{1} k[p,q]u[i-p+1, j-q+1]\right|_2^2$$

$$E[u] = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\left|g[i,j] - \sum_{p=0}^{1}\sum_{q=0}^{1} k[p,q]u[i-p+1, j-q+1]\right|_2^2$$
$$E[u] = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\left|g[i,j] - (\frac{1}{2}u[i-0+1, j-0+1] + \frac{1}{2}u[i-1+1, j-0+1])\right|_2^2$$
$$\nabla E[u] = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\left(g[i,j] - (\frac{1}{2}u[i-0+1, j-0+1] + \frac{1}{2}u[i-1+1, j-0+1])\right)^2$$

we finish here with the basis form where then we go further.

Now we can start with the spacial cases, there are 9 cases: center pixel (i, j), northern pixel (i, j - 1), southern pixel (i, j + 1), western pixel (i - 1, j), eastern pixel (i + 1, j), north-western pixel (i - 1, j - 1), north-eastern pixel (i + 1, j - 1), south-western pixel (i - 1, j + 1), south-eastern pixel (i + 1, j + 1).

Here we see the center pixel (i, j):

$$u_{\nabla}E[i,j] = 2(g[i,j] - (\frac{1}{2}u[i+1, j+1] + \frac{1}{2}u[i, j+1]))$$
$$u_{\nabla}E[i,j] = \frac{1}{2}2(g[i,j] - (\frac{1}{2}u[i+1, j+1] + \frac{1}{2}u[i, j+1]))$$
$$u_{\nabla}E[i,j] = g[i,j] - \frac{1}{2}u[i+1, j+1] + \frac{1}{2}u[i, j+1]$$

The others will "behave" the same, other u are hidden. Lets look at the boundaries where $i, j = 0$ and $i, j = N$
$i, j = 0$

$$u_{\nabla}E[i,j] = 2(g[0,0] - (\frac{1}{2}u[0+1, 0+1] + \frac{1}{2}u[0, 0+1]))$$
$$u_{\nabla}E[i,j] = g[0,0] - \frac{1}{2}u[1,1] + \frac{1}{2}u[0,1]$$

It stays the same as we have no negative value.

$i, j = N-1, M-1$

$$u_\nabla E[i,j] = 2(g[N,M] - (\frac{1}{2}u[N+1,M+1] + \frac{1}{2}u[N,M+1]))$$
$$u_\nabla E[i,j] = g[N,M] - \frac{1}{2}u[N+1,M+1] + \frac{1}{2}u[N,M+1]$$

Behaves good, as we set $N-1$ and $M-1$

2.2 Gaussian prior

The derivative of the data term stays the same, so we only have to look at the Gaussian prior which is:

$$R[u] = |\nabla u|_2^2 = \sum_{i=0}^{m}\sum_{j=0}^{n} |\nabla u[i,j]|_2^2$$
$$R[u] = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}(u[i+1,j]-u[i,j])^2 + (u[i,j+1]-u[i,j])^2 + \sum_{j=0}^{n-1}(u[m,j+1]-u[m,j])^2 + \sum_{i=0}^{m-1}(u[i+1,n]-u[i,n])^2$$

$$\nabla R[u] = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}(u[i+1,j]-u[i,j])^2 + (u[i,j+1]-u[i,j])^2 + \sum_{j=0}^{n-1}(u[m,j+1]-u[m,j])^2 + \sum_{i=0}^{m-1}(u[i+1,n]-u[i,n])^2$$
$$\nabla R[u] = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}2(u[i+1,j]-u[i,j]) + 2(u[i,j+1]-u[i,j]) + \sum_{j=0}^{n-1}2(u[m,j+1]-u[m,j]) + \sum_{i=0}^{m-1}2(u[i+1,n]-u[i,n])$$

Here we see the center pixel (i, j):

$$u_\nabla R[i,j] = 2(u[i+1,j]-u[i,j]) + 2(u[i,j+1]-u[i,j]) + 2(u[m,j+1]-u[m,j]) + 2(u[i+1,n]-u[i,n])$$
$$u_\nabla R[i,j] = 2(u[i+1,j]-u[i,j]) + 2(u[i,j+1]-u[i,j]) + 2(0-u[i,n])$$
$$u_\nabla R[i,j] = 2(u[i+1,j]-u[i,j]) + 2(u[i,j+1]-u[i,j]) - 2u[i,n]$$

Complete Equation:

$$u_\nabla R[i,j] = g[i,j] - \frac{1}{2}u[i+1,j+1] + \frac{1}{2}u[i,j+1] + \lambda(2(u[i+1,j]-u[i,j]) + 2(u[i,j+1]-u[i,j]) - 2u[i,n])$$

2.3 Anisotropic Total Variation

The derivative of the data term stays the same, so we only have to look at the anisotropic Gaussian prior which is:

$$R[u] = |\nabla u|_1 = \sum_{i=0}^{m}\sum_{j=0}^{n} |\nabla u[i,j]|_1$$
$$R[u] = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|u[i+1,j]-u[i,j]| + |u[i,j+1]-u[i,j]| + \sum_{j=0}^{n-1}|u[m,j+1]-u[m,j]| + \sum_{i=0}^{m-1}|u[i+1,n]-u[i,n]|$$

$$\nabla R[u] = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|u[i+1,j]-u[i,j]| + |u[i,j+1]-u[i,j]| + \sum_{j=0}^{n-1}|u[m,j+1]-u[m,j]| + \sum_{i=0}^{m-1}|u[i+1,n]-u[i,n]|$$
$$\nabla R[u] = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\frac{u[i+1,j]-u[i,j]}{|u[i+1,j]-u[i,j]|} + \frac{u[i,j+1]-u[i,j]}{|u[i,j+1]-u[i,j]|} + \sum_{j=0}^{n-1}\frac{u[m,j+1]-u[m,j]}{|u[m,j+1]-u[m,j]|} + \sum_{i=0}^{m-1}\frac{u[i+1,n]-u[i,n]}{|u[i+1,n]-u[i,n]|}$$

Here we see the center pixel (i, j):

$$u_{\nabla}R[i,j] = \frac{u[i+1,j]-u[i,j]}{|u[i+1,j]-u[i,j]|} + \frac{u[i,j+1]-u[i,j]}{|u[i,j+1]-u[i,j]|} + \frac{u[m,j+1]-u[m,j]}{|u[m,j+1]-u[m,j]|} + \frac{u[i+1,n]-u[i,n]}{|u[i+1,n]-u[i,n]|}$$

$$u_{\nabla}R[i,j] = \frac{u[i+1,j]-u[i,j]}{|u[i+1,j]-u[i,j]|} + \frac{u[i,j+1]-u[i,j]}{|u[i,j+1]-u[i,j]|} + 0 + \frac{0-u[i,n]}{|0-u[i,n]|}$$

$$u_{\nabla}R[i,j] = \frac{u[i+1,j]-u[i,j]}{|u[i+1,j]-u[i,j]|} + \frac{u[i,j+1]-u[i,j]}{|u[i,j+1]-u[i,j]|} - \frac{u[i,n]}{|u[i,n]|}$$

Complete Equation:

$$u_{\nabla}R[i,j] = g[i,j] - \frac{1}{2}u[i+1,j+1] + \frac{1}{2}u[i,j+1] + \lambda(\frac{u[i+1,j]-u[i,j]}{|u[i+1,j]-u[i,j]|} + \frac{u[i,j+1]-u[i,j]}{|u[i,j+1]-u[i,j]|} - \frac{u[i,n]}{|u[i,n]|})$$