Solution - Assignment 1

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October 27, 2020

Contents

Important: $g \in R^{m\times n}$ and $u \in R^{(m+1)\times(n+1)}$.

e.g. $m = n = 4$ we have $g \in \mathbb{R}^{4\times 4}$ and $u \in \mathbb{R}^{5\times 5}$

1 Calculation of the exact gradient of the discretized E

1.1 The gradient of the total variation - Anisotropic

Energy discretization of the total variation for the image of $m \times n$ (forward difference scheme):

$$R[u] = |\nabla u|_1 = \sum_{i=0}^{m-1}\sum_{j=0}^{n}\Big|u[i+1,j]-u[i,j]\Big| + \sum_{i=0}^{m}\sum_{j=0}^{n-1}\Big|u[i,j+1]-u[i,j]\Big| \tag{1}$$

if we unfold the first part of above expression for $m = n = 4$ image we will have:

$$\sum_{i=0}^{m-1}\sum_{j=0}^{n}\Big|u[i+1,j]-u[i,j]\Big| =$$

$$(0,0),(0,1),(0,2),(0,3),(0,4) = \Big|u[1,0]-u[0,0]\Big| + \Big|u[1,1]-u[0,1]\Big| + \Big|u[1,2]-u[0,2]\Big| + \Big|u[1,3]-u[0,3]\Big| + \Big|u[1,4]-u[0,4]\Big|$$

$$(1,0),(1,1),(1,2),(1,3),(1,4) = \Big|u[2,0]-u[1,0]\Big| + \Big|u[2,1]-u[1,1]\Big| + \Big|u[2,2]-u[1,2]\Big| + \Big|u[2,3]-u[1,3]\Big| + \Big|u[2,4]-u[1,4]\Big|$$

$$(2,0),(2,1),(2,2),(2,3),(2,4) = \Big|u[3,0]-u[2,0]\Big| + \Big|u[3,1]-u[2,1]\Big| + \Big|u[3,2]-u[2,2]\Big| + \Big|u[3,3]-u[2,3]\Big| + \Big|u[3,4]-u[2,4]\Big|$$

$$(3,0),(3,1),(3,2),(3,3),(3,4) = \Big|u[4,0]-u[3,0]\Big| + \Big|u[4,1]-u[3,1]\Big| + \Big|u[4,2]-u[3,2]\Big| + \Big|u[4,3]-u[3,3]\Big| + \Big|u[4,4]-u[3,4]\Big|$$

indices i/j	0	1	2	3	4
0	1	1	1	1	1
1	2	2	2	2	2
2	2	2	2	2	2
3	2	2	2	2	2
4	1	1	1	1	1

Table 1: The number of terms appearing in total variation for each coordinate (for $m = n = 4$)

if we unfold the second part of above expression for $m = n = 4$ image we will have:

$$\sum_{i=0}^{m}\sum_{j=0}^{n-1}\Big|u[i,j+1]-u[i,j]\Big| =$$

$$(0,0),(0,1),(0,2),(0,3) = \Big|u[0,1]-u[0,0]\Big| + \Big|u[0,2]-u[0,1]\Big| + \Big|u[0,3]-u[0,2]\Big| + \Big|u[0,4]-u[0,3]\Big|$$

$$(1,0),(1,1),(1,2),(1,3) = \Big|u[1,1]-u[1,0]\Big| + \Big|u[1,2]-u[1,1]\Big| + \Big|u[1,3]-u[1,2]\Big| + \Big|u[1,4]-u[1,3]\Big|$$

$$(2,0),(2,1),(2,2),(2,3) = \Big|u[2,1]-u[2,0]\Big| + \Big|u[2,2]-u[2,1]\Big| + \Big|u[2,3]-u[2,2]\Big| + \Big|u[2,4]-u[2,3]\Big|$$

$$(3,0),(3,1),(3,2),(3,3) = \Big|u[3,1]-u[3,0]\Big| + \Big|u[3,2]-u[3,1]\Big| + \Big|u[3,3]-u[3,2]\Big| + \Big|u[3,4]-u[3,3]\Big|$$

$$(4,0),(4,1),(4,2),(4,3) = \Big|u[4,1]-u[4,0]\Big| + \Big|u[4,2]-u[4,1]\Big| + \Big|u[4,3]-u[4,2]\Big| + \Big|u[4,4]-u[4,3]\Big|$$

indices i/j	0	1	2	3	4
0	1	2	2	2	1
1	1	2	2	2	1
2	1	2	2	2	1
3	1	2	2	2	1
4	1	2	2	2	1

Table 2: The number of terms appearing in total variation for each coordinate (for $m = n = 4$)

The gradient of the first term:

$$\begin{aligned}
i = 0 \text{ and } \forall j \in [0, n] \quad &\longrightarrow \quad \frac{\partial |\nabla u|_1}{\partial u[i,j]} = \frac{\partial \tau[i,j]}{\partial u[i,j]} = -\operatorname{sign}(u[i+1,j] - u[i,j]) \\
\forall i \in [1, m-1] \text{ and } \forall j \in [0, n] \quad &\longrightarrow \quad \frac{\partial |\nabla u|_1}{\partial u[i,j]} = \frac{\partial \tau[i,j]}{\partial u[i,j]} + \frac{\partial \tau[i-1,j]}{\partial u[i,j]} = \\
&\longrightarrow \quad = -\operatorname{sign}(u[i+1,j] - u[i,j]) + \operatorname{sign}(u[i,j] - u[i-1,j]) \\
\forall i = m \text{ and } \forall j \in [0, n] \quad &\longrightarrow \quad \frac{\partial |\nabla u|_1}{\partial u[i,j]} = \frac{\partial \tau[i-1,j]}{\partial u[i,j]} = \operatorname{sign}(u[i,j] - u[i-1,j])
\end{aligned} \tag{2}$$

where

$$\tau[i,j] = \Big| u[i+1,j] - u[i,j] \Big| \tag{3}$$

The gradient of the second term:

$$\begin{aligned}
\forall i \in [0, m] \text{and } \forall j = 0 \quad &\longrightarrow \quad \frac{\partial |\nabla u|_1}{\partial u[i,j]} = \frac{\partial \tau[i,j]}{\partial u[i,j]} = -\operatorname{sign}(u[i,j+1] - u[i,j]) \\
\forall i \in [0, m] \text{ and } \forall j \in [1, n-1] \quad &\longrightarrow \quad \frac{\partial |\nabla u|_1}{\partial u[i,j]} = \frac{\partial \tau[i,j]}{\partial u[i,j]} + \frac{\partial \tau[i-1,j]}{\partial u[i,j]} \\
&\longrightarrow \quad = -\operatorname{sign}(u[i,j+1] - u[i,j]) + \operatorname{sign}(u[i,j] - u[i,j-1]) \\
\forall i \in [0, m] \text{and } \forall j = n \quad &\longrightarrow \quad \frac{\partial |\nabla u|_1}{\partial u[i,j]} = \frac{\partial \tau[i-1,j]}{\partial u[i,j]} = \operatorname{sign}(u[i,j] - u[i,j-1])
\end{aligned} \tag{4}$$

where

$$\tau[i,j] = \Big| u[i,j+1] - u[i,j] \Big| \tag{5}$$

1.2 Gradient of the total variation - Gauss Prior

We will consider the following form of total variation:

$$R[u] = |\nabla u|_2^2 \cong \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} (u[i+1,j] - u[i,j])^2 + (u[i,j+1] - u[i,j])^2 + \sum_{j=0}^{n-1} (u[m,j+1] - u[m,j])^2 + \sum_{i=0}^{m-1} (u[i+1,n] - u[i,n])^2 \tag{6}$$

For $m = n = 4$ we have

$$
\begin{aligned}
R[u] = |\nabla u|_2^2 = & \\
(0,0) &= (u[1,0]-u[0,0])^2 + (u[0,1]-u[0,0])^2 \\
(0,1) &= (u[1,1]-u[0,1])^2 + (u[0,2]-u[0,1])^2 \\
(0,2) &= (u[1,2]-u[0,2])^2 + (u[0,3]-u[0,2])^2 \\
(0,3) &= (u[1,3]-u[0,3])^2 + (u[0,4]-u[0,3])^2 \\
(1,0) &= (u[2,0]-u[1,0])^2 + (u[1,1]-u[1,0])^2 \\
(1,1) &= (u[2,1]-u[1,1])^2 + (u[1,2]-u[1,1])^2 \\
(1,2) &= (u[2,2]-u[1,2])^2 + (u[1,3]-u[1,2])^2 \\
(1,3) &= (u[2,3]-u[1,3])^2 + (u[1,4]-u[1,3])^2 \\
(2,0) &= (u[3,0]-u[2,0])^2 + (u[2,1]-u[2,0])^2 \\
(2,1) &= (u[3,1]-u[2,1])^2 + (u[2,2]-u[2,1])^2 \\
(2,2) &= (u[3,2]-u[2,2])^2 + (u[2,3]-u[2,2])^2 \\
(2,3) &= (u[3,3]-u[2,3])^2 + (u[2,4]-u[2,3])^2 \\
(3,0) &= (u[4,0]-u[3,0])^2 + (u[3,1]-u[3,0])^2 \\
(3,1) &= (u[4,1]-u[3,1])^2 + (u[3,2]-u[3,1])^2 \\
(3,2) &= (u[4,2]-u[3,2])^2 + (u[3,3]-u[3,2])^2 \\
(3,3) &= (u[4,3]-u[3,3])^2 + (u[3,4]-u[3,3])^2 \\
\text{extra term}(4,0) &= (u[4,1]-u[4,0])^2 \\
\text{extra term}(4,1) &= (u[4,2]-u[4,1])^2 \\
\text{extra term}(4,2) &= (u[4,3]-u[4,2])^2 \\
\text{extra term}(4,3) &= (u[4,4]-u[4,3])^2 \\
\text{extra term}(0,4) &= (u[1,4]-u[0,4])^2 \\
\text{extra term}(1,4) &= (u[2,4]-u[1,4])^2 \\
\text{extra term}(2,4) &= (u[3,4]-u[2,4])^2 \\
\text{extra term}(3,4) &= (u[4,4]-u[3,4])^2
\end{aligned}
$$

indices i/j	0	1	2	3	4
0	1	2	2	2	1 + 1
1	2	3	3	3	1 + 2
2	2	3	3	3	1 + 2
3	2	3	3	3	1 + 2
4	1 + 1	1 + 2	1 + 2	1+ 2	1 + 1

Table 3: The number of terms appearing in total variation for each coordinate (for $m = n = 4$) . The red color indicates terms coming from an extra term in total variation.

The gradient of the total variation: **the first term**

$$
\begin{aligned}
\forall i \in [1, m-1] \text{ and } \forall j \in [1, n-1] \quad &\longrightarrow \quad \frac{\partial |\nabla u|_2^2}{\partial u[i,j]} = \frac{\partial \tau[i,j]}{\partial u[i,j]} + \frac{\partial \tau[i-1,j]}{\partial u[i,j]} + \frac{\partial \tau[i,j-1]}{\partial u[i,j]} \\
i = 0 \text{ and } j = 0 \quad &\longrightarrow \quad \frac{\partial ||\nabla u||_2}{\partial u[i,j]} = \frac{\partial \tau[i,j]}{\partial u[i,j]} \\
\forall i \in [1 : m-1] \text{ and } j = 0 \quad &\longrightarrow \quad \frac{\partial |\nabla u|_2^2}{\partial u[i,j]} = \frac{\partial \tau[i,j]}{\partial u[i,j]} + \frac{\partial \tau[i-1,j]}{\partial u[i,j]} \\
\forall i \in [0, m-1] \text{ and } j = n \quad &\longrightarrow \quad \frac{\partial |\nabla u|_2^2}{\partial u[i,j]} = \frac{\partial \tau[i,j-1]}{\partial u[i,j]} \\
i = m \text{ and } \forall j \in [0, n-1] \quad &\longrightarrow \quad \frac{\partial |\nabla u|_2^2}{\partial u[i,j]} = \frac{\partial \tau[i-1,j]}{\partial u[i,j]} \\
i = 0 \text{ and } \forall j \in [1, n-1] \quad &\longrightarrow \quad \frac{\partial |\nabla u|_2^2}{\partial u[i,j]} = \frac{\partial \tau[i,j]}{\partial u[i,j]} + \frac{\partial \tau[i,j-1]}{\partial u[i,j]} \\
i = m \text{ and } j = n \quad &\longrightarrow \quad 0
\end{aligned}
\tag{7}
$$

where

$$
\tau[i,j] = (u[i+1,j] - u[i,j])^2 + (u[i,j+1] - u[i,j])^2 + \epsilon \tag{8}
$$

Let us compute the derivative of each term. The gradient of the first term:

$$
\begin{aligned}
\frac{\partial \tau[i,j]}{\partial u[i,j]} &= \frac{\partial}{\partial u[i,j]} \Big((u[i+1,j] - u[i,j])^2 + (u[i,j+1] - u[i,j])^2 + \epsilon \Big) \\
&= \Big| \quad x := u[i,j] \qquad y := u[i+1,j] \qquad z := u[i,j+1] \quad \Big| \\
&= \frac{\partial}{\partial x} \Big((y-x)^2 + (z-x)^2 + \epsilon \Big) \\
&= \Big(-2(y-x) - 2(z-x) \Big) \\
&= \Big(4x - 2y - 2z \Big) \\
&= 4u[i,j] - 2u[i+1,j] - 2u[i,j+1]
\end{aligned}
$$

The gradient of the second term:

$$
\begin{aligned}
\frac{\partial \tau[i-1,j]}{\partial u[i,j]} &= \frac{\partial}{\partial u[i,j]} \Big((u[i,j] - u[i-1,j])^2 + (u[i-1,j+1] - u[i-1,j])^2 + \epsilon \Big) \\
&= \Big| \quad x := u[i,j] \qquad y := u[i-1,j] \qquad z := u[i-1,j+1] \quad \Big| \\
&= \frac{\partial}{\partial x} \Big((x-y)^2 + (z-y)^2 + \epsilon \Big) \\
&= \Big(2(x-y) \Big) \\
&= 2x - 2y \\
&= 2u[i,j] - 2u[i-1,j]
\end{aligned}
$$

The gradient of the third term:

$$\begin{aligned}
\frac{\partial \tau[i,j-1]}{\partial u[i,j]} &= \frac{\partial}{\partial u[i,j]}\Big((u[i+1,j-1]-u[i,j-1])^2+(u[i,j]-u[i,j-1])^2+\epsilon\Big) \\
&= \Big| \; x := u[i,j] \qquad y := u[i+1,j-1] \qquad z := u[i,j-1] \; \Big| \\
&= \frac{\partial}{\partial x}\Big((y-z)^2+(x-z)^2+\epsilon\Big) \\
&= 2x-2z \\
&= 2u[i,j]-2u[i,j-1]
\end{aligned}$$

The gradient of the total variation: **the first term**:

$$\begin{aligned}
\forall i \in [1,m-1] \text{ and } \forall j \in [1,n-1] \quad &\longrightarrow \quad \frac{\partial |\nabla u|_2^2}{\partial u[i,j]} = \frac{\partial \tau[i,j]}{\partial u[i,j]} + \frac{\partial \tau[i-1,j]}{\partial u[i,j]} + \frac{\partial \tau[i,j-1]}{\partial u[i,j]} \\
&= \quad 8u[i,j]-2u[i+1,j]-2u[i,j+1]-2u[i-1,j]-2u[i,j-1] \\
i=0 \text{ and } j=0 \quad &\longrightarrow \quad \frac{\partial |\nabla u|_2^2}{\partial u[i,j]} = \frac{\partial \tau[i,j]}{\partial u[i,j]} = 4u[i,j]-2u[i+1,j]-2u[i,j+1] \\
\forall i \in [1:m-1] \text{ and } j=0 \quad &\longrightarrow \quad \frac{\partial |\nabla u|_2^2}{\partial u[i,j]} = \frac{\partial \tau[i,j]}{\partial u[i,j]} + \frac{\partial \tau[i-1,j]}{\partial u[i,j]} = 6u[i,j]-2u[i+1,j]-2u[i,j+1]-2u[i-1,j] \\
\forall i \in [0,m-1] \text{ and } j=n \quad &\longrightarrow \quad \frac{\partial |\nabla u|_2^2}{\partial u[i,j]} = \frac{\partial \tau[i,j-1]}{\partial u[i,j]} = 2u[i,j]-2u[i,j-1] \\
i=m \text{ and } \forall j \in [0,n-1] \quad &\longrightarrow \quad \frac{\partial |\nabla u|_2^2}{\partial u[i,j]} = \frac{\partial \tau[i-1,j]}{\partial u[i,j]} = 2u[i,j]-2u[i-1,j] \\
i=0 \text{ and } \forall j \in [1,n-1] \quad &\longrightarrow \quad \frac{\partial |\nabla u|_2^2}{\partial u[i,j]} = \frac{\partial \tau[i,j]}{\partial u[i,j]} + \frac{\partial \tau[i,j-1]}{\partial u[i,j]} = 6u[i,j]-2u[i+1,j]-2u[i,j+1]-2u[i,j-1] \\
i=m \text{ and } j=n \quad &\longrightarrow \quad 0
\end{aligned} \tag{9}$$

The gradient of the total variation: **the second term**

$$\begin{aligned}
i=m \text{ and } \forall j \in [1,n-1] \quad &\longrightarrow \quad \frac{\partial |\nabla u|_2^2}{\partial u[i,j]} = \frac{\partial \tau[i,j]}{\partial [i,j]} + \frac{\partial \tau[i,j-1]}{\partial u[i,j]} = 2(u[i,j]-u[i,j+1])+2(u[i,j]-u[i,j-1]]) \\
&\longrightarrow \quad = 4u[i,j]-2u[i,j-1]-2u[i,j+1] \\
i=m \text{ and } j=0 \quad &\longrightarrow \quad \frac{\partial |\nabla u|_2^2}{\partial u[i,j]} = 2u[i,j]-2u[i,j+1] \\
i=m \text{ and } j=n \quad &\longrightarrow \quad \frac{\partial |\nabla u|_2^2}{\partial u[i,j]} = 2u[i,j]-2u[i,j-1]
\end{aligned} \tag{10}$$

The gradient of the total variation: **the third term**

$$\begin{aligned}
\forall i \in [1,m-1] \text{ and } j=n \quad &\longrightarrow \quad \frac{\partial |\nabla u|_2^2}{\partial u[i,j]} = \frac{\partial \tau[i,j]}{\partial [i,j]} + \frac{\partial \tau[i,j-1]}{\partial u[i,j]} = 2(u[i,j]-u[i+1,j])+2(u[i,j]-u[i-1,j]) \\
&\longrightarrow \quad = 4u[i,j]-2[i-1,j]-2u[i+1,j] \\
i=0 \text{ and } j=n \quad &\longrightarrow \quad \frac{\partial |\nabla u|_2^2}{\partial u[i,j]} = 2(u[i,j]-u[i+1,j]) \\
i=m \text{ and } j=n \quad &\longrightarrow \quad \frac{\partial |\nabla u|_2^2}{\partial u[i,j]} = 2(u[i,j]-u[i-1,j])
\end{aligned} \tag{11}$$

1.3 The gradient of the data term

The disctritized energy term is given as follows:

$$|u * k - g|^2 = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\left|g[i,j] - \sum_{p=0}^{1}\sum_{q=0}^{1} k[p,q]u[i-p+1, j-q+1]\right|_2^2 \tag{12}$$

Next we will compute the gradient for four different k choices. That is: $\begin{bmatrix} 1/2 & 1/2 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1/2 & 0 \\ 1/2 & 0 \end{bmatrix}$, $\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$, and $\begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$.

1.3.1 The gradient of the data term. Case 1. $k = \begin{bmatrix} 1/2 & 1/2 \\ 0 & 0 \end{bmatrix}$

Let us unfold the Eq. 12 for the above k:

$$\begin{aligned}|u * k - g|^2 &= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} |g[i,j] - k[0,0]u[i+1,j+1] - k[0,1]u[i+1,j] + k[1,0]u[i,j+1] + k[1,1]u[i,j]|_2^2 \\ &= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\left|g[i,j] - \frac{1}{2}u[i+1,j+1] - \frac{1}{2}u[i+1,j]\right|_2^2\end{aligned} \tag{13}$$

E.g. for $m = 4$ and $n = 4$

$$\begin{aligned}
&|u * k - g|^2 = \\
&(0,0),(0,1) = \left|g[0,0] - \frac{1}{2}u[1,1] - \frac{1}{2}u[1,0]\right|_2^2 + \left|g[0,1] - \frac{1}{2}u[1,2] - \frac{1}{2}u[1,1]\right|_2^2 \\
&(0,2),(0,3) = \left|g[0,2] - \frac{1}{2}u[1,3] - \frac{1}{2}u[1,2]\right|_2^2 + \left|g[0,3] - \frac{1}{2}u[1,4] - \frac{1}{2}u[1,3]\right|_2^2 \\
&(1,0),(1,1) = \left|g[1,0] - \frac{1}{2}u[2,1] - \frac{1}{2}u[2,0]\right|_2^2 + \left|g[1,1] - \frac{1}{2}u[2,2] - \frac{1}{2}u[2,1]\right|_2^2 \\
&(1,2),(1,3) = \left|g[1,2] - \frac{1}{2}u[2,3] - \frac{1}{2}u[2,2]\right|_2^2 + \left|g[1,3] - \frac{1}{2}u[2,4] - \frac{1}{2}u[2,3]\right|_2^2 \\
&(2,0),(2,1) = \left|g[2,0] - \frac{1}{2}u[3,1] - \frac{1}{2}u[3,0]\right|_2^2 + \left|g[2,1] - \frac{1}{2}u[3,2] - \frac{1}{2}u[3,1]\right|_2^2 \\
&(2,2),(2,3) = \left|g[2,2] - \frac{1}{2}u[3,3] - \frac{1}{2}u[3,2]\right|_2^2 + \left|g[2,3] - \frac{1}{2}u[3,4] - \frac{1}{2}u[3,3]\right|_2^2 \\
&(3,0),(3,1) = \left|g[3,0] - \frac{1}{2}u[4,1] - \frac{1}{2}u[4,0]\right|_2^2 + \left|g[3,1] - \frac{1}{2}u[4,2] - \frac{1}{2}u[4,1]\right|_2^2 \\
&(3,2),(3,3) = \left|g[3,2] - \frac{1}{2}u[4,3] - \frac{1}{2}u[4,2]\right|_2^2 + \left|g[3,3] - \frac{1}{2}u[4,4] - \frac{1}{2}u[4,3]\right|_2^2
\end{aligned}$$

indices i/j	0	1	2	3	4
0	0	0	0	0	0
1	1	2	2	2	1
2	1	2	2	2	1
3	1	2	2	2	1
4	1	2	2	2	1

Table 4: The number of terms appearing in data term for each coordinate (for $m = n = 4$)

The gradient of the data term:

$$\begin{aligned}
\forall i \in [1, m] \text{ and } \forall j \in [1, n-1] \quad &\longrightarrow \quad \frac{\partial |u * k - g|^2}{\partial u[i,j]} = \frac{\partial \tau[i-1, j-1]}{\partial u[i,j]} + \frac{\partial \tau[i-1, j]}{\partial u[i,j]} \\
i = 0 \text{ and } \forall j \in [0, n] \quad &\longrightarrow \quad 0 \\
\forall i \in [1, n] \text{ and } j = 0 \quad &\longrightarrow \quad \frac{\partial |u * k - g|^2}{\partial u[i,j]} = \frac{\partial \tau[i-1, j]}{\partial u[i,j]} \\
\forall i \in [1, n] \text{ and } j = m \quad &\longrightarrow \quad \frac{\partial |u * k - g|^2}{\partial u[i,j]} = \frac{\partial \tau[i-1, j-1]}{\partial u[i,j]}
\end{aligned} \tag{14}$$

where

$$\tau[i,j] = \left(g[i,j] - \frac{1}{2} u[i+1, j+1] - \frac{1}{2} u[i+1, j] \right)^2 \tag{15}$$

Let us compute the gradient of the each term separately. The gradient of the first term

$$\begin{aligned}
\frac{\partial \tau[i-1, j-1]}{\partial u[i,j]} &= \frac{\partial}{\partial u[i,j]} \left(g[i-1, j-1] - \frac{1}{2} u[i,j] - \frac{1}{2} u[i, j-1] \right)^2 \\
&= 2 \left(g[i-1, j-1] - \frac{1}{2} u[i,j] - \frac{1}{2} u[i, j-1] \right) \frac{\partial}{\partial u[i,j]} \left(g[i-1, j-1] - \frac{1}{2} u[i,j] - \frac{1}{2} u[i, j-1] \right) \\
&= - \left(g[i-1, j-1] - \frac{1}{2} u[i,j] - \frac{1}{2} u[i, j-1] \right) \\
&= \frac{1}{2} u[i,j] + \frac{1}{2} u[i, j-1] - g[i-1, j-1]
\end{aligned}$$

The gradient of the second term

$$\begin{aligned}
\frac{\partial \tau[i-1, j]}{\partial u[i,j]} &= \frac{\partial}{\partial u[i,j]} \left(g[i-1, j] - \frac{1}{2} u[i, j+1] - \frac{1}{2} u[i,j] \right)^2 \\
&= 2 \left(g[i-1, j] - \frac{1}{2} u[i, j+1] - \frac{1}{2} u[i,j] \right) \frac{\partial}{\partial u[i,j]} \left(g[i-1, j] - \frac{1}{2} u[i, j+1] - \frac{1}{2} u[i,j] \right) \\
&= - \left(g[i-1, j] - \frac{1}{2} u[i, j+1] - \frac{1}{2} u[i,j] \right) \\
&= \frac{1}{2} u[i, j+1] + \frac{1}{2} u[i,j] - g[i-1, j]
\end{aligned}$$

The full gradient is as

$$\begin{aligned}
\frac{\partial |u * k - g|^2}{\partial u[i,j]} &= \frac{\partial \tau[i-1, j-1]}{\partial u[i,j]} + \frac{\partial \tau[i-1, j]}{\partial u[i,j]} \\
&= - \left(g[i-1, j-1] - \frac{1}{2} u[i,j] - \frac{1}{2} u[i, j-1] \right) - \left(g[i-1, j] - \frac{1}{2} u[i, j+1] - \frac{1}{2} u[i,j] \right) \\
&= u[i,j] + \frac{1}{2} u[i, j-1] + \frac{1}{2} u[i, j+1] - g[i-1, j-1] - g[i-1, j]
\end{aligned}$$

$$
\begin{aligned}
&\forall i \in [1,m] \text{ and } \forall j \in [1,n-1] && \longrightarrow && \frac{\partial |u*k-g|^2}{\partial u[i,j]} = u[i,j] + \frac{1}{2}u[i,j-1] + \frac{1}{2}u[i,j+1] - g[i-1,j-1] - g[i-1,j] \\
&i = 0 \text{ and } \forall j \in [0,n] && \longrightarrow && 0 \\
&\forall i \in [1,m] \text{ and } j = 0 && \longrightarrow && \frac{\partial |u*k-g|^2}{\partial u[i,j]} = \frac{\partial \tau[i-1,j]}{\partial u[i,j]} = \frac{1}{2}u[i,j+1] + \frac{1}{2}u[i,j] - g[i-1,j] \\
&\forall i \in [1,m] \text{ and } j = n && \longrightarrow && \frac{\partial |u*k-g|^2}{\partial u[i,j]} = \frac{\partial \tau[i-1,j-1]}{\partial u[i,j]} = \frac{1}{2}u[i,j] + \frac{1}{2}u[i,j-1] - g[i-1,j-1]
\end{aligned}
\tag{16}
$$

1.3.2 The gradient of the data term. Case 2. $k = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 0 \end{bmatrix}$

For the above k, the data term takes the following form:

$$
\begin{aligned}
|u*k-g|^2 &= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} |g[i,j] - k[0,0]u[i+1,j+1] - k[0,1]u[i+1,j] + k[1,0]u[i,j+1] + k[1,1]u[i,j]|_2^2 \\
&= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} \left| g[i,j] - \frac{1}{2}u[i+1,j+1] - \frac{1}{2}u[i,j+1] \right|_2^2
\end{aligned}
\tag{17}
$$

E.g. for $m = 4$ and $n = 4$

$$
\begin{aligned}
&|u*k-g|^2 = \\
&(0,0),(0,1) = \left|g[0,0] - \frac{1}{2}u[1,1] - \frac{1}{2}u[0,1]\right|_2^2 + \left|g[0,1] - \frac{1}{2}u[1,2] - \frac{1}{2}u[0,2]\right|_2^2 \\
&(0,2),(0,3) = \left|g[0,2] - \frac{1}{2}u[1,3] - \frac{1}{2}u[0,3]\right|_2^2 + \left|g[0,3] - \frac{1}{2}u[1,4] - \frac{1}{2}u[0,4]\right|_2^2 \\
&(1,0),(1,1) = \left|g[1,0] - \frac{1}{2}u[2,1] - \frac{1}{2}u[1,1]\right|_2^2 + \left|g[1,1] - \frac{1}{2}u[2,2] - \frac{1}{2}u[1,2]\right|_2^2 \\
&(1,2),(1,3) = \left|g[1,2] - \frac{1}{2}u[2,3] - \frac{1}{2}u[1,3]\right|_2^2 + \left|g[1,3] - \frac{1}{2}u[2,4] - \frac{1}{2}u[1,4]\right|_2^2 \\
&(2,0),(2,1) = \left|g[2,0] - \frac{1}{2}u[3,1] - \frac{1}{2}u[2,1]\right|_2^2 + \left|g[2,1] - \frac{1}{2}u[3,2] - \frac{1}{2}u[2,2]\right|_2^2 \\
&(2,2),(2,3) = \left|g[2,2] - \frac{1}{2}u[3,3] - \frac{1}{2}u[2,3]\right|_2^2 + \left|g[2,3] - \frac{1}{2}u[3,4] - \frac{1}{2}u[2,4]\right|_2^2 \\
&(3,0),(3,1) = \left|g[3,0] - \frac{1}{2}u[4,1] - \frac{1}{2}u[3,1]\right|_2^2 + \left|g[3,1] - \frac{1}{2}u[4,2] - \frac{1}{2}u[3,2]\right|_2^2 \\
&(3,2),(3,3) = \left|g[3,2] - \frac{1}{2}u[4,3] - \frac{1}{2}u[3,3]\right|_2^2 + \left|g[3,3] - \frac{1}{2}u[4,4] - \frac{1}{2}u[3,4]\right|_2^2
\end{aligned}
$$

$$
\begin{aligned}
&\forall i \in [1,m-1] \text{ and } \forall j \in [1,n] && \longrightarrow && \frac{\partial |u*k-g|^2}{\partial u[i,j]} = \frac{\partial \tau[i-1,j-1]}{\partial u[i,j]} + \frac{\partial \tau[i,j-1]}{\partial u[i,j]} \\
&\forall i \in [0,m] \text{ and } j = 0 && \longrightarrow && 0 \\
&i = 0 \text{ and } \forall j \in [1,n] && \longrightarrow && \frac{\partial |u*k-g|^2}{\partial u[i,j]} = \frac{\partial \tau[i,j-1]}{\partial u[i,j]} \\
&i = m \text{ and } \forall j \in [1,n] && \longrightarrow && \frac{\partial |u*k-g|^2}{\partial u[i,j]} = \frac{\partial \tau[i-1,j-1]}{\partial u[i,j]}
\end{aligned}
\tag{18}
$$

indices i/j	0	1	2	3	4
0	0	1	1	1	1
1	0	2	2	2	2
2	0	2	2	2	2
3	0	2	2	2	2
4	0	1	1	1	1

Table 5: The number of terms appearing in data term for each coordinate (for $m = n = 4$)

where

$$\tau[i,j] = \Big(g[i,j] - \frac{1}{2}u[i+1,j+1] - \frac{1}{2}u[i,j+1]\Big)^2 \tag{19}$$

Let us compute again the gradient of the each term separately. The gradient of the first term

$$\begin{aligned}
\frac{\partial \tau[i-1,j-1]}{\partial u[i,j]} &= \frac{\partial}{\partial u[i,j]}\Big(g[i-1,j-1] - \frac{1}{2}u[i,j] - \frac{1}{2}u[i-1,j]\Big)^2 \\
&= 2\Big(g[i-1,j-1] - \frac{1}{2}u[i,j] - \frac{1}{2}u[i-1,j]\Big)\frac{\partial}{\partial u[i,j]}\Big(g[i-1,j-1] - \frac{1}{2}u[i,j] - \frac{1}{2}u[i-1,j]\Big) \\
&= -\Big(g[i-1,j-1] - \frac{1}{2}u[i,j] - \frac{1}{2}u[i-1,j]\Big)
\end{aligned}$$

The gradient of the second term

$$\begin{aligned}
\frac{\partial \tau[i,j-1]}{\partial u[i,j]} &= \frac{\partial}{\partial u[i,j]}\Big(g[i,j-1] - \frac{1}{2}u[i+1,j] - \frac{1}{2}u[i,j]\Big)^2 \\
&= \Big(g[i,j-1] - \frac{1}{2}u[i+1,j] - \frac{1}{2}u[i,j]\Big)\frac{\partial}{\partial u[i,j]}\Big(g[i,j-1] - \frac{1}{2}u[i+1,j] - \frac{1}{2}u[i,j]\Big) \\
&= -\Big(g[i,j-1] - \frac{1}{2}u[i+1,j] - \frac{1}{2}u[i,j]\Big)
\end{aligned}$$

The full gradient

$$\begin{aligned}
\frac{\partial |u*k-g|^2}{\partial u[i,j]} &= \frac{\partial \tau[i-1,j-1]}{\partial u[i,j]} + \frac{\partial \tau[i,j-1]}{\partial u[i,j]} \\
&= -\Big(g[i-1,j-1] - \frac{1}{2}u[i,j] - \frac{1}{2}u[i-1,j]\Big) - \Big(g[i,j-1] - \frac{1}{2}u[i+1,j] - \frac{1}{2}u[i,j]\Big) \\
&= u[i,j] + \frac{1}{2}u[i-1,j] + \frac{1}{2}u[i+1,j] - g[i-1,j-1] - g[i,j-1]
\end{aligned}$$

$$\begin{aligned}
\forall i \in [1, m-1] \text{ and } \forall j \in [1,n] \quad &\longrightarrow \quad \frac{\partial |u*k-g|^2}{\partial u[i,j]} = \frac{\partial \tau[i-1,j-1]}{\partial u[i,j]} + \frac{\partial \tau[i,j-1]}{\partial u[i,j]} \\
&\longrightarrow \quad u[i,j] + \frac{1}{2}u[i-1,j] + \frac{1}{2}u[i+1,j] - g[i-1,j-1] - g[i,j-1] \\
\forall i \in [0, m] \text{ and } j = 0 \quad &\longrightarrow \quad 0 \\
i = 0 \text{ and } \forall j \in [1,n] \quad &\longrightarrow \quad \frac{\partial |u*k-g|^2}{\partial u[i,j]} = \frac{\partial \tau[i,j-1]}{\partial u[i,j]} \\
&\longrightarrow \quad \frac{1}{2}u[i,j] + \frac{1}{2}u[i+1,j] - g[i,j-1] \\
i = m \text{ and } \forall j \in [1,n] \quad &\longrightarrow \quad \frac{\partial |u*k-g|^2}{\partial u[i,j]} = \frac{\partial \tau[i-1,j-1]}{\partial u[i,j]} \\
&\longrightarrow \quad \frac{1}{2}u[i-1,j] + \frac{1}{2}u[i,j] - g[i-1,j-1]
\end{aligned} \tag{20}$$

1.3.3 The gradient of the data term. Case 3. $k = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$

For above k, the data term takes the following form:

$$\begin{aligned} |u*k-g|^2 &= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} |g[i,j] - k[0,0]u[i+1,j+1] - k[0,1]u[i+1,j] + k[1,0]u[i,j+1] + k[1,1]u[i,j]|_2^2 \\ &= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} \left| g[i,j] - \frac{1}{2}u[i+1,j+1] - \frac{1}{2}u[i,j] \right|_2^2 \end{aligned} \tag{21}$$

E.g. for $m = 4$ and $n = 4$

$$\begin{aligned} &|u*k-g|^2 = \\ (0,0),(0,1) &= \left|g[0,0] - \frac{1}{2}u[1,1] - \frac{1}{2}u[0,0]\right|_2^2 + \left|g[0,1] - \frac{1}{2}u[1,2] - \frac{1}{2}u[0,1]\right|_2^2 \\ (0,2),(0,3) &= \left|g[0,2] - \frac{1}{2}u[1,3] - \frac{1}{2}u[0,2]\right|_2^2 + \left|g[0,3] - \frac{1}{2}u[1,4] - \frac{1}{2}u[0,3]\right|_2^2 \\ (1,0),(1,1) &= \left|g[1,0] - \frac{1}{2}u[2,1] - \frac{1}{2}u[1,0]\right|_2^2 + \left|g[1,1] - \frac{1}{2}u[2,2] - \frac{1}{2}u[1,1]\right|_2^2 \\ (1,2),(1,3) &= \left|g[1,2] - \frac{1}{2}u[2,3] - \frac{1}{2}u[1,2]\right|_2^2 + \left|g[1,3] - \frac{1}{2}u[2,4] - \frac{1}{2}u[1,3]\right|_2^2 \\ (2,0),(2,1) &= \left|g[2,0] - \frac{1}{2}u[3,1] - \frac{1}{2}u[2,0]\right|_2^2 + \left|g[2,1] - \frac{1}{2}u[3,2] - \frac{1}{2}u[2,1]\right|_2^2 \\ (2,2),(2,3) &= \left|g[2,2] - \frac{1}{2}u[3,3] - \frac{1}{2}u[2,2]\right|_2^2 + \left|g[2,3] - \frac{1}{2}u[3,4] - \frac{1}{2}u[2,3]\right|_2^2 \\ (3,0),(3,1) &= \left|g[3,0] - \frac{1}{2}u[4,1] - \frac{1}{2}u[3,0]\right|_2^2 + \left|g[3,1] - \frac{1}{2}u[4,2] - \frac{1}{2}u[3,1]\right|_2^2 \\ (3,2)+(3,3) &= \left|g[3,2] - \frac{1}{2}u[4,3] - \frac{1}{2}u[3,2]\right|_2^2 + \left|g[3,3] - \frac{1}{2}u[4,4] - \frac{1}{2}u[3,3]\right|_2^2 \end{aligned}$$

indices i/j	0	1	2	3	4
0	1	1	1	1	0
1	1	2	2	2	1
2	1	2	2	2	1
3	1	2	2	2	1
4	0	1	1	1	1

Table 6: The number of terms appearing in data term for each coordinate (for $m = n = 4$)

$$
\begin{aligned}
\forall i \in [1, m-1] \text{ and } \forall j \in [1, n-1] \quad &\longrightarrow \quad \frac{\partial |u * k - g|^2}{\partial u[i,j]} = \frac{\partial \tau[i-1, j-1]}{\partial u[i,j]} + \frac{\partial \tau[i,j]}{\partial u[i,j]} \\
i = 0 \text{ and } j = n \quad &\longrightarrow \quad 0 \\
i = m \text{ and } j = 0 \quad &\longrightarrow \quad 0 \\
\forall i \in [0, m-1] \text{ and } j = 0 \quad &\longrightarrow \quad \frac{\partial |u * k - g|^2}{\partial u[i,j]} = \frac{\partial \tau[i,j]}{\partial u[i,j]} \\
i = 0 \text{ and } \forall j \in [1, n-1] \quad &\longrightarrow \quad \frac{\partial |u * k - g|^2}{\partial u[i,j]} = \frac{\partial \tau[i,j]}{\partial u[i,j]} \\
\forall i \in [1, m-1] \text{ and } j = n \quad &\longrightarrow \quad \frac{\partial |u * k - g|^2}{\partial u[i,j]} = \frac{\partial \tau[i-1, j-1]}{\partial u[i,j]} \\
i = m \text{ and } \forall j \in [1, n] \quad &\longrightarrow \quad \frac{\partial |u * k - g|^2}{\partial u[i,j]} = \frac{\partial \tau[i-1, j-1]}{\partial u[i,j]}
\end{aligned} \tag{22}
$$

where

$$
\tau[i,j] = \left(g[i,j] - \frac{1}{2} u[i+1, j+1] - \frac{1}{2} u[i,j] \right)^2 \tag{23}
$$

The gradient of the first term

$$
\begin{aligned}
\frac{\partial \tau[i-1, j-1]}{\partial u[i,j]} &= \frac{\partial}{\partial u[i,j]} \left(g[i-1, j-1] - \frac{1}{2} u[i,j] - \frac{1}{2} u[i-1, j-1] \right)^2 \\
&= 2 \left(g[i-1, j-1] - \frac{1}{2} u[i,j] - \frac{1}{2} u[i-1, j-1] \right) \frac{\partial}{\partial u[i,j]} \left(g[i-1, j-1] - \frac{1}{2} u[i,j] - \frac{1}{2} u[i-1, j-1] \right) \\
&= - \left(g[i-1, j-1] - \frac{1}{2} u[i,j] - \frac{1}{2} u[i-1, j-1] \right)
\end{aligned}
$$

The gradient of the second term

$$
\begin{aligned}
\frac{\partial \tau[i,j]}{\partial u[i,j]} &= \frac{\partial}{\partial u[i,j]} \left(g[i,j] - \frac{1}{2} u[i+1, j+1] - \frac{1}{2} u[i,j] \right)^2 \\
&= 2 \left(g[i,j] - \frac{1}{2} u[i+1, j+1] - \frac{1}{2} u[i,j] \right) \frac{\partial}{\partial u[i,j]} \left(g[i,j] - \frac{1}{2} u[i+1, j+1] - \frac{1}{2} u[i,j] \right) \\
&= - \left(g[i,j] - \frac{1}{2} u[i+1, j+1] - \frac{1}{2} u[i,j] \right)
\end{aligned}
$$

The full gradient

$$
\begin{aligned}
\frac{\partial |u * k - g|^2}{\partial u[i,j]} &= \frac{\partial \tau[i-1, j-1]}{\partial u[i,j]} + \frac{\partial \tau[i,j]}{\partial u[i,j]} \\
&= - \left(g[i-1, j-1] - \frac{1}{2} u[i,j] - \frac{1}{2} u[i-1, j-1] \right) - \left(g[i,j] - \frac{1}{2} u[i+1, j+1] - \frac{1}{2} u[i,j] \right) \\
&= u[i,j] + \frac{1}{2} u[i-1, j-1] + \frac{1}{2} u[i+1, j+1] - g[i-1, j-1] - g[i,j]
\end{aligned}
$$

$$
\begin{aligned}
\forall i \in [1, m-1] \text{ and } \forall j \in [1, n-1] &\longrightarrow \quad \frac{\partial |u * k - g|^2}{\partial u[i,j]} = \frac{\partial \tau[i-1, j-1]}{\partial u[i,j]} + \frac{\partial \tau[i,j]}{\partial u[i,j]} \\
&= \quad u[i,j] + \frac{1}{2} u[i-1, j-1] + \frac{1}{2} u[i+1, j+1] - g[i-1, j-1] - g[i,j] \\
i = 0 \text{ and } j = n &\longrightarrow \quad 0 \\
i = m \text{ and } j = 0 &\longrightarrow \quad 0 \\
\forall i \in [0, m-1] \text{ and } j = 0 &\longrightarrow \quad \frac{\partial |u * k - g|^2}{\partial u[i,j]} = \frac{\partial \tau[i,j]}{\partial u[i,j]} = \frac{1}{2} u[i,j] + \frac{1}{2} u[i+1, j+1] - g[i,j] \\
i = 0 \text{ and } \forall j \in [1, n-1] &\longrightarrow \quad \frac{\partial |u * k - g|^2}{\partial u[i,j]} = \frac{\partial \tau[i,j]}{\partial u[i,j]} = \frac{1}{2} u[i,j] + \frac{1}{2} u[i+1, j+1] - g[i,j] \\
\forall i \in [1, m-1] \text{ and } j = n &\longrightarrow \quad \frac{\partial |u * k - g|^2}{\partial u[i,j]} = \frac{\partial \tau[i-1, j-1]}{\partial u[i,j]} = \frac{1}{2} u[i,j] + \frac{1}{2} u[i-1, j-1] - g[i-1, j-1] \\
i = m \text{ and } \forall j \in [1, n] &\longrightarrow \quad \frac{\partial |u * k - g|^2}{\partial u[i,j]} = \frac{\partial \tau[i-1, j-1]}{\partial u[i,j]} = \frac{1}{2} u[i,j] + \frac{1}{2} u[i-1, j-1] - g[i-1, j-1]
\end{aligned}
\tag{24}
$$

1.3.4 The gradient of the data term. Case 4. $k = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$

For the above k, the data term takes the following form:

$$
\begin{aligned}
|u * k - g|^2 &= \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} |g[i,j] - k[0,0]u[i+1, j+1] - k[0,1]u[i+1, j] + k[1,0]u[i, j+1] + k[1,1]u[i,j]|_2^2 \\
&= \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \left| g[i,j] - \frac{1}{2} u[i+1, j] - \frac{1}{2} u[i, j+1] \right|_2^2
\end{aligned}
\tag{25}
$$

E.g. for $m = 4$ and $n = 4$

$$
\begin{aligned}
&|u * k - g|^2 = \\
&(0,0),(0,1) = \left| g[0,0] - \frac{1}{2}u[1,0] - \frac{1}{2}u[0,1] \right|_2^2 + \left| g[0,1] - \frac{1}{2}u[1,1] - \frac{1}{2}u[0,2] \right|_2^2 \\
&\qquad = \\
&(0,2),(0,3) = \left| g[0,2] - \frac{1}{2}u[1,2] - \frac{1}{2}u[0,3] \right|_2^2 + \left| g[0,3] - \frac{1}{2}u[1,3] - \frac{1}{2}u[0,4] \right|_2^2 \\
&(1,0),(1,1) = \left| g[1,0] - \frac{1}{2}u[2,0] - \frac{1}{2}u[1,1] \right|_2^2 + \left| g[1,1] - \frac{1}{2}u[2,1] - \frac{1}{2}u[1,2] \right|_2^2 \\
&(1,2),(1,3) = \left| g[1,2] - \frac{1}{2}u[2,2] - \frac{1}{2}u[1,3] \right|_2^2 + \left| g[1,3] - \frac{1}{2}u[2,3] - \frac{1}{2}u[1,4] \right|_2^2 \\
&(2,0),(2,1) = \left| g[2,0] - \frac{1}{2}u[3,0] - \frac{1}{2}u[2,1] \right|_2^2 + \left| g[2,1] - \frac{1}{2}u[3,1] - \frac{1}{2}u[2,2] \right|_2^2 \\
&(2,2),(2,3) = \left| g[2,2] - \frac{1}{2}u[3,2] - \frac{1}{2}u[2,3] \right|_2^2 + \left| g[2,3] - \frac{1}{2}u[3,3] - \frac{1}{2}u[2,4] \right|_2^2 \\
&(3,0),(3,1) = \left| g[3,0] - \frac{1}{2}u[4,0] - \frac{1}{2}u[3,1] \right|_2^2 + \left| g[3,1] - \frac{1}{2}u[4,1] - \frac{1}{2}u[3,2] \right|_2^2 \\
&(3,2),(3,3) = \left| g[3,2] - \frac{1}{2}u[4,2] - \frac{1}{2}u[3,3] \right|_2^2 + \left| g[3,3] - \frac{1}{2}u[4,3] - \frac{1}{2}u[3,4] \right|_2^2
\end{aligned}
$$

indices i/j	0	1	2	3	4
0	0	1	1	1	1
1	1	2	2	2	1
2	1	2	2	2	1
3	1	2	2	2	1
4	1	1	1	1	0

Table 7: The number of terms appearing in data term for each coordinate (for $m = n = 4$)

$$\begin{aligned}
\forall i \in [1, m-1] \text{ and } \forall j \in [1, n-1] \quad &\longrightarrow \quad \frac{\partial |u * k - g|^2}{\partial u[i,j]} = \frac{\partial \tau[i-1,j]}{\partial u[i,j]} + \frac{\partial \tau[i,j-1]}{\partial u[i,j]} \\
i = 0 \text{ and } j = 0 \quad &\longrightarrow \quad 0 \\
i = m \text{ and } j = n \quad &\longrightarrow \quad 0 \\
\forall i \in [1, m] \text{ and } j = 0 \quad &\longrightarrow \quad \frac{\partial |u * k - g|^2}{\partial u[i,j]} = \frac{\partial \tau[i-1,j]}{\partial u[i,j]} \\
\forall i \in [0, m-1] \text{ and } j = n \quad &\longrightarrow \quad \frac{\partial |u * k - g|^2}{\partial u[i,j]} = \frac{\partial \tau[i,j-1]}{\partial u[i,j]} \\
i = 0 \text{ and } \forall j \in [1, n-1] \quad &\longrightarrow \quad \frac{\partial |u * k - g|^2}{\partial u[i,j]} = \frac{\partial \tau[i,j-1]}{\partial u[i,j]} \\
i = m \text{ and } \forall j \in [1, n-1] \quad &\longrightarrow \quad \frac{\partial |u * k - g|^2}{\partial u[i,j]} = \frac{\partial \tau[i-1,j]}{\partial u[i,j]}
\end{aligned} \tag{26}$$

where

$$\tau[i,j] = \left(g[i,j] - \frac{1}{2} u[i+1,j] - \frac{1}{2} u[i,j+1] \right)^2 \tag{27}$$

The gradient of the first term

$$\begin{aligned}
\frac{\partial \tau[i-1,j]}{\partial u[i,j]} &= \frac{\partial}{\partial u[i,j]} \left(g[i-1,j] - \frac{1}{2} u[i,j] - \frac{1}{2} u[i-1,j+1] \right)^2 \\
&= 2\left(g[i-1,j] - \frac{1}{2} u[i,j] - \frac{1}{2} u[i-1,j+1] \right) \frac{\partial}{\partial u[i,j]} \left(g[i-1,j] - \frac{1}{2} u[i,j] - \frac{1}{2} u[i-1,j+1] \right) \\
&= -\left(g[i-1,j] - \frac{1}{2} u[i,j] - \frac{1}{2} u[i-1,j+1] \right)
\end{aligned}$$

The gradient of the second term

$$\begin{aligned}
\frac{\partial \tau[i,j-1]}{\partial u[i,j]} &= \frac{\partial}{\partial u[i,j]} \left(g[i,j-1] - \frac{1}{2} u[i+1,j-1] - \frac{1}{2} u[i,j] \right)^2 \\
&= 2\left(g[i,j-1] - \frac{1}{2} u[i+1,j-1] - \frac{1}{2} u[i,j] \right) \frac{\partial}{\partial u[i,j]} \left(g[i,j-1] - \frac{1}{2} u[i+1,j-1] - \frac{1}{2} u[i,j] \right) \\
&= -\left(g[i,j-1] - \frac{1}{2} u[i+1,j-1] - \frac{1}{2} u[i,j] \right)
\end{aligned}$$

The full gradient

$$\begin{aligned}
\frac{\partial |u * k - g|^2}{\partial u[i,j]} &= \frac{\partial \tau[i-1,j]}{\partial u[i,j]} + \frac{\partial \tau[i,j-1]}{\partial u[i,j]} \\
&= -\left(g[i-1,j] - \frac{1}{2} u[i,j] - \frac{1}{2} u[i-1,j+1] \right) - \left(g[i,j-1] - \frac{1}{2} u[i+1,j-1] - \frac{1}{2} u[i,j] \right) \\
&= u[i,j] + \frac{1}{2} u[i-1,j+1] + \frac{1}{2} u[i+1,j-1] - g[i-1,j] - g[i,j-1]
\end{aligned}$$

The final gradients:

$$
\begin{aligned}
\forall i \in [1, m-1] \text{ and } \forall j \in [1, n-1] \quad &\longrightarrow \quad \frac{\partial |u*k-g|^2}{\partial u[i,j]} = \frac{\partial \tau[i-1,j]}{\partial u[i,j]} + \frac{\partial \tau[i,j-1]}{\partial u[i,j]} \\
&= \quad u[i,j] + \frac{1}{2}u[i-1,j+1] + \frac{1}{2}u[i+1,j-1] - g[i-1,j] - g[i,j-1] \\
i = 0 \text{ and } j = 0 \quad &\longrightarrow \quad 0 \\
i = m \text{ and } j = n \quad &\longrightarrow \quad 0 \\
\forall i \in [1, m] \text{ and } j = 0 \quad &\longrightarrow \quad \frac{\partial |u*k-g|^2}{\partial u[i,j]} = \frac{\partial \tau[i-1,j]}{\partial u[i,j]} = \frac{1}{2}u[i,j] + \frac{1}{2}u[i-1,j+1] - g[i-1,j] \\
\forall i \in [0, m-1] \text{ and } j = n \quad &\longrightarrow \quad \frac{\partial |u*k-g|^2}{\partial u[i,j]} = \frac{\partial \tau[i,j-1]}{\partial u[i,j]} = \frac{1}{2}u[i,j] + \frac{1}{2}u[i+1,j-1] - g[i,j-1] \\
i = 0 \text{ and } \forall j \in [1, n-1] \quad &\longrightarrow \quad \frac{\partial |u*k-g|^2}{\partial u[i,j]} = \frac{\partial \tau[i,j-1]}{\partial u[i,j]} = \frac{1}{2}u[i,j] + \frac{1}{2}u[i+1,j-1] - g[i,j-1] \\
i = m \text{ and } \forall j \in [1, n-1] \quad &\longrightarrow \quad \frac{\partial |u*k-g|^2}{\partial u[i,j]} = \frac{\partial \tau[i-1,j]}{\partial u[i,j]} = \frac{1}{2}u[i,j] + \frac{1}{2}u[i-1,j+1] - g[i-1,j]
\end{aligned} \tag{28}
$$

2 Calculation of the Hessian matrix of the discretized E

2.1 Hessian of the total variation - Gauss Prior

We will consider again the total variation define in Eq.6:

$$
R[u] = |\nabla u|_2^2 \cong \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}(u[i+1,j]-u[i,j])^2+(u[i,j+1]-u[i,j])^2+\sum_{j=0}^{n-1}(u[m,j+1]-u[m,j])^2+\sum_{i=0}^{m-1}(u[i+1,n]-u[i,n])^2
$$

The gradient of the total variation: the gradient for **the first, second and the third term** we given in Eq. 9, 10, 11 respectively. The gradient of the total variation: **the first term**:

$$
\begin{aligned}
\forall i \in [1, m-1] \text{ and } \forall j \in [1, n-1] \quad &\longrightarrow \quad \frac{\partial |\nabla u|_2^2}{\partial u[i,j]} = \frac{\partial \tau[i,j]}{\partial u[i,j]} + \frac{\partial \tau[i-1,j]}{\partial u[i,j]} + \frac{\partial \tau[i,j-1]}{\partial u[i,j]} \\
&= \quad 8u[i,j] - 2u[i+1,j] - 2u[i,j+1] - 2u[i-1,j] - 2u[i,j-1] \\
i = 0 \text{ and } j = 0 \quad &\longrightarrow \quad \frac{\partial |\nabla u|_2^2}{\partial u[i,j]} = \frac{\partial \tau[i,j]}{\partial u[i,j]} = 4u[i,j] - 2u[i+1,j] - 2u[i,j+1] \\
\forall i \in [1 : m-1] \text{ and } j = 0 \quad &\longrightarrow \quad \frac{\partial |\nabla u|_2^2}{\partial u[i,j]} = \frac{\partial \tau[i,j]}{\partial u[i,j]} + \frac{\partial \tau[i-1,j]}{\partial u[i,j]} = 6u[i,j] - 2u[i+1,j] - 2u[i,j+1] - 2u[i-1,j] \\
\forall i \in [0, m-1] \text{ and } j = n \quad &\longrightarrow \quad \frac{\partial |\nabla u|_2^2}{\partial u[i,j]} = \frac{\partial \tau[i,j-1]}{\partial u[i,j]} = 2u[i,j] - 2u[i,j-1] \\
i = m \text{ and } \forall j \in [0, n-1] \quad &\longrightarrow \quad \frac{\partial |\nabla u|_2^2}{\partial u[i,j]} = \frac{\partial \tau[i-1,j]}{\partial u[i,j]} = 2u[i,j] - 2u[i-1,j] \\
i = 0 \text{ and } \forall j \in [1, n-1] \quad &\longrightarrow \quad \frac{\partial |\nabla u|_2^2}{\partial u[i,j]} = \frac{\partial \tau[i,j]}{\partial u[i,j]} + \frac{\partial \tau[i,j-1]}{\partial u[i,j]} = 6u[i,j] - 2u[i+1,j] - 2u[i,j+1] - 2u[i,j-1] \\
i = m \text{ and } j = n \quad &\longrightarrow \quad 0
\end{aligned}
$$

The second derivatives of the first term:

case 0: $\forall i \in [1, m-1]$ and $\forall j \in [1, n-1]$

$$\frac{\partial^2 |\nabla u|_2^2}{\partial u[i,j]^2} = \frac{\partial}{\partial u[i,j]}\Big(8u[i,j] - 2u[i+1,j] - 2u[i,j+1] - 2u[i-1,j] - 2u[i,j-1]\Big) = 8$$

$$\frac{\partial^2 |\nabla u|_2^2}{\partial u[i,j]\partial u[i,j-1]} = \frac{\partial}{\partial u[i,j-1]}\Big(8u[i,j] - 2u[i+1,j] - 2u[i,j+1] - 2u[i-1,j] - 2u[i,j-1]\Big) = -2$$

$$\frac{\partial^2 |\nabla u|_2^2}{\partial u[i,j]\partial u[i,j+1]} = \frac{\partial}{\partial u[i,j+1]}\Big(8u[i,j] - 2u[i+1,j] - 2u[i,j+1] - 2u[i-1,j] - 2u[i,j-1]\Big) = -2$$

$$\frac{\partial^2 |\nabla u|_2^2}{\partial u[i,j]\partial u[i+1,j]} = \frac{\partial}{\partial u[i+1,j]}\Big(8u[i,j] - 2u[i+1,j] - 2u[i,j+1] - 2u[i-1,j] - 2u[i,j-1]\Big) = -2$$

$$\frac{\partial^2 |\nabla u|_2^2}{\partial u[i,j]\partial u[i-1,j]} = \frac{\partial}{\partial u[i-1,j]}\Big(8u[i,j] - 2u[i+1,j] - 2u[i,j+1] - 2u[i-1,j] - 2u[i,j-1]\Big) = -2$$

$$\frac{\partial^2 |\nabla u|_2^2}{\partial u[i,j]\partial u[i-1,j-1]} = \frac{\partial^2 |\nabla u|_2^2}{\partial u[i-1,j+1]\partial u[i-1,j]} = \frac{\partial^2 |\nabla u|_2^2}{\partial u[i,j]\partial u[i+1,j-1]} = \frac{\partial^2 |\nabla u|_2^2}{\partial u[i,j]\partial u[i+1,j+1]} = 0$$

case 1: $i = 0$ and $j = 0$

$$\frac{\partial^2 |\nabla u|_2^2}{\partial u[i,j]^2} = \frac{\partial}{\partial u[i,j]}\Big(4u[i,j] - 2u[i+1,j] - 2u[i,j+1]\Big) = 4$$

$$\frac{\partial^2 |\nabla u|_2^2}{\partial u[i,j]\partial u[i+1,j]} = \frac{\partial}{\partial u[i+1,j]}\Big(4u[i,j] - 2u[i+1,j] - 2u[i,j+1]\Big) = -2$$

$$\frac{\partial^2 |\nabla u|_2^2}{\partial u[i,j]\partial u[i,j+1]} = \frac{\partial}{\partial u[i,j+1]}\Big(4u[i,j] - 2u[i+1,j] - 2u[i,j+1]\Big) = -2$$

for all other second derivatives are zero

case 2: $\forall i \in [1 : m-1]$ and $j = 0$

$$\frac{\partial^2 |\nabla u|_2^2}{\partial u[i,j]^2} = \frac{\partial}{\partial u[i,j]}\Big(6u[i,j] - 2u[i+1,j] - 2u[i,j+1] - 2u[i-1,j]\Big) = 6$$

$$\frac{\partial^2 |\nabla u|_2^2}{\partial u[i,j]\partial u[i+1,j]} = \frac{\partial}{\partial u[i+1,j]}\Big(6u[i,j] - 2u[i+1,j] - 2u[i,j+1] - 2u[i-1,j]\Big) = -2$$

$$\frac{\partial^2 |\nabla u|_2^2}{\partial u[i,j]\partial u[i,j+1]} = \frac{\partial}{\partial u[i,j+1]}\Big(6u[i,j] - 2u[i+1,j] - 2u[i,j+1] - 2u[i-1,j]\Big) = -2$$

$$\frac{\partial^2 |\nabla u|_2^2}{\partial u[i,j]\partial u[i-1,j]} = \frac{\partial}{\partial u[i-1,j]}\Big(6u[i,j] - 2u[i+1,j] - 2u[i,j+1] - 2u[i-1,j]\Big) = -2$$

case 3: $\forall i \in [0, m-1]$ and $j = n$

$$\frac{\partial^2 |\nabla u|_2^2}{\partial u[i,j]^2} = \frac{\partial}{\partial u[i,j]}\Big(2u[i,j] - 2u[i,j-1]]\Big) = 2$$

$$\frac{\partial^2 |\nabla u|_2^2}{\partial u[i,j]\partial u[i,j-1]} = \frac{\partial}{\partial u[i,j-1]}\Big(2u[i,j] - 2u[i,j-1]]\Big) = -2$$

for all other cases the second derivative is zero

case 4: $i = m$ and $\forall j \in [0, n-1]$

$$\frac{\partial^2 |\nabla u|_2^2}{\partial u[i,j]^2} = \frac{\partial}{\partial u[i,j]}\Big(2u[i,j] - 2u[i-1,j]\Big) = 2$$

$$\frac{\partial^2|\nabla u|_2^2}{\partial u[i,j]\partial u[i-1,j]} = \frac{\partial}{\partial u[i-1,j]}\Big(2u[i,j] - 2u[i-1,j]\Big) = -2$$

for all other cases the second derivative is zero

case 5**:** $i = 0$ and $\forall j \in [1, n-1]$

$$\frac{\partial^2|\nabla u|_2^2}{\partial u[i,j]^2} = \frac{\partial}{\partial u[i,j]}\Big(6u[i,j] - 2u[i+1,j] - 2u[i,j+1] - 2u[i,j-1]\Big) = 6$$

$$\frac{\partial^2|\nabla u|_2^2}{\partial u[i,j]\partial u[i+1,j]} = \frac{\partial}{\partial u[i+1,j]}\Big(6u[i,j] - 2u[i+1,j] - 2u[i,j+1] - 2u[i,j-1]\Big) = -2$$

$$\frac{\partial^2|\nabla u|_2^2}{\partial u[i,j]\partial u[i,j+1]} = \frac{\partial}{\partial u[i,j+1]}\Big(6u[i,j] - 2u[i+1,j] - 2u[i,j+1] - 2u[i,j-1]\Big) = -2$$

$$\frac{\partial^2|\nabla u|_2^2}{\partial u[i,j]\partial u[i,j-1]} = \frac{\partial}{\partial u[i,j+1]}\Big(6u[i,j] - 2u[i+1,j] - 2u[i,j-1] - 2u[i,j-1]\Big) = -2$$

The gradient of the total variation: **the second term**

$$i = m \text{ and } \forall j \in [1, n-1] \quad \longrightarrow \quad \frac{\partial|\nabla u|_2^2}{\partial u[i,j]} = \frac{\partial\tau[i,j]}{\partial[i,j]} + \frac{\partial\tau[i,j-1]}{\partial u[i,j]} = 2(u[i,j] - u[i,j+1]) + 2(u[i,j] - u[i,j-1]])$$

$$\longrightarrow \quad = 4u[i,j] - 2u[i,j-1] - 2u[i,j+1]$$

$$i = m \text{ and } j = 0 \quad \longrightarrow \quad \frac{\partial|\nabla u|_2^2}{\partial u[i,j]} = 2u[i,j] - 2u[i,j+1]$$

$$i = m \text{ and } j = n \quad \longrightarrow \quad \frac{\partial|\nabla u|_2^2}{\partial u[i,j]} = 2u[i,j] - 2u[i,j-1]$$

The second derivatives of the second term:

case 0: $i = m$ and $\forall j \in [1, n-1]$

$$\frac{\partial^2|\nabla u|_2^2}{\partial u[i,j]^2} = \frac{\partial}{\partial u[i,j]}\big(4u[i,j] - 2u[i,j-1] - 2u[i,j+1]\big) = 4$$

$$\frac{\partial^2|\nabla u|_2^2}{\partial u[i,j]\partial u[i,j+1]} = \frac{\partial}{\partial u[i,j+1]}\big(4u[i,j] - 2u[i,j-1] - 2u[i,j+1]\big) = -2$$

$$\frac{\partial^2|\nabla u|_2^2}{\partial u[i,j]\partial u[i,j-1]} = \frac{\partial}{\partial u[i,j-1]}\big(4u[i,j] - 2u[i,j-1] - 2u[i,j+1]\big) = -2$$

case 1: $i = m$ and $j = 0$

$$\frac{\partial^2|\nabla u|_2^2}{\partial u[i,j]^2} = \frac{\partial}{\partial u[i,j]}\big(2u[i,j] - 2u[i,j+1]]\big) = 2$$

$$\frac{\partial^2|\nabla u|_2^2}{\partial u[i,j]\partial u[i,j+1]} = \frac{\partial}{\partial u[i,j+1]}\big(2u[i,j] - 2u[i,j+1]\big) = -2$$

case 2: $i = m$ and $j = n$

$$\frac{\partial^2|\nabla u|_2^2}{\partial u[i,j]^2} = \frac{\partial}{\partial u[i,j]}\big(2u[i,j] - 2u[i,j-1\big) = 2$$

$$\frac{\partial^2|\nabla u|_2^2}{\partial u[i,j]\partial u[i,j-1]} = \frac{\partial}{\partial u[i,j-1]}\big(2u[i,j] - 2u[i,j-1]\big) = 2$$

The gradient of the total variation: **the third term**

$$\forall i \in [1, m-1] \text{ and } j = n \quad \longrightarrow \quad \frac{\partial |\nabla u|_2^2}{\partial u[i,j]} = \frac{\partial \tau[i,j]}{\partial [i,j]} + \frac{\partial \tau[i,j-1]}{\partial u[i,j]} = 2(u[i,j] - u[i+1,j]) + 2(u[i,j] - u[i-1,j])$$

$$\longrightarrow \quad = 4u[i,j] - 2[i-1,j] - 2u[i+1,j]$$

$$i = 0 \text{ and } j = n \quad \longrightarrow \quad \frac{\partial |\nabla u|_2^2}{\partial u[i,j]} = 2(u[i,j] - u[i+1,j])$$

$$i = m \text{ and } j = n \quad \longrightarrow \quad \frac{\partial |\nabla u|_2^2}{\partial u[i,j]} = 2(u[i,j] - u[i-1,j])$$

The second derivatives of the third term:

case 0: $\forall i \in [1, m-1]$ and $j = n$

$$\frac{\partial^2 |\nabla u|_2^2}{\partial u[i,j]^2} = \frac{\partial}{\partial u[i,j]} \big(4u[i,j] - 2[i-1,j] - 2u[i+1,j]\big) = 4$$

$$\frac{\partial^2 |\nabla u|_2^2}{\partial u[i,j] \partial u[i-1,j]} = \frac{\partial}{\partial u[i-1,j]} \big(4u[i,j] - 2[i-1,j] - 2u[i+1,j]\big) = -2$$

$$\frac{\partial^2 |\nabla u|_2^2}{\partial u[i,j] \partial u[i+1,j]} = \frac{\partial}{\partial u[i+1,j]} \big(4u[i,j] - 2[i-1,j] - 2u[i+1,j]\big) = -2$$

case 1: $i = 0$ and $j = n$

$$\frac{\partial^2 |\nabla u|_2^2}{\partial u[i,j]^2} = \frac{\partial}{\partial u[i,j]} \big(2(u[i,j] - u[i+1,j])\big) = 2$$

$$\frac{\partial^2 |\nabla u|_2^2}{\partial u[i,j] \partial u[i+1,j]} = \frac{\partial}{\partial u[i+1,j]} \big(2(u[i,j] - u[i+1,j])\big) = -2$$

case 2: $i = m$ and $j = n$

$$\frac{\partial^2 |\nabla u|_2^2}{\partial u[i,j]^2} = \frac{\partial}{\partial u[i,j]} \big(2u[i,j] - 2u[i-1,j]\big) = 2$$

$$\frac{\partial^2 |\nabla u|_2^2}{\partial u[i,j] \partial u[i,j-1]} = \frac{\partial}{\partial u[i,j-1]} \big(2u[i,j] - 2u[i-1,j]\big) = -2$$

2.2 Hessian of the data term

2.2.1 Hessian of the data term. Case 1. $k = \begin{bmatrix} 1/2 & 1/2 \\ 0 & 0 \end{bmatrix}$

The gradient (from eq. 16):

$$\forall i \in [1, m] \text{ and } \forall j \in [1, n-1] \quad \longrightarrow \quad \frac{\partial |u * k - g|^2}{\partial u[i,j]} = u[i,j] + \frac{1}{2} u[i,j-1] + \frac{1}{2} u[i,j+1] - g[i-1,j-1] - g[i-1,j]$$

$$i = 0 \text{ and } \forall j \in [0, n] \quad \longrightarrow \quad 0$$

$$\forall i \in [1, m] \text{ and } j = 0 \quad \longrightarrow \quad \frac{\partial |u * k - g|^2}{\partial u[i,j]} = \frac{\partial \tau[i-1,j]}{\partial u[i,j]} = \frac{1}{2} u[i,j+1] + \frac{1}{2} u[i,j] - g[i-1,j]$$

$$\forall i \in [1, m] \text{ and } j = n \quad \longrightarrow \quad \frac{\partial |u * k - g|^2}{\partial u[i,j]} = \frac{\partial \tau[i-1,j-1]}{\partial u[i,j]} = \frac{1}{2} u[i,j] + \frac{1}{2} u[i,j-1] - g[i-1,j-1]$$

The second derivatives: Case 0**:** $\forall i \in [1,m]$ and $\forall j \in [1,n-1]$

$$\frac{\partial^2 |u*k-g|^2}{\partial u[i,j]^2} = \frac{\partial}{\partial u[i,j]}\left(u[i,j] + \frac{1}{2}u[i,j-1] + \frac{1}{2}u[i,j+1] - g[i-1,j-1] - g[i-1,j]\right) = 1$$

$$\frac{\partial^2 |u*k-g|^2}{\partial u[i,j]\partial u[i,j+1]} = \frac{\partial}{\partial u[i,j+1]}\left(u[i,j] + \frac{1}{2}u[i,j-1] + \frac{1}{2}u[i,j+1] - g[i-1,j-1] - g[i-1,j]\right) = \frac{1}{2}$$

$$\frac{\partial^2 |u*k-g|^2}{\partial u[i,j]\partial u[i,j-1]} = \frac{\partial}{\partial u[i,j-1]}\left(u[i,j] + \frac{1}{2}u[i,j-1] + \frac{1}{2}u[i,j+1] - g[i-1,j-1] - g[i-1,j]\right) = \frac{1}{2}$$

$$\frac{\partial^2 |u*k-g|^2}{\partial u[i,j]\partial u[i+1,j]} = \frac{\partial^2 |u*k-g|^2}{\partial u[i,j]\partial u[i-1,j]} = 0$$

$$\frac{\partial^2 |u*k-g|^2}{\partial u[i,j]\partial u[i-1,j-1]} = \frac{\partial^2 |u*k-g|^2}{\partial u[i-1,j+1]\partial u[i-1,j]} = \frac{\partial^2 |u*k-g|^2}{\partial u[i,j]\partial u[i+1,j-1]} = \frac{\partial^2 |u*k-g|^2}{\partial u[i,j]\partial u[i+1,j+1]} = 0$$

Case 1**:** $i = 0$ and $\forall j \in [0,n]$

All second derivatives are zero.

Case 2**:** $\forall i \in [1,m]$ and $j = 0$

$$\frac{\partial^2 |u*k-g|^2}{\partial u[i,j]^2} = \frac{\partial}{\partial u[i,j]}\left(\frac{1}{2}u[i,j+1] + \frac{1}{2}u[i,j] - g[i-1,j]\right) = \frac{1}{2}$$

$$\frac{\partial^2 |u*k-g|^2}{\partial u[i,j]\partial u[i,j+1]} = \frac{\partial}{\partial u[i,j]}\left(\frac{1}{2}u[i,j+1] + \frac{1}{2}u[i,j] - g[i-1,j]\right) = \frac{1}{2}$$

Case 3**:** $\forall i \in [1,n]$ and $j = m$

$$\frac{\partial^2 |u*k-g|^2}{\partial u[i,j]^2} = \frac{\partial}{\partial u[i,j]}\left(\frac{1}{2}u[i,j] + \frac{1}{2}u[i,j-1] - g[i-1,j-1]\right) = \frac{1}{2}$$

$$\frac{\partial^2 |u*k-g|^2}{\partial u[i,j]\partial u[i,j-1]} = \frac{\partial}{\partial u[i,j-1]}\left(\frac{1}{2}u[i,j] + \frac{1}{2}u[i,j-1] - g[i-1,j-1]\right) = \frac{1}{2}$$

2.2.2 Hessian of the data term. Case 2. $k = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 0 \end{bmatrix}$

The gradients (from Eq. 20):

$$\forall i \in [1,m-1] \text{ and } \forall j \in [1,n] \quad \longrightarrow \quad \frac{\partial |u*k-g|^2}{\partial u[i,j]} = \frac{\partial \tau[i-1,j-1]}{\partial u[i,j]} + \frac{\partial \tau[i,j-1]}{\partial u[i,j]}$$

$$\longrightarrow \quad u[i,j] + \frac{1}{2}u[i-1,j] + \frac{1}{2}u[i+1,j] - g[i-1,j-1] - g[i,j-1]$$

$$\forall i \in [0,m] \text{ and } j = 0 \quad \longrightarrow \quad 0$$

$$i = 0 \text{ and } \forall j \in [1,n] \quad \longrightarrow \quad \frac{\partial |u*k-g|^2}{\partial u[i,j]} = \frac{\partial \tau[i,j-1]}{\partial u[i,j]} = \frac{1}{2}u[i,j] + \frac{1}{2}u[i+1,j] - g[i,j-1]$$

$$i = m \text{ and } \forall j \in [1,n] \quad \longrightarrow \quad \frac{\partial |u*k-g|^2}{\partial u[i,j]} = \frac{\partial \tau[i-1,j-1]}{\partial u[i,j]} = \frac{1}{2}u[i-1,j] + \frac{1}{2}u[i,j] - g[i-1,j-1]$$

Let us compute the second derivatives for each case:

Case 0**:** $\forall i \in [1, m-1]$ and $\forall j \in [1, n]$

$$\frac{\partial^2 |u * k - g|^2}{\partial u[i,j]^2} = \frac{\partial}{\partial u[i,j]}\Big(u[i,j] + \frac{1}{2}u[i-1,j] + \frac{1}{2}u[i+1,j] - g[i-1,j-1] - g[i,j-1]\Big) = 1$$

$$\frac{\partial^2 |u * k - g|^2}{\partial u[i,j]\partial u[i-1,j]} = \frac{\partial}{\partial u[i,j]}\Big(u[i,j] + \frac{1}{2}u[i-1,j] + \frac{1}{2}u[i+1,j] - g[i-1,j-1] - g[i,j-1]\Big) = \frac{1}{2}$$

$$\frac{\partial^2 |u * k - g|^2}{\partial u[i,j]\partial u[i+1,j]} = \frac{\partial}{\partial u[i,j]}\Big(u[i,j] + \frac{1}{2}u[i+1,j] + \frac{1}{2}u[i+1,j] - g[i-1,j-1] - g[i,j-1]\Big) = \frac{1}{2}$$

Case 1**:** $i = 0$ and $\forall j \in [1, n]$

$$\frac{\partial^2 |u * k - g|^2}{\partial u[i,j]^2} = \frac{\partial}{\partial u[i,j]}\Big(\frac{1}{2}u[i,j] + \frac{1}{2}u[i+1,j] - g[i,j-1]\Big) = \frac{1}{2}$$

$$\frac{\partial^2 |u * k - g|^2}{\partial u[i,j]\partial [i+1,j]} = \frac{\partial}{\partial u[i,j]}\Big(\frac{1}{2}u[i,j] + \frac{1}{2}u[i+1,j] - g[i,j-1]\Big) = \frac{1}{2}$$

other second derivatives are zero. Case 2**:** $i = m$ and $\forall j \in [1, n]$

$$\frac{\partial^2 |u * k - g|^2}{\partial u[i,j]^2} = \frac{\partial}{\partial u[i,j]}\Big(\frac{1}{2}u[i-1,j] + \frac{1}{2}u[i,j] - g[i-1,j-1]\Big) = \frac{1}{2}$$

$$\frac{\partial^2 |u * k - g|^2}{\partial u[i,j]\partial [i-1,j]} = \frac{\partial}{\partial u[i-1,j]}\Big(\frac{1}{2}u[i-1,j] + \frac{1}{2}u[i,j] - g[i-1,j-1]\Big) = \frac{1}{2}$$

other second derivatives are zero.

2.2.3 Hessian of the data term. Case 3. $k = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$

The gradients (from Eq. 24):

$$\forall i \in [1, m-1] \text{ and } \forall j \in [1, n-1] \quad \longrightarrow \quad \frac{\partial |u * k - g|^2}{\partial u[i,j]} = \frac{\partial \tau[i-1,j-1]}{\partial u[i,j]} + \frac{\partial \tau[i,j]}{\partial u[i,j]}$$

$$= \quad u[i,j] + \frac{1}{2}u[i-1,j-1] + \frac{1}{2}u[i+1,j+1] - g[i-1,j-1] - g[i,j]$$

$$i = 0 \text{ and } j = n \quad \longrightarrow \quad 0$$

$$i = m \text{ and } j = 0 \quad \longrightarrow \quad 0$$

$$\forall i \in [0, m-1] \text{ and } j = 0 \quad \longrightarrow \quad \frac{\partial |u * k - g|^2}{\partial u[i,j]} = \frac{\partial \tau[i,j]}{\partial u[i,j]} = \frac{1}{2}u[i,j] + \frac{1}{2}u[i+1,j+1] - g[i,j]$$

$$i = 0 \text{ and } \forall j \in [1, n-1] \quad \longrightarrow \quad \frac{\partial |u * k - g|^2}{\partial u[i,j]} = \frac{\partial \tau[i,j]}{\partial u[i,j]} = \frac{1}{2}u[i,j] + \frac{1}{2}u[i+1,j+1] - g[i,j]$$

$$\forall i \in [1, m-1] \text{ and } j = n \quad \longrightarrow \quad \frac{\partial |u * k - g|^2}{\partial u[i,j]} = \frac{\partial \tau[i-1,j-1]}{\partial u[i,j]} = \frac{1}{2}u[i,j] + \frac{1}{2}u[i-1,j-1] - g[i-1,j-1]$$

$$i = m \text{ and } \forall j \in [1, n] \quad \longrightarrow \quad \frac{\partial |u * k - g|^2}{\partial u[i,j]} = \frac{\partial \tau[i-1,j-1]}{\partial u[i,j]} = \frac{1}{2}u[i,j] + \frac{1}{2}u[i-1,j-1] - g[i-1,j-1]$$

case 0**:** $\forall i \in [1, m-1]$ and $\forall j \in [1, n-1]$

$$\frac{\partial^2 |u * k - g|^2}{\partial u[i,j]^2} = \frac{\partial}{\partial u[i,j]}\Big(u[i,j] + \frac{1}{2}u[i-1,j-1] + \frac{1}{2}u[i+1,j+1] - g[i-1,j-1] - g[i,j]\Big) = 1$$

$$\frac{\partial^2|u*k-g|^2}{\partial u[i,j]\partial u[i-1,j-1]} = \frac{\partial}{\partial u[i-1,j-1]}\left(u[i,j]+\frac{1}{2}u[i-1,j-1]+\frac{1}{2}u[i+1,j+1]-g[i-1,j-1]-g[i,j]\right) = \frac{1}{2}$$

$$\frac{\partial^2|u*k-g|^2}{\partial u[i,j]\partial u[i+1,j+1]} = \frac{\partial}{\partial u[i+1,j+1]}\left(u[i,j]+\frac{1}{2}u[i-1,j-1]+\frac{1}{2}u[i+1,j+1]-g[i-1,j-1]-g[i,j]\right) = \frac{1}{2}$$

case 1: $\forall i \in [0, m-1]$ and $j = 0$

$$\frac{\partial^2|u*k-g|^2}{\partial u[i,j]^2} = \frac{\partial}{\partial u[i,j]}\left(\frac{1}{2}u[i,j]+\frac{1}{2}u[i+1,j+1]-g[i,j]\right) = \frac{1}{2}$$

$$\frac{\partial^2|u*k-g|^2}{\partial u[i,j]\partial u[i+1,j+1]} = \frac{\partial}{\partial u[i+1,j+1]}\left(\frac{1}{2}u[i,j]+\frac{1}{2}u[i+1,j+1]-g[i,j]\right) = \frac{1}{2}$$

case 2: $i = 0$ and $\forall j \in [1, n-1]$

$$\frac{\partial^2|u*k-g|^2}{\partial u[i,j]^2} = \frac{\partial}{\partial u[i,j]}\left(\frac{1}{2}u[i,j]+\frac{1}{2}u[i+1,j+1]-g[i,j]\right) = \frac{1}{2}$$

$$\frac{\partial^2|u*k-g|^2}{\partial u[i,j]\partial u[i+1,j+1]} = \frac{\partial}{\partial u[i+1,j+1]}\left(\frac{1}{2}u[i,j]+\frac{1}{2}u[i+1,j+1]-g[i,j]\right) = \frac{1}{2}$$

case 3: $\forall i \in [1, m-1]$ and $j = n$

$$\frac{\partial^2|u*k-g|^2}{\partial u[i,j]^2} = \frac{\partial}{\partial u[i,j]}\left(\frac{1}{2}u[i,j]+\frac{1}{2}u[i-1,j-1]-g[i-1,j-1]\right) = \frac{1}{2}$$

$$\frac{\partial^2|u*k-g|^2}{\partial u[i,j]\partial u[i-1,j-1]} = \frac{\partial}{\partial u[i-1,j-1]}\left(\frac{1}{2}u[i,j]+\frac{1}{2}u[i-1,j-1]-g[i-1,j-1]\right) = \frac{1}{2}$$

case 4: $i = m$ and $\forall j \in [1, n]$

$$\frac{\partial^2|u*k-g|^2}{\partial u[i,j]^2} = \frac{\partial}{\partial u[i,j]}\left(\frac{1}{2}u[i,j]+\frac{1}{2}u[i-1,j-1]-g[i-1,j-1]\right) = \frac{1}{2}$$

$$\frac{\partial^2|u*k-g|^2}{\partial u[i,j]\partial u[i-1,j-1]} = \frac{\partial}{\partial u[i-1,j-1]}\left(\frac{1}{2}u[i,j]+\frac{1}{2}u[i-1,j-1]-g[i-1,j-1]\right) = \frac{1}{2}$$

2.2.4 Hessian of the data term. Case 4. $k = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$

The gradients (from Eq. 2.2.4):

$$\forall i \in [1, m-1] \text{ and } \forall j \in [1, n-1] \quad \longrightarrow \quad \frac{\partial |u*k-g|^2}{\partial u[i,j]} = \frac{\partial \tau[i-1,j]}{\partial u[i,j]} + \frac{\partial \tau[i,j-1]}{\partial u[i,j]}$$

$$= \quad u[i,j] + \frac{1}{2}u[i-1,j+1] + \frac{1}{2}u[i+1,j-1] - g[i-1,j] - g[i,j-1]$$

$$i = 0 \text{ and } j = 0 \quad \longrightarrow \quad 0$$

$$i = m \text{ and } j = n \quad \longrightarrow \quad 0$$

$$\forall i \in [1, m] \text{ and } j = 0 \quad \longrightarrow \quad \frac{\partial |u*k-g|^2}{\partial u[i,j]} = \frac{\partial \tau[i-1,j]}{\partial u[i,j]} = \frac{1}{2}u[i,j] + \frac{1}{2}u[i-1,j+1] - g[i-1,j]$$

$$\forall i \in [0, m-1] \text{ and } j = n \quad \longrightarrow \quad \frac{\partial |u*k-g|^2}{\partial u[i,j]} = \frac{\partial \tau[i,j-1]}{\partial u[i,j]} = \frac{1}{2}u[i,j] + \frac{1}{2}u[i+1,j-1] - g[i,j-1]$$

$$i = 0 \text{ and } \forall j \in [1, n-1] \quad \longrightarrow \quad \frac{\partial |u*k-g|^2}{\partial u[i,j]} = \frac{\partial \tau[i,j-1]}{\partial u[i,j]} = \frac{1}{2}u[i,j] + \frac{1}{2}u[i+1,j-1] - g[i,j-1]$$

$$i = m \text{ and } \forall j \in [1, n-1] \quad \longrightarrow \quad \frac{\partial |u*k-g|^2}{\partial u[i,j]} = \frac{\partial \tau[i-1,j]}{\partial u[i,j]} = \frac{1}{2}u[i,j] + \frac{1}{2}u[i-1,j+1] - g[i-1,j]$$

case 0**:** $\forall i \in [1, m-1]$ and $\forall j \in [1, n-1]$

$$\frac{\partial^2 |u*k-g|^2}{\partial u[i,j]^2} = \frac{\partial}{\partial u[i,j]}\left(u[i,j] + \frac{1}{2}u[i-1,j+1] + \frac{1}{2}u[i+1,j-1] - g[i-1,j] - g[i,j-1]\right) = 1$$

$$\frac{\partial^2 |u*k-g|^2}{\partial u[i,j]\partial u[i-1,j+1]} = \frac{\partial}{\partial u[i-1,j+1]}\left(u[i,j] + \frac{1}{2}u[i-1,j+1] + \frac{1}{2}u[i+1,j-1] - g[i-1,j] - g[i,j-1]\right) = \frac{1}{2}$$

$$\frac{\partial^2 |u*k-g|^2}{\partial u[i,j]\partial u[i+1,j-1]} = \frac{\partial}{\partial u[i+1,j-1]}\left(u[i,j] + \frac{1}{2}u[i-1,j+1] + \frac{1}{2}u[i+1,j-1] - g[i-1,j] - g[i,j-1]\right) = \frac{1}{2}$$

case 1**:** $\forall i \in [1, m]$ and $j = 0$

$$\frac{\partial^2 |u*k-g|^2}{\partial u[i,j]^2} = \frac{\partial}{\partial u[i,j]}\left(\frac{1}{2}u[i,j] + \frac{1}{2}u[i-1,j+1] - g[i-1,j]\right) = 1$$

$$\frac{\partial^2 |u*k-g|^2}{\partial u[i,j]\partial u[i-1,j+1]} = \frac{\partial}{\partial u[i-1,j+1]}\left(\frac{1}{2}u[i,j] + \frac{1}{2}u[i-1,j+1] - g[i-1,j]\right) = \frac{1}{2}$$

case 2**:** $\forall i \in [0, m-1]$ and $j = n$

$$\frac{\partial^2 |u*k-g|^2}{\partial u[i,j]^2} = \frac{\partial}{\partial u[i,j]}\left(\frac{1}{2}u[i,j] + \frac{1}{2}u[i+1,j-1] - g[i,j-1]\right) = 1$$

$$\frac{\partial^2 |u*k-g|^2}{\partial u[i,j]\partial u[i+1,j-1]} = \frac{\partial}{\partial u[i+1,j-1]}\left(\frac{1}{2}u[i,j] + \frac{1}{2}u[i+1,j-1] - g[i,j-1]\right) = \frac{1}{2}$$

case 3**:** $i = 0$ and $\forall j \in [1, n-1]$

$$\frac{\partial^2 |u*k-g|^2}{\partial u[i,j]^2} = \frac{\partial}{\partial u[i,j]}\left(\frac{1}{2}u[i,j] + \frac{1}{2}u[i+1,j-1] - g[i,j-1]\right) = 1$$

$$\frac{\partial^2 |u * k - g|^2}{\partial u[i,j] \partial u[i+1,j-1]} = \frac{\partial}{\partial u[i+1,j-1]} \left(\frac{1}{2} u[i,j] + \frac{1}{2} u[i+1,j-1] - g[i,j-1] \right) = \frac{1}{2}$$

case 4**:** $i = m$ and $\forall j \in [1, n-1]$

$$\frac{\partial^2 |u * k - g|^2}{\partial u[i,j]^2} = \frac{\partial}{\partial u[i,j]} \left(\frac{1}{2} u[i,j] + \frac{1}{2} u[i-1,j+1] - g[i-1,j] \right) = 1$$

$$\frac{\partial^2 |u * k - g|^2}{\partial u[i,j] \partial u[i-1,j+1]} = \frac{\partial}{\partial u[i-1,j+1]} \left(\frac{1}{2} u[i,j] + \frac{1}{2} u[i-1,j+1] - g[i-1,j] \right) = \frac{1}{2}$$