Review Assignment 1

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1 Probability theory review

Solve each of the following problems and show all the steps of your working.

1. Show that the covariance matrix is always symmetric and positive semidefinite.

$$\begin{aligned}\sum_{ij} = \operatorname{cov}(x_i, x_j) &= E[x_i x_j] - E[x_i]E[x_j] \\ &= E[x_j x_i] - E[x_j]E[x_i]\end{aligned}$$

E is symmetric

2. $X \in \mathbb{R}^n$ and $Y \in \mathbb{R}^m$ are independent random variables. Their expectations and covariances are $E[X] = 0$, $Cov[X] = I$, $E[Y] = \mu$ and $Cov[Y] = \sigma I$, where I is the identity matrix of the appropriate size and s is a scalar. What is the expectation and covariance of the random variable $Z = AX + Y$, where $A \in \mathbb{R}^{m \times n}$?

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3. Thomas and Viktor are friends. It is Friday night and Thomas does not have a phone. Viktor knows that there is a 2/3 probability that Thomas goes to the party to downtown. There are 5 pubs in downtown and there is an equal probability of Thomas going to any of them if he goes to the party. Viktor already looked for Thomas in 4 of the bars. What is the probability of Viktor ending Thomas in the last bar?

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4. Derive the mean for the Beta Distribution, which is defined as

$$\text{Beta}(x|a, b) = \frac{1}{B(a, b)} a^{-1}(1 - x)b^{-1} \tag{1}$$

where $B(a,b), G(a)$ are Beta and Gamma functions respectively:

$$B(a,b) \triangleq \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \tag{2}$$

and

$$\Gamma(x) \triangleq \int_0^\infty u^{x-1} e^{-u} du \tag{3}$$

Hint: Use integration by parts.

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5. Let $A \in \mathbb{R}^{n\times n}$ be a positive definite square matrix, $b \in \mathbb{R}^n$, and c be a scalar. Prove that

$$\int_{x\in\mathbb{R}^n} e^{-\frac{1}{2}x^\intercal Ax - x^\intercal b - c} dx = \frac{(2\pi)^{n/2}|A|^{-1/2}}{e^{c-\frac{1}{2}b^\intercal A^{-1}b}}$$

Hint: Use the fact that the integral of the Gaussian probability density function of a random variable with mean μ and covariance $\sum$ is 1.

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6. From the definition of conditional probability of multiple random variables, show that

$$f(x_1, x_2, \ldots x_n) = f(x_1) \prod_{i=2}^{n} f(x_i|x_1, \ldots x_{i-1})$$

where $x_1, \ldots x_n$ are random variables and f is a probability density function of its arguments.
