Review Assignment 1

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1 Linear algebra review

1. S = $\{v_1, ..., v_n\}$ be an orthogonal set of non-zero vectors in $\mathbb{R}^n$. Prove that the vectors in S are linearly independent.

We assume a linear combination

$$c_1v_1 + c_2v_2 + ... + c_kv_k = 0$$

We want to show that

$$c_1 = c_2 = ... = 0$$

The dot product of v_i for each $i = 1, 2, ..., k$:

$$\begin{aligned} 0 &= v_i \cdot 0 \\ &= v_i \cdot (c_1v_1 + c_2v_2 + ... + c_kv_k) \\ &= c_1v_i \cdot v_1 + c_2v_i \cdot v_2 + ... + c_kv_i \cdot v_k \end{aligned}$$

S is an orthogonal set, we have $v_i \cdot v_j = 0 \, \text{if} \, i \neq j$, then we have:

$$0 = c_iv_i \cdot v_i = c_i\|v_i\|^2$$

v_i is nonzero and length $\|v_i\|$ is nonzero, following that $c_i = 0$
We conclude that $c_1v_1 + c_2v_2 + ... + c_kv_k = 0$ for every $i = 1, 2, ..., k$, so S is **linearly independent**

2. Given a square matrix $A \in \mathbb{R}^{n\times n}$ and a vector $x \in \mathbb{R}^n$ show that $x^\intercal Ax = x^\intercal(\frac{1}{2}A + \frac{1}{2}A^\intercal)x$.

From theory we know that $(cA)^\intercal = cA^\intercal$. Therefore we can transform the term $x^\intercal Ax$

$$x^\intercal Ax \rightarrow (x^\intercal Ax)^\intercal \rightarrow x^\intercal A^\intercal x$$

This is similar to the goal in parentheses.

$$x^\intercal A^\intercal x \rightarrow \frac{(x^\intercal Ax) + (x^\intercal A^\intercal x)}{2}$$

By writing the divisor 2 outsite of the brackets we become

$$\frac{(x^\intercal Ax) + (x^\intercal A^\intercal x)}{2} \rightarrow \frac{1}{2}(x^\intercal Ax + x^\intercal A^\intercal x) \rightarrow x^\intercal(\frac{1}{2}A + \frac{1}{2}A^\intercal)x$$

Finally

$$x^\intercal Ax = x^\intercal(\frac{1}{2}A + \frac{1}{2}A^\intercal)x$$

3. Show that if $(A+B)^{-1} = A^{-1} + B^{-1}$ then $AB^{-1}A = BA^{-1}B$

We knew from theory $AA^{-1} = A^{-1}A = I$ where I is equal to the identity matrix.

$$\begin{aligned}
(AB^{-1}A)^{-1}(BA^{-1}B) &= I \\
(A^{-1}BA^{-1})(BA^{-1}B) &= I \quad |\text{premultiply by } A \\
A(A^{-1}BA^{-1})(BA^{-1}B) &= AI \\
IBA^{-1}(BA^{-1}B) &= AI \\
BA^{-1}(BA^{-1}B) &= A \quad |\text{premultiply by } B^{-1} \\
&\vdots \\
BA^{-1}B &= AB^{-1}A
\end{aligned}$$

4. Use the definition of trace to show that $\text{tr}(A+B) = \text{tr}A + \text{tr}B$, where $A, B \in \mathbb{R}^{n\times n}$

$$\begin{aligned}
\text{tr}(A) &= \sum_{i=1}^{n} a_{ii} \quad \text{if } A = \text{squared matrix} \\
\text{tr}(A+B) &= \sum_{i=1}^{n} a_{ii} + b_{ii} \\
\text{tr}(A+B) &= \begin{bmatrix} a_{11}+b_{11} & \dots & \dots \\ \vdots & a_{22}+b_{22} & \vdots \\ \dots & \dots & a_{nn}+b_{nn} \end{bmatrix} \\
&= (a_{11}+b_{11}) + (a_{22}+b_{22}) + (\dots) + (a_{nn}+b_{nn}) \\
&= a_{11} + b_{11} + \dots + a_{nn} + a_{22} + b_{22} + \dots + b_{nn} \\
&= (a_{11} + b_{11} + \dots + a_{nn}) + (a_{22} + b_{22} + \dots + b_{nn}) \\
&= \text{tr}(A) + \text{tr}(B)
\end{aligned}$$

5. Show that if (λ_i, x_i) are the i-th eigenvalue and i-th eigenvector of a non-singular and symmetric matrix $A \in \mathbb{R}^{n\times n}$, then $(\frac{1}{\lambda_i}, x_i)$ are the i-th eigenvalue and i-th eigenvector of A^1. *Hint: use the eigendecomposition of A*

$$Ax = \lambda_i x_i \rightarrow x_i = \lambda_i A^{-1} x_i \rightarrow A^{-1} x_i = \frac{1}{\lambda_i} x$$

6. Show that $\text{rank}(A) \leqslant \min\{m, n\}$, where $A, B \in \mathbb{R}^{m\times n}$

We know that column rank and row rank of any matrix is equal. Also we know that the column rank is at most equal to the number of columns and the row rank is at most equal to the number of rows. These two consideration implies that $(rank(A)) \leqslant \min\{m, n\}$.

7. In each of the following cases, state whether the real matrix A is guaranteed to be singular or not. Justify your answer in each case.

   (a) $A \in \mathbb{R}^{(n+1)\times n}$ is a full rank matrix.

   No. A non-singular matrices should be square.

   (b) $|A| = 0$.

   No. An square matrix A is non-singular if and only $|A| \neq 0$

   (c) A is an orthogonal matrix.

   Yes, For an orthogonal matrix Q we have $Q^\intercal Q = QQ^\intercal = I$ so Q is non singular and $Q^{-1} = Q^\intercal$

   (d) A has no eigenvalue equal to zero.

   Yes, we know that $|A| = \prod \lambda_i$. So if A has no zero eigenvalue, then $|A| \neq 0$ so A is non-singular.

   (e) A is a symmetric matrix with non-negative eigenvalues.

   No. We know that if A is a symmetric matrix then $x^\intercal A = \sum_{i=1}^{n}$ and it is positive/negative for any x if and only if all the eigenvalues are positive/negative.