# Review Assignment 1

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## 1 Linear algebra review

1. S = $\{v_1, ..., v_n\}$ be an orthogonal set of non-zero vectors in $\mathbb{R}^n$. Prove that the vectors in $S$ are linearly independent.

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We assume a linear combination

$$c_1v_1 + c_2v_2 + ... + c_kv_k = 0$$

We want to show that

$$c_1 = c_2 = ... = 0$$

The dot product of $v_i$ for each $i = 1, 2, ..., k$:

$$\begin{aligned} 0 &= v_i \cdot 0 \\ &= v_i \cdot (c_1v_1 + c_2v_2 + ... + c_kv_k) \\ &= c_1v_i \cdot v_1 + c_2v_i \cdot v_2 + ... + c_kv_i \cdot v_k \end{aligned}$$

$S$ is an orthogonal set, we have $v_i \cdot v_j = 0 \,\text{if}\, i \neq j$, then we have:

$$0 = c_iv_i \cdot v_i = c_i\|v_i\|^2$$

$v_i$ is nonzero and length $\|v_i\|$ is nonzero, following that $c_i = 0$
We conclude that $c_1v_1 + c_2v_2 + ... + c_kv_k = 0$ for every $i = 1, 2, ..., k$, so $S$ is **linearly independent**

2. Given a square matrix $A \in \mathbb{R}^{n\times n}$ and a vector $x \in \mathbb{R}^n$ show that $x^\intercal Ax = x^\intercal(\frac{1}{2}A + \frac{1}{2}A^\intercal)x$.

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We assume that:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} x = \begin{pmatrix} b_1 & b_2 & \dots & b_m \end{pmatrix} \quad where\, m = n$$

The transposed values are:

$$A^\intercal = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix} x^\intercal = \begin{pmatrix} b_1 & b_2 & \dots & b_m \end{pmatrix}$$

We want to show that this equation is true:

$$x^\intercal A x = x^\intercal (\frac{1}{2} A + \frac{1}{2} A^\intercal) x$$

If we insert the matrices:

$$\begin{pmatrix} b_1 & b_2 & \dots & b_m \end{pmatrix} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix} =$$

$$\begin{pmatrix} b_1 & b_2 & \dots & b_m \end{pmatrix} (\frac{1}{2} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix}) \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix}$$

Calculating in two steps:

$$((a_{11} \cdot b_1 + a_{21} \cdot b_2 + a_{m1} \cdot b_3) \cdot b_1 + (a_{12} \cdot b_1 + a_{22} \cdot b_2$$
$$+ a_{32} \cdot b_3) \cdot b_2 + (a_{1n} \cdot b_1 + a_{2n} \cdot b_2 + a_{mn} \cdot b_3) \cdot b_3) =$$

$$\begin{pmatrix} b_1 & b_2 & \dots & b_m \end{pmatrix} ( \begin{bmatrix} a_{11} & \frac{a_{12}+a_{21}}{2} & \dots & \frac{a_{1n}+a_{m1}}{2} \\ \frac{a_{12}+a_{21}}{2} & a_{22} & \dots & \frac{a_{12}+a_{21}}{2} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{a_{1n}+a_{m1}}{2} & \frac{a_{2n}+a_{32}}{2} & \dots & a_{mn} \end{bmatrix} ) \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix}$$

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\begin{aligned}
&((a_{11} \cdot b_1 + a_{21} \cdot b_2 + a_{m1} \cdot b_3) \cdot b_1 + (a_{12} \cdot b_1 + a_{22} \cdot b_2 \\
&+a_{32} \cdot b_3) \cdot b_2 + (a_{1n} \cdot b_1 + a_{2n} \cdot b_2 + a_{mn} \cdot b_3) \cdot b_3) = \\
&(\frac{1}{2} \cdot (2 \cdot a_{11} \cdot b_1 + (a_{12} + a_{21}) \cdot b_2 + (a_{1n} + a_{m1}) \cdot b_3) \cdot b_1 + \\
&\frac{1}{2} \cdot ((a_{12} + a_{21}) \cdot b_1 + 2 \cdot a_{22} \cdot b_2 + (a_{2n} + a_{32}) \cdot b_3) \cdot b_2 + \\
&\frac{1}{2} \cdot ((a_{1n} + a_{m1}) \cdot b_1 + (a_{2n} + a_{32}) \cdot b_2 + 2 \cdot a_{mn} \cdot b_3) \cdot b_3)
\end{aligned}
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3. Show that if $(A+B)^{-1} = A^{-1} + B^{-1}$ then $AB^{-1}A = BA^{-1}B$

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\begin{aligned}
(AB^{-1}A)(BA^{-1}B) &= I \,|\, \text{premultiply by } A^{-1} \\
A^{-1}(AB^{-1}A)(BA^{-1}B) &= A^{-1}I \\
I(B^{-1}A)(BA^{-1}B) &= A^{-1} \\
(B^{-1}A)(BA^{-1}B) &= A^{-1} \,|\, \text{premultiply by } B \\
B(B^{-1}A)(BA^{-1}B) &= BA^{-1} \\
A(BA^{-1}B) &= BA^{-1} \text{ premultiply by } A^{-1} \\
A^{-1}A(BA^{-1}B) &= A^{-1}BA^{-1} \\
BA^{-1}B &= A^{-1}BA^{-1}
\end{aligned}
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4. Use the definition of trace to show that $\mathrm{tr}(A+B) = \mathrm{tr}A + \mathrm{tr}B$, where $A, B \in \mathbb{R}^{n \times n}$

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\begin{aligned}
\mathrm{tr}(A) &= \sum_{i=1}^{n} a_{ii} \quad \text{if } A = \text{squared matrix} \\
\mathrm{tr}(A+B) &\rightarrow \mathrm{tr}(C) \\
C &= \begin{bmatrix} a_{11}+b_{11} & \dots & \dots \\ \dots & a_{22}+b_{22} & \dots \\ \vdots & \vdots & \vdots \\ \dots & \dots & a_{nn}+b_{nn} \end{bmatrix} \\
\mathrm{tr}(C) &= (a_{11}+b_{11}) + (a_{22}+b_{22}) + (\dots) + (a_{nn}+b_{nn}) \\
\mathrm{tr}(C) &= a_{11} + b_{11} + \dots + a_{nn} + a_{22} + b_{22} + \dots + b_{nn} \\
\mathrm{tr}(C) &= (a_{11} + b_{11} + \dots + a_{nn}) + (a_{22} + b_{22} + \dots + b_{nn}) \\
\mathrm{tr}(C) &= \mathrm{tr}(A) + \mathrm{tr}(B)
\end{aligned}
$$

5. Show that if $(\lambda_i, x_i)$ are the $i$-th eigenvalue and $i$-th eigenvector of a non-singular and symmetric matrix $A \in \mathbb{R}^{n \times n}$, then $(\frac{1}{\lambda_i}, x_i)$ are the $i$-th eigenvalue and $i$-th eigenvector of $A^1$. *Hint: use the eigendecomposition of A*

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$$x_i y_i^\mathsf{T} = [x_i][y_i] = \begin{bmatrix} x_1y_1 & x_1y_2 & \dots \\ x_2y_1 & \dots & \dots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

For $A, B \in \mathbb{R}^{m \times n}$, $\operatorname{rank}(A + B) \leqslant \operatorname{rank}(A) + \operatorname{rank}(B)$

$$\sum_{i=1}^{m} x_i - y_i^\mathsf{T} \leqslant m$$

6. Show that $\operatorname{rank}(A) \leqslant \min\{m, n\}$, where $A, B \in \mathbb{R}^{m \times n}$

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where $m$ corresponds to the number of rows and $n$ to the number of columns

$$\text{row-rank}(A) \leqslant m \quad \textbf{and} \quad \text{column-rank}(A) \leqslant n$$
$$\text{row-rank}(A) = \text{column-rank}(A) = \operatorname{rank}(A) \leqslant n$$

7. In each of the following cases, state whether the real matrix A is guaranteed to be singular or not. Justify your answer in each case.

   (a) $A \in \mathbb{R}^{(n+1) \times n}$ is a full rank matrix.

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   A singular matrix is never a full-rank, because only if $\operatorname{rank}(A) \leqslant \min\{m, n\}$ is equal.

   (b) $|A| = 0$.

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   When the determinant is zero, then the matrix is singular.

   (c) $A$ is an orthogonal matrix.

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   The transpose of an orthogonal matrix is equal to its inverse, hence this matrix is non-singular and invertible.

   (d) $A$ has no eigenvalue equal to zero.

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   If a matrix has non-zero eigenvalues, then it is invertible.

   (e) $A$ is a symmetric matrix with non-negative eigenvalues.

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   When all eigenvalues are positive a matrix is invertible.