Review Assignment 1

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1 Probability theory review

Solve each of the following problems and show all the steps of your working.

1. Show that the covariance matrix is always symmetric and positive semidefinite.

$$\begin{aligned}\Sigma_{ij} = \operatorname{cov}(x_i, x_j) &= \Sigma[x_i x_j] - \Sigma[x_i]\Sigma[x_j] \\ &= \Sigma[x_j x_i] - \Sigma[x_j]\Sigma[x_i]\end{aligned}$$

Σ is symmetric

$$z^\intercal \Sigma z \geqslant 0$$

when all eigenvalues are larger then zero or equal, a matrix can be symmetric and positive semidefinite

$$[z_1 z_2 \dots z_n] \begin{bmatrix} \Sigma_{11} & \dots & \dots \\ \vdots & \ddots & \vdots \\ \dots & \dots & \Sigma_{nn} \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \dots \\ z_n \end{pmatrix}$$

$$
\begin{aligned}
z^\intercal \Sigma z &= \sum_{i=1}^{n}\sum_{j=1}^{n} \Sigma_{ij} z_i z_j \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \operatorname{cov}[x_i x_j] z_i z_j \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \Sigma[(x_i - E[x_i])(x_j - E[x_j])] z_i z_j \\
&= E[\sum_{i=1}^{n}\sum_{j=1}^{n} \Sigma[(x_i - E[x_i])(x_j - E[x_j])] z_i z_j] \\
&= E[(x^\intercal z)^2]
\end{aligned}
$$

$E[(x^\intercal z)^2] \geqslant 0$ when $z \neq 0$

2. $X \in \mathbb{R}^n$ and $Y \in \mathbb{R}^m$ are independent random variables. Their expectations and covariances are $E[X] = 0$, $\operatorname{cov}[X] = I$, $E[Y] = \mu$ and $\operatorname{cov}[Y] = \sigma I$, where I is the identity matrix of the appropriate size and s is a scalar. What is the expectation and covariance of the random variable $z = AX + Y$, where $A \in \mathbb{R}^{m\times n}$?

$$
\begin{aligned}
z_i &= \sum_{j=1}^{n}(A_{ij} x_j) + Y_i \\
\Sigma[z] &= \Sigma[Ax + Y] \\
&= [A]\Sigma[x] + \Sigma[Y] \\
&= 0 + \mu = \mu
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{var}(z) &= \operatorname{var}(Ax + Y) \\
&= \operatorname{var}(Ax) + \operatorname{var}(Y) \\
&= AA^\intercal \operatorname{var}(X) + \operatorname{var}(Y) \\
&= \mathbb{R}^{m\times n} + \gtrdot \times \gtrdot \\
&= AA^\intercal + \sigma I
\end{aligned}
$$

3. Thomas and Viktor are friends. It is Friday night and Thomas does not have a phone. Viktor knows that there is a 2/3 probability that Thomas goes to the party to downtown. There are 5 pubs in downtown and there is an equal probability of Thomas going to any of them if he goes to the party. Viktor already looked for Thomas in 4 of the bars. What is the probability of Viktor ending Thomas in the last bar?

The sample space is

$$S = f\{\text{home}, \text{pub } 1, \text{pub } 2, \text{pub } 3, \text{pub } 4, \text{pub } 5\}$$

and the probability of the events are P("home") = 1=3 and P("pub i") = 2=15. We need to compute P("pub 5"j"not in pub 1 ... 4"). Using the Bayes rule, P("pub 5"j"not in pub 1 ... 4") = P("pub 5" "not in pub 1 ... 4") P("not in pub 1 ... 4") = 2=15 7=15 = 2 7 : Note that : P("not in pub 1 ... 4") = P("home"and"not in pub 1 ... 4") + P("out"and"not in pub 1 ... 4") = 1 3 1 + 2 3 1 5 = 7 15

4. Derive the mean for the Beta Distribution, which is defined as

$$\text{Beta}(x|a, b) = \frac{1}{B(a, b)} a^{-1}(1 - x)b^{-1} \tag{1}$$

where $B(a, b), G(a)$ are Beta and Gamma functions respectively:

$$B(a, b) \triangleq \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)} \tag{2}$$

and

$$\Gamma(x) \triangleq \int_0^\infty u^{x-1}e^{-u}du \tag{3}$$

Hint: Use integration by parts.

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5. Let $A \in \mathbb{R}^{n\times n}$ be a positive definite square matrix, $b \in \mathbb{R}^n$, and c be a scalar. Prove that

$$\int_{x\in\mathbb{R}^n} e^{-\frac{1}{2}x^\intercal Ax - x^\intercal b - c}dx = \frac{(2\pi)^{n/2}|A|^{-1/2}}{e^{c-\frac{1}{2}b^\intercal A^{-1}b}}$$

Hint: Use the fact that the integral of the Gaussian probability density function of a random variable with mean μ and covariance $\sum$ is 1.

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6. From the definition of conditional probability of multiple random variables, show that

$$f(x_1, x_2, \ldots x_n) = f(x_1) \prod_{i=2}^{n} f(x_i|x_1, \ldots x_{i-1})$$

where $x_1, \ldots x_n$ are random variables and f is a probability density function of its arguments.

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a

$n = 2$

a