Problem 1

(a)

$A = \begin{pmatrix} 6 & -2 \\ -2 & 3 \end{pmatrix}$

$\det\left(\begin{pmatrix} 6 & -2 \\ -2 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$

$\det\begin{pmatrix} 6-\lambda & -2 \\ -2 & 3-\lambda \end{pmatrix} = 0$

$\lambda^2 - 9\lambda + 14 = 0$

$(\lambda - 7)(\lambda - 2) = 0 \rightarrow$

$\rightarrow \lambda_1 = 7,\ \lambda_2 = 2$

Eigenvector v_1 for $\lambda_1 = 7$:

$(\begin{pmatrix} 6 & -2 \\ -2 & 3 \end{pmatrix} - 7\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})v_1 = 0$

$\begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix} v_1 = 0 \rightarrow v_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$

Eigenvector v_2 for $\lambda_2 = 2$:

$(\begin{pmatrix} 6 & -2 \\ -2 & 3 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})v_2 = 0$

$\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} v_2 = 0 \rightarrow v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$

Normalized v_1: $\dfrac{\begin{pmatrix} -2 & 1 \end{pmatrix}^T}{\sqrt{5}} = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$

Normalized v_2: $\dfrac{\begin{pmatrix} 1 & 2 \end{pmatrix}^T}{\sqrt{5}} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$

Thus, $P = \begin{bmatrix} Normalized(v_1) & Normalized(v_2) \end{bmatrix} = \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$

$P^{-1} = P^T = \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$

$D = P^{-1}AP = \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 6 & -2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} -\frac{14}{\sqrt{5}} & \frac{7}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{4}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} =$

$= \begin{pmatrix} 7 & 0 \\ 0 & 2 \end{pmatrix}$. This result corresponds to our found eigenvalues.

(b)

$$A = \begin{pmatrix} 3 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 6 \end{pmatrix}$$

$$\det\left(\begin{pmatrix} 3 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 6 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) = 0$$

$$\det \begin{pmatrix} 3-\lambda & 0 & 2 \\ 0 & 3-\lambda & 0 \\ 2 & 0 & 6-\lambda \end{pmatrix} = 0$$

$$(3-\lambda)\det \begin{pmatrix} 3-\lambda & 0 \\ 0 & 6-\lambda \end{pmatrix} + 2\det \begin{pmatrix} 0 & 3-\lambda \\ 2 & 0 \end{pmatrix} = 0$$

$$(-\lambda+3)\left(\lambda^2 - 9\lambda + 18\right) - 4(-\lambda+3) = 0$$

$$-\lambda^3 + 12\lambda^2 - 41\lambda + 42 = 0$$

$$-(\lambda-2)(\lambda-3)(\lambda-7) = 0 \rightarrow$$

$$\rightarrow \lambda_1 = 7, \lambda_2 = 3, \lambda_3 = 2$$

Eigenvector v_1 for $\lambda_1 = 7$:

$$(\begin{pmatrix} 3 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 6 \end{pmatrix} - 7 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix})v_1 = 0$$

$$\begin{pmatrix} -4 & 0 & 2 \\ 0 & -4 & 0 \\ 2 & 0 & -1 \end{pmatrix} v_1 = 0 \rightarrow v_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

Eigenvector v_2 for $\lambda_2 = 3$:

$$(\begin{pmatrix} 3 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 6 \end{pmatrix} - 3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix})v_2 = 0$$

$$\begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 3 \end{pmatrix} v_2 = 0 \rightarrow v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Eigenvector v_3 for $\lambda_3 = 2$:

$$(\begin{pmatrix} 3 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 6 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix})v_3 = 0$$

$$\begin{pmatrix}1 & 0 & 2\\0 & 1 & 0\\2 & 0 & 4\end{pmatrix} v_3 = 0 \to v_3 = \begin{pmatrix}-2\\0\\1\end{pmatrix}$$

Normalized v_1: $\dfrac{\begin{pmatrix}1 & 0 & 2\end{pmatrix}}{\sqrt{5}} = \begin{pmatrix}\frac{1}{\sqrt{5}}\\0\\\frac{2}{\sqrt{5}}\end{pmatrix}$

Normalized v_2: $\hat{a} = \dfrac{\begin{pmatrix}0 & 1 & 0\end{pmatrix}}{1} = \begin{pmatrix}0\\1\\0\end{pmatrix}$

Normalized v_3: $\dfrac{\begin{pmatrix}-2 & 0 & 1\end{pmatrix}}{\sqrt{5}} = \begin{pmatrix}-\frac{2}{\sqrt{5}}\\0\\\frac{1}{\sqrt{5}}\end{pmatrix}$

Thus, $P = \begin{pmatrix}\frac{1}{\sqrt{5}} & 0 & -\frac{2}{\sqrt{5}}\\0 & 1 & 0\\\frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}}\end{pmatrix}$

$P^{-1} = P^T = \begin{pmatrix}\frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}}\\0 & 1 & 0\\-\frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}}\end{pmatrix}$

$$D = P^{-1}AP = \begin{pmatrix}\frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}}\\0 & 1 & 0\\-\frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}}\end{pmatrix}\begin{pmatrix}3 & 0 & 2\\0 & 3 & 0\\2 & 0 & 6\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{5}} & 0 & -\frac{2}{\sqrt{5}}\\0 & 1 & 0\\\frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}}\end{pmatrix} = \begin{pmatrix}\frac{7}{\sqrt{5}} & 0 & \frac{14}{\sqrt{5}}\\0 & 3 & 0\\-\frac{4}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{5}} & 0 & -\frac{2}{\sqrt{5}}\\0 & 1 & 0\\\frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}}\end{pmatrix} =$$

$= \begin{pmatrix}7 & 0 & 0\\0 & 3 & 0\\0 & 0 & 2\end{pmatrix}$. This result corresponds to our found eigenvalues.

(c)

$$\begin{pmatrix}3 & -1 & 0\\-1 & 4 & 1\\0 & 1 & 5\end{pmatrix}$$

$$\det\left(\begin{pmatrix}3 & -1 & 0\\-1 & 4 & 1\\0 & 1 & 5\end{pmatrix} - \lambda\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\right) = 0$$

$$\begin{pmatrix}3-\lambda & -1 & 0\\-1 & 4-\lambda & 1\\0 & 1 & 5-\lambda\end{pmatrix} = 0$$

$$(3-\lambda)\det\begin{pmatrix}4-\lambda & 1\\1 & 5-\lambda\end{pmatrix} - (-1)\det\begin{pmatrix}-1 & 1\\0 & 5-\lambda\end{pmatrix} = 0$$

$$(3-\lambda)\left(\lambda^2 - 9\lambda + 19\right) - (-1)(\lambda - 5) = 0$$

$$-\lambda^3+12\lambda^2-45\lambda+52=0$$

$$-(\lambda-4)\left(\lambda^2-8\lambda+13\right)=0\rightarrow$$

$$\rightarrow\lambda_1=4;\lambda_{2,\,3}=\frac{-(-8)\pm\sqrt{(-8)^2-4\cdot\,1\cdot\,13}}{2\cdot\,1}\rightarrow\lambda_2=4+\sqrt{3};\lambda_3=4-\sqrt{3}$$

Eigenvector v_1 for $\lambda_1=4$:

$$(\begin{pmatrix}3&-1&0\\-1&4&1\\0&1&5\end{pmatrix}-4\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix})v_1=0$$
$$\begin{pmatrix}-1&-1&0\\-1&0&1\\0&1&1\end{pmatrix}v_1=0\rightarrow$$
$$\rightarrow v_1=\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

Eigenvector v_2 for $\lambda_2=4+\sqrt{3}$:

$$(\begin{pmatrix}3&-1&0\\-1&4&1\\0&1&5\end{pmatrix}-\left(4+\sqrt{3}\right)\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix})v_2=0$$
$$\begin{pmatrix}-1-\sqrt{3}&-1&0\\-1&-\sqrt{3}&1\\0&1&1-\sqrt{3}\end{pmatrix}v_2=0\rightarrow$$
$$\rightarrow v_2=\begin{pmatrix}-2+\sqrt{3}\\\sqrt{3}-1\\1\end{pmatrix}$$

Eigenvector v_3 for $\lambda_3=4-\sqrt{3}$:

$$(\begin{pmatrix}3&-1&0\\-1&4&1\\0&1&5\end{pmatrix}-\left(4-\sqrt{3}\right)\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix})v_3=0$$
$$\begin{pmatrix}\sqrt{3}-1&-1&0\\-1&\sqrt{3}&1\\0&1&1+\sqrt{3}\end{pmatrix}v_3=0\rightarrow$$
$$\rightarrow v_3=\begin{pmatrix}-2-\sqrt{3}\\-1-\sqrt{3}\\1\end{pmatrix}$$

Let's normalize v_1,v_2,v_3:

$$v_1^{'}=\frac{\begin{pmatrix}1&-1&1\end{pmatrix}^T}{\sqrt{3}}=\begin{pmatrix}\frac{1}{\sqrt{3}}&-\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{3}}\end{pmatrix}^T$$

$$v_2^{'} = \frac{\begin{pmatrix} -2+\sqrt{3} & \sqrt{3}-1 & 1 \end{pmatrix}^T}{3-\sqrt{3}} = \begin{pmatrix} \frac{1-\sqrt{3}}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{\sqrt{3}+1}{2\sqrt{3}} \end{pmatrix}^T$$

$$v_3^{'} = \frac{\begin{pmatrix} -2-\sqrt{3} & -1-\sqrt{3} & 1 \end{pmatrix}^T}{\sqrt{3}+3} = \begin{pmatrix} -\frac{\sqrt{3}+1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{\sqrt{3}-1}{2\sqrt{3}} \end{pmatrix}^T$$

$$P = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1-\sqrt{3}}{2\sqrt{3}} & -\frac{\sqrt{3}+1}{2\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{\sqrt{3}+1}{2\sqrt{3}} & \frac{\sqrt{3}-1}{2\sqrt{3}} \end{pmatrix}.$$

$$D = P^{-1}AP = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1-\sqrt{3}}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{\sqrt{3}+1}{2\sqrt{3}} \\ -\frac{\sqrt{3}+1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{\sqrt{3}-1}{2\sqrt{3}} \end{pmatrix} \begin{pmatrix} 3 & -1 & 0 \\ -1 & 4 & 1 \\ 0 & 1 & 5 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1-\sqrt{3}}{2\sqrt{3}} & -\frac{\sqrt{3}+1}{2\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{\sqrt{3}+1}{2\sqrt{3}} & \frac{\sqrt{3}-1}{2\sqrt{3}} \end{pmatrix} =$$

$$\begin{pmatrix} \frac{4}{\sqrt{3}} & -\frac{4}{\sqrt{3}} & \frac{4}{\sqrt{3}} \\ \frac{1-3\sqrt{3}}{2\sqrt{3}} & \frac{\sqrt{3}+4}{\sqrt{3}} & \frac{7+5\sqrt{3}}{2\sqrt{3}} \\ \frac{-1-3\sqrt{3}}{2\sqrt{3}} & \frac{\sqrt{3}-4}{\sqrt{3}} & \frac{5\sqrt{3}-7}{2\sqrt{3}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1-\sqrt{3}}{2\sqrt{3}} & -\frac{\sqrt{3}+1}{2\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{\sqrt{3}+1}{2\sqrt{3}} & \frac{\sqrt{3}-1}{2\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4+\sqrt{3} & 0 \\ 0 & 0 & 4-\sqrt{3} \end{pmatrix}.$$

This result corresponds to our eigenvalues.

Problem 3

(a) $\vec{v}_1 = \begin{pmatrix}1\\0\\1\end{pmatrix}, \vec{v}_2 = \begin{pmatrix}1\\1\\-1\end{pmatrix}, \vec{v}_3 = \begin{pmatrix}a\\b\\0\end{pmatrix}; \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = \lambda.$

1) Possible values for $\lambda_3 = \lambda$, such that λ is distinct:

in this case (a, b) should satisfy:

1. $\vec{v}_1 \cdot \vec{v}_3 = 0 \longrightarrow \begin{pmatrix}1\\0\\1\end{pmatrix}^T \begin{pmatrix}a\\b\\0\end{pmatrix} = a \longrightarrow a = 0$

2. $\vec{v}_2 \cdot \vec{v}_3 = 0 \longrightarrow \begin{pmatrix}1\\1\\-1\end{pmatrix}^T \begin{pmatrix}a\\b\\0\end{pmatrix} = a + b \longrightarrow -a = b$

It means that $\vec{v}_3$ should be $\begin{pmatrix}0\\0\\0\end{pmatrix} = 0$, but this is impossible, which means that no a and b exist for $\lambda_3 \neq 1$, $\lambda_3 \neq 2$. Hense, no symmetric matrices exist for such value of λ_3.

2) Let's suppose that λ_3 is not distinct:

2.1) $\lambda_3 = \lambda_2 = 2$:

(a, b) should satisfy $\vec{v}_1 \cdot \vec{v}_3 = 0$;

$\begin{pmatrix}1\\0\\1\end{pmatrix}^T \begin{pmatrix}a\\b\\0\end{pmatrix} = 0; \longrightarrow$ a must be 0 and b can be any value, except for 0. $\longrightarrow$

$\longrightarrow \vec{v}_3 = \begin{pmatrix}0\\b\\0\end{pmatrix}$

2.2) $\lambda_3 = \lambda_1 = 1$:

(a, b) should satisfy $\vec{v}_2 \cdot \vec{v}_3 = 0$;

$$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}^T \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} = 0; \rightarrow a = -b.$$

(b)

Let's find symmetric matrices for two case of $\lambda_3 = \lambda_2 = 2$ and $\lambda_3 = \lambda_1 = 1$.

For $\lambda_3 = \lambda_2 = 2$: (let's suppose that $b = 1$ in $\begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix} \rightarrow \vec{v}_3 = \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$)

To find A we need to find P from $P D P^{-1}$. In order to find P we firstly need to apply Gram-Schmidt process to $\vec{v}_2$ and $\vec{v}_3$, as they are from the same eigenspace, so that $\vec{v}_2$ and $\vec{v}_3$ become orthogonal to each other. Thus:

$$u_2 = \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

We assumed $b = 1$ in $\vec{v}_3$, so $\vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

So, $\vec{u}_3$ $\vec{v}_3 - proj_{\vec{u}_2}(\vec{v}_3)$.

$$proj_{\vec{u}_2} \vec{v}_3 = \frac{\vec{u}_2 \cdot \vec{v}_3}{\|\vec{u}_2\|^2} \vec{u}_2 = \frac{1}{(\sqrt{3})^2} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \end{pmatrix} \longrightarrow$$

$$\longrightarrow \vec{u}_3 = \vec{v}_3 - \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}.$$

Now let's normalize $\vec{v}_1, \vec{v}_2, \vec{u}_3$:

$$\vec{v}_1{}' = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{\begin{pmatrix} 1 & 0 & 1 \end{pmatrix}^T}{\sqrt{2}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}^T$$

$$\vec{v}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \frac{\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}^T}{\sqrt{3}} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{pmatrix}^T$$

$$\vec{u}_3 = \frac{\vec{u}_3}{\| u_3 \|} = \frac{\begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}^T}{\sqrt{\frac{2}{3}}} = \begin{pmatrix} -\frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}^T.$$

Thus, $P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}$.

Now let's find symmetric matrice A:

$$A = PDP^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{3}} & -\sqrt{\frac{2}{3}} \\ 0 & \frac{2}{\sqrt{3}} & 2\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{2}} & -\frac{2}{\sqrt{3}} & \sqrt{\frac{2}{3}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & 0 & -\frac{1}{2} \\ 0 & 2 & 0 \\ -\frac{1}{2} & 0 & \frac{3}{2} \end{pmatrix} = A.$$

For $\lambda_3 = \lambda_1 = 1$:

As in $\lambda_3 = \lambda_2$, we will now apply Gram-Schmidt process to $\vec{v}_1$ and $\vec{v}_3$.

We will assume that $a = 1$ in $\vec{v}_3$ for $\lambda_3 = 1$. It means that $b = -1$, given facts established in (a) for $\lambda_3 = 1$, where $a = -b$ in $\vec{v}_3$.

Thus:

$$\vec{u}_1 = \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}^T$$

$$\vec{u}_3 = \vec{v}_3 - \text{proj}_{\vec{u}_1}(\vec{v}_3),$$ where $\vec{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, given above considerations.

$$\text{proj}_{\vec{u}_1} \vec{v}_3 = \frac{\vec{u}_1 \cdot \vec{v}_3}{\|\vec{u}_1\|^2} \vec{u}_1 = \frac{1}{(\sqrt{2})^2} \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}^T = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}^T.$$

Then $\vec{u}_3 = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}^T - \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}^T = \begin{pmatrix} \frac{1}{2} & -1 & -\frac{1}{2} \end{pmatrix}^T$.

Now let's normalize $\vec{v}_1, \vec{v}_2, \vec{u}_3$:

$$\vec{v}_1{}' = (\text{from } \lambda_3 = 2 \text{ solution above}) = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}^T$$

$$\vec{v}_2{}' = (\text{from } \lambda_3 = 2 \text{ solution above}) = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{pmatrix}^T$$

$$\vec{u}_3{}' = \frac{\vec{u}_3}{\|u_3\|} = \frac{\begin{pmatrix} \frac{1}{2} & -1 & -\frac{1}{2} \end{pmatrix}}{\sqrt{\frac{3}{2}}} = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} \end{pmatrix}^T.$$

$$\text{Thus, } P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{pmatrix}. \longrightarrow$$

$$\longrightarrow A = PDP^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{2}{\sqrt{3}} & -\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{2}} & -\frac{2}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} \frac{4}{3} & 1 & 0 \\ 1 & \frac{4}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{4}{3} \end{pmatrix} = A.$$

Problem 2

(a) $\lambda_1 = -1$, $\lambda_2 = 3$, $\lambda_3 = 7$; $\vec{v}_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$.

In (a) we have an eigenvector for each distinct eigenvalue of A. It means that $\rightarrow$ eigenvectors must be pairwise orthogonal automatically. $\rightarrow$ dot product between each two of them shall be by definition equal to 0.

Thus, let's check for what values of (a, b) in $\vec{v}_3 = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$ it's dot product with $\vec{v}_1$ and $\vec{v}_2$ equal to zero:

1. $\vec{v}_1 \cdot \vec{v}_3 = 0$;

$$\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}^T \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} = 0;$$

$b = 0$.

2. $\vec{v}_2 \cdot \vec{v}_3 = 0$;

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}^T \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} = 0;$$

$a + b = 0$;
$a = -b$.

We see that $\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ for $(\vec{v}_1 \perp \vec{v}_3) \cup (\vec{v}_2 \perp \vec{v}_3)$, but eigenvector CAN'T be a zero vector. So, no values a and b exist for which A is a symmetric, given conditions, described in (a).

PPLIED
IENCES
CULTY.DS

(b) For $\lambda_3 = 3$.

If $\lambda_3 = 3$ then $\lambda_3 = \lambda_2$. It means that $\vec{v}_2$ and $\vec{v}_3$ share the same eigenspace of eigenvalue 3 and we only need to check whether $\vec{v}_1$ and $\vec{v}_3$ are orthogonal for some values (a, b):

$$\vec{v}_1 \cdot \vec{v}_3 = 0$$

$$\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}^T \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} = 0$$

$b = 0 \longrightarrow$ for $v_1 \perp v_3$ value b has to be zero, while a take any real value, except for zero.

(c)

To find symmetric matrice A, where $A = PDP^{-1}$, we need to find P. In order to find P, in our case we need to guarantee orthogonality between $\vec{v}_2$ and $\vec{v}_3$ through Gram-Schmidt process, as $\vec{v}_2$ and $\vec{v}_3$ are in the same eigenspace.

Thus, let's perform Gram-Schmidt process. We will also assume that $a = 1$ in $\vec{v}_3 = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$ and $b = 0$, as it was established in (b). So, $\vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

APPLIED SCIENCES FACULTY.DS

Then let $\vec{u}_2 = \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

$$\vec{u}_3 = \vec{v}_3 - proj_{\vec{u}_2}(\vec{v}_3).$$

$$proj_{\vec{u}_2}\vec{v}_3 = \frac{\vec{u}_2 \cdot \vec{v}_3}{\|\vec{u}_2\|^2}\vec{u}_2 = \frac{1}{(\sqrt{3})^2}(1 \ \ 1 \ \ 1)^T = \left(\frac{1}{3} \ \ \frac{1}{3} \ \ \frac{1}{3}\right)^T \rightarrow$$

$$\rightarrow \vec{u}_3 = (1 \ \ 0 \ \ 0)^T - \left(\frac{1}{3} \ \ \frac{1}{3} \ \ \frac{1}{3}\right)^T = \left(\frac{2}{3} \ \ -\frac{1}{3} \ \ -\frac{1}{3}\right).$$

Now let's normalize $\vec{v}_1, \vec{v}_2, \vec{u}_3$:

$$\vec{v}_1{}' = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{(0 \ \ 1 \ \ -1)^T}{\sqrt{2}} = \left(0 \ \ \frac{1}{\sqrt{2}} \ \ -\frac{1}{\sqrt{2}}\right)^T;$$

$$\vec{v}_2{}' = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \frac{(1 \ \ 1 \ \ 1)^T}{\sqrt{3}} = \left(\frac{1}{\sqrt{3}} \ \ \frac{1}{\sqrt{3}} \ \ \frac{1}{\sqrt{3}}\right)^T;$$

$$\vec{u}_3{}' = \frac{\vec{u}_3}{\|\vec{u}_3\|} = \frac{\left(\frac{2}{3} \ \ -\frac{1}{3} \ \ -\frac{1}{3}\right)^T}{\sqrt{\frac{2}{3}}} = \left(\sqrt{\frac{2}{3}} \ \ -\frac{1}{\sqrt{6}} \ \ -\frac{1}{\sqrt{6}}\right)^T.$$

Then $P = \begin{pmatrix} 0 & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{pmatrix}$.

$$A = PDP^{-1} = \begin{pmatrix} 0 & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{3} & -\frac{1}{\sqrt{6}} \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & \sqrt{3} & \sqrt{6} \\ -\frac{1}{2} & \sqrt{3} & -\sqrt{\frac{3}{2}} \\ \frac{1}{\sqrt{2}} & \sqrt{3} & -\sqrt{\frac{3}{2}} \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix} = A.$$

Problem 4

By definition of symmetric matrix we know that if A is symmetric matrix, then A has orthogonal eigenvectors, real eigenvalues and is diagonalizable:

$A = VDV^T$, where V is a vector of orthogonal eigenvectors,
D is a diagonal matrix with eigenvalues.

$$\rightarrow A = VDV^T = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix} =$$

$$= \begin{bmatrix} \lambda_1 v_1 & \lambda_2 v_2 & \cdots & \lambda_n v_n \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix} = \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T + \cdots + \lambda_n v_n v_n^T,$$

where
- $(\lambda_1, \lambda_2 \ldots \lambda_n)$ is set of eigenvalues of A;
- $(v_1, v_2 \ldots v_n)$ is set of eigenvectors of A.

$\rightarrow$ Can A be non-orthogonal?
Only in case when $v_1, v_2 \ldots v_n$ are not orthogonal bases and $\lambda_1, \lambda_2 \ldots \lambda_n$ are not real.

But in 4 Problem $v_1, v_2 \ldots v_n$ are pairwise orthogonal and $\lambda_1, \lambda_2 \ldots \lambda_n$ are real $\rightarrow$ We proved that A is a symmetric matrix and determined its eigenvalues and eigenvectors.

Problem 5

$$A = I_n - vv^T.$$

(a)

$$vv^T = \begin{pmatrix} v_1 v_1 & v_1 v_2 & \cdots & v_1 v_n \\ v_2 v_1 & v_2 v_2 & \cdots & v_2 v_n \\ v_3 v_1 & v_3 v_2 & & \vdots \\ \vdots & \vdots & & v_n v_n \\ v_n v_1 & v_n v_2 & \cdots & \end{pmatrix}$$

→ $n \times n$ outer product matrix which is symmetric because off diagonal elements are equal. →

→ this means that vv^T is orthogonally diagonalizable due to being symmetric.

I is symmetric by definition.

$$A^T = (I_n - vv^T)^T = I^T - (vv^T)^T = I - (v^T)^T (v)^T = I - vv^T = A \rightarrow$$

$$\longrightarrow A^T = A. \longrightarrow$$ A is orthogonally diagonalizable due to being symmetric.

(b) Eigenvalues: If matrices A and B are both diagonalizable, symmetric and commutative, then EV of $A+B = EV(A) + EV(B)$:

$EV(vv^T) = \|v\|^2$, 0 with multiplicity of $n-1$;

$EV(I) = 1$ with multiplicity of n.

Thus, EV of $I - vv^T = 1 - \|v\|^2$, 1 with multiplicity of $n-1$.

(c)

If $v = (1\ 0\ 1)^T$ then from

(b) we know that eigenvalues of

A, where $A = I - vv^T$, will be:

$\lambda_1 = 1$ with multiplicity 2

and

$$\lambda_2 = 1 - \left\|\begin{pmatrix}1\\0\\1\end{pmatrix}\right\|^2 = 1 - 2 = -1.$$

Before finding eigenvectors let's find A itself:

$$A = I - vv^T = I - \begin{pmatrix}1\\0\\1\end{pmatrix}(1\ 0\ 1) = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - \begin{pmatrix}1&0&1\\0&0&0\\1&0&1\end{pmatrix} =$$

$$= \begin{pmatrix}0&0&-1\\0&1&0\\-1&0&0\end{pmatrix} \longrightarrow \text{For } \lambda_1 = 1 \text{ eigenvectors are:}$$

$$\left(\begin{pmatrix}0&0&-1\\0&1&0\\-1&0&0\end{pmatrix} - \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right)\vec{v} = 0;$$

$$\begin{pmatrix}-1&0&-1\\0&0&0\\-1&0&-1\end{pmatrix}\vec{v} = 0 \longrightarrow \vec{v}_1 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \vec{v}_2 = \begin{pmatrix}-1\\0\\1\end{pmatrix}.$$

For $\lambda_2 = -1$ eigenvector is:

$$\left(\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \vec{v}_3 = 0$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} \vec{v}_3 = 0 \longrightarrow \vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Let's apply Gram-Schmidt process to $\vec{v}_1$ and $\vec{v}_2$ as they share the same eigenspace, so we can't construct P yet.

$\vec{u}_1 = \vec{v}_1 = (0 \; 1 \; 0)^T$

$\vec{u}_2 = \vec{v}_2 - \text{proj}_{\vec{u}_1} \vec{v}_2$;

$\text{proj}_{\vec{u}_1} \vec{v}_2 = \frac{\vec{u}_1 \cdot \vec{v}_2}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 = \frac{0}{1^2} (0 \; 1 \; 0) = \vec{0}$

$\vec{u}_2 = \vec{v}_2 - 0 = \vec{v}_2 = (-1 \; 0 \; 1)^T.$

Let's normalize our eigenvectors:

$\vec{v}_1' = \frac{(0 \; 1 \; 0)^T}{1} = (0 \; 1 \; 0)^T$

$\vec{v}_2' = \frac{(-1 \; 0 \; 1)^T}{\sqrt{2}} = (-\frac{1}{\sqrt{2}} \; 0 \; \frac{1}{\sqrt{2}})^T$

$v_3' = \frac{(1 \; 0 \; 1)^T}{\sqrt{2}} = (\frac{1}{\sqrt{2}} \; 0 \; \frac{1}{\sqrt{2}})^T.$

It means, that $P = \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$

(Problem 6)

(a)

Matrix A is Hermitian if $A = A^*$

$$A = \begin{pmatrix} 1 & i & 0 \\ i & 2 & -i \\ 0 & -i & 1 \end{pmatrix}$$

$$A^* = \begin{pmatrix} 1 & -i & 0 \\ -i & 2 & i \\ 0 & i & 1 \end{pmatrix} \rightarrow A \neq A^*$$

(b)

Matrix A is Hermitian if $A = A^*$

$$A = \begin{pmatrix} 1 & i & 0 \\ -i & 2 & -i \\ 0 & i & 1 \end{pmatrix}$$

$$A^* = \begin{pmatrix} 1 & i & 0 \\ -i & 2 & -i \\ 0 & i & 1 \end{pmatrix} \rightarrow A = A^* \rightarrow \text{A is Hermitian.}$$

Let's now perform unitary diagonalization of A and construct $D = U^*AU$, where U is unitary matrice of eigenvectors of A and D is diagonal matrice of its eigenvalues:

$$EV(A) \rightarrow \det\left(\begin{pmatrix} 1 & i & 0 \\ -i & 2 & -i \\ 0 & i & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) = 0;$$

$$\det\begin{pmatrix} 1-\lambda & i & 0 \\ -i & 2-\lambda & -i \\ 0 & i & 1-\lambda \end{pmatrix} = 0;$$

$$(1-\lambda)\det\begin{pmatrix} 2-\lambda & -i \\ i & 1-\lambda \end{pmatrix} - i\det\begin{pmatrix} -i & -i \\ 0 & 1-\lambda \end{pmatrix} + 0\det\begin{pmatrix} -i & 2-\lambda \\ 0 & i \end{pmatrix} = 0;$$

$$(1-\lambda)\left(\lambda^2 - 3 + 1\right) - ii\left(-1+\lambda\right) + 0\cdot\ 1 = 0;$$

$$(-\lambda+1)\left(\lambda^2 - 3\lambda + 1\right) - \left(-\left(-1\right)\right) = 0;$$

$$-\lambda^3 + 4\lambda^2 - 4\lambda + 1 + \lambda - 1 = 0;$$

$$\lambda^3 + 4\lambda^2 - \lambda = 0;$$

$$-\lambda\left(\lambda-1\right)\left(\lambda-3\right) = 0;$$

$$\rightarrow \lambda_1 = 3,\ \lambda_2 = 1,\ \lambda_3 = 0$$

Eigenvector for $\lambda_1 = 3:$

$$\begin{pmatrix} 1 & i & 0 \\ -i & 2 & -i \\ 0 & i & 1 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & i & 0 \\ -i & -1 & -i \\ 0 & i & -2 \end{pmatrix} R_1 \leftrightarrow R_2 = \begin{pmatrix} -i & -1 & -i \\ -2 & i & 0 \\ 0 & i & -2 \end{pmatrix} R_2 \leftarrow$$

$R_2+2i\cdot R_1=\begin{pmatrix}-i&-1&-i\\0&-i&2\\0&i&-2\end{pmatrix} R_2\leftrightarrow R_3=\begin{pmatrix}-i&-1&-i\\0&i&-2\\0&-i&2\end{pmatrix} R_3\leftarrow R_3+1\cdot R_2=$
$\begin{pmatrix}-i&-1&-i\\0&i&-2\\0&0&0\end{pmatrix} R_2 \leftarrow -i\cdot R_2=\begin{pmatrix}-i&-1&-i\\0&1&2i\\0&0&0\end{pmatrix} R_1 \leftarrow R_1+1\cdot R_2=$
$\begin{pmatrix}-i&0&i\\0&1&2i\\0&0&0\end{pmatrix} R_1\leftarrow i\cdot R_1=\begin{pmatrix}1&0&-1\\0&1&2i\\0&0&0\end{pmatrix}\rightarrow\begin{pmatrix}1&0&-1\\0&1&2i\\0&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\rightarrow$
$\begin{Bmatrix}x-z=0\\y+2iz=0\end{Bmatrix}=\begin{Bmatrix}x=z\\y=-2iz\end{Bmatrix}\rightarrow v_1=\begin{pmatrix}z\\-2iz\\z\end{pmatrix}=\begin{pmatrix}z\\-2iz\\z\end{pmatrix}=\begin{pmatrix}1\\-2i\\1\end{pmatrix}$

Eigenvectors for $\lambda_2=1:$
$\begin{pmatrix}1&i&0\\-i&2&-i\\0&i&1\end{pmatrix}-1\cdot\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}0&i&0\\-i&1&-i\\0&i&0\end{pmatrix} R_1\leftrightarrow R_2=\begin{pmatrix}-i&1&-i\\0&i&0\\0&i&0\end{pmatrix} R_2\leftrightarrow$
$R_3=\begin{pmatrix}-i&1&-i\\0&i&0\\0&i&0\end{pmatrix} R_3\leftarrow R_3-1\cdot R_2=\begin{pmatrix}-i&1&-i\\0&i&0\\0&0&0\end{pmatrix} R_2\leftarrow -i\cdot R_2=$
$\begin{pmatrix}-i&1&-i\\0&1&0\\0&0&0\end{pmatrix} R_1\leftarrow R_1-1\cdot R_2=\begin{pmatrix}-i&0&-i\\0&1&0\\0&0&0\end{pmatrix} R_1\leftarrow i\cdot R_1=\begin{pmatrix}1&0&1\\0&1&0\\0&0&0\end{pmatrix}\rightarrow$
$\begin{pmatrix}1&0&1\\0&1&0\\0&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\rightarrow\begin{Bmatrix}x+z=0\\y=0\end{Bmatrix}=\begin{Bmatrix}y=0\\x=-z\end{Bmatrix} v_2=\begin{pmatrix}-z\\0\\z\end{pmatrix}, z\neq 0\rightarrow$
$z=1\rightarrow v_2=\begin{pmatrix}-1\\0\\1\end{pmatrix}$

Eigenvectors for $\lambda_3=0:$
$\begin{pmatrix}1&i&0\\-i&2&-i\\0&i&1\end{pmatrix}-0\cdot\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}1&i&0\\-i&2&-i\\0&i&1\end{pmatrix} R_1\leftrightarrow R_2=\begin{pmatrix}-i&2&-i\\1&i&0\\0&i&1\end{pmatrix} R_2\leftarrow$
$R_2-i\cdot R_1=\begin{pmatrix}-i&2&-i\\0&-i&-1\\0&i&1\end{pmatrix} R_2\leftrightarrow R_3=\begin{pmatrix}-i&2&-i\\0&i&1\\0&-i&-1\end{pmatrix} R_3\leftarrow R_3+1\cdot R_2=$
$\begin{pmatrix}-i&2&-i\\0&i&1\\0&0&0\end{pmatrix} R_2\leftarrow -i\cdot R_2=\begin{pmatrix}-i&2&-i\\0&1&-i\\0&0&0\end{pmatrix} R_1\leftarrow R_1-2\cdot R_2=$
$\begin{pmatrix}-i&0&i\\0&1&-i\\0&0&0\end{pmatrix} R_1\leftarrow i\cdot R_1=\begin{pmatrix}1&0&-1\\0&1&-i\\0&0&0\end{pmatrix}\rightarrow\begin{pmatrix}1&0&-1\\0&1&-i\\0&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\rightarrow$

$$\begin{Bmatrix} x-z=0 \\ y-iz=0 \end{Bmatrix} = \begin{Bmatrix} x=z \\ y=iz \end{Bmatrix} \rightarrow$$

$$\rightarrow v_3 = \begin{pmatrix} z \\ iz \\ z \end{pmatrix} , z \neq 0 \rightarrow z=1 \rightarrow v_3 = \begin{pmatrix} 1 \\ i \\ 1 \end{pmatrix}$$

$$\rightarrow So, v_1 = \begin{pmatrix} 1 \\ -2i \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ i \\ 1 \end{pmatrix}$$

$Unit:$

$$\hat{v_1} = \frac{\begin{pmatrix} 1 & -2i & 1 \end{pmatrix}}{\sqrt{6}} = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ -i\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$\hat{v_2} = \frac{\begin{pmatrix} -1 & 0 & 1 \end{pmatrix}}{\sqrt{2}} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\hat{v_3} = \frac{\begin{pmatrix} 1 & i & 1 \end{pmatrix}}{\sqrt{3}} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ i\frac{\sqrt{3}}{3} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$U = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -i\sqrt{\frac{2}{3}} & 0 & i\frac{\sqrt{3}}{3} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$U^* = \begin{pmatrix} \frac{1}{\sqrt{6}} & i\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -i\frac{\sqrt{3}}{3} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$\rightarrow D = U^*AU = \begin{pmatrix} \frac{1}{\sqrt{6}} & i\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -i\frac{\sqrt{3}}{3} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 1 & i & 0 \\ -i & 2 & -i \\ 0 & i & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -i\sqrt{\frac{2}{3}} & 0 & i\frac{\sqrt{3}}{3} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix} =$$

$$\begin{pmatrix} \sqrt{\frac{3}{2}} & \sqrt{6}i & \sqrt{\frac{3}{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -i\sqrt{\frac{2}{3}} & 0 & i\frac{\sqrt{3}}{3} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} =$$

$= D$, where diagonal entries are, as expected, our eigenvalues.

Problem 7

Matrix is positive definite when all pivots of the matrix in row echelon form are positive. So, let's use this pivot test to show that matrices in (a) and (b) are both positive:

(a)

$A = \begin{pmatrix} 5 & -2 \\ -2 & 5 \end{pmatrix}$ $R_2 \leftarrow R_2 + \frac{2}{5} \cdot R_1 = \begin{pmatrix} 5 & -2 \\ 0 & \frac{21}{5} \end{pmatrix}$ $\rightarrow$ all pivots are positive $\rightarrow$ A is positive definite.

Also all principal minors of $A > 0$:$D_1 = 5, D_2 = det(A) = 21 \rightarrow$ A is positive definite.

(b)

$A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 4 & 1 \\ 0 & 1 & 5 \end{pmatrix}$ $R_2 \leftarrow R_2 + \frac{1}{3} \cdot R_1 = \begin{pmatrix} 3 & -1 & 0 \\ 0 & \frac{11}{3} & 1 \\ 0 & 1 & 5 \end{pmatrix}$ $R_3 \leftarrow R_3 - \frac{3}{11} \cdot R_2 =$ $\begin{pmatrix} 3 & -1 & 0 \\ 0 & \frac{11}{3} & 1 \\ 0 & 0 & \frac{52}{11} \end{pmatrix}$ $\rightarrow$ all pivots are positive $\rightarrow$ A is positive definite.

Also all principal minors of $A > 0$:$D_1 = 3, D_2 = 11, D_3 = det(A) = 52 \rightarrow$ A is positive definite.

Problem 8

$B = \begin{pmatrix} 1 & -3 \\ -1 & 3 \\ 2 & c \end{pmatrix}$;

$B^\top B = \begin{pmatrix} 1 & -1 & 2 \\ -3 & 3 & c \end{pmatrix} \begin{pmatrix} 1 & -3 \\ -1 & 3 \\ 2 & c \end{pmatrix} = \begin{pmatrix} 6 & -6+2c \\ -6+2c & 18+c^2 \end{pmatrix}$

(a)

Let's find for which value of c $det(B^\top B)$ is equal to 0 in order to find for which values of c $B^\top B$ is invertible:

$\det \begin{pmatrix} 6 & -6+2c \\ -6+2c & 18+c^2 \end{pmatrix} = 0;$

$6\left(18+c^2\right) - (-6+2c)(-6+2c) = 0;$

$6\left(18+c^2\right) - (2c-6)^2 = 0;$

$6\left(18+c^2\right)-\left(36-24c+4c^2\right)=0;$

$108+6c^2-\left(36-24c+4c^2\right)=0;$

$108+6c^2-36+24c-4c^2=0;$

$2c^2+24c+72=0. \rightarrow$

$\rightarrow$ Roots of $2a^2+24a+72:$

$\frac{-24\pm\sqrt{24^2-4\cdot 2\cdot 72}}{2\cdot 2}=\frac{-24\pm\sqrt{0}}{2\cdot 2}=-6 \rightarrow B^\top B$ is invertible for all $c\in\mathbb{R} \mid c\neq -6$.

(b)

We can analyze principal minors of $B^\top B$ to look for which values of c $B^\top B$ is positive definite. For $B^\top B$ to be positive definite $D_1>0, D_2>0:$
$D_1=6$
$D_2: det(B^\top B)>0;$
$2c^2+24c+72>0;$
$c<-6 \quad \text{or} \quad c>-6.$

Problem 9

$$A=\begin{pmatrix} s & -4 & -4\\ -4 & s & 4\\ -4 & 4 & s\end{pmatrix}; B=\begin{pmatrix} t & -3 & 0\\ -3 & t & 4\\ 0 & 4 & t\end{pmatrix}$$

Let's use the leading principal minors of matrices A and B to determine for which values of c and t they are negative definite and positive definite.

1.Negative definite

Negative definite: $D_1<0, D_2>0, D_3<0$

Matrix A:

$D_1=s\rightarrow D_1<0$ only if $s<0$

$D_2=\begin{vmatrix} s & -4\\ -4 & s\end{vmatrix}=s^2-16\rightarrow$

$\rightarrow s^2-16>0;$

$s^2>16;$

$s<-4 \quad \text{or} \quad s>4\rightarrow D_2>0$ only if $s<-4$ or $s>4$,

$D_3=\begin{vmatrix} s & -4 & -4\\ -4 & s & 4\\ -4 & 4 & s\end{vmatrix}.=s^3-48s+128\rightarrow$

$\rightarrow s^3-48s+128<0;$

$(s-4)(s-4)(s+8)<0;$

$(s-4)^2\,(s+8) < 0; \rightarrow s < -8;$
Matrix A is negative definite for all $s \in \mathbb{R} \mid s < -8$.

For Matrix B:
$D_1 = t \rightarrow D_1 < 0$ only if $t < 0$.
$D_2 = \det\begin{pmatrix} t & -3 \\ -3 & t \end{pmatrix} = t^2 - 9 \rightarrow t^2 - 9 > 0;$
$t^2 > 9; \rightarrow D_2 > 0$ only if $t < -3$ or $t > 3$.
$D_3 = \det\begin{pmatrix} t & -3 & 0 \\ -3 & t & 4 \\ 0 & 4 & t \end{pmatrix}; t\det\begin{pmatrix} t & 4 \\ 4 & t \end{pmatrix} - (-3)\det\begin{pmatrix} -3 & 4 \\ 0 & t \end{pmatrix} + 0\cdot\det\begin{pmatrix} -3 & t \\ 0 & 4 \end{pmatrix} =$
$t^3 - 25t; \rightarrow$
$t^3 - 25t = 0;$
$t\,(t+5)\,(t-5) < 0; \rightarrow t < -5 \quad \text{or} \quad 0 < t < 5 \rightarrow$
$\rightarrow D_3 < 0$ only if $t < -5$ or $0 < t < 5$.
Matrix B is negative definite for all $t \in \mathbb{R} \mid t < -5$.

2.Positive definite

Positive definite: $D_1 > 0, D_2 > 0, D_3 > 0$
Matrix A:
$D_1 \rightarrow s > 0.$
$D_2 \rightarrow \det\begin{pmatrix} s & -4 \\ -4 & s \end{pmatrix} > 0;$
$s^2 - 16 > 0;$
$s < -4 \quad \text{or} \quad s > 4$
$D_3 \rightarrow \det\begin{pmatrix} s & -4 & -4 \\ -4 & s & 4 \\ -4 & 4 & s \end{pmatrix} > 0;$
$s^3 - 48s + 128 > 0;$
$-8 < s < 4 \quad \text{or} \quad s > 4$ Matrix A is positive definite for all $s \in \mathbb{R} \mid s > 4$.
Matrix B:
$D_1 \rightarrow t > 0.$
$D_2 \rightarrow \det\begin{pmatrix} t & -3 \\ -3 & t \end{pmatrix} > 0;$
$t^2 - 9 > 0;$
$t < -3 \quad \text{or} \quad t > 3.$
$D_3 \rightarrow \det\begin{pmatrix} t & -3 & 0 \\ -3 & t & 4 \\ 0 & 4 & t \end{pmatrix} > 0;$
$t^3 - 25t > 0;$
$t\,(t+5)\,(t-5) > 0;$
$-5 < t < 0 \quad \text{or} \quad t > 5.$
Matrix B is positive definite for all $t \in \mathbb{R} \mid t > 5$.

Problem 10

(a) $Q(x_1, x_2) = 2x_1^2 + 2x_2^2 - 2x_1x_2$

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$\det(A) = (2)(2) - (-1)(-1) = 3$; $tr(A) = 4 \longrightarrow$

$\longrightarrow \lambda_1 = 3,\ \lambda_2 = 1 \longrightarrow \quad D = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$

We know that if $x = Py \rightarrow y = P^{-1}x$.

Thus, quadtratic form of A:

$$x^T A x = x^T P D P^{-1} x = (Py)^T P D y = y^T P^T P D y =$$

$$= y^T I D y = y^T D y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^T \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} =$$

$$= 3y_1^2 + y_2^2.$$

Let's also find P as to know how to represent $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ in terms of $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, given that $y = P^{-1}x$.

For $\lambda_1 = 3$ we have $(A - \lambda I)\vec{v}_1 = 0$;

$$\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \vec{v}_1 = 0 \rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

For $\lambda_2 = 1$: $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \vec{v}_2 = 0 \longrightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Let's normalize $\vec{v}_1$ and $\vec{v}_2$:

$$\vec{v}_1{}' = \frac{(1 \quad -1)^T}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}} \quad -\frac{1}{\sqrt{2}}\right)^T$$

$$v_2' = \frac{(1 \quad 1)}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}}\right)^T.$$

then $P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$; $P^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$;

$$y = P^{-1}x = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \vec{x} = \begin{cases} y_1 = \frac{1}{\sqrt{2}}x_1 - \frac{1}{\sqrt{2}}x_2 \\ y_2 = \frac{1}{\sqrt{2}}x_1 + \frac{1}{\sqrt{2}}x_2 \end{cases}.$$

(b)

$$Q(x_1, x_2, x_3) = 3x_1^2 + 4x_2^2 + 5x_3^2 + 4x_1x_2 - 4x_2x_3$$

$$A = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 4 & -2 \\ 0 & -2 & 5 \end{pmatrix}.$$

Let's find eigenvalues:

$$\det(A - \lambda I) = 0,$$

$$\det\left(\begin{pmatrix} 3 & 2 & 0 \\ 2 & 4 & -2 \\ 0 & -2 & 5 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right) = 0;$$

$$\det\begin{pmatrix} 3-\lambda & 2 & 0 \\ 2 & 4-\lambda & -2 \\ 0 & -2 & 5-\lambda \end{pmatrix} = 0;$$

$$(3-\lambda)\det\begin{pmatrix}4-\lambda & -2\\ -2 & 5-\lambda\end{pmatrix} - 2\det\begin{pmatrix}2 & -2\\ 0 & 5-\lambda\end{pmatrix} = 0;$$

$$= (3-\lambda)(\lambda^2 - 9\lambda + 16) - 2(2)(-\lambda + 5) = 0;$$

$$= -\lambda^3 + 12\lambda^2 - 39\lambda + 28 = 0;$$

$$-(\lambda - 1)(\lambda - 4)(\lambda - 7) = 0.$$

$$\downarrow$$

$$\lambda_1 = 7, \ \lambda_2 = 4, \ \lambda_3 = 1.$$

Then (given (a) considerations) $\mathcal{D} = \begin{pmatrix}7&0&0\\0&4&0\\0&0&1\end{pmatrix}$ and

$$x^T A x = y^T \mathcal{D} y = y^T\begin{pmatrix}7&0&0\\0&4&0\\0&0&1\end{pmatrix}y =$$

$$= 7y_1^2 + 4y_2^2 + 1y_3^2.$$ Then we will find P:

$\vec{v}_1$ for $\lambda_1 = 7$: $\left(\begin{pmatrix}3&2&0\\2&4&-2\\0&-2&5\end{pmatrix} - \begin{pmatrix}7&0&0\\0&7&0\\0&0&7\end{pmatrix}\right)\vec{v}_1 = \begin{pmatrix}-4&2&0\\2&-3&-2\\0&-2&-2\end{pmatrix}\vec{v}_1 \rightarrow \vec{v}_1 = \begin{pmatrix}-1\\-2\\2\end{pmatrix}.$

$\vec{v}_2$ for $\lambda_2 = 4$: $\left(\begin{pmatrix}3&2&0\\2&4&-2\\0&-2&5\end{pmatrix} - \begin{pmatrix}4&0&0\\0&4&0\\0&0&4\end{pmatrix}\right)\vec{v}_2 = \begin{pmatrix}-1&2&0\\2&0&-2\\0&-2&1\end{pmatrix}\vec{v}_2 \Rightarrow \vec{v}_2 = \begin{pmatrix}2\\1\\2\end{pmatrix}.$

$\vec{v}_3$ for $\lambda_3 = 1$: $\begin{pmatrix}2&2&0\\2&3&-2\\0&-2&4\end{pmatrix}\vec{v}_3 \rightarrow \vec{v}_3 = \begin{pmatrix}-2\\2\\1\end{pmatrix}.$

Normalize: $\vec{v}_1' = \dfrac{(-1 \ -2 \ 2)^T}{\sqrt{9}} = \left(-\frac{1}{3} \ -\frac{2}{3} \ \frac{2}{3}\right)^T$

$\vec{v}_2' = \dfrac{(2 \ 1 \ 2)^T}{\sqrt{9}} = \left(\frac{2}{3} \ \frac{1}{3} \ \frac{2}{3}\right)^T$

$\vec{v}_3' = \left(-\frac{2}{3} \ \frac{2}{3} \ \frac{1}{3}\right)^T.$

$$P = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix} \rightarrow P^{-1} = \begin{pmatrix} -\frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$

Then $\vec{y} = P^{-1}\vec{x} = \begin{pmatrix} -\frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{cases} y_1 = -\frac{1}{3}x_1 - \frac{2}{3}x_2 + \frac{2}{3}x_3 \\ y_2 = \frac{2}{3}x_1 + \frac{1}{3}x_2 + \frac{2}{3}x_3 \\ y_3 = -\frac{2}{3}x_1 + \frac{2}{3}x_2 + \frac{1}{3}x_3 \end{cases}$

Problem 11

$3x^2 + 4xy + 6y^2 = 14$

Convert to standard form:

$$\frac{3}{14}x^2 + \frac{4}{14}xy + \frac{6}{14}y^2 = 1$$

or

We can see this as quadtratic form $x^T A x$ of matrix A, where $x = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $A = \frac{1}{14}\begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}$.

Let's find eigenvalues of A:

$$\det\begin{pmatrix} \frac{3-14\lambda}{14} & \frac{1}{7} \\ \frac{1}{7} & \frac{3-7\lambda}{7} \end{pmatrix} = 0;$$

$$\frac{3-14\lambda}{14} \cdot \frac{3-7\lambda}{7} - \frac{1}{7} \cdot \frac{1}{7} = 0;$$

$$\frac{9 - 63\lambda + 98\lambda^2}{98} - \frac{1}{49} = 0;$$

$$9 - 63\lambda + 98\lambda^2 - 2 = 0$$

$$98\lambda^2 - 63\lambda + 7 = 0$$

$$\lambda_{1,2} = \frac{-(-63) \pm \sqrt{(-63)^2 - 4 \cdot 98 \cdot 7}}{2 \cdot 98} \longrightarrow \lambda_1 = \frac{1}{2},\ \lambda_2 = \frac{1}{7}.$$

Eigevector for $\lambda_1 = \frac{1}{2}$: $\left(\begin{pmatrix} \frac{3}{14} & \frac{2}{14} \\ \frac{2}{14} & \frac{6}{14} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\vec{v}_1 = \begin{pmatrix} -\frac{2}{7} & \frac{1}{7} \\ \frac{1}{7} & -\frac{1}{14} \end{pmatrix}\vec{v}_1 \longrightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Eigenvector for $\lambda_2 = \frac{1}{7}$: $\begin{pmatrix} \frac{1}{14} & \frac{1}{7} \\ \frac{1}{7} & \frac{2}{7} \end{pmatrix}\vec{v}_2 \longrightarrow \vec{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

Let's normalize $\vec{v}_1$ and $\vec{v}_2$:

$$\vec{v}_1{}' = \frac{(1 \quad 2)^T}{\sqrt{5}} = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{3}}\right)^T$$

$$\vec{v}_2{}' = \frac{(-2 \quad 1)^T}{\sqrt{5}} = \left(-\frac{2}{\sqrt{5}} \quad \frac{1}{\sqrt{5}}\right)^T.$$

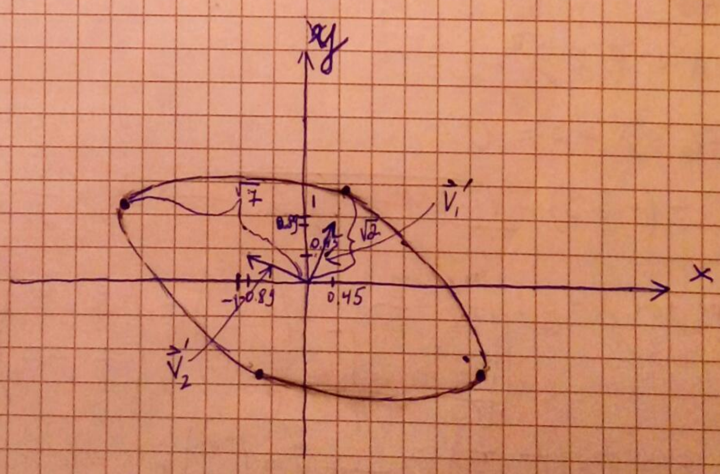
$$P = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}.$$

The major axis is in the direction of $\vec{v}_2{}'$ and has length $\frac{1}{\sqrt{\lambda_2}} = \frac{1}{\sqrt{\frac{1}{7}}} = \sqrt{7} \approx 2,64.$

The minor axis is in the direction of $\vec{v}_1{}'$ and has length $\frac{1}{\sqrt{\lambda_1}} = \frac{1}{\sqrt{\frac{1}{2}}} = \sqrt{2} \approx 1,41.$

$$\vec{v}_1{}' = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)^T \approx (0,45 \quad 0,89)^T$$

$$\vec{v}_2{}' = \left(-\frac{2}{\sqrt{5}} \quad \frac{1}{\sqrt{5}}\right)^T \approx (-0.89 \quad 0.45)^T.$$



Problem 12

(a)

$$A = \begin{pmatrix} 3 & -2 & 1 \\ -2 & 6 & -2 \\ 1 & -2 & 3 \end{pmatrix}; v_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Let's multiply v_1 by A in order to establish whether v_1 satisfy definition of eigenvector v such that $Av = \lambda v$:

$Av_1 = \begin{pmatrix} 3 & -2 & 1 \\ -2 & 6 & -2 \\ 1 & -2 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} \rightarrow v_1$ is an eigenvector by definition and it's eigenvalue $\lambda_1 = 2$.

Now let's find other eigenvalues and eigenvectors. Definitely, we can say that eigenvalue of 2 goes with multiplicity of 2 because we have simultaneously all rows sum and all columns sum equal to 2. So let's find out another eigenvector v_2 for $\lambda_2 = 2$:

$$\left(\begin{pmatrix} 3 & -2 & 1 \\ -2 & 6 & -2 \\ 1 & -2 & 3 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) v_2 = 0$$

$$\left(\begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix}\right) v_2 = 0 \rightarrow v_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

We know that $tr(A) = 12$. Thus, $\lambda_3 = tr(A) - 2 - 2 = 8$. Let's find out eigenvector v_3 for $\lambda_3 = 8$:

$$\left(\begin{pmatrix} 3 & -2 & 1 \\ -2 & 6 & -2 \\ 1 & -2 & 3 \end{pmatrix} - 8 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) v_3 = 0$$

$$\left(\begin{pmatrix} -5 & -2 & 1 \\ -2 & -2 & -2 \\ 1 & -2 & -5 \end{pmatrix}\right) v_3 = 0 \rightarrow v_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Let's perform Gram-Schmidt process for v_1 and v_2 to make vectors from the same eigenspace orthogonal :

$$u_1 = v_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix};$$

$$u_2 = v_2 - proj_{u_1} v_2$$

$$proj_{u_1} v_2 = \frac{\vec{u_1} \cdot \vec{v_2}}{|\vec{u_1}|^2} \cdot \vec{u_1} = \frac{-2}{(\sqrt{2})^2} \begin{pmatrix} -1 & 0 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}^T;$$

$u_2 = v_2 - \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}^T = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}.$

Let's now normalize our eigenvectors:

$v_1^{'} = \frac{\begin{pmatrix} -1 & 0 & 1 \end{pmatrix}^T}{\sqrt{2}} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}^T;$

$u_2^{'} = \frac{\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T}{\sqrt{3}} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}^T;$

$v_3^{'} = \frac{\begin{pmatrix} 1 & -2 & 1 \end{pmatrix}}{\sqrt{6}} = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}$

Thus, $P = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}.$

Thus, $D = P^{-1}AP = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 3 & -2 & 1 \\ -2 & 6 & -2 \\ 1 & -2 & 3 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix} =$

$= \begin{pmatrix} -\sqrt{2} & 0 & \sqrt{2} \\ \frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} \\ \frac{4\sqrt{2}}{\sqrt{3}} & -\frac{8\sqrt{2}}{\sqrt{3}} & \frac{4\sqrt{2}}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix}.$

This result corresponds to our eigenvalues.

(b)

As all eigenvalues of A are positive, we can conclude that it's quadratic form $Q(x) = x^T Ax$ is positive definite. The principal axes of quadratic form $Q(x)$ are simply column vectors of above-found matrice P.

If "transition matrix" represents transformation from x to y, such that $y = P^{-1}x$ then our transition matrix is simply $P^{-1} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}.$