Linear Algebra
Home assignment 4: Symmetric matrices and quadratic forms

Problem 1 (2 pt)**.** Find a matrix P that orthogonally diagonalizes A, and determine $P^{-1}AP$:

$$(a)\quad \begin{pmatrix} 6 & -2 \\ -2 & 3 \end{pmatrix}; \qquad (b)\quad \begin{pmatrix} 3 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 6 \end{pmatrix}; \qquad (c)\quad \begin{pmatrix} 3 & -1 & 0 \\ -1 & 4 & 1 \\ 0 & 1 & 5 \end{pmatrix}$$

Solution. Your solution comes here

Problem 2 (4 pt)**.** (a) Find all values of a and b for which there exists a 3×3 symmetric matrix A with eigenvalues $\lambda_1 = -1$, $\lambda_2 = 3$, $\lambda_3 = 7$ and the corresponding eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}.$$

(b) Reconsider the problem if $\lambda_3 = 3$.

(c) Find all symmetric matrices A satisfying the conditions of part (b).

Problem 3 (4 pt)**.** A 3×3 symmetric matrix A has eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = \lambda$ and the corresponding eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}.$$

(a) Find all possible values of a, b, and λ.

(b) Find the corresponding matrices A.

Problem 4 (2 pt)**.** Assume that $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ is an *orthogonal* basis of $\mathbb{R}^n$ and $\lambda_1, \lambda_2, \dots, \lambda_n$ are real numbers. Prove that

$$A = \lambda_1 \mathbf{v}_1 \mathbf{v}_1^\top + \lambda_2 \mathbf{v}_2 \mathbf{v}_2^\top + \cdots + \lambda_n \mathbf{v}_n \mathbf{v}_n^\top$$

is a symmetric matrix and determine its eigenvalues and eigenvectors.

Problem 5 (3 pt)**.** Assume that $\mathbf{v}$ is any $n \times 1$ vector, I_n is the identity $n \times n$ matrix, and $A = I_n - \mathbf{v}\mathbf{v}^\top$.

(a) Show that A is orthogonally diagonalizable;

(b) find the eigenvalues of A;

(c) find a matrix P that orthogonally diagonalizes A if $\mathbf{v} = (1, 0, 1)^\top$.

Problem 6 (2 pt)**.** Which of the two matrices below is Hermitian?

$$(a)\quad \begin{pmatrix} 1 & i & 0 \\ i & 2 & -i \\ 0 & -i & 1 \end{pmatrix} \qquad (b)\quad \begin{pmatrix} 1 & i & 0 \\ -i & 2 & -i \\ 0 & i & 1 \end{pmatrix}$$

For the Hermitian matrix, find eigenvalues, eigenvectors, and perform its unitary diagonalization.

Problem 7 (2 pt)**.** Show that the matrix A is positive definite:

$$(a) \quad \begin{pmatrix} 5 & -2 \\ -2 & 5 \end{pmatrix}; \qquad (b) \quad \begin{pmatrix} 3 & -1 & 0 \\ -1 & 4 & 1 \\ 0 & 1 & 5 \end{pmatrix}$$

Problem 8 (2 pt)**.** Let c be a constant and let

$$B = \begin{pmatrix} 1 & -3 \\ -1 & 3 \\ 2 & c \end{pmatrix}$$

(a) For what value(s) of c will the matrix $B^\top B$ be invertible?

(b) For what value(s) of c will the matrix $B^\top B$ be positive definite?

Justify your answers.

Problem 9 (4 pt)**.** Find all s and t for which the following matrices are (a) positive or (b) negative definite:

$$S = \begin{pmatrix} s & -4 & -4 \\ -4 & s & 4 \\ -4 & 4 & s \end{pmatrix}, \qquad T = \begin{pmatrix} t & -3 & 0 \\ -3 & t & 4 \\ 0 & 4 & t \end{pmatrix}$$

Problem 10 (3 pt)**.** Find an orthogonal change of variables that eliminates the cross product terms in the quadratic form Q, and express Q in terms of the new variables:

(a) $Q(x_1, x_2) = 2x_1^2 + 2x_2^2 - 2x_1x_2$;

(b) $Q(x_1, x_2, x_3) = 3x_1^2 + 4x_2^2 + 5x_3^2 + 4x_1x_2 - 4x_2x_3$

Problem 11 (2 pt)**.** Sketch the curve $3x^2 + 4xy + 6y^2 = 14$ in the xy-plane. Determine the axes of the ellipse and their length.

Problem 12 (3 pt)**.** Consider the matrix A and the vector $\mathbf{v}_1$, where

$$A = \begin{pmatrix} 3 & -2 & 1 \\ -2 & 6 & -2 \\ 1 & -2 & 3 \end{pmatrix}, \qquad \mathbf{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

(a) Show that $\mathbf{v}_1$ is an eigenvector of A, and find its corresponding eigenvalue. Then find the other two eigenvalues and eigenvectors and orthogonally diagonalize A.

(b) Is the quadratic form $Q(\mathbf{x}) = \mathbf{x}^\top A\mathbf{x}$ positive definite, negative definite or indefinite? Find the principal axes of Q and the corresponding transition matrix.

Problem 13 (Oblique projectors; 5pt)**.** Assume that $\mathbb{R}^n$ is represented as the *direct* (but not necessarily *orthogonal*) sum $M_1 \dot{+} M_2$ of two its subspaces M_1 and M_2. In particular, every $\mathbf{x} \in \mathbb{R}^n$ can be represented in a unique way as $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$ with some $\mathbf{x}_j \in M_j$, and the mapping $P_j : \mathbf{x} \mapsto \mathbf{x}_j$ is called the *(oblique) projector onto M_j parallel to M_{3-j}*. It is easy to show that P_j satisfy the following properties: $P_1 + P_2 = I_n$, $P_j^2 = P_j$ and $P_1P_2 = P_2P_1 = 0$.

(a) Show that any matrix P satisfying the relation $P^2 = P$ is a projector onto some subspace L parallel to M, and identify these L and M.

(b) Show that the projector P is an orthogonal projector if and only if the matrix P is symmetric.

(c) Assume that two transformations P_1 and P_2 of $\mathbb{R}^n$ satisfy the following conditions: $P_1 + P_2 = I_n$ and $P_1 P_2 = 0$. Prove that P_1 and P_2 are projectors and that $P_2 P_1 = 0$.

Problem 14 (Projectors; 5pt)**.** (a) Find a matrix of oblique projector in $\mathbb{R}^3$ onto the subspace $U = \operatorname{ls}\{(1,0,1)^\top\}$ parallel to the subspace $W = \operatorname{ls}\{(1,1,0)^\top, (0,1,1)^\top\}$.

(b) Find a projection matrix of $\mathbb{R}^3$ onto the subspace $U = \operatorname{ls}\{(1,2,1)^\top, (1,0,-1)^\top\}$ parallel to the subspace $W = \operatorname{ls}\{(1,0,1)^\top\}$.

(c) Is it possible to fill in the missing entries in the matrix

$$A = \begin{pmatrix} 1 & * & 0 \\ 0 & \frac{1}{2} & * \\ * & * & * \end{pmatrix}$$

to get a matrix of an orthogonal projection in $\mathbb{R}^3$? If so, find the subspace U of $\mathbb{R}^3$ such that A is an orthogonal projection onto U.

Hint: Do you see why only one of (a) or (b) needs to be worked out in detail?