Problem 8

$A = \begin{pmatrix} 10 & 6 \\ 0 & 8 \end{pmatrix} \frac{1}{\sqrt{10}} = \begin{pmatrix} \sqrt{10} & \frac{3\sqrt{2}}{\sqrt{5}} \\ 0 & \frac{4\sqrt{2}}{\sqrt{5}} \end{pmatrix} \; A = QS; Q = UV^\top; S = V\Sigma V^\top;$

$1. S = V\Sigma V^\top:$

$A^\top A = \begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix}$

$Eigenvalues(A^\top A):$

$\det\left(\begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0;$

$\det\begin{pmatrix} 10 - & 6 \\ 6 & 10 - \end{pmatrix} = 0;$

$(10 -)\,(10 -) - 6 \cdot\ 6 = 0;$

$\lambda^2 - 20\lambda + 64 = 0;$

$\lambda_1 = \frac{-(-20)+\sqrt{(-20)^2 - 4\cdot\ 1\cdot\ 64}}{2\cdot\ 1} = 16; \lambda_2 = \frac{-(-20)-\sqrt{(-20)^2 - 4\cdot\ 1\cdot\ 64}}{2\cdot\ 1} = 4. \rightarrow$

$\rightarrow \sigma_1 = \sqrt{\lambda_1} = 4; \sigma_1 = \sqrt{\lambda_2} = 2.$

$Eigenvector(\lambda_1 = 16):$

$\begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix} - 16 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -6 & 6 \\ 6 & -6 \end{pmatrix} R_2 \leftarrow R_2 + 1 \cdot\ R_1 = \begin{pmatrix} -6 & 6 \\ 0 & 0 \end{pmatrix} R_1 \leftarrow -\frac{1}{6} \cdot\ R_1 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \rightarrow x = y \rightarrow v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow \frac{v_1}{|v_1|} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

$Eigenvector(\lambda_2 = 4):$

$\begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix} - 4 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 6 \\ 6 & 6 \end{pmatrix} R_2 \leftarrow R_2 - 1 \cdot\ R_1 = \begin{pmatrix} 6 & 6 \\ 0 & 0 \end{pmatrix} R_1 \leftarrow \frac{1}{6} \cdot\ R_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \rightarrow x = -y \rightarrow v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \rightarrow \frac{v_2}{|v_2|} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

$Thus,$

$V = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}; \Sigma = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}.$

$S = V\Sigma V^\top = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

$Q = UV^\top.$

$U:$

$u_1 = \frac{Av_1}{\sigma_1} \begin{pmatrix} \sqrt{10} & \frac{3\sqrt{2}}{\sqrt{5}} \\ 0 & \frac{4\sqrt{2}}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \frac{1}{4} = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$

$$u_2 = \frac{Av_2}{\sigma_2} = \begin{pmatrix} \sqrt{10} & \frac{3\sqrt{2}}{\sqrt{5}} \\ 0 & \frac{4\sqrt{2}}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \frac{1}{2} = \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}.$$

$$U = \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$$

$$Q = UV^\top = \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{pmatrix}$$

Let's check values of S and Q:

$$A = QS = \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} \sqrt{10} & \frac{3\sqrt{2}}{\sqrt{5}} \\ 0 & \frac{4\sqrt{2}}{\sqrt{5}} \end{pmatrix} = A.$$

Problem 10

Matrix A

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \to A^\top A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

$Eigenvalues(A^\top A) : \lambda_1 = 4, \lambda_2, \lambda_3, \lambda_4 = 0. \to$
$\to \sigma_1 = \sqrt{\lambda_1} = 2, \sigma_2, \sigma_3, \sigma_4 = 0$

$Eigenvector(\lambda_1 = 4) : (1,1,1,1)^\top = v_1 \to \frac{\vec{v_1}}{\|v_1\|} = \frac{(1,1,1,1)^\top}{2} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})^\top$

$Eigenvectors(\lambda_2 = 0) :$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} ((R_2, R_3, R_4) \leftarrow (R_2, R_3, R_4) - 1 \cdot R_1) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$(A - 0I) \begin{pmatrix} x \\ y \\ z \\ v \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\to x + y + z + v = 0;$
$x = -y - z - v.$

$$\to \begin{pmatrix} -y - z - v \\ y \\ z \\ v \end{pmatrix} = \begin{pmatrix} -y \\ y \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -z \\ 0 \\ z \\ 0 \end{pmatrix} + \begin{pmatrix} -v \\ 0 \\ 0 \\ v \end{pmatrix}. \to$$

$For\ y, z, v = 1 :$

$$v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \to \frac{\vec{v_2}}{\|v_2\|} = \frac{(-1,1,0,0)^\top}{\sqrt{2}} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix}^\top$$

$$v_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \to \frac{\vec{v_3}}{\|v_3\|} = \frac{(-1,0,1,0)^\top}{\sqrt{2}} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}^\top$$

$$v_4 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \to \frac{\vec{v_4}}{\|v_4\|} = \frac{(-1,0,0,1)^\top}{\sqrt{2}} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}^\top$$

$$\to V = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

$$\to u_1 = \frac{Av_1}{\sigma_1} = \frac{1}{\sigma_1} \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2})^\top = 1.$$

$$\to A = U\Sigma V^\top = [1] \begin{pmatrix} 2 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}.$$

$$A^+ = V\Sigma^+ U^\top = \begin{pmatrix} \frac{1}{2} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix}$$

Matrix B

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \to B^\top B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$Eigenvalues(B^\top B) : \lambda_1, \lambda_2 = 1, \lambda_2 = 0. \to$
$\to \sigma_1 = 1, \sigma_2 = 1, \sigma_3 = 0.$
$Eigenvectors(\lambda_1 = 1) :$

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix};$$

$Eigenvector(\lambda_2 = 0) :$

$$v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

$V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$

$u_1 = \frac{Bv_1}{\sigma_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; u_2 = \frac{Bv_2}{\sigma_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix};$

$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$Thus,$

$B = U\Sigma V^\top = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = B$

$B^+ = V\Sigma^+ U^\top = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$

Matrix C

$C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$

$C^\top C = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$

$Eigenvalues(C^\top C) : \lambda_1 = 2, \lambda_2 = 0 \rightarrow$

$\rightarrow \sigma_1 = \sqrt{2}, \sigma_2 = 0$

$Eigenvector(\lambda_1 = 2) : v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \rightarrow \frac{\vec{v_1}}{\|v_1\|} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$

$Eigenvector(\lambda_2 = 0) : v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \rightarrow \frac{\vec{v_2}}{\|v_2\|} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

$V = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

$u_1 = \frac{Cv_1}{\sigma_1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$u_2 = 0$ because $\|Cv_2\| = \sigma_2 = 0$.

So, we need to find one non-zero u_2, orthogonal to $u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} :\rightarrow u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$C = U\Sigma V^\top = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = C.$

$C^+ = V\Sigma^+ U^\top = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{pmatrix}.$

Problem 12

(a)

$A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 10 & 10 \\ 2 & 10 & 14 \end{pmatrix}$

Lets use the leading principal minors of matrix A to determine whether A is a positive definite matrix.

For A to be positive definite all principal minors D_i should be: $D_1 > 0, D_2 > 0, D_3 > 0$:

$D_1 = 4 \rightarrow D_1 > 0$;

$D_2 = \det \begin{pmatrix} 4 & 2 \\ 2 & 10 \end{pmatrix} = 36 \rightarrow D_2 > 0$;

$D_3 = det(A) = 4 \cdot \det \begin{pmatrix} 10 & 10 \\ 10 & 14 \end{pmatrix} - 2 \cdot \det \begin{pmatrix} 2 & 10 \\ 2 & 14 \end{pmatrix} + 2 \cdot \det \begin{pmatrix} 2 & 10 \\ 2 & 10 \end{pmatrix} 36 = 144 \rightarrow$
$\rightarrow D_3 > 0.$

We have $D_1 > 0, D_2 > 0, D_3 > 0 \rightarrow$ We proved without finding eigenvalues that matrix A is positive definite.

(b)

Matrix A is symmetric as $A = A^\top$. That means that in $LDL^\top$ decomposition of A we have $U = L^\top$. Before finding $LDL^\top$ we shall firstly find LU decomposition for A: $A = LU$:

$U = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 10 & 10 \\ 2 & 10 & 14 \end{pmatrix} R_2 \leftarrow R_2 - \frac{1}{2} \cdot R_1 = \begin{pmatrix} 4 & 2 & 2 \\ 0 & 9 & 9 \\ 2 & 10 & 14 \end{pmatrix} R_3 \leftarrow R_3 - \frac{1}{2} \cdot R_1 =$
$\begin{pmatrix} 4 & 2 & 2 \\ 0 & 9 & 9 \\ 0 & 9 & 13 \end{pmatrix} R_3 \leftarrow R_3 - 1 \cdot R_2 = \begin{pmatrix} 4 & 2 & 2 \\ 0 & 9 & 9 \\ 0 & 0 & 4 \end{pmatrix} \rightarrow$
$\rightarrow l(2,1) = -(-\frac{1}{2}) = \frac{1}{2}; l(3,1) = -(-\frac{1}{2}) = \frac{1}{2}; l(3,2) = -(-1) = 1.$

Thus,

$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 1 & 1 \end{pmatrix} \rightarrow$

$\rightarrow A = LU = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 1 & 1 \end{pmatrix} \begin{pmatrix} 4 & 2 & 2 \\ 0 & 9 & 9 \\ 0 & 0 & 4 \end{pmatrix} \rightarrow checking \rightarrow \begin{pmatrix} 4 & 2 & 2 \\ 2 & 10 & 10 \\ 2 & 10 & 14 \end{pmatrix} =$
$= A.$

In $LDL^\top \rightarrow D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 4 \end{pmatrix}; L^\top = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Thus, $LDL^\top = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 1 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \to checking \to$

$\to \begin{pmatrix} 4 & 2 & 2 \\ 2 & 10 & 10 \\ 2 & 10 & 14 \end{pmatrix} = A.$

(c)

We can get a Cholesky decomposition by evenly splitting square root of diagonal elements of D between L and $L^\top$:
For L:
$\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 1 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{4} & 0 & 0 \\ 0 & \sqrt{9} & 0 \\ 0 & 0 & \sqrt{4} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 2 \end{pmatrix};$
For $L^\top$:
$\begin{pmatrix} \sqrt{4} & 0 & 0 \\ 0 & \sqrt{9} & 0 \\ 0 & 0 & \sqrt{4} \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{pmatrix}.$
$\to A = LL^\top = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{pmatrix} \to checking \to \begin{pmatrix} 4 & 2 & 2 \\ 2 & 10 & 10 \\ 2 & 10 & 14 \end{pmatrix} = A.$

(d)

$B = \begin{pmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{pmatrix}.$

Let's recall matrix definiteness dependence on value of its principal minors and respective criterions:
1) For positive definite: $D_1 > 0, D_2 > 0, D_3 > 0$;
2) Negative definite: $D_1 < 0, D_2 > 0, D_3 < 0$;
3) Positive semidefinite: $D_1 \geq 0, D_2 \geq 0, D_3 \geq 0$ for all principal minors;
4) Negative semidefinite: $D_1 \leq 0, D_2 \geq 0, D_3 \leq 0$.
For matrix B its principal minors are:
$D_1 = 1; D_2 = -15.$ $\to$ without computing D_3 we see that combination of D_1, D_2 of matrix B doesn't match any of above-mentioned criterions and this means that matrix B is indefinite.
$LU decomposition$:
$\begin{pmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{pmatrix} R_2 \leftarrow R_2 + 4 \cdot R_1 = \begin{pmatrix} 1 & -4 & 2 \\ 0 & -15 & 6 \\ 2 & -2 & -2 \end{pmatrix} R_3 \leftarrow R_3 + (-2) \cdot R_1 =$

$$\begin{pmatrix}1 & -4 & 2\\ 0 & -15 & 6\\ 0 & 6 & -6\end{pmatrix} R_3 \leftarrow R_3 + (\tfrac{2}{5})\cdot R_2 = \begin{pmatrix}1 & -4 & 2\\ 0 & -15 & 6\\ 0 & 0 & -\frac{18}{5}\end{pmatrix} = U \rightarrow$$

$$\rightarrow L = \begin{pmatrix}1 & 0 & 0\\ -4 & 1 & 0\\ 2 & -\frac{2}{5} & -6\end{pmatrix};$$

$$LU = \begin{pmatrix}1 & 0 & 0\\ -4 & 1 & 0\\ 2 & -\frac{2}{5} & 1\end{pmatrix}\begin{pmatrix}1 & -4 & 2\\ 0 & -15 & 6\\ 0 & 0 & -\frac{18}{5}\end{pmatrix} \rightarrow checking \rightarrow \begin{pmatrix}1 & -4 & 2\\ -4 & 1 & -2\\ 2 & -2 & -2\end{pmatrix} =$$

$$= B \rightarrow D = \begin{pmatrix}1 & 0 & 0\\ 0 & -15 & 0\\ 0 & 0 & -\frac{18}{5}\end{pmatrix} \rightarrow$$

$$\rightarrow LDL^\top = \begin{pmatrix}1 & 0 & 0\\ -4 & 1 & 0\\ 2 & -\frac{2}{5} & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & -15 & 0\\ 0 & 0 & -\frac{18}{5}\end{pmatrix}\begin{pmatrix}1 & -4 & 2\\ 0 & 1 & -\frac{2}{5}\\ 0 & 0 & 1\end{pmatrix} \rightarrow checking \rightarrow$$

$$\begin{pmatrix}1 & -4 & 2\\ -4 & 1 & -2\\ 2 & -2 & -2\end{pmatrix} = B.$$

Problem 11

$$A = \begin{pmatrix}1 & 0 & 0\\ 1 & 0 & 0\\ 1 & 1 & 1\end{pmatrix}$$

$$b = \begin{pmatrix}0\\ 2\\ 2\end{pmatrix}$$

$A^+ = V\Sigma^+U^\top$

The shortest solution $x^{'} = (V\Sigma^+U^\top)b$

$$\rightarrow A^\top A = \begin{pmatrix}1 & 1 & 1\\ 0 & 0 & 1\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 1 & 0 & 0\\ 1 & 1 & 1\end{pmatrix} = \begin{pmatrix}3 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}.$$

$Eigenvalues(A^\top A):$

$$\det\left(\begin{pmatrix}3 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix} - \lambda\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right) = 0;$$

$-\lambda^3 + 5\lambda^2 - 4\lambda = 0;$

$-\lambda\,(\lambda - 1)\,(\lambda - 4) = 0; \rightarrow \lambda_1 = 0; \lambda_2 = 1; \lambda_3 = 4;$

$Eigenvector(\lambda_1 = 0):$

$$(A - \lambda\, I): \begin{pmatrix}3 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix} - 0\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}3 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix} \rightarrow via{-}row{-}reduction \rightarrow$$

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 0\end{pmatrix} \to v_1 = (A - 0I)\begin{pmatrix}x\\ y\\ z\end{pmatrix} \to \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix} \to \left\{\begin{matrix}x = 0\\ y = -z\end{matrix}\right\} \to$$

$$v_1 = \begin{pmatrix}0\\ -z\\ z\end{pmatrix} = \begin{pmatrix}0\\ -1\\ 1\end{pmatrix}$$

$Eigenvector(\lambda_2 = 1):$

$$\begin{pmatrix}3 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix} - 1 \cdot \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}2 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{pmatrix} \to via - row - reduction \to$$

$$\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & -1\\ 0 & 0 & 0\end{pmatrix} \to v_2 = \begin{pmatrix}1 & 0 & 1\\ 0 & 1 & -1\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix} \to \left\{\begin{matrix}x = -z\\ y = z\end{matrix}\right\} \to$$

$$v_2 = \begin{pmatrix}-1\\ 1\\ 1\end{pmatrix}$$

$Eigenvector(\lambda_3):$

$$\begin{pmatrix}3 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix} - 4\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}-1 & 1 & 1\\ 1 & -3 & 1\\ 1 & 1 & -3\end{pmatrix} \to via - row - reduction \to$$

$$\begin{pmatrix}1 & 0 & -2\\ 0 & 1 & -1\\ 0 & 0 & 0\end{pmatrix} \to v_3 = \begin{pmatrix}1 & 0 & -2\\ 0 & 1 & -1\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix} \to \left\{\begin{matrix}x = 2z\\ y = z\end{matrix}\right\} \to v_3 = \begin{pmatrix}2\\ 1\\ 1\end{pmatrix}$$

$$\frac{\vec{v_1}}{\|v_1\|} = \begin{pmatrix}0\\ -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$$

$$\frac{\vec{v_2}}{\|v_2\|} = \begin{pmatrix}-\frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\end{pmatrix}$$

$$\frac{\vec{v_3}}{\|v_3\|} = \begin{pmatrix}\sqrt{\frac{2}{3}}\\ \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{6}}\end{pmatrix}$$

Let's sort eigenvalues from in descending order and arrange respective eigenvectors, so that λ_3 interchange with λ_1 and the same goes for eigenvectors v_1 and v_3:

$\lambda_1 = 4; \lambda_2 = 1; \lambda_3 = 0; \to \sigma_1 = 2; \sigma_2 = 1; \sigma_3 = 0.$

$$V = \begin{pmatrix}\sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{3}} & 0\\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}\end{pmatrix}$$

$u_1 = \frac{Av_1}{\sigma_1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix} \frac{1}{2} = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \sqrt{\frac{2}{3}} \end{pmatrix}$

$u_2 = \frac{Av_2}{\sigma_2} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}.$

We need to find vector u_3 orthogonal to both u_1 and u_2 to form a matrix U of SVD decomposition. To satisfy these two orthogonality conditions, the vector u_3 must be a solution of the homogeneous linear system:

$\begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow reduced - row - echelon - form \rightarrow$

$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow x_3 = 0; x_1 = -x_2 \rightarrow u_3 = \begin{pmatrix} -x_2 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \rightarrow$

$\frac{\vec{u_3}}{\|u_3\|} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$

Let's check orthogonality:

$u_2 \cdot u_3 = \begin{pmatrix} -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix} = 0$

$u_1 \cdot u_3 = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = 0 \rightarrow$

$\rightarrow u_3$ is orthogonal to u_1 and u_2. $\rightarrow U = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \end{pmatrix}$

$\Sigma = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Let's check $\rightarrow A = U\Sigma V^\top = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} =$

$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} = A \rightarrow$ we performed SVD rightly. Now let's find pseudo-inverse in order to find minimum length solution:

$$A^{+} = V\Sigma^{+}U^{\top} = \begin{pmatrix} \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} =$$
$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \end{pmatrix}. \to x^{'} = (V\Sigma^{+}U^{\top})b = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \to$$
$\to$ minimum-length least square solution.

Problem 3

(a)

$A = U\Sigma V^{\top} \to A^{\top} = (U\Sigma V^{\top})^{\top} = V\Sigma^{\top}U^{\top}.$

After transposing Σ, in Σ^{T} we have that main diagonal remains the same as in Σ, while other entries are zero. We also know that matrices V and U are orthogonal. As a conclusion, $V\Sigma^{\top}U^{T}$ is an SVD of A.

(b)

$A = uv^{\top} = (\|u\| \frac{u}{\|u\|})(\|v\| \frac{v}{\|v\|})^{\top} = (\|u\|\|v\|)\frac{u}{\|u\|}\frac{v}{\|v\|}^{\top} = \sigma\hat{u}\hat{v^{\top}}$
$, where$
$\sigma = \|u\|\|v\|;$
$\hat{u} = \frac{u}{\|u\|};$
$\hat{v} = \frac{v}{\|v\|}.$

(c)

(i)

$A = \begin{pmatrix} 0 & 1 & 2 \end{pmatrix} \to A = U\Sigma V^{\top}.$
$A^{\top}A = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{pmatrix}$
Eigenvalues $(A^{\top}A)$: $\det\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) = 0;$
$\det \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & 2 & 4-\lambda \end{pmatrix} = 0;$
$-\ \det \begin{pmatrix} 1- & 2 \\ 2 & 4- \end{pmatrix} - 0 \cdot \det \begin{pmatrix} 0 & 2 \\ 0 & 4- \end{pmatrix} + 0 \cdot \det \begin{pmatrix} 0 & 1- \\ 0 & 2 \end{pmatrix} = 0;$
$-\lambda\left(\lambda^{2} - 5\lambda\right) = 0;$
$-\lambda^{2}(\lambda - 5) = 0 \to \lambda_1 = 5; \lambda_2, \lambda_3 = 0. \to$

$\rightarrow \sigma_1 = \sqrt{5}; \sigma_2 = 0; \sigma_3 = 0.$

$Eigevector(\lambda_1 = 5):$
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{pmatrix} - 5 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -5 & 0 & 0 \\ 0 & -4 & 2 \\ 0 & 2 & -1 \end{pmatrix} R_3 \leftarrow R_3 + \frac{1}{2} \cdot R_2 = \begin{pmatrix} -5 & 0 & 0 \\ 0 & -4 & 2 \\ 0 & 0 & 0 \end{pmatrix} R_2 \leftarrow$
$-\frac{1}{4} \cdot R_2 = \begin{pmatrix} -5 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} R_1 \leftarrow -\frac{1}{5} \cdot R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{Bmatrix} x = 0 \\ y = \frac{1}{2} z \end{Bmatrix} \rightarrow$
$v_1 = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 1 \end{pmatrix} \rightarrow \frac{v_1}{|v_1|} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$
$Eigevectors(\lambda_2 = 0):$
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{pmatrix} - 0 \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{pmatrix} R_2 \leftrightarrow R_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{pmatrix} R_3 \leftarrow$
$R_3 - \frac{1}{2} \cdot R_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{pmatrix} R_2 \leftarrow \frac{1}{2} \cdot R_2 = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow y = -2z$
$\rightarrow v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \rightarrow \frac{v_2}{|v_2|} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; v_3 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \rightarrow \frac{v_3}{|v_3|} = \begin{pmatrix} 0 \\ -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$
$u_1 = \frac{Av_1}{\sigma_1} = \begin{pmatrix} 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} \frac{1}{\sqrt{5}} = 1 = U.$
$V = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \end{pmatrix};$
$A = U\Sigma V^\top = 1 \begin{pmatrix} \sqrt{5} & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 1 & 0 & 0 \\ 0 & -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \end{pmatrix} = A.$

(ii)

$A = \begin{pmatrix} 0 & 1 & 2 \end{pmatrix}^\top$
Matrix from (ii) is a transpose matrix of matrix from (i). As we know: $A^\top = (U\Sigma V^\top)^\top = V\Sigma^\top U^\top$. $\rightarrow$ we can conclude that SVD of (ii) is:
$(1 \cdot \begin{pmatrix} \sqrt{5} & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 1 & 0 & 0 \\ 0 & -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix})^\top = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \sqrt{5} \\ 0 \\ 0 \end{pmatrix} \cdot 1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = A$

(iii)

$$A = \begin{pmatrix} 2 & 1 & -2 \\ -2 & -1 & 2 \end{pmatrix} \rightarrow A^\top A = \begin{pmatrix} 8 & 4 & -8 \\ 4 & 2 & -4 \\ -8 & -4 & 8 \end{pmatrix}$$

Eigenvalues($A^\top A$):

$$\det\left(\begin{pmatrix} 8 & 4 & -8 \\ 4 & 2 & -4 \\ -8 & -4 & 8 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) = 0;$$

$$\det\begin{pmatrix} 8-\lambda & 4 & -8 \\ 4 & 2-\lambda & -4 \\ -8 & -4 & 8-\lambda \end{pmatrix} = 0;$$

$(8-\lambda)\left(\lambda^2 - 10\lambda\right) - 4\left(-4\lambda\right) - 8\left(-8\lambda\right) = 0;$
$-\lambda^2\left(\lambda - 18\right) = 0; \rightarrow \lambda_1 = 18, \lambda_2, \lambda_3 = 0; \rightarrow$
$\rightarrow \sigma_1 = \sqrt{18}; \sigma_2 = 0; \sigma_3 = 0.$

Eigenvector($\lambda_1 = 18$): $\begin{pmatrix} 8 & 4 & -8 \\ 4 & 2 & -4 \\ -8 & -4 & 8 \end{pmatrix} - 18 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -10 & 4 & -8 \\ 4 & -16 & -4 \\ -8 & -4 & -10 \end{pmatrix} R_2 \leftarrow$
$R_2 + \frac{2}{5} \cdot R_1 = \begin{pmatrix} -10 & 4 & -8 \\ 0 & -\frac{72}{5} & -\frac{36}{5} \\ -8 & -4 & -10 \end{pmatrix} R_3 \leftarrow R_3 - \frac{4}{5} \cdot R_1 = \begin{pmatrix} -10 & 4 & -8 \\ 0 & -\frac{72}{5} & -\frac{36}{5} \\ 0 & -\frac{36}{5} & -\frac{18}{5} \end{pmatrix} R_3 \leftarrow$
$R_3 - \frac{1}{2} \cdot R_2 = \begin{pmatrix} -10 & 4 & -8 \\ 0 & -\frac{72}{5} & -\frac{36}{5} \\ 0 & 0 & 0 \end{pmatrix} R_2 \leftarrow -\frac{5}{72} \cdot R_2 = \begin{pmatrix} -10 & 4 & -8 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} R_1 \leftarrow$
$R_1 - 4 \cdot R_2 = \begin{pmatrix} -10 & 0 & -10 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} R_1 \leftarrow -\frac{1}{10} \cdot R_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \left\{\begin{matrix} x = -z \\ y = -\frac{1}{2}z \end{matrix}\right\} \rightarrow$
$v_1 = \begin{pmatrix} -1 \\ -\frac{1}{2} \\ 1 \end{pmatrix} \rightarrow \frac{v_1}{|v_1|} = \begin{pmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$

Eigenvectors($\lambda_2 = 0$):
$\begin{pmatrix} 8 & 4 & -8 \\ 4 & 2 & -4 \\ -8 & -4 & 8 \end{pmatrix} - 0 \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 4 & -8 \\ 4 & 2 & -4 \\ -8 & -4 & 8 \end{pmatrix} \rightarrow$ via row-reduction
$\rightarrow \begin{pmatrix} 1 & \frac{1}{2} & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow x + \frac{1}{2}y - z = 0 \rightarrow \begin{pmatrix} -\frac{1}{2}y \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} z \\ 0 \\ z \end{pmatrix} \rightarrow v_2 = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} \rightarrow$
$\frac{v_2}{|v_2|} = \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{pmatrix} v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \rightarrow \frac{v_3}{|v_3|} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

$V = \begin{pmatrix} -\frac{2}{3} & -\frac{1}{\sqrt{5}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{3} & \frac{2}{\sqrt{5}} & 0 \\ \frac{2}{3} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$

$\Sigma = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$u_1 = \frac{Av_1}{\sigma_1} = \begin{pmatrix} 2 & 1 & -2 \\ -2 & -1 & 2 \end{pmatrix} \begin{pmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix} \frac{1}{\sqrt{18}} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

u_2 should be orthogonal to u_1: $u_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

$U = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

$A = U\Sigma V^\top = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{18} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -3 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} =$

$= \begin{pmatrix} 2 & 1 & -2 \\ -2 & -1 & 2 \end{pmatrix} = A.$

Problem 2

(a)

From Problem 3(i) we already concluded that $V\Sigma^\top U^\top$ is an SVD of $A^\top$. Since after transposing A, we have that $\Sigma^\top$ contains the same main diagonal as Σ, non-zero singular values of A and $A^\top$ are the same.

(b.i)

From Problem 3(i) we conclude that singular values of (i) are:
$\sigma_1 = \sqrt{5}; \sigma_2 = 0; \sigma_3 = 0.$

(b.ii)

From (a) and Problem 3(i) we concluded that non-zero singular values of (i) and (ii) are the same: $\sigma_1 = \sqrt{5}$.
However here we wouldn't have 3 by 3 matrix from $A^T A$ as in (b.ii), but only 1 by 1 from AA^T. That's why we don't have zero singular values here.

(b.iii)

$A = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \end{pmatrix} \to A^\top A = \begin{pmatrix} 2 & 5 & 5 \\ 5 & 13 & 13 \\ 5 & 13 & 13 \end{pmatrix}$

Eigenvalues($A^\top A$):

$\det\left(\begin{pmatrix}2 & 5 & 5\\ 5 & 13 & 13\\ 5 & 13 & 13\end{pmatrix} - \lambda\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right) = 0;$
$\det\begin{pmatrix}2-\lambda & 5 & 5\\ 5 & 13-\lambda & 13\\ 5 & 13 & 13-\lambda\end{pmatrix} = 0;$
$(2-\lambda)\left(\lambda^2 - 26\lambda\right) - 5\left(-5\lambda\right) + 5\cdot\ 5\lambda = 0;$
$-\lambda^3 + 28\lambda^2 - 2\lambda = 0;$
$-\lambda\left(\lambda^2 - 28\lambda + 2\right) = 0 = 0;$
$\rightarrow \lambda^2 - 28\lambda + 2 = 0: \quad \lambda = 14 + \sqrt{194},\ \lambda = 14 - \sqrt{194}$
$Thus,$
$\lambda_1 = 14 + \sqrt{194}; \lambda_2 = 14 - \sqrt{194}; \lambda_3 = 0$
$Thus,$ singular values are equal $\sigma_1 = \sqrt{14 + \sqrt{194}}; \sigma_2 = \sqrt{14 - \sqrt{194}}; \sigma_3 = 0.$

(b.iv)

$A = \begin{pmatrix}2 & 0\\ 1 & 2\end{pmatrix} \rightarrow A^\top A = \begin{pmatrix}5 & 2\\ 2 & 4\end{pmatrix}$
$Eigenvalues(A^\top A) : \det\left(\begin{pmatrix}5 & 2\\ 2 & 4\end{pmatrix} - \lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right) = 0$
$\begin{pmatrix}5- & 2\\ 2 & 4-\lambda\end{pmatrix} = 0$
$(5-\lambda)(4-\lambda) - 2\cdot\ 2 = 0$
$\lambda^2 - 9\lambda + 16 = 0 \rightarrow \lambda_1 = \frac{9+\sqrt{17}}{2}, \lambda_2 = \frac{9-\sqrt{17}}{2}.$
$\rightarrow \sigma_1 = \sqrt{\frac{9+\sqrt{17}}{2}}; \sigma_2 = \sqrt{\frac{9-\sqrt{17}}{2}}.$

Problem 4

$A = \begin{pmatrix}3 & 2 & 2\\ 2 & 3 & -2\end{pmatrix}$

(a) and (b)

Given that A^TA is 3 by 3 matrix, we will have 3 singular values. Number of non-zero singular values of matrix A equals to the rank of the matrix A, which can be determined by number of linearly independent columns. In any 2 by 3 matrix such number at maximum equals 2. In matrix A we have exactly 2 linearly independent vectors. It means that $rank(A) = 2 \rightarrow A$ has 2 non-zero singular values.
Non-zero singular values of A are determined by non-zero eigenvalues of A^TA:
$A^\top A = \begin{pmatrix}3 & 2\\ 2 & 3\\ 2 & -2\end{pmatrix}\begin{pmatrix}3 & 2 & 2\\ 2 & 3 & -2\end{pmatrix} = \begin{pmatrix}13 & 12 & 2\\ 12 & 13 & -2\\ 2 & -2 & 8\end{pmatrix} \rightarrow$
$\rightarrow Eigenvalues(A^\top A):$

$$\det\left(\begin{pmatrix}13 & 12 & 2\\ 12 & 13 & -2\\ 2 & -2 & 8\end{pmatrix} - \lambda\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right) = 0;$$
$$\det\begin{pmatrix}13-\lambda & 12 & 2\\ 12 & 13-\lambda & -2\\ 2 & -2 & 8-\lambda\end{pmatrix} = 0;$$
$$(13-\lambda)\left(\lambda^2 - 21\lambda + 100\right) - 12\left(-12\lambda + 100\right) + 2\left(2\lambda - 50\right) = 0;$$
$$-\lambda^3 + 34\lambda^2 - 225\lambda = 0;$$
$$-\lambda\left(\lambda - 9\right)\left(\lambda - 25\right) = 0; \rightarrow \lambda_1 = 25; \lambda_2 = 9; \lambda_3 = 0; \rightarrow$$
$$\rightarrow \sigma_1 = \sqrt{25} = 5; \sigma_2 = \sqrt{9} = 3.$$

Right-singular vectors of A are orthonormal eigenvectors of $A^\top A$:
$Eigenvector(\lambda_1 = 25):$
$$\begin{pmatrix}13 & 12 & 2\\ 12 & 13 & -2\\ 2 & -2 & 8\end{pmatrix} - 25\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}-12 & 12 & 2\\ 12 & -12 & -2\\ 2 & -2 & -17\end{pmatrix} R_2 \leftarrow R_2 + 1 \cdot R_1 =$$
$$\begin{pmatrix}-12 & 12 & 2\\ 0 & 0 & 0\\ 2 & -2 & -17\end{pmatrix} R_3 \leftarrow R_3 + \tfrac{1}{6} \cdot R_1 = \begin{pmatrix}-12 & 12 & 2\\ 0 & 0 & 0\\ 0 & 0 & -\frac{50}{3}\end{pmatrix} R_3 \leftarrow -\tfrac{3}{50} \cdot$$
$$R_3 = \begin{pmatrix}-12 & 12 & 2\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix} R_1 \leftarrow R_1 - 2 \cdot R_3 = \begin{pmatrix}-12 & 12 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix} R_2 \leftrightarrow R_3 =$$
$$\begin{pmatrix}-12 & 12 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix} R_1 \leftarrow -\tfrac{1}{12} \cdot R_1 = \begin{pmatrix}1 & -1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix} \rightarrow$$
$$\rightarrow (A - 25I)\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix} \rightarrow \begin{pmatrix}1 & -1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix} \rightarrow \begin{Bmatrix}z = 0\\ x = y\end{Bmatrix} \rightarrow v_1 =$$
$$= \begin{pmatrix}y\\ y\\ 0\end{pmatrix} \rightarrow v_1 = \begin{pmatrix}1\\ 1\\ 0\end{pmatrix} \rightarrow \tfrac{v_1}{\|v_1\|} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\\ 0\end{pmatrix}$$
$Eigenvector(\lambda_1 = 9):$
$$\begin{pmatrix}13 & 12 & 2\\ 12 & 13 & -2\\ 2 & -2 & 8\end{pmatrix} - 9\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}4 & 12 & 2\\ 12 & 4 & -2\\ 2 & -2 & -1\end{pmatrix} R_1 \leftrightarrow R_2 = \begin{pmatrix}12 & 4 & -2\\ 4 & 12 & 2\\ 2 & -2 & -1\end{pmatrix} R_2 \leftarrow$$
$$R_2 - \tfrac{1}{3} \cdot R_1 = \begin{pmatrix}12 & 4 & -2\\ 0 & \frac{32}{3} & \frac{8}{3}\\ 2 & -2 & -1\end{pmatrix} R_3 \leftarrow R_3 - \tfrac{1}{6} \cdot R_1 = \begin{pmatrix}12 & 4 & -2\\ 0 & \frac{32}{3} & \frac{8}{3}\\ 0 & -\frac{8}{3} & -\frac{2}{3}\end{pmatrix} R_3 \leftarrow$$

$R_3+\frac{1}{4}\cdot R_2 = \begin{pmatrix} 12 & 4 & -2 \\ 0 & \frac{32}{3} & \frac{8}{3} \\ 0 & 0 & 0 \end{pmatrix} R_2 \leftarrow \frac{3}{32}\cdot R_2 = \begin{pmatrix} 12 & 4 & -2 \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \end{pmatrix} R_1 \leftarrow R_1-4\cdot R_2 =$
$\begin{pmatrix} 12 & 0 & -3 \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \end{pmatrix} R_1 \leftarrow \frac{1}{12}\cdot R_1 = \begin{pmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \end{pmatrix} \rightarrow$
$\rightarrow (A-9I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{Bmatrix} x = \frac{1}{4}z \\ y = -\frac{1}{4}z \end{Bmatrix} \rightarrow$
$\rightarrow v_2 = \begin{pmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ 1 \end{pmatrix} \rightarrow \frac{v_2}{\|v_2\|} = \begin{pmatrix} \frac{1}{3\sqrt{2}} \\ -\frac{1}{3\sqrt{2}} \\ \frac{2\sqrt{2}}{3} \end{pmatrix}$
$Eigenvector(\lambda_3 = 0):$
$\begin{pmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{pmatrix} - 0\cdot\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{pmatrix} R_2 \leftarrow R_2 - \frac{12}{13}\cdot R_1 =$
$\begin{pmatrix} 13 & 12 & 2 \\ 0 & \frac{25}{13} & -\frac{50}{13} \\ 2 & -2 & 8 \end{pmatrix} R_3 \leftarrow R_3 - \frac{2}{13}\cdot R_1 = \begin{pmatrix} 13 & 12 & 2 \\ 0 & \frac{25}{13} & -\frac{50}{13} \\ 0 & -\frac{50}{13} & \frac{100}{13} \end{pmatrix} R_2 \leftrightarrow R_3 =$
$\begin{pmatrix} 13 & 12 & 2 \\ 0 & -\frac{50}{13} & \frac{100}{13} \\ 0 & \frac{25}{13} & -\frac{50}{13} \end{pmatrix} R_3 \leftarrow R_3 + \frac{1}{2}\cdot R_2 = \begin{pmatrix} 13 & 12 & 2 \\ 0 & -\frac{50}{13} & \frac{100}{13} \\ 0 & 0 & 0 \end{pmatrix} R_2 \leftarrow -\frac{13}{50}\cdot$
$R_2 = \begin{pmatrix} 13 & 12 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} R_1 \leftarrow R_1 - 12\cdot R_2 = \begin{pmatrix} 13 & 0 & 26 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} R_1 \leftarrow \frac{1}{13}\cdot$
$R_1 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{Bmatrix} x = -2z \\ y = 2z \end{Bmatrix} \rightarrow v_3 =$
$\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \rightarrow \frac{v_3}{\|v_3\|} = \begin{pmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$

Thus, right singular vectors are:
$v_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}; v_2 = \begin{pmatrix} \frac{1}{3\sqrt{2}} \\ -\frac{1}{3\sqrt{2}} \\ \frac{2\sqrt{2}}{3} \end{pmatrix}; v_3 = \begin{pmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}.$

(c)

Left singular vectors are:

$$u_1 = \frac{Av_1}{\sigma_1} = \frac{Av_1}{\sqrt{\lambda_1}} = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \frac{1}{5} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$u_2 = \frac{Av_2}{\sqrt{\lambda_2}} = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix} \begin{pmatrix} \frac{1}{3\sqrt{2}} \\ -\frac{1}{3\sqrt{2}} \\ \frac{2\sqrt{2}}{3} \end{pmatrix} \frac{1}{3} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

(d)

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$V = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & -\frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} & \frac{2}{3} \\ 0 & \frac{2\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}$$

$$A = U\Sigma V^\top = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} & \frac{2\sqrt{2}}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix} =$$

$$= A.$$

Problem 1

(a)

Let $A = [a_1, \dots, a_n]$, where a_i is a column vector of A.

For any A with rank r there exists a set of vectors, which comprises a basis of column space of A:

$$\left(\forall A, \exists \{u_1, \dots, u_r\} \mid \{u_1, \dots, u_r\} \text{ is a basis for } C(A)\right),$$

where r – rank of A.

It means that every column vector of A can be created as a linear combination of $\{u_1, \dots u_z\}$:

$$a_i = v_1 u_1 + \dots + v_n u_r = \sum_{j=1}^{r} v_j^{(i)} u_j \,,$$

where

- $v_j^{(i)}$ is a scalar for certain basis vector u_j;
- v_j is a vector of scalars for a col. vector of A;
- i is an index of the column vector of A.

Thus,

$$A = [a_1 \dots a_n] = \left[\left(v_1^{(1)} u_1 + \dots + v_r^{(1)} u_r\right) \dots \left(v_1^{(n)} u_1 + \dots v_r^{(n)} u_r\right)\right] =$$

$$= \left[\sum_{j=1}^{r} v_j^{(1)} u_j \dots \sum_{j=1}^{r} v_j^{(n)} u_j\right] = \sum_{j=1}^{r} u_j v_j^T.$$

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Given that there is an infinite ways to construct matrix A from different sets of basis vectors, we can conclude that representation of A as r summands $u_i v_i^T$ of rank 1 is not unique.

(b)

$$A = \begin{pmatrix} 1 & -2 & 3 & -4 \\ -2 & 4 & -6 & 8 \\ 3 & -6 & 9 & -12 \\ -4 & 8 & -12 & 16 \end{pmatrix} \longrightarrow$$ we see that all columns of A are just multiples of first column.

It means, that A has rank of 1 with $u = \begin{pmatrix} 1 \\ -2 \\ 3 \\ -4 \end{pmatrix}$ being a basis.

Thus, we need only to find a vector v, such that $A = uv^T$.

We see that vector v of coefficients is $\begin{pmatrix} 1 \\ -2 \\ 3 \\ -4 \end{pmatrix}$ because first column is just a multiple of itself $a_1 = 1u$, while other cols are $a_2 = -2u$, $a_3 = 3u$, $a_4 = -4u$.

$$\text{So, } uv^T = \begin{pmatrix} 1 \\ -2 \\ 3 \\ -4 \end{pmatrix} \begin{pmatrix} 1 & -2 & 3 & -4 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 3 & -4 \\ -2 & 4 & -6 & 8 \\ 3 & -6 & 9 & -12 \\ -4 & 8 & -12 & 16 \end{pmatrix} = A.$$

$$B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix}$$

It seems like we can construct matrix B from 2 basis vectors: $(1\ 1\ 1\ 1)^T$ and first column vector of B, that is, use their linear combination.

Thus, let $\underbrace{\text{Basis}}_{\text{of column space of } B} = \{u_1, u_2\} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \right\}$.

Then: $0u_1 + 1u_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$;

$1u_1 + 1u_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$;

$2u_1 + 1u_2 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$;

$3u_1 + 1u_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$.

Thus, for v_1 we will use $\begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$ and for $v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

Therefore $B = u_1 v_1^T + u_2 v_2^T$, where

$u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, u_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, v_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

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Problem 9

(a)

Let $A = U\Sigma V^T.$

$A^+ =$ [illegible] $(A^TA)^{-1}A^T = ((V\Sigma^T U^T)(U\Sigma V^T))^{-1} V\Sigma U^T =$

$= (V\Sigma^2 V^T)^{-1} V\Sigma U^T = (V^T)^{-1}\Sigma^{-2}V^{-1}V\Sigma U^T = V\Sigma^{-2}\Sigma U^T =$

$= V\Sigma^{-1}U^T = V\Sigma^+U^T \longrightarrow (V\Sigma^+U^T)A = V\Sigma^+U^TU\Sigma V^T =$

$= V\Sigma^+\Sigma V^T = VV^T = I \longrightarrow A^+$ is a left inverse of A.

$AA^+ = (U\Sigma V^T)(VS^+U^T) = U\Sigma\Sigma^+U^T.$

(b)(1).

$AA^+A = (U\Sigma V^T)(V\Sigma^+U^T)(U\Sigma V^T) =$

$= U\Sigma V^TV\Sigma^+\Sigma V^T = U\Sigma V^T = A.$

(b)(2).

$A^+AA^+ = (V\Sigma^+U^T)(U\Sigma V^T)(V\Sigma^+U^T) =$

$V\Sigma^+\Sigma\Sigma^+U^T = V\Sigma^+U^T = A^+$

(B)(3)

$$(AA^+)^T = ((U\Sigma V^T)(V\Sigma^+U^T))^T =$$

$$= (U\Sigma\Sigma^+U^T)^T = U(\Sigma\Sigma^+)^T U^T = U\Sigma\Sigma^+U^T =$$

$$= AA^+ \rightarrow AA^+ \text{ is symmetric.}$$

$$(A^+A)^T = ((V\Sigma^+U^T)(U\Sigma V^T))^T = (V\Sigma^+\Sigma V^T)^T =$$

$$= V(\Sigma^+\Sigma)^T V^T = V\Sigma^+\Sigma V^T = A^+A \rightarrow$$

$$\rightarrow A^+A \text{ is symmetric.}$$

(c)

Let $A = QR$ and $A^+ = (A^TA)^{-1}A^T \rightarrow$

$\rightarrow$ Then

$$A^+ = ((QR)^T(QR))^{-1}(QR)^T = (R^TQ^TQR)^{-1}R^TQ^T =$$

by $(Q^TQ)=I$

$$= (R^TR)^{-1}R^TQ^T = R^{-1}(R^T)^{-1}R^TQ^T = R^{-1}IQ^T =$$

$$= R^{-1}Q^T.$$

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