## Problem 1

### (a)

$A = \left(\begin{array}{cc|c} 1 & 2 & 4 \\ 3 & k & 8 \end{array}\right)$.

$\left(\begin{array}{cc|c} 1 & 2 & 4 \\ 3 & k & 8 \end{array}\right) R_1 \leftrightarrow R_2 = \left(\begin{array}{cc|c} 3 & k & 8 \\ 1 & 2 & 4 \end{array}\right) R_2 \leftarrow R_2 - \frac{1}{3} \cdot R_1 = \left(\begin{array}{cc|c} 3 & k & 8 \\ 0 & \frac{6-k}{3} & \frac{4}{3} \end{array}\right)$.

A is a matrix of inconsistent linear system if its left-hand side is zero while right-hand side is not zero or if one equation of the system contradicts other. Let's solve for which k LHS is zero in the second row:

$\frac{6-k}{3} = 0$
$6 - k = 0$
$k = 6$.

Answer: linear system is consistent for all $k \neq 6$.

Geometrical interpretation: both equations can be represented in xy-plane($R^2$) as two distinct lines. Solution would represent a point of intersection between these two lines.

### (b)

I will use the same procedure as in (a) for (b):
$A = \left(\begin{array}{cc|c} 1 & 2 & -2 \\ -2 & k & 4 \end{array}\right)$.

$\left(\begin{array}{cc|c} 1 & 2 & -2 \\ -2 & k & 4 \end{array}\right) R_1 \leftrightarrow R_2 = \left(\begin{array}{cc|c} -2 & k & 4 \\ 1 & 2 & -2 \end{array}\right) R_2 \leftarrow R_2 + \frac{1}{2} \cdot R_1 = \left(\begin{array}{cc|c} -2 & k & 4 \\ 0 & \frac{4+k}{2} & 0 \end{array}\right)$.

RHS is zero,which means that system has solution for any k.

Thus, system is consistent for every k $\in R$.

Geometrical interpretation: same situation - both equations are two distinct lines in $R^2$, with a solution in the form of intersection point.

### (c)

I will use the same procedure as in (a) for (c):
$A = \left(\begin{array}{cc|c} 4 & -2 & k \\ -2 & 1 & -3 \end{array}\right)$.

$\left(\begin{array}{cc|c} 4 & -2 & k \\ -2 & 1 & -3 \end{array}\right) R_2 \leftarrow R_2 + \frac{1}{2} \cdot R_1 = \left(\begin{array}{cc|c} 4 & -2 & k \\ 0 & 0 & \frac{-6+k}{2} \end{array}\right)$

$\frac{-6+k}{2} = 0$;

$-6+k=0$;
$k=6$.
Thus, system is consistent only when $k=6$.

Geometrical interpretation: both equations for $k=6$ represent coincident lines in $R^2$ with infinitely many solutions because they overlap with each other.

## Problem 2

$A = \left(\begin{array}{ccc|c} a & 0 & b & 1 \\ a & a & 2 & 2 \\ 0 & a & 1 & b \end{array}\right).$

$\left(\begin{array}{ccc|c} a & 0 & b & 1 \\ a & a & 2 & 2 \\ 0 & a & 1 & b \end{array}\right) R_1 \leftrightarrow R_2 = \left(\begin{array}{ccc|c} a & a & 2 & 2 \\ a & 0 & b & 1 \\ 0 & a & 1 & b \end{array}\right) R_2 \leftarrow R_2 - 1 \cdot R_1 =$

$= \left(\begin{array}{ccc|c} a & a & 2 & 2 \\ 0 & -a & b-2 & -1 \\ 0 & a & 1 & b \end{array}\right) R_2 \leftrightarrow R_3 =$

$= \left(\begin{array}{ccc|c} a & a & 2 & 2 \\ 0 & a & 1 & b \\ 0 & -a & b-2 & -1 \end{array}\right) R_3 \leftarrow R_3 + 1 \cdot R_2 = \left(\begin{array}{ccc|c} a & a & 2 & 2 \\ 0 & a & 1 & b \\ 0 & 0 & b-1 & b-1 \end{array}\right)$

## (d) - no solution case

For our system this occurs only when $a=0$ and $b=c_2$, such that $c_2 \neq 1$. In this case from 1st,2d,3d rows we have:

$\begin{cases} x_3 = 1 \\ x_3 \neq 1 \\ x_3 = 1 \end{cases}$ ,that is a contradiction.

Such system clearly doesn't have any solutions.

## (c) - two-parameter solution set

A linear system has two-parameter solution set when it has two free variables. For our system this occurs only when $a=0$ and $b=1$. This way last row is zero, while first and second are:

$\begin{cases} 2x_3 = 2 \\ x_3 = 1 \end{cases}$

Thus,general two-parameter solution set for $a=0$ and $b=1$ is:

$\begin{cases} x_3 = 1 \\ x_2 - free \\ x_1 - free \end{cases}$

## (b) - one-parameter solution set

A linear system has one-parameter solution set when it has one free variable. For our system this occurs only when $a = c_1$, such that $c_1 \neq 0$ and $b = 1$. This way last row is zero, while first and second are:

$$\begin{cases} ax_1 + ax_2 + 2x_3 = 2 \\ ax_2 + x_3 = 1 \end{cases} = \begin{cases} ax_1 + ax_2 + 2x_3 = 2 \\ x_2 = \frac{1-x_3}{a} \end{cases} = \begin{cases} x_1 = \frac{2-a\frac{1-x_3}{a}-2x_3}{a} \\ x_2 = \frac{1-x_3}{a} \end{cases}$$

Thus, for $a \neq 0$ and $b = 1$ general one-parameter solution set is:

$$\begin{cases} x_1 = \frac{2-a\frac{1-x_3}{a}-2x_3}{a} \\ x_2 = \frac{1-x_3}{a} \\ x_3 - free \end{cases}$$

## (a) - unique solution

A linear system has a unique solution when it has no free variables. For our system this occurs only when $a = c_1$ and $b = c_2$,such that $a \neq 0$ and $b \neq 1$. This way linear system is:

$$\begin{cases} c_1x_1 + c_1x_2 + 2x_3 = 2 \\ c_1x_2 + x_3 = c_2 \\ (c_2 - 1)x_3 = c_2 - 1 \end{cases} = \begin{cases} c_1x_1 + c_1x_2 + 2x_3 = 2 \\ c_1x_2 + x_3 = c_2 \\ x_3 = 1 \end{cases} = \begin{cases} x_1 = \frac{c_1\frac{c_2-1}{c_1}}{c_1} \\ x_2 = \frac{c_2-1}{c_1} \\ x_3 = 1 \end{cases}$$

Thus, for $a \neq 0$ and $b \neq 1$ unique solution is:

$$\begin{cases} x_1 = \frac{c_1\frac{c_2-1}{c_1}}{c_1} \\ x_2 = \frac{c_2-1}{c_1} \\ x_3 = 1 \end{cases}$$

We see that there is no free variables in such a linear system.

## Problem 4

## (i) - homogeneous case

## (a)

$A = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$

We see that column vectors are independent (can't be constructed from scalar multiples of each other), which means that coefficient matrix in $\left(\begin{array}{cc|c} 1 & 3 & 0 \\ 2 & 1 & 0 \end{array}\right)$ is non-singular, providing in homogeneous case only trivial unique solution - zero vector in $R^2$.

## (b)

$A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 0 \end{pmatrix}$

We see that coefficient matrix has 1 dependent vector (free column), providing in homogeneous case non-trivial solutions that depend on free variable $x_3$. Let's for fun see it by row-reduction of coefficient matrix:

$\begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 0 \end{pmatrix} R_1 \leftrightarrow R_2 = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 2 \end{pmatrix} R_2 \leftarrow R_2 - \frac{1}{2} \cdot R_1 = \begin{pmatrix} 2 & 1 & 0 \\ 0 & \frac{5}{2} & 2 \end{pmatrix} R_2 \leftarrow \frac{2}{5} \cdot R_2 = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & \frac{4}{5} \end{pmatrix} R_1 \leftarrow R_1 - 1 \cdot R_2 = \begin{pmatrix} 2 & 0 & -\frac{4}{5} \\ 0 & 1 & \frac{4}{5} \end{pmatrix} R_1 \leftarrow \frac{1}{2} \cdot R_1 = \begin{pmatrix} 1 & 0 & -\frac{2}{5} \\ 0 & 1 & \frac{4}{5} \end{pmatrix}$

Thus, there can be infinitely many solutions.
General solution is:

$\begin{cases} x_1 = \frac{2}{5} x_3 \\ x_2 = -\frac{4}{5} x_3 \\ x_3 = free \end{cases}$

## (c)

$A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 0 \end{pmatrix}$

As in (a), coefficient matrix is non-singular, providing only trivial unique solution in homogeneous case - zero vector in $R^2$.

## (d)

$A = \begin{pmatrix} 1 & 4 & 3 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

As in (a) and (c), coefficient matrix is non-singular, providing only trivial unique solution for homogeneous case - zero vector in $R^3$. But let's for fun row reduce it:

$\begin{pmatrix} 1 & 4 & 3 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} R_1 \leftrightarrow R_2 = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 4 & 3 \\ 1 & 1 & 1 \end{pmatrix} R_2 \leftarrow R_2 - \frac{1}{2} \cdot R_1 = \begin{pmatrix} 2 & 1 & 0 \\ 0 & \frac{7}{2} & 3 \\ 1 & 1 & 1 \end{pmatrix} R_3 \leftarrow R_3 - \frac{1}{2} \cdot R_1 = \begin{pmatrix} 2 & 1 & 0 \\ 0 & \frac{7}{2} & 3 \\ 0 & \frac{1}{2} & 1 \end{pmatrix} R_3 \leftarrow R_3 - \frac{1}{7} \cdot R_2 = \begin{pmatrix} 2 & 1 & 0 \\ 0 & \frac{7}{2} & 3 \\ 0 & 0 & \frac{4}{7} \end{pmatrix} R_3 \leftarrow \frac{7}{4} \cdot R_3 = \begin{pmatrix} 2 & 1 & 0 \\ 0 & \frac{7}{2} & 3 \\ 0 & 0 & 1 \end{pmatrix} R_2 \leftarrow R_2 - 3 \cdot R_3 = \begin{pmatrix} 2 & 1 & 0 \\ 0 & \frac{7}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} R_2 \leftarrow \frac{2}{7} \cdot R_2 = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} R_1 \leftarrow$

$$R_1 - 1 \cdot R_2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} R_1 \leftarrow \frac{1}{2} \cdot R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus, homogeneous case has only trivial solution.

## (ii) - generic RHS case

### (a)

$A = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$

We have established in (i)(a) that column vectors of coefficient matrix A are independent. For 2 vectors being independent in $R^2$ means they form basis for $R^2$. This means that there exist unique solution for any generic RHS.

### (b)

$A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 0 \end{pmatrix}$

We established in (i)(b) that this coefficient matrix is dependent and provide infinitely many solutions for homogeneous case. Having instead generic right-hand side won't change that fact, as $x_3$ remains free variable.

I can do row reduction to find general solution, but I don't see it is necessary as a wording of the Problem 4 demands only answer about number of solution, which, as we see, is infinite.

### (c)

$A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 0 \end{pmatrix}$

Let's row reduce RHS to see number of solutions:

$$\left(\begin{array}{cc|c} 1 & 2 & a \\ 3 & 1 & b \\ 2 & 0 & c \end{array}\right) R_1 \leftrightarrow R_2 = \left(\begin{array}{cc|c} 3 & 1 & b \\ 1 & 2 & a \\ 2 & 0 & c \end{array}\right) R_2 \leftarrow R_2 - \frac{1}{3} \cdot R_1 = \left(\begin{array}{cc|c} 3 & 1 & b \\ 0 & \frac{5}{3} & \frac{3a-b}{3} \\ 2 & 0 & c \end{array}\right) R_3 \leftarrow$$
$$R_3 - \frac{2}{3} \cdot R_1 = \left(\begin{array}{cc|c} 3 & 1 & b \\ 0 & \frac{5}{3} & \frac{3a-b}{3} \\ 0 & -\frac{2}{3} & \frac{3c-2b}{3} \end{array}\right) R_3 \leftarrow R_3 + \frac{2}{5} \cdot R_2 = \left(\begin{array}{cc|c} 3 & 1 & b \\ 0 & \frac{5}{3} & \frac{3a-b}{3} \\ 0 & 0 & \frac{-4b+2a+5c}{5} \end{array}\right)$$

This system can have solutions only when $-4b + 2a + 5c = 0$

This means that for generic RHS no solution exists. Additionally, this can be explained by saying that span of two column vectors here is a plane in $R^3$. So only for those vectors that are in vectors' span (on this plane) can be solution found, but we have an infinitely many vectors in $R^3$ , for which there will be no solution.

### (d)

$A = \begin{pmatrix} 1 & 4 & 3 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

As we established in (i)(d), coefficient matrix of A has 3 independent vectors in $R^3$. That means that such vectors form a basis for $R^3$ and provide a unique solution for any generic RHS.

## Problem 6

### (a)

We will solve for which $k$ given column vectors are linearly independent by finding in what case determinant of matrix composed of such vectors equals zero.
Row expansion formula for determinant of any 3 x 3 matrix:
$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a \cdot \det \begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \cdot \det \begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \cdot \det \begin{pmatrix} d & e \\ g & h \end{pmatrix}$
$A = \begin{pmatrix} 1 & -2 & 4 \\ 2 & 3 & k \\ -2 & 5 & 2 \end{pmatrix}$
Given above formula:

$det(A) = 1 \cdot \det \begin{pmatrix} 3 & k \\ 5 & 2 \end{pmatrix} - (-2) \det \begin{pmatrix} 2 & k \\ -2 & 2 \end{pmatrix} + 4 \cdot \det \begin{pmatrix} 2 & 3 \\ -2 & 5 \end{pmatrix} =$
$= 1 \cdot (6 - 5k) - (-2)\,(4 + 2k) + 4 \cdot\ 16 = 6 - 5k + 8 + 4k + 64 = -k + 78$

Thus, $det(A) = -k + 78$

Let's solve for root:
$-k + 78 = 0$
$k = 78$

Answer: column vectors of matrix A are linearly independent when k = 78.

### (b)

It is obvious that second column vector $(-2, -4, 2)^T$ of this task can be created by multiplication $(1, 2, 1)^T$ by two. Therefore, no k can lead to this set of vectors to be linearly independent.
However, I will confirm it by the same technical approach as in (a):

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & -4 & k \\ -1 & 2 & -3 \end{pmatrix}$$
$$det(A) = 1 \cdot \det \begin{pmatrix} -4 & k \\ 2 & -3 \end{pmatrix} - (-2) \det \begin{pmatrix} 2 & k \\ -1 & -3 \end{pmatrix} + 3 \cdot \det \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} =$$
$$= 1 \cdot (12 - 2k) - (-2)(-6 + k) + 3 \cdot \ 0 = 12 - 2k - 12 + 2k + 0 = 0.$$

Answer: there is no k for which column vectors of matrix A are linearly independent

## Problem 8

As all edges of the given parallelepiped come out from the same origin $(0, 0, 0)^T$ we can represent edges as three vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}$ in 3-dimensional space:
$$v_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}; v_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}; v_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$
Volume of the parallelepiped with given edges is an absolute value of determinant of the matrix A with column vectors, that represent those edges:
$$A = [\vec{v_1} \ \vec{v_2} \ \vec{v_3}] = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$
$$Volume = det(A) = 2 \cdot \det \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - 1 \cdot \det \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} + 1 \cdot \det \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = 2 \cdot \ 3 -$$
$$1 \cdot \ 1 + 1 \cdot (-1) =$$
$$= 4.$$
Volume of parallelepiped is 4.

Parallelogram has 6 faces. As opposite faces of parallelepiped are congruent, we need to find only areas of 3 faces $(A_1, A_2, A_3)$.
Area of each face is equal to Euclidean norm of cross-product between vectors corresponding to each face.
Formula of cross-product:
$$\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2, \ a_3 b_1 - a_1 b_3, \ a_1 b_2 - a_2 b_1)^T \ .$$
$$area(A_1) = \|\vec{v_1} \times \vec{v_2}\| = \left\| \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix} \right\| = \sqrt{11}.$$
$$area(A_2) = \|\vec{v_1} \times \vec{v_3}\| = \left\| \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \right\| = \sqrt{11}.$$
$$area(A_3) = \|\vec{v_2} \times \vec{v_3}\| = \left\| \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} \right\| = \sqrt{11}.$$
Answer: Volume of parallelepiped is 4 and area of each its face is $\sqrt{11}$.

## Problem 9

## 1.Finding det(B)

$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$
$det(A) = 5;$
$a \det \begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \det \begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \det \begin{pmatrix} d & e \\ g & h \end{pmatrix} = 5;$
$a\,(ei - fh) - b\,(di - fg) + c\,(dh - eg) = 5;$
$aei - afh - bdi + bfg + cdh - ceg = 5;$

$B = \begin{pmatrix} a+d & b+e & c+f \\ d+g & e+h & f+i \\ g+a & h+b & i+c \end{pmatrix}$
$det(B) = (a{+}d) \det \begin{pmatrix} e+h & f+i \\ h+b & i+c \end{pmatrix} - (b{+}e) \det \begin{pmatrix} d+g & f+i \\ g+a & i+c \end{pmatrix} + (c{+}f) \det \begin{pmatrix} d+g & e+h \\ g+a & h+b \end{pmatrix} =$

$= (a{+}d)((e+h)\,(i+c) - (f+i)\,(h+b)) - (b{+}e)((d+g)\,(i+c) - (f+i)\,(g+a)) +$
$(c+f)((d+g)\,(h+b) - (e+h)\,(g+a)) =$

$= (a+d)(ch - fh + ec + ei - bi - bf) - (b+e)(cg - fg + dc + di - ai - af) +$
$(c+f)(-eg + bg - ea + db + dh - ah) =$

$= (ach - afh + eac + eai - abi - abf + dch - dfh + edc + edi - dbi - dbf) -$
$(-bcg + bfg - dbc - dbi + abi + abf - ecg + efg - edc - edi + eai + eaf) + (-ecg +$
$bcg - eac + dbc + dch - ach - efg + bfg - eaf + dbf + dfh - afh) =$

$= 2eai - 2afh - 2bdi + 2bfg + 2cdh - 2ecg =$

$= 2(eai - afh - bdi + bfg + cdh - ecg) =$

$= 2(det(A)) = 2(5) =$

$= 10$

## 2.Finding det(C)

$C = \begin{pmatrix} a-d & b-e & c-f \\ d-g & e-h & f-i \\ g-a & h-b & i-c \end{pmatrix}$
$det(C) = (a{-}d) \det \begin{pmatrix} e-h & f-i \\ h-b & i-c \end{pmatrix} - (b{-}e) \det \begin{pmatrix} d-g & f-i \\ g-a & i-c \end{pmatrix} + (c{-}f) \det \begin{pmatrix} d-g & e-h \\ g-a & h-b \end{pmatrix} =$

$= (a-d)(ch - fh - ec + ei - bi + bf) - (b-e)(cg - fg - dc + di - ai + af) +$

$(c-f)(-eg+bg+ea-db+dh-ah) =$

$= (ach - afh - eac + eai - abi + abf - dch + dfh + edc - edi + dbi - dbf) - (bcg - bfg - dbc + dbi - abi + abf - ecg + efg + edc - edi + eai - eaf) + (-ecg + bcg + eac - dbc + dch - ach + efg - bfg - eaf + dbf - dfh + afh) =$

$= bfg - efg + efg - bfg + dfh - dfh - edi + edi - bcg + ecg - ecg + bcg - dch + dch =$

$= -efg + efg + ecg - ecg =$

$= 0.$

## Problem 13

To determine for what $c$ a set of vectors is a basis for $R^3$ we will check:

1) Whether there are exactly 3 vectors in such set (as two or one vectors can only span plane or line in $R^3$ and more than 3 vectors in $R^3$ are not linearly independent by definition);

2) For what $c$ such set is linearly independent, namely for what $c$ matrix constructed from such vectors has determinant not equal to zero, as such matrix will be invertible, having only trivial solution and independent column vectors, which span $R^n$ (in our case - $R^3$).

### (a)

$V = (c,1,1)^T, (1,-1,2)^T, (3,4,-1)^T.$

$A = \begin{pmatrix} c & 1 & 3 \\ 1 & -1 & 4 \\ 1 & 2 & -1 \end{pmatrix}$

$det(A) = c\det\begin{pmatrix} -1 & 4 \\ 2 & -1 \end{pmatrix} - 1\cdot\det\begin{pmatrix} 1 & 4 \\ 1 & -1 \end{pmatrix} + 3\cdot\det\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} = c\,(-7) - 1\cdot(-5) + 3\cdot\ 3 = -7c + 14$

Let's solve for c:
$-7c + 14 = 0;$
$c = 2$
Answer: set of vectors V is a basis for $R^3$ for all $c \neq 2$

### (b)

$V = (c,1,1)^T, (1,-1,2)^T, (-2,2,-4)^T.$
This set of vectors can't be a basis for $R^3$ for any value of $c$ because third vector

$\begin{pmatrix} -2 \\ 2 \\ -4 \end{pmatrix}$ can be constructed from second $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ by multiplication by $-2$.
Answer: set of vectors is linearly dependent and can't be a basis for $R^3$ for any $c$.

### (c)

$V = (c, 1, 1)^T, (1, 1, 0)^T, (0, 1, 2)^T, (3, 0, -1)^T$.

Answer: this set of vectors also can't be a basis for $R^3$ for any value of $c$ as any basis of $R^n$ will contain exactly n linearly independent vectors but we have 4 vectors here, instead of 3.

### (d)

$V = (c, 1, 1)^T, (1, 0, 1)^T$.

Answer: the number of vectors in this set is less than three. Hence, it can't span a 3 dimensional space for any value of $c$.

## Problem 10

$$A - \lambda I = \begin{pmatrix} a-\lambda & b & c & d \\ a & b-\lambda & c & d \\ a & b & c-\lambda & d \\ a & b & c & d-\lambda \end{pmatrix}$$

Let's just find determinant of given matrix and find its roots to find what values of $\lambda$ make determinant equal zero, making matrix singular. Before that, let's reduce matrix to upper-triangular form to quickly compute determinant:

$$\begin{pmatrix} a-\lambda & b & c & d \\ a & b-\lambda & c & d \\ a & b & c-\lambda & d \\ a & b & c & d-\lambda \end{pmatrix} R_1 \leftrightarrow R_4 = \begin{pmatrix} a & b & c & d-\lambda \\ a & b-\lambda & c & d \\ a & b & c-\lambda & d \\ a-\lambda & b & c & d \end{pmatrix} R_2 \leftarrow$$

$$R_2 - 1 \cdot R_1 = \begin{pmatrix} a & b & c & d-\lambda \\ 0 & -\lambda & 0 & \lambda \\ 0 & 0 & -\lambda & \lambda \\ a-\lambda & b & c & d \end{pmatrix} R_4 \leftarrow R_4 - \tfrac{a-\lambda}{a} \cdot R_1 = \begin{pmatrix} a & b & c & d-\lambda \\ 0 & -\lambda & 0 & \lambda \\ 0 & 0 & -\lambda & \lambda \\ 0 & \frac{\lambda b}{a} & \frac{\lambda c}{a} & \frac{a\lambda+\lambda d-^2}{a} \end{pmatrix} R_2 \leftrightarrow$$

$$R_4 = \begin{pmatrix} a & b & c & d-\lambda \\ 0 & \frac{\lambda b}{a} & \frac{\lambda c}{a} & \frac{a\lambda+\lambda d-\lambda^2}{a} \\ 0 & 0 & -\lambda & \lambda \\ 0 & -\lambda & 0 & \lambda \end{pmatrix} R_4 \leftarrow R_4 + \tfrac{a}{b} \cdot R_2 = \begin{pmatrix} a & b & c & d-\lambda \\ 0 & \frac{\lambda b}{a} & \frac{\lambda c}{a} & \frac{a\lambda+\lambda d-\lambda^2}{a} \\ 0 & 0 & -\lambda & \lambda \\ 0 & 0 & \frac{\lambda c}{b} & \frac{\lambda b+a\lambda+\lambda d-\lambda^2}{b} \end{pmatrix} R_3 \leftrightarrow$$

$$R_4 = \begin{pmatrix} a & b & c & d-\lambda \\ 0 & \frac{\lambda b}{a} & \frac{\lambda c}{a} & \frac{a\lambda+\lambda d-\lambda^2}{a} \\ 0 & 0 & \frac{\lambda c}{b} & \frac{\lambda b+a\lambda+\lambda d-\lambda^2}{b} \\ 0 & 0 & -\lambda & \lambda \end{pmatrix} R_4 \leftarrow R_4 + \frac{b}{c} \cdot R_3 = \begin{pmatrix} a & b & c & d-\lambda \\ 0 & \frac{\lambda b}{a} & \frac{\lambda c}{a} & \frac{a\lambda+\lambda d-\lambda^2}{a} \\ 0 & 0 & \frac{\lambda c}{b} & \frac{\lambda b+a\lambda+\lambda d-\lambda^2}{b} \\ 0 & 0 & 0 & \frac{\lambda c+\lambda b+a\lambda+\lambda d-\lambda^2}{c} \end{pmatrix}.$$

We know that determinant of upper-triangular matrix is just a product of its diagonals:
$det(A) = a\frac{\lambda b}{a} \cdot \frac{\lambda c}{b} \cdot \frac{\lambda c+\lambda b+a\lambda+\lambda d-\lambda^2}{c} = \lambda^3 \left(a - \lambda + b + c + d\right).$

Let's solve for roots, i.e what values of $\lambda$ make determinant equal zero:
$\lambda^3 \left(a - \lambda + b + c + d\right) = 0; \rightarrow \lambda^3 = 0, a - \lambda + b + c + d = 0 \rightarrow \lambda = 0, \lambda = a+b+c+d$

Thus, matrix is singular for $\lambda = 0$ or $\lambda = a + b + c + d$. It can also be said that matrix A has eigenvalue of 0 with multiplicity of 3 and eigenvalue a+b+c+d.

## Problem 14

- $B = \{v_1, v_2, v_3\}$, where $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $B$ is basis for $\mathbb{R}^3$.

- $B' = \{v_1', v_2', v_3'\}$, where $v_1' = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, v_2' = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, v_3' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $B'$ is basis for $\mathbb{R}^3$.

(a) $\underline{P_{B \to B'} \text{ transition matrix finding.}}$

We will use row reduction, so that:

$$[\, B' \mid B \,] \sim [\, I \mid P_{B \to B'} \,]$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) R_2 \leftarrow R_2 - 1 \cdot R_1 =$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) R_3 \leftarrow R_3 - 1 \cdot R_1 = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 & -1 & 0 \end{array}\right) R_3 \to R_3 - (1 \cdot R_2) =$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{array}\right). \quad P_{B \to B'} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

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(b) <u>Computing coordinate vector $(\vec{u})_B$ for $\vec{u} = (1, 1, -1)^T$.</u>

$$B(\vec{u})_B = \vec{u};$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} (\vec{u})_B = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix};$$

Let's solve above equation by row-reduction of augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right) R_2 \leftarrow R_2 + (R_3 \cdot (-1)) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right) R_1 \leftarrow R_1 - R_3 =$$

$$= \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right) R_1 \leftarrow R_1 - R_2 = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right).$$

Thus, $(\vec{u})_B = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$.

(c). <u>Using transition matrix $P_{B \to B'}$ to compute $(\vec{u})_{B'}$.</u>

$$(\vec{u})_{B'} = P_{B \to B'} (\vec{u})_B;$$

$$(\vec{u})_{B'} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}.$$

(d) Computing $(u)_{B'}$ directly.

$$B'(\vec{u})_{B'} = \vec{u}.$$

We will use row-reduction (as in (b)) to solve this equation:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} (\vec{u})_{B'} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix};$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & -1 \end{array}\right) \underset{R_2 \leftarrow R_2 - R_1}{=} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 \end{array}\right) \underset{R_3 \leftarrow R_3 - R_1}{=}$$

$$= \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 \end{array}\right) \underset{R_3 \leftarrow R_3 - R_2}{=} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

Thus, $(\vec{u})_{B'} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$.

This result is equal to result in (c). So, our transition matrix is correct.

## Problem 5

Let's construct coefficient matrix $A$ from our vectors ($n \times (n+1)$ matrix)
(~~p. 57 of David Lay~~)

The columns of arbitrary matrix $A$ are linearly dependent iff $Ax = 0$ has at least one non-trivial solution.
(~~p. 43 of David Lay~~)

$Ax = 0$ has a non-trivial solution iff the equation has at least one free variable, having at least one column free of pivot.

Let our $n \times (n+1)$ matrix be:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{nn} & a_{1\,n+1} \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & a_{n+1\,n+1} \end{pmatrix}. \qquad \begin{pmatrix} a_{1\,n+1} \\ \vdots \\ a_{n+1\,n+1} \end{pmatrix}$$

After Gauss-Jordan row-reduction applied to $A$, we are guaranteed to have at least one column be free of pivot, because there would be no row left below $a_{nn}$ (reduced to pivot), which by definition leads to at least one free column in such type of matrices. Thus, any $n \times (n+1)$ matrix has a non-trivial solution in homogeneous case $\rightarrow$ such matrices have linearly dependent column vectors. $\longrightarrow$

$\longrightarrow$ any $n+1$ vectors in $\mathbb{R}^n$ are linearly dependent.

Problem 7

(a)

Yes, such matrix exist.

Let us construct next matrix:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix},$$ where every entry, except by default $a_{11}, a_{1n}, a_{n1}, a_{nn}$, is a zero entry and $n$ is arbitrarily large or $n = 100$

Let's assume, that $a_{11} = 0$, $a_{1n} = -1$, $a_{n1} = 0$, $a_{nn} = 0$.

Then our matrix will look like:

$$A = \begin{pmatrix} 0 & \cdots & -1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \neq 0$$

$$\text{Then } A^2 = \begin{pmatrix} 0 & \cdots & -1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}\begin{pmatrix} 0 & \cdots & -1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} = 0.$$

In this case $a_{11} = 0$, $a_{1n} = 0$, $a_{n1} = 0$, $a_{nn} = 0$ and all other entries by definition of matrix multiplication are zero.

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# (b)

We will use here in (b) the same procedure as By default. in (a), i.e. all entries in our matrix will be zero, except $a_{1n}, a_{n1}, a_{n1}, a_{nn}$.

Yes, such matrix exist for arbitrarily large $n$.

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \neq 0 \neq I_n.$$

Then $A^2 = \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} =$

$= A.$

# (c)

Yes, such matrix exists. It is a well-known ~~matrix~~ matrix that represent linear transformation, which swaps axes and where all entries are zero, except ones on the diagonal from lower left to up right:

$$A = \begin{pmatrix} 0 & \cdots & & 1 \\ \vdots & & 1 & 0 \\ 0 & 1 & & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}. \quad A^2 = \begin{pmatrix} 0 & \cdots & & 1 \\ \vdots & & 1 & 0 \\ & 1 & & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 0 & \cdots & & 1 \\ \vdots & & 1 & 0 \\ & 1 & & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ & 1 & & \vdots \\ & & 1 & \\ 0 & \cdots & & 1 \end{pmatrix}$$

$= I.$

If such antidiagonal $A$ is multiplied by itself, then we literally swap already swapped axes and end up at $I$.

(d)

Yes, such matrices $A$ and $B$ exist for arbitrarily large $n$, where all entries are by default zeros, except from $a_{11}, a_{1n}, a_{n1}, a_{nn}$ in $A$ and $b_{11}, b_{1n}, b_{n1}, b_{nn}$ in $B$.

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} 0 & \cdots & -1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -1 \end{pmatrix}$$

$$B = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix} = \begin{pmatrix} -1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ -1 & \cdots & 1 \end{pmatrix}$$

$$AB = \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix} \begin{pmatrix} b_{11} & \cdots & b_{1n} \end{pmatrix} + \cdots + \begin{pmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{pmatrix} \begin{pmatrix} b_{n1} & \cdots & b_{nn} \end{pmatrix} =$$

$$= \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \begin{pmatrix} -1 & \cdots & 1 \end{pmatrix} + \cdots 0 \cdots + \begin{pmatrix} -1 \\ \vdots \\ -1 \end{pmatrix} \begin{pmatrix} -1 & \cdots & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} + \begin{pmatrix} 1 & \cdots & -1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & -1 \end{pmatrix} = \begin{pmatrix} 1 & \cdots & -1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & -1 \end{pmatrix} \neq 0;$$

$$BA = \begin{pmatrix} b_{11} \\ \vdots \\ b_{n1} \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1n} \end{pmatrix} + \cdots + \begin{pmatrix} b_{1n} \\ \vdots \\ b_{nn} \end{pmatrix} \begin{pmatrix} a_{n1} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} -1 \\ \vdots \\ -1 \end{pmatrix} \begin{pmatrix} 0 & \cdots & -1 \end{pmatrix} + \cdots 0 \cdots + \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \begin{pmatrix} 0 & \cdots & -1 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ -1 & \cdots & 1 \end{pmatrix} + \begin{pmatrix} 0 & \cdots & -1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & -1 \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} = 0.$$

## Problem 12

No, there are no such matrices.

From the properties of trace we know that trace of $I_n = n$ and $trace(AB) = trace(BA)$. We also know that $trace(A_1 - A_2) = trace(A_1) - trace(A_2)$.

Let's construct and solve equation from Problem:

$$AB - BA = I_n;$$

Let's take trace of both sides and use above-defined properties of trace:

$$AB - BA = I_n;$$
$$trace(AB - BA) = trace(I_n);$$
$$trace(AB) - trace(BA) = n;$$
$$0 = n,$$ which is a contradiction.

Therefore, there is no $n \times n$ matrices $A$ and $B$, which satisfy an equation $AB - BA = I_n$.

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## Problem 11

(a)

I will show that $Col(AB)$ is a subset of $Col(A)$ by implication.

Let $b \in Col(AB)$, which means that $\exists x : (AB)x = b$.

By association property of matrix multiplication we see that:

$(AB)x = b$;

$A(Bx) = b$; which means that $b \in Col(A)$.

Thus, $b \in Col(AB) \to b \in Col(A)$, which means that $Col(AB) \subseteq Col(A)$.

Thus, $rank(AB) \leq rank(A)$.

Let's also transpose:

$rank(AB)^T = rank(B^T A^T)$.

By (a) above proven property we know that $rank(B^T A^T) \leq rank(B^T)$.

We know that column rank and row rank are equal.

Thus, $rank(B^T) = rank(B)$.

Thus, $rank(B^T A^T) \leq rank(B)$

Thus, $rank(B^T A^T)^T = rank(AB) \leq rank(B) = rank(B^T)$.

(b)

Let our matrices be:

- $A$ is $m \times n$ matrix;
- $B$ is $n \times m$ matrix;

Let's assume that $m < n$, but it's just for the sake of beginning proof, which can also be applied when $m > n$.

In case $m < n$:

$rank(A), rank(B) \leq min(m, n)$;

$rank(A), rank(B) \leq m$

$BA$ is $n \times n$ matrix.

$Rank(BA) \leq rank(B) \leq m < n$.

Thus, we have $n \times n$ matrix $BA$ with rank less than $n$. This means that such matrix is singular.

In case of $m > n$ we have:

$rank(A), rank(B) \leq n$

$AB$ is $m \times m$ matrix.

$Rank(AB) \leq rank(A) \leq n < m$

Thus, we have $m \times m$ matrix $AB$ with rank less than $m$. This means that such matrix is singular.

Conclusion: for non-square matrices $A$ and $B$, if their products $AB$ and $BA$ are defined, at least one such products is singular.

# Problem 3

I will create all systems of equations for each case (a),(b),(c). For no solution (a) I'll provide geometrical interpretation with plots and without general solution because geometrical interpretation is sufficient according to the wording of the Problem. For (b) and (c) plotting would be two costly in terms of resourses and in such cases I will go without geometrical interpretation, but with a general solution.

In [151]:

```
#importing necessary libraries
import numpy as np
import matplotlib.pyplot as plt
import matplotlib as mpl
from mpl_toolkits import mplot3d
from mpl_toolkits.mplot3d import Axes3D
import matplotlib.patches as mpatches
```

## (a) no solution

$(iii)$
$m = 2, n = 3$

Let our system of linear equations be:

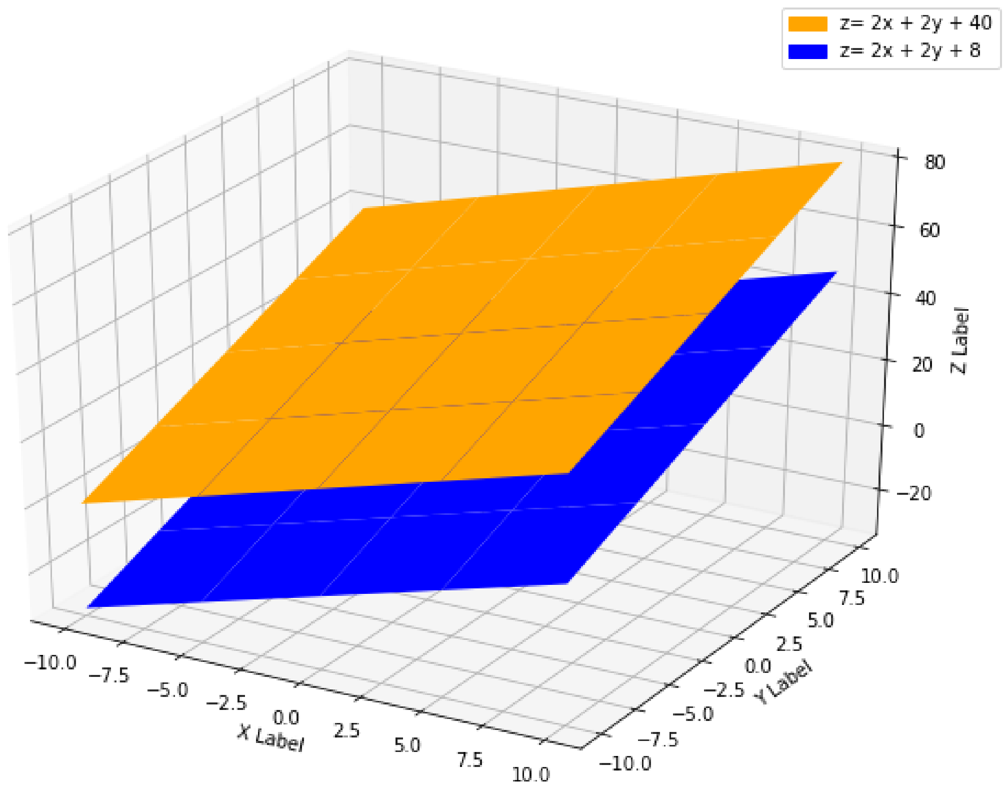
$$\begin{cases} -2x - 2y + z = 8 \\ -2x - 2y + z = 40 \end{cases} \rightarrow \begin{cases} z = 8 + 2x + 2y \\ z = 40 + 2x + 2y \end{cases}$$

These equations represent parallel planes in 3-dimensional space which have no intersection between each other. Such system of equations has no solutions. I will plot these two equations in order to confirm that statement.

I will generate few values of x and y and plot respective planes.

In [152]:

```
x = np.linspace(-10,10,5)
y = np.linspace(-10,10,5)
#first equation of the matrix
X_1,Y_1 = np.meshgrid(x,y)
Z_1= 2*X_1 + 2*Y_1 + 8
#second equation of the matrix
X_2,Y_2 = np.meshgrid(x,y)
Z_2= 2*X_2 + 2*Y_2 + 40

#plt.figure(figsize=(20,10))
fig = plt.figure()
ax = fig.gca(projection='3d')
fig.set_size_inches(11,8)
#ax.plot(X_1, Y_1, Z_1)
#ax.legend()
surf_1 = ax.plot_surface(X_1, Y_1, Z_1,color='b')
surf_2 = ax.plot_surface(X_2, Y_2, Z_2, color='orange')
ax.set_xlabel('X Label')
ax.set_ylabel('Y Label')
ax.set_zlabel('Z Label')

orange_patch = mpatches.Patch(color='orange', label='z= 2x + 2y + 40')
blue_patch = mpatches.Patch(color='blue', label='z= 2x + 2y + 8')

plt.legend(handles=[orange_patch,blue_patch])

plt.show()
```




As we see, planes of above-defined systems of equations are parallel and have no intersection, which means that the system has no solutions.

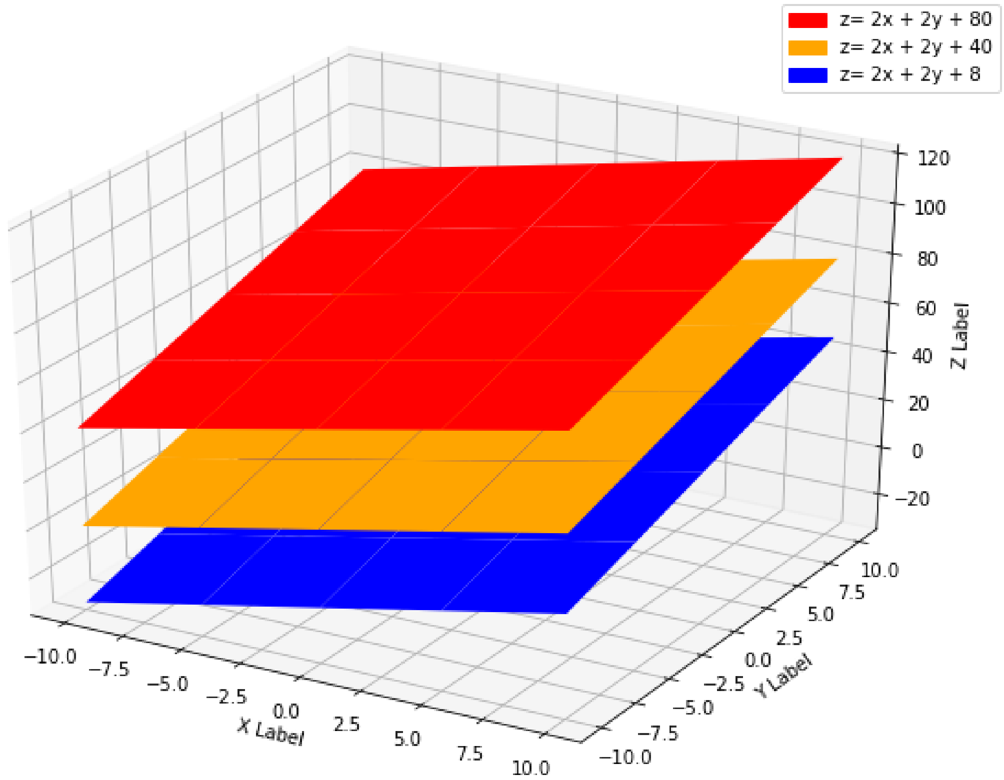
$(i)$
$m = 3, n = 3$

The same approach goes for (iii) for m equations in n unknowns - we just add third parallel plane:

$$\begin{cases} -2x - 2y + z = 8 \\ -2x - 2y + z = 40 \\ -2x - 2y + z = 80 \end{cases} \to \begin{cases} z = 8 + 2x + 2y \\ z = 40 + 2x + 2y \\ z = 80 + 2x + 2y \end{cases}$$

In [84]:

```
x = np.linspace(-10,10,5)
y = np.linspace(-10,10,5)
#first equation of the matrix
X_1,Y_1 = np.meshgrid(x,y)
Z_1= 2*X_1 + 2*Y_1 + 8
#second equation of the matrix
X_2,Y_2 = np.meshgrid(x,y)
Z_2= 2*X_2 + 2*Y_2 + 40
#third equation of the matrix
X_3,Y_3 = np.meshgrid(x,y)
Z_3= 2*X_3 + 2*Y_3 + 80

#plt.figure(figsize=(20,10))
fig = plt.figure()
ax = fig.gca(projection='3d')
fig.set_size_inches(11,8)
#ax.plot(X_1, Y_1, Z_1)
#ax.legend()
surf_1 = ax.plot_surface(X_1, Y_1, Z_1,color='b')
surf_2 = ax.plot_surface(X_2, Y_2, Z_2, color='orange')
surf_3 = ax.plot_surface(X_3, Y_3, Z_3, color='red')


ax.set_xlabel('X Label')
ax.set_ylabel('Y Label')
ax.set_zlabel('Z Label')

orange_patch = mpatches.Patch(color='orange', label='z= 2x + 2y + 40')
blue_patch = mpatches.Patch(color='blue', label='z= 2x + 2y + 8')
red_patch = mpatches.Patch(color='red', label='z= 2x + 2y + 80')


plt.legend(handles=[red_patch,orange_patch,blue_patch])

plt.show()
```




$(ii)$
$m = 3, n = 2$

This time we have system of 3 equations and 2 unknowns with no solution.

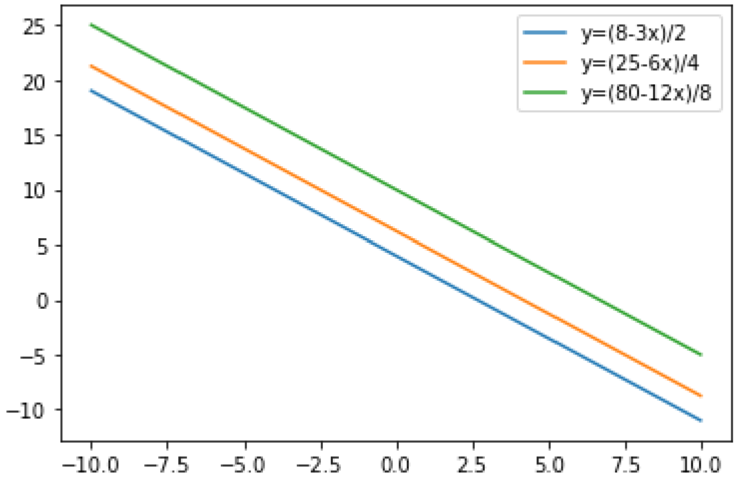
Let such system be:

$$\begin{cases} 3x + 2y = 8 \\ 6x + 4y = 25 \\ 12x + 8y = 80 \end{cases} \rightarrow \begin{cases} y = (8 - 3x)/2 \\ y = (25 - 6x)/4 \\ y = (80 - 12x)/8 \end{cases}$$

We can represent these as 3 parallel lines in 2-dimensinal space.

In [93]:

```
plt.plot(x, (8-3*x)/2, linestyle='solid',label='y=(8-3x)/2')
plt.plot(x, (25-6*x)/4, linestyle='solid',label='y=(25-6x)/4')
plt.plot(x, (80-12*x)/8, linestyle='solid',label='y=(80-12x)/8')
plt.legend()
plt.show()
```




## (b) exactly one solution

$(ii)$
$m = 3, n = 2$

For generic RHS we can't have a solution when m>n, however unique solution exists when it lies in the same plane as two vectors, beaing a basis for such plane.

Let our system be:

$$\begin{cases} -2x + y = 0 \\ 5x + y = 7 \\ 3x + 0y = 3 \end{cases}$$

General unique solution for such system is:

$$\begin{cases} x = 1 \\ y = 2 \end{cases}$$

$(i)$
$m = 3, n = 3$

Let our system be:

$$\begin{cases} 1x + 0y + 0z = 4 \\ 0x + 1y + 0z = 6 \\ 0x + 0y + 1z = 10 \end{cases}$$

General solution for such system is:

$$\begin{cases} x = 4 \\ y = 6 \\ z = 10 \end{cases}$$

$(iii)$
$m = 2, n = 3$

In generic case underdetermined system can't have a unique solution. But in concrete case we can construct artificially such solution.

Let our system be:

$$\begin{cases} 1x + 0y + 0z = 5 \\ 0x + 0y + 1z = 10 \end{cases}$$

General solution:

$$\begin{cases} x = 5 \\ z = 10 \\ y - free \end{cases}$$

## (c) infinitely many solutions

$(iii)$
$m = 2, n = 3$

Let our system be:

$$\begin{cases} -2x + 5y - 2z = 3 \\ 4x - 10y + 4z = -6 \end{cases}$$

General solution is:

$$\begin{cases} z = \frac{3+2x-5y}{-2} \\ x - free \\ y - free \end{cases}$$

Equation of this system can be represented as 2 overlapping planes in 3-dimensional space with infinitely many solutions (two free variables x and y).

$(ii)$
$m = 3, n = 2$

Let our system be:

$$\begin{cases} -2x + y = 4 \\ 6x - 3y = -12 \quad \rightarrow y = 2x + 4 \\ -12x + 6y = 24 \end{cases}$$

General solution is:

$$\begin{cases} y = 2x + 4 \\ x - free \end{cases}$$

Equation of this system can be represented as 3 overlapping lines in 2-dimensional space with infinitely many solutions (one free variable x)

$(i)$
$m = 3, n = 3$

Let our system be:

$$\begin{cases} 1x + 1y + 2z = 4 \\ 2x + 0y + 2z = 6 \\ 1x + 1y + 2z = 10 \end{cases}$$

General solution is:

$$\begin{cases} z = 3 - x \\ y = 10 + 1x \\ x - free \end{cases}$$