Linear Algebra
Home assignment 6: Iterative methods

- You should solve all problems in this assignment to get the maximum 10 points
- The problems below require many matrix computations. You can try to solve those involving 3×3 matrices by hand, but you'll certainly need to use software of your choice for some of them
- Preferably, you should prepare your solutions as `R` or `Python` notebooks, with all comments included in the notebook or in the companion pdf file. In any case, you should submit your code and results in a form that allows reproducibility and commenting

Problem 1 (Jacobi and Gauss–Seidel iteration scheme)**.** Consider the system of linear equations $A\mathbf{x} = \mathbf{b}$, with

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

(a) Solve the system using Gaussian elimination

(b) Determine the D, L, and U matrices of the Jacobi and Gauss–Seidel method and determine the spectral radius of $D^{-1}(L+U)$ and $(L+D)^{-1}U$. Comment on whether the Jacobi and Gauss–Seidel methods will always converge.

(c) Perform 10 steps of the Jacobi method starting with $\mathbf{x}_0 = (0, 0, 0)^\top$, and evaluate the Euclidean norm of the difference $\mathbf{x}_{k+1} - \mathbf{x}_k$ of two successive approximate solutions for $k = 0, 1, \ldots, 9$. Does the decay rate agree with the predicted one based on the spectral radius?

(d) Perform 10 steps of the Gauss–Seidel method starting with $\mathbf{x}_0 = (0, 0, 0)^\top$, and evaluate the Euclidean norms of the difference $\mathbf{x}_{k+1} - \mathbf{x}_k$ of two successive approximate solutions for $k = 0, 1, \ldots, 9$. Does the decay rate agree with the predicted one based on the spectral radius?

(e) Perform 10 steps of the successive overrelaxation (SOR) method using $\omega = 0.1$ and then $\omega = 1.1$, starting with $\mathbf{x}_0 = (0, 0, 0)^\top$, and evaluate the Euclidean norm of the difference $\mathbf{x}_{k+1} - \mathbf{x}_k$ of two successive approximate solutions for $k = 0, 1, \ldots, 9$. Does the decay rate agree with the predicted one based on the spectral radius?

Problem 2 (Jacobi and Gauss–Seidel iteration scheme)**.** Use the Jacobi and Gauss–Seidel methods to solve the 3×3 system $A\mathbf{x} = \mathbf{b}$ with

$$A = \alpha I_3 + \mathbf{u}\mathbf{u}^\top, \qquad \mathbf{b} = \mathbf{u},$$

where $\mathbf{u} = (1, -1, 1)^\top$.

(a) Find the closed form solution (if any) for each real α

(b) For what α do the methods work? Justify your answer by referring to the spectral radius or the norm of the corresponding B.

(c) Write the corresponding iteration scheme for the Jacobi method and find the solution starting with $\mathbf{x}_0 = \mathbf{0}$. How many iteration are required to achieve 0.001 accuracy? Test three different α in the convergence range.

(d) Write the corresponding iteration scheme for the Gauss–Seidel method and find the solution starting with $\mathbf{x}_0 = (0,0,0)^\top$. How many iteration are required to achieve 0.001 accuracy? Test three different α in the convergence range.

(e) Write the corresponding iteration scheme for the SOR method using $\omega = 0.1$ and $\omega = 1.1$ and find the solution starting with $\mathbf{x}_0 = (0,0,0)^\top$. How many iteration are required to achieve 0.001 accuracy? Test three different α in the convergence range.

Problem 3 (Conjugate gradient method)**.** Solve the system $A\mathbf{x} = \mathbf{b}$ with A

(a) of Problem 1

(b) of Problem 2 with $\alpha = 1$

and $\mathbf{b} = (1,0,1)^\top$ using the conjugate gradient method. Clearly identify the conjugate gradients and the iteration steps.

Problem 4 (Power method)**.** Set $D = \operatorname{diag}(1, 2, \dots, 10)$, take a **random** 10×10 matrix P, and then set $A = PDP^{-1}$.

(a) Realize the power method to find the dominating eigenvalue of A and the corresponding eigenvector.

(b) Apply the inverse power method to find several (e.g. 4) other eigenvalues and the corresponding eigenvectors. To that end, determine the range in which one should search for the remaining eigenvalues, and then try several random approximations λ.

Problem 5 (QR eigenvalue/eigenvector method)**.** (a) With the matrix D of the above problem, form $A = D + \mathbf{u}\mathbf{u}^\top$ with $\mathbf{u}$ being a random vector of size 10. Realize the QR algorithm to find the eigenvalues and eigenvectors of A.

(b) Then apply the power method to find the dominating eigenvalue and the corresponding eigenvector and compare the results.

You can use the QR-decomposition procedure implemented in the software but should implement the QR eigenvalue method by yourselves