Linear Algebra
Homework 1: Prerequisites

- The aim of this self-study introductory module is to review basic concepts and facts in linear algebra without which any further progress could be problematic. After having worked out this home assignment, you will be able to proceed with more advanced topics that are a must for any data science master student
- A complete solution of the home assignment amounts to 8% of the final grade
- There are 15 problems (49 points total). You do not have to solve all problems but can choose any subset whose total is 40 points (the best combination will be graded if you solve more). There is an extra problem that can bring you an extra 2%.
- If you can solve all the problems without referring to the textbook or online course, just do that. Otherwise, read the textbook and/or watch the online lectures and then try again
- If you still do not know how to solve some problem(s), ask your coursemates or us (i.e. Nadiya or Rostyslav). We encourage you to discuss the problems or questions you have with your colleagues; however, be sure to obey the academic code of honour of UCU. If you are not sure what is considered to be plagiarism, contact us
- A *solution* to a problem is not just a numerical or yes/no answer; a complete solution must include a concise explanation of the main steps and reasoning leading to the answer. Do not forget to examine all possible cases!
- Solutions should be submitted to the `cms` online platform (ask if you are not sure what to do) in electronic form. The (strongly) preferred format is `pdf` because it offers convenient annotation tools for our feedback. The best way to get a professionally-looking `pdf` of small size is to generate it from LaTeX; other options include conversion from R or `Python` notebooks, `Word` or similar editors etc. If needed, we can share the LaTeX source file which you could use as a template. As a contingent option, you can scan/take picture of your handwriting and then convert the result to `pdf`; however, your handwriting must be very readable, images of good quality, and the file size not larger than 10 MB. Please name your files as "`Name-Surname-LA-HW1-#.pdf`", where `#` stands for the corresponding file number in the case of multiple files.
- The deadline for submission is 21:00 of **Mon, 23 Sep 2019**
- Reading: *David C. Lay, Linear Algebra and its Applications*; see `cms` section
- MIT online course:
  `https://ocw.mit.edu/courses/mathematics/18-06-linear-algebra-spring-2010/`

Problem 1 (System of linear equations; 3pt)**.** Determine all the values of k for which the matrix below is the augmented matrix of a consistent linear system.

$$(a)\quad \left(\begin{array}{cc|c} 1 & 2 & 4 \\ 3 & k & 8 \end{array}\right) \qquad (b)\quad \left(\begin{array}{cc|c} 1 & 2 & -2 \\ -2 & k & 4 \end{array}\right) \qquad (c)\quad \left(\begin{array}{cc|c} 4 & -2 & k \\ -2 & 1 & -3 \end{array}\right)$$

Give a geometric interpretation of the results.

Problem 2 (System of linear equations; 4pt)**.** Let

$$\left(\begin{array}{ccc|c} a & 0 & b & 1 \\ a & a & 2 & 2 \\ 0 & a & 1 & b \end{array}\right)$$

be the augmented matrix of a linear system. Find all values of a and b for which the system has

(a) a unique solution;
(b) a one-parameter solution set;
(c) a two-parameter solution set;
(d) no solution.

Geometric interpretation of this problem is as follows: the 3×3 system gives intersection of three planes in $\mathbb{R}^3$; this intersection can be (a) a point; (b) a line; (c) a plane; (d) empty

Problem 3 (System of linear equations; 6pt)**.** Write a system of linear equations consisting of m equations in n unknowns with

(a) no solutions; (b) exactly one solution; (c) infinitely many solutions

for (i) $m = n = 3$; (ii) $m = 3$ and $n = 2$; (iii) $m = 2$, $n = 3$.

Hint: try to avoid using trivia such as $0x + 0y + 0z = 1$; this is rather not an equation in x, y, z. In each case, think of a geometric interpretation of a system as describing the intersection of lines or planes in $\mathbb{R}^2$ or $\mathbb{R}^3$. You do not have to actually solve the systems if such a geometric interpretation is provided

Problem 4 (System of linear equations; 4pt)**.** The following are coefficient matrices of linear systems. For each system, what can you say about the number of solutions to the corresponding system **(i)** in the homogeneous case (when $b_1 = \cdots = b_m = 0$) and **(ii)** for a generic RHS?

$$\text{(a)} \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}, \qquad \text{(b)} \begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 0 \end{pmatrix}, \qquad \text{(c)} \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 0 \end{pmatrix}, \qquad \text{(d)} \begin{pmatrix} 1 & 4 & 3 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Hint: again, geometric interpretation could be very helpful and instructive here

Problem 5 (System of linear equations; linear dependence; 3pt)**.** Prove that any $n+1$ vectors in $\mathbb{R}^n$ are linearly dependent using what you know about solution sets of linear systems.

Hint: Do not use bases or statements like "it is known that in $\mathbb{R}^n$ any linearly independent system has at most n vectors". Possible approach: regard a linear combination of the given vectors resulting in a zero vector as a homogeneous linear system and show that it possesses a non-trivial solution

Problem 6 (Gauss elimination; determinants; 2pt)**.** Determine all the values of k for which the column vectors below are linearly dependent:

$$\text{(a)} \quad \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 4 \\ k \\ 2 \end{pmatrix}; \qquad \text{(b)} \quad \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ -4 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ k \\ -3 \end{pmatrix}$$

Problem 7 (Matrix algebra; 4pt)**.** Let $\mathbf{0}_n$ and I_n denote respectively the zero and identity matrices of size n (say, $n = 100$).

(a) Is there an $n \times n$ matrix A such that $A \neq \mathbf{0}_n$ and $A^2 = \mathbf{0}_n$? Justify your answer.

(b) Is there an $n \times n$ matrix A such that $A \neq \mathbf{0}_n, I_n$ and $A^2 = A$? Justify your answer.

(c) Is there an $n \times n$ matrix A such that $A \neq I_n$ and $A^2 = I_n$? Justify your answer.

(d) Are there $n \times n$ matrices A and B such that $AB \neq \mathbf{0}_n$ but $BA = \mathbf{0}_n$?

Hint: analyse the case $n = 2$ to guess the answer and then try to see the pattern in higher dimensions

Problem 8 (Determinants and cross-products; 2pt)**.** A parallelepiped has edges from $(0; 0; 0)$ to $(2; 1; 1)$, $(1; 2; 1)$, and $(1; 1; 2)$. Find its volume and also find the area of each parallelogram face.

Hint: a cross-, or vector-product in $\mathbb{R}^3$ is handy here. Also, recall the geometric meaning of determinant

Problem 9 (Determinants and matrix algebra; 2pt)**.** Assume that 3×3 matrices A, B and C are as follows

$$A = \begin{pmatrix} \text{row } 1 \\ \text{row } 2 \\ \text{row } 3 \end{pmatrix} \qquad B = \begin{pmatrix} \text{row } 1 + \text{row } 2 \\ \text{row } 2 + \text{row } 3 \\ \text{row } 3 + \text{row } 1 \end{pmatrix} \qquad C = \begin{pmatrix} \text{row } 1 - \text{row } 2 \\ \text{row } 2 - \text{row } 3 \\ \text{row } 3 - \text{row } 1 \end{pmatrix}$$

Given that $\det(A) = 5$, find $\det(B)$ and $\det(C)$.

Hint: use the elementary row operations to produce B from A; an alternative (and more elegant) way is to find a matrix B' such that $B = B'A$; the same for C

Problem 10 (Determinants; eigenvalues and their properties; 3pt)**.** Find all λ for which the matrix below is singular:

$$A - \lambda I = \begin{pmatrix} a - \lambda & b & c & d \\ a & b - \lambda & c & d \\ a & b & c - \lambda & d \\ a & b & c & d - \lambda \end{pmatrix}$$

Hint: one approach is to calculate the determinant and find its roots. An alternative approach is to identify the λ's looked for as eigenvalues of A. Note A is of rank 1; what conclusions on eigenvalues can you derive? How many eigenvalues does A have? How many of them are zero? Is there any relation between the eigenvalues and the matrix that could help to identify the non-zero eigenvalues?

Problem 11 (Rank of a matrix; 4pt)**.** (a) Assume that A and B are matrices such that AB is well defined. By comparing the column spaces of A and AB, show that $\operatorname{rank}(AB) \leq \operatorname{rank}(A)$. Transpose to conclude that also $\operatorname{rank}(AB) \leq \operatorname{rank}(B)$.

(b) Assume that A and B are non-square matrices such that both AB and BA exist. Show that at least one of AB or BA is singular.

Hint: in (b), use (a) to show that at least one of AB and BA is not of full rank

Problem 12 (Trace of a matrix; 2pt)**.** Are there $n \times n$ matrices A and B such that $AB - BA = I_n$?

Problem 13 (Bases; 3pt)**.** For what numbers c are the following sets of vectors bases for $\mathbb{R}^3$?

(a) $(c,1,1)^\top$, $(1,-1,2)^\top$, $(3,4,-1)^\top$;

(b) $(c,1,1)^\top$, $(1,-1,2)^\top$, $(-2,2,-4)^\top$;

(c) $(c,1,1)^\top$, $(1,1,0)^\top$, $(0,1,2)^\top$, $(3,0,-1)^\top$;

(d) $(c,1,1)^\top$, $(1,0,1)^\top$

Problem 14 (Bases; transition matrices; 4pt)**.** Consider the bases $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $B' = \{\mathbf{v}'_1, \mathbf{v}'_2, \mathbf{v}'_3\}$ for $\mathbb{R}^3$, where

$$\mathbf{v}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix}1\\1\\0\end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix}1\\1\\1\end{pmatrix}$$
$$\mathbf{v}'_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}, \qquad \mathbf{v}'_2 = \begin{pmatrix}0\\1\\1\end{pmatrix}, \qquad \mathbf{v}'_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

(a) Find the transition matrix $P_{B\to B'}$ from B to B'.

(b) Compute the coordinate vector $(\mathbf{u})_B$ for $\mathbf{u} = (1,1,-1)^\top$.

(c) Use the transition matrix $P_{B\to B'}$ to compute the coordinate vector $(\mathbf{u})_{B'}$.

(d) Check your work by computing $(\mathbf{u})_{B'}$ directly.

Problem 15 (Linear transformations; 3pt)**.** (a) For what real values of parameters a and b is there a linear transformation of the space $\mathbb{R}^3$ sending the vectors $\mathbf{u}_1 = (1,0,0)^T, \mathbf{u}_2 = (1,a,0)^T$, and $\mathbf{u}_3 = (0,1,b)^T$ into the vectors $\mathbf{v}_1 = (0,0,1)^T, \mathbf{v}_2 = (0,b,1)^T$, and $\mathbf{v}_3 = (a,1,0)^T$ respectively?

(b) For what a and b is such a transformation unique?

(c) For what a and b does there exist an orthogonal transformation with this property?

Linear Algebra
Homework 1: Bonus problem

Problem 16 (extra 2%)**.** Undergraduate students like linear systems whose coefficient matrices have integer entries and whose determinant is ± 1 (do you see why?).

(a) Suggest a method generating randomly 10×10 matrices of this type with relatively small entries (e.g. at most 100 in absolute value). How can one easily find the inverse of such a matrix? Think of efficiency of your algorithm (e.g. as compared to the brute force approach); what is its complexity?

(b) Suggest a method generating 3×3 matrices with integer entries between -100 and 100 having pairwise orthogonal columns (no restriction on the determinant this time!).