Linear Algebra
Home assignment 2: Orthogonality

Problem 1 (Parallel and orthogonal planes; 2pt)**.** Determine whether the given planes are:

(a) parallel:

(i) $4x - y + 2z = 5$ and $7x - 3y + 4z = 8$;

(ii) $x - 4y - 3z - 2 = 0$ and $3x - 12y - 9z - 7 = 0$.

(b) perpendicular:

(i) $3x - y + z = 0$ and $x + 2z = -1$;

(ii) $x - 2y + 3z = 4$ and $-2x + 5y + 4z = -1$.

Problem 2 (Orthogonal complement; 3pt)**.** (a) Let W be the plane in $\mathbb{R}^3$ given by the equation $x - 2y - 3z = 0$. Find parametric equations for $W^\perp$.

(b) Let W be the line in $\mathbb{R}^3$ with parametric equations $x = 2t$, $y = -5t$, $z = 4t$. Find an equation for $W^\perp$.

(c) Let W be the intersection of the two planes $x + y + z = 0$ and $x - y + z = 0$ in $\mathbb{R}^3$. Find an equation for $W^\perp$.

Problem 3 (Distance from a point; 4pt)**.** (a) Find the distance from the point $P = (1, 1, 0)$ to the line $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z+1}{2}$.

Hint: take a point P_0 on the line; then decompose the vector $\overrightarrow{P_0P}$ into the projection onto the line and its orthogonal component

(b) Let π be a plane given by the equation $ax + by + cz + d = 0$ and $P(x_0, y_0, z_0)$ be a point outside it. Prove that the distance from P to π is given by the formula

$$\frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Hint: if Q is the point on π realizing the distance, then $\overrightarrow{PQ}$ is collinear to $\mathbf{n} = (a, b, c)$ (why?). Take now any point Q' on π and find a projection of $\overrightarrow{PQ'}$ onto direction $\mathbf{n}$

(c) Find the distance between the point $P = (1, 0, 1)$ and the plane $2x + 2y - z = 2$.

Problem 4 (Cross product; 3pt)**.** (a) For any two vectors $\mathbf{u} = (u_1, u_2, u_3)^\top$ and $\mathbf{v} = (v_1, v_2, v_3)^\top$, their ***vector product***, or ***cross product*** $\mathbf{u} \times \mathbf{v}$ is the vector $\mathbf{w} = (w_1, w_2, w_3)^\top$ with entries

$$w_1 = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, \qquad w_2 = -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \qquad w_3 = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}.$$

Prove that $\mathbf{w}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ in the sense that $\mathbf{w}^\top\mathbf{u} = \mathbf{w}^\top\mathbf{v} = 0$.

Hint: these products are cofactor expansions of some 3×3 matrices

(b) Assume that $\mathbf{u}_1, \dots, \mathbf{u}_{n-1}$ are linearly independent vectors in $\mathbb{R}^n$. Find a formula analogous to that in part (a) for a vector that is orthogonal to the subspace spanned by $\mathbf{u}_1, \dots, \mathbf{u}_{n-1}$.

Problem 5 (Orthogonal matrices; 2pt)**.** (a) If Q_1 and Q_2 are orthogonal matrices, show that Q_1^{-1} and Q_1Q_2 are orthogonal as well.

(b) Prove that an orthogonal matrix that is also upper-triangular must be diagonal.

Problem 6 (Projection matrices; 3pt)**.** For the vectors $\mathbf{a}_1 = (1, 0, 1)$, $\mathbf{a}_2 = (0, 1, 2)$ and $\mathbf{b} = (-1, 2, 1)$

(a) find the matrix of the orthogonal projection P_W onto the plane $W := \mathrm{ls}\{\mathbf{a}_1, \mathbf{a}_2\}$;

(b) find the matrix of the orthogonal projection $P_{W^\perp}$ onto the line $W^\perp$;

(c) find the components of the vector $\mathbf{b}$ with respect to the decomposition $\mathbb{R}^3 = W \oplus W^\perp$.

Problem 7 (Least squares solution; 3pt)**.** Is there any value of s for which $x_1 = 1$ and $x_2 = 2$ is the least squares solution of the linear system below? Explain your reasoning.

$$\begin{aligned} x_1 - x_2 &= 1,\\ 2x_1 + 3x_2 &= 1,\\ 4x_1 + 5x_2 &= s. \end{aligned}$$

Problem 8 (Regression; 6pt)**.** (a) Find the least squares straight line fit to the four points $(0, 1)$, $(2, 0)$, $(3, 1)$, and $(3, 2)$.

(b) Find the quadratic polynomial that best fits the four points $(2, 0)$, $(3, -10)$, $(5, -48)$, and $(6, -76)$.

(c) Find the cubic polynomial that best fits the five points $(-1, -14)$, $(0, -5)$, $(1, -4)$, $(2, 1)$, and $(3, 22)$.

Hint: the numbers are chosen so that $A^\top A$ can easily be inverted. If, however, this is not so, write and run a script on Python (or any other programming language)

Problem 9 (Least square solution; 4pt)**.** Assume $\mathbf{u}_1$ and $\mathbf{u}_2$ are two orthogonal vectors in $\mathbb{R}^n$ and set $\mathbf{a}_1 = \mathbf{u}_1$, $\mathbf{a}_2 = \mathbf{u}_1 + \varepsilon\mathbf{u}_2$ for $\varepsilon > 0$. Let also A be the matrix with columns $\mathbf{a}_1$ and $\mathbf{a}_2$ and $\mathbf{b}$ a vector linearly independenet of $\mathbf{a}_1$ and $\mathbf{a}_2$. In this problem, we discuss the least square solution to the system $A\mathbf{x} = \mathbf{b}$ as $\varepsilon \to 0$.

(a) Calculate the matrix $A^\top A$, its inverse, and then $\hat{\mathbf{x}} = (A^\top A)^{-1}A^\top\mathbf{b}$ explicitly. Show that $\hat{\mathbf{x}}$ explodes as $\varepsilon \to 0$.

(b) Calculate the projection $A\hat{\mathbf{x}}$ of $\mathbf{b}$ onto $\mathrm{col}(A)$ and check that it does not depend on $\varepsilon > 0$. Explain the result.

Problem 10 (Gram–Schmidt; 3pt)**.** Use the Gram–Schmidt process to transform the basis $\mathbf{u}_1, \dots, \mathbf{u}_k$ into an orthonormal basis.

(a) $\mathbf{u}_1 = (1, 3)$, $\mathbf{u}_2 = (2, -2)$;

(b) $\mathbf{u}_1 = (1, 0, 1)$, $\mathbf{u}_2 = (1, 3, -2)$, $\mathbf{u}_3 = (0, 2, 1)$

Problem 11 (QR; 4pt)**.** Write a code (in Python or any other language) for the QR-decomposition using the Gram–Schmidt algorithm. Find the QR-decomposition of the matrices below by hands and check your work by running the code:

$$(a)\ \begin{pmatrix} 3 & -1 \\ 4 & 3 \end{pmatrix}; \qquad (b)\ \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 2 & 4 \end{pmatrix}; \qquad (c)\ \begin{pmatrix} 1 & 2 & 2 \\ 2 & -1 & 1 \\ 2 & 0 & 1 \end{pmatrix}$$

Problem 12 (Householder reflection and QR; 5 pts)**.** (a) Find the unit vector $\mathbf{u} \in \mathbb{R}^2$ such that the *Householder reflection* $Q_{\mathbf{u}} := I - 2\mathbf{u}\mathbf{u}^\top$ maps the vector $(1,2)^\top$ onto a vector collinear to $(1,0)^\top$

(b) Explain how $Q_{\mathbf{u}}$ can be used to derive the QR-factorization of the matrix (a) of Problem 11. Explain a general approach to QR-factorization through Householder reflections.

(c) Write a code for QR-factorization using the Householder reflection method.
Hint: use recursion

(d) Use the above code to find the QR-factorization of matrices in (b) and (c) of Problem 11 using the Householder reflections approach. Compare the results with those obtained in Problem 11.