Problem 1

(a)(i)

For plane $4x - y + 2z = 5 \rightarrow$ normal vector $\vec{a} = (4, -1, 2)^\top$.
For plane $7x - 3y + 4z = 8 \rightarrow$ normal $\vec{b} = (7, -3, 4)^\top$.
If $\left\|\vec{a} \times \vec{b}\right\| = 0 \rightarrow$ planes are parallel.

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 4 & -1 & 2 \\ 7 & -3 & 4 \end{vmatrix} = \begin{vmatrix} -1 & 2 \\ -3 & 4 \end{vmatrix} i - \begin{vmatrix} 4 & 2 \\ 7 & 4 \end{vmatrix} j + \begin{vmatrix} 4 & -1 \\ 7 & -3 \end{vmatrix} k =$$

$= (-4 + 6)i - (16 - 14)j + ((-12) + 7)k =$
$= 2i - 2j - 5k \rightarrow \left\|\vec{a} \times \vec{b}\right\| = \left\|(2, -2, -5)^\top\right\| = \sqrt{33} \neq 0 \rightarrow$ planes are not parallel.

(a)(ii)

Let's use in (a)(ii) the same procedure as in (a)(i):

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 1 & -4 & -3 \\ 3 & -12 & -9 \end{vmatrix} = \begin{vmatrix} -4 & -3 \\ -12 & -9 \end{vmatrix} i - \begin{vmatrix} 1 & -3 \\ 3 & -9 \end{vmatrix} j + \begin{vmatrix} 1 & -4 \\ 3 & -12 \end{vmatrix} k =$$

$= (36 - 36)i - (-9 + 9)j + (-12 + 12)k =$
$= 0i - 0j - 0k \rightarrow \left\|(0, 0, 0)^\top\right\| = 0 \rightarrow$ planes are parallel.

(b)(i)

For $\vec{a} \in A$ and $\vec{b} \in B$:
If $\vec{a} \cdot \vec{b} = 0 \rightarrow A = B^\perp \mid \vec{a}$ is normal vector of A and $\vec{b}$ is normal vector of B.
Thus,
$\vec{a} \cdot \vec{b} = (3, -1, 1) \cdot (1, 0, 2) = 5 \neq 0 \rightarrow A \neq B^\top \perp\rightarrow$ plane A is not perpendicular to plane B.

(b)(ii)

Let's use in (b)(ii) the same procedure as in (b)(i):
$\vec{a} \cdot \vec{b} = (1, -2, 3) \cdot (-2, 5, 4) = -2 - 10 + 12 = 0 \rightarrow A = B^\top \perp\rightarrow$ plane A is perpendicular to plane B.

Problem 2

(a)

$W^\perp$ of plane W is a line, that is normal vector to the plane. In equation of the plane: $ax + by + cy = d$ its coefficients represent components of normal vector to the plane.
Therefore, parametric equation of line: $x = t, y = -2t, z = -3t \mid t \in \mathbb{R}$.

(b)

If W is line, then $W^\perp$ is a plane, which has an equation $\rightarrow 2x - 5y + 4z = 0$.

(c)

To find $W^\perp$ let's firstly find W, e.g. find equations for the intersection line:

$$(1,1,1) \times (1,-1,1) = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 2i - 0j - 2k \rightarrow W = \begin{cases} x = 2t \\ y = 0 \\ z = -2t \end{cases} \rightarrow$$

$\rightarrow W^\perp : 2x - 2z = 0 \rightarrow$ equation for $W^\perp$.

Problem 3

(a)

Let's convert symmetric equation of the line into parametric in order to know how to find any point P_0 on the line L:

$$L : \tfrac{x-1}{2} = \tfrac{y-2}{1} = \tfrac{z+1}{2} \to \begin{cases} x = 2t+1 \\ y = t+2 \\ z = 2t-1 \end{cases} = P_0$$

As we know, $P = (1, 0, 0)$.

We will proceed in the following manner:

1. Firsty, we will find vector $\vec{P_0P}$, (e.g. any vector from line L to point P_0);
2. Then we will find when $\vec{P_0P} \perp L$;
3. After that we just can find such vector's Euclidian norm, which is a distance from P_0 to P.

$\vec{P_0P} = (2t+1, t+2, 2t-1)^\top - (1, 1, 0)^\top = (2t, t+1, 2t-1)^\top$.

If $\vec{P_0P} \perp L$ and $\vec{u} \in L$ then $\vec{P_0P} \cdot \vec{u} = 0$.
Here, $\vec{u}$ is a direction vector of line L.

Denominators (a, b, c) represent components of direction vector $\vec{u}$ in symmetric equation of $L : \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z+z_0}{c} \to \vec{u} = (a, b, c)^\top = (2, 1, 2)^\top$.

$\vec{P_0P} \cdot \vec{u} = 0 \to (2t, t+1, 2t-1) \cdot (2, 1, 2) = 0$.

$(2t, t+1, 2t-1) \cdot (2, 1, 2) = 0;$
$4t + t + 1 + 4t - 2 = 0;$
$9t = 1;$
$t = \frac{1}{9}. \to \vec{P_0P} = (2 \cdot \frac{1}{9}, \frac{1}{9} + 1, 2 \cdot \frac{1}{9} - 1)^\top = (\frac{2}{9}, \frac{10}{9}, -\frac{7}{9})^\top$

We only have to find magnitude of $\vec{P_0P}$:
$\left\|\vec{P_0P}\right\| = \sqrt{(\frac{2}{9})^2 + (\frac{10}{9})^2 + \frac{7}{9})^2} = \frac{\sqrt{17}}{3} \to$ distance from point P to line L.

Problem 3

(b)

$\pi : ax + by + cz + d = 0$ - plane;
$d = -ax - by - cz$;
$P(x_0, y_0, z_0)$ - outside point;
$Q(x, y, z)$ - point on π, realizing distance;
D - distance from P to Q - ?

Let:
$\vec{n}(a, b, c)$ - normal vector to π;
$Q_0(x_1, y_1, z_1)$ - any point on π;
L - line that contains distance from P to Q;
$\vec{PQ_0} = (x_0 - x_1, y_0 - y_1, z_0 - z_1)$

$\vec{PQ}$ represents a direction vector of L. Therefore, $\vec{PQ}$ is by definition perpendicular to π.

Thus,
$\vec{n} = \pi^{\perp} \to \vec{n} \parallel \vec{PQ} \to \vec{n}$ is collinear to $\vec{PQ}$.

Thus, projection of $\vec{PQ_0}$ onto $\vec{n}$ is $\vec{PQ}$. $\to$ D $= \left\|\vec{PQ}\right\| = \left\|proj_n \vec{PQ_0}\right\|$.

D $= \left\|proj_n \vec{PQ_0}\right\| = \frac{\left\|(\vec{PQ_0} \cdot \vec{n})\right\|}{\|\vec{n}\|} = \frac{\left\|(x_0 - x_1, y_0 - y_1, z_0 - z_1) \cdot (a, b, c)\right\|}{\sqrt{a^2 + b^2 + c^2}} = \frac{\|ax_0 - ax_1 + by_0 - by_1 + cz_0 - cz_1\|}{\sqrt{a^2 + b^2 + c^2}} =$
$= \frac{\|ax_0 + by_0 + cz_0 + d\|}{\sqrt{a^2 + b^2 + c^2}}$.

(c)

$P = (1, 0, 1)$;
$\pi : 2x + 2y - z = 2$;
D $= \frac{\|ax_0 + by_0 + cz_0 + d\|}{\sqrt{a^2 + b^2 + c^2}} = \frac{\left\|2(1) + 2(0) - 1(1) - 2\right\|}{\sqrt{(2)^2 + (2)^2 + (-1)^2}} = \frac{\|2 - 1 - 2\|}{\sqrt{4 + 4 + 1}} = \frac{1}{3}$.

Problem 6

There are 3 vectors:

$$a_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} ; a_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} ; b = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

(a)

Let's construct then matrix A from bases of plane W, which are above-defined vectors a_1 and a_2:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}$$

Formula for orthogonal projection matrix onto W is:

$$P_W = A(A^T A)^{-1} A^T =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} (\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix})^{-1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} (\begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix})^{-1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} (\frac{1}{\det \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}} \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}) \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \frac{5}{6} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{5}{6} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{5}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{5}{6} \end{pmatrix}$$

(b)

To find orthogonal projection matrix onto $W^\perp$ we firstly need to find basis vector $w^\perp$ for $W^\perp$ itself. Basis vector of orthogonal complement to subspace W can be found by cross-product of a_1 and a_2, which are basis vectors for W:

$$w^\perp = a_1 \times a_2 = \begin{pmatrix} 0 \cdot\ 2 - 1 \cdot\ 1 \\ 1 \cdot\ 0 - 1 \cdot\ 2 \\ 1 \cdot\ 1 - 0 \cdot\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}.$$

Now orthogonal projection matrix can be found by formula in (a), where A consist solely of our $w^\perp$:

$$P_{W^\perp} = A(A^TA)^{-1}A^T = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} \frac{1}{6} \begin{pmatrix} -1 & -2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{pmatrix}$$

(c)

$\vec{b} = proj_W \vec{b} + proj_{W^\perp} \vec{b}$

Thus, components of $\vec{b}$ with respect to decomposition $W \bigoplus W^\perp$ are:

$$proj_W \vec{b} = P_W \vec{b} = \begin{pmatrix} \frac{5}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{5}{6} \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{4}{3} \\ \frac{4}{3} \\ \frac{4}{3} \end{pmatrix}$$

$$proj_{W^\perp} \vec{b} = P_{W^\perp} \vec{b} = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$$

Now let's check the result:

$$proj_W \vec{b} + proj_{W^\perp} \vec{b} = \begin{pmatrix} -\frac{4}{3} \\ \frac{4}{3} \\ \frac{4}{3} \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \vec{b}$$

Thus, our result is correct.

Problem 7

$A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}; \vec{b} = \begin{bmatrix} 1 \\ 1 \\ s \end{bmatrix}.$

Let's solve for what s in $\vec{b}$ A has a least squares solution $\hat{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$:

$A^T A\hat{x} = A^T b;$

$$\begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ s \end{bmatrix};$$

$$\begin{bmatrix} 21 & 25 \\ 25 & 35 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4s+3 \\ 5s+2 \end{bmatrix};$$

$$\begin{pmatrix} 71 \\ 95 \end{pmatrix} = \begin{bmatrix} 4s+3 \\ 5s+2 \end{bmatrix} \rightarrow \begin{cases} 68 = 4s \\ 93 = 5s \end{cases} \rightarrow \begin{cases} 17 = s \\ 18.6 = s \end{cases}$$

Definitely s can't be equal to 17 and 18.6 simultaneously, that's a contradiction. Thus, there are no values of s for which $\hat{x}$ is $(1, 2)^T$.

Problem 8

(a)

$\hat{y} = \beta_0 + \beta_1 x;$

$$X = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 3 \\ 1 & 3 \end{bmatrix}; \vec{y} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix};$$

$$\vec{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = (X^T X)^{-1} X^T \vec{y};$$

$$X^T X = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 3 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ 8 & 22 \end{pmatrix};$$

$$(X^T X)^{-1} = \frac{1}{\det \begin{pmatrix} 4 & 8 \\ 8 & 22 \end{pmatrix}} \begin{pmatrix} 22 & -8 \\ -8 & 4 \end{pmatrix} = \frac{1}{24} \begin{pmatrix} 22 & -8 \\ -8 & 4 \end{pmatrix} = \begin{pmatrix} \frac{11}{12} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{6} \end{pmatrix};$$

$$\vec{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} \frac{11}{12} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 3 \end{pmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} = \begin{pmatrix} \frac{11}{12} & \frac{1}{4} & -\frac{1}{12} & -\frac{1}{12} \\ -\frac{1}{3} & 0 & \frac{1}{6} & \frac{1}{6} \end{pmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} =$$

$= \begin{pmatrix} \frac{2}{3} \\ \frac{1}{6} \end{pmatrix}$; $\rightarrow$ best least squares straight line: $\hat{y} = \frac{2}{3} + \frac{2}{3}x$.

(b)

$\hat{y} = \beta_0 + \beta_1 x + \beta_2 x^2;$

$$X = \begin{pmatrix} 1 & 2 & 2^2 \\ 1 & 3 & 3^2 \\ 1 & 5 & 5^2 \\ 1 & 6 & 6^2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 5 & 25 \\ 1 & 6 & 36 \end{pmatrix}; \vec{y} = \begin{pmatrix} 0 \\ -10 \\ -48 \\ -76 \end{pmatrix}$$

$$\vec{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} = (X^T X)^{-1} X^T \vec{y};$$

$$X^T X = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 5 & 6 \\ 4 & 9 & 25 & 36 \end{pmatrix} \begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 5 & 25 \\ 1 & 6 & 36 \end{pmatrix} = \begin{pmatrix} 4 & 16 & 74 \\ 16 & 74 & 376 \\ 74 & 376 & 2018 \end{pmatrix};$$

$$X^T X)^{-1} = \begin{bmatrix} 4 & 16 & 74 & | & 1 & 0 & 0 \\ 16 & 74 & 376 & | & 0 & 1 & 0 \\ 74 & 376 & 2018 & | & 0 & 0 & 1 \end{bmatrix} R_1 \leftrightarrow R_3 =$$

$$= \left[\begin{array}{ccc|ccc} 74 & 376 & 2018 & 0 & 0 & 1 \\ 16 & 74 & 376 & 0 & 1 & 0 \\ 4 & 16 & 74 & 1 & 0 & 0 \end{array}\right] R_2 \leftarrow R_2 - \tfrac{8}{37} \cdot R_1 =$$

$$= \left[\begin{array}{ccc|ccc} 74 & 376 & 2018 & 0 & 0 & 1 \\ 0 & -\frac{270}{37} & -\frac{2232}{37} & 0 & 1 & -\frac{8}{37} \\ 4 & 16 & 74 & 1 & 0 & 0 \end{array}\right] R_3 \leftarrow R_3 - \tfrac{2}{37} \cdot R_1 =$$

$$= \left[\begin{array}{ccc|ccc} 74 & 376 & 2018 & 0 & 0 & 1 \\ 0 & -\frac{270}{37} & -\frac{2232}{37} & 0 & 1 & -\frac{8}{37} \\ 0 & -\frac{160}{37} & -\frac{1298}{37} & 1 & 0 & -\frac{2}{37} \end{array}\right] R_3 \leftarrow R_3 - \tfrac{16}{27} \cdot R_2 =$$

$$= \left[\begin{array}{ccc|ccc} 74 & 376 & 2018 & 0 & 0 & 1 \\ 0 & -\frac{270}{37} & -\frac{2232}{37} & 0 & 1 & -\frac{8}{37} \\ 0 & 0 & \frac{2}{3} & 1 & -\frac{16}{27} & \frac{2}{27} \end{array}\right] R_3 \leftarrow \tfrac{3}{2} \cdot R_3 =$$

$$= \left[\begin{array}{ccc|ccc} 74 & 376 & 2018 & 0 & 0 & 1 \\ 0 & -\frac{270}{37} & -\frac{2232}{37} & 0 & 1 & -\frac{8}{37} \\ 0 & 0 & 1 & \frac{3}{2} & -\frac{8}{9} & \frac{1}{9} \end{array}\right] R_2 \leftarrow R_2 + \tfrac{2232}{37} \cdot R_3 =$$

$$= \left[\begin{array}{ccc|ccc} 74 & 376 & 2018 & 0 & 0 & 1 \\ 0 & -\frac{270}{37} & 0 & \frac{3348}{37} & -\frac{1947}{37} & \frac{240}{37} \\ 0 & 0 & 1 & \frac{3}{2} & -\frac{8}{9} & \frac{1}{9} \end{array}\right] R_1 \leftarrow R_1 - 2018 \cdot R_3 =$$

$$= \left[\begin{array}{ccc|ccc} 74 & 376 & 0 & -3027 & \frac{16144}{9} & -\frac{2009}{9} \\ 0 & -\frac{270}{37} & 0 & \frac{3348}{37} & -\frac{1947}{37} & \frac{240}{37} \\ 0 & 0 & 1 & \frac{3}{2} & -\frac{8}{9} & \frac{1}{9} \end{array}\right] R_2 \leftarrow -\tfrac{37}{270} \cdot R_2 =$$

$$= \left[\begin{array}{ccc|ccc} 74 & 376 & 0 & -3027 & \frac{16144}{9} & -\frac{2009}{9} \\ 0 & 1 & 0 & -\frac{62}{5} & \frac{649}{90} & -\frac{8}{9} \\ 0 & 0 & 1 & \frac{3}{2} & -\frac{8}{9} & \frac{1}{9} \end{array}\right] R_1 \leftarrow R_1 - 376 \cdot R_2 =$$

$$= \left[\begin{array}{ccc|ccc} 74 & 0 & 0 & \frac{8177}{5} & -\frac{4588}{5} & 111 \\ 0 & 1 & 0 & -\frac{62}{5} & \frac{649}{90} & -\frac{8}{9} \\ 0 & 0 & 1 & \frac{3}{2} & -\frac{8}{9} & \frac{1}{9} \end{array}\right] R_1 \leftarrow \tfrac{1}{74} \cdot R_1 =$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{221}{10} & -\frac{62}{5} & \frac{3}{2} \\ 0 & 1 & 0 & -\frac{62}{5} & \frac{649}{90} & -\frac{8}{9} \\ 0 & 0 & 1 & \frac{3}{2} & -\frac{8}{9} & \frac{1}{9} \end{array}\right] \rightarrow \begin{pmatrix} \frac{221}{10} & -\frac{62}{5} & \frac{3}{2} \\ -\frac{62}{5} & \frac{649}{90} & -\frac{8}{9} \\ \frac{3}{2} & -\frac{8}{9} & \frac{1}{9} \end{pmatrix};$$

$$\vec{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} \frac{221}{10} & -\frac{62}{5} & \frac{3}{2} \\ -\frac{62}{5} & \frac{649}{90} & -\frac{8}{9} \\ \frac{3}{2} & -\frac{8}{9} & \frac{1}{9} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 5 & 6 \\ 4 & 9 & 25 & 36 \end{pmatrix} \begin{pmatrix} 0 \\ -10 \\ -48 \\ -76 \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{33}{10} & -\frac{8}{5} & -\frac{12}{5} & \frac{17}{10} \\ -\frac{23}{15} & \frac{37}{30} & \frac{43}{30} & -\frac{17}{15} \\ \frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 0 \\ -10 \\ -48 \\ -76 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix};$$ $\rightarrow$ best fit quadratic polynomial: $\hat{y} = 2 + 5x - 3x^2$.

(c)

$\hat{y} = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3;$

$$X = \begin{pmatrix} 1 & -1 & -1^2 & -1^3 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1^2 & 1^3 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{pmatrix}; \vec{y} = \begin{pmatrix} -14 \\ -5 \\ -4 \\ 1 \\ 22 \end{pmatrix}$$

$$\vec{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = (X^T X)^{-1} X^T \vec{y};$$

$$X^T X = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 4 & 9 \\ -1 & 0 & 1 & 8 & 27 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{pmatrix} = \begin{pmatrix} 5 & 5 & 15 & 35 \\ 5 & 15 & 35 & 99 \\ 15 & 35 & 99 & 275 \\ 35 & 99 & 275 & 795 \end{pmatrix};$$

$$X^T X)^{-1} = \left[\begin{array}{cccc|cccc} 5 & 5 & 15 & 35 & 1 & 0 & 0 & 0 \\ 5 & 15 & 35 & 99 & 0 & 1 & 0 & 0 \\ 15 & 35 & 99 & 275 & 0 & 0 & 1 & 0 \\ 35 & 99 & 275 & 795 & 0 & 0 & 0 & 1 \end{array}\right] R_1 \leftrightarrow R_4 =$$

$$= \left[\begin{array}{cccc|cccc} 35 & 99 & 275 & 795 & 0 & 0 & 0 & 1 \\ 5 & 15 & 35 & 99 & 0 & 1 & 0 & 0 \\ 15 & 35 & 99 & 275 & 0 & 0 & 1 & 0 \\ 5 & 5 & 15 & 35 & 1 & 0 & 0 & 0 \end{array}\right] R_2 \leftarrow R_2 - \tfrac{1}{7} \cdot R_1 =$$

$$= \left[\begin{array}{cccc|cccc} 35 & 99 & 275 & 795 & 0 & 0 & 0 & 1 \\ 0 & \frac{6}{7} & -\frac{30}{7} & -\frac{102}{7} & 0 & 1 & 0 & -\frac{1}{7} \\ 15 & 35 & 99 & 275 & 0 & 0 & 1 & 0 \\ 5 & 5 & 15 & 35 & 1 & 0 & 0 & 0 \end{array}\right] R_3 \leftarrow R_3 - \tfrac{3}{7} \cdot R_1 =$$

$$= \left[\begin{array}{cccc|cccc} 35 & 99 & 275 & 795 & 0 & 0 & 0 & 1 \\ 0 & \frac{6}{7} & -\frac{30}{7} & -\frac{102}{7} & 0 & 1 & 0 & -\frac{1}{7} \\ 0 & -\frac{52}{7} & -\frac{132}{7} & -\frac{460}{7} & 0 & 0 & 1 & -\frac{3}{7} \\ 5 & 5 & 15 & 35 & 1 & 0 & 0 & 0 \end{array}\right] R_4 \leftarrow R_4 - \tfrac{1}{7} \cdot R_1 =$$

$$= \left[\begin{array}{cccc|cccc} 35 & 99 & 275 & 795 & 0 & 0 & 0 & 1 \\ 0 & \frac{6}{7} & -\frac{30}{7} & -\frac{102}{7} & 0 & 1 & 0 & -\frac{1}{7} \\ 0 & -\frac{52}{7} & -\frac{132}{7} & -\frac{460}{7} & 0 & 0 & 1 & -\frac{3}{7} \\ 0 & -\frac{64}{7} & -\frac{170}{7} & -\frac{550}{7} & 1 & 0 & 0 & -\frac{1}{7} \end{array}\right] R_2 \leftrightarrow R_4 =$$

$$= \left[\begin{array}{cccc|cccc} 35 & 99 & 275 & 795 & 0 & 0 & 0 & 1 \\ 0 & -\frac{64}{7} & -\frac{170}{7} & -\frac{550}{7} & 1 & 0 & 0 & -\frac{1}{7} \\ 0 & -\frac{52}{7} & -\frac{132}{7} & -\frac{460}{7} & 0 & 0 & 1 & -\frac{3}{7} \\ 0 & \frac{6}{7} & -\frac{30}{7} & -\frac{102}{7} & 0 & 1 & 0 & -\frac{1}{7} \end{array}\right] R_3 \leftarrow R_3 - \tfrac{13}{16} \cdot R_2 =$$

$$= \left[\begin{array}{cccc|cccc} 35 & 99 & 275 & 795 & 0 & 0 & 0 & 1 \\ 0 & -\frac{64}{7} & -\frac{170}{7} & -\frac{550}{7} & 1 & 0 & 0 & -\frac{1}{7} \\ 0 & 0 & \frac{7}{8} & -\frac{15}{8} & -\frac{13}{16} & 0 & 1 & -\frac{5}{16} \\ 0 & \frac{6}{7} & -\frac{30}{7} & -\frac{102}{7} & 0 & 1 & 0 & -\frac{1}{7} \end{array}\right] R_4 \leftarrow R_4 + \frac{3}{32} \cdot R_2 =$$

$$= \left[\begin{array}{cccc|cccc} 35 & 99 & 275 & 795 & 0 & 0 & 0 & 1 \\ 0 & -\frac{64}{7} & -\frac{170}{7} & -\frac{550}{7} & 1 & 0 & 0 & -\frac{1}{7} \\ 0 & 0 & \frac{7}{8} & -\frac{15}{8} & -\frac{13}{16} & 0 & 1 & -\frac{5}{16} \\ 0 & 0 & -\frac{105}{16} & -\frac{351}{16} & \frac{3}{32} & 1 & 0 & -\frac{5}{32} \end{array}\right] R_3 \leftrightarrow R_4 =$$

$$= \left[\begin{array}{cccc|cccc} 35 & 99 & 275 & 795 & 0 & 0 & 0 & 1 \\ 0 & -\frac{64}{7} & -\frac{170}{7} & -\frac{550}{7} & 1 & 0 & 0 & -\frac{1}{7} \\ 0 & 0 & -\frac{105}{16} & -\frac{351}{16} & \frac{3}{32} & 1 & 0 & -\frac{5}{32} \\ 0 & 0 & \frac{7}{8} & -\frac{15}{8} & -\frac{13}{16} & 0 & 1 & -\frac{5}{16} \end{array}\right] R_4 \leftarrow R_4 + \frac{2}{15} \cdot R_3 =$$

$$= \left[\begin{array}{cccc|cccc} 35 & 99 & 275 & 795 & 0 & 0 & 0 & 1 \\ 0 & -\frac{64}{7} & -\frac{170}{7} & -\frac{550}{7} & 1 & 0 & 0 & -\frac{1}{7} \\ 0 & 0 & -\frac{105}{16} & -\frac{351}{16} & \frac{3}{32} & 1 & 0 & -\frac{5}{32} \\ 0 & 0 & 0 & -\frac{24}{5} & -\frac{4}{5} & \frac{2}{15} & 1 & -\frac{1}{3} \end{array}\right] R_4 \leftarrow -\frac{5}{24} \cdot R_4 =$$

$$= \left[\begin{array}{cccc|cccc} 35 & 99 & 275 & 795 & 0 & 0 & 0 & 1 \\ 0 & -\frac{64}{7} & -\frac{170}{7} & -\frac{550}{7} & 1 & 0 & 0 & -\frac{1}{7} \\ 0 & 0 & -\frac{105}{16} & -\frac{351}{16} & \frac{3}{32} & 1 & 0 & -\frac{5}{32} \\ 0 & 0 & 0 & 1 & \frac{1}{6} & -\frac{1}{36} & -\frac{5}{24} & \frac{5}{72} \end{array}\right] R_3 \leftarrow R_3 + \frac{351}{16} \cdot R_4 =$$

$$= \left[\begin{array}{cccc|cccc} 35 & 99 & 275 & 795 & 0 & 0 & 0 & 1 \\ 0 & -\frac{64}{7} & -\frac{170}{7} & -\frac{550}{7} & 1 & 0 & 0 & -\frac{1}{7} \\ 0 & 0 & -\frac{105}{16} & 0 & \frac{15}{4} & \frac{25}{64} & -\frac{585}{128} & \frac{175}{128} \\ 0 & 0 & 0 & 1 & \frac{1}{6} & -\frac{1}{36} & -\frac{5}{24} & \frac{5}{72} \end{array}\right] R_2 \leftarrow R_2 + \frac{550}{7} \cdot R_4 =$$

$$= \left[\begin{array}{cccc|cccc} 35 & 99 & 275 & 795 & 0 & 0 & 0 & 1 \\ 0 & -\frac{64}{7} & -\frac{170}{7} & 0 & \frac{296}{21} & -\frac{275}{126} & -\frac{1375}{84} & \frac{1339}{252} \\ 0 & 0 & -\frac{105}{16} & 0 & \frac{15}{4} & \frac{25}{64} & -\frac{585}{128} & \frac{175}{128} \\ 0 & 0 & 0 & 1 & \frac{1}{6} & -\frac{1}{36} & -\frac{5}{24} & \frac{5}{72} \end{array}\right] R_1 \leftarrow R_1 - 795 \cdot R_4 =$$

$$= \left[\begin{array}{cccc|cccc} 35 & 99 & 275 & 0 & -\frac{265}{2} & \frac{265}{12} & \frac{1325}{8} & -\frac{1301}{24} \\ 0 & -\frac{64}{7} & -\frac{170}{7} & 0 & \frac{296}{21} & -\frac{275}{126} & -\frac{1375}{84} & \frac{1339}{252} \\ 0 & 0 & -\frac{105}{16} & 0 & \frac{15}{4} & \frac{25}{64} & -\frac{585}{128} & \frac{175}{128} \\ 0 & 0 & 0 & 1 & \frac{1}{6} & -\frac{1}{36} & -\frac{5}{24} & \frac{5}{72} \end{array}\right] R_3 \leftarrow -\frac{16}{105} \cdot R_3 =$$

$$= \left[\begin{array}{cccc|cccc} 35 & 99 & 275 & 0 & -\frac{265}{2} & \frac{265}{12} & \frac{1325}{8} & -\frac{1301}{24} \\ 0 & -\frac{64}{7} & -\frac{170}{7} & 0 & \frac{296}{21} & -\frac{275}{126} & -\frac{1375}{84} & \frac{1339}{252} \\ 0 & 0 & 1 & 0 & -\frac{4}{7} & -\frac{5}{84} & \frac{39}{56} & -\frac{5}{24} \\ 0 & 0 & 0 & 1 & \frac{1}{6} & -\frac{1}{36} & -\frac{5}{24} & \frac{5}{72} \end{array}\right] R_2 \leftarrow R_2 + \frac{170}{7} \cdot$$

$R_3 =$

$$= \left[\begin{array}{cccc|cccc} 35 & 99 & 275 & 0 & -\frac{265}{2} & \frac{265}{12} & \frac{1325}{8} & -\frac{1301}{24} \\ 0 & -\frac{64}{7} & 0 & 0 & \frac{32}{147} & -\frac{1600}{441} & \frac{80}{147} & \frac{16}{63} \\ 0 & 0 & 1 & 0 & -\frac{4}{7} & -\frac{5}{84} & \frac{39}{56} & -\frac{5}{24} \\ 0 & 0 & 0 & 1 & \frac{1}{6} & -\frac{1}{36} & -\frac{5}{24} & \frac{5}{72} \end{array}\right] R_1 \leftarrow R_1 - 275 \cdot R_3 =$$

$$= \left[\begin{array}{cccc|cccc} 35 & 99 & 0 & 0 & \frac{345}{14} & \frac{1615}{42} & -\frac{725}{28} & \frac{37}{12} \\ 0 & -\frac{64}{7} & 0 & 0 & \frac{32}{147} & -\frac{1600}{441} & \frac{80}{147} & \frac{16}{63} \\ 0 & 0 & 1 & 0 & -\frac{4}{7} & -\frac{5}{84} & \frac{39}{56} & -\frac{5}{24} \\ 0 & 0 & 0 & 1 & \frac{1}{6} & -\frac{1}{36} & -\frac{5}{24} & \frac{5}{72} \end{array}\right] R_2 \leftarrow -\frac{7}{64} \cdot R_2 =$$

$$= \left[\begin{array}{cccc|cccc} 35 & 99 & 0 & 0 & \frac{345}{14} & \frac{1615}{42} & -\frac{725}{28} & \frac{37}{12} \\ 0 & 1 & 0 & 0 & -\frac{1}{42} & \frac{25}{63} & -\frac{5}{84} & -\frac{1}{36} \\ 0 & 0 & 1 & 0 & -\frac{4}{7} & -\frac{5}{84} & \frac{39}{56} & -\frac{5}{24} \\ 0 & 0 & 0 & 1 & \frac{1}{6} & -\frac{1}{36} & -\frac{5}{24} & \frac{5}{72} \end{array}\right] R_1 \leftarrow R_1 - 99 \cdot R_2 =$$

$$= \left[\begin{array}{cccc|cccc} 35 & 0 & 0 & 0 & 27 & -\frac{5}{6} & -20 & \frac{35}{6} \\ 0 & 1 & 0 & 0 & -\frac{1}{42} & \frac{25}{63} & -\frac{5}{84} & -\frac{1}{36} \\ 0 & 0 & 1 & 0 & -\frac{4}{7} & -\frac{5}{84} & \frac{39}{56} & -\frac{5}{24} \\ 0 & 0 & 0 & 1 & \frac{1}{6} & -\frac{1}{36} & -\frac{5}{24} & \frac{5}{72} \end{array}\right] R_1 \leftarrow \frac{1}{35} \cdot R_1 =$$

$$= \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{27}{35} & -\frac{1}{42} & -\frac{4}{7} & \frac{1}{6} \\ 0 & 1 & 0 & 0 & -\frac{1}{42} & \frac{25}{63} & -\frac{5}{84} & -\frac{1}{36} \\ 0 & 0 & 1 & 0 & -\frac{4}{7} & -\frac{5}{84} & \frac{39}{56} & -\frac{5}{24} \\ 0 & 0 & 0 & 1 & \frac{1}{6} & -\frac{1}{36} & -\frac{5}{24} & \frac{5}{72} \end{array}\right] \to \begin{pmatrix} \frac{27}{35} & -\frac{1}{42} & -\frac{4}{7} & \frac{1}{6} \\ -\frac{1}{42} & \frac{25}{63} & -\frac{5}{84} & -\frac{1}{36} \\ -\frac{4}{7} & -\frac{5}{84} & \frac{39}{56} & -\frac{5}{24} \\ \frac{1}{6} & -\frac{1}{36} & -\frac{5}{24} & \frac{5}{72} \end{pmatrix};$$

$$\vec{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} \frac{27}{35} & -\frac{1}{42} & -\frac{4}{7} & \frac{1}{6} \\ -\frac{1}{42} & \frac{25}{63} & -\frac{5}{84} & -\frac{1}{36} \\ -\frac{4}{7} & -\frac{5}{84} & \frac{39}{56} & -\frac{5}{24} \\ \frac{1}{6} & -\frac{1}{36} & -\frac{5}{24} & \frac{5}{72} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 4 & 9 \\ -1 & 0 & 1 & 8 & 27 \end{pmatrix} \begin{pmatrix} -14 \\ -5 \\ -4 \\ 1 \\ 22 \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{2}{35} & \frac{27}{35} & \frac{12}{35} & -\frac{8}{35} & \frac{2}{35} \\ -\frac{19}{42} & -\frac{1}{42} & \frac{2}{7} & \frac{13}{42} & -\frac{5}{42} \\ \frac{11}{28} & -\frac{4}{7} & -\frac{1}{7} & \frac{3}{7} & -\frac{3}{28} \\ -\frac{1}{12} & \frac{1}{6} & 0 & -\frac{1}{6} & \frac{1}{12} \end{pmatrix} \begin{pmatrix} -14 \\ -5 \\ -4 \\ 1 \\ 22 \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \\ -4 \\ 2 \end{pmatrix} \to$$

$\to$ best fit cubic polynomial: $\hat{y} = -5 + 3x - 4x^2 + 2x^3$.

Problem 10

(a)

$\vec{u_1} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}; \vec{u_2} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$

Let $\vec{u_1} = \vec{v_1} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

Then:

$\vec{v_2} = \vec{u_2} - proj_{\vec{v_1}} \vec{u_2} = \vec{u_2} - \frac{\vec{u_2} \cdot \vec{v_1}}{\vec{v_1} \cdot \vec{v_1}} \vec{v_1} =$

$$= \begin{pmatrix} 2 \\ -2 \end{pmatrix} - \frac{\begin{pmatrix} 2 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix}}{\begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix}} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} + \begin{pmatrix} \frac{2}{5} \\ \frac{6}{5} \end{pmatrix} = \begin{pmatrix} \frac{12}{5} \\ \frac{-4}{5} \end{pmatrix} = \vec{v_2}$$

Normalizing $\vec{v_1}$ and $\vec{v_2}$ to $\vec{v_1'}$ and $\vec{v_2'}$:

$\vec{v_1'} = \frac{\vec{v_1}}{\|\vec{v_1}\|} = \frac{\vec{v_1}}{\sqrt{10}} = \begin{pmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{pmatrix};$

$\vec{v_2'} = \frac{\vec{v_2}}{\|\vec{v_2}\|} = \frac{\vec{v_2}}{\sqrt{\frac{160}{25}}} = \begin{pmatrix} \frac{1}{\sqrt{\frac{160}{25}}} \cdot \frac{12}{5} \\ \frac{1}{\sqrt{\frac{160}{25}}} \left(-\frac{4}{5}\right) \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} \end{pmatrix};$

Thus, orthonormal basis for $\{\vec{u_1}, \vec{u_2}\}$ is $\{\begin{pmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{pmatrix}, \begin{pmatrix} \frac{3}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} \end{pmatrix}\}$.

(b)

$\vec{u_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}; \vec{u_2} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}; \vec{u_3} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix};$

Let $\vec{u_1} = \vec{v_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

Then:

$\vec{v_2} = \vec{u_2} - proj_{\vec{v_1}} \vec{u_2} = \vec{u_2} - \frac{\vec{u_2} \cdot \vec{v_1}}{\vec{v_1} \cdot \vec{v_1}} \vec{v_1} =$

$$= \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} - \frac{\begin{pmatrix} 1 & 3 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} - \left(\frac{-1}{2}\right) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ 3 \\ -\frac{3}{2} \end{pmatrix} = \vec{v_2}$$

If $W = span(\vec{v_1}, \vec{v_2})$ then:

$\vec{v_3} = \vec{u_3} - proj_W \vec{u_3} = \vec{u_3} - proj_{\vec{v_1}} \vec{u_3} - proj_{\vec{v_2}} \vec{u_3} =$

$$= \begin{pmatrix}0\\2\\1\end{pmatrix} - \frac{\begin{pmatrix}0 & 2 & 1\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix}}{\begin{pmatrix}1 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix}} \begin{pmatrix}1\\0\\1\end{pmatrix} - \frac{\begin{pmatrix}0 & 2 & 1\end{pmatrix}\begin{pmatrix}\frac{3}{2}\\3\\-\frac{3}{2}\end{pmatrix}}{\begin{pmatrix}\frac{3}{2} & 3 & -\frac{3}{2}\end{pmatrix}\begin{pmatrix}\frac{3}{2}\\3\\-\frac{3}{2}\end{pmatrix}} \begin{pmatrix}\frac{3}{2}\\3\\-\frac{3}{2}\end{pmatrix} =$$

$$= \begin{pmatrix}0\\2\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\0\\1\end{pmatrix} - \frac{1}{3}\begin{pmatrix}\frac{3}{2}\\3\\-\frac{3}{2}\end{pmatrix} = \begin{pmatrix}-1\\1\\1\end{pmatrix} = \vec{v_3};$$

Normalizing vectors $\vec{v_1}$, $\vec{v_2}$, $\vec{v_3}$ to $\vec{v_1'}$, $\vec{v_2'}$, $\vec{v_3'}$:

$$\vec{v_1'} = \frac{\vec{v_1}}{\|\vec{v_1}\|} = \frac{\vec{v_1}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{pmatrix};$$

$$\vec{v_2'} = \frac{\vec{v_2}}{\|\vec{v_2}\|} = \frac{\vec{v_2}}{\sqrt{\frac{54}{4}}} = \frac{1}{\sqrt{\frac{54}{4}}}\begin{pmatrix}\frac{3}{2}\\3\\-\frac{3}{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{6}}\\\sqrt{\frac{2}{3}}\\-\frac{1}{\sqrt{6}}\end{pmatrix};$$

$$\vec{v_3'} = \frac{\vec{v_3}}{\|\vec{v_3}\|} = \frac{\vec{v_3}}{\sqrt{3}} = \frac{1}{\sqrt{3}}\begin{pmatrix}-1\\1\\1\end{pmatrix} = \begin{pmatrix}-\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{pmatrix};$$

Thus, orthonormal basis for $\{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$ is $\{\begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{pmatrix}, \begin{pmatrix}\frac{1}{\sqrt{6}}\\\sqrt{\frac{2}{3}}\\-\frac{1}{\sqrt{6}}\end{pmatrix}, \begin{pmatrix}-\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{pmatrix}\}$.

Problem 9

(a)

$A = \begin{bmatrix} u_1 & u_1 + \epsilon u_2 \end{bmatrix}$

$A^T A = \begin{bmatrix} u_1^T \\ (u_1 + \epsilon u_2)^T \end{bmatrix} \begin{bmatrix} u_1 & u_1 + \epsilon u_2 \end{bmatrix} = \begin{bmatrix} u_1 \cdot u_1 & u_1 \cdot u_1 + u_1 \cdot \epsilon u_2 \\ u_1 \cdot u_1 + u_1 \cdot \epsilon u_2 & u_1 \cdot u_1 + u_1 \cdot \epsilon u_2 + u_1 \cdot \epsilon u_2 + \epsilon u_2 \cdot \epsilon u_2 \end{bmatrix} =$

$= \begin{bmatrix} u_1 \cdot u_1 & u_1 \cdot u_1 \\ u_1 \cdot u_1 & u_1 \cdot u_1 + \epsilon u_2 \cdot \epsilon u_2 \end{bmatrix}$

$(A^T A)^{-1} = \frac{1}{det(A^T A)} \begin{bmatrix} u_1 \cdot u_1 + \epsilon u_2 \cdot \epsilon u_2 & -(u_1 \cdot u_1) \\ -(u_1 \cdot u_1) & u_1 \cdot u_1 \end{bmatrix}$

$= \frac{1}{\|u_1\|^2(\|u_1\|^2 + \epsilon^2 \|u_2\|^2) - \|u_1\|^2 \|u_1\|^2} \begin{bmatrix} \|u_1\|^2 + \epsilon^2 \|u_2\|^2 & -(\|u_1\|^2) \\ -(\|u_1\|^2) & \|u_1\|^2 \end{bmatrix} =$

$= \frac{1}{\|u_1\|^2 \epsilon^2 \|u_2\|^2} \begin{bmatrix} \|u_1\|^2 + \epsilon^2 \|u_2\|^2 & -(\|u_1\|^2) \\ -(\|u_1\|^2) & \|u_1\|^2 \end{bmatrix}.$

$\hat{x} = (A^T A)^{-1} A^T \vec{b} = \frac{1}{\|u_1\|^2 \epsilon^2 \|u_2\|^2} \begin{bmatrix} \|u_1\|^2 + \epsilon^2 \|u_2\|^2 & -(\|u_1\|^2) \\ -(\|u_1\|^2) & \|u_1\|^2 \end{bmatrix} \begin{bmatrix} u_1^T \\ (u_1 + \epsilon u_2)^T \end{bmatrix} \vec{b}.$

In above expression we have a fraction $\frac{1}{\|u_1\|^2 \epsilon^2 \|u_2\|^2}$. If $\epsilon \to 0$ then value of this fraction goes to positive infinity. Thus, for any vector resulting from other part of expression we would have every entry of such vector going to positive infinity, because it is necessarily multiplied by mentioned fraction value. So, x is exploding in a sense that:

$\lim_{\epsilon \to 0} \hat{x} = \begin{bmatrix} \infty \\ \infty \end{bmatrix}$ OR $\lim_{\epsilon \to 0} \|\hat{x}\|_2 = \infty$.

Problem 9.

(b)

From (a) we know $\hat{x}$.

We also know that $A\hat{x}$ is a projection of b onto column space of A. Let's calculate $A\hat{x}$:

$$A\hat{x} = \begin{bmatrix} \vec{u}_1 & \vec{u}_1+\varepsilon\vec{u}_2 \end{bmatrix} \frac{1}{\|u_1\|^2\varepsilon^2\|u_2\|^2} \begin{bmatrix} \|u_1\|^2+\varepsilon^2\|u_2\|^2 & -\|u_1\|^2 \\ -\|u_1\|^2 & \|u_1\|^2 \end{bmatrix} \begin{bmatrix} \vec{u}_1^T \\ (\vec{u}_1+\varepsilon\vec{u}_2)^T \end{bmatrix} \vec{b} =$$

$$= \begin{bmatrix} \vec{u}_1 & \vec{u}_1+\varepsilon\vec{u}_2 \end{bmatrix} \frac{1}{\|u_1\|^2\varepsilon^2\|u_2\|^2} \begin{bmatrix} (\|u_1\|^2+\varepsilon^2\|u_2\|^2)\vec{u}_1^T - (\|u_1\|^2)(\vec{u}_1+\varepsilon\vec{u}_2)^T \\ (-\|u_1\|^2)\vec{u}_1^T + (\|u_1\|^2)(\vec{u}_1+\varepsilon\vec{u}_2)^T \end{bmatrix} \vec{b} =$$

$$= \begin{bmatrix} \vec{u}_1 & \vec{u}_1+\varepsilon\vec{u}_2 \end{bmatrix} \begin{bmatrix} \frac{\|u_1\|^2+\varepsilon^2\|u_2\|^2}{\|u_1\|^2\varepsilon^2\|u_2\|^2}\vec{u}_1^T - \frac{\|u_1\|^2}{\|u_1\|^2\varepsilon^2\|u_2\|^2}(\vec{u}_1+\varepsilon\vec{u}_2)^T \\ -\frac{\|u_1\|^2}{\|u_1\|^2\varepsilon^2\|u_2\|^2}\vec{u}_1^T + \frac{\|u_1\|^2}{\|u_1\|^2\varepsilon^2\|u_2\|^2}(\vec{u}_1+\varepsilon\vec{u}_2)^T \end{bmatrix} \vec{b} =$$

$$= \begin{bmatrix} \vec{u}_1 & \vec{u}_1+\varepsilon\vec{u}_2 \end{bmatrix} \begin{bmatrix} \left(\frac{\|u_1\|^2}{\|u_1\|^2\varepsilon^2\|u_2\|^2} + \frac{\varepsilon^2\|u_2\|^2}{\|u_1\|^2\varepsilon^2\|u_2\|^2}\right)\vec{u}_1^T - \left(\frac{\|u_1\|^2}{\|u_1\|^2\varepsilon^2\|u_2\|^2}\right)(\vec{u}_1+\varepsilon\vec{u}_2)^T \\ \left(-\frac{\|u_1\|^2}{\|u_1\|^2\varepsilon^2\|u_2\|^2}\right)\vec{u}_1^T + \left(\frac{\|u_1\|^2}{\|u_1\|^2\varepsilon^2\|u_2\|^2}\right)\vec{u}_1^T + \left(\frac{\|u_1\|^2}{\|u_1\|^2\varepsilon^2\|u_2\|^2}\right)\varepsilon\vec{u}_2^T \end{bmatrix} \vec{b} =$$

$$= \begin{bmatrix} \vec{u}_1 & \vec{u}_1+\varepsilon\vec{u}_2 \end{bmatrix} \begin{bmatrix} \left(\frac{1}{\varepsilon^2\|u_2\|^2}\right)\vec{u}_1^T + \left(\frac{1}{\|u_1\|^2}\right)\vec{u}_1^T - \left(\frac{1}{\varepsilon^2\|u_2\|^2}\right)\vec{u}_1^T - \left(\frac{1}{\varepsilon^2\|u_2\|^2}\right)\varepsilon\vec{u}_2^T \\ -\left(\frac{1}{\varepsilon\|u_2\|^2}\right)\vec{u}_1^T + \left(\frac{1}{\varepsilon\|u_2\|^2}\right)\vec{u}_1^T + \left(\frac{1}{\varepsilon^2\|u_2\|^2}\right)\varepsilon\vec{u}_2^T \end{bmatrix} \vec{b} =$$

APPLIED SCIENCES FACULTY.

$$= [\vec{u}_1 \quad \vec{u}_1 + \varepsilon \vec{u}_2] \begin{bmatrix} \frac{1}{\|u_1\|^2} \vec{u}_1^T - \frac{1}{\varepsilon \|u_2\|^2} \vec{u}_2^T \\ \frac{1}{\varepsilon \|u_2\|^2} \vec{u}_2^T \end{bmatrix} \vec{b}. =$$

Let $[\vec{u}_1 \quad \vec{u}_1 + \varepsilon \vec{u}_2] = \begin{bmatrix} (\vec{u}_1)_1 & (\vec{u}_1)_1 + \varepsilon (\vec{u}_2)_1 \\ \vdots & \vdots + \vdots \\ (\vec{u}_1)_m & (\vec{u}_1)_m + \varepsilon (\vec{u}_2)_m \end{bmatrix}$.

Then:

$$[\vec{u}_1 \quad \vec{u}_1 + \varepsilon \vec{u}_2] \begin{bmatrix} \frac{1}{\|u_1\|^2} \vec{u}_1^T - \frac{1}{\varepsilon \|u_2\|^2} \vec{u}_2^T \\ \frac{1}{\varepsilon \|u_2\|^2} \vec{u}_2^T \end{bmatrix} \vec{b} =$$

$$= \begin{bmatrix} (\vec{u}_1)_1 & (\vec{u}_1)_1 + \varepsilon (\vec{u}_2)_1 \\ \vdots & \vdots + \vdots \\ (\vec{u}_1)_m & (\vec{u}_1)_m + \varepsilon (\vec{u}_2)_m \end{bmatrix} \begin{bmatrix} \frac{1}{\|u_1\|^2} \vec{u}_1^T - \frac{1}{\varepsilon \|u_2\|^2} \vec{u}_2^T \\ \frac{1}{\varepsilon \|u_2\|^2} \vec{u}_2^T \end{bmatrix} \vec{b} =$$

$$= \begin{bmatrix} (\vec{u}_1)_1 \cdot \left(\frac{1}{\|u_1\|^2} \vec{u}_1^T - \frac{1}{\varepsilon \|u_2\|^2} \vec{u}_2^T \right) + \left((\vec{u}_1)_1 + \varepsilon (\vec{u}_2)_1 \right) \left(\frac{1}{\varepsilon \|u_2\|^2} \vec{u}_2^T \right) \\ (\vec{u}_1)_m \cdot \left(\frac{1}{\|u_1\|^2} \vec{u}_1^T - \frac{1}{\varepsilon \|u_2\|^2} \vec{u}_2^T \right) + \left((\vec{u}_1)_m + \varepsilon (\vec{u}_2)_m \right) \left(\frac{1}{\varepsilon \|u_2\|^2} \vec{u}_2^T \right) \end{bmatrix} \vec{b} =$$

$$=\begin{bmatrix} \left((\vec{u}_1)_1 \frac{1}{\|u_1\|^2}\right)\vec{u}_1^T - \left((\vec{u}_1)_1 \frac{1}{\varepsilon\|u_2\|^2}\right)\vec{u}_2^T + \left((\vec{u}_1)_1 \frac{1}{\varepsilon\|u_2\|^2}\right)\vec{u}_2^T + \left(\varepsilon(\vec{u}_2)_1 \frac{1}{\varepsilon\|u_2\|^2}\right)\vec{u}_2^T \\ \vdots \\ \left((u_1)_m \frac{1}{\|u_1\|^2}\right)\vec{u}_1^T - \left((u_1)_m \frac{1}{\varepsilon\|u_2\|^2}\right)\vec{u}_2^T + \left((\vec{u}_1)_m \frac{1}{\varepsilon\|u_2\|^2}\right)\vec{u}_2^T + \left(\varepsilon(\vec{u}_2)_m \frac{1}{\varepsilon\|u_2\|^2}\right)\vec{u}_2^T \end{bmatrix}\vec{b} =$$

$$= \begin{bmatrix} \left((u_1)_1 \frac{1}{\|u_1\|^2}\right)\vec{u}_1^T + \left((\vec{u}_2)_1 \frac{1}{\|u_2\|^2}\right)\vec{u}_2^T \\ \vdots \\ \left((u_1)_m \frac{1}{\|u_1\|^2}\right)\vec{u}_1^T + \left((\vec{u}_2)_m \frac{1}{\|u_2\|^2}\right)\vec{u}_2^T \end{bmatrix}\vec{b}.$$

We see that all „ε" terms were cancelled out in $A\hat{x}$. That means that $A\hat{x}$ does not depend on $\varepsilon > 0$.

Problem 4

(a). 1

$$w^T u = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} u_1 - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} u_2 + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} u_3 =$$

$$= (u_2 \cdot v_3 - u_3 \cdot v_2) u_1 - (u_1 v_3 - u_3 v_1) u_2 + (u_1 v_2 - u_2 v_1) u_3 =$$

$$= u_1 u_2 v_3 - u_1 u_3 v_2 - u_1 u_2 v_3 + u_2 u_3 v_1 + u_1 u_3 v_2 - u_2 u_3 v_1 =$$

$= 0 \longleftrightarrow$ w and u are orthogonal.

(a). 2.

$$w^T v = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} v_1 - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} v_2 + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} v_3 =$$

$$= v_1 v_3 u_2 - v_1 v_2 u_3 - v_2 v_3 u_1 + v_1 v_2 u_3 + v_2 v_3 u_1 - v_1 v_3 u_2 =$$

$= 0 \longleftrightarrow$ w and v are orthogonal.

(b)

Cross-product is not applicable in general case when n in $\mathbb{R}^n \neq 3$.

We know that if $C(A) : V$ then $V^{\perp} = N(A^T)$.

So, to find vector that is orthogonal to subspace V, spanned by lin. independent col. vectors of A: $u_1, \dots u_{n-1}$ we need to find non-trivial solution to:

$$A^T x = 0.$$

Problem 5

(a).1.

Matrix Q^{-1} is orthogonal if $(Q^{-1})^{-1} = (Q^{-1})^T$.

By Q being orthogonal we know that $Q^{-1} = Q^T$.

It means that $(Q^{-1})^{-1} = (Q^T)^{-1}$.

We also know that for an arbitrary matrix A it is true that $(A^{-1})^T = (A^T)^{-1}$.

Thus, $(Q^T)^{-1} = (Q^{-1})^T \longleftrightarrow (Q^{-1})^{-1} = (Q^{-1})^T \Leftrightarrow$

$\Leftrightarrow$ matrix Q^{-1} is orthogonal.

(a).2.

Matrix $Q_1 Q_2$ is orthogonal if $(Q_1 Q_2)^T (Q_1 Q_2) = (Q_1 Q_2)(Q_1 Q_2)^T$.

We can prove that by using such facts that $Q_1^T Q_1 = I$, $Q_1^T Q_1 = Q_1 Q_1^T$ and $Q_2^T Q_2 = I$, $Q_2^T Q_2 = Q_2 Q_2^T$:

- $(Q_1 Q_2)^T (Q_1 Q_2) = Q_2^T Q_1^T Q_1 Q_2 = Q_2^T I Q_2 = I$
- $(Q_1 Q_2)(Q_1 Q_2)^T = Q_1 Q_2 Q_2^T Q_1^T = Q_1 I Q_1^T = I$.

Thus, $(Q_1 Q_2)^T (Q_1 Q_2) = (Q_1 Q_2)(Q_1 Q_2)^T \longleftrightarrow Q_1 Q_2$ is orthogonal.

(b)

If Q is upper-triangular then of the form $Q = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}$

then Q^T is necessarily a lower-triangular.

By the fact that Q is orthogonal we can conclude:

Q is orthogonal $\longrightarrow$ Q is invertible $\longrightarrow$ Q^{-1} is upper-triangular.

For Q to satisfy being orthogonal and $Q^T = Q^{-1}$ it must necessarily be diagonal, because other way we have $Q^T \neq Q^{-1}$.