MATH 109: Homework 6

Professor Rabin

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1 Problem 5.56

Let there exist a real number x such that $x^6 + x^4 + 1 = 2x^2 \implies x^6 + x^4 - 2x^2 + 1 = 0$.
However, for all $x \in \mathbb{R}$, $x^6 + (x^2 - 1)^2 = (x^3)^2 + (x^2 - 1)^2$. Since for all $a \in \mathbb{R}, a^2 \geq 0$, $(x^3)^2 + (x^2 - 1)^2 \geq 0$.
Now two cases arise:

1.1 $(x^3)^2 + (x^2 - 1)^2 = 0$

However, for $a, b \in \mathbb{R}$, if $a^2 + b^2 = 0$, a and b must be 0. Therefore, $x^3 = 0 \implies x = 0$ and $x^2 - 1 = 0 \implies x = 1$, which is a contradiction.

1.2 $(x^3)^2 + (x^2 - 1)^2 > 0$

In this case, $x^6 + x^4 - 2x^2 + 1 > 0$, therefore, we have a contradiction.
Since we encountered a contradiction in both cases, the statement must be false.

2 Problem 5.62

(a) Let $a^2 + 1 = 2^n$ for even integer $a \geq 2$ and $n \geq 1$.

Since a is even, $a \equiv 0 \mod 2 \implies a^2 \equiv 0 \mod 2 \implies a^2 + 1 \equiv 1 \mod 2$.

However, $2^n = 2 * 2^{n-1}$, therefore $2^n = a^2 + 1 \equiv 0 \mod 2$, which is a contradiction.

Therefore, a must be odd.

(b) Let $a^2 + 1 = 2^n$ for integers $a \geq 2$ and $n \geq 1$.

Four cases arise:

$$a \equiv 0 \mod 4 \implies a^2 \equiv 0 \mod 4 \implies a^2 + 1 \equiv 1 \mod 4 \tag{1}$$
$$a \equiv 1 \mod 4 \implies a^2 \equiv 1 \mod 4 \implies a^2 + 1 \equiv 2 \mod 4 \tag{2}$$
$$a \equiv 2 \mod 4 \implies a^2 \equiv 0 \mod 4 \implies a^2 + 1 \equiv 1 \mod 4 \tag{3}$$
$$a \equiv 3 \mod 4 \implies a^2 \equiv 1 \mod 4 \implies a^2 + 1 \equiv 2 \mod 4 \tag{4}$$
$$\tag{5}$$

In all cases, either $a^2 + 1 \equiv 1 \mod 4$ or $a^2 + 1 \equiv 2 \mod 4$.

Since $a \geq 2 \implies a^2 + 1 \geq 5$, $2^n \geq 5 \implies n \geq 2$.

For $n \geq 2$, $4 | 2^n \implies 2^n = 4k$ for some $k \in \mathbb{Z}$. Therefore, $a^2 + 1 = 4k \implies a^2 + 1 \equiv 0 \mod 4$, which is a contradiction.

3 Problem 5.65

When suitor 1 says that they do not know, it must imply that at least one of Suitor 2 and Suitor 3 is wearing a gold crown (since if they were both silver crown wearers, suitor 1 must be wearing gold). Therefore, one of suitor 2 and suitor 3 is wearing gold and the other silver. When suitor 2 says that they don't know either, it must mean that suitor 1 and suitor 3 are both wearing gold crowns. Clearly, not both of them are wearing silver crowns (same reasoning as suitor 1). If exactly one of them is wearing a silver crown, then suitor 2 must themselves be wearing gold, which he did not assert. Therefore, it must be that all three are wearing gold crowns. And the third suitor does not require any extra information to deduce this.

4 Problem 6.6

(a) The number of cubes in an $n \times n \times n$ cube composed of n^3 $1 \times 1 \times 1$ cubes.

(b) Since at $1^3 = 1 = \frac{1^2(1+1)^2}{4}$, the formula holds true for $n = 1$.

Assume that for some $k \in \mathbb{Z}^+$, $1^3 + 2^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}$.

$$1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3$$
$$1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$$
$$1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \frac{(k+1)^2(k^2 + 4k + 4)}{4}$$
$$1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}$$

Hence, by the principle of mathematical induction, $1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$ for all $n \in \mathbb{Z}^+$.

5 Problem 6.10

Since $a = \frac{a(1-r)}{1-r}$, the formula holds true for $n = 1$.
Assume that for some $k \in \mathbb{Z}^+$, $a + ar + \cdots + ar^{k-1} = \frac{a(1-r^k)}{1-r}$. Then,

$$a + ar + \cdots + ar^{k-1} + ar^k = \frac{a(1-r^k)}{1-r} + ar^k$$
$$a + ar + \cdots + ar^{k-1} + ar^k = \frac{a(1-r^k + r^k(1-r))}{1-r}$$
$$a + ar + \cdots + ar^{k-1} + ar^k = \frac{a(1-r^{k+1})}{1-r}$$

Hence, by the principle of mathematical induction, $a + ar + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$ for all $n \in \mathbb{Z}^+$.

6 Problem 6.12

(a) $P(k) : 9 + 13 + \cdots + (4k + 5) = \frac{4k^2 + 14k + 1}{2}$.

$$9 + 13 + \cdots + (4k+5) + (4(k+1)+5) = \frac{4k^2 + 14k + 1}{2} + 4(k+1) + 5$$
$$9 + 13 + \cdots + (4k+5) + (4(k+1)+5) = \frac{4k^2 + 14k + 1 + 8k + 8 + 10}{2}$$
$$9 + 13 + \cdots + (4k+5) + (4(k+1)+5) = \frac{4(k+1)^2 + 14(k+1) + 1}{2}$$

Therefore, $P(k+1)$ is true. $P(k) \implies P(k+1)$.

(b) $P(1) : 9 = \frac{19}{2}$ is false. Therefore, $\forall n \in \mathbb{N}, P(n)$ is false.

7 Problem 6.18

7.1 Lemma 1: For $k \geq 10$, $k^3 > 3k^2 + 3k + 1$

Since $k \geq 10$, $k^3 > 10k^2$, $k^3 > 3k^2 + 7k^2$ and $7k^2 > 7 \cdot 10 \cdot k > 3k + 67k$.
Since $k \geq 10$, $67k \geq 670 > 1 \implies 67k + 3k > 3k + 1 \implies 70k > 3k + 1 \implies 7k^2 > 3k + 1 \implies 7k^2 + 3k^2 > 3k^2 + 3k + 1 \implies k^3 > 3k^2 + 3k + 1$.

Since $2^{10} = 1024 > 1000 = 10^3$, $2^n > n^3$ for $n = 10$.
Assume that for some $k \geq 10$, $2^k > k^3$.
Then, $2^{k+1} = 2^k \cdot 2 > 2k^3 \implies 2^k \cdot 2 > k^3 + k^3$. By lemma 1, $2^k \cdot 2 > k^3 + 3k^2 + 3k + 1 \implies 2^{k+1} > (k+1)^3$.
By the principle of mathematical induction, $2^n > n^3$ for all $n \in \mathbb{Z}^+ \geq 10$.

8 Problem 6.23

For $n = 0$, $7|3^0 - 2^0 = 7|0$ is true.
Assume that $7|(3^{2k} - 2^k)$ for $k \in \mathbb{Z}^+ + \{0\}$.
Then, $3^{2k} - 2^k = 7a$ for some $a \in \mathbb{Z}$.

$$3^{2k+2} - 2^{k+1} = 3^{2k} \cdot 9 - 2^k \cdot 2$$
$$3^{2k+2} - 2^{k+1} = 2 \cdot (3^{2k} - 2^k) + 7 \cdot 3^{2k}$$
$$3^{2k+2} - 2^{k+1} = 2 \cdot 7a + 7 \cdot 3^{2k}$$
$$3^{2k+2} - 2^{k+1} = 7 \cdot (3^{2k} + 2a)$$

Since $2a + 3^{2k} \in \mathbb{Z}$, $7|3^{2k+2} - 2^{k+1}$. Hence, by principle of mathematical induction, $7|(3^{2n} - 2^n)$ for all $n \in \mathbb{Z}^+ + \{0\}$.

9 Problem 6.24

9.1 Lemma 1: For every real number x and positive integer k, $1 + (k+1)x + kx^2 \geq 1 + (k+1)x$

For all reals x, $x^2 \geq 0 \implies kx^2 \geq 0 \implies 1 + (k+1)x + kx^2 \geq 1 + (k+1)x$.

Since $x + 1 \geq 1 + x$, the formula is true for $n = 1$.
Assume that for $k \in \mathbb{Z}^+$, $(x+1)^k \geq 1 + kx$.
Then, since $x + 1 \geq 0$, $(x+1)^k(x+1) \geq (1+kx)(x+1)$, by lemma 1, $(x+1)^{k+1} \geq x + 1 + kx^2 + kx \geq 1 + (k+1)x \implies (x+1)^{k+1} \geq 1 + (k+1)x$.
Therefore, by the principle of mathematical induction, $(x+1)^n > 1 + nx$ for all $n \in \mathbb{Z}^+$.

10 Problem 6.34

$a_1 = 1, a_2 = 2, a_3 = 4, a_4 = 8 \cdots$
Conjecture: $a_n = 2^{n-1}$
Since $a_1 = 1 = 2^0$, the formula is true for $n = 1$.
Assume that the formula is true for $a_i \forall i \leq k$. Then $a_{k+1} = a_k + 2a_{k-1} = 2^{k-1} + 2 \cdot 2^{k-2} \implies a_{k+1} = 2 \cdot 2^{k-1} \implies a_{k+1} = 2^k$.
Hence, by the strong principle of mathematical induction, $a_n = 2^n$ for all $n \in \mathbb{N}$.

11 Problem 6.37

Since $12 = 3 \cdot 4 + 7 \cdot 0$, $13 = 3 \cdot 2 + 7 \cdot 1$, $14 = 3 \cdot 0 + 7 \cdot 2$.

Assume that $i = 7x_i + 3y_i$ for some $x_i, y_i \in \mathbb{Z}^+ + \{0\}$ for all $12 \leq i \leq k$ for some $k \geq 14$.

Then $k + 1 = (k - 2) + 3$. Let $k - 2 = 3x_{k-2} + 7y_{k-2}$ for some $x_{k-2}, y_{k-2} \in \mathbb{Z}^+ + \{0\}$. Therefore, $k + 1 = 3(x_{k-2} + 1) + 7y_{k-2}$. Since $x_{k-2} + 1 and y_{k-2} \in \mathbb{Z}^+ + \{0\}$, by the strong principle of mathematical induction, all $n \geq 12$ can be represented as $n = 3a + 7b$ for some $a, b \in \mathbb{Z}^+ + \{0\}$.