# MATH 109: Homework 3

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# Problem 3.64

To prove: For $a, b \in \mathbb{Z}$, if $ab = 4$, then $(a-b)^3 - 9(a-b) = 0$

### Lemma 1: If $a - b = 0$ or $(a-b)^2 - 9 = 0$, then $(a-b)^3 - 9(a-b) = 0$

If $a - b = 0$ or $(a-b)^2 - 9 = 0$, then $(a-b)((a-b)^2 - 9) = 0$.
Then, $(a-b)^3 - 9(a-b) = 0$.

### Proof

If $a, b \in \mathbb{Z}$ and $ab = 4$, then the following cases arise:

1. $a = 1, b = 4$: $(1-4)^2 - 9 = 0 \implies (a-b)^2 - 9 = 0$
2. $a = 2, b = 2$: $(2-2) = 0 \implies a - b = 0$
3. $a = 4, b = 1$: $(4-1)^2 - 9 = 0 \implies (a-b)^2 - 9 = 0$
4. $a = -1, b = -4$: $(-1+4)^2 - 9 = 0 \implies (a-b)^2 - 9 = 0$
5. $a = -2, b = -2$: $(-2+2) = 0 \implies a - b = 0$
6. $a = -4, b = -1$: $(4-1)^2 - 9 = 0 \implies (a-b)^2 - 9 = 0$

In all cases, either $a - b = 0$ or $(a-b)^2 - 9 = 0$. Therefore, using Lemma 1, $(a-b)^3 - 9(a-b) = 0$.

# Problem 3.68

To prove: For $n \in \mathbb{N}$, If $n^3 - 5n - 10 > 0$, then $n \geq 3$.
Contrapositive: If $n < 3$, then $n^3 - 5n - 10 \leq 0$.
Since $n \in \mathbb{N}$, $1 \leq n \leq 2$.
Therefore, $n^3 \leq 8$ and $3 \leq n + 2 \leq 4 \implies 15 \leq 5(n+2) \leq 20$.
Therefore, $n^3 \leq 8 \leq 15 \leq 5(n+2) \implies n^3 \leq 5(n+2) \implies n^3 - 5(n+2) \leq 0 \implies n^3 - 5n - 10 \leq 0$.

# Problem 3.74

To prove: For $a, b, c \in \mathbb{Z}$, if $a^2 + b^2 = c^2$, then $abc$ is even.

### Lemma 1: If $x^2$ is even, then $x$ is even

Contrapositive: If $x$ is odd, then $x^2$ is odd.
Let $x = 2k + 1$ for some $k \in \mathbb{Z}$.
$x^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. Since $2k^2 + 2k \in \mathbb{Z}$, $x^2$ is odd.

### Case 1: $a$ or $b$ is even.

WLOG, let $a$ be even.
Then, $a = 2k$ for some $k \in \mathbb{Z}$.
$abc = 2kbc = 2(kbc)$. Since $kbc \in \mathbb{Z}$, $abc$ is even.

### Case 2: $a$ and $b$ are odd.

Let $a = 2k_1 + 1$ and $b = 2k_2 + 1$ for some $k_1, k_2 \in \mathbb{Z}$.
Then, $c^2 = a^2 + b^2 = 4k_1^2 + 1 + 4k_1 + 4k_2^2 + 1 + 4k_2 = 2(2k_1^2 + 1 + 2k_1 + 2k_2^2 + 2k_2)$.
Since $2k_1^2 + 1 + 2k_1 + 2k_2^2 + 2k_2 \in \mathbb{Z}$, $c^2$ is even.
Since $c^2$ is even, using lemma 1, $c$ is even.
Let $c = 2k$ for $k \in \mathbb{Z}$.
$abc = ab2k = 2(abk)$. Since $abk \in \mathbb{Z}$, $abc$ is even.

## Problem 4.5

To prove: For $a, b, c \in \mathbb{Z}$ such that $a \neq 0$, if $a \nmid b$, then $a \nmid b$ and $a \nmid c$.
Contrapositive: If $a \mid b$ or $a \mid c$, then $a \mid bc$.
WLOG, let $a \mid b$.
Then, $b = an$ for some $n \in \mathbb{Z}$.
$bc = anc = a(nc)$. Since $nc \in \mathbb{Z}$, $a \mid bc$.

## Problem 4.10

### Lemma 1: If $2 \mid n^4 - 3$, then $2 \nmid n$

Contrapositive: If $2 \mid n$, then $2 \nmid n^4 - 3$.
Let $n = 2k$ for $k \in \mathbb{Z}$.
$n^4 - 3 = 16k^4 - 3 = 16k^4 - 4 + 1 = 2(8k^4 - 2) + 1$. Since $8k^4 - 2 \in \mathbb{Z}$, $2 \nmid n^4 - 3$.

### Lemma 2: If $4 \mid n^2 + 3$, then $2 \nmid n$

Contrapositive: If $2 \mid n$, then $4 \nmid n^2 + 3$.
Let $n = 2k$ for $k \in \mathbb{Z}$.
$n^2 + 3 = 4k^2 + 3$. Since $k \in \mathbb{Z}$, $4 \nmid n^2 + 3$.

### Proof

By lemma 1, if $2 \mid n^4 - 3$, then $2 \nmid n$.
Then, $n = 2k + 1$ for $k \in \mathbb{Z}$. $n^2 + 3 = 4(k^2 + k + 1)$. Since $k^2 + k + 1 \in \mathbb{Z}$, $4 \mid n^2 + 3$.
By lemma 2, if $4 \mid n^2 + 3$, then $2 \nmid n$
Then, $n = 2k + 1$ for $k \in \mathbb{Z}$. $n^4 - 3 = 16k^4 + 32k^3 + 24k^2 + 8k + 2 = 2(8k^4 + 16k^3 + 12k^2 + 4k + 1)$. Since $8k^4 + 16k^3 + 12k^2 + 4k + 1 \in \mathbb{Z}$, $2 \mid n^4 - 3$. ’