MATH 109: Homework 5

Professor Rabin

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1 Problem 4.43

(a) $A = \{1, 2\}, B = \{1, 4\}, C = \{1, 5\}$

(b) $A = \{1\}, B = \{2\}, C = \{1, 2\}$

(c) Let $x \in B$. Two cases arise:

1.1 $x \in A$

Then $x \in A \cap B \implies x \in A \cap C \implies x \in A$ and $x \in C \implies x \in C$.

1.2 $x \notin A$

Then $x \in A \cup B \implies x \in A \cup C \implies x \in A$ or $x \in C \implies x \in C$.

Therefore, $B \subseteq C$.

Similarly, WLOG, let $x \in C$. Therefore, $C \subseteq B$.

Therefore, $B = C$.

2 Problem 4.48

$A = \{n \in \mathbb{Z} : 2|n\}, B = \{n \in \mathbb{Z} : 4|n\}$

2.1 If $n \neq 2k$ for some odd integer k, then $n \notin A - B$

Case 1: $n = 2k + 1$ for some $k \in \mathbb{Z}$.
Then, $n \notin A \implies n \notin A - B$
Case 2: $n = 2k$ for some even $k \in \mathbb{Z}$.
Then, $k = 2l$ and $n = 4l$ for some $l \in \mathbb{Z}$.
Therefore, $n \in A$ and $n \in B \implies n \notin A - B$.

2.2 If $n = 2k$ for some odd k, then $n \in A - B$

$k = 2l + 1$ for some $l \in \mathbb{Z}$. Therefore, $n = 2(2l + 1) \implies n = 4l + 2$
Since $2|2(2l + 1)$, $2|n \implies n \in A$ and $4 \not| 4l + 2 \implies n \notin B$. Therefore, $n \notin A - B$.

3 Problem 4.56

Let $x \in (A - B) \cup (A - C)$.
WLOG, let $x \in (A - B)$. $x \in (A - B) \implies x \in A$ and $x \notin B \implies x \notin B \cap C \implies x \in A - (B \cap C)$.
Therefore, $(A - B) \cup (A - C) \subseteq A - (B \cap C)$.
Let $x \in A - (B \cap C)$. Therefore, $x \in A$ and $x \notin B \cap C$.
WLOG, let $x \notin B$. $x \notin B \implies x \in A - B \implies x \in (A - B) \cup (A - C)$.
Therefore, $A - (B \cap C) \subseteq (A - B) \cup (A - C)$.
Therefore, $A - (B \cap C) = (A - B) \cup (A - C)$.

4 Problem 5.4

Let $n = 2$. $\frac{n(n+1)}{2} = 3$ and $\frac{(n+1)(n+2)}{2} = 6$. 3 is odd, but 6 is not odd. Hence, the implication is false.

5 Problem 5.15

Let a and b be odd integers such that $4|a^2+b^2$.
Let $a^2+b^2 = 4k_1, a = 2k_2+1, b = 2k_3+1$ for some $k_1, k_2, k_3 \in \mathbb{Z}$.
$(2k_2+1)^2 + (2k_3+1)^2 = 4k_1 \implies 4(k_2^2+k_2+k_3^2+k_3)+2 = 4k_1 \implies 2 = 4(k_1 - k_2^2 - k_2 + k_3^2 + k_3)$.
Since $k_1 - k_2^2 - k_2 + k_3^2 + k_3 \in \mathbb{Z}$, we proved $4|2$, which is a contradiction.

6 Problem 5.18

Let the irrational number be k_1 and the rational number be $\frac{p}{q}$ such that $p, q \in \mathbb{Z}$ and $q \neq 0$.
Since $\frac{p}{q} \neq 0 \implies p \neq 0$.
Let product k_2 be a rational number.
Then, $k_2 = \frac{k_1 p}{q}$. Since $p \neq 0$, $k_1 = \frac{k_2 q}{p}$. Since $\frac{k_2 q}{p}$ is rational, we have shown that k_1 is rational, which is a contradiction.

7 Problem 5.20

Let both $ar+s$ and $ar-s$ be rational.
Then, $ar+s = \frac{k_1}{k_2}$ for $k_1, k_2 \in \mathbb{Z}$ and $k_2 \neq 0$ and $ar - s = \frac{k_3}{k_4}$ for $k_3, k_4 \in \mathbb{Z}$ and $k_4 \neq 0$.
Adding, $2ar = \frac{k_1}{k_2} + \frac{k_3}{k_4} \implies a = \frac{k_1k_4+k_2k_3}{2rk_2k_4}$ and $r, k_2, k_4 \neq 0 \implies 2rk_2k_4 \neq 0$. Thus, we have shown that a is rational, which is a contradiction.

8 Problem 5.22

Let $\sqrt{2}+\sqrt{3}$ be rational.
Then, $\sqrt{2}+\sqrt{3} = \frac{p}{q}$ for $p, q \in \mathbb{Z}$ and $q \neq 0$. Further, since $\sqrt{2}+\sqrt{3} > 0, p \neq 0$.

$$\begin{aligned} 2 &= (\frac{p}{q} - \sqrt{3})^2 \\ 2 &= \frac{p^2}{q^2} + 3 - \frac{2\sqrt{3}p}{q} \\ \frac{p^2}{q^2} + 1 &= \frac{2\sqrt{3}p}{q} \\ \sqrt{3} &= \frac{p^2+1}{2pq}, 2pq \neq 0 \end{aligned}$$

By the result of exercise 5.21, we know that $\sqrt{3}$ is irrational. However, we have shown that $\sqrt{3}$ is rational, which is a contradiction.

9 Problem 5.28

9.1 Lemma: For all $x \in \mathbb{Z}$, either $x^2 \equiv 0 \mod 4$ or $x^2 \equiv 1 \mod 4$

4 cases arise:

$$\begin{aligned} x \equiv 0 \mod 4 &\implies x^2 \equiv 0 \mod 4 \\ x \equiv 1 \mod 4 &\implies x^2 \equiv 1 \mod 4 \\ x \equiv 2 \mod 4 &\implies x^2 \equiv 0 \mod 4 \\ x \equiv 3 \mod 4 &\implies x^2 \equiv 1 \mod 4 \end{aligned}$$

9.2 Proof

Let there exist integer c such that for some odd integers a and b, $a^2 + b^2 = c^2$.
Then, since a and b are odd, the following cases arise:

9.2.1 $a \equiv 1 \mod 4$ and $b \equiv 1 \mod 4$

$a^2 \equiv 1 \mod 4$ and $b^2 \equiv 1 \mod 4 \implies a^2 + b^2 \equiv 2 \mod 4 \implies c^2 \equiv 2 \mod 4$. However, by lemma 10.1, this is a contradiction.

9.2.2 WLOG, $a \equiv 1 \mod 4$ and $b \equiv 3 \mod 4$

$a^2 \equiv 1 \mod 4$ and $b^2 \equiv 1 \mod 4 \implies a^2 + b^2 \equiv 2 \mod 4 \implies c^2 \equiv 2 \mod 4$. However, by lemma 10.1, this is a contradiction.

9.2.3 $a \equiv 3 \mod 4$ and $b \equiv 3 \mod 4$

$a^2 \equiv 1 \mod 4$ and $b^2 \equiv 1 \mod 4 \implies a^2 + b^2 \equiv 2 \mod 4 \implies c^2 \equiv 2 \mod 4$. However, by lemma 10.1, this is a contradiction.
Since we arrive at a contradiction is all cases, c must not be an integer.

10 Problem 5.34

Let n and $m \in \mathbb{Z}^+$ such that $m^2 + m + 1 = n^2$.
Since for any positive integer $m + 1 > 0$, $m^2 + m + 1 > m^2 \implies n^2 > m^2 \implies n > m$.
Since $m^2 + m + 1 = n^2$, adding $m > 0$ to the left side, $m^2 + 2m + 1 > n^2 \implies (m+1)^2 > n^2 \implies m + 1 > n$.
Therefore, $m < n < m + 1$. Since $m \in \mathbb{Z}^+$, we have shown that n is not an integer (since there are no integers in $(m, m + 1)$), which is a contradiction.

11 Problem 5.45

Let there exist nonzero a, b such that $\sqrt{a^2 + b^2} = \sqrt[3]{a^3 + b^3}$.
Then,

$$\begin{aligned} (a^2 + b^2)^3 &= (a^3 + b^3)^2 \\ a^6 + b^6 + 3a^2b^2(a^2 + b^2) &= a^6 + b^6 + 2a^3b^3 \\ 3a^2b^2(a^2 + b^2) &= 2a^3b^3 \\ 3(a^2 + b^2) &= 2ab \text{ since } a, b \neq 0 \\ 2(a^2 + b^2) + (a - b)^2 &= 0 \\ \frac{(a - b)^2}{a^2 + b^2} &= -2 \end{aligned}$$

Clearly, all nonzero solutions are complex, not real, which is a contradiction.