MATH 154: Homework 1

Professor Kane

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Question 1

(a) Consider $G = (V, E)$. By definition, E = multiset of unordered *pairs* from V. Let $deg(v) = |\{e = \{x, y\} \in E : v = x \text{ or } v = y\}|$. Therefore, if there are multiple edges between two nodes, each of them contributes to the degree of the endpoints. Therefore, for the example in the problem statement, $2 + 2 = 2 \cdot 2$ holds.

Proof: Since each edge $e = \{x, y\}$ has 2 endpoints x and y, it is counted exactly twice in $\sum_{v \in V} deg(v)$, once in $deg(x)$ and once in $deg(y)$. Therefore, $\sum_{v \in V} deg(v) = 2 \cdot |E|$.

(b) As before, consider $G = (V, E)$. By definition, E = multiset of unordered *multisets* of size two from V. Let $deg(v) = |\{e = \{x, y\} \in E : \text{either } v = x \text{ or } v = y \text{ but not both}\}| + 2 \cdot |\{e = \{x, y\} \in E : v = x \text{ and } v = y\}|$. Therefore, all self loops on node v contribute 2 to $deg(v)$.

Proof: Each edge $e = \{x, y\}$ s.t. $x \neq y$ (i.e. e is not a self-loop) has 2 endpoints x and y, it is counted twice in $\sum_{v \in V} deg(v)$, once in $deg(x)$ and once in $deg(y)$. Each self loop $e = \{x, x\}$ is also counted twice in $deg(x)$. Therefore, $\sum_{v \in V} deg(v) = 2 \cdot |E|$.

(c) Let there exist edge $e = x, y \in E$ s.t e points away from x and towards y. Then, this edge is counted exactly once in $\sum_{v \in V} d_{in}(v)$, in $d_{in}(y)$. Similarly, it is counted exactly once $\sum_{v \in V} d_{out}(v)$, in $d_{out}(x)$. Therefore, $\sum_{v \in V} d_{in}(v) = |E| = \sum_{v \in V} d_{out}(v)$.

Question 2

Let the graph be $G = (V, E)$. $deg(v)$ = degree of vertices $v \in V$.

(a) **Lemma 2.1: For any vertex $v \in V$, $deg(v) \geq 3$.**

Let us assume, for the sake of contradiction, that $deg(v) < 3$. Let E_v = the set of edges with v as one of their endpoints. Since $deg(v) < 3 \implies |E_v| \leq 2$. So, if we remove E_v from G, vertex v is disconnected from the rest of the graph. However, the existence of E_v contradicts the fact that G was 3-connected. Therefore, $deg(v) \geq 3$.

Using Lemma 2.1, $\sum_{v \in V} deg(v) \geq 3 \cdot |V|$. Further, using handshake lemma, $\sum_{v \in V} deg(v) = 2 \cdot |E| \geq 3 \cdot |V|$. Rearranging, $|E| \geq \frac{3 \cdot |V|}{2}$.

(b) $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $E = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 7), (7, 8), (8, 1), (1, 5), (2, 6), (3, 7), (4, 8)\}$

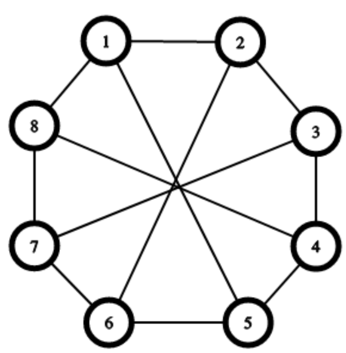

Figure 1: Rough sketch

Proof: We show that deleting any 2 edges from the graph yields a connected graph. Since the graph is symmetric across all "outer ring" edges (i.e. labels of the nodes can be swapped to obtain an isomorphic graph), consider WLOG edge $(1, 2)$ as one of the edges being deleted. On deleting this, in the new subgraph $E - \{(1, 2)\}$, we have a cycle $1, 8, 7, 6, 5, 1$ and another cycle $2, 3, 4, 5, 6, 2$. Deleting an edge, therefore, cannot disconnect these two sets within themselves. Further, since there are multiple connections between these two cycles (edges $(5, 6)$, $(1, 5)$ and $(2, 6)$), deleting any one of these edges cannot disconnect the two cycles. Alternately, if no edges on the "outer ring" are deleted, since the outer ring is a cycle connecting all nodes, the graph is also connected. Therefore, no 2 edges can be deleted to obtain a disconnected graph, i.e. the graph is 3-connected.

Question 3

(a) WLOG, let $d(u, v) \leq d(u, w)$. Then, consider the shortest walk corresponding to $d(u, v)$. Let this walk be denoted by the sequence of vertices $A_0 = u, A_1, \cdots, A_{d(u,v)} = v$. Then, since v and w are adjacent, $A_0 = u, A_1, \cdots, A_{d(u,v)} = v, A_{d(u,v)+1} = w$ is a walk of length $d(u, v) + 1$ between u and w. Therefore, it must be that the length of shortest walk $d(u, w) \leq d(u, v) + 1 \implies d(u, v) \leq d(u, w) \leq d(u, v) + 1$.

(b) WLOG, let us fix $u \in V$. Further, overloading notation, let $d(v) = d(u, v) \quad \forall v \in V$. Let the maximum distance $d(v) = k$ and let $A_1, A_2, \cdots, A_k$ be the sets of nodes $A_i = \{v \in V : d(v) = i\} \quad \forall i$. Further, let's define $B_1, B_2, \cdots, B_k$ s.t. $B_i = |A_{i-1}| + |A_i| + |A_{i+1}| \quad \forall i$.

Lemma 3.1 $|A_i| \geq 1 \quad \forall i \leq k$

Consider the shortest path to the farthest node from u, $x \in V$. Since $d(x) = k$, let the shortest path vertices be $C_0 = u, C_1, \cdots, C_k = x$. We will show that $d(C_i) = i$.

First, notice that $C_0 = u, C_1, \cdots, C_i$ denotes a valid path from u to C_i. So, $d(C_i) \leq i$.

Now, assume, for the sake of contradiction that $d(C_i) < i$. Then, consider the shortest path $D_0 = u, D_1, \cdots, D_{d(C_i)} = C_i$ corresponding to $d(C_i)$. Then, $D_0 = u, D_1, \cdots, D_{d(C_i)}, C_{i+1}, C_{i+2}, \cdots, C_k$ is also a valid path to node x, and since we assumed $d(C_i) < i$, $d(x) \leq d(C_i) + k - i < k$, which contradicts the fact that $d(x) = k$. Therefore, it must be $d(C_i) \geq i$.

Therefore, $d(C_i) = i \implies C_i \in A_i \implies |A_i| \geq 1 \quad \forall i \leq k$.

Since $\forall i \quad A_i \neq \phi$ (using lemma 3.1), let there exist some vertex $v \in A_i$. Then, using the result from Question 3(a), since the distance of all vertices in $neighbourhood(v)$ differs by at most 1, $B_i \geq deg(v) + 1 \geq \delta(G) + 1$. Further, $\sum_{i \leq k} B_i \geq k \cdot (\delta(G) + 1)$.

Now, using a different way to count $\sum_{i \leq k} B_i$, notice that for every vertex $v \in V$, it is counted at most thrice, in $B_{d(v)-1}, B_{d(v)}$ and $B_{d(v)+1}$. Therefore, $\sum_{i \leq k} B_i \leq 3 \cdot |V|$.

Combining these results, $k \cdot (\delta(G) + 1) \leq 3 \cdot |V| \implies k \leq \frac{3 \cdot |V|}{\delta(G)+1} \implies d(u, w) \leq 3 \cdot \left\lceil \frac{|V|}{\delta(G)+1} \right\rceil \forall w \in V$.

Question 4

$\sim$ 5 hours.