MATH 109: Homework 5

Professor Rabin

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Problem 4.43

(a) $A = \{1, 2\}, B = \{1, 4\}, C = \{1, 5\}$

(b) $A = \{1\}, B = \{2\}, C = \{1, 2\}$

(c) Let $x \in B$. Two cases arise:

Case 1: $x \in A$

Then $x \in A \cap B \implies x \in A \cap C \implies x \in A$ and $x \in C \implies x \in C$.

Case 2: $x \notin A$

Then $x \in A \cup B \implies x \in A \cup C \implies x \in A$ or $x \in C \implies x \in C$.
Therefore, $B \subseteq C$.
Similarly, WLOG, let $x \in C$. Therefore, $C \subseteq B$.
Therefore, $B = C$.

Problem 4.48

$A = \{n \in \mathbb{Z} : 2|n\}, B = \{n \in \mathbb{Z} : 4|n\}$

Part 1: If $n \neq 2k$ for some odd integer k, then $n \notin A - B$

Case 1: $n = 2k + 1$ for some $k \in \mathbb{Z}$.
Then, $n \notin A \implies n \notin A - B$
Case 2: $n = 2k$ for some even $k \in \mathbb{Z}$.
Then, $k = 2l$ and $n = 4l$ for some $l \in \mathbb{Z}$.
Therefore, $n \in A$ and $n \in B \implies n \notin A - B$.

Part 2: If $n = 2k$ for some odd k, then $n \in A - B$

$k = 2l + 1$ for some $l \in \mathbb{Z}$. Therefore, $n = 2(2l + 1) \implies n = 4l + 2$
Since $2|2(2l + 1)$, $2|n \implies n \in A$ and $4 \not| 4l + 2 \implies n \notin B$. Therefore, $n \notin A - B$.

Problem 4.56

Let $x \in (A - B) \cup (A - C)$.
WLOG, let $x \in (A - B)$. $x \in (A - B) \implies x \in A$ and $x \notin B \implies x \notin B \cap C \implies x \in A - (B \cap C)$.
Therefore, $(A - B) \cup (A - C) \subseteq A - (B \cap C)$.
Let $x \in A - (B \cap C)$. Therefore, $x \in A$ and $x \notin B \cap C$.
WLOG, let $x \notin B$. $x \notin B \implies x \in A - B \implies x \in (A - B) \cup (A - C)$.
Therefore, $A - (B \cap C) \subseteq (A - B) \cup (A - C)$.
Therefore, $A - (B \cap C) = (A - B) \cup (A - C)$.

Problem 5.4

Let $n = 2$. $\frac{n(n+1)}{2} = 3$ and $\frac{(n+1)(n+2)}{2} = 6$. 3 is odd, but 6 is not odd. Hence, the implication is false.

Problem 5.15

Let a and b be odd integers such that $4|a^2+b^2$.
Let $a^2+b^2 = 4k_1, a = 2k_2+1, b = 2k_3+1$ for some $k_1, k_2, k_3 \in \mathbb{Z}$.
$(2k_2+1)^2+(2k_3+1)^2 = 4k_1 \implies 4(k_2^2+k_2+k_3^2+k_3)+2 = 4k_1 \implies 2 = 4(k_1-k_2^2-k_2+k_3^2+k_3)$.
Since $k_1-k_2^2-k_2+k_3^2+k_3 \in \mathbb{Z}$, we proved $4|2$, which is a contradiction.

Problem 5.18

Let the irrational number be k_1 and the rational number be $\frac{p}{q}$ such that $q \neq 0$.
Since $\frac{p}{q} \neq 0 \implies p \neq 0$.
Let product k_2 be a rational number.
Then, $k_2 = \frac{k_1 p}{q}$. Since $p \neq 0$, $k_1 = \frac{k_2 q}{p}$. Since $\frac{k_2 q}{p}$ is rational, we have shown that k_1 is rational, which is a contradiction.

Problem 5.20

Let both $ar+s$ and $ar-s$ be rational.
Then, $ar+s = \frac{k_1}{k_2}$ for $k_2 \neq 0$ and $ar-s = \frac{k_3}{k_4}$ for $k_4 \neq 0$.
Adding, $2ar = \frac{k_1}{k_2}+\frac{k_3}{k_4} \implies a = \frac{k_1k_4+k_2k_3}{2rk_2k_4}$. Thus, we have shown that a is rational, which is a contradiction.

Problem 5.22

Let $\sqrt{2}+\sqrt{3}$ be rational.
Then, $\sqrt{2}+\sqrt{3} = \frac{p}{q}$ for $q \neq 0$. Further, since $\sqrt{2}+\sqrt{3} > 0, p \neq 0$.

$$2 = (\frac{p}{q}-\sqrt{3})^2$$
$$2 = \frac{p^2}{q^2}+3-\frac{2\sqrt{3}p}{q}$$
$$\frac{p^2}{q^2}+1 = \frac{2\sqrt{3}p}{q}$$
$$\sqrt{3} = \frac{p^2+1}{2pq}, 2pq \neq 0$$

By the result of exercise 5.21, we know that $\sqrt{3}$ is irrational. However, we have shown that $\sqrt{3}$ is rational, which is a contradiction.

Problem 5.34

Let n and $m \in \mathbb{Z}^+$ such that $m^2+m+1 = n^2$.
Since for any positive integer a, $\sqrt{a} < a$, $\sqrt{m^2+m+1} < m^2+m+1 \implies n < m^2+m+1$.
$n^2-m^2 = m+1$. $m+1 > 0 \implies n^2-m^2 > 0 \implies n^2 > m^2 \implies n > m$.