MATH 154: Midterm 2

Professor Kane

Nalin Bhardwaj
A16157819
nibnalin@ucsd.edu

Question 1

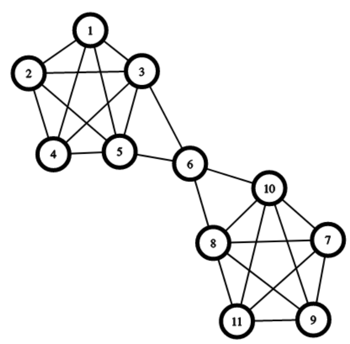
Let the number of vertices be v, edges be e and faces be f. Further, let number of triangle faces be x and square faces be y. Then, $x + y = f$. Then, for any vertex $a \in V$, note that $deg(a) = 5$. This is because each node is connected to 5 faces. Then, by the handshake lemma, $\sum_{a \in V} deg(v) = 2 \cdot e \implies e = \frac{5 \cdot v}{2}$. Further, since triangles has 3 sides and a square has 4 sides, using the dual-handshake lemma, $\sum_{a \in F} sides(a) = 2 \cdot e \implies 3 \cdot x + 4 \cdot y = 5 \cdot v$. Now, since polyhedra are planar graphs, using euler's formula $v - e + f = 2 \implies v - \frac{5 \cdot v}{2} + f = 2 \implies f = x + y = 2 + \frac{3 \cdot v}{2}$. Using this system of two equations, we solve for the 2 variables: $3x + 3y = 110 * 3$ and $3x + 4y = 72 * 5$. Subtracting, $y = 30$. Plugging back in, $x = 110 - 30 = 80$. Therefore, polyhedra has 80 triangle faces and 30 square faces.

Question 2

The set of nodes with color 1 is $\{A, E, F, H\}$. The set of nodes with color 2 is $\{B, C, D, G\}$. The set of nodes with color 3 is $\{I, J\}$.

Question 3

TODO



Question 4

Let V_G represent the set of vertices in a graph G and E_G represent the set of edges in G, i.e. $G = (V_G, E_G)$. Also, for $u \in V_G$, let $color(u, G) =$ the color of node u in a coloring corresponding to chromatic number of graph G.

Lemma 1: For a complete graph H and any graph G', in any valid colouring of $H \cup G'$, for any $u, v \in V_H, color(u, H \cup G') \neq color(v, H \cup G')$.

This is because for any $u, v \in V_H$, since H is a complete subgraph, there is an edge between vertex u and v. Therefore, any valid coloring cannot use the same color for both the vertices.

WLOG, let $\chi(H \cup E) \leq \chi(H \cup F)$, because otherwise we can swap E and F to obtain the same result.
Now, consider the coloring of $H \cup E$ corresponding to $\chi(H \cup E)$. Using lemma 1, we know that for any $u, v \in V_H, color(u, H \cup E) \neq color(v, H \cup E)$. Therefore, each node $u \in V_H$ has a different color, i.e. $|\{color(u, H \cup E) \forall u \in V_H\}| = |V_H|$. WLOG, let the set of colors $\{color(u, H \cup E) \forall u \in V_H\}$ be the set

$\{1, 2, \cdots, |V_H|\}$. This is always possible because we can just permute the set of colors so the labels/colors $\forall u \in V_H$ corresponds to $\{1, 2, \cdots, |V_H|\}$.
Similarly, consider the coloring of $H \cup F$ corresponding to $\chi(H \cup F)$. Using the same reasoning as $H \cup E$, let the set of colors $\{color(u, H \cup F) \forall u \in V_H\}$ be the set $\{1, 2, \cdots, |V_H|\}$. Further, we can permute the set $\{color(u, H \cup F) \forall u \in V_H\}$ such that $color(u, H \cup E) = color(u, H \cup F) \quad \forall u \in V_H$. This is always possible because, using lemma 1, we can set the color of u such that $color(u, H \cup E) = color(u, H \cup F)$ and permute the color of all vertices except u to obtain an equivalent valid coloring.
Now, using these two independent colorings, we can obtain a coloring of G based on three properties as $\forall u \in V_G$: If $u \in V_H$, then $color(u, G) = color(u, H \cup E) = color(u, H \cup F)$. Else if $u \in E$, $color(u, G) = color(u, H \cup E)$. Else, $u \in F$, and $color(u, G) = color(u, H \cup F)$. In this coloring, since there are no edges (u, v) such that $u \in E$ and $v \in F$, the set of neighbours for any node $u \in E \cup F$ is the same as the previous coloring, where it had a valid color. For all nodes $u \in V_H$, since we obtained a coloring in both subgraphs $H \cup E$ and $H \cup F$ such that u had the same color, it must be a valid color in $G = H \cup E \cup F$.
Therefore, we have shown $\chi(G) \leq \chi(H \cup F)$.
Now, assume, for the sake of contradiction, that $\chi(G) < \chi(H \cup F)$. Then, consider the coloring of nodes in G corresponding to $\chi(G)$. Since $H \cup F$ is an induced subgraph of G, any coloring valid in G will also be valid in $H \cup F$. This is because the set of constraints defined by edges $E_{H \cup F} \subseteq E_G$. Therefore, we have obtained a coloring of $H \cup F$ such that we use fewer than $\chi(H \cup F)$ colors, which is a contradiction.
Therefore, $\chi(G) \geq \chi(H \cup F)$.
Combining the two equations, $\chi(G) = \chi(H \cup F)$ when $\chi(H \cup F) \geq \chi(H \cup E)$.