# MATH 154: Homework 3

*Professor Kane*

**Nalin Bhardwaj**
A16157819
nibnalin@ucsd.edu

# Question 1

Let the tree be $T = (V, E)$. Let the nodes be labelled $1, 2, \cdots, |V|$. Let the degree of a node $u$ in tree $T$ be $deg(u, T)$ and the number of occurrences of node $v \in V$ in the Cayley sequence $L$ be $occ(v, L)$. Consider the algorithm for converting a tree into a Cayley sequence (from the lecture slides):

1. Take the lowest labeled leaf, $v \in V$
2. Record label of $v$'s neighbor
3. Remove $v$ from $T$
4. Repeat until $T$ has only 2 vertices

Let us proceed to prove using the principle of mathematical induction. We have to prove that $\forall v \in V occ(v, L) = deg(v, T) + 1$.

## Base case: $|V| = 2$

In this case, there are only 2 vertices in the tree $T$, and based on step 4, we obtain an empty Cayley sequence $\phi$. Since $deg(1, T) = 1 = occ(1, \phi) + 1$ and $deg(2, T) = 1 = occ(2, \phi) + 1$. Therefore, $\forall v \in V occ(v, L) = deg(v, T) + 1$.

## Inductive step

Assume that the statement is true for a tree $T = (V, E)$ and it's corresponding Cayley sequence $L$. Add a node $u$ to this tree $T$ to obtain $T' = (V', E')$ and Cayley sequence $L'$. WLOG, let us relabel all existing nodes of $T$ such that newly added node $u$ is the lowest labeled leaf. (Note that this can be done WLOG because otherwise, there must exist a sequence of node insertions into the tree such that node $u$ was the lowest labelled leaf when inserted. This follows from the Cayley's sequence generation algorithm itself.) Further, let $v \in V$ be the neighbour of $u$ in $T'$.
First, note that for all nodes $k \notin \{u, v\}$, the degree of the node is unchanged. Since only node $v$ has a new neighbour, $deg(k, T') = deg(k, T)$. By the inductive hypothesis, $deg(k, T) = occ(k, L)$. Further, $occ(k, L') = occ(k, L)$. Combining results, $deg(k, T') = occ(k, L') + 1 \quad \forall k \notin \{u, v\}$.
Secondly, for node $u$, since it is the lowest labelled leaf, $deg(u, T') = 1$ and the first step of our algorithm removes it from $T'$ to obtain $T$ (and we know $occ(u, L) = 0$). Therefore, $deg(u, T') = 1 = occ(u, L') + 1$.
Finally, since $u$ is lowest labelled leaf of $T'$, the first step of the algorithm will record the label of it's neighbour ($v$) and remove it from $T'$ to obtain $T$. Therefore, $occ(v, L') = 1 + occ(v, L)$. We also know that $deg(v, T') = deg(v, T)$ (since only $u$ is added as a neighbour). By the inductive hypothesis, $deg(v, T) = occ(v, L) + 1 \implies deg(v, T') = deg(v, T) + 1 = occ(v, L') + 1$.
Therefore, $\forall v \in V occ(v, L') = deg(v, T') + 1$.

# Question 2

Let the connected, directed graph be $G' = (V', E')$. Let the set of nodes $u$ s.t. $d_{in}(u) = d_{out}(u) = 0$ be $S$. Construct the graph $G = G' \setminus S$. Any eulerian circuit in $G$ is also an eulerian circuit in $G'$ (by definition, it covers all edges of $G'$, since no edges are connected to $S$). Therefore, let us prove that for connected, directed graphs $G = (V, E)$ such that $\forall v \in V d_{in}(v) = d_{out}(v) > 0$, $G$ has an Eulerian circuit if and only if for every vertex $v$ of $G$, $d_{in}(v) = d_{out}(v)$, and our results follow for all graphs of the form $G'$.

### Part 1: If $G$ has an Eulerian circuit, then $d_{in}(v) = d_{out}(v) \quad \forall v \in V$.

Let the Eulerian circuit of $G$ be denoted as the sequence of edges $A_1, A_2, \cdots, A_{|E|}$. Further, let $\forall e \in E, next(e) =$ the edge following $e$ in $A$ (i.e., $next(A_i) = A_{i+1} \quad \forall i < |E|$ and $next(A_{|E|}) = A_1$). Similarly, define the inverse function of $next$, $prev$.
For any vertex $v \in V$, let $E_{in}(v)$ denote the set of incoming edges into $v$ and $E_{out}(v)$ denote the set of outgoing edges from $v$. Then, since every edge $e \in E_{in}(v)$ it part of the eulerian circuit of $G$, it must have a $next(e) \in E_{out}(v)$. Therefore, $next : E_{in}(v) \mapsto E_{out}(v)$ is one-to-one. Similarly, every edge $e \in E_{out}(v)$ must have $prev(e) = next^{-1}(e) \in E_{in}(v)$. Therefore, $next : E_{in}(v) \mapsto E_{out}(v)$ is onto. Therefore, $next$ is a bijection between $E_{in}(v)$ and $E_{out}(v)$, and $|E_{in}(v)| = d_{in}(v) = d_{out}(v) = |E_{out}(v)|$.

### Part 2: If $d_{in}(v) = d_{out}(v) \quad \forall v \in V$, then $G$ has an Eulerian circuit

**Lemma 1: If $\forall v \in V d_{in}(v) = d_{out}(v) > 0$, then there must be a cycle in the graph.**

Let us assume, for the sake of contradiction, that there are no cycles in $G$. Then consider the longest path starting at an arbitrary vertex $v \in V$. Let this path be denoted by the sequence of vertices $A_1 = v, A_2, \cdots A_k$. Note that $k > 1$ since $d_{out}(A_1) > 0$. Since $d_{out}(A_k) > 0$, there must be some node $x$ such that there is an edge $(A_k, x)$. Then, two cases arise: Either $x \notin A$, in which case $A + \{x\}$ is a longer path than $A$, and $A$ was not the longest path starting at $v$, which is a contradiction. Or $x \in A$, in which case there is a cycle in $G$. Therefore, if $\forall v \in V d_{in}(v) = d_{out}(v) > 1$, there must be a cycle in $G$.

Now, let us proceed by the principle of strong induction on $|E|$.

**Base case: $\forall v \in V d_{in}(v) = d_{out}(v) = 1$ and $|E| = |V|$.**

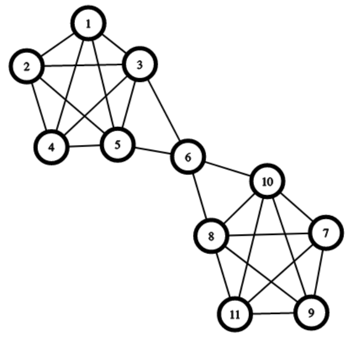
In this case, the only connected, directed graph that can be constructed is a cycle. This is because each node has exactly one incoming edge and one outgoing edge, and this is only possible if the graph is a cycle containing all nodes $\in V$.
By construction, this cycle contains all the edges in the graph, and therefore, it is an Eulerian circuit.

**Inductive step: For any graph $G$ with $d_{in}(v) = d_{out}(v) > 0 \quad \forall v \in V$ and $|E| > |V|$, $G$ contains an Eulerian cycle.**

By Lemma 1, we know that there must be a cycle in the graph. Let the smallest cycle in $G$ be denoted by the sequence of edges $A_1, A_2, \cdots, A_k$. Consider the graph $G'' = (V'', E'')$ where $V'' = V$ and $E'' = E \setminus A$. Since $|A| > 0, |E''| < |E|$. By the inductive hypothesis, graph $G''$ has an Eulerian cycle. Let this cycle be $B_1, B_2, \cdots, B_{|E''|}$. Note that $G$ must contain a node $v$ such that $d_{in}(v) = d_{out}(v) > 1$ (otherwise, $|E| = |V|$, which is the base case). This implies that $B$ must contain some $i$ such that $B_i = (x, v)$ and $B_{i+1} = (v, y)$ for some $x, y \in V$. Let's also relabel the cycle $A$ to start at $v$. Now, consider the eulerian circuit $B_1, B_2, \cdots B_i, A_1, A_2, \cdots A_k, B_{i+1}, \cdots B_k$. This covers are edges $\in E''$ and it covers all edges $\in A$ and therefore, it covers all edges in $E$. Therefore, we have found an eulerian cycle in graph $G$.

# Question 3

Let $G = (V, E)$, $V = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ and $E = \{(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4),$ $(3, 5), (4, 5), (5, 6), (3, 6), (7, 8), (7, 9), (7, 10), (7, 11), (8, 9), (8, 10), (8, 11), (9, 10), (9, 11), (10, 11), (6, 8), (6, 10)\}$.




Proof: In this graph, note that node 6 is an articulation point (i.e. deleting node 6 disconnects the graph). Let us assume, for the sake of contradiction, that there exists a hamiltonian cycle in this graph. WLOG, let that cycle that starts at a node labelled $k$ where $k \leq 6$ (since the graph is isomorphic in the other case). The cycle must cross over to nodes with label $l > 6$, and return back. However, if the path crosses node label 6 once, finding a hamiltonian cycle is equivalent to finding a path with node 6 deleted from the graph. However, we know that deleting an articulation point from the graph disconnects the graph. Therefore, there exist no hamiltonian cycle in this graph.

## Question 4

$\sim$ 6 hours.