MATH 154: Homework 2

Professor Kane

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Question 1

Let the graph be $G = (V, E)$.

First, let us generate a partitioning into two sets from a spanning tree T. Let us root the tree arbitrarily at any node *root*, then assign $height(u)$ = length of path from *root* to u using only tree edges. Then, consider sets $A = \{u \in V : height(u) \equiv 0 \pmod 2\}$ and $B = \{u \in V : height(u) \equiv 1 \pmod 2\}$. Firstly, since G is connected, T must also be connected. Therefore, $A \cup B = V$. Secondly, since there is exactly one path between any pair of vertices in a tree, either $height(u) \equiv 0 \pmod 2$ or $height(u) \equiv 1 \pmod 2$. Therefore, $A \cap B = \phi$. Therefore, $\{A, B\}$ is a valid partition of V.

Next, let us show that WLOG, there are no edge (u, v) s.t. $u \in A$ and $v \in A$. Let $d(u, v)$ = length of path between u and v in T.

Lemma 1.1: For all $u, v \in A$, $d(u, v) \equiv 0 \pmod 2$

Consider the path corresponding to $d(u, v)$ can be broken up as path from u to l followed by the path from l to v, where l is the lowest common ancestor of u and v, i.e. $l = \arg\max height(w) \forall w \in$ the path from *root* to u and the path from *root* to v. Using this l, $d(u, v) = height(u) + height(v) - 2 \cdot height(l) \implies d(u, v) \equiv height(u) + height(v) - 2 \cdot height(l) \pmod 2 \implies d(u, v) \equiv height(u) + height(v) \pmod 2$. Since $height(u) \equiv height(v) \equiv 0 \pmod 2$, $d(u, v) \equiv 0 \pmod 2$.

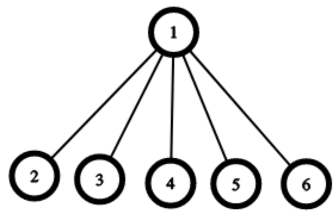
Assume, for the sake of contradiction, that there exists an edge (u, v) s.t. $u \in A$ and $v \in A$. If $(u, v) \in T$, $d(u, v) = 1$ which contradicts Lemma 1.1. Alternately, if $(u, v) \notin T$, consider the cycle formed by the path in T corresponding to $d(u, v)$ and the edge $(u, v) \notin T$. Since $d(u, v) \equiv 0 \pmod 2$, this cycle has odd length, which contradicts the fact that G was a bipartite graph.
Now, let us show that this partitioning is unique. WLOG, for any node $u \in A$, it must have neighbours $adj_u \subseteq \overline{A} = B$. Since G is connected, $|adj_u| \geq 1$. If we arbitrarily fix $u \in A$, we fix the set each other node belongs to: If $d(u, v) \equiv 0 \pmod 2$, $v \in A$ and if $d(u, v) \equiv 1 \pmod 2$, $v \in B$. Therefore, the partitioning is unique.

Question 2

Let us arbitrarily label the nodes $1, 2, 3, 4, 5, 6$. Let $deg(u)$ = degree of node u. For construction, let's fix the labels WLOG such that in the degree sequence, $\forall i, j$ s.t. $i < j$, $deg(i) \geq deg(j)$ in the isomorphic equivalent of the labelled tree we pick. Since we are generating a connected tree, number of edges = 5 and we also know $1 \leq deg(v) \leq 5 \quad \forall v$. Now, let us use case bashing. 5 possibilities arise for node 1:

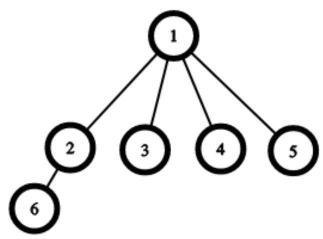
$deg(1) = 5$

Then, since total number of edges = 5, all of which have node 1 as an endpoint, only one possibility arises:



$deg(1) = 4$

Since 4 of the 5 edges have node 1 as an endpoint, the final edge must be connected to node 2 (since $deg(2) \geq deg(k) \quad \forall k > 2$

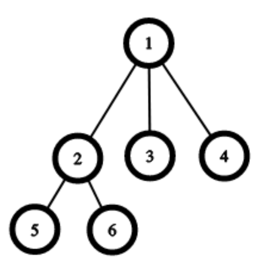


$deg(1) = 3$

Then, consider $deg(2)$. If $deg(2) = 1$, $deg(k) = 1 \quad \forall k \geq 2 \implies \sum_v deg(v) = 3 + 5 \cdot 1 = 8 \neq 10 = 2 \cdot |E|$, which violates handshake lemma. Therefore, $2 \leq deg(2) \leq 3$, i.e. 2 subcases arise for node 2:

$deg(2) = 3$

In this case, $\sum_{k \geq 3} deg(k) = 10 - (deg(1) + deg(2)) = 10 - 6 = 4 \implies deg(k) = 1 \quad \forall k \geq 3$.



$deg(2) = 2$

In this case, $\sum_{k\geq 3} deg(k) = 10 - (deg(1) + deg(2)) = 10 - 5 = 5 \implies deg(3) > 1$ (otherwise, $\sum_{k\geq 3} deg(k) = 4$). Therefore, $deg(3) = 2$. Since $deg(2) = deg(3)$, labels 2 and 3 are interchangeable. Note that either node 2 or node 3 must be connected to node 1 (Otherwise, since at least 3 nodes are connected to node 2 and 3, $deg(1) \neq 3$.) WLOG, let node 2 be the node connected to node 1.
2 distinct cases arise based on location of node 3:

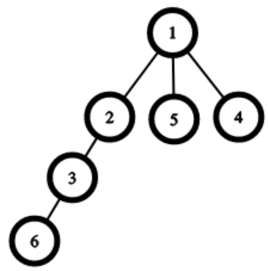


Figure 1: Node 3 is connected to node 2

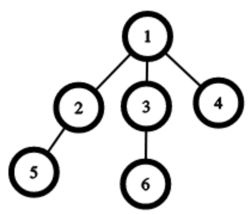
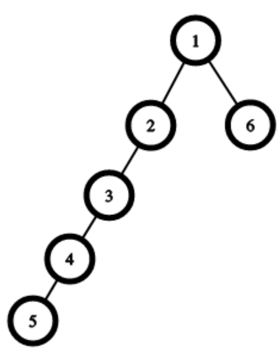


Figure 2: Node 3 is not connected to node 2

$deg(1) = 2$

In this case, since $deg(k) \leq 2 \quad \forall k$, the graph must be a chain.



$deg(1) = 1$

This case is not possible, because if $deg(k) = 1 \quad \forall k$, $\sum_k deg(k) = 6 \neq 10 = 2 \cdot |E|$, which contradicts the handshake lemma.

Therefore, there exist 6 different non-isomorphic trees of 6 nodes, as drawn above.

Question 3

Let the graph $G = (V, E)$. Let the weight of an edge e be denoted by w_e. WLOG, consider two vertices u and v. Let the minimum weight edge between u and v in T have weight α and connect (x, y). Let the largest truck weight that can pass between two vertices u and v be β.
Firstly, consider the path denoted by nodes $a_1, a_2, \cdots, a_k \in T$. Then, since, $w_{(a_i, a_{i+1})} \geq \alpha \quad \forall i < k$, using this path, a truck of weight at least α may pass between u and v, i.e. $\beta \geq \alpha$.
Next, assume, for the sake of contradiction, that $\beta > \alpha$, and the path a truck of weight β can take be $b_1, b_2, \cdots, b_l$. Compare this path with the path $a_1, a_2, \cdots, a_k \in T$. For the minimum weight edge (x, y) (having weight α), there must be some $i < k$ for which $a_i = x$ and $a_{i+1} = y$. Corresponding to this, the path b must **skip** this edge, and there must be some p, q such that $b_p = a_r$ and $b_q = a_s$ but $b_{p+1} \neq a_{r+1}$ and $b_{q-1} \neq a_{s-1}$ for the largest $r \leq i$, and the smallest $s \geq i + 1$. Informally, we pick the part of the path $b_p, \cdots, b_q$ that is used by trucks to skip over the minimum weight edge, using edges not in T. Now, Note that there are two paths between a_r and a_s, $a_r, a_{r+1}, \cdots, a_s \in T$ and $b_p = a_r, b_{p+1}, \cdots, b_q = a_s \notin T$, i.e., the edge (x, y) (with weight α) is in a cycle. Further, all edges on b have weight $\geq \beta$. Therefore, we can delete edge (x, y) (with weight α) and replace it with a new edge not in T from the cycle(i.e., from the path b) to obtain a new spanning tree T'. Now, total weight of $T' \geq$ total weight of $T + \beta - \alpha \implies$ total weight of $T' >$ total weight of T. which contradicts our assumption that T was the maximum spanning tree of G, and therefore, $\beta \leq \alpha$.
Combining these results, $\beta = \alpha$, that is, the largest truck that can make it from v to u is always the same as the minimum edge weight on the unique path from v to u in T.

Question 4

$\sim$ 5 hours.