MATH 109: Homework 3

Professor Rabin

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Problem 2.70

(a) There exists rational number r, such that the number $\frac{1}{r}$ is not rational.

(b) For every rational number r, $r^2 \neq 2$.

Problem 2.72

(a) True, for $x = 0$, $0^2 - 0 = 0$.

(b) True, since $\forall n \in \mathbb{N} \quad n \geq 1 \implies n + 1 \geq 2$.

(c) False, since for $x = -1$, $\sqrt{(-1)^2} = 1 \neq -1$.

(d) True, for $x = 3$, $3 \cdot (3)^2 - 27 = 0$ is true.

(e) True, for $x = 5, y = 0$, $x + y + 3 = 8$ is true.

(f) False, for $x = y = 0$, $x + y + 3 = 8$ is false.

(g) True, for $x = 3, y = 0$, $x^2 + y^2 = 9$ is true.

(h) False, for $x = y = 1$, $x^2 + y^2 = 9$ is false.

Problem 2.93a

P	Q	R	$(P \wedge Q)$	$(P \wedge Q) \implies R$	$\sim R$	$P \wedge \sim R$	$\sim Q$	$P \wedge (\sim R) \implies (\sim Q)$
T	T	T	T	T	F	F	F	T
T	T	F	T	F	T	T	F	F
T	F	T	F	T	F	F	T	T
T	F	F	F	T	T	T	T	T
F	T	T	F	T	F	F	F	T
F	T	F	F	T	T	F	F	T
F	F	T	F	T	F	F	T	T
F	F	F	F	T	T	F	T	T

Since the truth values of $(P \wedge Q) \implies R$ and $P \wedge (\sim R) \implies (\sim Q)$ are the same for all possible truth values of P, R and Q, they are logically equivalent.

Problem 2.103

(a) The real number r either has the property $r < 3$ or $r \geq \pi$.

(b) There exists integer n such that $|r - n| < \frac{1}{2}$ for the real number r.

(c) There exists real number s such that $r \cdot s \neq s$ for the real number r.

Problem 3.2

To prove: If $|n - 1| + |n + 1| \leq 1$, then $|n^2 - 1| \leq 4$.

Proof: Let $n \in \mathbb{N}$, since $n \geq 1 \implies n - 1 \geq 0 \implies |n - 1| \geq 0$.
Similarly, $n \geq 1 \implies n + 1 \geq 2 \implies |n + 1| \geq 2$,
Adding, $|n - 1| + |n + 1| \geq 0 + 2 \implies |n - 1| + |n + 1| \geq 2$.
Therefore, $|n - 1| + |n + 1| \leq 1$ is false $\forall n \in \mathbb{N}$ and the result follows vacuously.

Problem 3.10

To prove: If a and c are odd integers, then $a \cdot b + b \cdot c$ is even for every integer b.

Proof: Since a and c are odd, let $a = 2 \cdot k_1 + 1$ and $c = 2 \cdot k_2 + 1$ for some $k_1, k_2 \in \mathbb{Z}$.

$$\begin{aligned} a \cdot b + b \cdot c &= (a + c) \cdot b \\ &= (2 \cdot k_1 + 1 + 2 \cdot k_2 + 1) \cdot b \\ &= 2 \cdot ((k_1 + k_2 + 1) \cdot b) \end{aligned}$$

Since $(k_1 + k_2 + 1) \cdot b \in \mathbb{Z}$, $a \cdot b + b \cdot c$ must be even.

Problem 3.20

To prove: For $x \in \mathbb{Z}$, $3x + 1$ is even if and only if $5x - 2$ is odd.

Lemma 1: If $3x + 1$ is even, then x is odd.

Proof by contrapositive: Let x be an even integer, therefore $x = 2k$ for some $k \in \mathbb{Z}$.
Then, $3x + 1 = 2(3k) + 1$.
Since $3k \in \mathbb{Z}$, $3x + 1$ is odd.

Theorem 1: If $3x + 1$ is even, then $5x - 2$ is odd.

Using Lemma 1, x must be odd. Therefore, let $x = 2k + 1$ for some $k \in \mathbb{Z}$.
Then, $5x - 2 = 5(2k + 1) - 2 = 2 \cdot 5k + 3 = 2(5k + 1) + 1$.
Since $5k + 1 \in \mathbb{Z}$, $5x - 2$ is odd.

Lemma 2: If $5x - 2$ is odd, then x is odd.

Proof by contrapositive: Let x be an even integer, $x = 2k$ for some $k \in \mathbb{Z}$.
Then, $5 \cdot 2k - 2 = 2(5k - 1)$.
Since $5k - 1 \in \mathbb{Z}$, $5x - 2$ is even.

Theorem 2: If $5x - 2$ is odd, then $3x + 1$ is even.

Using Lemma 2, x must be odd. Therefore, let $x = 2k + 1$ for some $k \in \mathbb{Z}$.
Then, $3x + 1 = 3(2k + 1) + 1 = 2(3k + 2)$.
Since $3k + 2 \in \mathbb{Z}$, $3x - 1$ is even.

Problem 3.31

Contrapositive: If a is odd or b is odd, then $a + b$ and ab are of different parity.

Case 1

WLOG, Let a be odd and b be even. Therefore $a = 2k_1 + 1$ and $b = 2k_2$ for some $k_1, k_2 \in \mathbb{Z}$.
Then, $a + b = 2k_1 + 1 + 2k_2 = 2(k_1 + k_2) + 1$. Since $k_1 + k_2 \in \mathbb{Z}$, $a + b$ is odd.
$ab = (2k_1 + 1) \cdot (2k_2) = 2 \cdot ((2k_1 + 1) \cdot k_2)$. Since $((2k_1 + 1) \cdot k_2) \in \mathbb{Z}$, ab is even.
Therefore, $a + b$ and ab have opposite parity.

Case 2

Let a be odd and b be odd. Therefore $a = 2k_1 + 1$ and $b = 2k_2 + 1$ for some $k_1, k_2 \in \mathbb{Z}$.
Then, $a + b = 2k_1 + 1 + 2k_2 + 1 = 2(k_1 + k_2 + 1)$. Since $k_1 + k_2 + 1 \in \mathbb{Z}$, $a + b$ is even.
$ab = (2k_1 + 1) \cdot (2k_2 + 1) = 4k_1k_2 + 2(k_1 + k_2) + 1 = 2(2k_1k_2 + k_1 + k_2) + 1$. Since $2k_1k_2 + k_1 + k_2 \in \mathbb{Z}$, ab is odd.
Therefore, $a + b$ and ab have opposite parity.

Problem 3.36

Contrapositive: If x is odd or y is odd, then $3x + 4y$ is odd or $4x + 5y$ is odd.

Case 1

Let x and y be odd. Therefore, $x = 2k_1 + 1$ and $y = 2k_2 + 1$ for some $k_1, k_2 \in \mathbb{Z}$.
$3x + 4y = 3 \cdot 2k_1 + 3 + 4 \cdot 2k_2 + 4 = 2(3k_1 + 4k_2 + 3) + 1$. Since $3k_1 + 4k_2 + 3 \in \mathbb{Z}$, $3x + 4y$ is an odd integer and the implication is true.

Case 2

Let x be odd and y be even. Therefore, $x = 2k_1 + 1$ and $y = 2k_2$ for some $k_1, k_2 \in \mathbb{Z}$.
$3x + 4y = 3 \cdot 2k_1 + 3 + 4 \cdot 2k_2 = 2(3k_1 + 4k_2 + 1) + 1$. Since $3k_1 + 4k_2 + 1 \in \mathbb{Z}$, $3x + 4y$ is an odd integer and the implication is true.

Case 3

Let x be even and y be odd. Therefore, $x = 2k_1$ and $y = 2k_2 + 1$ for some $k_1, k_2 \in \mathbb{Z}$.
$4x + 5y = 4 \cdot 2k_1 + 5 \cdot 2k_2 + 5 = 2(4k_1 + 5k_2 + 2) + 1$. Since $4k_1 + 5k_2 + 2 \in \mathbb{Z}$, $4x + 5y$ is an odd integer and the implication is true.

Problem 3.42

The first part proves the implication: If x is even, then $3x^2 - 4x - 5$ is odd.
The second part proves: If x is odd, then $3x^2 - 4x - 5$ is even. By contrapositive, this is equivalent to: If $3x^2 - 4x - 5$ is odd, x is even, which is the converse of the first result.

Therefore, overall result is: x is even if and only if $3x^2 - 4x - 5$ is odd.

Extra Problem

To prove: Every integer is either even or odd, but not both.

By the fundamental theorem of arithmetic, for division by 2 of any number, remainder on division must be 0 or 1. Therefore, by definition, every integer can be expressed as $2k$ or $2k+1$ for some $k \in \mathbb{Z}$.
Now, let us prove that no integer can be both even and odd at the same time.
Let an integer x be both even and odd. Therefore, $x = 2k_1$ and $x = 2k_2 + 1$ for some $k_1, k_2 \in \mathbb{Z}$. Therefore,

$$\begin{aligned} 2k_1 &= 2k_2 + 1 \\ 2 \cdot (k_1 - k_2) &= 1 \\ k_1 - k_2 &= \frac{1}{2} \end{aligned}$$

However, subtraction is closed in $\mathbb{Z}$. Therefore, our assumption must be false, and it is impossible for an integer to be even and odd simultaneously.