Joint Chance Constrained Games

Nalin Gupta and Eshan Balachandar

Prof. Vikas Vikram Singh

Indian Institute of Technology, Delhi

June 29, 2020

Abstract

We consider a n-player game where each player's strategy set contains some stochastic linear constraints. The existence of a Nash Equilibrium under certain conditions has been proved earlier. We further analyse this problem via a constructed example of a Cournot Competition among electricity firms over a Network. We run simulations of an Iterative scheme to find the Nash Equilibrium in this context and describe the results.

1 Introduction

Certain practical game theoretic situations are best modeled using random variables (to account for uncertain parameters) that restrict the strategy space. One way to model these is using chance constraints, in particular, Stochastic linear constraints.

John Nash proved the existence the existence of a mixed strategy Nash equilibrium for a n-player finite strategy game. The proof required the condition of convexity on the strategy spaces of the players. This condition is violated, in general, by the Stochastic linear constraints. Hence, the existence of a Nash Equilibrium is not guaranteed in general for these stochastic games. An earlier work by Prof. Vikas Vikram Singh provided a reformulation of the strategy space when the random matrix of the stochastic linear constraint follows a normal distribution. This was used to prove the existence of Nash Equilibrium under the stochastic constraints.

We construct an experimental setup of a Cournot Competition among Electricity Firms, which would emulate the Stochastic Linear Constraints that we are interested in, and use this as a test bed for an Iterative Algorithm to find the Nash Equilibrium, and analyse it's properties.

2 Model

We consider a n-player non-cooperative game. Let $I = [1, 2..., n]$ be the set of players, where i denotes a generic element of the set I. The payoffs of player i are defined by $u_i : \mathbb{R}^{m_1}_{++} \times \mathbb{R}^{m_2}_{++} \times ... \times \mathbb{R}^{m_n}_{++} \to \mathbb{R}$ where $\mathbb{R}^{m_i}_{++}$ denotes the positive orthant of $\mathbb{R}^{m_1}$. The set $X^i \subset \mathbb{R}^{m_i}_{++}$ denotes the set of all strategies of player i. We assume this set to be convex and compact.

We consider the case where the strategies of player i are further constrained by $A^i x^i \leq b^i$ where A^i is a $K_i \times m_i$ random matrix, and $b^i \in \mathbb{R}^{K_i}$and the K_i constraints are jointly satisfied with probability α_i. Hence the strategy set of player i can be written as

$$S^i_{\alpha_i} = \{x \in X^i | P(A^i x \leq b^i) \geq \alpha_i\}, i \in I \tag{1}$$

Further, we assume that the rows of the random matrices A^i are independent and each row vector A^i_k follows a multivariate normal distribution with mean μ^i_k and co-variance matrix Σ^i_k.

2.1 Deterministic Reformulation

$S^i_{\alpha_i}$ is a probabilistic formulation of the space, and the joint satisfaction of K_i constraints is not easy to handle directly. If $X \in S^i_{\alpha_i}$, then for each $k \in [1, 2...K_i]$ there must exist a z_k such that the constraint corresponding to kth row is satisfied with probability $\geq \alpha_i^{z_k}$ and z_k sum to 1. Here F $(\cdot)$ is the one-dimensional standard normal distribution function.

$$Q^i_{\alpha_i} = \{(x, z) \in X^i \times \mathbb{R}^{K_i} | \tag{2a}$$

$$(u^i_k)^T x + F^{-1}(\alpha_i^{z_k})||(\Sigma^i_k)^{1/2} x|| \leq b^i_k \;\forall k \in [1, 2...K_i]\} \tag{2b}$$

$$\sum_{k=1}^{K_i} z_k = 1 \tag{2c}$$

$$z_k \geq 0 \;\forall k \in [1, 2...K_i] \tag{2d}$$

For each x in $S^i_{\alpha_i}$, there must exist a pair (x, z) in $Q^i_{\alpha_i}$. Conversely, if (x, z) is in $Q^i_{\alpha_i}$, then $x \in S^i_{\alpha_i}$.

Additionally, constraints (2c) and (2d) are always convex. Constraint (2b) can be written as $f(x, z) \leq h$ and if f is proved a convex function, then set $Q^i_{\alpha_i}$ would be proved a convex set. This can be done with an additional logarithmic transformation of x in $Q^i_{\alpha_i}$ to create $\tilde{Q}^i_{\alpha_i}$ when $\alpha_i \geq 0.5$.

$$\tilde{Q}^i_{\alpha_i} = \{(y, z) \in logX^i \times \mathbb{R}^{K_i} | \tag{3a}$$

$$(u^i_k)^T e^y + ||(\Sigma^i_k)^{1/2} e^{\log F^{-1}(\alpha_i^{z_k}) 1_m + y}|| \leq b^i_k \;\forall k \in [1, 2...K_i]\} \tag{3b}$$

$$\sum_{k=1}^{K_i} z_k = 1 \tag{3c}$$

$$z_k \geq 0 \;\forall k \in [1, 2...K_i] \tag{3d}$$

The reformulated strategy space set $\tilde{Q}^i_{\alpha_i}$ can be proved to be convex set (for $\alpha \geq 0.84$ for all players) [1].

2.2 Existence of a Nash Equilibrium

The logarithmic transformation from $Q^i_{\alpha_i}$ to $\tilde{Q}^i_{\alpha_i}$ also changes the utility function.
For each $i \in I$, define the composition function $C_i = -u_i \circ d_i$ where $d_i : \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times ... \times \mathbb{R}^{m_n} \to \mathbb{R}^{m_1}_{++} \times \mathbb{R}^{m_2}_{++} \times ... \times \mathbb{R}^{m_n}_{++}$ such that $d_i(y^1, y^2, ...y^n) = (e^{y^1}, e^{y^2} ..., e^{y^n})$. This is the new effective utility function.

The final list of assumptions for the existence of Nash Equilibrium can be summarized as follows [1]

Assumption 1. The transformed utility function $C_i(., y^{-i})$ is a convex function of $y^i \;\forall\; i \in I$

Assumption 2. The mean μ^i_k and covariance matrix Σ^i_k have all components positive $\forall\; i \in I \;\forall\; k \in [1, 2...K_i]$

Assumption 3. The set X_i is convex and compact $\forall\; i \in I$

Assumption 4. The set $Y_i = \log X_i$ is convex (compactness can be derived) $\forall\; i \in I$

Assumption 5. The probability threshold $\alpha_i \in [F(1), 1] \;\forall\; i \in I \;(F(1) \approx 0.84)$

3 Cournot Competition among Electricity Firms

3.1 Description

Consider an Electricity market where firms compete over an Electricity Network comprised of a set of nodes. There are several generation nodes where firms have setup their generation facilities to produce electricity. There are also several distribution nodes from where the electricity is provided to the consumers. The firms generate the electricity at their facilities and transmit it to the distribution nodes. The transmission over long distances creates power losses, which are best modelled as random variables. The components of the Electricity market are described as follows.

I – the set of firms called as players
N – the set of generation nodes
N_i – the subset of generation nodes where firm i has installed its generation facilities
I_k – the set of firms who have generation nodes at node k
M – the set of distribution nodes

3.2 Pricing

Let x_{ij}^k be the quantity transmitted from generation node i to distribution node j by firm k, and 0 if such transmission doesn't happen. The price for this depends on the total electricity transmitted from node i to node j, i.e. the firms have a Cournot competition for the electricity transmitted between two nodes.

$$p_{ij}(x_{ij}) = \beta_{ij} - \delta_{ij} \sum_{k \in I} x_{ij}^k, \ i \in N, \ j \in M \tag{4}$$

where $\delta_{ij} \geq 0$ for all $i \in N, j \in M$.

3.3 Utility

Each firm encounters a cost in the generation and transmission of electricity. Let $c_{ij}^k(x_{ij}^k)$ be the cost for firm k to generate and transmit x_{ij}^k units of electricity. We assume a linear cost function for each firm, i.e., $c_{ij}^k(x_{ij}^k) = c_{ij}^k x_{ij}^k$ for all $k \in I$, $i \in N$, $j \in M$.
The payoff function for firm k is then given by

$$u_k(x^k, x^{-k}) = \sum_{i \in N} \sum_{j \in M} (\beta_{ij} - \delta_{ij} \sum_{k \in I} x_{ij}^k - c_{ij}^k) \times x_{ij}^k \tag{5}$$

3.4 Transmission Constraints

The strategy set of firm k is defined as $X^k = \{x^k = (x_{ij}^k)_{i \in N_i, j \in M} \mid x_{ij}^k \in [\varepsilon_{ij}^k, C_{ij}^k]\}$, where ε_{ij}^k and C_{ij}^k denote the minimum and maximum output of firm k from node i to node j.
Let A^k be a $|N_i| \times |M|$ random matrix where a_{ij}^k per unit electricity is lost during transmission from generation node i to distribution node j. Each firm wants to keep it's loss corresponding to the generation facilities under a certain threshold. Let b^k denote the threshold vector for player k. Then firm k faces the following stochastic constraint, which it must jointly satisfy with probability α_k.

$$A^k x^k \leq b^k \tag{6}$$

We consider the case where we have partial information on the distribution of A^k. Therefore, using the distributionally robust joint chance constraint formulation the feasible strategy set of player k is given by

$$S_{\alpha_k}^k = \{x^k \in X^k \mid \inf_{F^k \in \mathcal{D}_k} \mathbb{P}\{A^k x^k \leq b^k\} \geq \alpha_k\}.$$

3.5 Modelled as a Joint Chance Constraint game

We consider the case when the electricity losses at all the generation nodes are independent, i.e., the row vectors $(A_i^k)_{i=1}^{|Ni|}$ are independent and we have the information of their mean vector μ_i^k and co-variance matrix Σ_i^k. The reformulation of $S_{\alpha_k}^k$ in this case is given by

$$\widetilde{S}_{\alpha_k}^k = \Big\{(y^k, z^k) \in Y^k \times \mathbb{R}^{N_k} \mid (y^k, z^k) \in \widetilde{Q}^k(\alpha_k)\Big\},$$

where $Y^k = \big\{y^k = (y_{ij}^k)_{i \in N_i, j \in M} \mid y_{ij}^k \in [\ln \varepsilon_{ij}^k, \ln C_{ij}^k]\big\}$, $k \in I$, and

$$\widetilde{Q}_{\alpha_k}^k = \begin{cases} (\mu_i^k)^T e^{y^k} + \left\| \left(\Sigma_k^k\right)^{1/2} e^{\log F^{-1}\left(\alpha_k^{z_i^k}\right) 1_{|M|} + y^k} \right\| \leq b_i^k, & \\ \qquad\qquad \forall\ i = 1, 2, \ldots, |N_k|, & (i) \\ \sum_{i=1}^{|N_k|} z_i^k = 1 & (ii) \\ z_i^k \geq 0,\ \forall\ i = 1, 2, \ldots, |N_i| & (iii) \end{cases} \tag{7}$$

Let us define a transformed payoff function $C_i(y^i, y^{-i}) = -u_i(e^{y^i}, e^{y^{-i}})$. Consider an optimization problem for firm i for fixed y^{-i}

$$\begin{aligned} [P_i] \quad & \min_{(y^i, z^i)} C_i(y^i, y^{-i}) \\ \text{s.t.} \quad & (y^i, z^i) \in \widetilde{S}_{\alpha_i}^i \end{aligned}$$

The set of optimal solution of $[P_i]$ is given by

$$BR_i(y^{-i}) = \{(\bar{y}^i, \bar{z}^i) \mid C_i(\bar{y}^i, y^{-i}) \leq C_i(y^i, y^{-i})\ \forall (y^i, z^i) \in \widetilde{S}_{\alpha_i}^i\}.$$

It is clear that if $(y^{i*}, z^{i*}) \in BR_i(y^{-i*})$ for all $i \in I$, $x^* = e^{y^*}$ is a Nash equilibrium of the game.

3.6 Existence of a Nash Equilibrium

As we have modelled this game as an extension of a general Joint Chance Constrained game, we only need to prove that the 5 assumptions mentioned in section 2.2 are satisfied.
Assumptions 3 and 4 can be shown to be true trivially [1].
We will restrict this game to having μ^k and Σ^k having positive components, which will satisfy Assumption 2.
Assumption 5 will similarly hold as we constrain ourselves to $\alpha_k \geq 0.84$ for all $k \in I$.

Assumption 1 : The transformed utility function $C_i(., y^{-i})$ is a convex function of y^i for all $i \in I$.
Proof in Appendix

4 Simulation

We have run a suite of simulations to experimentally find the Nash Equilibrium in randomly generated scenarios of a two-firm Electricity Market.

Complete simulation data is available in the Github repository. Refer to Appendix

4.1 Iterative Scheme to find Nash Equilibrium

As described in the Midterm report, we worked to solve the non-linear convex optimization problem $[P_i]$ using Karush–Kuhn–Tucker conditions, and find conditions under which solving for the optimal strategy profile offline would be possible. Since the feasible set is convex and the utility function chosen is also convex, the KKT conditions become sufficient (rather than just a necessary condition in the general case).

However, since the resulting KKT equations turn out to be intractable, we use an Iterative algorithm instead. For computational purpose, we consider the case of two firms and use the best response algorithm as outlined below:

Algorithm 1. **Step-1** *Select initial feasible point* $(y^{2(0)}, z^{2(0)}) \in \widetilde{S}^i_{\alpha_i}$ *for player 2. Set* $k := 0$.

Step-2 *Solve convex optimization problem* $[P_1]$ *and find a point* $(y^{1(k)}, z^{1(k)}) \in BR_1(y^{2(k)})$.

Step-3 *If* $(y^{2(k)}, z^{2(k)}) \in BR_1(y^{1(k)})$*, then set* $(x^{1*}, x^{2*}) = (e^{y^{1(k)}}, e^{y^{2(k)}})$ *and stop. Otherwise, solve convex optimization problem* $[P_2]$ *and find a point* $(y^{2(k)}, z^{2(k)}) \in BR_1(y^{1(k)})$*, set* $k = k + 1$ *and go to step 2.*

If the Algorithm 1 stops, (x^{1*}, x^{2*}) is a Nash equilibrium of the game. This follows from the fact that we have a convex optimization problem in finding the Best Response, and hence cannot have any local maxima. The proof that Algorithm 1 never cycles is still an open problem.

4.2 Simulation 1

For this simulation we have taken 4 Generation nodes and 3 Distribution nodes. For suitable set of values for different variables, nash equilibrium is obtained.

To show it will always provide a solution we simulate the same setup for 20 different instances where we change Chance Matrix (A) each time. We can also randomize all other variables but for simplicity we have ignored that.

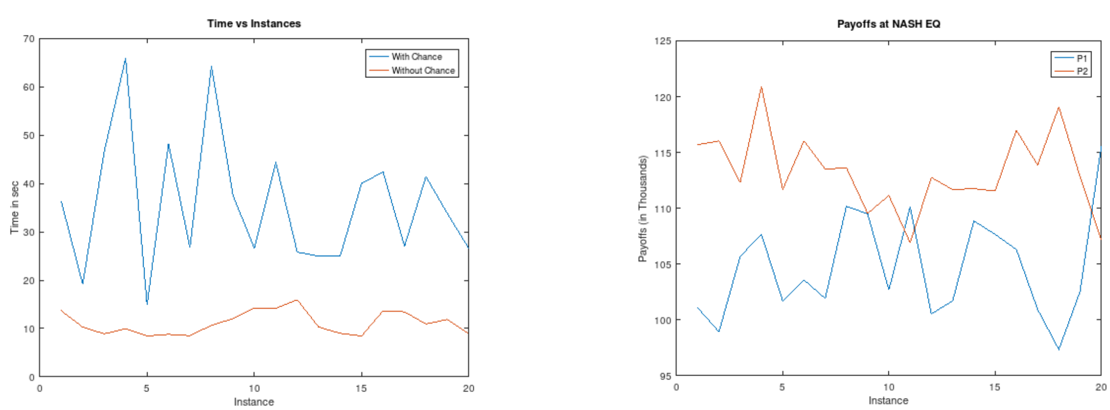

Figure 1

From above figure we can infer that time varies significantly for chance constrained problem in comparison to non-chance constrained problem.

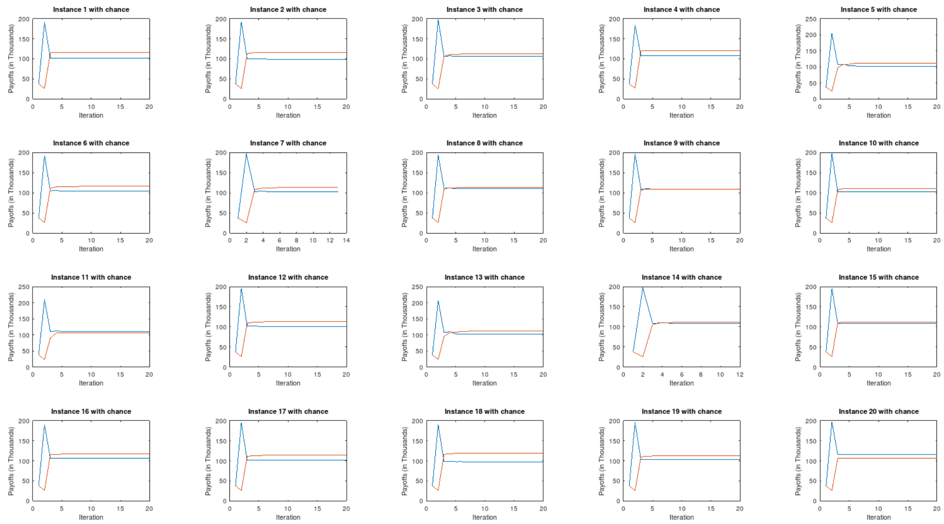

Figure 2: Convergence with Chance Constraints

4.3 Simulation 2

The convergence also depends on the initial point. For this simulation we have considered a specific instance of the above simulation and iterated over different initial points (25). We can further analyse differences in Nash Equilibrium if initial point changes.

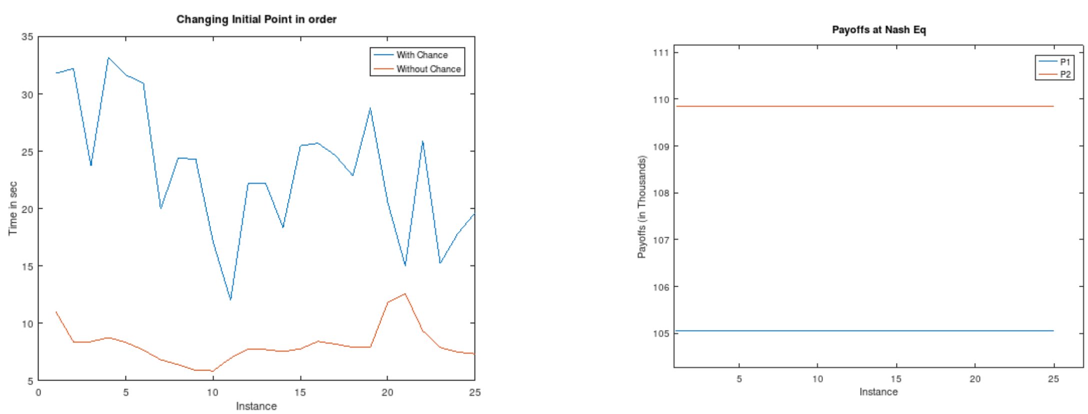

Figure 3

From the above figures we can infer that we achieve same Nash Equilibrium for each case. Time varies and some starting points are time effective for both chance and non-chance constrained problem. Also for all cases we attain Nash Equilibrium, which ensures to provide solution irrespective of the initial point.

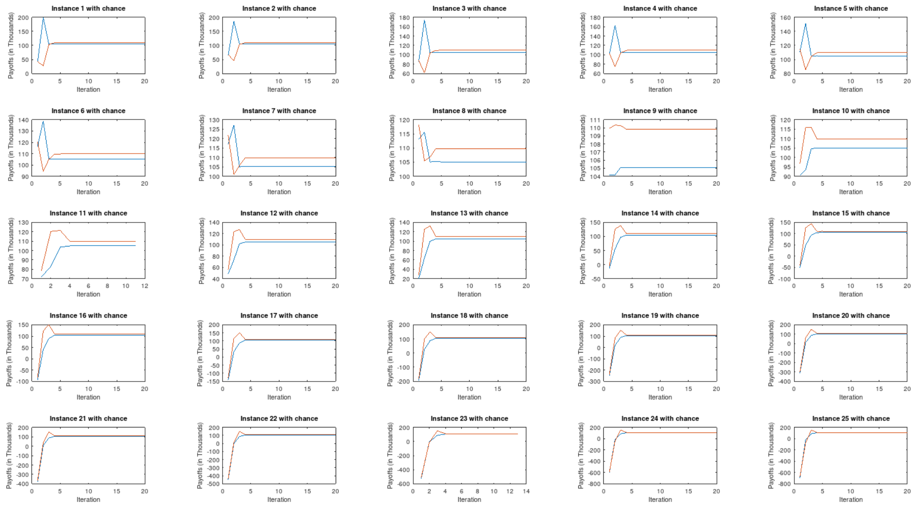

Figure 4: Convergence with Chance Constraints with different initial points

.

Appendix: Proof of Convexity, Section 3.6

A twice differentiable function of several variables is convex on a convex set iff its Hessian matrix of partial second-derivatives is Positive Semi-definite on the interior of the convex set.

$$C_k(y^k, y^{-k}) = -\sum_{i\in N}\sum_{j\in M}(\beta_{ij} - \delta_{ij}\sum_{k\in I} e^{y^k_{ij}} - c^k_{ij}) \times e^{y^k_{ij}} \tag{8a}$$

$$\frac{\partial C_k(y^k, y^{-k})}{\partial y^k_{ij}} = -[(\beta_{ij} - \delta_{ij}\sum_{k\in I} e^{y^k_{ij}} - c^k_{ij}) \times e^{y^k_{ij}}] + [e^{y^k_{ij}} \times \delta_{ij}] \tag{8b}$$

$$\frac{\partial^2 C_k(y^k, y^{-k})}{(\partial y^k_{ij})^2} = -[(\beta_{ij} - \delta_{ij}\sum_{k\in I} e^{y^k_{ij}} - c^k_{ij}) \times e^{y^k_{ij}}] + [3 \times e^{y^k_{ij}} \times \delta_{ij}] \tag{8c}$$

$$\frac{\partial^2 C_k(y^k, y^{-k})}{\partial y^k_{i_1 j_1}\partial y^k_{i_2 j_2}} = 0 \tag{8d}$$

So the Hessian Matrix is a diagonal matrix, which is positive semidefinite iff each diagonal element is ≥ 0. Hence we get the requirement that

$$3e^{y^k_{ij}} + \sum_{t\in I} e^{y^t_{ij}} \geq \beta_{ij} - c^k_{ij}$$

which can be satisfied by choosing an appropriate value of ε^k_{ij}, a lower bound on $e^{y^k_{ij}}$.

Appendix: Github Repository github.com/naling98/Joint-Chance-Constraints

References

[1] Shen Peng, V.V. Singh, Abdel Lisser, General Sum Games with Joint Chance Constraints