Joint Chance Constrained Games

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Abstract

We consider an n-player game where each player's strategy set contains some stochastic linear constraints. The existence of a Nash Equilibrium under certain conditions has been proved earlier. We further analyse this problem via a constructed example of a Cournot Competition among electricity firms over a Network. We run simulations of an Iterative scheme to find the Nash Equilibrium in this context and describe the results.

1 Introduction

The games with continuous strategy sets and are deterministic in nature have been extensively studied in the literature. However, certain practical game theoretic situations are best modeled using random variables (to account for uncertain parameters) that restrict the strategy space. One way to model these is using chance constraints, in particular, Stochastic linear constraints. Such games are called Stochastic Nash games.

John Nash proved the existence the existence of a mixed strategy Nash equilibrium for a n-player finite strategy game [2][3] . The proof required the condition of convexity on the strategy spaces of the players. This condition is violated, in general, by the Stochastic linear constraints. Hence, the existence of a Nash Equilibrium is not guaranteed in general for these stochastic games. Recently, [[7], [8], [9], [10], [11]] considered a finite strategic game with random payoffs. They introduced chance-constrained games by defining the players' payoff function using a chance constraint.

We formulate the stochastic linear constraints of each player as a joint chance constraint. We assume that the row vectors of the matrix defining stochastic linear constraints are independent and each row vector follows a multivariate normal distribution. Under certain conditions, we propose a new convex reformulation for the joint chance constraints in this case. There always exists a Nash equilibrium of such a chance constrained game if the payoff function of each player satisfies certain assumptions [1].

We construct an experimental setup of a Cournot Competition among Electricity Firms, which would emulate the Stochastic Linear Constraints that we are interested in, and use this as a test bed for an Iterative Algorithm to find the Nash Equilibrium, and analyse it's properties.

2 Model

We consider a n-player non-cooperative game. Let $I = [1, 2..., n]$ be the set of players, where i denotes a generic element of the set I. The payoffs of player i are defined by $u_i : \mathbb{R}^{m_1}_{++} \times \mathbb{R}^{m_2}_{++} \times ... \times \mathbb{R}^{m_n}_{++} \to \mathbb{R}$ where $\mathbb{R}^{m_i}_{++}$ denotes the positive orthant of $\mathbb{R}^{m_1}$. The set $X^i \subset \mathbb{R}^{m_i}_{++}$ denotes the set of all strategies of player i. We assume this set to be convex and compact.

We consider the case where the strategies of player i are further constrained by $A^i x^i \leq b^i$ where A^i is a $K_i \times m_i$ random matrix, and $b^i \in \mathbb{R}^{K_i}$and the K_i constraints are jointly satisfied with probability α_i. Hence the strategy set of player i can be written as

$$S^i_{\alpha_i} = \{x \in X^i | P(A^i x \leq b^i) \geq \alpha_i\}, i \in I \tag{1}$$

Further, we assume that the rows of the random matrices A^i are independent and each row vector A^i_k follows a multivariate normal distribution with mean vector μ^i_k and co-variance matrix Σ^i_k i.e. $A^i_k \backsim N(\mu^i_k, \Sigma^i_k)$. We assume Σ^i_k to be a Positive Definite matrix.

2.1 Deterministic Reformulation

$S^i_{\alpha_i}$ is a probabilistic formulation of the space, and the joint satisfaction of K_i constraints is not easy to handle directly. If $X \in S^i_{\alpha_i}$, then for each $k \in [1, 2...K_i]$ there must exist a z_k such that the constraint corresponding to kth row is satisfied with probability $\geq \alpha_i^{z_k}$ and z_k sum to 1. $F^{-1}(\cdot)$ is a Quantile function for one-dimensional standard normal distribution.

$$Q^i_{\alpha_i} = \{(x, z) \in X^i \times \mathbb{R}^{K_i} | \tag{2a}$$

$$(\mu^i_k)^T x + F^{-1}(\alpha_i^{z_k})||(\Sigma^i_k)^{1/2} x|| \leq b^i_k \ \forall k \in [1, 2...K_i]\} \tag{2b}$$

$$\sum_{k=1}^{K_i} z_k = 1 \tag{2c}$$

$$z_k \geq 0 \ \forall k \in [1, 2...K_i] \tag{2d}$$

For each x in $S^i_{\alpha_i}$, there must exist a pair (x, z) in $Q^i_{\alpha_i}$. Conversely, if (x, z) is in $Q^i_{\alpha_i}$, then $x \in S^i_{\alpha_i}$.

Additionally, constraints (2c) and (2d) are always convex. Constraint (2b) can be written as $f(x, z) \leq h$ and if f is proved a convex function, then set $Q^i_{\alpha_i}$ would be proved a convex set. This can be done with an additional logarithmic transformation of x in $Q^i_{\alpha_i}$ to create $\tilde{Q}^i_{\alpha_i}$ when $\alpha_i \geq 0.5$.

$$\tilde{Q}^i_{\alpha_i} = \{(y, z) \in log X^i \times \mathbb{R}^{K_i} | \tag{3a}$$

$$(u^i_k)^T e^y + ||(\Sigma^i_k)^{1/2} e^{\log F^{-1}(\alpha_i^{z_k}) 1_m + y}|| \leq b^i_k \ \forall k \in [1, 2...K_i]\} \tag{3b}$$

$$\sum_{k=1}^{K_i} z_k = 1 \tag{3c}$$

$$z_k \geq 0 \ \forall k \in [1, 2...K_i] \tag{3d}$$

Lemma 1. *The reformulated strategy space set $\tilde{Q}^i_{\alpha_i}$ can be proved to be a convex set (for $\alpha \geq 0.84$ for all players) [1].*

2.2 Existence of a Nash Equilibrium

The logarithmic transformation from $Q^i_{\alpha_i}$ to $\tilde{Q}^i_{\alpha_i}$ also changes the utility function.
For each $i \in I$, define the transformed function $C_i = -u_i \circ d_i$ where $d_i : \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times ... \times \mathbb{R}^{m_n} \to \mathbb{R}^{m_1}_{++} \times \mathbb{R}^{m_2}_{++} \times ... \times \mathbb{R}^{m_n}_{++}$ such that $d_i(y^1, y^2, ...y^n) = (e^{y^1}, e^{y^2} ..., e^{y^n})$.

Theorem. *If we consider an n-player non-cooperative game where :*

Assumption 1. The set X_i is convex and compact $\forall\ i \in I$

Assumption 2. The set $Y_i = \log X_i$ is convex (compactness can be derived) $\forall\ i \in I$

Assumption 3. The players face stochastic constraints of the form $P(A^i x \leq b^i) \geq \alpha_i\ \forall i \in I$.

Assumption 4. A^i is a random matrix and the mean μ^i_k and covariance matrix Σ^i_k of the rows A^i_k are Independent and have all components non-negative $\forall\ i \in I\ \forall\ k \in [1, 2...K_i]$

Assumption 5. The probability threshold $\alpha_i \in [F(1), 1]\ \forall\ i \in I\ (F(1) \approx 0.84)$

Assumption 6. The transformed utility function $C_i(., y^{-i})$ is a convex function of $y^i\ \forall\ i \in I$

Then, there exists a Nash Equilibrium of the Chance Constrained Game.

Proof. .

Define the Best Response set for player i for fixed $y^i \in Y^{-i}$

$$BR_i(y^{-i}) = \{(\bar{y}^i, \bar{z}^i) \mid C_i(\bar{y}^i, y^{-i}) \leq C_i(y^i, y^{-i})\ \forall (y^i, z^i) \in \widetilde{Q}^i_{\alpha_i}\}.$$

Denote $\widetilde{S}(\alpha) = \Pi_{i \in I} \widetilde{Q}^i_{\alpha_i}$.
Define $\widetilde{G}^\alpha : \widetilde{S}(\alpha) \to \mathcal{P}(\widetilde{S}(\alpha))$ such that $\widetilde{G}^\alpha(y, z) = \Pi_{i \in I} BR_i(y^{-i})$

Assumption 6 says that the Composition function $C_i(., y^{-i})$ is a convex function of y^i, which implies that the Best Response set $BR_i(y^{-i})$ is a convex set. Hence, $\widetilde{G}^\alpha(y, z)$ is a non-empty and convex function for each (y, z). The closed graph condition for $\widetilde{G}^\alpha(.)$ follows from the continuity of the functions $C_i, i \in I$.
Therefore, from Kakutani's Fixed Point theorem [Appendix 3], there exists a fixed point (y^*, z^*) for the set-valued map $\widetilde{G}^\alpha(.)$.

Then for each $i \in I$ $C_i(y^{i*}, y^{-i*}) \leq C_i(y^i, y^{-i})$ $\forall (y^i, z^i) \in \widetilde{Q}^i_{\alpha_i}$.

Since $\widetilde{S}^i_{\alpha_i}$ is a reformulation of $S^i_{\alpha_i}$, we get $x^* = e^{y^*}$ such that $u_i(x^{i*}, x^{-i*}) \geq u_i(x^i, x^{-i*})$ $\forall x^i \in S^i_{\alpha_i}$.

Hence x^* is a Nash Equilibrium for the Chance Constrained Game under the given Assumptions. □

3 Cournot Competition among Electricity Firms

3.1 Description

Consider an Electricity market where firms compete over an Electricity Network comprised of a set of nodes. There are several generation nodes where firms have setup their generation facilities to produce electricity. There are also several distribution nodes from where the electricity is provided to the consumers. The firms generate the electricity at their facilities and transmit it to the distribution nodes. The transmission over long distances creates power losses, which are best modelled as random variables. The components of the Electricity market are described as follows.

I - the set of firms called as players

N - the set of generation nodes

N_k - the subset of generation nodes where firm k has installed its generation facilities

I_i - the set of firms who have generation nodes at node i

M - the set of distribution nodes

3.2 Pricing

Let x^k_{ij} be the quantity transmitted from generation node i to distribution node j by firm k. The price for this depends on the total electricity transmitted from node i to node j, i.e. the firms have a Cournot competition for the electricity transmitted between two nodes.

$$p_{ij}(x_{ij}) = \beta_{ij} - \delta_{ij} \sum_{k \in I_i} x^k_{ij}, \; i \in N, \; j \in M \tag{4}$$

where $\delta_{ij} \geq 0$ for all $i \in N, j \in M$.

3.3 Utility

Each firm encounters a cost in the generation and transmission of electricity. Let $c_{ij}^k(x_{ij}^k)$ be the cost for firm k to generate and transmit x_{ij}^k units of electricity. We assume a linear cost function for each firm, i.e., $c_{ij}^k(x_{ij}^k) = c_{ij}^k x_{ij}^k$ for all $k \in I$, $i \in N$, $j \in M$.

The payoff function for firm k is then given by

$$u_k(x^k, x^{-k}) = \sum_{i \in N} \sum_{j \in M} (\beta_{ij} - \delta_{ij} \sum_{k \in I_i} x_{ij}^k - c_{ij}^k) \times x_{ij}^k \tag{5}$$

3.4 Transmission Constraints

The strategy set of firm k is defined as $X^k = \{x^k = (x_{ij}^k)_{i \in N_k, j \in M} \mid x_{ij}^k \in [\varepsilon_{ij}^k, \gamma_{ij}^k]\}$, where ε_{ij}^k and γ_{ij}^k denote the minimum and maximum output of firm k from node i to node j.

Let A^k be a $|N_k| \times |M|$ random matrix where a_{ij}^k per unit electricity is lost during transmission from generation node i to distribution node j. Each firm wants to keep it's loss corresponding to the generation facilities under a certain threshold. Let b^k denote the threshold vector for player k. Then firm k faces the following stochastic constraint, which it must jointly satisfy with probability α_k.

$$A^k x^k \leq b^k \tag{6}$$

We consider the case where we have partial information on the distribution of A^k. Therefore, using the distributionally robust joint chance constraint formulation the feasible strategy set of player k is given by

$$S_{\alpha_k}^k = \{x^k \in X^k \mid \inf_{F^k \in \mathcal{D}_k} \mathbb{P}\{A^k x^k \leq b^k\} \geq \alpha_k\}.$$

3.5 Modelled as a Joint Chance Constraint game

We consider the case when the electricity losses at all the generation nodes are independent, i.e., the row vectors $(A_i^k)_{i=1}^{|Ni|}$ are independent and we have the information of their mean vector μ_i^k and co-variance matrix Σ_i^k. The reformulation of $S_{\alpha_k}^k$ in this case is given by

$$\widetilde{S}_{\alpha_k}^k = \Big\{(y^k, z^k) \in Y^k \times \mathbb{R}^{N_k} \mid (y^k, z^k) \in \widetilde{Q}^k(\alpha_k)\Big\},$$

where $Y^k = \Big\{y^k = (y_{ij}^k)_{i \in N_i, j \in M} \mid y_{ij}^k \in [\ln \varepsilon_{ij}^k, \ln \gamma_{ij}^k]\Big\}$, $k \in I$, and

$$\widetilde{Q}_{\alpha_k}^k = \begin{cases} (\mu_i^k)^T e^{y^k} + \left\| \left(\Sigma_i^k\right)^{1/2} e^{\log F^{-1}\left(\alpha_k^{z_i^k}\right) 1_{|M|} + y^k} \right\| \leq b_i^k, & \\ \qquad \forall\ i = 1, 2, \ldots, |N_k|, & (i) \\ \sum_{i=1}^{|N_k|} z_i^k = 1 & (ii) \\ z_i^k \geq 0,\ \forall\ i = 1, 2, \ldots, |N_k| & (iii) \end{cases} \tag{7}$$

Let us define a transformed payoff function $C_k(y^k, y^{-k}) = -u_k(e^{y^k}, e^{y^{-k}})$. Consider an optimization problem for firm k for fixed y^{-k}

$$\begin{aligned} [P_k] \quad & \min_{(y^k, z^k)} C_k(y^k, y^{-k}) \\ \text{s.t.} \quad & (y^k, z^k) \in \widetilde{S}_{\alpha_k}^k \end{aligned}$$

The set of optimal solution of $[P_k]$ is given by

$$BR_k(y^{-k}) = \{(\bar{y}^k, \bar{z}^k) \mid C_k(\bar{y}^k, y^{-k}) \leq C_k(y^k, y^{-k})\ \forall (y^k, z^k) \in \widetilde{S}_{\alpha_k}^k\}.$$

It is clear that if $(y^{k*}, z^{k*}) \in BR_k(y^{-k*})$ for all $k \in I$, $x^* = e^{y^*}$ is a Nash equilibrium of the game.

3.6 Existence of a Nash Equilibrium

As we have modelled this game as an extension of a general Joint Chance Constrained game, we only need to prove that the 6 assumptions mentioned in section 2.2 are satisfied.

Assumptions 1, 2 and 3 can be shown to be true trivially [1].

We will restrict this game to having μ_i^k and Σ_i^k having positive components $\forall\ k \in I, \forall\ i \in [1, 2...|N_k|]$, which will satisfy Assumption 4.

Assumption 5 will similarly hold as we constrain ourselves to $\alpha_k \geq 0.84$ for all $k \in I$.

Assumption 6 : The transformed utility function $C_k(., y^{-k})$ is a convex function of y^k for all $k \in I$.

Proof : A twice differentiable function of several variables is convex on a convex set iff its Hessian matrix of partial second-derivatives is Positive Semi-definite on the interior of the convex set.

$$C_k(y^k, y^{-k}) = -\sum_{i\in N}\sum_{j\in M}(\beta_{ij} - \delta_{ij}\sum_{t\in I_i} e^{y^t_{ij}} - c^k_{ij}) \times e^{y^k_{ij}} \tag{8a}$$

$$\frac{\partial C_k(y^k, y^{-k})}{\partial y^k_{ij}} = -[(\beta_{ij} - \delta_{ij}\sum_{t\in I_i} e^{y^t_{ij}} - c^k_{ij}) \times e^{y^k_{ij}}] + [e^{2y^k_{ij}} \times \delta_{ij}] \tag{8b}$$

$$\frac{\partial^2 C_k(y^k, y^{-k})}{(\partial y^k_{ij})^2} = -[(\beta_{ij} - \delta_{ij}\sum_{t\in I_i} e^{y^t_{ij}} - c^k_{ij}) \times e^{y^k_{ij}}] + [3 \times e^{2y^k_{ij}} \times \delta_{ij}] \tag{8c}$$

$$\frac{\partial^2 C_k(y^k, y^{-k})}{\partial y^k_{i_1j_1}\partial y^k_{i_2j_2}} = 0 \;\text{ if } (i_1, j_1) \neq (i_2, j_2) \tag{8d}$$

So the Hessian Matrix is a diagonal matrix, which is positive semi-definite iff each diagonal element is ≥ 0. Hence we get the requirement that

$$3e^{y^k_{ij}} + \sum_{t\in I_i} e^{y^t_{ij}} \geq \frac{\beta_{ij} - c^k_{ij}}{\delta_{ij}} \tag{9}$$

which can be satisfied by choosing an appropriate value of ε^k_{ij}, a lower bound on $e^{y^k_{ij}}$ $\forall$ $k \in I$, $\forall$ $i \in [1, 2...|N_k|]$, $\forall$ $j \in [1, 2...M]$

4 Simulation

We run a suite of simulations to experimentally find the Nash Equilibrium in randomly generated scenarios of a two-firm Electricity Market.

Complete simulation data is available in the Github repository. Refer to Appendix

4.1 Iterative Scheme to find Nash Equilibrium

As described in the Midterm report, we worked to solve the non-linear convex optimization problem $[P_i]$ using Karush–Kuhn–Tucker conditions, and find conditions under which solving for the optimal strategy profile offline would be possible. Since the feasible set is convex and the utility function chosen is also convex, the KKT conditions become sufficient (rather than just a necessary condition in the general case).[4]

However, since the resulting KKT equations turn out to be intractable, we use an Iterative algorithm instead. For computational purpose, we consider the case of two firms and use the best response algorithm as outlined below:

Algorithm 1. **Step-1** *Select initial feasible point* $(y^{2(0)}, z^{2(0)}) \in \widetilde{S}^i_{\alpha_i}$ *for player 2. Set* $k := 0$.

Step-2 *Solve convex optimization problem* $[P_1]$ *and find a point* $(y^{1(k)}, z^{1(k)}) \in BR_1(y^{2(k)})$.

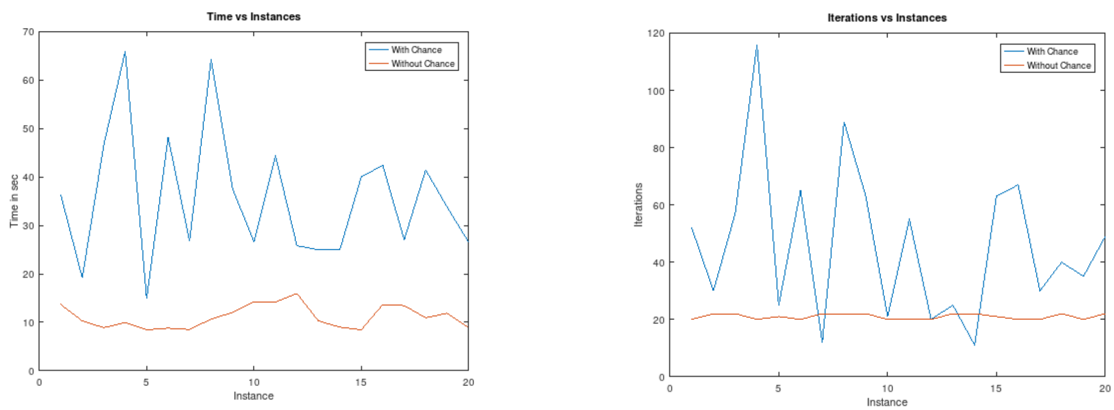
Step-3 *If* $(y^{2(k)}, z^{2(k)}) \in BR_1(y^{1(k)})$, *then set* $(x^{1*}, x^{2*}) = (e^{y^{1(k)}}, e^{y^{2(k)}})$ *and stop. Otherwise, solve convex optimization problem* $[P_2]$ *and find a point* $(y^{2(k)}, z^{2(k)}) \in BR_1(y^{1(k)})$, *set* $k = k+1$ *and go to step 2.*

If the Algorithm 1 stops, (x^{1*}, x^{2*}) is a Nash equilibrium of the game. This follows from the fact that we have a convex optimization problem in finding the Best Response, and hence cannot have any local maxima. The proof that Algorithm 1 never cycles is still an open problem.

4.2 Simulation 1

For this simulation we have taken 4 Generation nodes (All shared between both players) and 3 Distribution nodes. For suitable set of values for different parameters, nash equilibrium is obtained.

To show it will always provide a solution we simulate the same setup for 20 different instances where we change Chance Matrix (A) and Threshold Vector (b) each time. We can also randomize all other parameters but for simplicity we have ignored that.



(a) **(b)**

Figure 1: (a) Time vs Instances (b) Total Iterations vs Instances

Time and Iteration Analysis (Chance Constrained Problem)

Mean Time - 35.8819 sec
Standard Deviation in Time : 13.3053 sec
Mean Total Iterations - 46.25
Standard Deviation in Total Iterations : 26.08

Time and Iteration Analysis (Non-Chance Constrained Problem)

Mean Time - 11.098 sec
Standard Deviation in Time : 2.3036 sec
Mean Total Iterations - 21
Standard Deviation in Total Iterations : 0.94

From above figures we can infer that time varies significantly for chance constrained problem in comparison to non-chance constrained problem. Total Iterations gives a clear picture of the behaviour of Algorithm 1 for both problems. *Step-2 to Step-3 and vice versa refers to 1 iteration*

Considering Instance 1 (*refer to Appendix 1 for data corresponding to all instances*) with different parameter values, we obtain the following Nash Equilibrium. Rows correspond to 4 generation nodes and columns correspond to 3 distribution nodes.

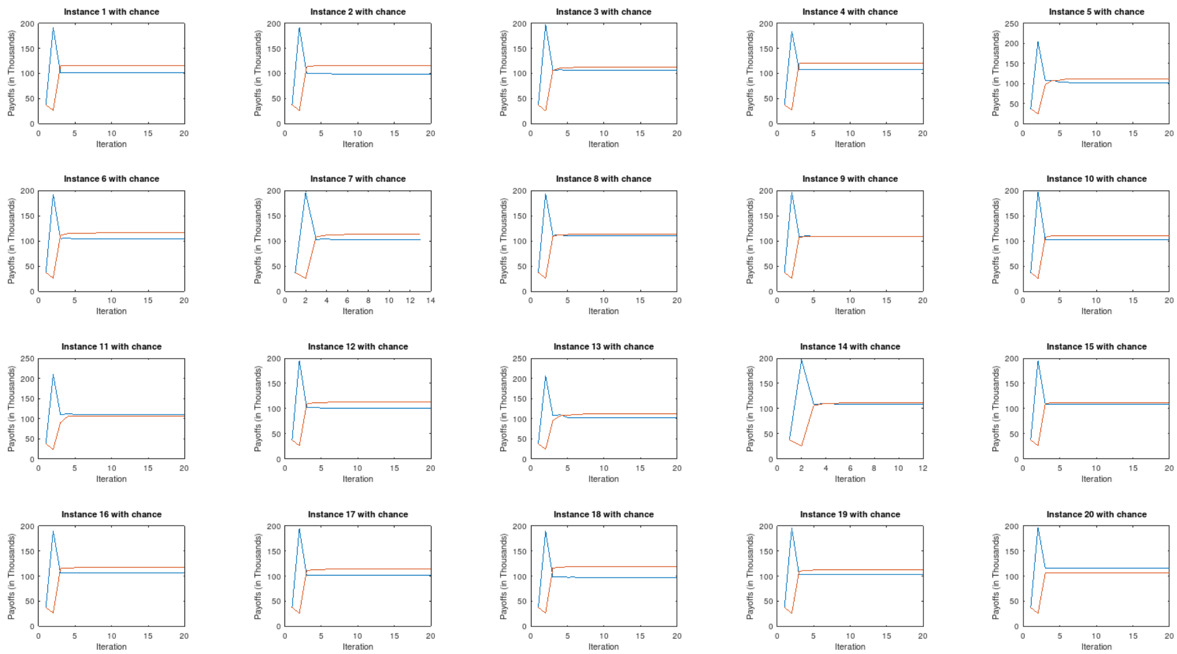


Figure 2: Convergence with Chance Constraints

86.569	85.530	87.164
89.221	89.188	90.011
89.105	90.052	89.459
88.134	89.489	88.974

(a) x^1

99.098	99.632	98.783
96.671	96.178	95.766
95.765	94.636	95.334
94.988	93.960	95.150

(b) x^2

Table 1: Nash Equilibrium

300	300	300
300	300	300
300	300	300
300	300	300

(a) β

1	1	1
1	1	1
1	1	1
1	1	1

(b) δ

20	20	20
20	20	20
20	20	20
20	20	20

(c) c^1 - Marginal Cost P1

15	15	15
15	15	15
15	15	15
15	15	15

(d) c^2 - Marginal Cost P2

Table 2

0.28	0.23	0.22
0.24	0.20	0.22
0.21	0.24	0.23
0.23	0.24	0.29

(a) μ^1

0.21	0.26	0.21
0.25	0.24	0.26
0.29	0.22	0.24
0.21	0.25	0.23

(b) μ^2

167.73
152.82
152.04
165.91

(c) b^1

156.56
185.71
173.38
186.66

(d) b^2

Table 3: Mean Matrices and Threshold Vectors

0.036	0.032	0.042
0.032	0.059	0.063
0.042	0.063	0.069

(a) Σ_1^1

0.032	0.031	0.013
0.031	0.051	0.016
0.013	0.016	0.009

(b) Σ_2^1

0.025	0.020	0.022
0.020	0.025	0.028
0.022	0.028	0.049

(c) Σ_3^1

0.069	0.039	0.043
0.039	0.038	0.025
0.043	0.025	0.033

(d) Σ_4^1

Table 4: Co-variance Matrices P1

0.002	0.003	0.003
0.003	0.009	0.009
0.003	0.009	0.018

(a) Σ_1^2

0.029	0.033	0.011
0.033	0.067	0.034
0.011	0.034	0.027

(b) Σ_2^2

0.042	0.031	0.029
0.031	0.045	0.041
0.029	0.041	0.038

(c) Σ_3^2

0.052	0.055	0.066
0.055	0.069	0.086
0.066	0.086	0.111

(d) Σ_4^2

Table 5: Co-variance Matrices P2

0.9	0.9

Table 6: α

56.7	56.7	56.7
56.7	56.7	56.7
56.7	56.7	56.7
56.7	56.7	56.7

(a) ε^1

57.3	57.3	57.3
57.3	57.3	57.3
57.3	57.3	57.3
57.3	57.3	57.3

(b) ε^2

208.66	207.40	208.95
209.12	207.61	201.60
200.93	208.92	207.37
201.66	201.49	203.64

(c) γ^1

204.27	201.49	209.74
203.03	203.87	200.63
200.64	202.71	209.26
201.84	209.19	204.77

(d) γ^2

Table 7: Lower and Upper Production Bounds

4.3 Simulation 2 - Varying the Initial Point

The convergence also depends on the starting point. For this simulation we have considered a specific instance (instance 3) from the above simulation and iterated over different initial points (25). We can further analyse differences in Nash Equilibrium if the starting point changes.

Starting points are chosen by multiplying a constant 'I' to a all-ones matrix of dimension 4*3. We iterate 'I' from 10 to 250 with some randomness over 25 instances.

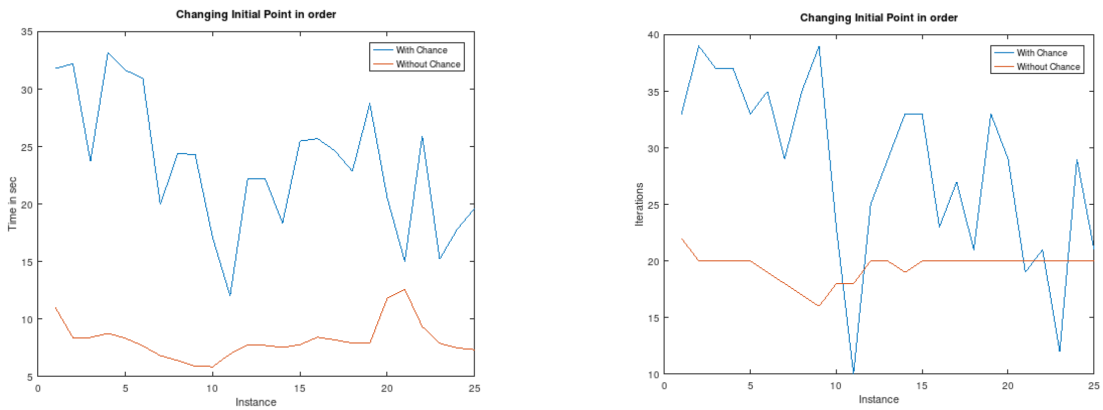


Figure 3: (a) Time vs Instances (b) Total Iterations vs Instances

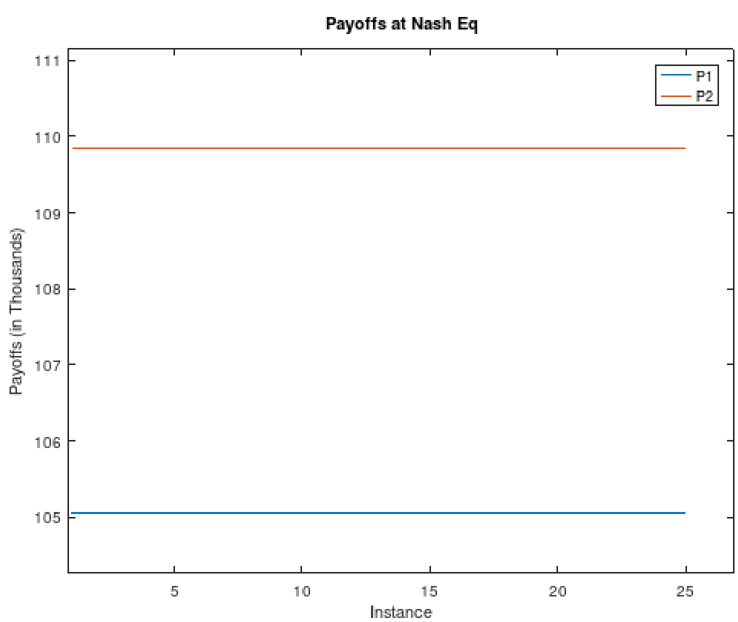


Figure 4: Payoffs at Nash Equilibrium for both Players

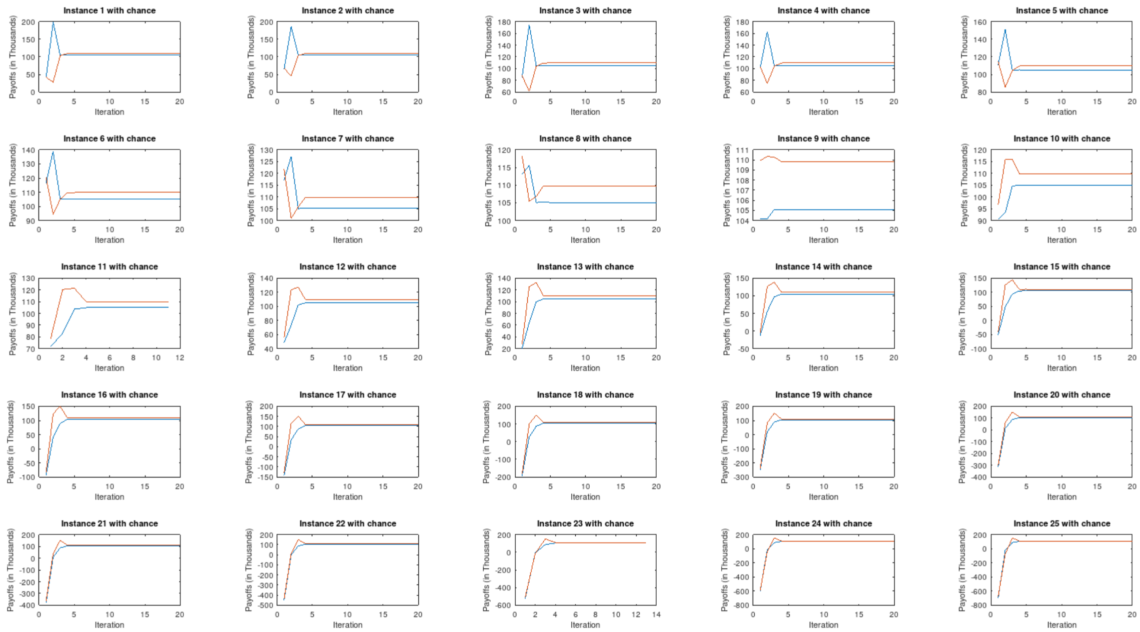


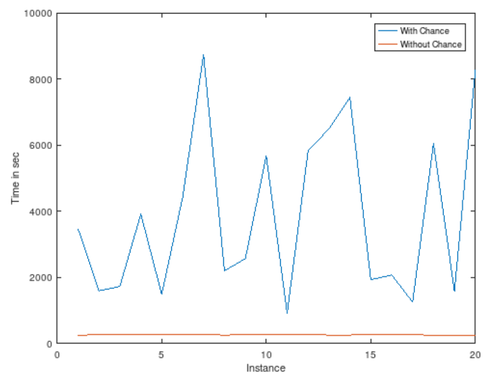
Figure 5: Convergence with Chance Constraints with different initial points

From the above figures we can infer that we achieve same Nash Equilibrium for each case. Some starting points are time effective for both Chance as well as Non-Chance constrained problem. Few starting points even solve the problem in lesser iterations than the corresponding Non-chance problem. Also for all cases we attain Nash Equilibrium, which mostly ensures to provide solution irrespective of the initial point.

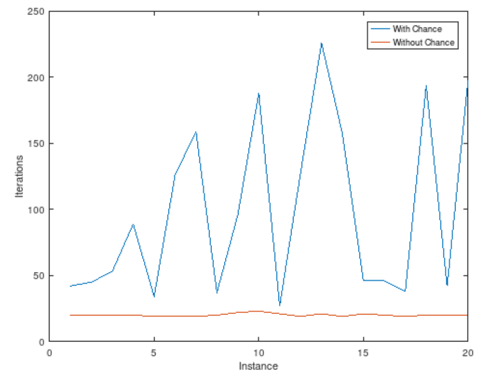
4.4 Simulation 3 - Larger Model

For this simulation we have taken 10 Generation nodes (Solo P1 : 2, Solo P2: 2, Shared: 6) and 12 Distribution nodes, which is a slightly larger model as compared to previous model.

To show it will always provide a solution we simulate the same setup for 20 different instances where we change Chance Matrix (A) and Threshold Vector (b) each time. We can also randomize all other parameters but for simplicity we have ignored that.



(a)



(b)

Figure 6: (a) Time vs Instances (b) Total Iterations vs Instances

Time and Iteration Analysis (Chance Constrained Problem)

Mean Time - 3885.6 sec Standard Deviation in Time : 2481.5 sec
Mean Total Iterations - 98.55 Standard Deviation in Total Iterations : 65.43

Time and Iteration Analysis (Non - Chance Constrained Problem)

Mean Time - 262.05 sec Standard Deviation in Time : 8.68
Mean Total Iterations - 20.1 Standard Deviation in Total Iterations : 1.04

We can observe that in compared to Simulation 1, time and iterations taken to solve the Chance Constrained problem are significantly higher. Also the deviation in time and total iteration taken is quite high.

Note : We cannot directly compare time between Simulation 1 and 3 as parameters have changed which can result in completely different strategy sets.

Appendix 1: Github Repository github.com/naling98/Joint-Chance-Constraints

Appendix 2: Convex Optimization Solver

We use the SQP (Sequential Quadratic Programming) solver to solve the Non-Linear Optimization problem in each Iteration.

The SQP implementation consists of three main stages, which are discussed briefly in the following subsections:

- **Updating the Hessian Matrix** : At each major iteration a positive definite quasi-Newton approximation of the Hessian of the Lagrangian function, H, is calculated using the BFGS (Broyden–Fletcher–Goldfarb–Shanno) method.

- **Quadratic Programming Solution** : At each major iteration of the SQP method, a QP problem of the following form is solved. The solution procedure involves two phases. The first phase involves the calculation of a feasible point (if one exists). The second phase involves the generation of an iterative sequence of feasible points that converge to the solution.

- **Initialization** : The algorithm requires a feasible point to start. If the current point from the SQP method is not feasible, then you can find a point by solving the linear programming problem. *For more details refer to Sequential Quadratic Programming Solver* [6]

Appendix 3: Kakutani's Fixed Point Theorem

Theorem. *Kakutani's Fixed Point Theorem states:*

Let S be a non-empty compact and convex subset of some Euclidean Space R^n.

Let ϕ: $S \rightarrow 2^S$ be a set-valued function on S with the following properties:

ϕ has a closed graph.

$\phi(x)$ is non-empty and convex for all $x \in S$.

Then ϕ has a fixed point. [5]

References

[1] Shen Peng, V.V. Singh, Abdel Lisser, General Sum Games with Joint Chance Constraints, Oper. Res. Lett. 46(2018) 482-486

[2] J.F. Nash, Equilibrium points in n-person games, Proc. Natl. Acad. Sci. (1950) 36 (1) 48-49

[3] T. Basar, G.J. Olsder, Dynamic Non-Cooperative Game Theory, second ed., SIAM, Philadelphia, PA, 1999

[4] Edwin P. Chong, Stanislaw H. Zak, An Introduction to Optimization Second Edition, Wiley Student Edition, John Wiley and Sons, 2005

[5] Kakutani, Shizuo (1941). ”A generalization of Brouwer’s fixed point theorem”. Duke Mathematical Journal. 8 (3): 457–459. doi:10.1215/S0012-7094-41-00838-4.

[6] Constrained Non-Linear Optimization Algorithms, Mathworks Help Centre, Mathworks, https://www.mathworks.com/help/optim/ug/constrained-nonlinear-optimization-algorithms.html

[7] V.V. Singh, O. Jouini, A. Lisser, Equivalent nonlinear complementarity problem for chance-constrained games, Electron. Notes Discrete Math. 55 (2016)

[8] V.V. Singh, O. Jouini, A. Lisser, Existence of Nash equilibrium for chance constrained games, Oper. Res. Lett. 44 (2016) 640–644.

[9] V.V. Singh, O. Jouini, A. Lisser, Distributionally robust chance-constrained games: Existence and characterization of Nash equilibrium, Optim. Lett. 11 (2017) 1385–1405.

[10] V.V. Singh, O. Jouini, A. Lisser, Solving Chance-Constrained Games using Complementarity Problems, Springer International Publishing, Cham, 2017, pp. 52–67.

[11] V.V. Singh, A. Lisser, A characterization of Nash equilibrium for the games with random payoffs J. Optim. Theory Appl. (2018). http://dx.doi.org/10.1007/ s10957-018-1343-0.