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General sum games with joint chance constraints

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ABSTRACT

We consider an n-player non-cooperative game with continuous strategy sets. The strategy set of each player contains a set of stochastic linear constraints. We model the stochastic linear constraints of each player as a joint chance constraint. We assume that the row vectors of a matrix defining the stochastic constraints of each player are independent and each row vector follows a multivariate normal distribution. Under certain conditions, we show the existence of a Nash equilibrium for this game.

1. Introduction

The notion of the Nash equilibrium was introduced by John Nash in 1950 [20]. He showed that there exists a mixed strategy Nash equilibrium for an n-player finite strategic game. The games with continuous strategy sets have also been extensively studied in the literature. A Nash equilibrium in this case exists under certain conditions on the players' strategy sets and payoff functions [1]. The games considered in [1,20] are deterministic in nature, i.e., the payoff function and strategy set of each player are deterministic. However, there could be some practical game theoretic situations which are better modeled using random variables due to the presence of various uncertain parameters. Such games are called stochastic Nash games.

One way to study stochastic Nash games is using expected payoff criterion [13,22,29]. Ravat and Shanbhag [22] considered the stochastic Nash games where each player is interested in minimizing the expected cost function subject to expected value constraints. They showed the existence of a Nash equilibrium in various cases. A sample average approximation method has been used in [29] to solve a stochastic Nash equilibrium problem. In [13], a variational inequality approach on probabilistic Lebesgue spaces has been considered to study a stochastic Nash equilibrium problem. The stochastic Nash games using stochastic variational inequalities are considered in [14,17,30].

The expected payoff criterion is more suitable for the risk neutral cases. The risk averse situations arising from electricity market using risk measures CVaR and variance are considered in [16,22] and [9], respectively. Recently, Singh et al. [23,24,25–27] considered a finite strategic game with random payoffs. They introduced chance-constrained games by defining the players' payoff function using a chance constraint [4,21]. These games are appropriate for the cases where each player is interested in payoffs which can be obtained with a certain confidence. For the case of normal, Cauchy, and elliptical distributions the existence of a mixed strategy Nash equilibrium for a chance-constrained game is shown in [24], and some computational approaches using mathematical programming techniques are proposed in [23,26,27]. The games with partially known probability distributions using distributionally robust approach are considered in [25]. The chance-constrained games have also received some attention in electricity market [10,19]. There is a scarce literature on zero sum chance-constrained games available [2,3,5,6].

In this paper, we consider an n-player non-cooperative game with continuous strategy sets. We investigate the case where the strategy sets are stochastic in nature. We assume that the strategy set of each player contains a set of stochastic linear constraints. We formulate the stochastic linear constraints of each player as a joint chance constraint. We assume that the row vectors of the matrix defining stochastic linear constraints are independent and each row vector follows a multivariate normal distribution. Under certain conditions, we propose a new convex reformulation for the joint chance constraints in this case. We show that there always exists a Nash equilibrium of such a chance constrained game if the payoff function of each player satisfies certain assumptions. Such game theoretic situations arise in a renewable energy markets based on wind turbine and solar panels with n-players. Each player aims at maximizing his payoff subject to technical, operational and

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budget constraints. Both energy technologies are highly concerned by uncertainties. For the wind turbine case, the constraints are related to tensile strength, tip deflection rate, blade natural frequency, turbulence, and turbine size. For the solar energy, the constraints are related to generation sites, storage and inter-regional power transmission, and the size of the panels. Such constraints could be modeled as chance constraints and considered either as individual chance constraints or joint chance constraints [28]. However, it is well known that joint chance constraints are highly reliable compared to individual ones [7,8]. To the best of our knowledge, joint chance constraints have not been yet considered in the literature for such problems.

The remainder of the paper is organized as follows. Section 2 contains the definition of a non-cooperative game with joint chance constraints. Section 3 presents a new convex reformulation of the joint chance constraints and discusses the existence of a Nash equilibrium.

2. The model

We consider an n-player non-cooperative game. Let $I = \{1, 2, \ldots, n\}$ be the set of players. A generic element of the set I is denoted by i. The payoffs of player i is defined by a function $u_i : \mathbb{R}^{m_1}_{++} \times \mathbb{R}^{m_2}_{++} \times \cdots \times \mathbb{R}^{m_n}_{++} \to \mathbb{R}$, where $\mathbb{R}^{m_i}_{++}$ $\left(\mathbb{R}^{m_i}_{+}\right)$ denotes the positive (non-negative) orthant of $\mathbb{R}^{m_i}$. The set $X^i \subset \mathbb{R}^{m_i}_{++}$ denotes the set of all strategies of player i. We assume X^i to be a convex and compact set. The product set $X = \prod_{i\in I} X^i$ denotes the set of all strategy profiles. We denote the set of all vectors of strategies of all the players except player i by $X^{-i} = \prod_{j=1;j\neq i}^{n} X^j$. The generic elements of X^i, X^{-i}, and X are denoted by x^i, x^{-i}, and x, respectively. We define (y^i, x^{-i}) to be a strategy profile where player i chooses a strategy y^i and each player $j \in I, j \neq i$, chooses a strategy x^j. We consider the case where the strategies of player i are further constrained by the following stochastic linear constraints:

$$A^i x^i \le b^i, \tag{2.1}$$

where $A^i = [A^i_1, A^i_2, \ldots, A^i_{K_i}]^T$ is a $K_i \times m_i$ random matrix, and $b^i \in \mathbb{R}^{K_i}$; T denotes the transposition. For each $k = 1, 2, \ldots, K_i$, A^i_k is the kth row of A^i. We consider the case where the constraints of player i given by (2.1) are jointly satisfied with at least a given probability level. Let α_i be a given probability level of player i. We formulate the stochastic linear constraints (2.1) as a joint chance constraint given by

$$\mathbb{P}\left\{A^i x^i \le b^i\right\} \ge \alpha_i, \tag{2.2}$$

where $\mathbb{P}$ is a probability measure. Therefore, for an $\alpha_i \in [0, 1]$, the feasible strategy set of player i is defined by

$$S^i_{\alpha_i} = \left\{x^i \in X^i \mid \mathbb{P}\left\{A^i x^i \le b^i\right\} \ge \alpha_i\right\}, \ i \in I.$$

We assume that the set $S^i_{\alpha_i}$ is non-empty, and the probability distribution of the random matrix A^i and the probability level vector $(\alpha_i)_{i\in I}$ are known to all the players. Then, the above chance-constrained game is a non-cooperative game with complete information. A strategy profile x^* is said to be a Nash equilibrium of a chance-constrained game at α if and only if for each $i \in I$

$$u_i(x^{i*}, x^{-i*}) \ge u_i(x^i, x^{-i*}), \ \forall\, x^i \in S^i_{\alpha_i}.$$

Assumption 2.1. For each player i, $i \in I$, the following conditions hold.

1. The payoff function $u_i(\cdot, x^{-i})$ is a concave function of x^i for every $x^{-i} \in X^{-i}$.
2. The payoff function $u_i(\cdot)$ is a continuous function of x.

3. Existence of Nash equilibrium

We consider the case where for each $i \in I$, the row vector A^i_k, $k = 1, 2, \ldots, K_i$, follows a multivariate normal distribution with mean $\mu^i_k = (\mu^i_{k1}, \mu^i_{k2}, \ldots, \mu^i_{km_i})$ and a covariance matrix Σ^i_k, i.e., $A^i_k \sim \mathcal{N}\left(\mu^i_k, \Sigma^i_k\right)$. We assume Σ^i_k to be a positive definite matrix. Moreover, the row vectors are also independent. In this case we have the following results.

Lemma 3.1. *For each $i \in I$, let the row vector $A^i_k \sim \mathcal{N}\left(\mu^i_k, \Sigma^i_k\right)$ with positive definite covariance matrix Σ^i_k, $k = 1, 2, \ldots, K_i$. Moreover, the row vectors of A^i, $i \in I$, are independent. Then, $S^i_{\alpha_i}$, $i \in I$, is a convex set when $\alpha_i > F\left(\max\left\{\sqrt{3}, \bar{v}_i\right\}\right)$, where $F(\cdot)$ is the one-dimensional standard normal distribution function,*

$$\bar{v}_i = \max_{k=1,2,\ldots,K_i} 4\lambda^{i,k}_{\max}\left(\lambda^{i,k}_{\min}\right)^{-\frac{3}{2}} \left\|\mu^i_k\right\|,$$

and $\lambda^{i,k}_{\max}$, $\lambda^{i,k}_{\min}$ refer to the largest and smallest eigenvalues of Σ^i_k; $\|\cdot\|$ denotes the Euclidean norm.

Proof. The proof follows from Theorem 5.1 of [12]. □

Remark 3.2. If the value of $\bar{v}_i$ is smaller than $\sqrt{3}$, $S^i_{\alpha_i}$ is a convex set when $\alpha_i > F(\sqrt{3}) \approx 0.958$.

Lemma 3.3. *For each $i \in I$, let the row vector $A^i_k \sim \mathcal{N}\left(\mu^i_k, \Sigma^i_k\right)$ with positive definite covariance matrix Σ^i_k, $k = 1, 2, \ldots, K_i$. Moreover, the row vectors of A^i, $i \in I$, are independent. Then, $S^i_{\alpha_i}$, $i \in I$, is a compact set provided*

$$\alpha_i > \min_{k=1,2,\ldots,K_i} F\left(\left\|(\Sigma^i_k)^{-1/2}\mu^i_k\right\|\right).$$

Proof. The proof directly follows from Theorem 2.3 of [11]. □

For each $i \in I$, the set of best response strategies of player i for a fixed strategy profile x^{-i} of other players is given by

$$\mathcal{B}^{\alpha_i}_i(x^{-i}) = \left\{\bar{x}^i \in S^i_{\alpha_i} \mid u_i(\bar{x}^i, x^{-i}) \ge u_i(x^i, x^{-i}), \ \forall\, x^i \in S^i_{\alpha_i}\right\}.$$

Denote, $S_\alpha = \prod_{i\in I} S^i_{\alpha_i}$, where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T$. Let $\mathcal{P}(S_\alpha)$ be the power set of S_α. Then, for an $\alpha \in [0, 1]^n$, define a set-valued map

$$G^\alpha : S_\alpha \to \mathcal{P}(S_\alpha)$$

such that

$$G^\alpha(x) = \prod_{i\in I} \mathcal{B}^{\alpha_i}_i(x^{-i}).$$

A point x is said to be a fixed point of $G^\alpha(\cdot)$ if $x \in G^\alpha(x)$. It is clear that a fixed point of $G^\alpha(\cdot)$ is a Nash equilibrium of a chance-constrained game.

Theorem 3.4. *If we consider an n-player non-cooperative game where*

1. *the payoff function of player i, $i \in I$, satisfies Assumption 2.1,*
2. *the stochastic linear constraints of each player i are jointly satisfied with at least a given probability α_i, $i \in I$*
3. *for each $i \in I$, let the row vector $A^i_k \sim \mathcal{N}\left(\mu^i_k, \Sigma^i_k\right)$ with positive definite covariance matrix Σ^i_k, $k = 1, 2, \ldots, K_i$,*
4. *the row vectors of A^i, $i \in I$, are independent,*

Then, there exists a Nash equilibrium for a chance-constrained game for all $\alpha \in (\hat{\alpha}_1, 1] \times (\hat{\alpha}_2, 1] \times \cdots \times (\hat{\alpha}_n, 1]$, where for each $i \in I$

$$\hat{\alpha}_i = \max\{\bar{\alpha}_i, \tilde{\alpha}_i\},$$

and

$$\bar{\alpha}_i = F\left(\max\left\{\sqrt{3}, \bar{v}_i\right\}\right), \text{ where}$$

$$\bar{v}_i = \max_{k=1,2,\ldots,K_i} 4\lambda^k_{\max}\left(\lambda^k_{\min}\right)^{-\frac{3}{2}} \left\|\mu^i_k\right\|,$$

$$\tilde{\alpha}_i = \min_{k=1,2,\ldots,K_i} F\left(\left\|(\Sigma^i_k)^{-1/2}\mu^i_k\right\|\right).$$

Proof. Fix $\alpha \in (\hat{\alpha}_1, 1]\times(\hat{\alpha}_2, 1]\times\cdots\times(\hat{\alpha}_n, 1]$. To show the existence of a Nash equilibrium for a chance-constrained game, it is enough to show that $G^\alpha(\cdot)$ has a fixed point. We show that $G^\alpha(\cdot)$ satisfies all the conditions of Kakutani fixed point theorem [15] as given below:

(i) S_α is a non-empty, convex, and compact subset of a finite dimensional Euclidean space.
(ii) $G^\alpha(x)$ is a non-empty and convex set for all $x \in S_\alpha$.
(iii) The graph of set-valued map $G^\alpha(\cdot)$, defined by $\{(x, y) \mid y \in G^\alpha(x)\}$, is a closed subset of $S_\alpha \times S_\alpha$: if $(x_n, \bar{x}_n) \to (x, \bar{x})$ with $\bar{x}_n \in G^\alpha(x_n)$ for all n, then $\bar{x} \in G^\alpha(x)$.

S_α is a non-empty set because for each $i \in I$, $S^i_{\alpha_i}$ is a non-empty set. The convexity of S_α follows from Lemma 3.1 and the compactness of S_α follows from Lemma 3.3. To show the condition (ii), it is enough to show that $\mathcal{B}^{\alpha_i}_i(x^{-i})$ is a non-empty and convex set for all $i \in I$. The set $\mathcal{B}^{\alpha_i}_i(x^{-i})$ in non-empty because $u_i(\cdot, x^{-i})$ is a continuous function of x^i and $S^i_{\alpha_i}$ is a compact set. The set $\mathcal{B}^{\alpha_i}_i(x^{-i})$ is convex because $u_i(\cdot, x^{-i})$ is a concave function of x^i. Since the payoff function $u_i(\cdot)$, $i \in I$, is a continuous function of x, then the closed graph condition can be proved using the similar arguments given in the proof of Theorem 3.2. [24] (see also Theorem 4.4 of [1]). □

From Theorem 3.4, a Nash equilibrium for a chance-constrained game exists for sufficiently large values for α_i, $i \in I$. Therefore, in most of the cases we do not have an answer for the existence of a Nash equilibrium for a chance-constrained game defined in Section 2. In order to answer this question we first propose a new reformulation for (2.2).

Under independent and normally distributed assumption on matrix A^i, we have the following equivalent deterministic reformulation for the joint chance-constraint (2.2)

$$Q^i_{\alpha_i} = \begin{cases} (\mu^i_k)^T x^i + F^{-1}\left(\alpha_i^{z^i_k}\right)\left\|\left(\Sigma^i_k\right)^{1/2} x^i\right\| \\ \quad \le b^i_k, \ \forall\, k = 1, 2, \ldots, K_i, & (i) \\ \sum_{k=1}^{K_i} z^i_k = 1, & (ii) \\ z^i_k \ge 0, \ \forall\, k = 1, 2, \ldots, K_i, & (iii) \end{cases}$$

where $F^{-1}(\cdot)$ is a quantile function for a standard normal distribution [7]. For an $\alpha_i \in [0.5, 1]$, the set $Q^i_{\alpha_i}$ is a bi-convex set as it is a convex set in x^i (resp. $(z^i_k)^{K_i}_{k=1}$) for a fixed $(z^i_k)^{K_i}_{k=1}$ (resp. x^i).

We propose a new convex reformulation of the set $Q^i_{\alpha_i}$. Let $\alpha_i \in [0.5, 1]$, $i \in I$. For $0 \le z^i_k \le 1$, $F^{-1}\left(\alpha_i^{z^i_k}\right) \ge 0$ for all $\alpha_i \ge 0.5$. Therefore, the constraint (i) of the set $Q^i_{\alpha_i}$ can be written as

$$\left(\mu^i_k\right)^T x^i + \left\|\left(\Sigma^i_k\right)^{1/2}\left(F^{-1}\left(\alpha_i^{z^i_k}\right) x^i\right)\right\| \le b^i_k, \quad \forall\, k = 1, 2, \ldots, K_i. \tag{3.1}$$

We use a change of variables technique under logarithmic transformation [18]. The logarithmic transformation is well defined because $X^i \subset \mathbb{R}^{m_i}_{++}$. We transform the vector $x^i \in X^i$ into a vector $y^i \in \mathbb{R}^{m_i}$, where $y^i_j = \log x^i_j$, $j = 1, 2, \ldots, m_i$. Then, constraint (3.1) can be written as

$$(\mu^i_k)^T e^{y^i} + \left\|\left(\Sigma^i_k\right)^{1/2} e^{\log F^{-1}\left(\alpha_i^{z^i_k}\right)\mathbb{1}_{m_i} + y^i}\right\| \le b^i_k, \ \forall\, k = 1, 2, \ldots, K_i,$$

where $\mathbb{1}_{m_i}$ is an $m_i \times 1$ vector of ones, and $e^{y^i} = \left(e^{y^i_1}, \ldots, e^{y^i_{m_i}}\right)^T$ and

$$e^{\log F^{-1}\left(\alpha_i^{z^i_k}\right)\mathbb{1}_{m_i} + y^i} = \left(e^{\log F^{-1}\left(\alpha_i^{z^i_k}\right) + y^i_1}, \ldots, e^{\log F^{-1}\left(\alpha_i^{z^i_k}\right) + y^i_{m_i}}\right)^T.$$

Therefore, we have the following deterministic reformulation for (2.2):

$$\widetilde{Q}^i_{\alpha_i} = \begin{cases} (\mu^i_k)^T e^{y^i} + \left\|\left(\Sigma^i_k\right)^{1/2} e^{\log F^{-1}\left(\alpha_i^{z^i_k}\right)\mathbb{1}_{m_i} + y^i}\right\| \le b^i_k, \\ \qquad \forall\, k = 1, 2, \ldots, K_i, & (i) \\ \sum_{k=1}^{K_i} z^i_k = 1 & (ii) \\ z^i_k \ge 0, \ \forall\, k = 1, 2, \ldots, K_i. & (iii) \end{cases}$$

Let Y^i be an image of X^i under logarithmic function. Since the logarithmic function is continuous and X^i is a compact set, Y^i is also a compact set. Broadly speaking, the convexity may not be preserved under logarithmic transformation. In this section, we consider the sets X^i for which the sets Y^i remain convex. We give hereafter few examples of convex sets X^i which are invariant under logarithmic transformation.

Example 3.5. Consider a set

$$X^i = \left\{x^i \in \mathbb{R}^{m_i}_{++} \mid c^T x^i \le h\right\},$$

where $c = (c_1, \ldots, c_{m_i})^T \in \mathbb{R}^{m_i}_+$ and $h \in \mathbb{R}_+$ are all constant. Then,

$$Y^i = \left\{y^i \in \mathbb{R}^{m_i} \mid \sum_{j=1}^{m_i} c_j e^{y^i_j} \le h\right\},$$

is a convex set.

Example 3.6. Consider a set

$$X^i = \left\{x^i \in \mathbb{R}^{m_i}_{++} \mid \sum_{j=1}^{m_i - 1} x^i_j \le x^i_{m_i}\right\}.$$

Then, Y^i can be reformulated as

$$Y^i = \left\{y^i \in \mathbb{R}^{m_i} \mid \sum_{j=1}^{m_i - 1} e^{y^i_j - y^i_{m_i}} \le 1\right\},$$

which is also a convex set.

Example 3.7. Consider a set

$$X^i = \left\{x^i \in \mathbb{R}^{m_i}_{++} \mid \sum_{j=1}^{m_i} (x^i_j)^2 \le h\right\},$$

where $h \in \mathbb{R}_{++}$ is a constant. Then,

$$Y^i = \left\{y^i \in \mathbb{R}^{m_i} \mid \sum_{j=1}^{m_i} e^{2y^i_j} \le h\right\},$$

is a convex set.

Example 3.8. Consider a set

$$X^i = \left\{ x^i \in \mathbb{R}^{m_i}_{++} \mid f(x^i) \leq h \right\},$$

where $f : \mathbb{R}^{m_i}_{++} \to \mathbb{R}_{++}$ is a log-convex and non-decreasing function of x^i, and $h \in \mathbb{R}_{++}$ is a constant. Then,

$$Y^i = \left\{ y^i \in \mathbb{R}^{m_i} \mid f(e^{y^i}) \leq h \right\},$$

is a convex set.

The reformulation of feasible strategy set $S^i_{\alpha_i}$ of player i, $i \in I$, is given by

$$\widetilde{S}^i_{\alpha_i} = \left\{ (y^i, z^i) \in Y^i \times \mathbb{R}^{K_i} \mid (y^i, z^i) \in \widetilde{Q}^i_{\alpha_i} \right\}.$$

Assumption 3.9. For each player i, $i \in I$, $u_i(\cdot, x^{-i})$ is a non-increasing function for every $x^{-i} \in X^{-i}$.

Assumption 3.10. For each $i \in I$ and $k = 1, 2, \ldots, K_i$, all the components of Σ^i_k and μ^i_k are non-negative.

Under Assumption 3.10, we show that the set $\widetilde{S}^i_{\alpha_i}$ is convex. It is enough to show that constraint (i) of $\widetilde{Q}^i_{\alpha_i}$ is convex. We present Lemma 3.11 on the composition of convex functions whose proof is given in the Appendix. It is used to prove Lemma 3.12 which is the key to prove the convexity of the sets $\widetilde{S}^i_{\alpha_i}$, $i \in I$.

Lemma 3.11. *Let $\Gamma_1, \Gamma_2 \subseteq \mathbb{R}$ and Γ_1, Γ_2 are convex sets. Suppose $f_1 : \Gamma_1 \to \Gamma_2$ is a convex function on Γ_1, and $f_2 : \Gamma_2 \to \mathbb{R}$ is a non-decreasing and convex function on Γ_2. Then, the composition function $f_2 \circ f_1$ is a convex function on Γ_1.*

Lemma 3.12. *For each $i \in I$ and $k = 1, 2, \ldots, K_i$, $\log F^{-1}\left(\alpha_i^{z^i_k}\right)$ is a convex function of z^i_k on $[0, 1]$ for all $\alpha_i \in [F(1), 1]$ where $F(1) \approx 0.84$.*

Proof. Let $g_1 : [0, 1] \to [F(1), 1]$ such that $g_1(z^i_k) = \alpha_i^{z^i_k}$, and $g_2 : [F(1), 1] \to \mathbb{R}$ such that $g_2(p) = \log F^{-1}(p)$ be two functions. Then, the composition function $(g_2 \circ g_1)(z^i_k) = \log F^{-1}\left(\alpha_i^{z^i_k}\right)$. From Lemma 3.11, $\log F^{-1}\left(\alpha_i^{z^i_k}\right)$ is a convex function of z^i_k if $g_1(\cdot)$ is a convex function and $g_2(\cdot)$ is convex and non-decreasing function on their respective domains.

Since $0 \leq z^i_k \leq 1$ and $F(1) \leq \alpha_i \leq 1$, then $\alpha_i^{z^i_k} \geq \alpha_i$. Therefore, the function $g_1(\cdot)$ is well defined and it is also a convex function of z^i_k. The function $g_2(\cdot)$ is a non-decreasing function because the quantile function $F^{-1}(\cdot)$ as well as $\log(\cdot)$ are non-decreasing functions. It remains to show that $\log F^{-1}(p)$ is a convex function. It is enough to show that the second order derivative of $\log F^{-1}(p)$ is non-negative. The second order derivative of $\log F^{-1}(p)$ can be written as

$$-\frac{\psi(y) + y\psi'(y)}{(y\psi(y))^2 \psi(y)}.$$

where $y = F^{-1}(p)$, and $\psi(\cdot)$ is the probability density function of a standard normal distribution, and $\psi'(\cdot)$ is the first order derivate of $\psi(\cdot)$. We have

$$\psi(y) + y\psi'(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} (1 - y)(1 + y) \leq 0,$$

because $1 + y = 1 + F^{-1}(p) \geq 0$ and $1 - y = 1 - F^{-1}(p) \leq 0$ for all $p \in [F(1), 1]$. Hence, the second order derivative of $\log F^{-1}(p)$ is non-negative for all $p \in [F(1), 1]$. Therefore, $\log F^{-1}\left(\alpha_i^{z^i_k}\right)$ is a convex function of z^i_k on $[0, 1]$ for all $\alpha_i \in [F(1), 1]$. □

Lemma 3.13. *For each $i \in I$, let the convex set X^i be such that Y^i is a convex set. Let Assumption 3.10 hold. Then, the set $\widetilde{S}^i_{\alpha_i}$, $i \in I$, is a convex set for all $\alpha_i \in [F(1), 1]$.*

Proof. Fix $i \in I$ and $\alpha_i \in [F(1), 1]$.

Then, $\left(\left(\Sigma^i_k\right)^{1/2} e^{\log F^{-1}\left(\alpha_i^{z^i_k}\right) \mathbb{1}_{m_i} + y^i} \right)$ is an $m_i \times 1$ vector. Each component of the vector is a non-negative linear combination of the convex functions. Hence, it is a vector of non-negative convex functions. The Euclidean norm is a convex function and it is also a non-decreasing function in each argument when the arguments are non-negative. Therefore, the composition function $\left\| \left(\Sigma^i_k\right)^{1/2} e^{\log F^{-1}\left(\alpha_i^{z^i_k}\right) \mathbb{1}_{m_i} + y^i} \right\|$ is a convex function. The term $(\mu^i_k)^T e^{y^i}$ is a convex function because $\mu^i_k \geq 0$. Hence, the constraints

$$(\mu^i_k)^T e^{y^i} + \left\| \left(\Sigma^i_k\right)^{1/2} e^{\log F^{-1}\left(\alpha_i^{z^i_k}\right) \mathbb{1}_{m_i} + y^i} \right\| \leq b^i_k, \ \forall\, k = 1, 2, \ldots, K_i,$$

are convex. It is easy to see that the other constraints of $\widetilde{S}^i_{\alpha_i}$ are convex. Hence, $\widetilde{S}^i_{\alpha_i}$ is a convex set. □

Theorem 3.14. *If we consider an n-player non-cooperative game where*

1. *the payoff function of player i, $i \in I$, satisfies Assumptions 2.1 and 3.9,*
2. *the stochastic linear constraints of each player i are jointly satisfied with a given probability α_i, $i \in I$*
3. *for each $i \in I$, let the row vector $A^i_k \sim \mathcal{N}\left(\mu^i_k, \Sigma^i_k\right)$ where mean vector μ^i_k and positive definite covariance matrix Σ^i_k, $k = 1, 2, \ldots, K_i$, satisfies Assumption 3.10,*
4. *the row vectors of A^i, $i \in I$, are independent,*

Then, there exists a Nash equilibrium of a chance-constrained game for all $\alpha \in [F(1), 1]^n$.

Proof. Let $\alpha \in [F(1), 1]^n$. For each $i \in I$, define a composition function $C_i = -u_i \circ d_i$, where $d_i : \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \cdots \times \mathbb{R}^{m_n} \to \mathbb{R}^{m_1}_{++} \times \mathbb{R}^{m_2}_{++} \times \cdots \times \mathbb{R}^{m_n}_{++}$, such that

$$d_i\left(y^1, y^2, \ldots, y^n\right) = \left(e^{y^1}, e^{y^2}, \ldots, e^{y^n}\right).$$

Then, define a best response set for player i for a fixed $y^{-i} \in Y^{-i}$

$$\widetilde{\mathcal{B}}^{\alpha_i}_i(y^{-i}) = \left\{ (\bar{y}^i, \bar{z}^i) \in \widetilde{S}^i_{\alpha_i} \mid C_i(\bar{y}^i, y^{-i}) \leq C_i(y^i, y^{-i}), \ \forall\, (y^i, z^i) \in \widetilde{S}^i_{\alpha_i} \right\}.$$

Denote $\widetilde{S}(\alpha) = \prod_{i \in I} \widetilde{S}^i_{\alpha_i}$. Then, define a set-valued map

$$\widetilde{G}^\alpha : \widetilde{S}_\alpha \to \mathcal{P}(\widetilde{S}_\alpha)$$

such that

$$\widetilde{G}^\alpha(y, z) = \prod_{i \in I} \widetilde{\mathcal{B}}^{\alpha_i}_i(y^{-i}).$$

It follows from Lemma 3.13 that $\widetilde{S}_\alpha$ is a convex set. It is also a closed and bounded set. Both the functions $u_i(\cdot)$ and $d_i(\cdot)$ are continuous functions. Therefore, the composition function $C_i(\cdot)$ is also a continuous function. Since $\widetilde{S}^i_{\alpha_i}$ is a compact set, then the best response set $\widetilde{\mathcal{B}}^{\alpha_i}_i(y^{-i})$ is non-empty. The function $d_i(\cdot)$ is a convex function and $-u_i(\cdot, x^{-i})$ is a convex and non-decreasing function of x^i. Then, the composition function $C_i(\cdot, y^{-i})$ is a convex function of y^i which, in turn, implies that the best response set $\widetilde{\mathcal{B}}_i^{\alpha_i}(y^{-i})$ is a convex set. Hence, $\widetilde{G}^\alpha(y, z)$ is a non-empty and convex set for each (y, z). The closed graph condition for $\widetilde{G}^\alpha(\cdot)$ follows from the continuity of the functions $C_i(\cdot)$, $i \in I$. Therefore, from Kakutani fixed point theorem there exists a fixed point (y^*, z^*) for the set-valued map $\widetilde{G}^\alpha(\cdot)$. Then, for each $i \in I$

$$C_i(y^{i*}, y^{-i*}) \leq C_i(y^i, y^{-i*}), \ \forall (y^i, z^i) \in \widetilde{S}^i_{\alpha_i}.$$

Under the hypothesis of Theorem 3.14, $\widetilde{S}^i_{\alpha_i}$ is a reformulation of $S^i_{\alpha_i}$, where $x^i = e^{y^i}$. This implies

$$u_i(x^{i*}, x^{-i*}) \geq u_i(x^i, x^{-i*}), \ \forall x^i \in S^i_{\alpha_i}.$$

Hence, x^* is a Nash equilibrium of a chance-constrained game for all $\alpha \in [F(1), 1]^n$. $\square$

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Appendix. Proof of Lemma 3.11

Proof. Consider any two points $x_1, x_2 \in \Gamma_1$ and $\lambda \in [0, 1]$. Since f_1 is a convex function on Γ_1, we have

$$f_1(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f_1(x_1) + (1-\lambda) f_1(x_2).$$

Using the non-decreasing and convexity properties of f_2 on Γ_2, we have

$$(f_2 \circ f_1)(\lambda x_1 + (1-\lambda)x_2) \leq \lambda (f_2 \circ f_1)(x_1) + (1-\lambda)(f_2 \circ f_1)(x_2).$$

Hence, $f_2 \circ f_1$ is a convex function on Γ_1.

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