Problem Set III: Linear Transformations - II

Note: The basis $e_1, e_2, \dots e_k$ refer to the standard basis vectors in $\mathbf{R}^k$ for some positive integer k, where the value of k will vary. Vector spaces considered are finite dimensional. The variables m, n and k will generally be used to indicate arbitrary positive integers. Let V, W be finite dimensional vector spaces over $\mathbf{R}$ of dimension n and m respectively. Let $T : V \mapsto W$ be a linear Transformation of rank r.

1. a Show that there exist vectors $b_1, b_2 \dots b_r \in V$ such that $T(b_1), T(b_2), \dots, T(b_r)$ is a basis of Img(T).
   b Let $v \in V$ and $x \in W$ such that $T(v) = x$. Suppose $x = \alpha_1 T(b_1) + \alpha_2 T(b_2) + \dots + \alpha_r T(b_r)$, where $\alpha_i \in \mathbf{R}$. Then, show that $T(v - \alpha_1 b_1 - \alpha_2 b_2 - \dots - \alpha_r b_r) = 0$.
   c Let $c_1, c_2 \dots c_k$ be a basis of Nspace(T). Then show that $\{c_1, c_2, \dots c_k, b_1, b_2, \dots b_r\}$ forms a basis of V. Hence conclude that $k = n - r$.

2. Consider the matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix}$ Let $b = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$. Find real numbers $\alpha, \beta, \gamma, \delta$ such that for all $t \in \mathbf{R}$, the parametrized solution vector $x = \begin{bmatrix} \alpha + \beta t \\ \gamma + \delta t \end{bmatrix}$ is a solution to $Ax = b$.

3. Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & \alpha & \beta \end{bmatrix}$.
   a Find real numbers α, β such that there exists at least one vector $b \in \mathbf{R}^2$ such that $Ax = b$ has no solution. (Hint: What is the condition on the $Rank(A)$ for the requirement to be satisfied.)
   b For the α, β values found above, find a vector b for which $Ax = b$ is solvable. Find the general parametric form for all solutions for $Ax = b$.

4. Consider the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$.
   a Find a vector $d \in \mathbf{R}^2$ such that $Ax = b$ has a solution if and only if $b = td$ for some $t \in \mathbf{R}$. Justify your answer. (Hint: What is the dimension of the image of A? How many vectors are needed to span that space?)
   b Find two vectors $u, v \in \mathbf{R}^3$ such that $Ax = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ if and only if $x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha u + \beta v$ for some $\alpha, \beta \in \mathbf{R}$.

5. Consider the linear transformation T whose matrix is $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix}$ with respect to the standard basis on both $\mathbf{R}^3$ and $\mathbf{R}^2$. Suppose the basis or $\mathbf{R}^3$ is changed to $b_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $b_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $b_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, and the basis of $\mathbf{R}^2$ is changed to $c_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $c_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$; how will the matrix of T change with respect to the new bases?

6. Consider the linear transformation T whose matrix with respect to the standard basis on $\mathbf{R}^2$ is $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Find a basis b_1, b_2 of $\mathbf{R}^2$ such that, with respect to this basis, the matrix of T takes the form $\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$ for some real numbers α, β. Find the values of α and β.

7. Suppose that $W = V$. That, is $T : V \mapsto V$. Suppose $b_1, b_2, \dots b_n$ be a basis of V such that $T(b_i) = \alpha_i b_i$ for some real numbers α_i, $1 \le i \le n$, then what will be the matrix of T with respect to the basis $b_1, b_2, \dots b_n$?

8. Suppose that dim(V)=dim(W). Prove that T is injective $\Leftrightarrow$ T is surjective $\Leftrightarrow$ T is bijective $\Leftrightarrow$ Nspace$(T) = \{0\}$. (Hint: Use the Rank Nullity Theorem.)