ME/AE 5481 SPECIAL ASSIGNMENT (Due December 4th) Fall 2018
(250 points)

(F-8 longitudinal dynamics.)

Any aircraft in level flight can be characterized by a set of Linear Time Invariant equations. For the design of a longitudinal stability augmentation system the following perturbed variables define the state vector $\underline{x}(t)$

$$\underline{x}(t)\underline{\underline{\Delta}}\begin{bmatrix} q(t) \\ v(t) \\ \alpha(t) \\ \theta(t) \end{bmatrix} \tag{1}$$

where

q(t): Pitch rate (rad/sec)
v(t): Velocity error from constant horizontal velocity (ft/sec)
α(t): Angle of attack perturbation from trimmed value (rad)
θ(t): Pitch perturbation from trimmed value (rad)

The control variable is:
δ_e(t): Elevator angle perturbation from trimmed value (rad)
The general structure of the linearized state equations takes the form

$$\underline{x}(t) = \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ 0 & a_{22} & a_{23} & -g \\ 1 & a_{32} & a_{33} & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{\underline{A}} \underline{x}(t) + \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ 0 \end{bmatrix}}_{\underline{B}} \delta_e(t) \tag{2}$$

If we ignore servo dynamics so that the control system can instantaneously position the elevator, we can use the general structure of Eq. (2) for the design of the control system.

For the F-8 aircraft flying at 20,000 ft at a Mach number of 0.6 the following numerical values for the elements of the $\underline{A}$ matrix are given.

$$a_{11} = -0.49 \qquad g = 32.2$$
$$a_{12} = .5\times10^{-4} \qquad a_{32} = -0.19\times10^{-3}$$
$$a_{13} = -4.79 \qquad a_{33} = -0.836$$
$$a_{22} = -0.148\times10^{-1}$$
$$a23 = -13.877$$

and for the $\underline{B}$ vector

$$b_1 = -8.743$$
$$b_2 = -1.096$$
$$b_3 = -.01115$$

The next step is to establish a class of performance criteria. From human factor experiments we know that the following variables are important to the pilot

(1) Pitch, $\theta(t)$
(2) Pitch rate, q(t)
(3) Normal acceleration $a_{nz}(t)$

Next we define normal acceleration (in g's)

$$a_{nz}(t) \underline{\underline{\Delta}} \frac{V_0}{g}(q(t) - \dot{\alpha}(t)) \tag{3}$$

where

V_0 = (mach no.) x (speed of sound) (4)
= (0.6) x (1036.93 ft/sec)
g = 32.2 ft/sec^2 (5)

From Eqs. (1) and (2) we know that

$$\dot{\alpha}(t) = q(t) + a_{32}v(t) + a_{33}\alpha(t) + b_3\delta(t) \tag{6}$$

Hence, substituting Eq. (6) into Eq. (3) we obtain the following expression for normal acceleration.

$$a_{nz}(t) = \frac{V_0}{g}\left(-a_{32}v(t) - a_{33}\alpha(t) - b_3\delta(t)\right) \tag{7}$$

To construct the performance criterion we ask the following question; what are the worst values of
Pitch, θ_{max}

Pitch rate, q_{max}
Normal acceleration a_{nzmax}

So that if they occurred one would be willing to use the maximum elevator deflection available, δ_{emax}.

For the design problem suppose that

$$\begin{aligned}
\delta_{e\max} &= \left(\frac{8}{57.3}\right)\ rads \\
\theta_{\max} &= 5g / V_0 a_{33} \\
q_{\max} &= 5g / V_0 \\
a_{nz\max} &= 7 \ \ (g's)
\end{aligned} \tag{8}$$

The structure of the cost functional J to be minimized is

$$J = \int_0^\infty \left\{ \left[\frac{\theta(t)}{\theta_{\max}}\right]^2 + \left[\frac{q(t)}{q_{\max}}\right]^2 + \left[\frac{a_{nz}(t)}{a_{nz\max}}\right]^2 + \left[\frac{\delta_e(t)}{\delta_{e\max}}\right]^2 \right\} dt \tag{9}$$

(a) Determine the solutions of the optimization problem. Determine the control gains and the eigenvalues of the open loop and closed loop system.

(b) Plot on the same graph the open loop response and the closed loop response to the following initial conditions (all others are equal to zero)

(1) $\alpha(0) = (5/57.3)$ rads
(2) $\alpha(0) = \theta(0) = (5/57.3)$ rads

(c) Change q_{max} to
$q_{\max} = 3g/V_0$

This increases the penalty on pitch rate so that you should expect the response to be more sluggish. Repeat parts (a) and (b).

d) Comment on your results obtained from the previous parts. Your detailed analysis on what was expected and the results obtained is very important.

Optimal Control Solutions to F-8 Longitudinal Dynamics

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December 4, 2018

Introduction

This paper seeks to find a control solution for the augmented stability of an F-8 aircraft in level flight. The state variables that describe the system are defined as the pitch rate (q) in rad/sec, the velocity error from constant horizontal velocity (v) in ft/sec, the angle of attack perturbation from trimmed value (a) in radians, and the pitch perturbation from trimmed value (θ) in radians and can be described in vector form as follows:

$$x(t) = \begin{bmatrix} q(t) \\ v(t) \\ a(t) \\ \theta(t) \end{bmatrix} \tag{1}$$

The control variable for this system is the elevator angle perturbation from trimmed value ($\delta_e(t)$) in radians. Combining the control with the states gives a generalized state-space structure of the system in the form of:

$$x(t) = \begin{bmatrix} -0.49 & .5x10^{-4} & -4.79 & 0 \\ 0 & -0.148x10^{-1} & -13.877 & -32.2 \\ 1 & -0.19x10^{-3} & -0.836 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} -8.743 \\ -1.096 \\ -0.1115 \\ 0 \end{bmatrix} \delta e(t) \tag{2}$$

Where the numerical values are for an F-8 aircraft flying at Mach 0.6 at an altitude of 20,000 ft.

It is important to consider the pilot of this aircraft and the amount of force and acceleration that a human can handle. Such limitations shape the above problem into an optimal control problem; the control should be optimized to produce the best performance given certain constraints concerning the state variables. The state variables having the most impact on the pilot are: the pitch θ(t), the pitch rate $q(t)$, and the normal acceleration $a_{nz}(t)$ which is a function of certain state variables taking the form:

$$a_{nz}(t) = \frac{V_0}{g}(q(t) - \dot{a}(t)) \tag{3}$$

Where V_o is defined as the mach number multiplied by the speed of sound and g is the acceleration due to gravity. Substituting the relationship from $\dot{a}(t)$ to the other state variables allows us to achieve:

$$a_{nz}(t) = \frac{V_0}{g}(-a_{32}v(t) - a_{33}a(t) - b_3\delta(t)) \tag{4}$$

The maximum values allowable for the critical variables are defined in table 1. The critical variables coupled with their maximum values allow us to construct a cost function to be optimized having the form:

$$J = \int_0^\infty \left\{ \left[\frac{\theta(t)}{\theta_{max}}\right]^2 + \left[\frac{q(t)}{q_{max}}\right]^2 + \left[\frac{a_{nz}(t)}{a_{nzmax}}\right]^2 + \left[\frac{\delta_e(t)}{\delta_{emax}}\right]^2 \right\} dt \tag{5}$$

Minimizing this cost function will provide the gains that will result in optimal performance.

Table 1

Variable	Maximum Value
δ_e	0.14 rads
θ	-3.09 rads
q	0.26 rads/sec
a_{nz}	225.4 rads/sec^2

Results

To solve for the optimal control gains, first the a_{nz} term of the cost function was evaluated in terms of state variables and this result substituted to form the new cost function. Then the cost function terms were grouped such that the Q, R and N weights could be quantified. Once these matrices were determined, Matlab was used to solve the steady state Ricatti equation and thus find the optimal gains and the resulting closed loop poles. These gains, closed loop poles and open loop poles of the system can be found in table 2. The Matlab code and hand calculations have been attached as appendix A.

Table 2

Value	Real	Imaginary
Open Loop Poles	-0.6636	2.18
	-0.6636	-2.18
	-0.0068	0.0765
	-0.0068	-0.0765
Gains	-.5063	0
	-5.0045e-6	0
	0.3464	0
	-0.4488	0
Closed Loop Poles	-2.6569	0.6262
	-2.6569	-0.6262
	-0.4348	0
	-0.0152	0

The system state variable responses were then graphed to compare results between the open loop system and the closed loop with the optimal gains for two different sets of initial conditions for a time of fifty seconds. The first initial condition was to set the angle of attack (a(t)) to a fixed value of 5/57.3 radians. The second initial condition was to set the angle of attack and the pitch perturbation ($\theta(t)$) to the fixed value of 5/57.3 radians. For both initial conditions, all other initial variable values were zero. The state response graphs to the first set of initial conditions can be found in figures 1-4. The responses to the second set of initial conditions can be found in figures 5-8.

The controller effect on pitch response (q) for both initial conditions seems to improve the transient response mainly as open loop q is stable for both initial conditions. The controller limits the number of oscillations experienced and improves the settling time drastically. The oscillation overshoot is also significantly corrected in both cases. The transient response for q is acceptable and the steady state value is very good.

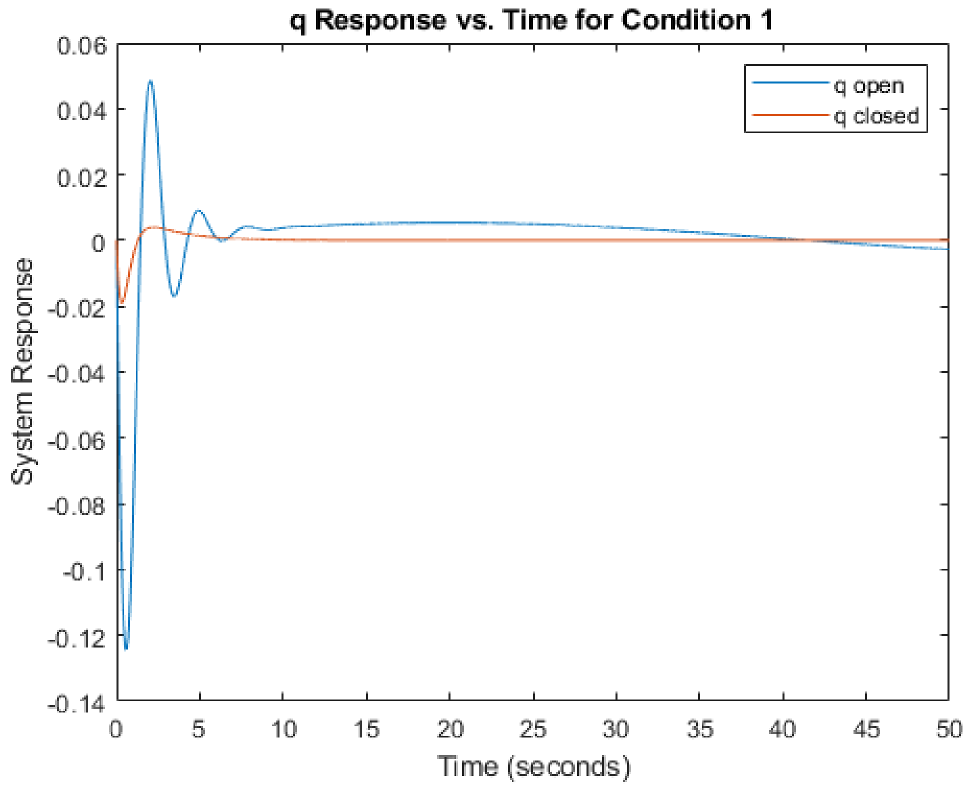

Figure 1: q(t) response to initial condition 1

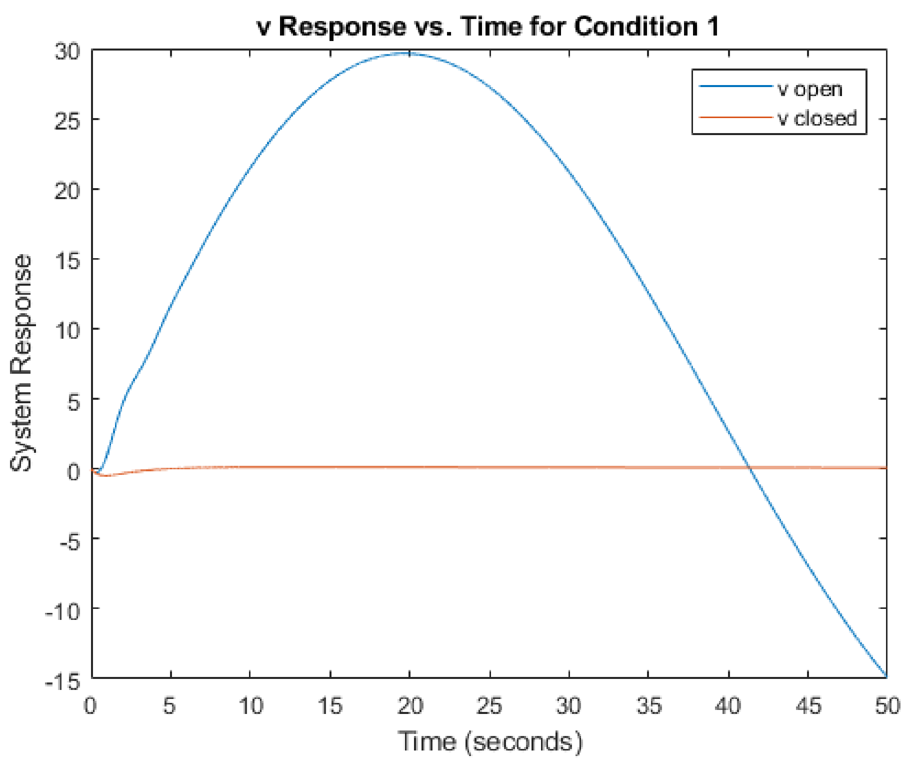

Figure 2: v(t) response to initial condition 1

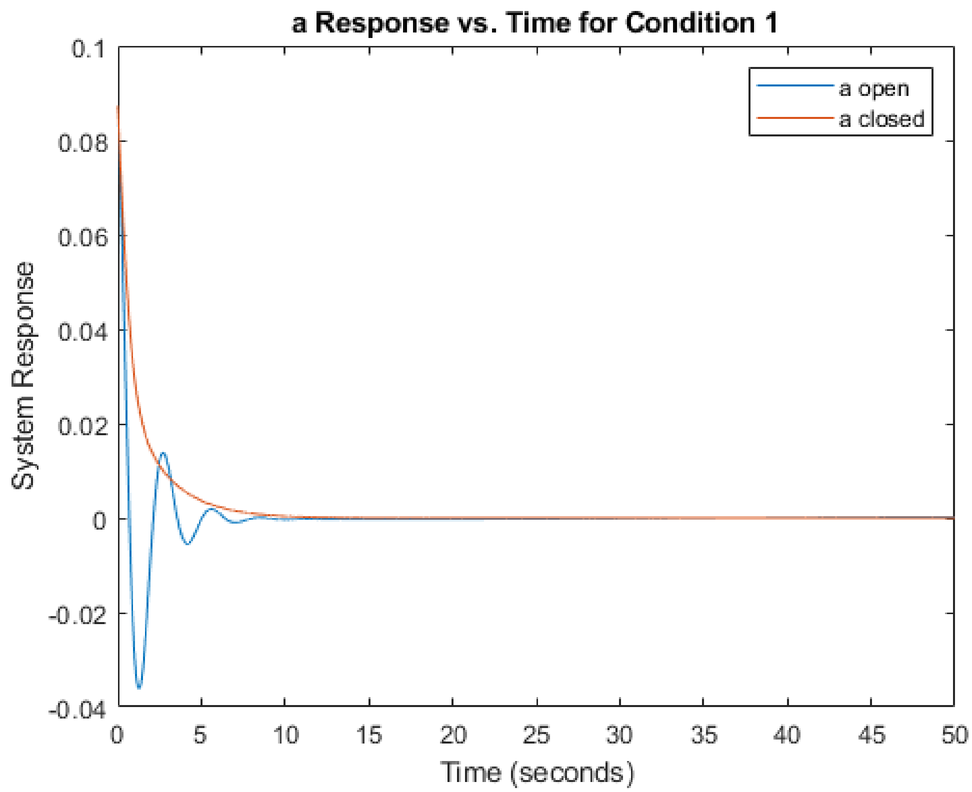

Figure 3: a(t) response to initial condition 1

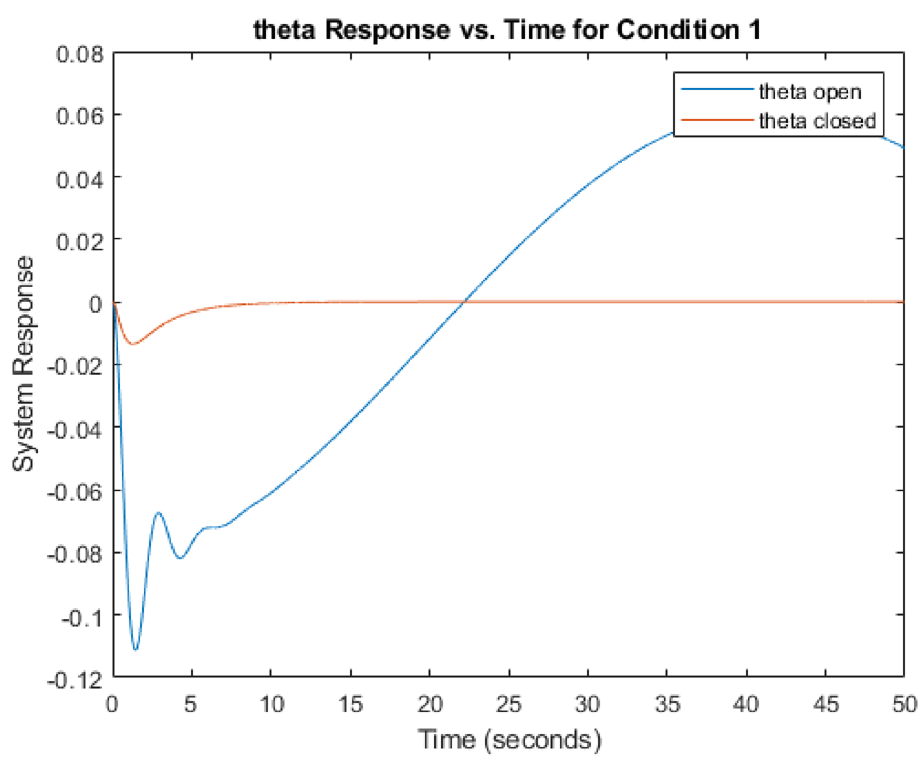

Figure 4: θ(t) response to initial condition 1

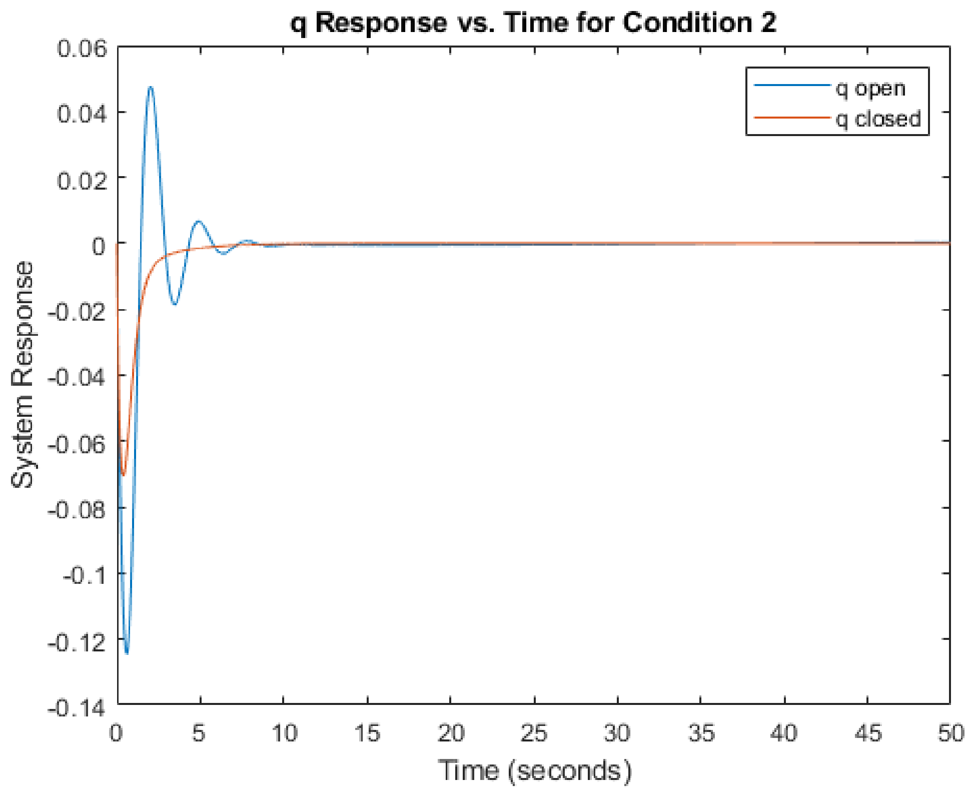

Figure 5: q(t) response to initial condition 2

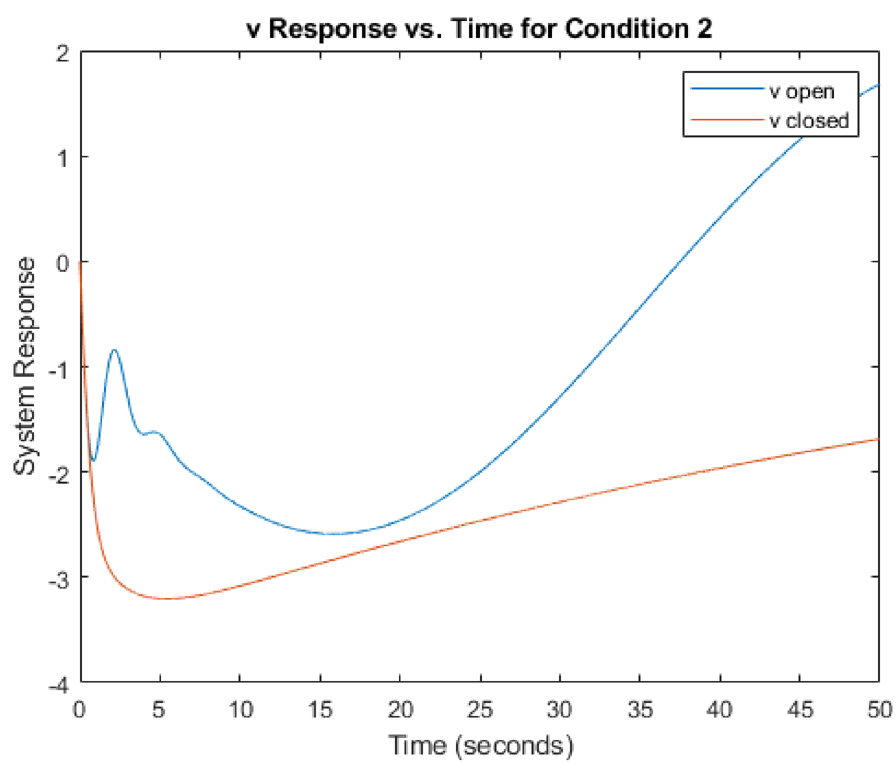

Figure 6: v(t) response to initial condition 2

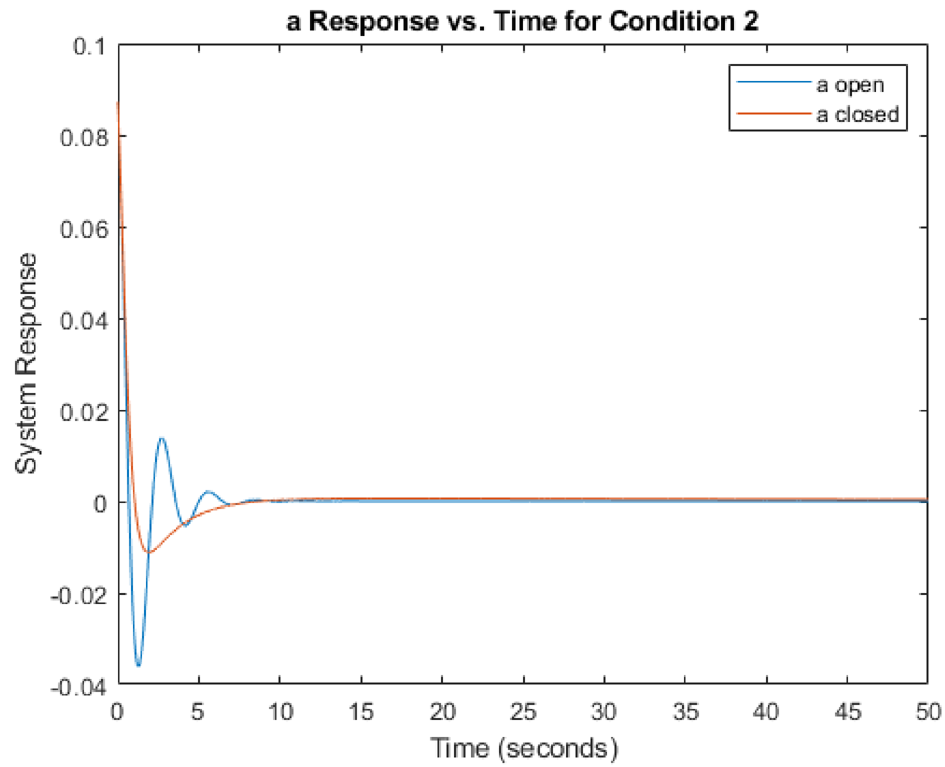

Figure 7: a(t) response to initial condition 2

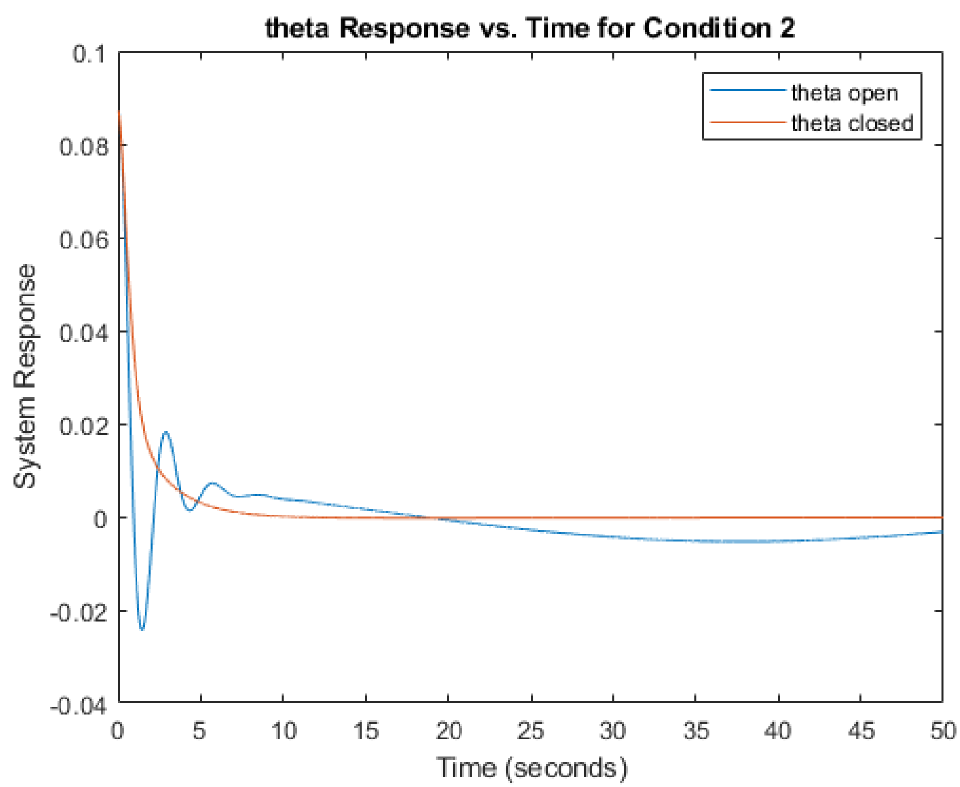

Figure 8: θ(t) response to initial condition 2

The effect of the controller on the velocity error (v) is very profound. For the first set of initial conditions, v is effectively stabilized and experiences very little transient deviation. Its steady state error is very near zero at t=50. For the second set of initial conditions, the controller performance is less drastic than the first but succeeds in stabilizing v. The closed loop response shows a higher overshoot than the open loop but begins to stabilize as the open loop oscillates. The closed loop response is also much smoother than the open loop which shows rapid oscillation during the first several seconds.

The angle of attack response (a) is another state that shows open loop stability for both sets of initial conditions. As such, the addition of the controller does not do much in the way of correcting steady state error, however the transient response for both conditions is greatly improved for the closed loop response. The controller eliminates the oscillations in both conditions and improves the convergence time for each as well. The closed loop response for condition set 2 shows a slight dip around t=3 seconds but the magnitude of this defect is much smaller than that of its open loop equivalent.

Lastly, open loop theta varies widely between the two sets of initial conditions. The response to condition set 1 shows large oscillations that don't appear to converge while the response to condition set 2 experiences some early oscillatory behavior before starting to converge to zero. In both cases, the response is greatly improved through the addition of the controller. The transient response is reduced to a smooth curve or a very slight overshoots that converges to zero around the five second mark quite nicely.

Results-Part 2

The maximum allowable value of the pitch rate was changed to observe any effects this change would have on the system. The value was decreased to 0.155 rad/sec. The optimal control gains, closed loop and open loop poles were found in the same manner as the previous methodology. They're values can be found in table 3. It is interesting to note that the only appreciable change in gains is gain 1 which would be associated with the pitch rate. This makes sense as the weight criteria for the pitch rate was the only weight that changed. Another interesting point is that the closed loop poles are completely on the real axis and thus have no imaginary component. I am unsure why the change in pitch rate weight would cause such a change.

Table 3

Value	Real	Imaginary
Open Loop Poles	-0.6636	2.18
	-0.6636	-2.18
	-0.0068	0.0765
	-0.0068	-0.0765
Gains	-.8567	0
	-3.9018e-6	0
	0.3863	0
	-0.4494	0
Closed Loop Poles	-7.2163	0
	-1.2301	0
	-0.3650	0
	-0.0152	0

The state variable responses were then plotted to the same set of initial conditions as part 1. The responses to the first set of conditions can be seen in figures 9-12 and to the second in figure 13-16.

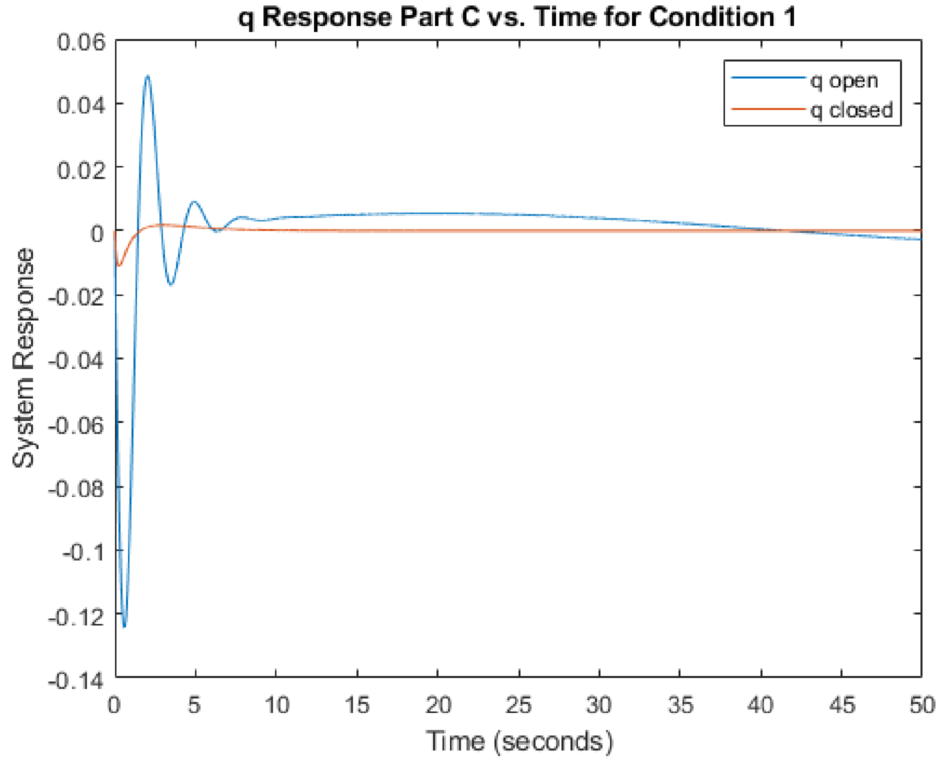

Figure 9: q response to initial condition 1 part 2

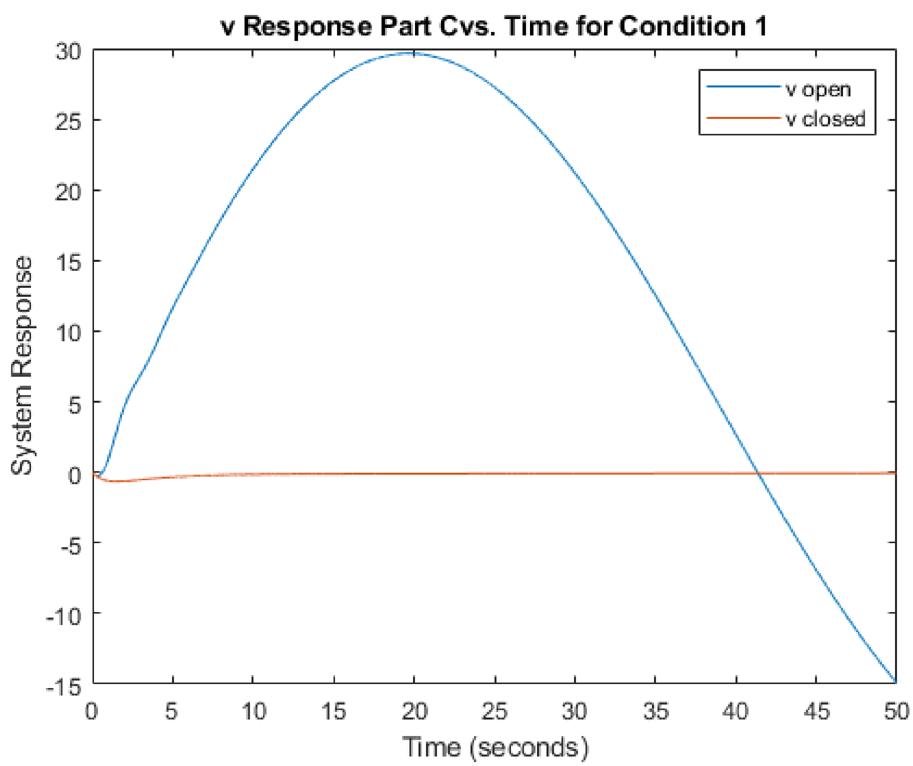

Figure 10: v response to initial condition 1 part 2

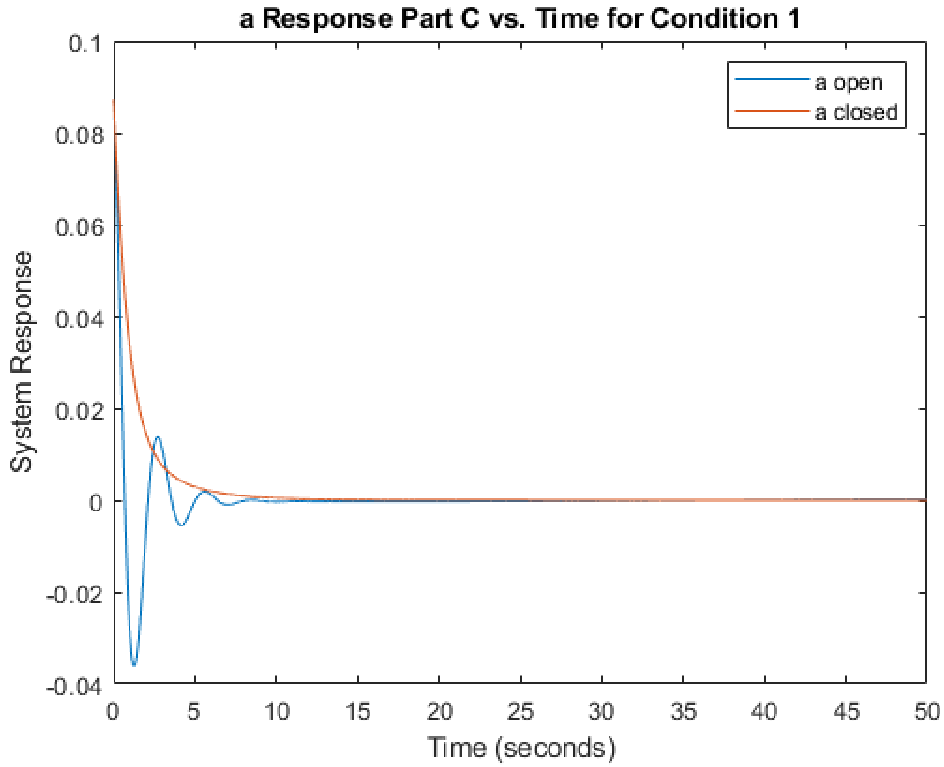

Figure 11: a response to initial condition 1 part 2

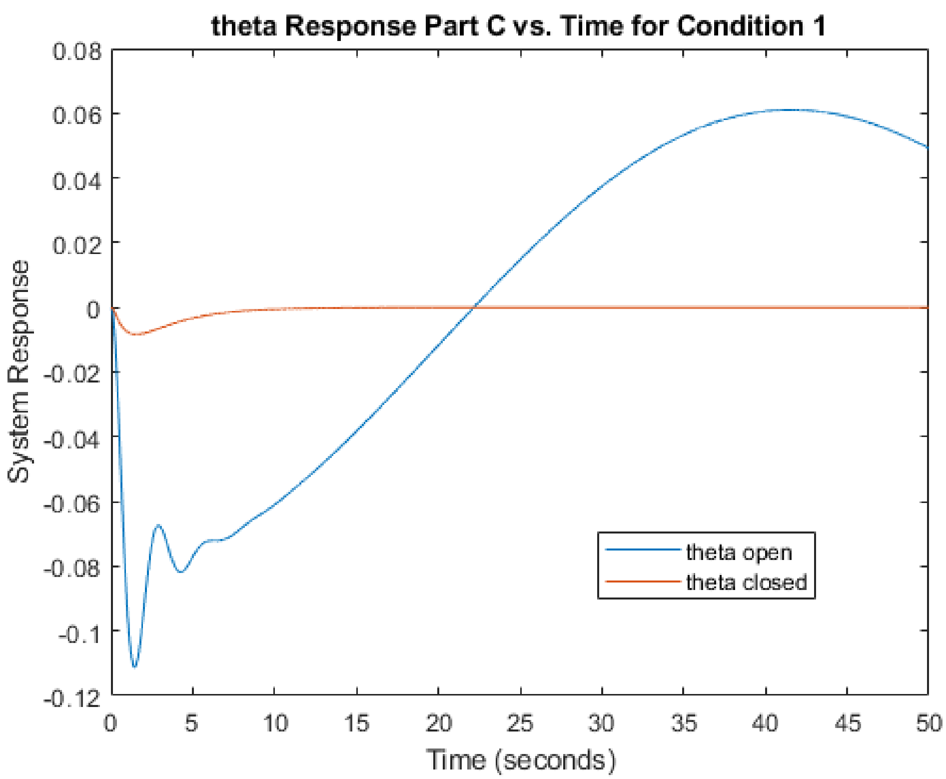

Figure 12: θ response to initial condition 1 part 2

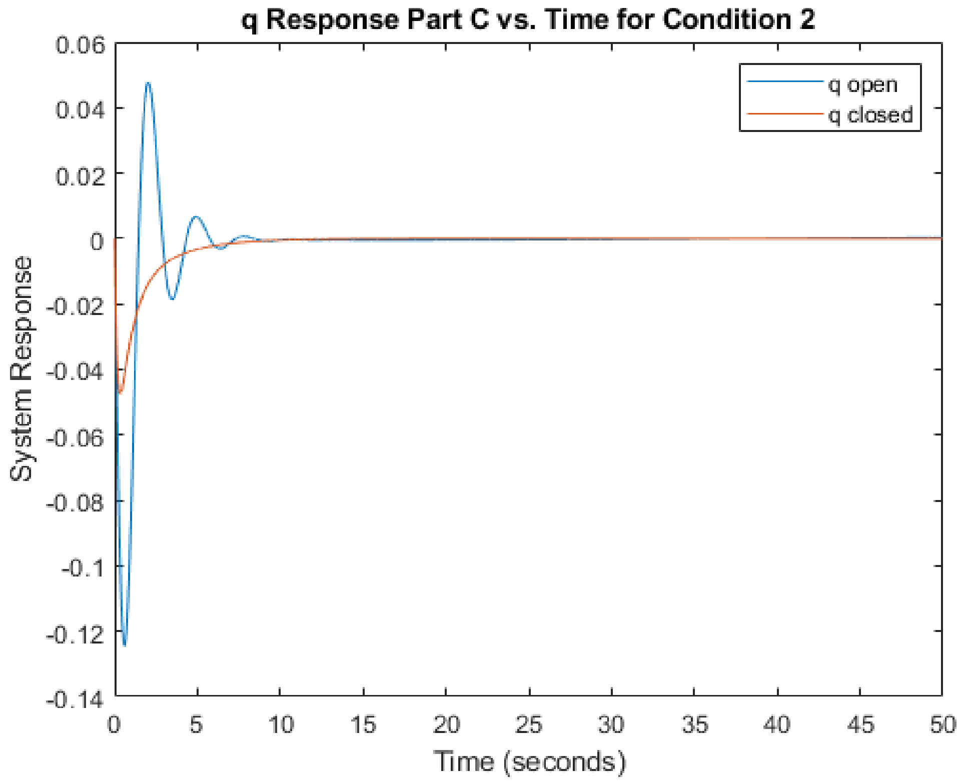

Figure 13: q response to initial condition 2 part 2

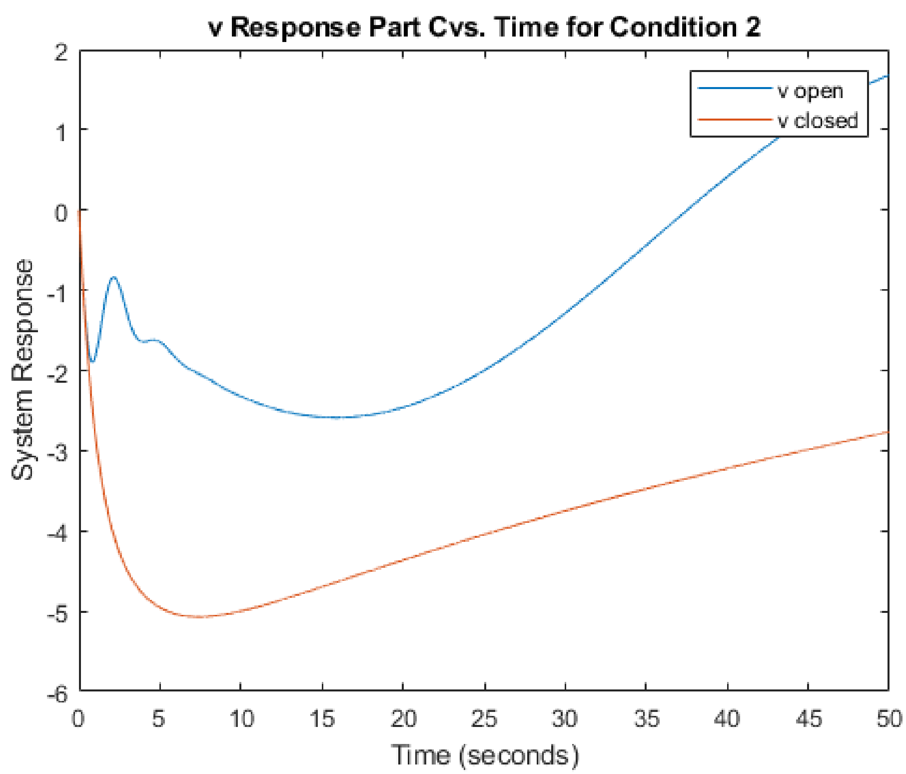

Figure 14: v response to initial condition 2 part 2

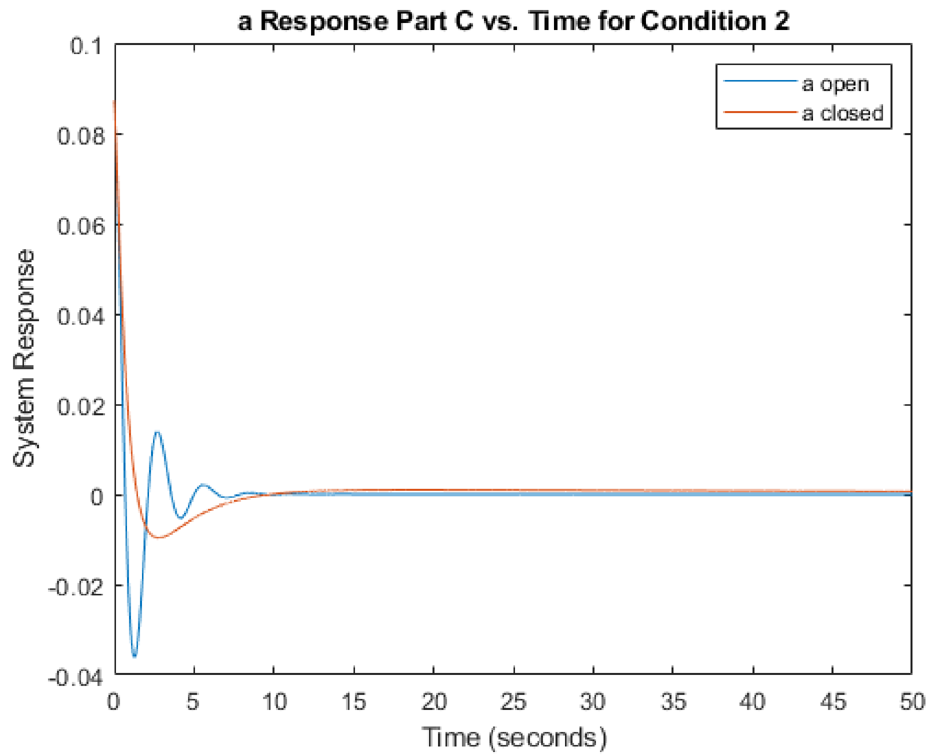

Figure 15: a response to initial condition 2 part 2

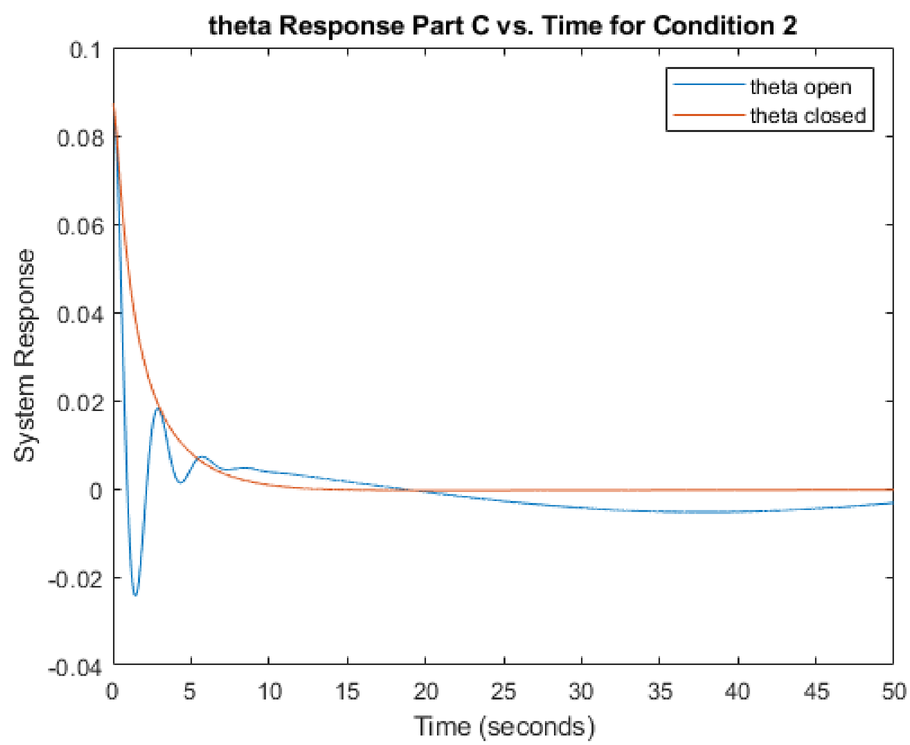

Figure 16: θ response to initial condition 2 part 2

The open loop response for q between part 1 and part 2 for initial condition set 1 don't appear to have many discrepancies. This makes sense because increasing the maximum allowed value should not change the open loop response. The control signal for part 2 seems to have a better transient response with less deviation than the control signal from part 1. Both experience excellent steady state values. The same story can be observed for the second set of initial conditions. The transient response is improved and less deviation is observed. I believe this result makes sense since the maximum allowed value for part 2 was decreased, resulting in more weight being distributed to this variable in the cost function.

The responses for v for condition set 1 between parts 1 and 2 showed no discernable difference between the closed and open loop response. The responses for condition set 2 however resulted in a much larger deviation in the transient response and a longer time to converge for part 2. It is interesting how changing a parameter concerning one variable can cause such a noticeable difference in the response of another.

The responses for a remained largely unchanged between parts 1 and 2 for both sets of initial conditions. The responses for theta however showed greater variation. For condition set 1, the transient response improved between parts 1 and 2 but for condition set 2, the closed loop response was more sluggish. It took more time to converge to a steady state value.

Conclusion

Optimal control solutions were found for an F-18 aircraft flying in level flight for two different cost functions. The optimal gains calculated for each controller showed much improved system response for every state variable for two different condition sets. It was shown that decreasing the maximum allowed value for a variable causes a sluggish response for some of the other variables. This assignment was very helpful in understanding optimal control and the effects of the controller on improving the system were interesting to observe.

```
>> type ctrlsspecialassign2

%Nathan Lutes
%ctrlsspecialassign2
%12/1/2018

%Part a
A=[-0.49 0.5e-4 -4.79 0; 0 -0.148e-1 -13.877 -32.2; 1 -0.19e-3 -0.836 0; 1 0 0 0];
B=[-8.743; -1.096; -0.01115; 0];
C=zeros(4,4);
D=zeros(4,1);
Q=[14.79 0 0 0; 0 2.7e-10 1.182e-6 0; 0 1.182e-6 0.00514 0; 0 0 0 10.41];
R=51.02+9.101e-7;
N=[0; 1.57e-8; 6.85e-5; 0];
T=linspace(0,50,10000);

%calculate open loop pole locations
sysO=ss(A, B, C, D);
Eopen=pole(sysO);
Eopen

%calculate gains and closed loop poles
[Kcl, Scl, Ecl]=lqr(A,B,Q,R,N);
Acl=A-B*Kcl;
syscl=ss(Acl,B,C,D);
Kcl
Ecl

%part B response plot
%Initial condition 1
[yopen,t,xopen]=initial(sysO,[0;0;5/57.3;0],T);
[ycl,tcl,xcl]=initial(syscl,[0;0;5/57.3;0],T);

%q
plot(T,xopen(:,1))
hold on
plot(T,xcl(:,1))
hold off
legend('q open','q closed')
xlabel('Time (seconds)')
ylabel('System Response')
title('q Response vs. Time for Condition 1')
%v
figure()
plot(T,xopen(:,2))
hold on
plot(T,xcl(:,2))
hold off
legend('v open','v closed')
xlabel('Time (seconds)')
ylabel('System Response')
title('v Response vs. Time for Condition 1')
%a
figure()
plot(T,xopen(:,3))
hold on
plot(T,xcl(:,3))
```

```
hold off
legend('a open','a closed')
xlabel('Time (seconds)')
ylabel('System Response')
title('a Response vs. Time for Condition 1')
%theta
figure()
plot(T,xopen(:,4))
hold on
plot(T,xcl(:,4))
hold off
legend('theta open','theta closed')
xlabel('Time (seconds)')
ylabel('System Response')
title('theta Response vs. Time for Condition 1')

%part B response plot
%Initial conditions 2

[yopen2,t2,xopen2]=initial(sysO,[0;0;5/57.3;5/57.3],T);
[ycl2,tcl2,xcl2]=initial(syscl,[0;0;5/57.3;5/57.3],T);

%q
figure()
plot(T,xopen2(:,1))
hold on
plot(T,xcl2(:,1))
hold off
legend('q open','q closed')
xlabel('Time (seconds)')
ylabel('System Response')
title('q Response vs. Time for Condition 2')
%v
figure()
plot(T,xopen2(:,2))
hold on
plot(T,xcl2(:,2))
hold off
legend('v open','v closed')
xlabel('Time (seconds)')
ylabel('System Response')
title('v Response vs. Time for Condition 2')
%a
figure()
plot(T,xopen2(:,3))
hold on
plot(T,xcl2(:,3))
hold off
legend('a open','a closed')
xlabel('Time (seconds)')
ylabel('System Response')
title('a Response vs. Time for Condition 2')
%theta
figure()
plot(T,xopen2(:,4))
hold on
plot(T,xcl2(:,4))
```

```
hold off
legend('theta open','theta closed')
xlabel('Time (seconds)')
ylabel('System Response')
title('theta Response vs. Time for Condition 2')

%Part C(a)
Qc=[41.62 0 0 0; 0 2.7e-10 1.182e-6 0; 0 1.182e-6 0.00514 0; 0 0 0 10.41];

%calculate open loop pole locations
sysOc=ss(A, B, C, D);
Eopenc=pole(sysOc);
Eopenc

%calculate gains and closed loop poles
[Kclc, Sclc, Eclc]=lqr(A,B,Qc,R,N);
Aclc=A-B*Kclc;
sysclc=ss(Aclc,B,C,D);
Kclc
Eclc

%Part C(B)
%initial conditions 1
[yopenc,tc,xopenc]=initial(sysOc,[0;0;5/57.3;0],T);
[yclc,tclc,xclc]=initial(sysclc,[0;0;5/57.3;0],T);

%q
figure()
plot(T,xopenc(:,1))
hold on
plot(T,xclc(:,1))
hold off
legend('q open','q closed')
xlabel('Time (seconds)')
ylabel('System Response')
title('q Response Part C vs. Time for Condition 1')
%v
figure()
plot(T,xopenc(:,2))
hold on
plot(T,xclc(:,2))
hold off
legend('v open','v closed')
xlabel('Time (seconds)')
ylabel('System Response')
title('v Response Part Cvs. Time for Condition 1')
%a
figure()
plot(T,xopenc(:,3))
hold on
plot(T,xclc(:,3))
hold off
legend('a open','a closed')
xlabel('Time (seconds)')
ylabel('System Response')
title('a Response Part C vs. Time for Condition 1')
%theta
```

```
figure()
plot(T,xopenc(:,4))
hold on
plot(T,xclc(:,4))
hold off
legend('theta open','theta closed')
xlabel('Time (seconds)')
ylabel('System Response')
title('theta Response Part C vs. Time for Condition 1')

%part B response plot
%Initial conditions 2

[yopen2c,t2c,xopen2c]=initial(sysOc,[0;0;5/57.3;5/57.3],T);
[ycl2c,tcl2c,xcl2c]=initial(sysclc,[0;0;5/57.3;5/57.3],T);

%q
figure()
plot(T,xopen2c(:,1))
hold on
plot(T,xcl2c(:,1))
hold off
legend('q open','q closed')
xlabel('Time (seconds)')
ylabel('System Response')
title('q Response Part C vs. Time for Condition 2')
%v
figure()
plot(T,xopen2c(:,2))
hold on
plot(T,xcl2c(:,2))
hold off
legend('v open','v closed')
xlabel('Time (seconds)')
ylabel('System Response')
title('v Response Part Cvs. Time for Condition 2')
%a
figure()
plot(T,xopen2c(:,3))
hold on
plot(T,xcl2c(:,3))
hold off
legend('a open','a closed')
xlabel('Time (seconds)')
ylabel('System Response')
title('a Response Part C vs. Time for Condition 2')
%theta
figure()
plot(T,xopen2c(:,4))
hold on
plot(T,xcl2c(:,4))
hold off
legend('theta open','theta closed')
xlabel('Time (seconds)')
ylabel('System Response')
title('theta Response Part C vs. Time for Condition 2')
>> ctrlsspecialassign2
```

```
Eopen =

  -0.6636 + 2.1800i
  -0.6636 - 2.1800i
  -0.0068 + 0.0765i
  -0.0068 - 0.0765i

Warning: The [Q N;N' R] matrix should be positive semi-definite. Type "help lqr" for more
information.
> In ss/lqr (line 82)
  In lqr (line 41)
  In ctrlsspecialassign2 (line 21)

Kcl =

   -0.5063   -0.0000    0.3464   -0.4488


Ecl =

  -2.6569 + 0.6262i
  -2.6569 - 0.6262i
  -0.4348 + 0.0000i
  -0.0152 + 0.0000i


Eopenc =

  -0.6636 + 2.1800i
  -0.6636 - 2.1800i
  -0.0068 + 0.0765i
  -0.0068 - 0.0765i

Warning: The [Q N;N' R] matrix should be positive semi-definite. Type "help lqr" for more
information.
> In ss/lqr (line 82)
  In lqr (line 41)
  In ctrlsspecialassign2 (line 128)

Kclc =

   -0.8567    0.0000    0.3863   -0.4494


Eclc =

   -7.2163
   -1.2301
   -0.3650
   -0.0152

>>
```