Nasser Alrashi

HW5.2.2:

I need to show $E[S_b^2] = \frac{\sigma^2(n-1)}{n}$

$\because\ S_b^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$ and $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$

I will tak E for both side

$\therefore\ E[S_b^2] = E\left[\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2\right]$

$E[S_b^2] = E\left[\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x} + u - u)^2\right]$

$E[S_b^2] = E\left[\frac{1}{n}\sum_{i=1}^{n}\left((x_i - u) - (\bar{x} - u)\right)^2\right]$

$E[S_b^2] = E\left[\frac{1}{n}\sum_{i=1}^{n}\left((x_i - u)^2 - 2(\bar{x} - u)(x_i - u) + (\bar{x} - u)^2\right)\right]$

$E[S_b^2] = E\left[\frac{1}{n}\sum_{i=1}^{n}(x_i - u)^2 - \frac{2}{n}(\bar{x} - u)\sum_{i=1}^{n}(x_i - u) + \frac{1}{n}(\bar{x} - u)^2\sum_{i=1}^{n} 1\right]$

$E[S_b^2] = E\left[\frac{1}{n}\sum_{i=1}^{n}(x_i - u)^2 - \frac{2}{n}(\bar{x} - u)\sum_{i=1}^{n}(x_i - u) + \frac{1}{n}(\bar{x} - u)^2 n\right]$

$E[S_b^2] = E\left[\frac{1}{n}\sum_{i=1}^{n}(x_i - u)^2 - \frac{2}{n}(\bar{x} - u)\sum_{i=1}^{n}(x_i - u) + (\bar{x} - u)^2\right]$

$E[S_b^2] = E\left[\frac{1}{n}\sum_{i=1}^{n}(x_i - u)^2 - \frac{2}{n}(\bar{x} - u)\sum_{i=1}^{n}(x_i - u) + (\bar{x} - u)^2\right]$ (1)

from above we know $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$

$\therefore\ \bar{x} - u = \frac{1}{n}\sum_{i=1}^{n} x_i - u$

$\bar{x} - u = \frac{1}{n}\left[\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} u\right] = \frac{1}{n}\sum_{i=1}^{n}(x_i - u)$ (2)

we will apply 2 in 1

$E[S_b^2] = E\left[\frac{1}{n}\sum_{i=1}^{n}(x_i - u)^2 - \frac{2}{n}(\bar{x} - u)\cdot n(\bar{x} - u) + (\bar{x} - u)^2\right]$

$E[S_b^2] = E\left[\frac{1}{n}\sum_{i=1}^{n}(x_i - u)^2 - 2(\bar{x} - u)^2 + (\bar{x} - u)^2\right]$

HW 5.2.2 Nasser Alrasbi

$$E[S_b^2] = E\left[\frac{1}{n}\sum_{i=1}^{n}(x_i-u)^2 - (\bar{x}-u)^2\right]$$

$$E[S_b^2] = E\left[\frac{1}{n}\sum_{i=1}^{n}(x_i-u)^2\right] - E[(\bar{x}-u)^2] \quad (3)$$

$$\because \sigma^2 = E[(x-u)^2] \quad (4)$$

we will apply 4 in 3

$$E[S_b^2] = \frac{1}{n}\left[\sum_{i-1}^{n}\sigma^2\right] - E[(\bar{x}-u)^2]$$

$$E[S_b^2] = \frac{n\sigma^2}{n} - E[(\bar{x}-u)^2] \quad (5)$$

$$\because E(\bar{x})^2 = \frac{\sigma^2}{n} + u^2 \quad (6)$$

we will apply 6 in 5

$$E[S_b^2] = \sigma^2 - \left[\frac{\sigma^2}{n}\right]$$

$$E[S_b^2] = \frac{\sigma^2 n - \sigma^2}{n} = \sigma^2\left[\frac{n-1}{n}\right]$$

$$\therefore \boxed{E[S_b^2] = \frac{\sigma^2}{n}(n-1)}$$