Homework 1

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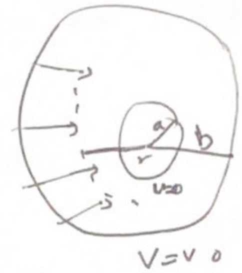
1

Method 1 : Gauss' law



$$\oint \vec{E} \cdot ds = \frac{Q_{encl}}{\varepsilon_0}$$

$$E \, 4\pi r^2 = \frac{Q}{\varepsilon_0}$$

$$E = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r} \Rightarrow \boxed{E = \frac{V_0}{r^2}\left(\frac{ab}{a-b}\right)}$$

$$V(b) - V(a) = -\int_a^b \vec{E} \cdot \vec{dl}$$

$$V_0 - 0 = -\int_a^b \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} dr = \frac{-Q}{4\pi\varepsilon_0} \int_a^b r^{-2} dr = \frac{-Q}{4\pi\varepsilon_0} (-1) \frac{1}{r} \Big|_a^b$$

$$V_0 = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{b} - \frac{1}{a}\right) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{a-b}{ab}\right) \Rightarrow Q = 4\pi V_0 \varepsilon_0 \left(\frac{ab}{a-b}\right)$$

$$V(r) - 0 = -\int_a^r \vec{E} \cdot \vec{dl} = -\int_a^r \frac{1}{4\pi\varepsilon_0} \frac{Q}{r'^2} dr' = \frac{Q}{4\pi\varepsilon_0} \frac{1}{r'} \Big|_a^r$$

$$V(r) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{a}\right) \qquad Q = 4\pi\varepsilon_0 V_0 \left(\frac{ab}{a-b}\right)$$

$$\boxed{V(r) = V_0 \left(\frac{ab}{a-b}\right)\left(\frac{1}{r} - \frac{1}{a}\right)}$$

Method 2: Laplace

$$\nabla^2 V = 0 \qquad \frac{\partial V}{\partial \theta} = \frac{\partial V}{\partial \phi} = 0$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0$$

$$r^2 \frac{\partial V}{\partial r} = C_1$$

$$\frac{\partial V}{\partial r} = C_1 r^{-2}$$

$$V(r) = \frac{-C_1}{r} + C_2$$

- Apply Boundary conditions

$$V(a) = 0 = C_2 - \frac{C_1}{a} \qquad V(b) = V_0 = C_2 - \frac{C_1}{b}$$

$$C_2 = \frac{C_1}{a} \qquad V_0 = \frac{C_1}{a} - \frac{C_1}{b} = C_1\left(\frac{b-a}{ab}\right)$$

$$C_1 = \left(\frac{ab}{b-a}\right) V_0$$

$$C_2 = \left(\frac{ab}{b-a}\right)\frac{V_0}{a}$$

$$V(r) = -\left(\frac{ab}{b-a}\right)\frac{V_0}{r} + \left(\frac{ab}{b-a}\right)\frac{V_0}{a}$$

$$V(r) = V_0\left(\frac{ab}{b-a}\right)\left(\frac{1}{a} - \frac{1}{r}\right)$$

$$\boxed{V(r) = V_0\left(\frac{ab}{a-b}\right)\left(\frac{1}{r} - \frac{1}{a}\right)}$$

$$\vec{E} = -\nabla V$$

$$E_r = -\frac{\partial V}{\partial r} = \frac{\partial}{\partial r}\left(V_0\left(\frac{ab}{a-b}\right)\left(\frac{1}{r} - \frac{1}{a}\right)\right)$$

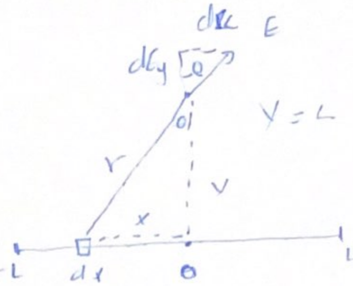
$$E_r = -V_0\left(\frac{ab}{a-b}\right)(-1)\frac{1}{r^2}$$

$$E_r = \frac{V_0}{r^2}\left(\frac{ab}{a-b}\right)$$

$$\boxed{E_r = \frac{V_0}{r^2}\left(\frac{ab}{a-b}\right)\hat{r}}$$

② Exact

$\cancel{E}\ d\vec{E}$



$\cancel{dE}_y \quad dE_y = \underbrace{\frac{1}{4\pi\varepsilon_0} \frac{\lambda_0 \, dx}{r^2}} \cos\theta$

$$dE_y = \frac{1}{4\pi\varepsilon_0} \frac{\lambda_0 \, dx \, y}{r^3} \quad \cancel{\cos\theta} \quad \cos\theta = \frac{y}{r} \Rightarrow y = L$$

$$dE_y = \frac{\lambda_0 L}{4\pi\varepsilon_0} \frac{dx}{r^3} = \frac{\lambda_0 L}{4\pi\varepsilon_0} \frac{dx}{(x^2+L^2)^{3/2}}$$

$x = L \tan u$

$dx = L\sec^2 u \, du$

$$E_y = \int_{-L}^{L} \frac{\lambda_0 L}{4\pi\varepsilon_0} \frac{dx}{(x^2+L^2)^{3/2}} = \frac{\lambda_0 L}{4\pi\varepsilon_0} \int_{-L}^{L} \frac{dx}{(x^2+L^2)^{3/2}}$$

$$= \frac{\lambda_0 L}{4\pi\varepsilon_0} \int_{-L}^{L} \frac{L \sec^2 u \, du}{(L^2 \tan^2 u + L^2)^{3/2}} = \frac{\lambda_0 L}{4\pi\varepsilon_0} \int_{-L}^{L} \frac{L \quad \sec^2 u \, du}{L^2 \quad (\tan^2 u + 1)^{3/2}}$$

$(\tan^2 u + 1) \to \sec^2 u$

$$= \frac{\lambda_0}{4\pi\varepsilon_0 L} \int_{-L}^{L} \frac{\sec^2 u \, du}{(\sec^2 u)^{3/2}} = \frac{\lambda_0}{4\pi\varepsilon_0 L} \int_{-L}^{L} \frac{du}{\sec u}$$

$$= \frac{\lambda_0}{4\pi\varepsilon_0 L} \int_{-L}^{L} \cos u \, du$$

$$= \frac{\lambda_0}{4\pi\varepsilon_0 L} \sin u \Big|_{x=-L}^{x=L} = \frac{\lambda_0}{4\pi\varepsilon_0 L} \sin\left(\arctan\left(\frac{x}{L}\right)\right) \Big|_{-L}^{L}$$

$$\sin(\arctan(x)) = \frac{x}{\sqrt{1+x^2}}$$

$$E_y = \frac{\lambda_0}{4\pi\varepsilon_0 L} \left. \frac{x/L}{\sqrt{1+x^2/L^2}} \right|_{-L}^{L}$$

$$= \frac{\lambda_0}{4\pi\varepsilon_0 L} \left. \frac{x}{\sqrt{L^2+x^2}} \right|_{-L}^{L}$$

$$E_y = \frac{\lambda_0}{4\pi\varepsilon_0 L} \left[\frac{L}{\sqrt{L^2+L^2}} + \frac{+L}{\sqrt{L^2+L^2}} \right]$$

$$E_y = \frac{\lambda_0}{4\pi\varepsilon_0 L} \left[\frac{2L}{\sqrt{2L^2}} \right.$$

$$\boxed{E_y = \frac{2\lambda_0}{4\pi\varepsilon_0 \sqrt{2} L}}$$

Approximation

$\cdot\ y = L$

$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$

$-L=-2\Delta \quad -\Delta \quad 0 \quad \Delta \quad 2\Delta = L$

each point charge $Q = \dfrac{2L\lambda_0}{5}$

$$E_y = E_{y_0} + 2E_{y_1} + 2E_2$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{Q}{L^2} + 2\left[\frac{1}{4\pi\varepsilon_0}\frac{QL}{(\Delta^2+L^2)^{\frac{3}{2}}}\right) + 2\left(\frac{1}{4\pi\varepsilon_0}\frac{QL}{(4\Delta^2+L^2)^{3/2}}\right)$$

For n charges (n is odd)

$$E_y = \frac{1}{4\pi\varepsilon_0}\frac{Q}{L^2} + \sum_{i=1}^{\frac{n-1}{2}} 2\left(\frac{1}{4\pi\varepsilon_0}\frac{QL}{((i\Delta)^2+L^2)^{\frac{3}{2}}}\right) \qquad \boxed{Q = \frac{2L\lambda_0}{n}}$$

$$E_y = \frac{1}{4\pi\varepsilon_0}\left(\frac{2L\lambda_0}{n}\right)\left[\frac{1}{L^2} + \sum_{m=1}^{\frac{n-1}{2}}\frac{2L}{((m\Delta)^2+L^2)^{\frac{3}{2}}}\right]$$

$\boxed{\text{when } n=51 \text{ charges, the error drops below } 1\%}$