Midterm
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1) Region 1 $(a < r < 2a)$

$$\nabla^2 V = \frac{\rho_0}{\epsilon_0}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = \frac{\rho_0}{\epsilon_0}$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = \frac{\rho_0 r^2}{\epsilon_0}$$

$$r^2 \frac{\partial V}{\partial r} = \frac{\rho_0 r^3}{3\epsilon_0} + A_1$$

$$\frac{\partial V}{\partial r} = \frac{\rho_0 r}{3\epsilon_0} + A_1 r^{-2}$$

$$\boxed{V_1(r) = \frac{\rho_0 r^2}{6\epsilon_0} - \frac{A_1}{r} + B_1}$$

Region 2 $(2a < r < 3a)$

$$\nabla^2 V = 0$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0$$

$$r^2 \frac{\partial V}{\partial r} = A_2$$

$$\frac{\partial V}{\partial r} = A_2 r^{-2}$$

$$V_2(r) = -\frac{A_2}{r} + B_2$$

Inside central conductor

$(r < a)$

$$\oint \vec{E}\cdot d\vec{a} = \frac{Q_{encl}}{\epsilon_0} = 0$$

$$\Downarrow$$

$$E = 0 \Rightarrow \boxed{V = 0}$$

outside spheres $(r > 3a)$

$$\oint \vec{E}\cdot d\vec{a} = \frac{Q_{encl}}{\epsilon_0} = 0$$

$$\boxed{V = 0}$$

BCs
① $V_1(a)=0$
② $V_2(3a)=V_0$
③ $V_1(2a)=V_2(2a)$
④ $D_1(2a)=D_2(2a)$

① $V_1(a)=0$

$$V_1(a)=0=\frac{\rho_0 a^2}{6\epsilon_0}-\frac{A_1}{a}+B_1$$

$$B_1=\frac{A_1}{a}-\frac{\rho_0 a^2}{6\epsilon_0}$$

$$V_1(r)=\frac{\rho_0 r^2}{6\epsilon_0}-\frac{A_1}{r}-\frac{\rho_0 a^2}{6\epsilon_0}+\frac{A_1}{a}=\frac{\rho_0}{6\epsilon_0}(r^2-a^2)+A_1\left(\frac{1}{a}-\frac{1}{r}\right)$$

$$V_2(r)=-\frac{A_2}{r}+B_2$$

② $V_2(3a)=V_0$

$$V_2(3a)=V_0=\frac{-A_2}{3a}+B_2$$

$$B_2=V_0+\frac{A_2}{3a}$$

$$\rightarrow V_2(r)=\frac{-A_2}{r}+V_0+\frac{A_2}{3a}$$

$$V_2(r)=V_0+A_2\left(\frac{1}{3a}-\frac{1}{r}\right)$$

$$V_1(r)=\frac{\rho_0}{6\epsilon_0}(r^2-a^2)+A_1\left(\frac{1}{a}-\frac{1}{r}\right)$$

$$V_2(r)=V_0+A_2\left(\frac{1}{3a}-\frac{1}{r}\right)$$

③ $V_1(2a)=V_2(2a)$

$$\frac{\rho_0}{6\epsilon_0}(4a^2-a^2)+A_1\left(\frac{1}{a}-\frac{1}{2a}\right)=V_0+A_2\left(\frac{1}{3a}-\frac{1}{2a}\right)$$

$$\frac{\rho_0}{6\epsilon_0} 3a^2 + A_1 \frac{1}{2a} = V_0 + A_2\left(\frac{-1}{6a}\right)$$

$$\frac{A_2}{6a} = V_0 - \frac{\rho_0 a^2}{2\epsilon_0} - \frac{A_1}{2a}$$

$$A_2 = 6aV_0 - \frac{3\rho_0 a^3}{\epsilon_0} - 3A_1$$

$$V_2(r) = V_0 + \left(6aV_0 - \frac{3\rho_0 a^3}{\epsilon_0} - 3A_1\right)\left(\frac{1}{3a} - \frac{1}{r}\right)$$

$$V_1(r) = \frac{\rho_0}{6\epsilon_0}(r^2 - a^2) + A_1\left(\frac{1}{a} - \frac{1}{r}\right)$$

$$V_2(r) = V_0 + \left(6aV_0 - \frac{3\rho_0 a^3}{\epsilon_0} - 3A_1\right)\left(\frac{1}{3a} - \frac{1}{r}\right)$$

④ $D_1(2a) = D_2(2a)$ $\qquad D = \epsilon E$

$$\epsilon_0 E_1(2a) = 2\epsilon_0 E_2(2a)$$

$$\left.\frac{\partial V_1}{\partial r}\right|_{2a} = 2\left.\frac{\partial V_2}{\partial r}\right|_{2a}$$

$$\left.\frac{\rho_0}{6\epsilon_0}(2r) + A_1\frac{1}{r^2}\right|_{2a} = 2\left.\left[\left(6aV_0 - \frac{3\rho_0 a^3}{\epsilon_0} - 3A_1\right)\frac{1}{r^2}\right]\right|_{2a}$$

$$\frac{\rho_0}{6\epsilon_0}(4a) + \frac{A_1}{4a^2} = \frac{1}{2a^2}\left(6aV_0 - \frac{3\rho_0 a^3}{\epsilon_0} - 3A_1\right)$$

$$4a^2\left[\frac{\rho_0}{6\epsilon_0}(4a)+\frac{A_1}{4a^2}\right]=4a^2\left[\frac{1}{2a^2}\left(6aV_0-\frac{3\rho_0 a^3}{\epsilon_0}-3A_1\right)\right]$$

$$\frac{16a^3\rho_0}{6\epsilon_0}+A_1=12aV_0-\frac{6\rho_0 a^3}{\epsilon_0}-6A_1$$

$$7A_1=12aV_0-\frac{6\rho_0 a^3}{\epsilon_0}-\frac{16a^3\rho_0}{6\epsilon_0}$$

$$A_1=\frac{12}{7}aV_0-\frac{6}{7}\frac{\rho_0 a^3}{\epsilon_0}-\frac{16a^3\rho_0}{42\epsilon_0}$$

$$A_1=\frac{12aV_0}{7}-\frac{\rho_0 a^3}{\epsilon_0}\left(\frac{6}{7}+\frac{8}{21}\right)$$

$$A_1=\frac{12aV_0}{7}-\frac{\rho_0 a^3}{\epsilon_0}\left(\frac{26}{21}\right)$$

$\therefore$

$$\boxed{\begin{aligned}
V_1(r)&=\frac{\rho_0}{6\epsilon_0}(r^2-a^2)+A_1\left(\frac{1}{a}-\frac{1}{r}\right)\\
V_2(r)&=V_0+\left(6V_0-\frac{3\rho_0 a^3}{\epsilon_0}-3A_1\right)\left(\frac{1}{3a}-\frac{1}{r}\right)\\
&\because \epsilon_0=8.85\times10^{-12},\ a=1,\ V_0=1,\ \rho_0=\frac{V_0\epsilon_0}{a^2}=\epsilon_0,\\
A_1&=\frac{12aV_0}{7}-\frac{\rho_0 a^3}{\epsilon_0}\left(\frac{26}{21}\right)=0.47619
\end{aligned}}$$

2) $\nabla^2 V = 0$

$$\therefore \frac{\partial^2 V}{\partial r^2} = \frac{V(r+h) - 2V(r) + V(r-h)}{h^2}$$

$$\therefore \frac{\partial V}{\partial r} = \frac{V(r+h) - V(r-h)}{2h}$$

from homework3 we got

$$\therefore V(r) = \left(\frac{1}{2} + \frac{h}{2r}\right) V(r+h) + \left(\frac{1}{2} - \frac{h}{2r}\right) V(r-h)$$

$$\nabla^2 V = \frac{\rho_0}{\epsilon_0}$$

$$\frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = \frac{\rho_0}{\epsilon_0}$$

$$\frac{1}{r^2}\left(2r \frac{\partial V}{\partial r} + r^2 \frac{\partial^2 V}{\partial r^2}\right) = \frac{\rho_0}{\epsilon_0}$$

$$\frac{2}{r} \frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial r^2} = \frac{\rho_0}{\epsilon_0}$$

$$\frac{2}{r} \frac{1}{2h}\left(V(r+h) - V(r-h)\right) + \frac{1}{h^2}\left(V(r+h) - 2V(r) + V(r-h)\right) = \frac{\rho_0}{\epsilon_0}$$

$$\frac{1}{rh} V(r+h) - \frac{1}{rh} V(r-h) + \frac{1}{h^2} V(r+h) + \frac{2}{h^2} V(r) + \frac{1}{h^2} V(r-h) = \frac{\rho_0}{\epsilon_0}$$

$$\frac{2}{h^2} V(r) = V(r+h)\left(\frac{1}{rh} + \frac{1}{h^2}\right) + V(r-h)\left(\frac{1}{h^2} - \frac{1}{rh}\right) - \frac{\rho_0}{\epsilon_0}$$

$$\boxed{V(r) = \frac{1}{2} V(r+h)\left(\frac{h}{r} + 1\right) + \frac{1}{2} V(r-h)\left(1 - \frac{h}{r}\right) - \frac{\rho_0 h^2}{2\epsilon_0}}$$

At boundary

$$\epsilon_1 = \epsilon_0 \ , \ \epsilon_2 = 2\epsilon_0$$

$$D_1(2a) = D_2(2a)$$

$$\epsilon_0 E_1(2a) = 2\epsilon_0 E_2(2a)$$

$$\frac{\partial V_1}{\partial r}\Big|_{2a} = 2\frac{\partial V_2}{\partial r}\Big|_{2a}$$

$$V_1(r-h) \approx V_1(r) - h\frac{\partial V_1}{\partial r}$$

$$V_2(r+h) \approx V_2(r) + h\frac{\partial V_2}{\partial r}$$

$$\Downarrow$$

$$\begin{cases} \dfrac{\partial V_1}{\partial r} = \dfrac{V_1(r) - V(r-h)}{h} \\ \dfrac{\partial V_2}{\partial r} = \dfrac{V_2(r+h) - V_2(r)}{h} \end{cases}$$

$$\frac{V_1(r) - V_1(r-h)}{h} = \frac{2(V_2(r+h) - V_2(r))}{h}$$

$$\because V_1(2a) = V_2(2a)$$

$$V(r) - V_1(r-h) = 2V_2(r+h) - 2V(r)$$

$$3V(r) = 2V_2(r+h) + V_1(r-h)$$

$$\boxed{V(r) = \frac{2V_2(r+h) + V_1(r-h)}{3}\Big|_{r=2a}}$$