STAT3553 - A2

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Question 1

We want to show that

$$\frac{MS_{reg}}{MS_{res}} : F_0 \sim F_{1,n-2}.$$

From class we know that

$$\frac{\chi_n^2/n}{\chi_m^2/m} \sim F_{n,m}.$$

Thus we want to show that under the null hypothesis,

$$MS_{reg} \sim \chi_1^2$$

and

$$MS_{res} \sim \chi_{n-2}^2.$$

Starting with MS_{res}, we have,

$$\epsilon_i \sim N(0, \sigma^2)$$

.

Thus normalizing we get,

$$\frac{\epsilon_i}{\sigma} \sim N(0, 1).$$

Now,

$$\frac{\epsilon_i^2}{\sigma^2} \sim \chi_1^2$$

and it follows that

$$\frac{\sum_{i=1}^n \epsilon_i^2}{\sigma^2} \sim \chi_n^2.$$

From here, we algebraically manipulate the definition of ϵ_i to get our desired end result. Since $\epsilon_i = yi - \hat{y}_i$, we get

$$\frac{\sum (y_i - \hat{y}_i)^2}{\sigma^2}.$$

We also know from class that

$$MS_{res} = \hat{\sigma}^2 = \frac{RSS}{n-2} = \frac{\sum(y_i - \hat{y}_i)^2}{n-2}.$$

So we can rewrite $\sum(y_i - \hat{y}_i)^2$ as $\hat{\sigma}^2(n-2)$.
Finally, we get that

$$\frac{\hat{\sigma}^2(n-2)}{\sigma^2} \sim \chi^2_{n-2}.$$

Now we want to show that under the null hypothesis $MS_{reg} \sim \chi^2_1$.
We know from class that

$$\hat{\beta}_1 \sim N(\beta_1, \sigma^2/Sxx).$$

Standardizing we get

$$Z = \frac{\hat{\beta}_1 - \beta_1}{SE(\hat{\beta}_1)} \sim N(0,1).$$

Under the null hypothesis $\beta_1 = 0$, so we can rewrite the above as

$$Z = \frac{\hat{\beta}_1}{SE(\hat{\beta}_1)}.$$

From class, we know that where $Z : N(0,1)$, $Z^2 \sim \chi^2_1$.
We square Z to get,

$$Z^2 = \frac{(\hat{\beta}_1)^2}{(SE(\hat{\beta}_1))^2} = \frac{(\hat{\beta}_1)^2}{Var(\hat{\beta}_1)} = \frac{\hat{\beta}_1^{\,2} Sxx}{\sigma^2}$$

We proved in class and on the midterm that $MS_{reg} = SS_{reg} = \hat{\beta}_1^{\,2} Sxx$.
Thus we can conclude that

$$Z^2 = \frac{SS_{reg}}{\sigma^2} \sim \chi^2_1.$$

Finally,

$$\frac{\frac{SS_{reg}}{\sigma^2}/1}{\frac{MS_{res}(n-2)}{\sigma^2}/n-2} = \frac{MS_{reg}}{MS_{res}} \sim F_0 : F_{1,n-2}$$

Question 2

I will start by simplifying TSS and then I will simplify $RSS + SS_{reg}$. I'll finish by showing that they are equal.

$$TSS = (Y - \frac{1}{n}11^TY)^T(Y - \frac{1}{n}11^TY)$$

I start by distributing, making sure to follow an matrix properties:

$$= Y^TY - Y^T\frac{1}{n}11^TY - Y^T11^T\frac{1}{n}Y + Y^T11^T\frac{1}{n}\cdot\frac{1}{n}11^TY$$

$$= Y^TY - Y^T\frac{1}{n}11^TY - Y^T11^T\frac{1}{n}Y + Y^T11^T\frac{1}{n^2}11^TY$$

We can take out any scalars (ie. $\frac{1}{n}$)

$$= Y^TY - \frac{1}{n}Y^T11^TY - \frac{1}{n}Y^T11^TY + \frac{1}{n^2}Y^T11^T11^TY$$

$$= Y^TY - \frac{2}{n}Y^T11^TY + \frac{1}{n^2}Y^T11^T11^TY$$

From the hint, we can turn 1^T1 into n.

$$= Y^TY - \frac{2}{n}Y^T11^TY + \frac{1}{n^2}Y^T11^T11^TY = Y^TY - \frac{2}{n}Y^T11^TY - \frac{n}{n^2}Y^T11^TY = \boxed{Y^TY - \frac{1}{n}Y^T11^TY}$$

Thus,

$$TSS = \boxed{Y^TY - \frac{1}{n}Y^T11^TY}.$$

$$RSS + SS_{reg} = Y^TY - Y^THY + (HY - \frac{1}{n}11^TY)^T(Y - \frac{1}{n}11^TY)$$

I distribute the above following any necessary matrix rules:

$$= Y^TY - Y^THY + Y^TH^THY - Y^TH^T\frac{1}{n}11^TY - Y^T11^T\frac{1}{n}HY + Y^T11^T\frac{1}{n}\frac{1}{n}11^TY$$

By the symmetric property of the hat matrix, $H^T = H$ and idempotency of the hat matrix $HH = H$, we can simplify further. I also move any scalars out front. I box out any changes I make for clarity.

$$= Y^TY - Y^THY + (Y^T\boxed{H^TH}Y - Y^TH^T\boxed{\frac{1}{n}}11^TY - Y^T11^T\boxed{\frac{1}{n}}HY + Y^T11^T\boxed{\frac{1}{n}\frac{1}{n}}11^TY)$$

$$= Y^TY \boxed{-Y^THY + Y^THY} - \frac{1}{n}Y^TH^T11^TY - \frac{1}{n}Y^T11^THY + \frac{1}{n^2}Y^T11^T11^TY$$

I can clean up my equation a little bit

$$= Y^TY - \frac{1}{n}Y^TH^T11^TY - \frac{1}{n}Y^T1\boxed{1^THY} + \frac{1}{n^2}Y^T1\boxed{1^T1}1^TY$$

Using the hint again that $1^T1 = n$ and that $1^THY = 1^TY$,

$$= Y^TY - \frac{2}{n}Y^TH^T11^TY + \frac{n}{n^2}Y^T11^TY$$

$$= Y^TY - \frac{2}{n}Y^TH^T11^TY + \frac{1}{n}Y^T11^TY$$

$$= Y^TY - \frac{1}{n}Y^TH^T11^TY$$

Note that $Y^TH^T1 = (1^THY)^T$

$$= Y^TY - \frac{1}{n}(1^THY)^T1^TY$$

$$= Y^TY - \frac{1}{n}(1^TY)^T1^TY$$

$$= Y^TY - \frac{1}{n}Y^T11^TY$$

$$RSS + SS_{reg} = \boxed{Y^TY - \frac{1}{n}Y^T11^TY}.$$

Therefore, we have shown that

$$TSS = RSS + SS_{reg}.$$