Artificial Intelligence

CS4365 --- Fall 2022

Non-Zero-Sum Games

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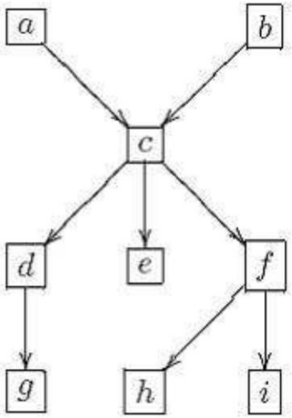
Bayes Net

- P(c|g, h)



at node a:

Prob(a)
0.7

at node b:

Prob(b)
0.8

at node c:

a	b	Prob(c)
0	0	0.1
0	1	0.6
1	0	0.7
1	1	0.9

at node d:

c	Prob(d)
0	0.6
1	0.8

at node e:

c	Prob(e)
0	0.1
1	0.7

at node f:

c	Prob(f)
0	0.8
1	0.3

at node g:

d	Prob(g)
0	0.15
1	0.75

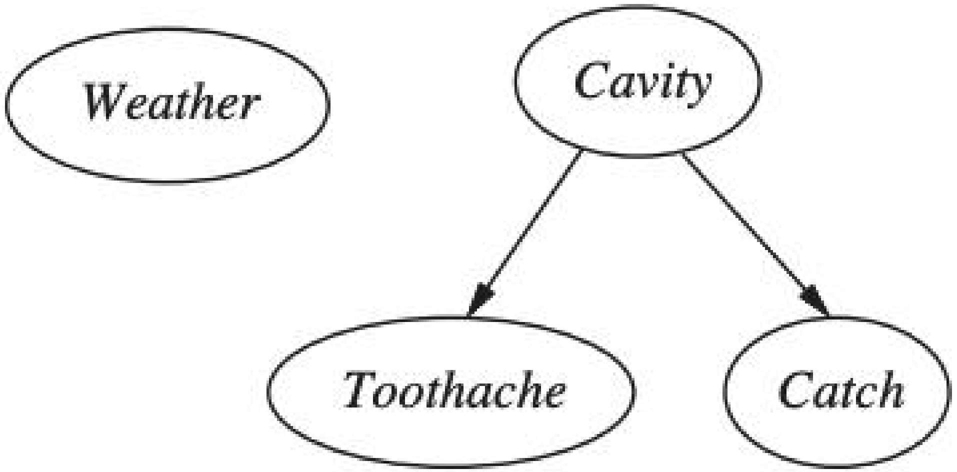
at node h:

f	Prob(h)
0	0.25
1	0.65

at node i:

f	Prob(i)
0	0.35
1	0.55

Bayes Net

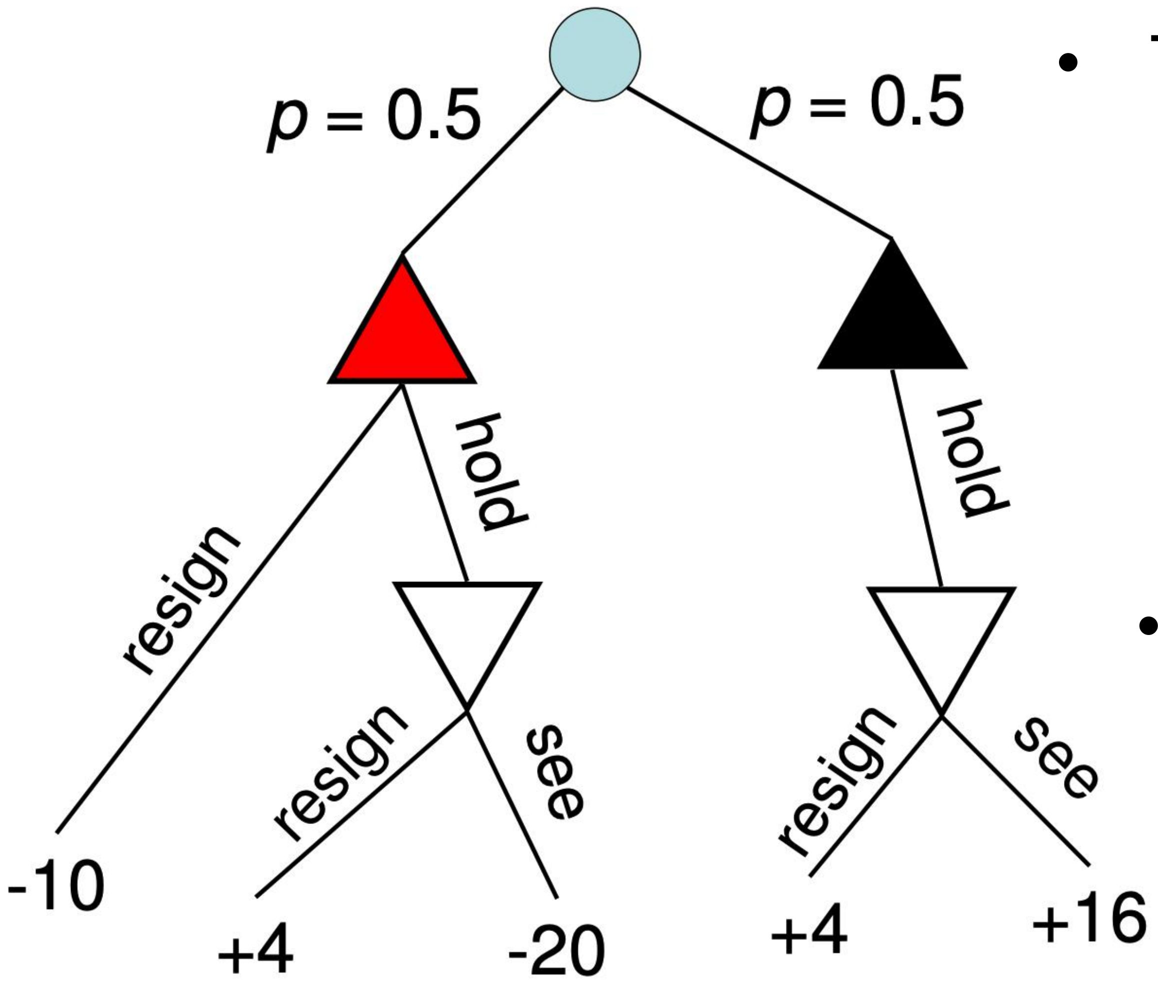


- P(Toochache, Catch) = P(Toochache)P(Catch)?
- If true, then P(Toochache) = P(Toochache|Catch)

Another Example: Poker

- Players A and B play with two types of cards: Red and Black
- Player A is dealt one card at random (50% probability of being Red)
- If the card is red, Player A may *resign* and loses $10
- Or Player A may *hold*
  - B may then *resign* A wins $4
  - B may *see*
    - A loses $20 if the card is Red
    - A wins $16 otherwise

Another Example: Poker



- The game is non-deterministic

- Hidden information: Player B cannot know which of these 2 states it's in

Another Example: Poker

Player B

Player A

	Resign	See
Resign	-5	-5
Hold	4	-2

- Generate the matrix form of the game (be careful: It's not a deterministic game)
- Find the expected payoff for Player A
- Find the optimal mixed strategy

Types of Games

- **Assumptions so far**
  - **Two-player game**: Players A and B.
  - **Perfect information**: Both players see all the states and decisions. Each decision is made sequentially.
  - **Zero-sum**: Player’s A gain is exactly equal to player B’s loss

- We are going to eliminate the third assumption
  - Non zero-sum game

Prisoner's Dilemma

- Two persons (A and B) are arrested with enough evidence for a minor crime, but not enough for a major crime.
- If they both confess to the crime, they each know that they will serve 5 years in prison
- If only one of them testify, he will go free and the other prisoner will serve 10 years.
- If neither of them confess, they'll each spend 1 year in prison

Matrix Normal Form for Non-Zero-Sum Games

Player B

Player A

	Testify	Refuse
Testify	-5,-5	0,-10
Refuse	-10,0	-1,-1

Why This Example?

- This example models a huge variety of situations in which participants have similar rewards as in this game.

- Duopoly:
  - Two firms compete for producing the same product and they both try to maximize profit. They can set two prices, “High” and “Low”. If both firms choose Low, they both make a profit of $1000. If they both choose High, they both make a lower profit of $600. Otherwise, the High firm makes a profit of $1200 and the Low firm takes a loss of $200.

Dominant Strategies

Player B

Player A

	Testify	Refuse
Testify	-5,-5	0,-10
~~Refuse~~	~~-10,0~~	~~-1,-1~~

- Player A's payoff is greater if he testifies than if he refuses, no matter what strategy B chooses
- Therefore Player A does not need to consider strategy "refuse" since it cannot possibly yield a higher payoff

Dominant Strategies

Player B

Player A

	Testify	~~Refuse~~
Testify	-5,-5	~~0,-10~~
~~Refuse~~	~~-10,0~~	~~-1,-1~~

- The same reasoning can be applied to Player B:
  - Player B’s payoff is greater if he testifies than if he refuses, no matter what strategy A chooses
  - Therefore Player B does not need to consider strategy “refuse” since it cannot possibly yield a higher payoff

Dominant Strategies

Player B

Player A

	Testify	~~Refuse~~
Testify	-5,-5	~~0,-10~~
~~Refuse~~	~~-10,0~~	~~-1,-1~~

- A strategy strictly dominates if it yields a higher payoff than any other strategy for every one of the possible actions of the other player.
- Dominant strategy equilibrium.

Dominant Strategies

Player B

Player A

	Testify	~~Refuse~~
Testify	-5,-5	~~0,-10~~
~~Refuse~~	~~-10,0~~	~~-1,-1~~

- Can the players get the higher payoff?
  - Yes!
  - Each player refuses
  - Cannot be achieved by rational play

Side Note: More than 2 Players?

- The formalism extends directly to more than 2 players.
- If we have n players, we need to define n payoff functions u_i, $i=1,..,n$.
- Payoff function u_i maps a tuple of n strategies to the corresponding payoff for player i
- $u_i(s_1 ,..,s_n)$ = payoff for player i if players 1,..,n use pure strategy $s_1,\ldots,s_n$.
- Everything else (definition of dominating strategies, etc. remains the same)

Nash Equilibrium

- A tuple of pure strategies (s_1*,s_2*,..,s_n*) is a pure equilibrium if, for all i's:

$$u_i\left(s_1^*,\cdots,s_{i-1}^*,s_i\ ,s_{i+1}^*,\cdots,s_n^*\right)\leq u_i\left(s_1^*,\cdots,s_{i-1}^*,s_i^*,s_{i+1}^*,\cdots,s_n^*\right)$$

- In words: Player i cannot find a better strategy than s_i* if the other player use the remaining strategies in the equilibrium

- Technically, called a pure Nash Equilibrium (NE)

Nash Equilibrium

- A dominant strategy equilibrium is always a Nash equilibrium

- A Nash equilibrium is not always a dominant strategy equilibrium

	$Acme{:}bluray$	$Acme{:}dvd$
$Best{:}bluray$	$A = +9, B = +9$	$A = -4, B = -1$
$Best{:}dvd$	$A = -3, B = -1$	$A = +5, B = +5$

- Nash equilibrium can be thought as a local optimum

More Formal Definition

- So, we’ve generalized our concepts for solving games to non zero-sum games $\rightarrow$ NEs

- Basic questions:
    - – Is there always a NE?
    - – Is it unique?

Example with Multiple NEs

	Left	Right
Left	+1,+1	-1,-1
Right	-1,-1	+1,+1

- Two vehicles are driving toward each other. They have 2 choices: Move right or move left
- Why is having multiple NEs a problem?
  - The player does not know which one to play

Example with Multiple NEs

	Hockey	Movie
Hockey	+2,+1	0,0
Movie	0,0	+1,+2

- Two friends have different tastes
- A likes to watch hockey games but B prefers to go see a movie. Neither likes to go to his preferred choice alone; each would rather go the other’s preferred choice rather than go alone to its own.

Example with No Pure NE

	I	II
I	0,1	1,0
II	1,0	0,1

- Even very simple games may not have a pure strategy equilibrium
  - This is not surprising since we saw earlier that we had a similar problem with zero-sum games, which did not necessarily have a pure strategy solution
- Solution: Same trick as with zero-sum games → Allow the players to randomize and to use mixed strategies

Mixed Strategy Equilibrium

- The concept of equilibrium can be extended to mixed strategies.
- In that case, a mixed strategy for each player i is a vector of probabilities $\boldsymbol{p}_i = (p_{ij})$, such that player i chooses pure strategy j with probability p_{ij}
- A set of mixed strategies $(\boldsymbol{p}^*_1,..,\boldsymbol{p}^*_n)$ is a mixed strategy equilibrium if player i (for any i) gets a lower payoff by changing $\boldsymbol{p}^*_i$ to any other mixed strategy $\boldsymbol{p}_i$

Example

	Hockey	Movie
Hockey	+2,+1	0,0
Movie	0,0	+1,+2

- Let A choose Hockey with probability p and B choose Hockey with probability q
- The expected payoff for Player A is:

  $u_A = (+2)xpq + (+1)x(1\text{-}p)x(1\text{-}q) = 1\text{-}p\text{-}q+3pq$
- The expected payoff for Player B is:

  $u_B = (+1)xpq + (+2)x(1\text{-}p)x(1\text{-}q) = 2\text{-}2p\text{-}2q+3pq$
- At the equilibrium, the derivative of u_A with respect to p is zero
- (because u_A (p^*,q^*) is greater than u_A for any other value (p,q^*)), therefore: $3q^*-1 = 0$ $q^* = 1/3$
- Similarly, the derivative of u_B with respect to q must be 0 at the equilibrium, therefore: $3p^*-2 = 0$ $p^* = 2/3$

Example

	Hockey	Movie
Hockey	+2,+1	0,0
Movie	0,0	+1,+2

- An example mixed strategy is:
  - A chooses Hockey with probability: $p = 2/3$
  - B chooses Hockey with probability: $q = 1/3$
- In fact, this is a mixed strategy equilibrium for this game
- The expected payoff is 2/3 for both A and B

Key Results

- Theorem (Nash): For any game with a finite number of players, there exists at least one equilibrium

- There might not exist an equilibrium with only pure strategies, but at least one mixed strategy equilibrium exists

- A game may have both pure-strategy and mixed strategy equilibria

How to compute the equilibrium: Example

- The same product is produced by two firms A and B
- The unit production cost is c, so the cost to produce q_A units for firm A is $C = cq_A$
- The market price depends on the total number of units produced: $P = a – (q_A + q_B)$
- Therefore firm A's revenue is $q_A(a – c - (q_A + q_B))$
- Problem: How to figure out the "optimal" output for firm A and B?
- If they produce too much, the price will go down and so would the revenue for each firm
- If they produce too little, the revenue will be small

Example

- Each possible value of q_A is a pure strategy for firm A (and similarly for B).
- At equilibrium, A's revenue is maximum as we vary q_A
  - The derivative of $q_A(a - c - (q_A + q_B))$ with respect to q_A is zero at the NE
- Similarly, B's revenue is maximum as we vary q_B
  - The derivative of $q_B(a - c - (q_A + q_B))$ with respect to q_B is zero at the NE
- Therefore (q^*_A, q^*_B) is solution of the system: $a - c - 2q_A - q_B = 0$ $a - c - 2q_B - q_A = 0$
- With the solution: $q^*_A = q^*_B = (a - c)/3$
- And revenue for each firm: $(a - c)^2/9$
- Note: We ignored the fact that the price must be set to 0 for $q_A + q_B > a$

Is NE really the best that the 2 players can do?

- Suppose that instead of trying to find an equilibrium for A and B independently, we try to maximize the total revenue.

- The maximum is reached for $q_A = q_B = (a - c)/4$ (just take the derivative of the total revenue with respect to the total output $q_A + q_B$)

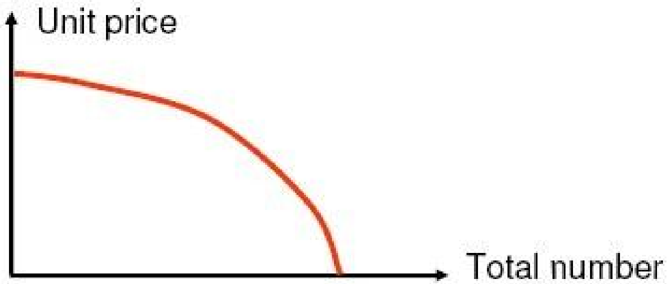
- This corresponds to a revenue per firm of $(a - c)^2/8$, which is *greater* than the revenue we get from the NEs.

Coordination vs. No Coordination

- In general, in any game, the players would get a greater payoff if they agree to cooperate (coordinate, communicate).

- For example, in the prisoner's dilemma, the obvious solution is for the prisoners to both refuse to testify, *if* they agree in advance to coordinate their actions.

Tragedy of the Commons



- The previous example is one example of a more general situation, illustrated by the canonical example:
  - *n* farmers use a common field for grazing goats
  - Because the common field is a *finite resource shared* among all the farmers, the larger the total number of goats, the less food there is, and their unit value goes down
  - Each individual farmer gets a higher profit if they all cooperate (maximize total profit) than if they use the NE equilibrium, acting "rationally" → In the latter case, they tend to each try to "exhaust" the common resource.
- Note: Replace the silly example by changing common field → energy resources, communication bandwidth, oil,.. and farmers → customers, robots, vehicles, firms,…

Summary

- Matrix form of non-zero-sum games and basic concepts for those games
- Strict dominance and its use
- Definition of game equilibrium
- Key result: Existence of (possibly mixed) equilibrium for any finite game
- Understand the difference between cooperating and non-cooperating situations