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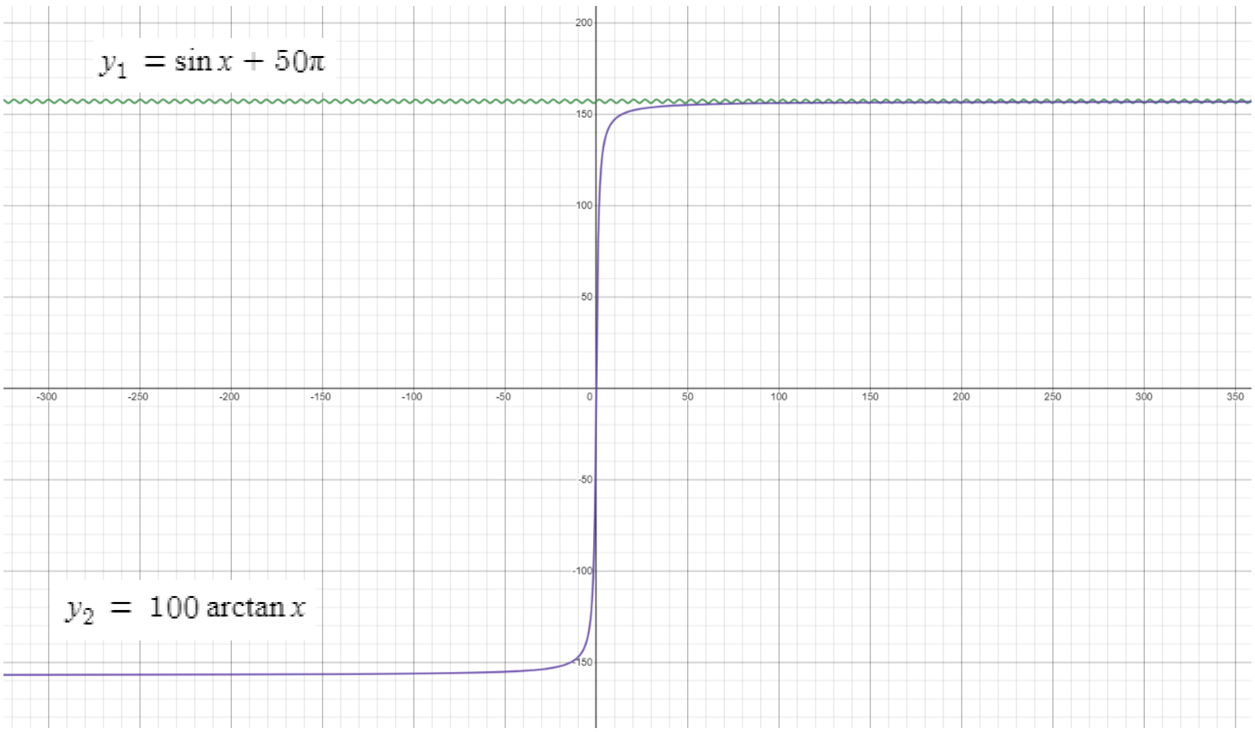
Exercises

Section 3.1: 3, 8, 14, 15, 16, 19 (b,c,d)

3. Demonstrate graphically that the equation $50\pi + \sin x = 100 \arctan x$ has infinitely many solutions.

Answer:



Let $y_1 = 50\pi + \sin x$ and $y_2 = 100 \arctan x$. In the picture above, we can consider that have infinitely many solutions. Because observe that two equations intersect at infinite many points.

8. If a = 0.1 and b = 1.0, how many steps of the bisection method are needed to determine the root with an error of at most $1/2 \times 10^{-8}$?

Answer:

. a = 0.1 and b = 1.0

. Determine the root with an error of at most $(1/2) \times 10^{-8}$

. If $f(a)f(b) < 0$ then after n step an appropriate root error $(b-a)/2^{n+1}$ is obtained.

We Have: $(b-a)/2^{n+1} < (1/2) \times 10^{-8}$ (with a = 0.1 and b = 1.0)

Then $0.9/2^{n+1} < (1/2) \times 10^{-8}$

Then $0.9/2^{n} < 10^{-8}$

Then $2^{n} > 0.9*10^{8}$

Then $n > \log_2(0.9*10^{8})$

Then $n > 26.43$

Hence, after 27 steps then the root will be with an error of at most 0.5* 10−8

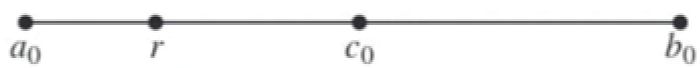
14. Denote the successive intervals that arise in the bisection method by $[a_0, b_0]$, $[a_1, b_1]$, $[a_2, b_2]$, and so on.

a. Show that $a_0 \le a_1 \le a_2 \cdots$ and that $b_0 \ge b_1 \ge b_2 \cdots$.

b. Show that $b_n - a_n = 2^{-n}(b_0 - a_0)$.

c. Show that, for all n, $a_nb_n + a_{n-1}b_{n-1} = a_{n-1}b_n + a_nb_{n-1}$.

Answer:



. The bisection of $[a_0, b_0]$. Let $c_0 = (a_0 + b_0)/2$ is the middle.

. successive intervals that arise in the bisection method by $[a_0, b_0]$, $[a_1, b_1]$, $[a_2, b_2]$, … $[a_n, b_n]$

. So two possible interval $[a_1, c_0]$ or $[c_0, b_1]$.

If $f(b_0)f(c_0) < 0$ then $[c_0, b_1]$, then $a_1 = c_0$ and $b_1 = b_0$.

If $f(a_0)*f(c_0) < 0$ then $[a_1, c_0]$, then $b_1 = c_0$ and $a_1 = a_0$.

a. Show that $a_0 \le a_1 \le a_2 \cdots$ and that $b_0 \ge b_1 \ge b_2 \cdots$.

By taking $[a_1, c_0] = [a_1, b_1]$, That mean $a_0 \le a_1$ and $b_0 \ge c_0 = b_1$.

Similarly we have: $a_1 \le a_2$ and $b_1 \ge b_2$.

So $\{a_0, a_1, .. a_n\}$ will be the monotonic increasing sequence.

So $\{b_0, b_1, .. b_n\}$ will be the monotonic decreasing sequence.

Hence: Show that $a_0 \le a_1 \le a_2 \cdots$ and that $b_0 \ge b_1 \ge b_2 \cdots$.

b. Show that $b_n - a_n = 2^{-n}(b_0 - a_0)$.

If $f(b_0)f(c_0) < 0$ then $[c_0, b_1]$, then $a_1 = c_0$ and $b_1 = b_0$.

$b_1 - a_1 = (b_0 - a_0)/2$

If $f(a_0)*f(c_0) < 0$ then $[a_1, c_0]$, then $b_1 = c_0$ and $a_1 = a_0$.

$b_1 - a_1 = (b_0 - a_0)/ 2$

In both cases, $b_1 - a_1 = (b_0 - a_0)/ 2$. Now consider

$b_1 - a_1 = 2^{-1}(b_0 - a_0)$.

$b_2 - a_2 = 2^{-1}(b_1 - a_1) = 2^{-2}(b_0 - a_0)$. …

$b_n - a_n = 2^{n-1}(b_1 - a_1) = 2^{-n}(b_0 - a_0)$.

Hence $b_n - a_n = 2^{-n}(b_0 - a_0)$.

c. Show that, for all n, $a_n b_n + a_{n-1} b_{n-1} = a_{n-1} b_n + a_n b_{n-1}$.

$a_n b_n + a_{n-1} b_{n-1} = a_{n-1} b_n + a_n b_{n-1}$

Then $a_n b_n + a_{n-1} b_{n-1} - a_{n-1} b_n - a_n b_{n-1} = 0$

Then $a_n (b_n + b_{n-1}) - a_{n-1} (b_n + b_{n-1}) = 0$

Then $(b_n + b_{n-1}) (a_n + a_{n-1}) = 0$

When $[a_{n-1}, b_{n-1}]$, then next step will remain either a_{n-1} or b_{n-1}, So the equation above will always true when one of both b or a size will remain 0, then the equation always true for $n \geq 1$.

15. (Continuation) Can it happen that $a_0 = a_1 = a_2 = \cdots$

Answer:

Only If $f(a_n)*f(c_n) < 0$ then $[a_{n+1}, c_n]$, then $b_{n+1} = c_n$ and $a_{n+1} = a_n$.

In this case, a_0 can remain then $a_0 = a_1 = a_2 = \cdots$ can happen.

16. (Continuation) Let cn = (an + bn)/2. Show that

$\lim_{n\to\infty} c_n = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

Answer:

From above:

Consider: $b_n - a_n = 2^{-n}(b_0 - a_0)$.

Then: $\lim_{n\to\infty} (b_n) - \lim_{n\to\infty} (a_n) = \lim_{n\to\infty} 2^{-n}(b_0 - a_0) = 0$

Then: $\mathbf{\lim_{n\to\infty} (b_n) = \lim_{n\to\infty} (a_n)}$

Now, consider: $c_n = (a_n + b_n)/2$

Then $\mathbf{\lim_{n\to\infty} c_n} = \lim_{n\to\infty} (a_n + b_n)/2 = 0.5 (\lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n) = 0.5(2* \lim_{n\to\infty} a_n) = \mathbf{\lim_{n\to\infty} a_n}$

Hence: $\lim_{n\to\infty} c_n = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

19. (True–False) Using the notation of the text, determine which of these assertions are true and which are generally false:

b. $a_n \le r \le c_n$

c. $c_n \le r \le b_n$

d. $|r - a_n| \le 2^{-n}4$

Answer:

b. $a_n \le r \le c_n$

False – Since r can lie in the interval (c_n, b_n) if $\text{lf}(b_n)*f(c_n) < 0$

c. $c_n \le r \le b_n$

False – Since r can lie in the interval (a_n, c_n) if $\text{lf}(a_n)*f(c_n) < 0$

d. $|r - a_n| \le 2^{-n}$

True - Since $r - a_n \le b_n - a_n = 2^{-n}(b_0 - a_0)$. Hence: $|r - a_n| \le 2^{-n}$.

Section 3.2: 5, 6, 14

5. The equation $x - Rx^{-1} = 0$ has $x = \pm R^{1/2}$ for its solution. Establish Newton's iterative scheme, in simplified form, for this situation. Carry out five steps for $R = 25$ and $x_0 = 1$.

Answer:

$f(x) = x - Rx^{-1} = 0$

Then $f(x) = x^2 - R = 0$ (when multiple x into the equation we still get $f(x) = 0$)

Newton's method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$f(x) = x^2 - R$, then $f'(x) = 2x$

$x_{n+1} = x_n - ((x_n^{\ 2} - R)/(2x_n)) = (x_n^{\ 2} + R)/(2x_n)$

$R = 25$ and $x_0 = 1$

Step 1: $x_1 = (x_0^{\ 2} + R)/(2x_0) = 13$

Step 2: $x_2 = (x_1^{\ 2} + R)/(2x_1) = 194/23 = 97/13$

Step 3: $x_3 = (x_2^{\ 2} + R)/(2x_2) = 6817/1261 \sim 5.406$

Step 4: $x_4 = (x_3{}^2 + R)/(2x_3)$ ~ 5.015

Step 5: $x_5 = (x_4{}^2 + R)/(2x_4)$ ~ 5

Hence: $x_1 = 13$; $x_2 = 97/13$; x_3 ~ 5.406; x_4 ~ 5.015; x_5 ~ 5;

6. Using a calculator, observe the sluggishness with which Newton's method converges in the case of f (x) = $(x - 1)^m$ with m = 8 or 12. Reconcile this with the theory. Use x0 = 1.1.

Answer:

f (x) = $(x - 1)^m$ with m = 8 or 12 Use $x_0 = 1.1$

then $f'(x) = m (x - 1)^{m-1}$.

m = 8				
n	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}
0	1.1	10^{-8}	$8*10^{-7}$	1.088
1	1.088	$3.436*10^{-8}$	$3.142*10^{-7}$	1.077
2	1.077	$1.185*10^{-8}$	$1.238*10^{-7}$	1.067
3	1.067	$4.061*10^{-8}$	$4.849*10^{-8}$	1.089
4	1.059	$1.391*10^{-8}$	$1.898*10^{-8}$	1.051

m = 12				
n	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}
0	1.1	10^{-12}	$8 * 10^{-11}$	1.088
1	1.088	$2.014*10^{-13}$	$2.762*10^{-11}$	1.08
2	1.08	$7.081*10^{-14}$	$1.06*10^{-11}$	1.074
3	1.074	$2.486*10^{-14}$	$4.058*10^{-12}$	1.067
4	1.067	$8.789*10^{-15}$	$1.565*10^{-12}$	1.062

When exponent is large then it moving towards the solution with slow speed. Otherwise.

14. Determine the formulas for Newton's method for finding a root of the function f (x) = x – (e/x). What is the behavior of the iterates?

Answer:

f (x) = x – (e/x).

then f' (x) = 1 + (e/x^2).

Newton's method: $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$x_{n+1} = x_n - ((x_n - (e/x_n))/(1 + (e/x_n^2)))$

$= x_n - (((x_n^2 - e) x_n)/(x_n^2 + e))$

$= (x_n^3 + ex_n - x_n^3 + e x_n)/(x_n^2 + e)$

$= (2ex_n)/(x_n^2 + e)$

Then $x_{n+1} = (2ex_n)/(x_n^2 + e)$. When $x_n > 1$, then use the formula to get the iterates decrease when $0 < x_n \leq 1$, then the iterates increase when $x_n < 0$.

Section 3.3: 2, 5, 13 (b,d)

2. If we use the secant method on $f(x) = x^3 - 2x + 2$ starting with $x_0 = 0$ and $x_1 = 1$, what is x_2?

Answer:

$$x_{n+1} = x_n - \left(\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right) f(x_n).$$

Secant method:

with $x_0 = 0$ and $x_1 = 1$.

$f(x) = x^3 - 2x + 2$ then $f(0) = 2$; and $f(1) = 1$

$x_2 = x_1 - ((x_1 - x_0)/(f(x_1)-f(x_0))) = 1 - ((1-0)/(1-2)) = 2$

Hence, $x_2 = 2$.

5. Using the bisection method, Newton's method, and the secant method, find the largest positive root correct to three decimal places of $x^3 - 5x + 3 = 0$. (All roots are in [−3, +3].)

Answer:

$x^3 - 5x + 3 = 0$. (All roots are in [−3, +3].)

$f(1) = -1$, $f(2) = 1$

As $f(1)f(2) = -1 < 0$

Use Bisection interval [1,2]

$c_1 = (1+2)/2 = 1.5$

$f(c_1) = -1.125 < 0$. Then interval [1.5,2]

$c_2 = (1.5+2)/2 = 1.75$

$f(c_2) = -0.391 < 0$. Then interval [1.75,2]

$c_3 = (1.75+2)/2 = 1.875$

$f(c_3) = 0.217 > 0$. Then interval [1.75,1.875]

$c_4 = (1.75+1.875)/2 = 1.8125$

$f(c_4) = -0.1082 > 0$. Then interval [1.8125,1.875]

$c_5 = (1.8125+1.875)/2 = 1.844$

$f(c_5) = 0.05 > 0$. Then interval [1.8125,1.844]

So on…

Newton's Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Let $x_0 = 2$

$f(x) = x^3 - 5x + 3$

$f'(x) = 3X^2 - 5$

$x_1 = 1.875$ then $f(x_1) = 0.216$

$x_2 = 1.835$ then $f(x_2) = 0.004$

$x_3 = 1.834$ then $f(x_3) = -0.001$

$x_4 = 1.834$ then $f(x_4) = -0.001$

$x_5 = 1.834$ then $f(x_5) = -0.001$

Secant method

$$x_{n+1} = x_n - \left(\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right) f(x_n).$$

$x_0 = 1$; $x_1 = 2$.

$x_2 = 1.5$ then $f(x_2) = -1.125$

$x_3 = 1.765$ then $f(x_3) = -0.3266$

$x_4 = 1.873$ then $f(x_4) = 0.2057$

$x_5 = 1.831$ then $f(x_5) = -0.0165$

13. Test the following sequences for different types of convergence (i.e., linear, superlinear, or quadratic), where n = 1, 2, 3 ... .

b. $x_n = 2^{-n}$

d. $x_n = 2^{-a_n}$ with $a_0 = a_1 = 1$ and $a_{n+1} = a_n + a_{n-1}$ for $n \geq 2$.

Answer:

b. $x_n = 2^{-n}$

$$\frac{|x_{n+1}|}{|x_n|^{\alpha}} = \frac{2^{-(n+1)}}{2^{-\alpha n}} = \frac{2^{n(\alpha-1)}}{2}$$

If α < 1, the exponent is negative, and the ratio converges to 0.

If α = 1, the exponent is 0, and the ratio converges to 1/2.

If α > 1, the exponent is diverges.

Hence the order of convergence is linear.

d. $x_n = 2^{-a_n}$ with $a_0 = a_1 = 1$ and $a_{n+1} = a_n + a_{n-1}$ for $n \geq 2$.

$$\frac{|x_{n+1}|}{|x_n|^{\alpha}} = \frac{2^{-a_{n+1}}}{2^{-\alpha a_n}} = \frac{2^{-a_n - a_{n-1}}}{2^{-\alpha a_n}} = \frac{2^{a_n(\alpha-1)}}{2^{a_{n-1}}}$$

If α < 1, the exponent is negative, and the ratio converges to 0.

If α = 1, the exponent is 0, and the ratio converges to $\frac{1}{2^{a_{n-1}}}$.

If α > 1, the exponent is diverges.

Hence the order of convergence is linear.

Computing Exercises

Section 3.1: 3, 7

3. Find a root of the equation tan x = x on the interval [4, 5] by using the bisection method. What happens on the interval [1, 2]?

Answer:

f(x) = tanx -x = 0

Code:

```
clc;

% Display answer
disp('the interval [4, 5]');
root = bisectionMethod(4, 5)

disp('the interval [1, 2]');
root = bisectionMethod(1, 2)

% Bisection function
function root = bisectionMethod(a,b)
    f = @(x) (tan(x) - x);      % Function
    ermin = 0.000001;             % limit of answer
    error = 3;                  % check error
    c = (a + b)/2;              % Middle point
    while (error > ermin)
        if(f(c)==0)
            root = c;
            break;
        end
        if (f(a)*f(c) < 0)
            b = c;
            error = abs(b - a);
            c = (a+b)/2;
        else
            a = c;
            error = abs(b - a);
            c = (a+b)/2;
        end
    end
    root = c;
    disp('f(root)');
    f(root)
end
```

Result:

```
Command Window
the interval [4, 5]
f(root)

ans =

     3.548067384784304e-06


root =

   4.493409633636475

the interval [1, 2]
f(root)

ans =

    -6.137957800452815e+06


root =

   1.570796489715576
```

7. Use the bisection method to determine roots of these functions on the intervals indicated. Process all three functions in one computer run.

$f(x) = x^3 + 3x - 1$ on [0, 1]

$g(x) = x^3 - 2 \sin x$ on [0.5, 2]

$h(x) = x + 10 - x \cosh(50/x)$ on [120, 130]

Answer:

Code:

```
clc;
f = @(x) (x*x*x + 3*x -1);
a = 0;
b = 1;
disp('f(x) = (x*x*x + 3*x -1) on [0,1]')
root = bisection(f,a,b)

g = @(x) (x*x*x - 2*sin(x));
e = 0.5;
f = 2;
disp('g(x) = (x*x*x - 2*sin(x)) on [0.5, 2]')
root = bisection(g,e,f)
```

```matlab
h = @(x) (x + 10 - x*cosh(50/x));
k = 120;
i = 130;
disp('g(x) = (x + 10 - x*cosh(50/x)) on [120, 130]')
root = bisection(h,k,i)

function root = bisection(f,a,b)
    ermin = 0.000001;              % limit of answer
    error = 3;                   % check error
    c = (a + b)/2;               % Middle point
    while (error > ermin)
        if(f(c)==0)
            root = c;
            break;
        end
        if (f(a)*f(c) < 0)
            b = c;
            error = abs(b - a);
            c = (a+b)/2;
        else
            a = c;
            error = abs(b - a);
            c = (a+b)/2;
        end
    end
    root = c;
    disp('f(root)');
    f(root)
end
```

Result:

```
f(x) = (x*x*x + 3*x -1) on [0,1]
f(root)

ans =

    -1.043444539439164e-06


root =

   0.322185039520264

g(x) = (x*x*x - 2*sin(x)) on [0.5, 2]
f(root)

ans =

    -6.502003953023916e-07


root =

   1.236183762550354

g(x) = (x + 10 - x*cosh(50/x)) on [120, 130]
f(root)

ans =

     1.424967877028394e-08


root =

     1.266324362158775e+02

>>
```

Section 3.2: 2, 6, 15, 18

2. Write a simple, self-contained program to apply Newton's method to the equation $x^3 + 2x^2 + 10x = 20$, starting with $x_0 = 2$. Evaluate the appropriate f (x) and f' (x), using nested multiplication. Stop the computation when two successive points differ by $1/2 \times 10^{-5}$ or some other convenient tolerance close to your machine's capability. Print all intermediate points and function values. Put an upper limit of ten on the number of steps.

Answer:

Code:

```
clc;
% Given
x = [2];
```

```
f = @(x) (x^3 + 2*x^2 + 10*x - 20);
h = @(x) (x^3 + 2*x^2 + 10*x - 20)/ (3*x^2 + 4*x + 10);

% Check
stop = 3;
while(stop > 0.0000005)
n = length(x);                 % Get lenght of x
temp =  x(n) - h(x(n));        % Get new xi
x_new = [x(1:n) temp] ;        % add to new list
n_new = length(x_new);         % get index of the last value in the new index
stop = abs(x(n)-x_new(n_new)); % get The diff
x = x_new;                     % save the list
stop;
end

% Print
for i = 1: length(x)
     fprintf('Index %i  have x= %i then f(x)= %i \n', i, x(i),f(x(i)));
end
```

Result:

```
Index 1  have x= 2 then f(x)= 16
Index 2  have x= 1.466667e+00 then f(x)= 2.123852e+00
Index 3  have x= 1.371512e+00 then f(x)= 5.708664e-02
Index 4  have x= 1.368810e+00 then f(x)= 4.461441e-05
Index 5  have x= 1.368808e+00 then f(x)= 2.731326e-11
Index 6  have x= 1.368808e+00 then f(x)= 0
```

6. In celestial mechanics, **Kepler's equation** is important. It reads x = y − ε sin y, in which x is a planet's mean anomaly, y its eccentric anomaly, and ε the eccentricity of its orbit. Taking ε = 0.9, construct a table of y for 30 equally spaced values of x in the interval 0 ≤ x ≤ π. Use Newton's method to obtain each value of y. The y corresponding to an x can be used as the starting point for the iteration when x is changed slightly.

Answer:

Code:

```
clc;
% Given
Y = [0.1];
f = @(Y) (Y - 0.9*sin(Y));
h = @(Y) (Y - 0.9*sin(Y))/ (1 - 0.9*cos(Y));

% Check
stop = 1;
while(stop <= 30)
n = length(Y);                 % Get lenght of x
```

```
temp =  Y(n) - h(Y(n));          % Get new xi
Y_new = [Y(1:n) temp+0.1] ;  % add to new list
n_new = length(Y_new);           % get index of the last value in the new index
stop = stop +1;  % get The diff
Y = Y_new;                       % save the list
end

% Print
for i = 1: length(Y)
     fprintf('Index %i  have Y= %i then f(y)= %i \n', i, Y(i),f(Y(i)));
end
```

Result:

```
Index 1  have Y= 1.000000e-01 then f(y)= 1.014993e-02
Index 2  have Y= 1.028680e-01 then f(y)= 1.045000e-02
Index 3  have Y= 1.031140e-01 then f(y)= 1.047577e-02
Index 4  have Y= 1.031357e-01 then f(y)= 1.047804e-02
Index 5  have Y= 1.031376e-01 then f(y)= 1.047824e-02
Index 6  have Y= 1.031378e-01 then f(y)= 1.047826e-02
Index 7  have Y= 1.031378e-01 then f(y)= 1.047826e-02
Index 8  have Y= 1.031378e-01 then f(y)= 1.047826e-02
Index 9  have Y= 1.031378e-01 then f(y)= 1.047826e-02
Index 10  have Y= 1.031378e-01 then f(y)= 1.047826e-02
Index 11  have Y= 1.031378e-01 then f(y)= 1.047826e-02
Index 12  have Y= 1.031378e-01 then f(y)= 1.047826e-02
Index 13  have Y= 1.031378e-01 then f(y)= 1.047826e-02
Index 14  have Y= 1.031378e-01 then f(y)= 1.047826e-02
Index 15  have Y= 1.031378e-01 then f(y)= 1.047826e-02
Index 16  have Y= 1.031378e-01 then f(y)= 1.047826e-02
Index 17  have Y= 1.031378e-01 then f(y)= 1.047826e-02
Index 18  have Y= 1.031378e-01 then f(y)= 1.047826e-02
Index 19  have Y= 1.031378e-01 then f(y)= 1.047826e-02
Index 20  have Y= 1.031378e-01 then f(y)= 1.047826e-02
Index 21  have Y= 1.031378e-01 then f(y)= 1.047826e-02
Index 22  have Y= 1.031378e-01 then f(y)= 1.047826e-02
Index 23  have Y= 1.031378e-01 then f(y)= 1.047826e-02
Index 24  have Y= 1.031378e-01 then f(y)= 1.047826e-02
Index 25  have Y= 1.031378e-01 then f(y)= 1.047826e-02
Index 26  have Y= 1.031378e-01 then f(y)= 1.047826e-02
Index 27  have Y= 1.031378e-01 then f(y)= 1.047826e-02
Index 28  have Y= 1.031378e-01 then f(y)= 1.047826e-02
Index 29  have Y= 1.031378e-01 then f(y)= 1.047826e-02
Index 30  have Y= 1.031378e-01 then f(y)= 1.047826e-02
Index 31  have Y= 1.031378e-01 then f(y)= 1.047826e-02
>>
```

15. Find the root of the equation $1/2\, x^2 + x + 1 - e^x = 0$ by Newton's method, starting with $x_0 = 1$, and account for the slow convergence.

Answer:

Code:

```
clc;
% Given
x = [1];
f = @(x) (0.5*x^2 + x + 1 - exp(x));
f1 = @(x)(x + 1 + - exp(x));
h = @(x) (0.5*x^2 + x + 1 - exp(x))/ (x + 1 + - exp(x));

% Check
stop = 3;
while(stop > 0.0000005)
n = length(x);                  % Get lenght of x
temp =  x(n) - h(x(n));         % Get new xi
x_new = [x(1:n) temp] ;         % add to new list
n_new = length(x_new);          % get index of the last value in the new index
stop = abs(x(n)-x_new(n_new));  % get The diff
x = x_new;                      % save the list
stop;
end

% Print
for i = 1: length(x)
    fprintf('Index %i  have x= %i then f(x)= %i \n', i, x(i),f(x(i)));
    if (abs(f1(x(i))) < 0.00005 && i ~= length(x))
        fprintf('Small derivative \n ');
    end
    if(i == length(x))
        fprintf('Convergence \n ');
    end
end
```

Result:

```
Index 1  have x= 1 then f(x)= -2.182818e-01
Index 2  have x= 6.961056e-01 then f(x)= -6.753850e-02
Index 3  have x= 4.781129e-01 then f(x)= -2.061871e-02
Index 4  have x= 3.252851e-01 then f(x)= -6.234993e-03
Index 5  have x= 2.198578e-01 then f(x)= -1.873025e-03
Index 6  have x= 1.479339e-01 then f(x)= -5.601354e-04
Index 7  have x= 9.923640e-02 then f(x)= -1.670002e-04
Index 8  have x= 6.643295e-02 then f(x)= -4.968763e-05
Index 9  have x= 4.441177e-02 then f(x)= -1.476321e-05
Index 10  have x= 2.966279e-02 then f(x)= -4.382407e-06
Index 11  have x= 1.979969e-02 then f(x)= -1.300099e-06
Index 12  have x= 1.321069e-02 then f(x)= -3.855329e-07
Index 13  have x= 8.811982e-03 then f(x)= -1.142949e-07
Small derivative
 Index 14  have x= 5.876813e-03 then f(x)= -3.387760e-08
Small derivative
 Index 15  have x= 3.918835e-03 then f(x)= -1.004027e-08
Small derivative
 Index 16  have x= 2.612983e-03 then f(x)= -2.975380e-09
Small derivative
```

```
 Index 17   have x= 1.742179e-03 then f(x)= -8.816901e-10
Small derivative
 Index 18   have x= 1.161537e-03 then f(x)= -2.612603e-10
Small derivative
 Index 19   have x= 7.743954e-04 then f(x)= -7.741430e-11
Small derivative
 Index 20   have x= 5.162802e-04 then f(x)= -2.293832e-11
Small derivative
 Index 21   have x= 3.441940e-04 then f(x)= -6.796785e-12
Small derivative
 Index 22   have x= 2.294640e-04 then f(x)= -2.013945e-12
Small derivative
 Index 23   have x= 1.529721e-04 then f(x)= -5.966339e-13
Small derivative
 Index 24   have x= 1.019813e-04 then f(x)= -1.767475e-13
Small derivative
 Index 25   have x= 6.799318e-05 then f(x)= -5.240253e-14
Small derivative
 Index 26   have x= 4.532369e-05 then f(x)= -1.554312e-14
Small derivative
 Index 27   have x= 3.019118e-05 then f(x)= -4.440892e-15
Small derivative
 Index 28   have x= 2.044721e-05 then f(x)= -1.332268e-15
Small derivative
 Index 29   have x= 1.407412e-05 then f(x)= -6.661338e-16
Small derivative
 Index 30   have x= 7.348277e-06 then f(x)= -2.220446e-16
Small derivative
 Index 31   have x= -8.760156e-07 then f(x)= 0
Small derivative
 Index 32   have x= -8.760156e-07 then f(x)= 0
Convergence
```

18. Using the bisection method, find the positive root of $2x(1 + x^2)^{-1} = \arctan x$. Using the root as x_0, apply Newton's method to the function arctan x. Interpret the results.

Answer:

Code:

```
clc;
f = @(x) (2*x*(1/(1+x*x)) - atan(x));
h = @(x) (2*x*(1/(1+x*x)) - atan(x))/(1/(x*x +1));

a = 0;
b = 1;
disp('f(x) = (x*x*x + 3*x -1) on [0,1]')
root = bisection(f,a,b)

x = [root];
```

```
final = NewTon(f,h,x);

for i = 1: length(final)
    fprintf('Index %i  have x= %i then f(x)= %i \n', i, final(i),f(final(i)));
end

function final = NewTon (f,h,x)
    % Check
    stop = 3;
    while(stop > 0.0000005)
        n = length(x);                    % Get lenght of x
        temp =  x(n) - h(x(n));           % Get new xi
        x_new = [x(1:n) temp] ;           % add to new list
        n_new = length(x_new);            % get index of the last value in the new
index
        stop = abs(x(n)-x_new(n_new));   % get The diff
        x = x_new;                        % save the list
        final = x;
    end
end

function root = bisection(f,a,b)
    ermin = 0.000001;               % limit of answer
    error = 3;                    % check error
    c = (a + b)/2;                % Middle point
    while (error > ermin)
        if(f(c)==0)
            root = c;
            break;
        end
        if (f(a)*f(c) < 0)
            b = c;
            error = abs(b - a);
            c = (a+b)/2;
        else
            a = c;
            error = abs(b - a);
            c = (a+b)/2;
        end
    end
    root = c;
    disp('f(root)');
    f(root)
end
```

Result:

```
f(x) = (x*x*x + 3*x -1) on [0,1]
f(root)

ans =

    0.2146
```

```
root =

    1.0000

Index 1  have x= 9.999995e-01 then f(x)= 2.146021e-01
Index 2  have x= 5.707956e-01 then f(x)= 3.423846e-01
Index 3  have x= 1.168595e-01 then f(x)= 1.142384e-01
Index 4  have x= 1.061010e-03 then f(x)= 1.061008e-03
Index 5  have x= 7.962825e-10 then f(x)= 7.962825e-10
Index 6  have x= 0 then f(x)= 0
```

Section 3.3: 7, 10

7. Write a simple program to find the root of f (x) = x ^{3} + 2x ^{2} + 10x – 20 using the secant method with starting values x_0 = 2 and x_1 = 1. Let it run at most 20 steps, and include a stopping test as well. Compare the number of steps needed here to the number needed in Newton's method. Is the convergence quadratic?

Answer:

Code:

```
clc;
f= @(x) (x*x*x + 2*x*x + 10*x -20 );
f1 = @(x) (3*x*x + 4*x + 10);
h = @(x) (x*x*x + 2*x*x + 10*x -20 )/ (3*x*x + 4*x + 10);

fprintf('Secant method \n');
x = [2,1];
final1 = Secant(f,x);
for i = 1: length (final)
    fprintf('Index %i  have x= %i then f(x)= %i \n', i, final(i),f(final(i)));
     if (abs(f1(final(i))) < 0.00005 && i ~= length(final))
        fprintf('Small derivative \n ');
    end
    if(i == length(final))
        fprintf('Convergence \n ');
    end
end

fprintf('\nNewton method \n');
final2 = NewTon (f,h,x);

% Print
for i = 1: length(final2)
    fprintf('Index %i  have x= %i then f(x)= %i \n', i, final2(i),f(final2(i)));
    if (abs(f1(final2(i))) < 0.00005 && i ~= length(final2))
        fprintf('Small derivative \n ');
    end
    if(i == length(final2))
        fprintf('Convergence \n ');
```

```
    end
end

function final = NewTon (f,h,x)
    % Check
    stop = 3;
    while(stop > 0.0000005)
        n = length(x);                      % Get lenght of x
        temp =  x(n) - h(x(n));             % Get new xi
        x_new = [x(1:n) temp] ;             % add to new list
        n_new = length(x_new);              % get index of the last value in the new
index
        stop = abs(x(n)-x_new(n_new));  % get The diff
        x = x_new;                          % save the list
        final = x;
    end
end

function final = Secant(f,x)
n = length(x);
i = 1;
    while (i <20)
        temp = x(n) - ((x(n)-(x(n-1)))/(f(x(n)) - f(x(n-1))))* f(x(n));
        x_new = [x(1:n) temp] ;
        x = x_new;
        n = length(x);
        i = i +1;
        if(f(x(n))==0)
            final = x;
            break;
        end
    end
    final = x;
end
```

Result:

```
Secant method
Index 1  have x= 2 then f(x)= 16
Index 2  have x= 1 then f(x)= -7
Index 3  have x= 1.304348e+00 then f(x)= -1.334758e+00
Index 4  have x= 1.376054e+00 then f(x)= 1.531733e-01
Index 5  have x= 1.368672e+00 then f(x)= -2.872217e-03
Index 6  have x= 1.368808e+00 then f(x)= -6.018647e-06
Index 7  have x= 1.368808e+00 then f(x)= 2.372076e-10
Index 8  have x= 1.368808e+00 then f(x)= 0
Convergence

Newton method
Index 1  have x= 2 then f(x)= 16
Index 2  have x= 1 then f(x)= -7
Index 3  have x= 1.411765e+00 then f(x)= 9.175656e-01
Index 4  have x= 1.369336e+00 then f(x)= 1.114812e-02
```

```
Index 5  have x= 1.368808e+00 then f(x)= 1.704487e-06
Index 6  have x= 1.368808e+00 then f(x)= 3.907985e-14
Convergence
```

Explant:

Both the method give the same root of the equation. Hence, Newton method give the fast convergence while secant method give the slow convergence. The above convergence is quadratic.

10. Test the secant method on an example in which r, f′ (r), and f′′ (r) are known in advance. Monitor the ratios $e_{n+1}/(e_n e_{n-1})$ to see whether they converge to −1/ 2 f′′ (r)/ f′ (r). The function f (x) = arctan x is suitable for this experiment.

Answer:

Code:

```
clc;
f= @(x) (atan(x) );

fprintf('Secant method \n');
x = [0,1];
final1 = Secant(f,x);
for i = 1: length (final)
    fprintf('Index %i  have x= %i then f(x)= %i \n', i, final(i),f(final(i)));
     if (abs(f1(final(i))) < 0.00005 && i ~= length(final))
        fprintf('Small derivative \n ');
    end
    if(i == length(final))
        fprintf('Convergence \n ');
    end
end

function final = Secant(f,x)
n = length(x);
i = 1;
    while (i <20)
        temp = x(n) - ((x(n)-(x(n-1)))/(f(x(n)) - f(x(n-1))))* f(x(n));
        x_new = [x(1:n) temp] ;
        x = x_new;
        n = length(x);
        i = i +1;
        if(f(x(n))==0)
            final = x;
            break;
        end
    end
    final = x;
end
```

Result:

```
Secant method
Index 1  have x= 2 then f(x)= 1.107149e+00
Index 2  have x= 1 then f(x)= 7.853982e-01
Index 3  have x= 1.304348e+00 then f(x)= 9.167136e-01
Index 4  have x= 1.376054e+00 then f(x)= 9.423644e-01
Index 5  have x= 1.368672e+00 then f(x)= 9.398043e-01
Index 6  have x= 1.368808e+00 then f(x)= 9.398516e-01
Index 7  have x= 1.368808e+00 then f(x)= 9.398517e-01
Index 8  have x= 1.368808e+00 then f(x)= 9.398517e-01
Convergence
```

Explant:

f(x) = arctan x

Here r is a root of function f(x).

$f'(x) = 1/(1+x^2) \to f''(x) = -(2x)/(1+x^2)^2$

$f'(r) = 1/(1+r^2) \to f''(r) = -(2r)/(1+r^2)^2$

Now $\frac{-1}{2}\frac{f''(r)}{f'(r)} = \frac{-1}{2}\cdot\frac{\frac{-2r}{(1+r^2)^2}}{\frac{1}{1+r^2}} = \frac{r}{1+r^2}$

Since $\frac{-1}{2}\frac{f''(r)}{f'(r)}$ converges to $\frac{e_{n+1}}{e_n e_{n-1}}$ if $lim_{r->0}\left(\frac{-1}{2}\frac{f''(r)}{f'(r)}\right) = lim_{r->0}\left(\frac{r}{1+r^2}\right) = 1$

That is $lim_{r->0}\left(\frac{-1}{2}\frac{f''(r)}{f'(r)}\right) = 1$

Therefore $\frac{-1}{2}\frac{f''(r)}{f'(r)}$ converges to $\frac{e_{n+1}}{e_n e_{n-1}}$. Hence f(x) = arctan x is a convergence function