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Exercises

Section 4.1: 4, 10, 17, 20, 21, 23, 34, 35

4. Verify that the polynomials $p(x) = 5x^3 - 27x^2 + 45x - 21$, $q(x) = x^4 - 5x^3 + 8x^2 - 5x + 3$

interpolate the data

x	1	2	3	4
y	2	1	6	47

and explain why this does not violate the uniqueness part of the theorem on existence of polynomial interpolation.

Answer:

$p(x) = 5x^3 - 27x^2 + 45x - 21$,

$p(1) = 2$

$p(2) = 1$

$p(3) = 6$

$p(4) = 47$

$q(x) = x^4 - 5x^3 + 8x^2 - 5x + 3$

$q(1) = 2$

$q(2) = 1$

$q(3) = 6$

$q(4) = 47$

Hence: It does not violate the uniqueness, part of the existence theorem because two polynomials are not of same degree.

10. a. Construct Newton's interpolation polynomial for the data shown.

x	0	2	3	4
y	7	11	28	63

b. Without simplifying it, write the polynomial obtained in nested form for easy evaluation.

Answer:

a.

x	f[ ]	f[,]	f[, ,]	f[, , ,]
0	7	f[0,2] = (f[2] – f[0])/ (2 -0) = 2	f[0,2,3] = (f[2,3]- f[0,2])/ (3-0) = 5	f[0,2,3,4] = (f[2,3,4] – f[0,2,3]) / (4-0) = 1
2	11	f[2,3] = (f[3] – f[2])/ (3 -2) = 17	f[2,3,4] = (f[3,4]- f[2,3])/ (4-2) = 9	
3	28	f[3,4] = (f[4] – f[3])/ (4 -3) = 35		
4	63			

$P_3(x)$ = 7 + 2(x - 0) + 5(x -0)(x-2) + 1(x-0)(x-2)(x-3)

= x^3 -2x + 7

b. $P_3(x)$ = 7 + 2(x - 0) + 5(x -0)(x-2) + 1(x-0)(x-2)(x-3)

= 7 + (x - 0) (2 + 5(x-2) + (x-2)(x-3))

= 7 + (x - 0) (2 + (x - 2)(5 + (x-3)))

= 7 + x(2 + (x-2)(5 + (x -3)))

17. Determine by two methods the polynomial of degree 2 or less whose graph passes through the points (0, 1.1), (1, 2), and (2, 4.2). Verify that they are the same

Answer:

x	f[ ]	f[,]	f[, ,]
0	1.1	f[0,1] = (f[1] – f[0])/ (1 -0) = 0.9	f[0,1,2] = (f[1,2] – f[0,1])/ (2 -0) = 0.65
1	2	f[1,2] = (f[2] – f[1])/ (2 -1) = 2.2	
2	4.2		

P(x) = 1.1 + 0.9 (x - 0) + 0.65 (x – 0) (x -1)

= 1.1 + 0.25x + 0.65x^2

20. Without using a divided-difference table, derive and simplify the polynomial of least degree that assumes these values:

x	−2	−1	0	1	2
y	2	14	4	2	2

Answer:

$$l_0(x) = \frac{(x-x_1)\cdot(x-x_2)\cdot(x-x_3)\cdot(x-x_4)}{(x_0-x_1)\cdot(x_0-x_2)\cdot(x_0-x_3)\cdot(x_0-x_4)} = \frac{(x+1)(x)(x-1)(x-2)}{24}$$

$$l_1(x) = \frac{(x-x_0)\cdot(x-x_2)\cdot(x-x_3)\cdot(x-x_4)}{(x_1-x_0)\cdot(x_1-x_2)\cdot(x_1-x_3)\cdot(x_1-x_4)} = \frac{-(x+2)(x)(x-1)(x-2)}{6}$$

$$l_2(x) = \frac{(x-x_0)\cdot(x-x_1)\cdot(x-x_3)\cdot(x-x_4)}{(x_2-x_0)\cdot(x_2-x_1)\cdot(x_2-x_3)\cdot(x_2-x_4)} = \frac{(x+2)(x+1)(x-1)(x-2)}{4}$$

$$l_3(x) = \frac{(x-x_0)\cdot(x-x_1)\cdot(x-x_2)\cdot(x-x_4)}{(x_3-x_0)\cdot(x_3-x_1)\cdot(x_3-x_2)\cdot(x_3-x_4)} = \frac{(x+2)(x+1)(x)(x-2)}{-6}$$

$$l_4(x) = \frac{(x-x_0)\cdot(x-x_1)\cdot(x-x_2)\cdot(x-x_3)}{(x_4-x_0)\cdot(x_4-x_1)\cdot(x_4-x_2)\cdot(x_4-x_3)} = \frac{(x+2)(x+1)(x)(x-1)}{24}$$

$$p(x) = \sum_{i=0}^{n} l_i(x) f(x_i) = 2\frac{(x+1)(x)(x-1)(x-2)}{24} + 14\frac{-(x+2)(x)(x-1)(x-2)}{6} + 4\frac{(x+2)(x+1)(x-1)(x-2)}{4} + 2\frac{(x+2)(x+1)(x)(x-2)}{-6} + 2\frac{(x+2)(x+1)(x)(x-1)}{24} =$$

$$= \frac{(x+1)(x)(x-1)(x-2)}{12} + 7\frac{-(x+2)(x)(x-1)(x-2)}{3} + (x+2)(x+1)(x-1)(x-2) + -\frac{(x+2)(x+1)(x)(x-2)}{3} + \frac{(x+2)(x+1)(x)(x-1)}{12}$$

$$= 4 - 8x + \frac{11x^2}{2} + 2x^3 - \frac{3x^4}{2}$$

21. (Continuation) Find a polynomial that takes the values shown in the preceding problem and has at x = 3 the value 10. Hint: Add a suitable polynomial to the p(x) of the previous problem.

Answer:

q(x) = p(x) + c(x+2)(x+1)(x)(x-1)(x-2)

q(3) = p(3) + 120c

We have p(3) = -38 and q(3) = 10

Then q(3) = p(3) + 120c

10 = -38 + 120c then c = 2/5

Now: q(x) = p(x) + c(x+2)(x+1)(x)(x-1)(x-2)

$$q(x) = 4 - 8x + \frac{11x^2}{2} + 2x^3 - \frac{3x^4}{2} + \frac{2}{5}(x+2)(x+1)x(x-1)(x-2)$$

$$Hence,\ q(x) = 4 - \frac{32}{5}x + \frac{11}{2}x^2 - \frac{3}{2}x^4 + \frac{2}{5}x^5$$

23. Form a divided-difference table for the following and explain what happened

x	1	2	3	1
y	3	5	5	7

Answer:

x	f[ ]	f[,]	f[, ,]	f[, , ,]
1	3	f[1,2]=(5-3)/(2-1)=2	f[1,2,3] = (0-2)/(3-1) = -1	f[1,2,3,1] = (1+1)/(1-1) = undefined
2	5	f[2,3] = (5-5)/(3-2) = 0	f[2,3,1]= (-1-0)(1-2) = 1	
3	5	f[3,1] = (7-5)/(1-3) = -1		
1	7			

We cannot division by zero since the nodes are not unique. Interpolating polynomials are functions, but the give data set is not from a function, but a relation

34. Write the Lagrange form (1) of the interpolating polynomial of degree at most 2 that interpolates f (x) at x0, x1, and x2, where x0 < x1 < x2.

Answer:

p(x) = $\sum_{i=0}^{n} l_i(x) f(x_i)$

l_i(x)= $\prod_{j \neq i, j=0}^{n} \frac{x - x_j}{x_i - x_j}$

$$l_0(x) = \frac{(x - x_1) \cdot (x - x_2)}{(x_0 - x_1) \cdot (x_0 - x_2)}$$

$$l_1(x) = \frac{(x - x_0) \cdot (x - x_2)}{(x_1 - x_0) \cdot (x_1 - x_2)}$$

$$l_2(x) = \frac{(x - x_0) \cdot (x - x_1)}{(x_2 - x_0) \cdot (x_2 - x_1)}$$

$$p(x) = f(x_0) l_0(x) + f(x_1) l_1(x) + f(x_2) l_2(x)$$

$$p(x) = f(x_0) \frac{(x - x_1) \cdot (x - x_2)}{(x_0 - x_1) \cdot (x_0 - x_2)} + f(x_1) \frac{(x - x_0) \cdot (x - x_2)}{(x_1 - x_0) \cdot (x_1 - x_2)} + f(x_2) \frac{(x - x_0) \cdot (x - x_1)}{(x_2 - x_0) \cdot (x_2 - x_1)}$$

35. (Continuation) Write the Newton form of the interpolating polynomial p2(x), and show that it is equivalent to the Lagrange form.

Answer:

p(x) = $\sum_{i=0}^{n} f[x_0, x_1, x_2, x_3, \ldots, x_n] \prod_{j=0}^{i-1} (x - x_j)$.

$$p(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) + \frac{1}{x_2 - x_0}\left(\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}\right)(x - x_0)(x - x_1)$$

$$p(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) + \frac{1}{x_2 - x_0}\left(\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}\right)(x - x_0)(x - x_1)$$

$$p(x) = f(x_0)\left(1 - \frac{(x - x_0)}{x_1 - x_0} + \frac{(x - x_0)(x - x_1)}{x_1 - x_0}\right) + f(x_1)\left(\frac{(x - x_0)}{x_1 - x_0} - \frac{(x - x_0)(x - x_1)}{(x_2 - x_1)(x_2 - x_0)} - \frac{(x - x_0)(x - x_1)}{(x_1 - x_0)(x_2 - x_0)}\right) + f(x_2)\left(\frac{(x - x_0)(x - x_1)}{(x_2 - x_1)(x_2 - x_0)}\right)$$

$$p(x) = f(x_0)\frac{(x - x_1)\cdot(x - x_2)}{(x_0 - x_1)\cdot(x_0 - x_2)} + f(x_1)\frac{(x - x_0)\cdot(x - x_2)}{(x_1 - x_0)\cdot(x_1 - x_2)} + f(x_2)\frac{(x - x_0)\cdot(x - x_1)}{(x_2 - x_0)\cdot(x_2 - x_1)}$$

From Lagrange:

$$q(x) = f(x_0)l_0(x) + f(x_1)l_1(x) + f(x_2)l_2(x)$$

$$q(x) = f(x_0)\frac{(x - x_1)\cdot(x - x_2)}{(x_0 - x_1)\cdot(x_0 - x_2)} + f(x_1)\frac{(x - x_0)\cdot(x - x_2)}{(x_1 - x_0)\cdot(x_1 - x_2)} + f(x_2)\frac{(x - x_0)\cdot(x - x_1)}{(x_2 - x_0)\cdot(x_2 - x_1)}$$

Now we can consider p(x) = q(x).

Hence, Newton form of the interpolating polynomial p2(x), and it is equivalent to the Lagrange form.

Section 4.2: 6, 7, 10, 11

6. How accurately can we determine sin x by linear interpolation, given a table of sin x to ten decimal places, for x in [0, 2] with h = 0.01?

Answer:

Second Interpolation Error Theorem

Let f be a function such that $f^{(n+1)}$ is continuous on $[a, b]$ and satisfies $|f^{(n+1)}(x)| \leq M$. Let p be the polynomial of degree $\leq n$ that interpolates f at $n + 1$ equally spaced nodes in $[a, b]$, including the endpoints. Then on $[a, b]$,

$$|f(x) - p(x)| \leq \frac{1}{4(n+1)}Mh^{n+1} \tag{6}$$

where $h = (b - a)/n$ is the spacing between nodes.

f(x) = sin x

f’(x) = cos x

f’’(x) = -sin x

|f’’(x)| ≤1 = M

$$|f(x) - p(x)| \leq \frac{1}{4(n+1)}Mh^{n+1}$$

h = (b – a)/n is space between nodes. Since we want to approximate f(x) = sin x linear interpolation. We have n =1

$$|f(x) - p(x)| \leq \frac{1}{4(n+1)}Mh^{n+1} = \frac{1\cdot(0.01)^{1+1}}{4(1+1)} = 1.25\cdot 10^{-5}$$

7. (Continuation) Given the data

x	$\sin x$	$\cos x$
0.70	0.64421 76872	0.76484 21873
0.71	0.65183 37710	0.75836 18760

$\sin(0.705) \approx 0.6442176872 + \frac{0.6518337710 - 0.6442176872}{0.71 - 0.7}(0.705 - 0.7) \approx 0.6480258$:

$$p(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

$$\sin(0.705) \approx 0.6442176872 + \frac{0.6518337710 - 0.6442176872}{0.71 - 0.7}(0.705 - 0.7) \approx 0.6480258$$

$$\cos(0.702) \approx 0.7648421873 + \frac{0.7583618760 - 0.7648421873}{0.71 - 0.7}(0.702 - 0.7) \approx 0.7635461$$

Using direct trigonometry computation:

sin(0.705) = 0.6480338

cos(0.702) = 0.7635522

Error on sin = |0.6480338 – 0.6480258| = $8.1*10^{-6}$

Error on cos = |0.7635461-0.7635522| = $6.123*10^{-6}$.

10. Let the function f (x) = ln x be approximated by an interpolation polynomial of degree 9 with ten nodes uniformly distributed in the interval [1, 2]. What bound can be placed on the error?

Answer:

n = 9

$$|f(x) - p(x)| \leq \frac{1}{4(n+1)} M h^{n+1}$$

Since f(x) = lnx and x ∈ [1.2]

$|f'(x)| = |x^{-1}|$

$|f''(x)| = |-x^{-2}|$

$|f'''(x)| = |2x^{-3}|$

….

$|f^{(10)}(x)| = |9!x^{-10}|$

With $x^{-10} \leq 1$ then $|f^{(10)}x| \leq 9! = M$

$$|f(x) - p(x)| \leq \frac{1}{4(9+1)}(9!)\left(\frac{2-1}{9}\right)^{9+1} \approx 2.6018 \cdot 10^{-6}$$

11. In the first theorem on interpolation errors, show that if x0 < x1 < ··· < xn and x0 < x < xn, then x0 <ξ< x_n.

Answer:

Define $x_0 < x < x_n$,

INTERPOLATION ERRORS I

If p is the polynomial of degree at most n that interpolates f at the $n+1$ distinct nodes $x_0, x_1, \ldots, x_n$ belonging to an interval $[a, b]$ and if $f^{(n+1)}$ is continuous, then for each x in $[a, b]$, there is a ξ in (a, b) for which

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i=0}^{n} (x - x_i) \qquad (2)$$

$$w(t) = \prod_{i=0}^{n} (t - x_i) \qquad \text{(polynomial in the variable } t\text{)}$$

$$c = \frac{f(x) - p(x)}{w(x)} \qquad \text{(constant)}$$

$$\varphi(t) = f(t) - p(t) - cw(t) \qquad \text{(function in the variable } t\text{)}$$

Where p(t) is the polynomial degree at most n that interpolates n+ 1 points x_0 through x_n evaluated with f(x).

Note also that φ takes the value 0 at the n + 2 points x0, x1,..., xn, and x. Now invoke Rolle's Theorem, ∗ which states that between any two roots of φ, there must occur a root of φ . Thus, φ has at least n + 1 roots. By similar reasoning, φ has at least n roots, φ has at least n − 1 roots, and so on. Finally, it can be inferred that φ(n+1) must have at least one root. Let ξ be a root of $\varphi^{(n+1)}$. All the roots being counted in this argument are in (a, b). Thus, ξ between x_0 and x_n. Thus $x_0 < \xi < x_n$

Computing Exercises

Section 4.2: 1, 2

1. Using 21 equally spaced nodes on the interval [−5, 5], find the interpolating polynomial p of degree 20 for the function f (x) = $(x^2 + 1)^{-1}$. Print the values of f (x) and p(x) at 41 equally spaced points, including the nodes. Observe the large discrepancy between f (x) and p(x).

Answer:

Mathlab code:

```
close all
clc

x = linspace(-5,5,21);
fx = 1./(x.^2+1);
```

```
n = 20;
p = polyfit(x,fx,n);    % interpolating polynomial p of degree 20 for the function f
(x)
x = linspace(-5,5,41);
fx = 1./(x.^2+1);
px = polyval(p,x);      % Count p(x)

fprintf("\nDisplay lists fx || px\n")
[fx' px']

figure;plot(x,fx)
hold on
plot(x,px)
legend('f(x)','p(x)')
ylim([-1 4.5])
```

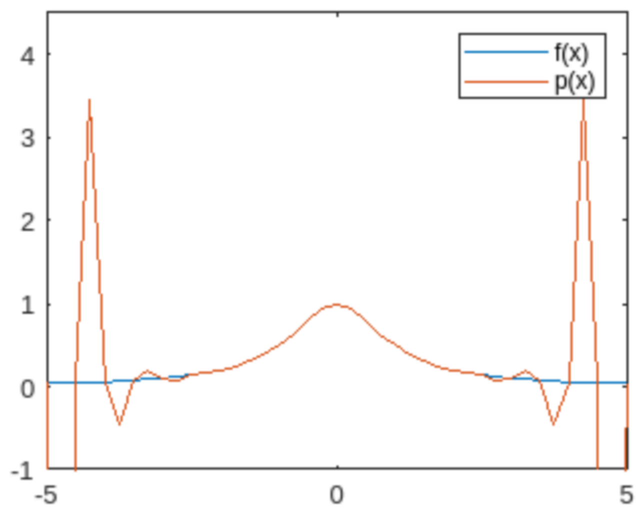
Display:

```
Display lists fx || px


ans =


    0.0385    0.0385
    0.0424  -39.9524
    0.0471    0.0471
    0.0525    3.4550
    0.0588    0.0588
    0.0664   -0.4471
    0.0755    0.0755
    0.0865    0.2024
    0.1000    0.1000
    0.1168    0.0807
    0.1379    0.1379
    0.1649    0.1798
    0.2000    0.2000
    0.2462    0.2384
    0.3077    0.3077
    0.3902    0.3951
    0.5000    0.5000
    0.6400    0.6368
    0.8000    0.8000
    0.9412    0.9425
    1.0000    1.0000
    0.9412    0.9425
    0.8000    0.8000
    0.6400    0.6368
    0.5000    0.5000
    0.3902    0.3951
    0.3077    0.3077
    0.2462    0.2384
    0.2000    0.2000
```

```
    0.1649    0.1798
    0.1379    0.1379
    0.1168    0.0807
    0.1000    0.1000
    0.0865    0.2024
    0.0755    0.0755
    0.0664   -0.4471
    0.0588    0.0588
    0.0525    3.4550
    0.0471    0.0471
    0.0424  -39.9524
    0.0385    0.0385
```



Another way to code:

```
clc

f=@(x) (x.^2+1).^(-1);

%create 21 point

x=linspace(-5,5,21);

y=fun(x);

%function to interpolate values

array=coef(x,y);

%create 41 point

x_p=linspace(-5,5,41);

fprintf('   Index |x_values |Actual_values |interpolated values |abs difference\n');

for i=1:1:length(x_p)
```

```
x_p_Index=x_p(i);

%interpolate values

inter_value=Evaluation(x,array,x_p_Index);

Actual_value=fun(x_p_Index);
% Display

fprintf('%5.0f %10.5f %15.5f %15.5f
%15.5f\n',i,x_p_Index,Actual_value,inter_value,abs(Actual_value-inter_value));

end

% Get funtion coef mean a given function to interpolate values
function[array]=coef(x,y)

n=length(x);

m=length(y);

if n~=m,error('same length vector applicable');end

F=zeros(n,n);

F(:,1)=y';

for j=2:n

for i=1:(n-j+1)

F(i,j)=(F(i+1,j-1)-F(i,j-1))/(x(i+j-1)-x(i));

end

end

array=F(1,:);

end



function inter_value=Evaluation(x,array,x_p_Index)

%Performs approximation of the values

z=length(array);

sum=0;

for i=1:z

value_prodx=1;
```

```
for j=1:i-1

value_prodx=value_prodx*(x_p_Index-x(j));

end

sum=sum+array(i)*value_prodx;

end

inter_value=sum;

end
```

Display

Index	x_values	Actual_values	interpolated values	abs difference
1	-5.00000	0.03846	0.03846	0.00000
2	-4.75000	0.04244	-39.95245	39.99489
3	-4.50000	0.04706	0.04706	0.00000
4	-4.25000	0.05246	3.45496	3.40250
5	-4.00000	0.05882	0.05882	0.00000
6	-3.75000	0.06639	-0.44705	0.51344
7	-3.50000	0.07547	0.07547	0.00000
8	-3.25000	0.08649	0.20242	0.11594
9	-3.00000	0.10000	0.10000	0.00000
10	-2.75000	0.11679	0.08066	0.03613
11	-2.50000	0.13793	0.13793	0.00000
12	-2.25000	0.16495	0.17976	0.01481
13	-2.00000	0.20000	0.20000	0.00000
14	-1.75000	0.24615	0.23845	0.00771
15	-1.50000	0.30769	0.30769	0.00000
16	-1.25000	0.39024	0.39509	0.00485
17	-1.00000	0.50000	0.50000	0.00000
18	-0.75000	0.64000	0.63676	0.00324
19	-0.50000	0.80000	0.80000	0.00000
20	-0.25000	0.94118	0.94249	0.00131
21	0.00000	1.00000	1.00000	0.00000
22	0.25000	0.94118	0.94249	0.00131
23	0.50000	0.80000	0.80000	0.00000
24	0.75000	0.64000	0.63676	0.00324
25	1.00000	0.50000	0.50000	0.00000
26	1.25000	0.39024	0.39509	0.00485
27	1.50000	0.30769	0.30769	0.00000
28	1.75000	0.24615	0.23845	0.00771
29	2.00000	0.20000	0.20000	0.00000
30	2.25000	0.16495	0.17976	0.01481
31	2.50000	0.13793	0.13793	0.00000
32	2.75000	0.11679	0.08066	0.03613
33	3.00000	0.10000	0.10000	0.00000
34	3.25000	0.08649	0.20242	0.11594

35	3.50000	0.07547	0.07547	0.00000
36	3.75000	0.06639	-0.44705	0.51344
37	4.00000	0.05882	0.05882	0.00000
38	4.25000	0.05246	3.45496	3.40250
39	4.50000	0.04706	0.04706	0.00000
40	4.75000	0.04244	-39.95245	39.99489
41	5.00000	0.03846	0.03846	0.00000

2. (Continuation) Perform the experiment in the preceding computer problem, using Chebyshev nodes $x_i = 5\cos(i\pi/20)$, where $0 \le i \le 20$, and nodes $x_i = 5\cos[(2i+1)\pi/42]$, where $0 \le i \le 20$. Record your conclusions.

Answer:

Mathlab code:

```
clc

f=@(x) (x.^2+1).^(-1);
i = 0:1:20;
c_p = 5*cos(i*pi/20);

x_p = linspace(-5,5,20);
ChebyshevInterpolate(f,c_p,x_p);


function ChebyshevInterpolate(fun,c_p,x_p)

n =length(c_p);

T = zeros(n,n);
F = zeros(n,1);

for i = 1:1:n
    c_p_index = c_p(i);
    for j = 1:1:n
        if j ==1
            T(i,j)=1;
        else
            T(i,j) = chebyshevT(j-1,c_p_index);
        end
    end
    F(i) = fun(c_p_index);
end
coeff = T\F;

fprintf('   Index     Exact     Interpolated      Difference(Error)\n');

for i = 1:1:length(x_p)
    sum = coeff(1);
    x_index = x_p(i);
    for j = 2:1:n
```

```
        Tj = chebyshevT(j-1,x_index);
        sum = sum + coeff(j)*Tj;
    end
    y_p_c = sum;
    y_p_e = fun(x_index);
    fprintf('%5.0f  %15.5f %15.5f %15.5f\n',i,y_p_e,y_p_c,abs(y_p_e - y_p_c));

end
end
```

Display

```
   Index     Exact     Interpolated      Difference(Error)
    1          0.03846          0.03846          0.00000
    2          0.04759          0.04701          0.00058
    3          0.06031          0.05704          0.00326
    4          0.07872          0.08303          0.00431
    5          0.10662          0.10461          0.00200
    6          0.15130          0.14629          0.00501
    7          0.22762          0.24132          0.01370
    8          0.36613          0.35211          0.01402
    9          0.61604          0.61552          0.00052
   10          0.93523          0.94311          0.00788
   11          0.93523          0.94311          0.00788
   12          0.61604          0.61552          0.00052
   13          0.36613          0.35211          0.01402
   14          0.22762          0.24132          0.01370
   15          0.15130          0.14629          0.00501
   16          0.10662          0.10461          0.00200
   17          0.07872          0.08303          0.00431
   18          0.06031          0.05704          0.00326
   19          0.04759          0.04701          0.00058
   20          0.03846          0.03846          0.00000
```