Artificial Intelligence

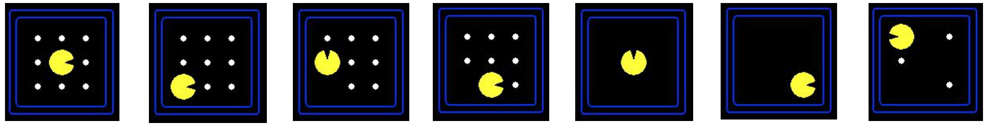
CS4365 --- Fall 2022

Informed Search

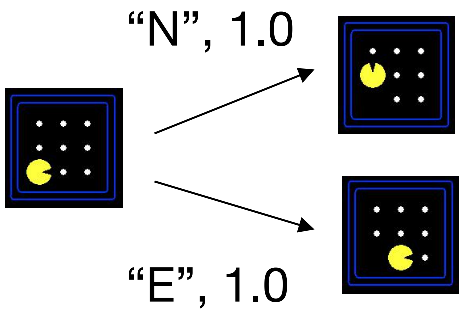
Instructor: Yunhui Guo

Define a Search Problem

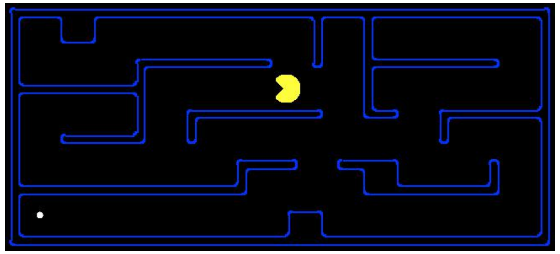
- A search problem consists of:
  - State space:



  - A successor function: (action + cost)



  - A start state and a goal test



Summary

	Complete?	Optimal?	Time	Space
BFS	Finte b	Yes	$O(b^d)$	$O(b^d)$
DFS	Finte d	No	$O(b^m)$	O(bm)
UCS	Finite b , step cost $\geq \epsilon$	Yes	$O(b^{1+\lfloor C^*/\epsilon \rfloor})$	$O(b^{1+\lfloor C^*/\epsilon \rfloor})$
IDS	Finte b	Yes	$O(b^d)$	O(bd)

b: branching factor d: depth of the shallowest solution m: maximum depth

Limitations

- What are the problems of all the methods? Slow!
- The search is blind in the sense that the information of the goal state is not used
- Informed search:
  - with the guidance of the goal state

Informed Methods: Heuristic Search

- Informed methods use problem-specific knowledge
  - The location of the goal
- Humans rely on informed search!

Informed Methods: Heuristic Search

- We want to have some estimate of the distance from the states in the frontier to the goal state.

- Why estimate? Because the states we can reach are based on the actions we can take.

Informed Methods: Heuristic Search

- We use an evaluation function f(n) as our estimate
- Best-first search:
  - Nodes are selected for expansion based on the evaluation function, f(n).
  - Expand the node with the lowest evaluation
  - The evaluation function can be complex: $f(n) = f_1(n) + f_2(n) + \ldots$

Greedy Best-First Search

- One natural component of f(n):
  - Heuristic function:
    - h(n) = estimated cost of the cheapest path from the state at node n to a goal state

Heuristic search is an attempt to search the most promising paths first

Greedy Best-First Search

- Greedy Best-First Search
  - Expands the node that is “closest” to the goal as measured by h(n)

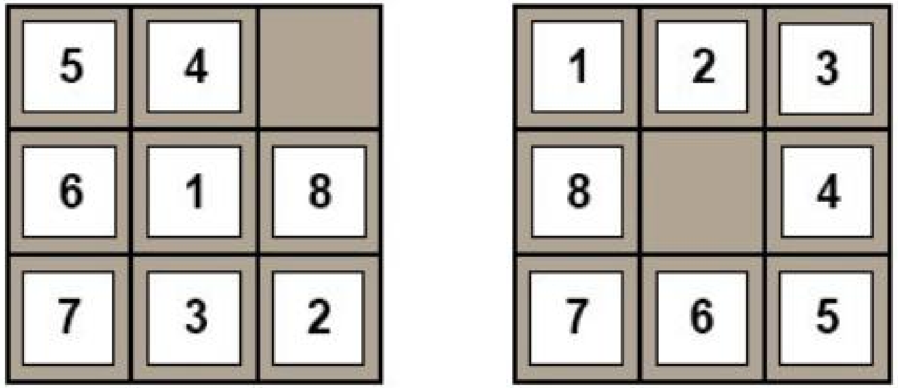
- A common case:
  - Best-first takes you straight to the (wrong) goal

- Worst-case: like a badly-guided DFS

Example: 8-puzzle problem

- Design of heuristic function is important

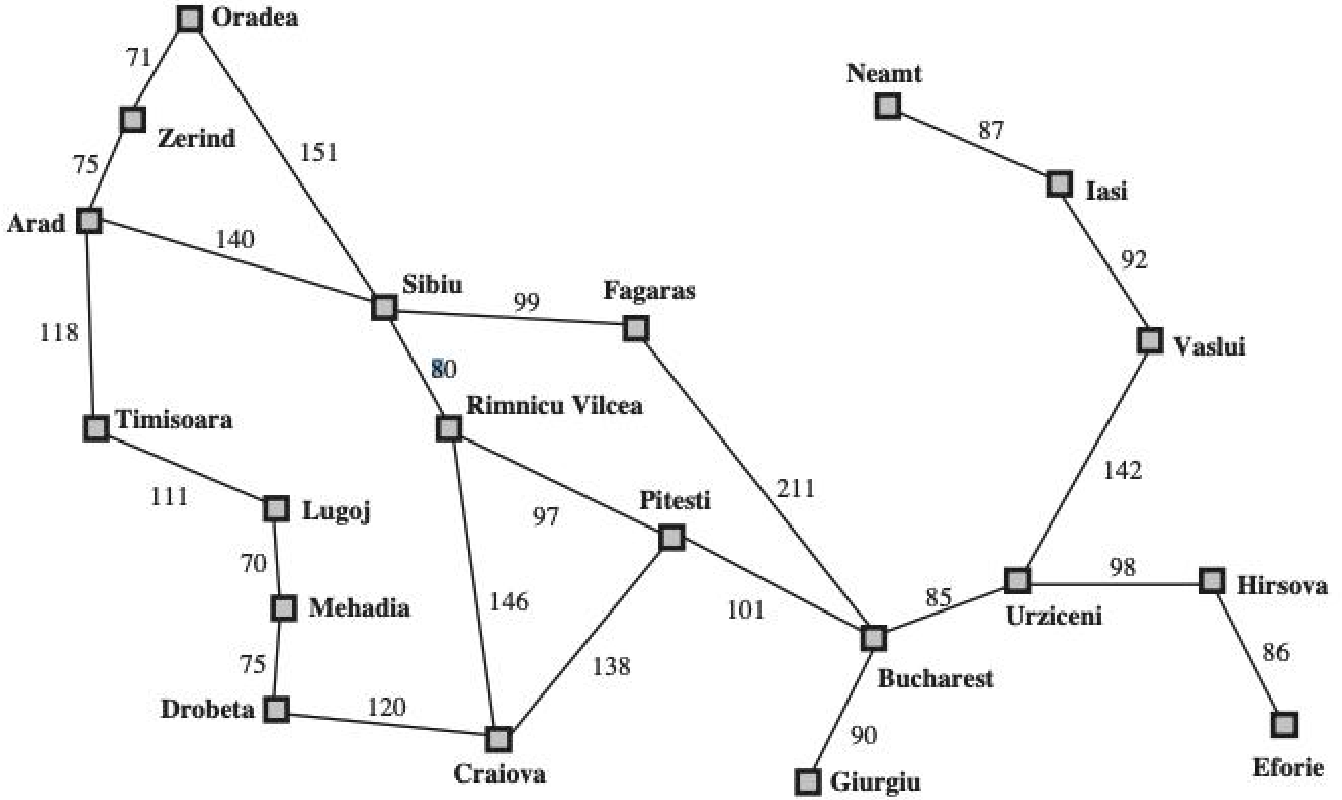
- One possible heuristic function:
    - The number of tiles misplaced



Start State Goal State

Example: Find a Path from Arad to Bucharest

- One possible heuristic function:
  - the straight-line distance to Bucharest



Arad	366	Mehadia	241
Bucharest	0	Neamt	234
Craiova	160	Oradea	380
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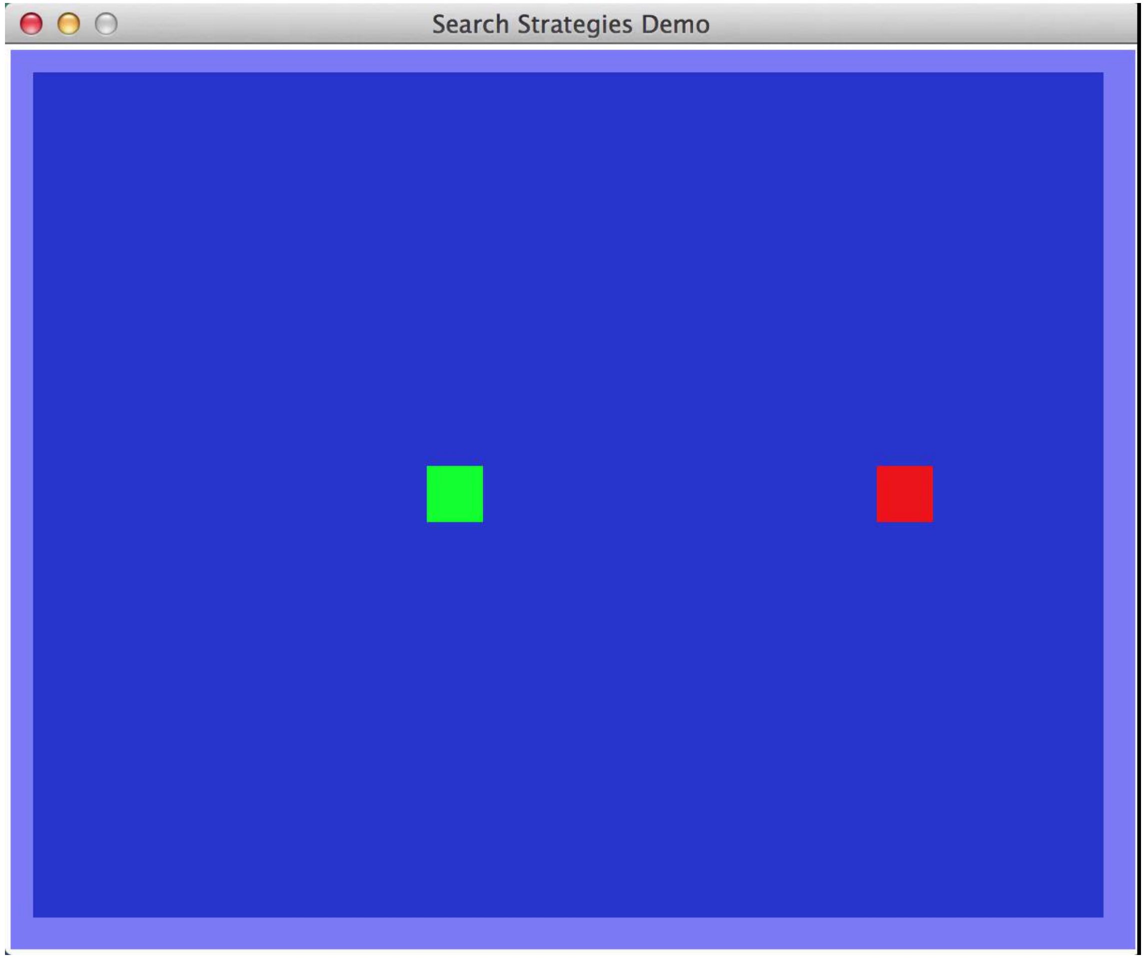
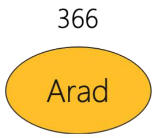
Eforie	161	Rimnicu Vilcea	193
Fagaras	176	Sibiu	253
Giurgiu	77	Timisoara	329
Hirsova	151	Urziceni	80
Iasi	226	Vaslui	199
Lugoj	244	Zerind	374

Greedy Best-First Search



Example: Find a Path from Arad to Bucharest

- One possible heuristic function:
  - the straight-line distance to Bucharest
- Expand the node seems "closest"



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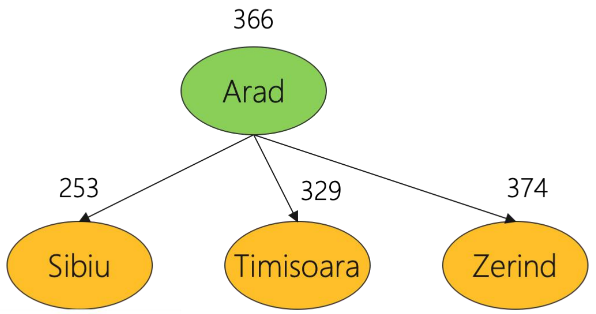

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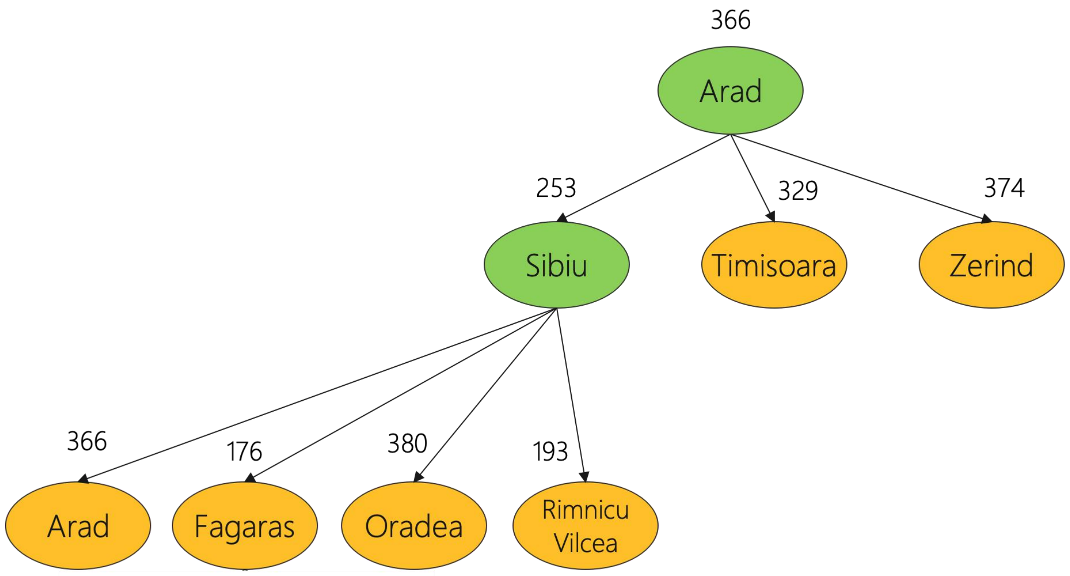

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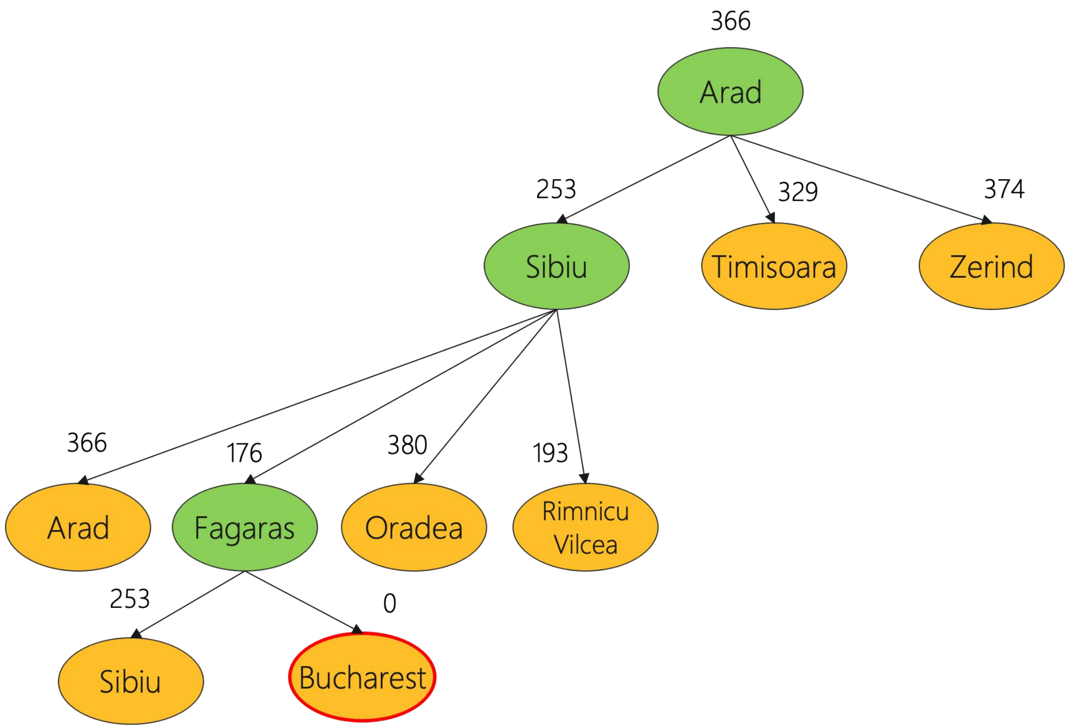
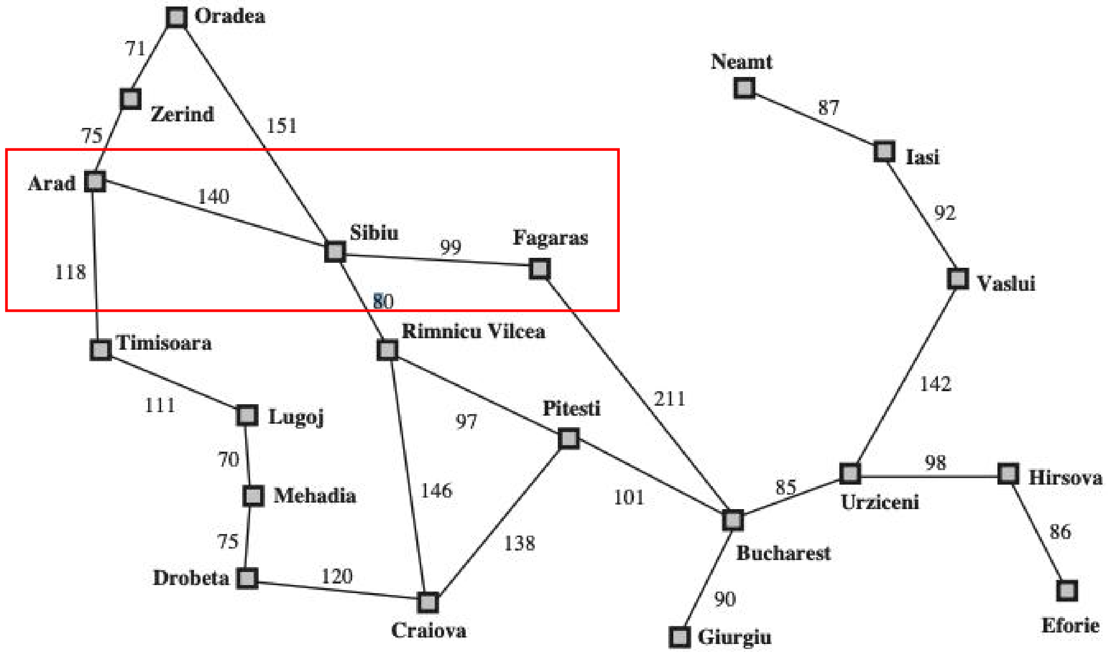


Greedy Best-First Search Can be Suboptimal

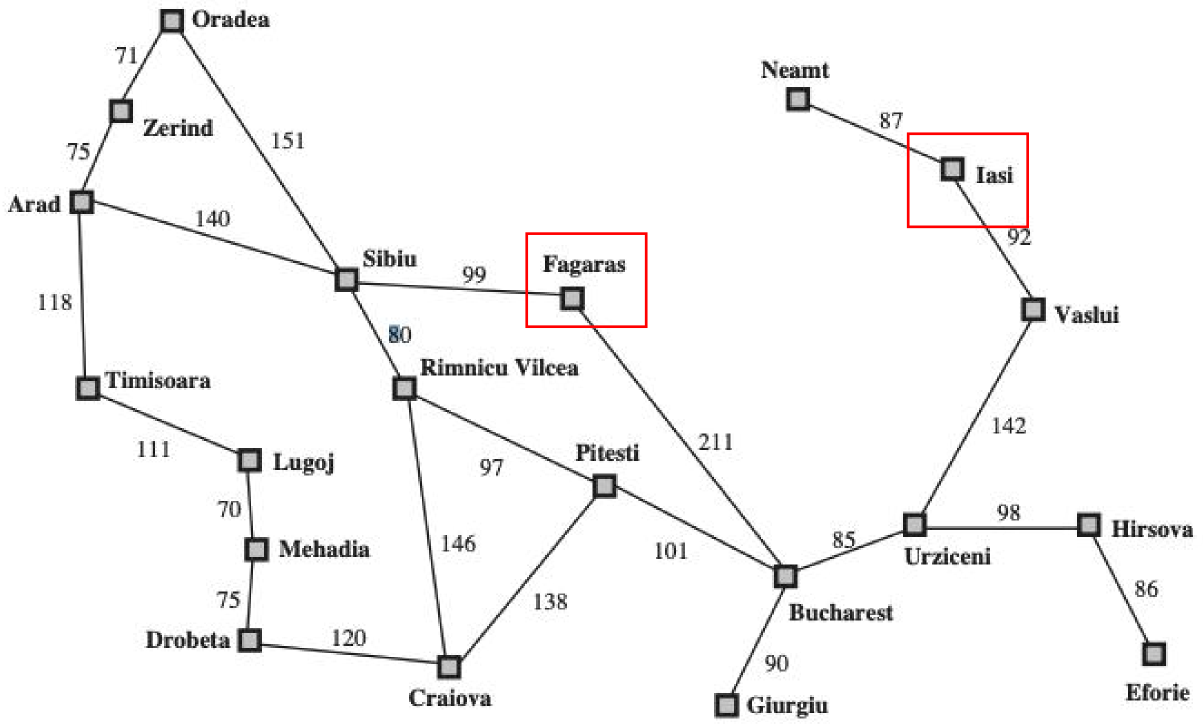
- From Arad to Sibiu to Fagaras -- but to Rimnicu would have been better



- What is missing?
    - The cost of getting from the start node (Arad) to intermidiate nodes!

Greedy Best-First Search is Incomplete

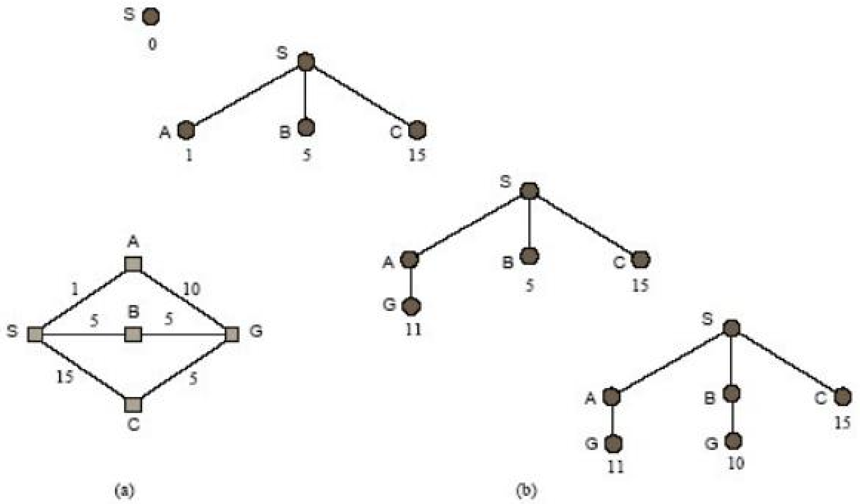
- Start state: Iasi
- Goal state: Fagaras



A* Search

- Proposed in 1968 by Peter Hart, Nils Nilsson and Bertram Raphael
- Most widely known form of Best-First Search.
- Combining Uniform-Cost Search and Greedy Best-First Search

UCS



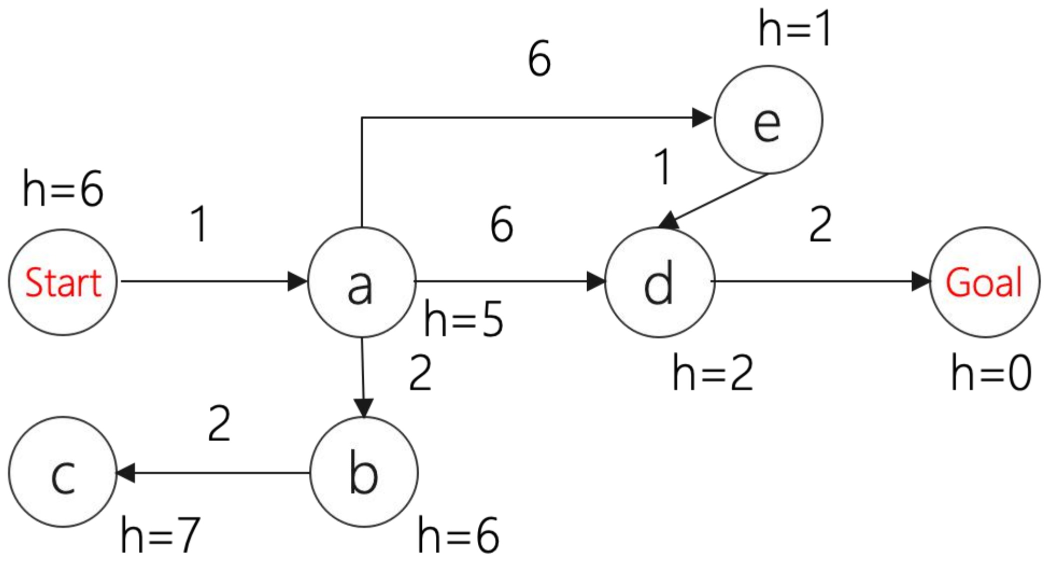
A* Search

- Combining Uniform-Cost Search and Greedy Best-First Search

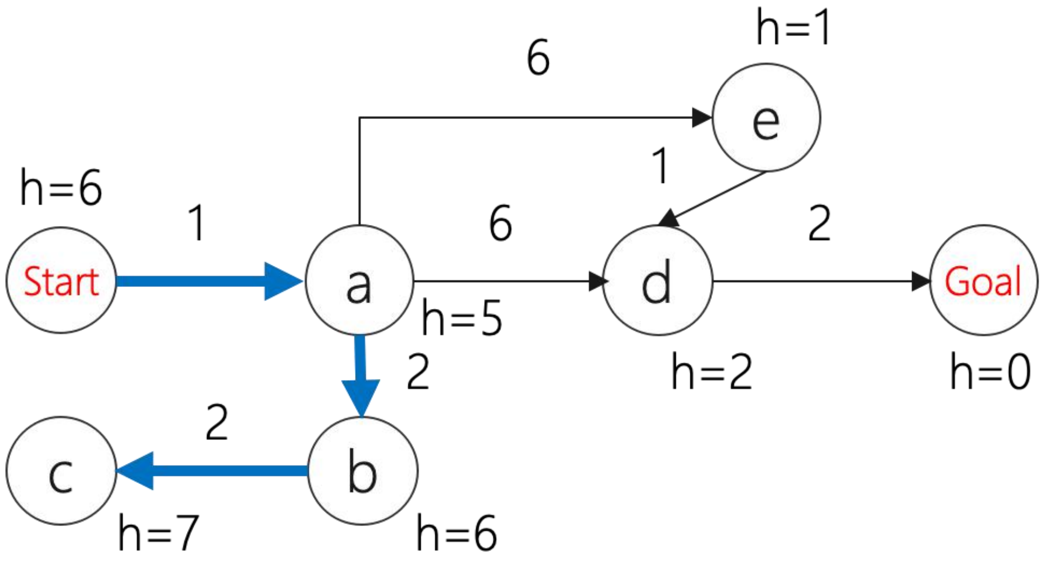
- $f(n) = g(n) + h(n)$
  - $g(n)$: the path cost from the start node to node n
  - $h(n)$: the estimated cost of the cheapest path from node n to the goal node

- When $h(n)$ satisfies certain properties, A* is both complete and optimal!

A* Search

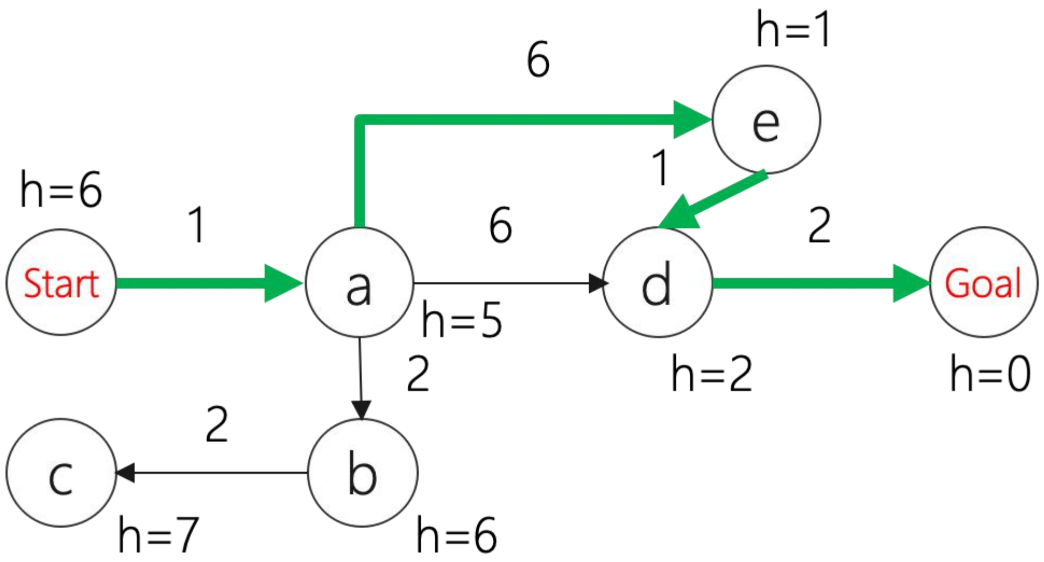


A* Search



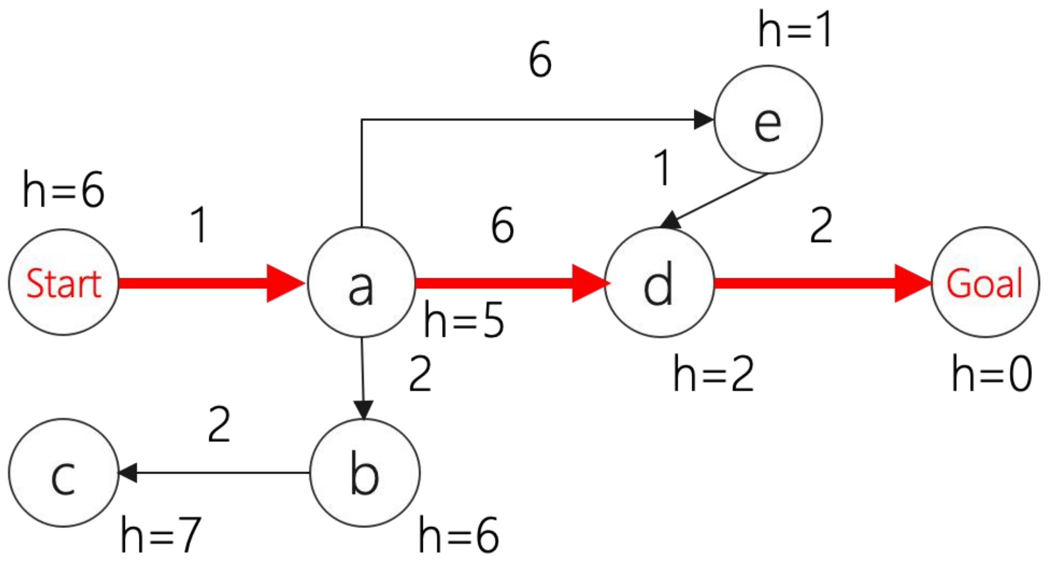
- Uniform-cost orders by path cost g(n)

A* Search



- Uniform-cost orders by path cost g(n)
- Greedy best first search orders by estimated goal proximity h(n)

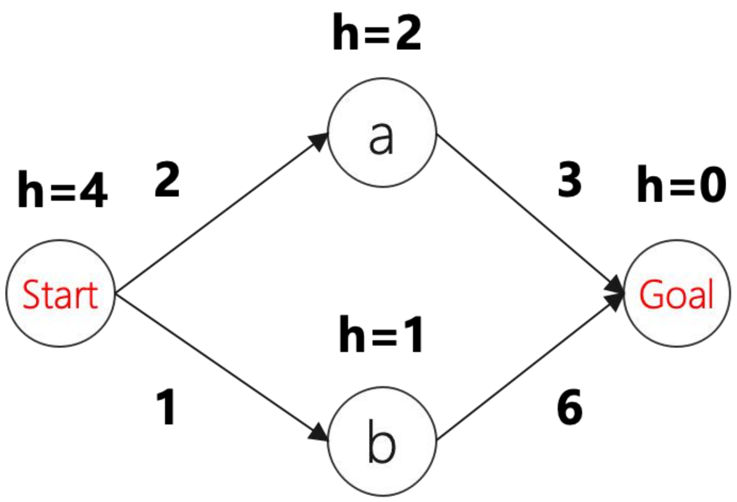
A* Search



- Uniform-cost orders by path cost g(n)
- Greedy best first search orders by estimated goal proximity h(n)
- A* search combines g(n) and h(n)

When to terminate in A* Search

- Similar to Uniform Cost Search, the goal test is applied to a node when it is selected for expansion,



- The path cost to the goal state may get updated.

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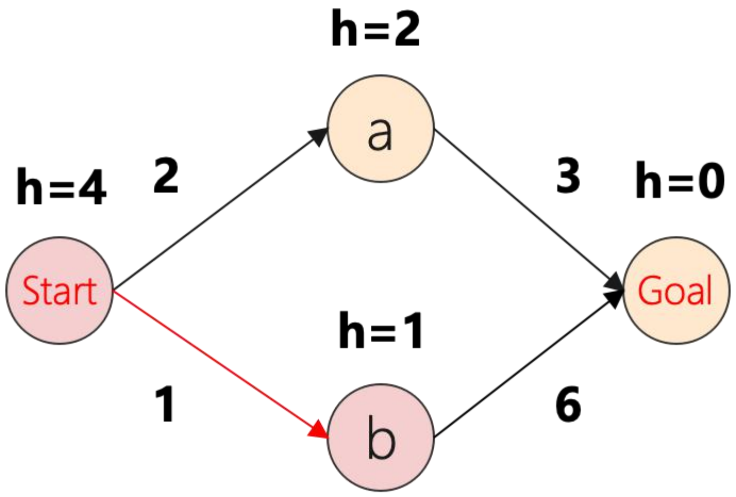

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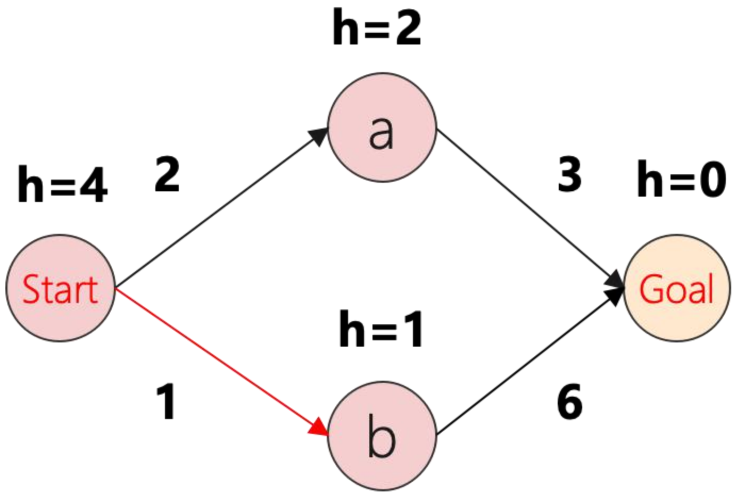
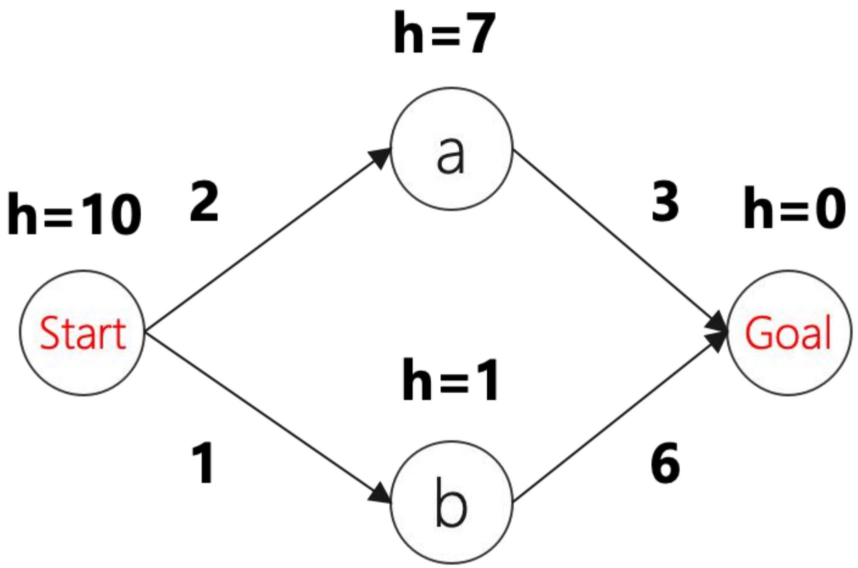
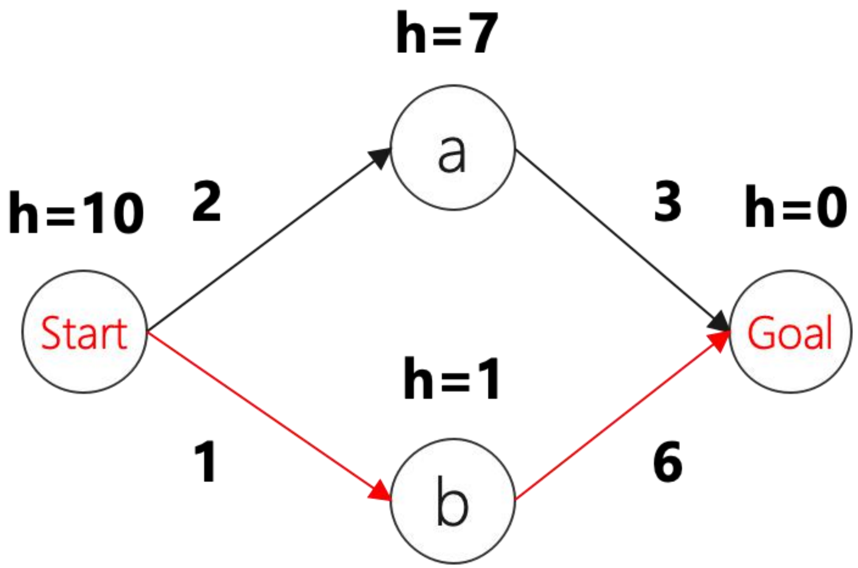


- The path cost to the goal state may get updated.

Is A* Search Optimal?



Is A* Search Optimal?



- What went wrong?
- Actual goal cost < estimated goal cost
- Solution?

Need Some Conditions

- To guarantee that A* finds an optimal solution, we need that h(n) never overestimates the cost of reaching the goal

- Called an admissible heuristic

- Transfer to f, i.e., f also doesn't overestimate.

Formal Definition of Admissibility

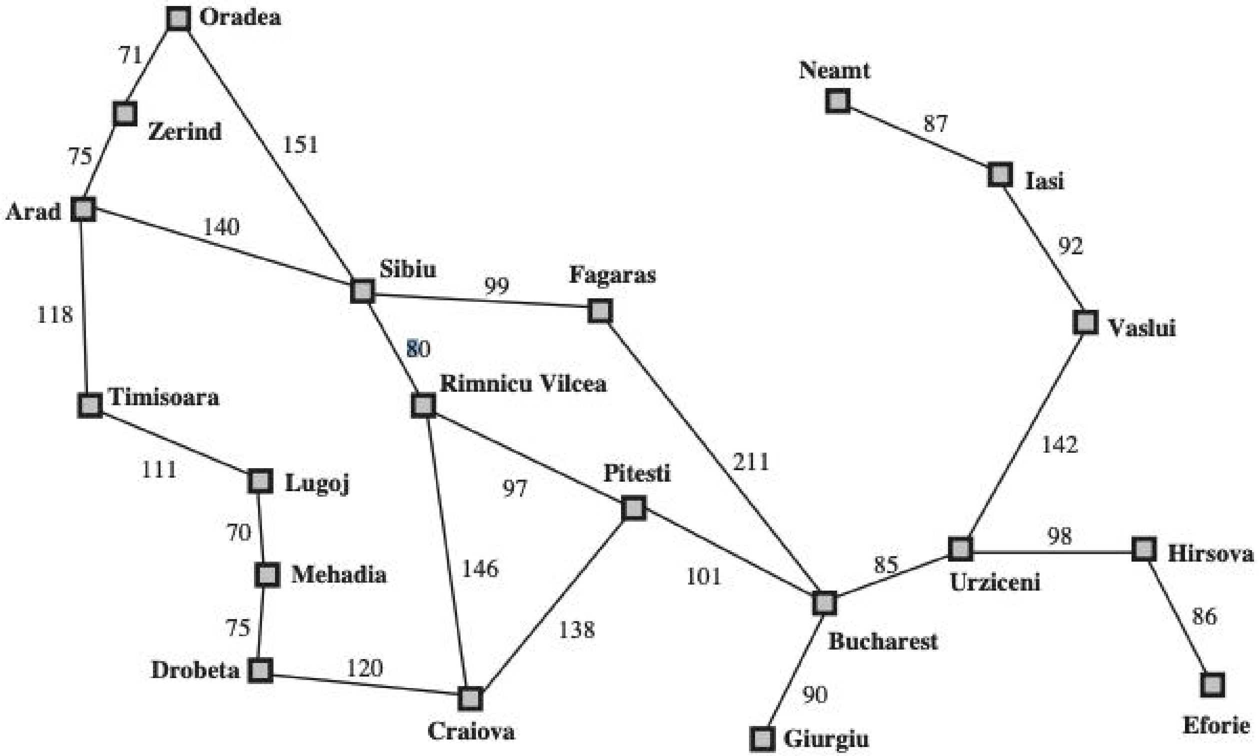
- Let $h^*(n)$ be the actual cost to reach a goal from n.

- A heuristic function h is optimistic or admissible if $0 \leq h(n) \leq h^*(n)$ all nodes n.

- If h is admissible, then the A* tree search will never return a sub-optimal goal node.

Example: Admissible Heuristic

- Path finding:
  - the straight-line distance to Bucharest



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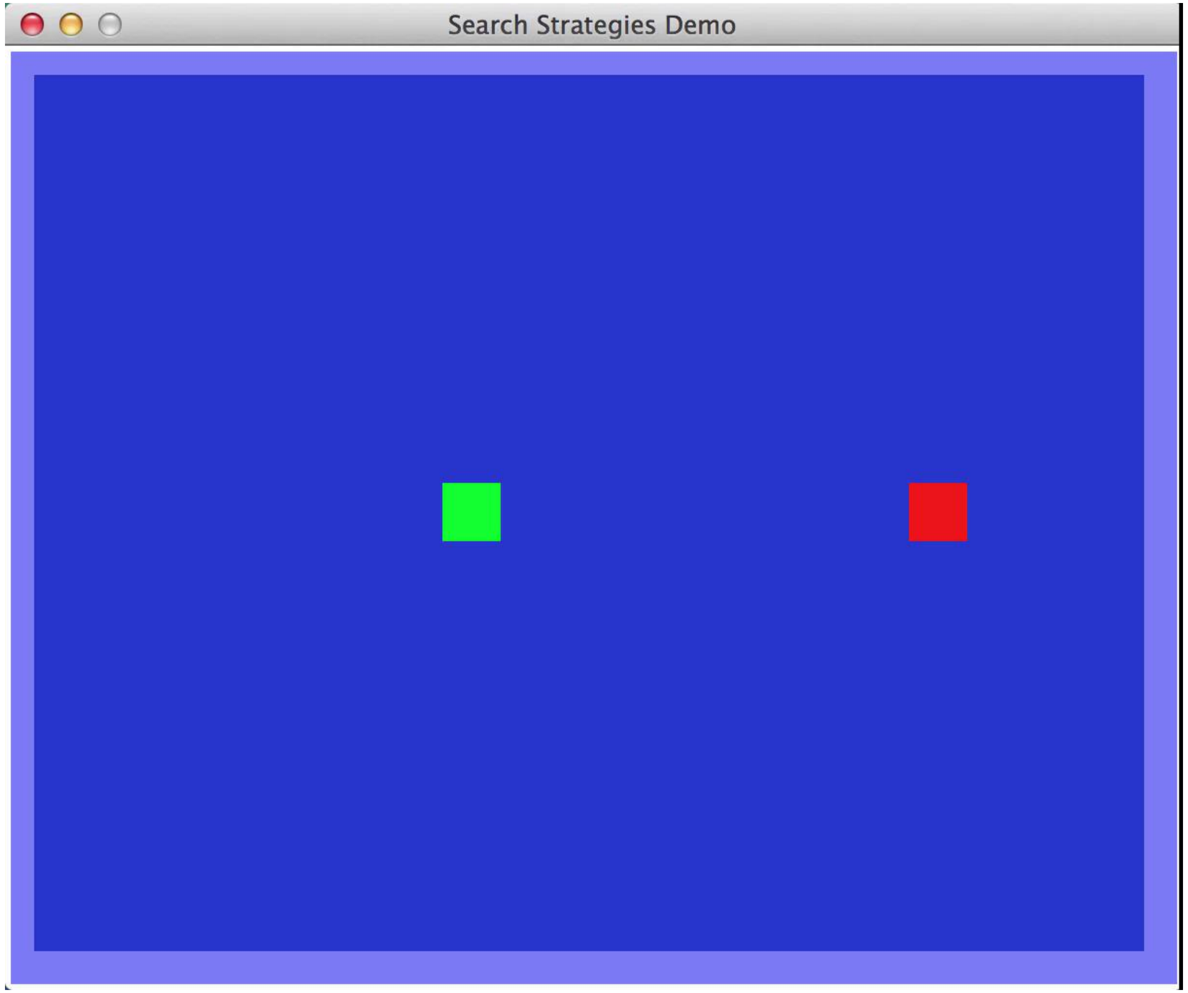
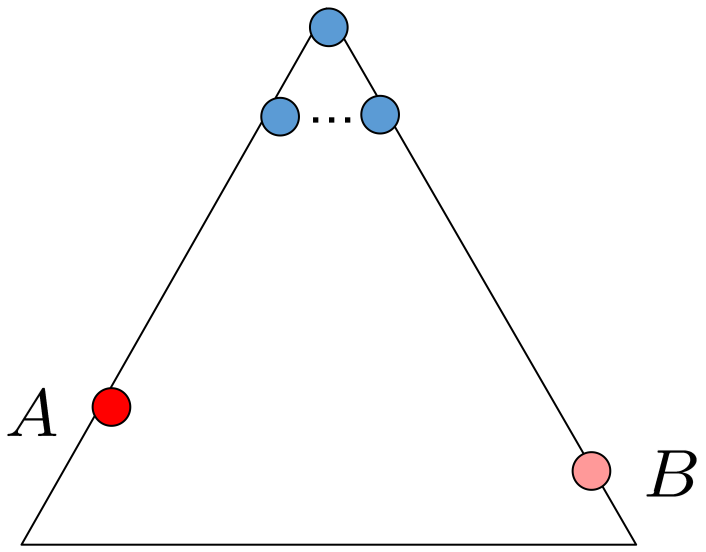
A* Search



Optimality of A* Tree Search

Assume:

- A is an optimal goal node
- B is a sub-optimal goal node
- h is admissible



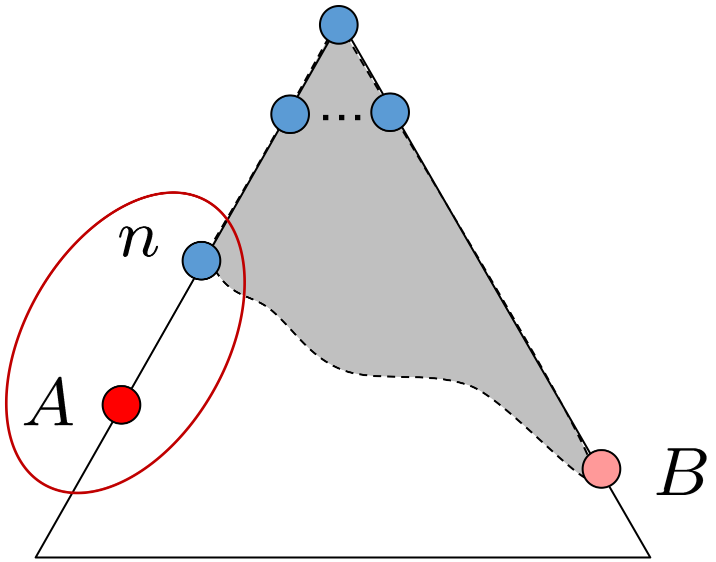
Claim:

A will be expanded before B

Optimality of A* Tree Search

Proof:

- Imagine B is on the frontier
- Some ancestor n of A is on the frontier, too (maybe A!)
- Claim: n will be expanded before B
  1. f(n) is less or equal to f(A)



$f(n) = g(n) + h(n)$	Definition of f-cost
$f(n) \leq g(A)$	Admissibility of h
$g(A) = f(A)$	h = 0 at a goal

Optimality of A* Tree Search

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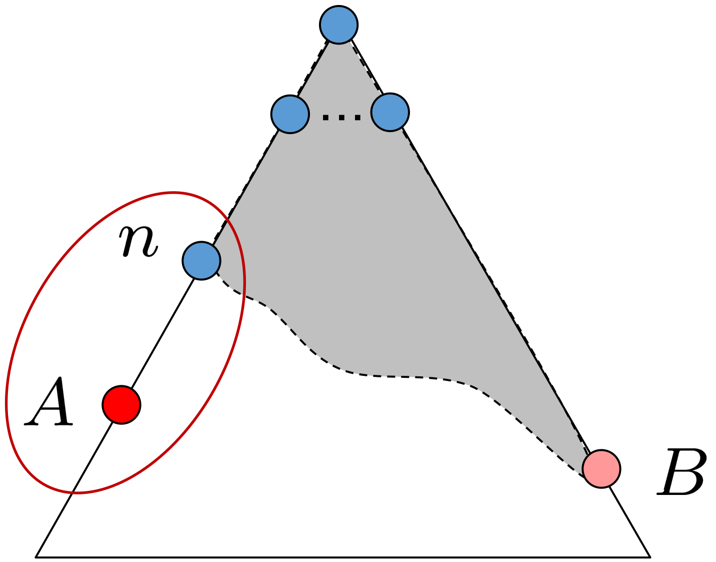


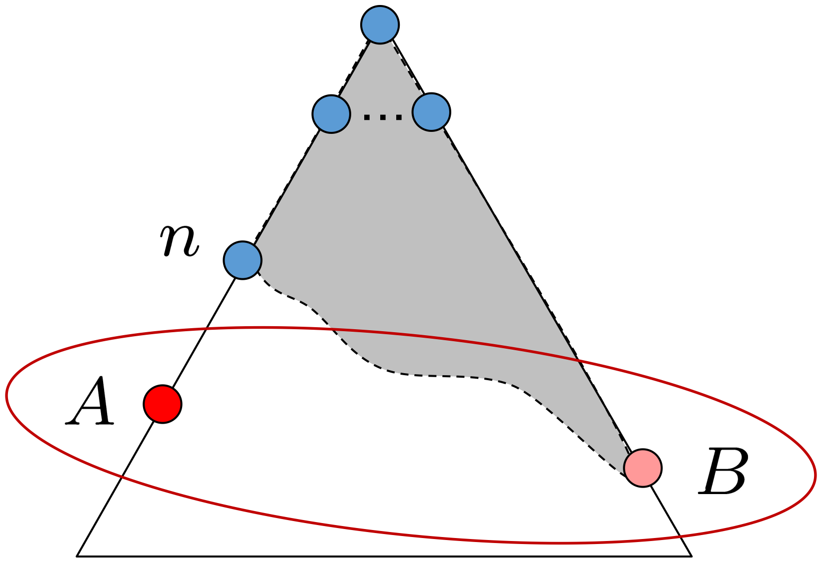
$$f(n) = g(n) + h(n)$$ Definition of f-cost

$$f(n) \leq g(A)$$ Admissibility of h

$$g(A) = f(A)$$ h = 0 at a goal

- Let h*(n) be the cheapest cost of getting to A from n
- h is admissible -> h(n) ≤ h*(n)
- h*(n) = g(A) - g(n) -> h(n) ≤ g(A) - g(n)
- -> f(n) ≤ f(A)

Optimality of A* Tree Search



Proof:

- Imagine B is on the frontier
- Some ancestor n of A is on the frontier, too (maybe A!)
- Claim: n will be expanded before B
  1. f(n) is less or equal to f(A)
  2. f(A) is less than f(B)

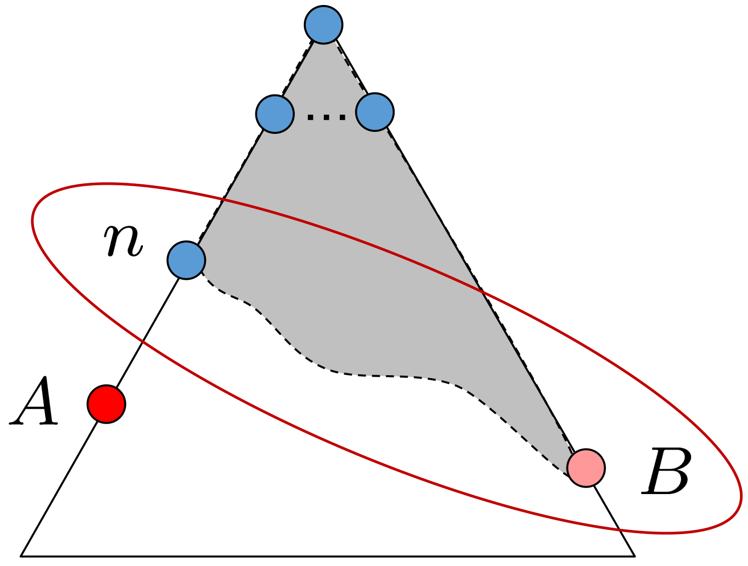
$g(A) < g(B)$ B is sub-optimal

$f(A) < f(B)$ h = 0 at a goal

Optimality of A* Tree Search

Proof:

- Imagine B is on the frontier
- Some ancestor n of A is on the frontier, too (maybe A!)
- Claim: n will be expanded before B
    1. f(n) is less or equal to f(A)
    2. f(A) is less than f(B)
    3. n expands before B

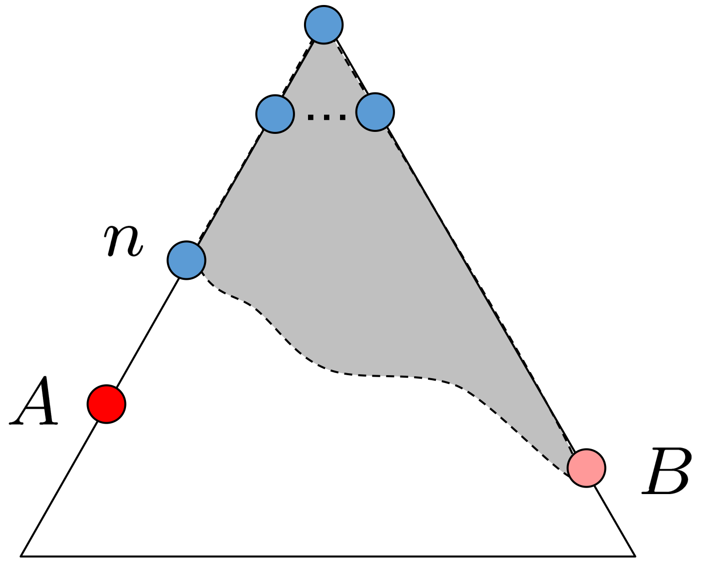


$$f(n) \leq f(A) < f(B)$$

Optimality of A* Tree Search

Proof:

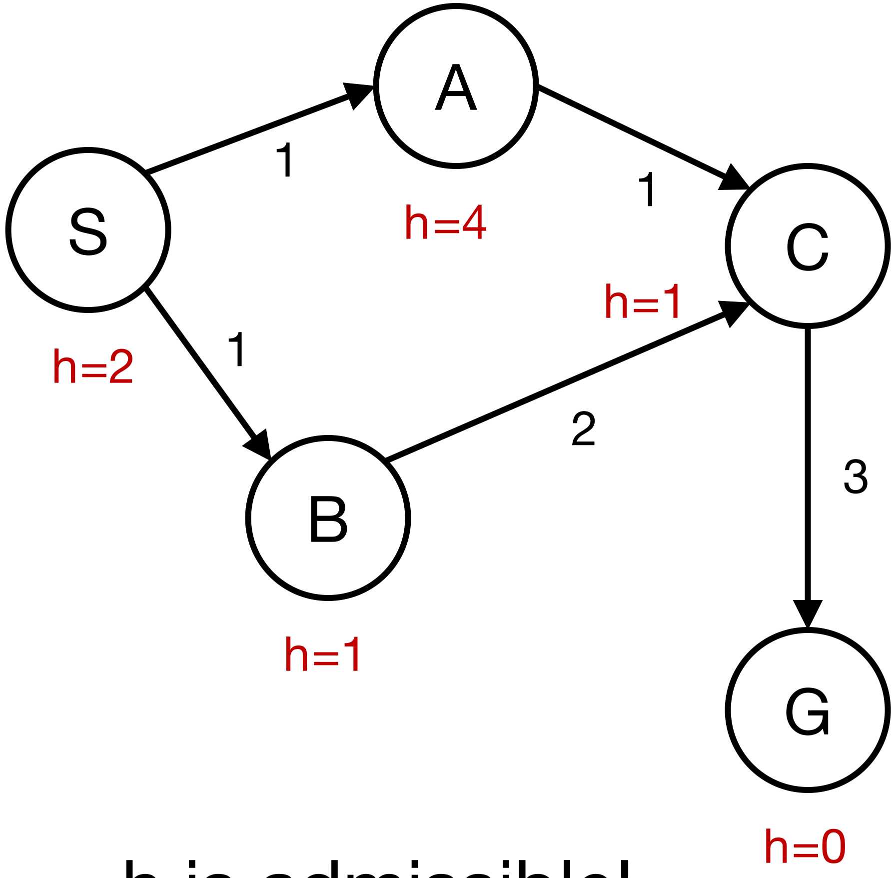
- Imagine B is on the frontier
- Some ancestor n of A is on the frontier, too (maybe A!)
- Claim: n will be expanded before B
    1. f(n) is less or equal to f(A)
    2. f(A) is less than f(B)
    3. n expands before B
- All ancestors of A expand before B --> A expands before B



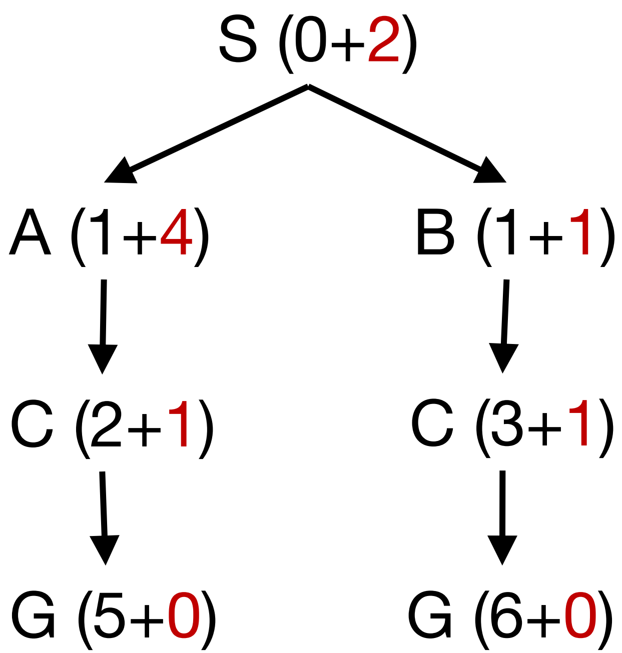
$$f(n) \leq f(A) < f(B)$$

A* Graph Search Gone Wrong?

State space graph

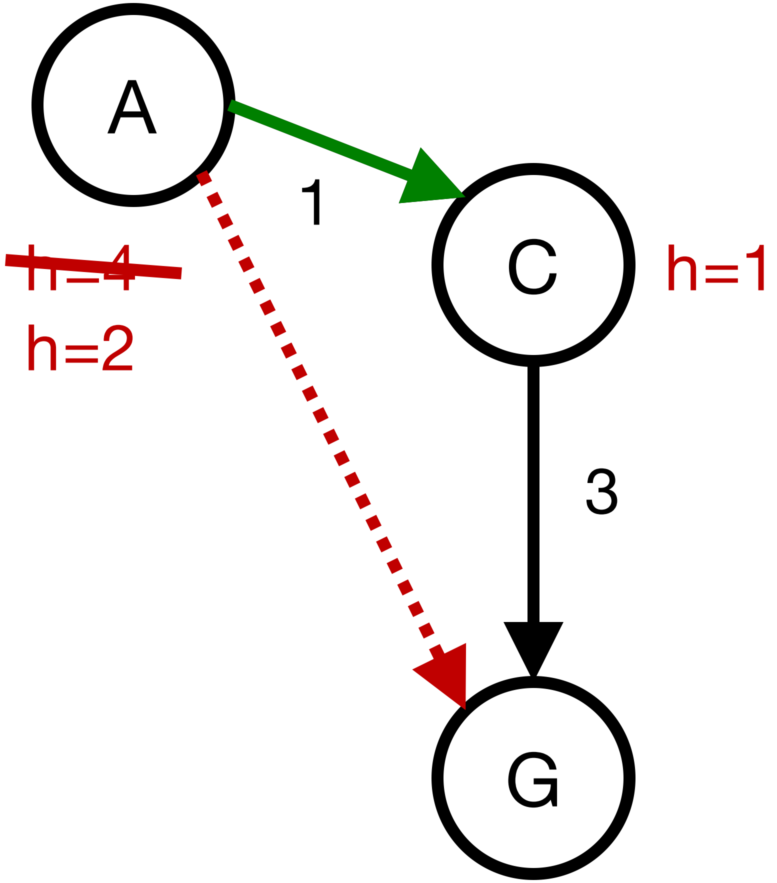


Search tree



h is admissible!

Consistency of Heuristics



- Main idea: estimated heuristic costs ≤ actual costs
  - Admissibility: heuristic cost ≤ actual cost to goal

    h(A) ≤ actual cost from A to G

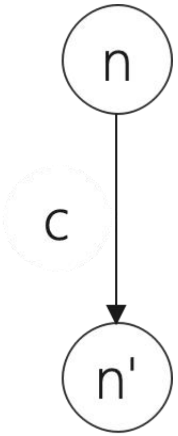
- Consistency: heuristic “arc” cost ≤ actual cost for each arc

    h(A) – h(C) ≤ cost(A to C)

Consequence of Consistency

- The f value along a path never decreases

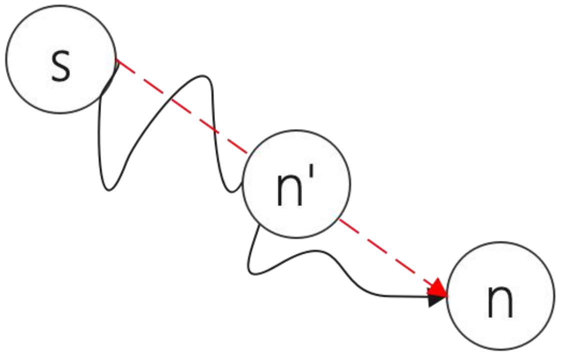
$$
\begin{aligned}
f(n') &= g(n') + h(n') \\
&= g(n) + c + h(n') \\
&\geq g(n) + h(n) \qquad \text{consistency} \\
&= f(n)
\end{aligned}
$$



Consequence of Consistency

- When A* selects a node for expansion, the optimal path to that node has been found
- Proof:

1. Assume $g(n) > g^*(n)$
2. Let n' be the shallowest node in frontier on the optimal path from s to n
3. $g(n') = g^*(n')$ and $f(n') = g^*(n') + h(n')$
4. We have $f(n') \leq g^*(n') + c(n', n) + h(n)$ consistency
5. $f(n') \leq g^*(n) + h(n)$
6. $f(n') < f(n)$ contradiction



Optimality of A* Graph Search

- Consider what A* does with a consistent heuristic:
  - Fact 1: A* expands nodes in nondecreasing total f value
  - Fact 2: For every state s, nodes that reach s optimally are expanded before nodes that reach s suboptimally
  - Result: A* graph search is optimal

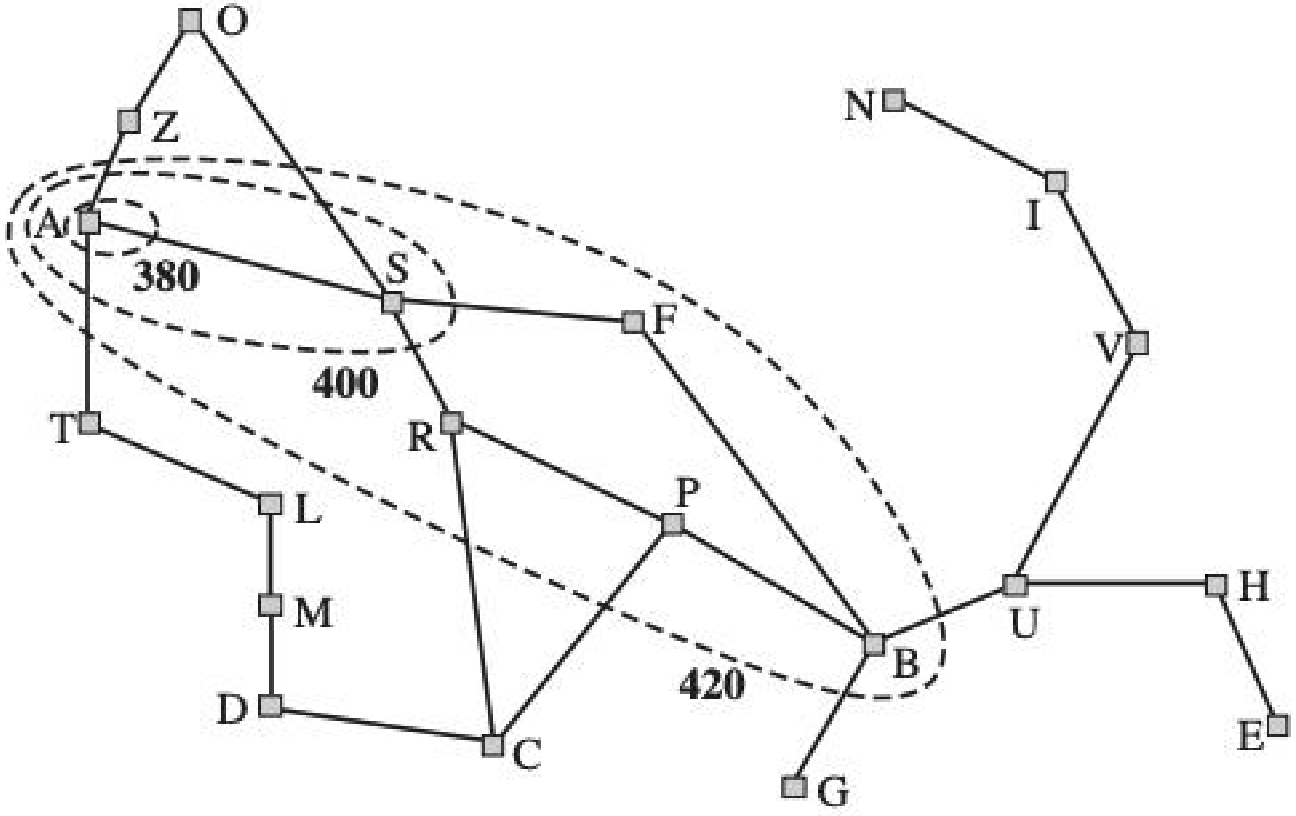
Summary

- Tree search:
  - A* is optimal if heuristic is admissible
  - UCS is a special case ($h = 0$)
- Graph search:
  - A* optimal if heuristic is consistent
  - UCS optimal ($h = 0$ is consistent)
- Consistency implies admissibility
- In general, most natural admissible heuristics tend to be consistent.

Intuition A*

- Let f* be the cost of optimal solution path.
- We have
  - A* expands all nodes with $f(n) < f^*$
  - A* may then expand some nodes right on “goal contour”, with $f(n) = f^*$ before selecting a goal node.

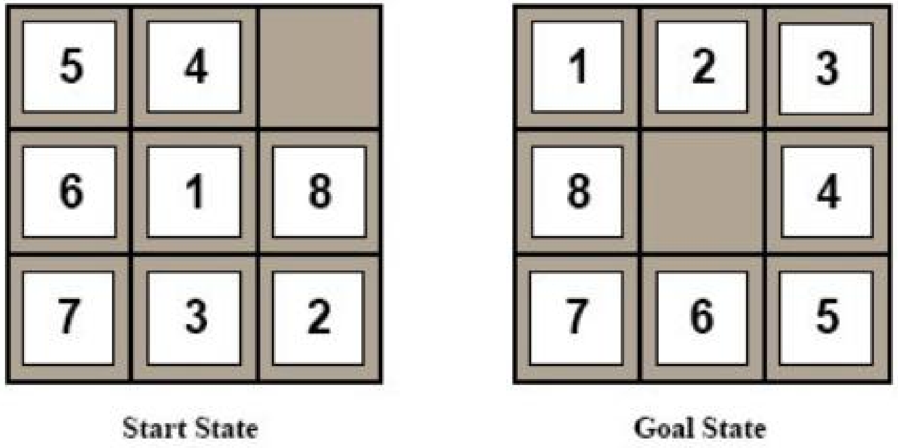
Intuition A*



- A* gradually adds "f-contours" of nodes
- Contour i has all nodes with $f=f_i$, where $f_i < f_{i+1}$

Design of Heuristics

1. h_1 = Number of misplaced tiles
2. h_2 = Manhattan distance
   - the sum of the distances of the tiles from their goal positions



Which one should we use?

$$h_1 \leq h_2 \leq h^*$$

Importance of h(n)

$$h_1 \leq h_2 \leq h^*$$

- Prefer h_2
- Note: Expand all nodes with $f(n) = g(n) + h(n) < f^*$
- Every node with $h(n) < f^* - g(n)$ will be expanded
- Since $h_1 \leq h_2$, every node expanded with h_2 will be expanded by h_1
- Aside. How would we get an h_{opt}?

Comparison of Search Costs on 8-Puzzle

	Search Cost (nodes generated)			Effective Branching Factor		
d	IDS	$A^*(h_1)$	$A^*(h_2)$	IDS	$A^*(h_1)$	$A^*(h_2)$
2	10	6	6	2.45	1.79	1.79
4	112	13	12	2.87	1.48	1.45
6	680	20	18	2.73	1.34	1.30
8	6384	39	25	2.80	1.33	1.24
10	47127	93	39	2.79	1.38	1.22
12	3644035	227	73	2.78	1.42	1.24
14	–	539	113	–	1.44	1.23
16	–	1301	211	–	1.45	1.25
18	–	3056	363	–	1.46	1.26
20	–	7276	676	–	1.47	1.27
22	–	18094	1219	–	1.48	1.28
24	–	39135	1641	–	1.48	1.26

- Effective branch factor b*: characterize the quality of a heuristic

$$N + 1 = 1 + b^* + (b^*)^2 + \ldots + (b^*)^d$$

Where Do the Heuristics Come From?

- Most of the work in solving hard search problems optimally is in coming up with admissible heuristics

- Admissible heuristics:

$$h(n) \leq h^*(n)$$

- But if $h^*(n)$ is unknown, how can we verify the condition?

Where Do the Heuristics Come From?

- Often, admissible heuristics are solutions to relaxed problems, where new actions are available -> fewer constraints

- The cost of an optimial solution to a relaxed problem is an admissible heuristic for the original problem

- The models obtained by such constraint-deletion processes are called relaxed models.

Example: 8 Queen

- Description of actions:

A tile can move from square A to square B if A is adjacent to B and B is blank

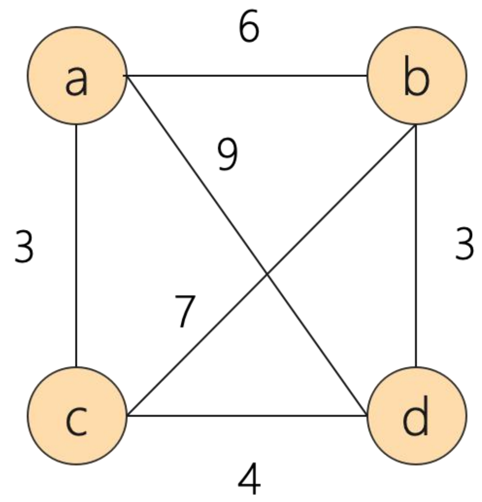
(a) A tile can move from square A to square B if A is adjacent to B.

(b) A tile can move from square A to square B if B is blank

(c) A tile can move from square A to square B

Example: Traveling Salesman Problem

- Find the shortest route that visits each city exactly once and returns to the origin city



- Find an estimate for the cheapest path that starts at a city X, ends at city Y, and go through each unvisited city.

Example: Traveling Salesman Problem

- Find an estimate for the cheapest path that starts at a city X, ends at city Y, and go through each unvisited city.

- Three conditions for the path:
  - Being a graph
  - Being connected
  - Being degree 2

Example: Traveling Salesman Problem

- Find an estimate for the cheapest path that starts at a city X, ends at city Y, and go through each unvisited city.

- Three conditions for the path:
  - Being a graph

    -> Optimal Assignment Problem

  - Being degree 2

Example: Traveling Salesman Problem

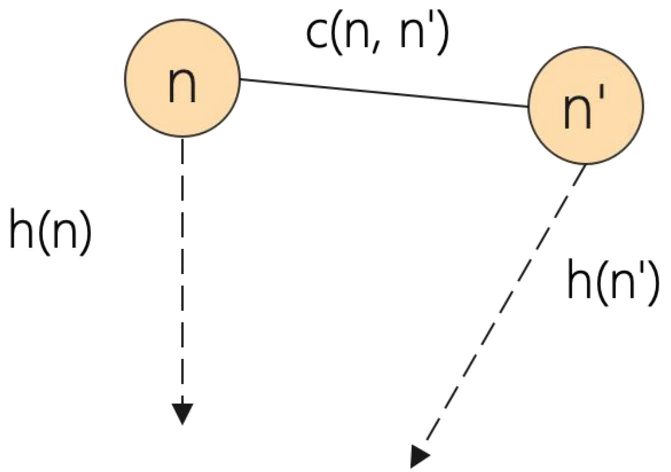
- Find an estimate for the cheapest path that starts at a city X, ends at city Y, and go through each unvisited city.

- Three conditions for the path:
  - Being a graph
  - Being connected

-> Minimum Spanning Tree Problem

Consistency

- Heuristics come from relaxed problems are guaranteed to be consistent.



- h(n) and h(n') stand for the minimum cost of finding solution for some relaxed problem. We have

$$h(n) \leq c'(n, n') + h(n') \quad \text{relaxed cost}$$

$$\rightarrow \quad h(n) \leq c(n, n') + h(n)$$

What if We Have Multple Heuristics

- Suppose we have heuristics $h_1(n)$, $h_2(n)$, $h_3(n)$

- Every node with $h(n) < f^* - g(n)$ will be expanded

- The final heuristic $h(n) = \max(h_1(n), h_2(n), h_3(n))$

A* Applications

- Video games
- Pathing / routing problems
- Resource planning problems
- Robot motion planning
- Language analysis
- Machine translation
- Speech recognition
- …

A*: Summary

Optimal: Yes

A* is optimally efficient: given the information in h, no other optimal search method can expand fewer nodes

Complete: Unless there are infinitely many nodes with $f(n) < f^*$. Asussme locally finite: (1) finite branching, (2) every operator costs at least $\epsilon > 0$.

Complexity (time and space): still exponential because of breadth-first nature. Unless $|h(n) - h^*| <= O(\log(h^*(n))$, with h true cost of getting to goal, the time complexity is polynomial.