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Today's agenda

Review theoretical tools for numerical analysis:

- Taylor series/Theorem
- Ratio test
- Alternating Theorem

Computational tools:

- Horner's algorithm

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Taylor series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad (|x| < \infty)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \qquad (|x| < \infty)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \qquad (|x| < \infty)$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots = \sum_{k=0}^{\infty} x^k \qquad (|x| < 1)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k} \qquad (-1 < x \leq 1)$$

Use Taylor for computation

- Use the Taylor series for the natural logarithm

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

- With $x = 1$

$$\ln\ 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots$$

- Add the eight terms

$$\ln\ 2 \approx 0.63452 \qquad \text{(poor approx.)}$$

$$\ln\ 2 = 0.69315\ldots \qquad \text{(exact value)}$$

Alternative

- Use a different Taylor series

$$\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\right)$$

- With $x = 1/3$

$$\ln\ 2 = 2\left(3^{-1} + \frac{3^{-3}}{3} + \frac{3^{-5}}{5} + \frac{3^{-7}}{7} + \cdots\right)$$

- Add the four terms and multiply by 2

$$\ln\ 2 \approx 0.69313$$

$$\ln\ 2 = 0.69315\ldots \quad \text{(exact value)}$$

Take-home message

Fast convergence of a Taylor series can be expected near the point of expansion.

- Taylor series for $f(x)$ at a point c

$$f(x) \sim f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots$$

$$f(x) \sim \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

- Maclaurin series if $c = 0$.
- How to compute? Horner's algorithm.

Deflation

Given a polynomial

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n$$

and a number r find another polynomial s.t.

$$p(x) = (x - r) q(x) + p(r)$$

$$q(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_{n-1} x^{n-1}$$

- A special case of polynomial long division.
- If $p(r) = 0$, r is a root of the polynomial.

Pseudocodes

Synthetic division

integer i, n; **real** r
real array $(a_i)_{0:n}, (b_i)_{0:n-1}$
$b_{n-1} \leftarrow a_n$
for $i = n - 1$ **to** 0
    $b_{i-1} \leftarrow a_i + r b_i$
end for

Derivative

integer i, n; **real** p, r
real array $(a_i)_{0:n}, (b_i)_{0:n-1}$
$\alpha \leftarrow a_n$; $\beta \leftarrow 0$
for $i = n - 1$ **to** 0
    $\beta \leftarrow \alpha + r\beta$
    $\alpha \leftarrow a_i + r\alpha$
end for

Back to Taylor expansion

- Recall

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$$
$$= c_n (x-r)^n + c_{n-1}(x-r)^{n-1} + \cdots + c_1(x-r) + c_0$$

- Deflating the polynomial

$$q(x) = \frac{p(x) - p(r)}{x - r} = c_n (x-r)^{n-1} + c_{n-1}(x-r)^{n-2} + \cdots + c_1$$

- Pseudocode

```
integer n, k, j;
real r; real array (a_i)_{0:n}
for k = 0 to n − 1
    for j = n − 1 to k
        a_j ← a_j + r a_{j+1}
    end for
end for
```

Taylor Theorem

Taylor's Theorem for $f(x)$

If the function f possesses continuous derivatives of orders $0, 1, 2, \ldots, (n+1)$ in a closed interval $I = [a, b]$, then for any c and x in I,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + E_{n+1} \tag{8}$$

where the error term E_{n+1} can be given in the form

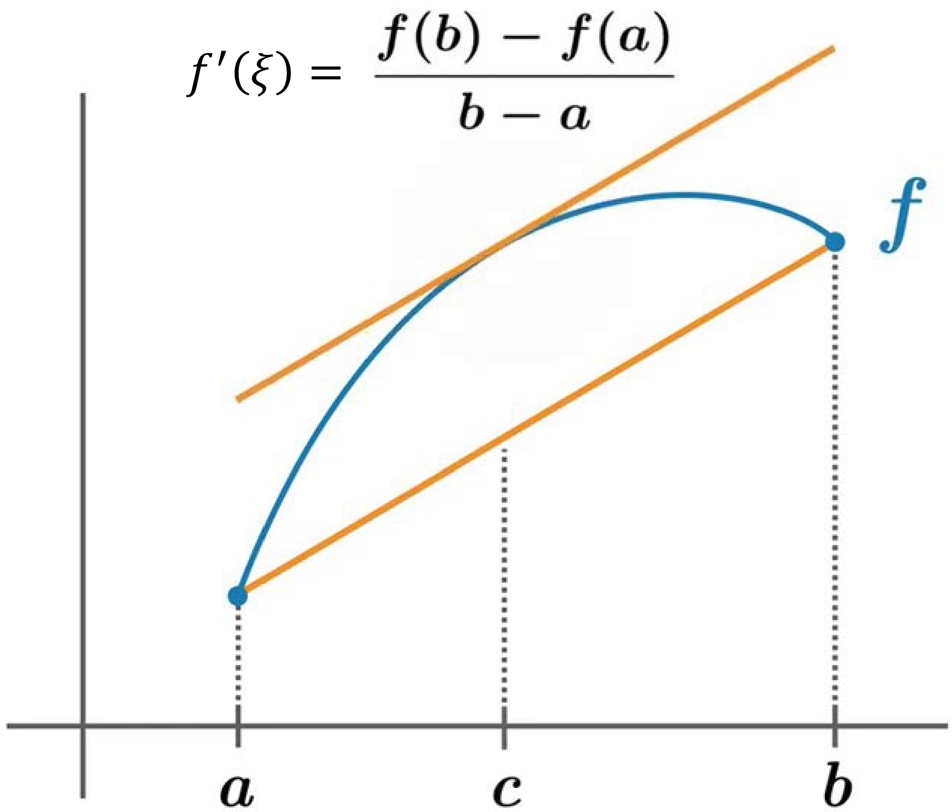
$$E_{n+1} = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}$$

Here ξ is a point that lies between c and x and depends on both.

- In practice, it is necessary to truncate→partial sum.
- E is called the remainder or error term.
- Convergence can be established in some cases.

Mean Value Theorem

A special case of Taylor Theorem



Taylor Theorem for $f(x+h)$

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + E_{n+1}$$

$$x \to x + h$$

$$f(x+h) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} h^k + E_{n+1}$$

with

$$E_{n+1} = \frac{f^{(n+1)}(\xi)}{(n+1)!} h^{n+1}$$

- Error term converges to zero with the same rate as h^{n+1}.
- Introduce big O notation, $E_{n+1} = O(h^{n+1})$, which means

$$|E_{n+1}| \le C|h|^{n+1}$$

Examples

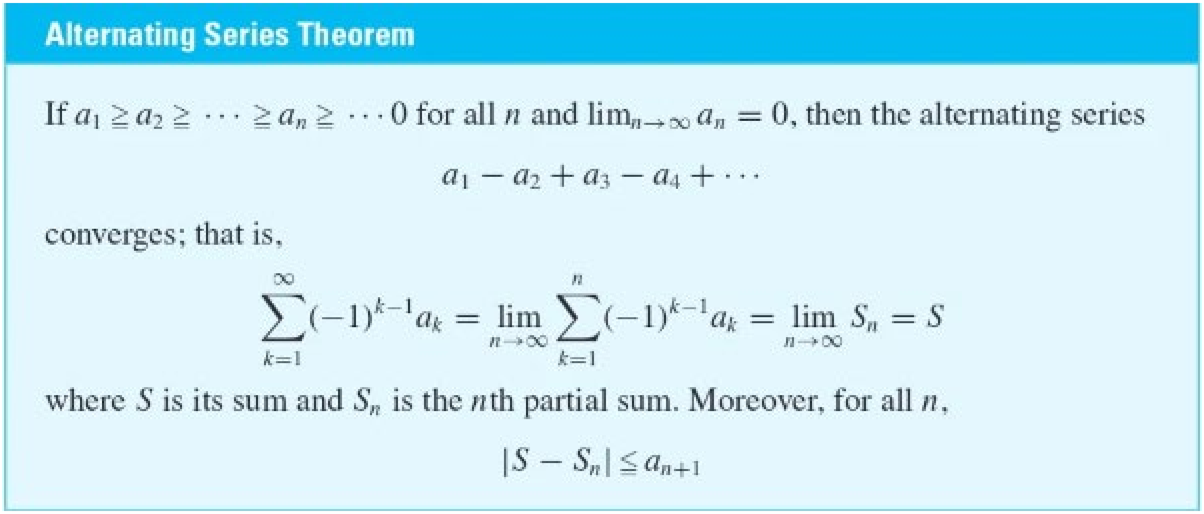
- It holds for every n

$$f(x+h) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} h^k + E_{n+1}$$

- Some commonly used ones:

$$\begin{aligned}
f(x+h) &= f(x) + f'(\xi_1)h \\
&= f(x) + \mathcal{O}(h) \\
f(x+h) &= f(x) + f'(x)h + \frac{1}{2!} f''(\xi_2)h^2 \\
&= f(x) + f'(x)h + \mathcal{O}(h^2) \\
f(x+h) &= f(x) + f'(x)h + \frac{1}{2!} f''(x)h^2 + \frac{1}{3!} f'''(\xi_3)h^3 \\
&= f(x) + f'(x)h + \frac{1}{2!} f''(x)h^2 + \mathcal{O}(h^3)
\end{aligned}$$

Alternating series



Alternating Series Theorem

If $a_1 \geqq a_2 \geqq \cdots \geqq a_n \geqq \cdots 0$ for all n and $\lim_{n\to\infty} a_n = 0$, then the alternating series

$$a_1 - a_2 + a_3 - a_4 + \cdots$$

converges; that is,

$$\sum_{k=1}^{\infty}(-1)^{k-1}a_k = \lim_{n\to\infty}\sum_{k=1}^{n}(-1)^{k-1}a_k = \lim_{n\to\infty} S_n = S$$

where S is its sum and S_n is the nth partial sum. Moreover, for all n,

$$|S - S_n| \leqq a_{n+1}$$

- It only applies to alternating series.
- It gives an upper bound for the error.
- Back to $\ln 2$ for an example.