Recap on Newton's method

- Newton's recursive definition

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- Today's agenda
  - Pseudocode of Newton
  - Quadratic Convergence
  - Secant method

Pseudocode

- Input: f(x), df(x), x0 (initial)

- Output: root of f(x)

Example

- Given $f(x) = x^3 - 2x^2 + x - 3$

$$f'(x) = 3x^2 - 4x + 1$$

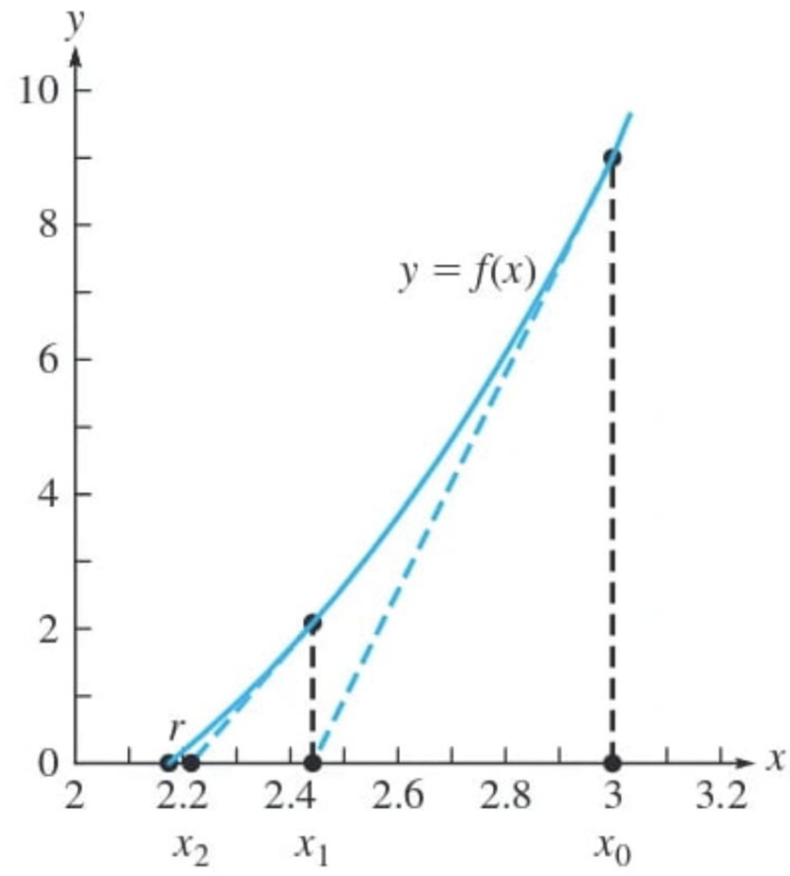
- For efficiency, nested multiplication

$$f(x) = ((x-2)x+1)x - 3$$

$$f'(x) = (3x-4)x + 1$$

Example (cont'd)

n	x_n	$f(x_n)$
0	3.0	9.0
1	2.4375	2.04
2	2.21303 27224 73144 5	2.56×10^{-1}
3	2.17555 49386 14368 4	6.46×10^{-3}
4	2.17456 01006 55071 4	4.48×10^{-6}
5	2.17455 94102 93284 1	1.97×10^{-12}



- Doubling of the accuracy in f(x)
- Rapid convergence!
- 5-10 iterations are generally sufficient.

Convergence analysis

If f, f', f'' are continuous in a neighborhood of a root r of f and if $f'(r) \neq 0$ (simple zero) , then Newton's method converges quadratically!

Recall Taylor Theorem

$$\begin{aligned} f(x+h) &= f(x) + f'(x)h + \frac{1}{2!} f''(\xi_2)h^2 \\ &= f(x) + f'(x)h + \mathcal{O}(h^2) \end{aligned}$$

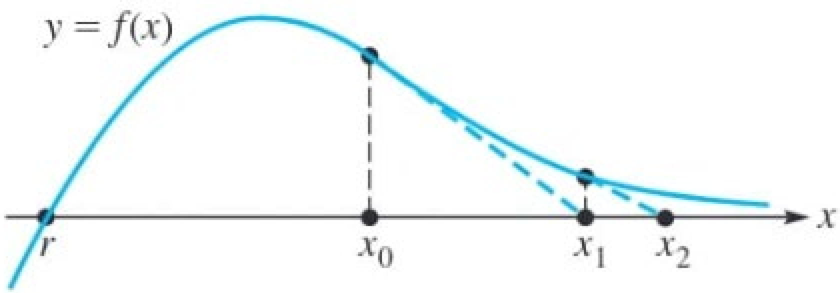
Define

$$c(\delta) = \frac{1}{2} \frac{\max\limits_{|x-r| \leqq \delta} |f''(x)|}{\min\limits_{|x-r| \leqq \delta} |f'(x)|} \qquad (\delta > 0)$$

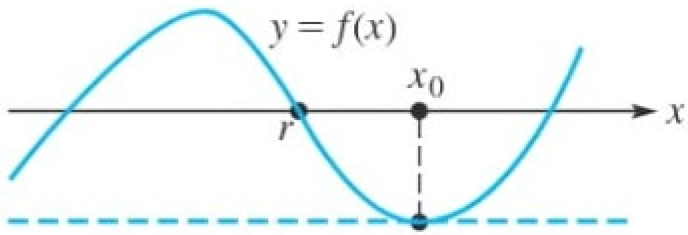
Remarks

- Newton's method relies on a starting point.
- Bisection (initial) + Newton (improve accuracy)
- Convergence depends upon hypotheses that are difficult to verify *a priori*.

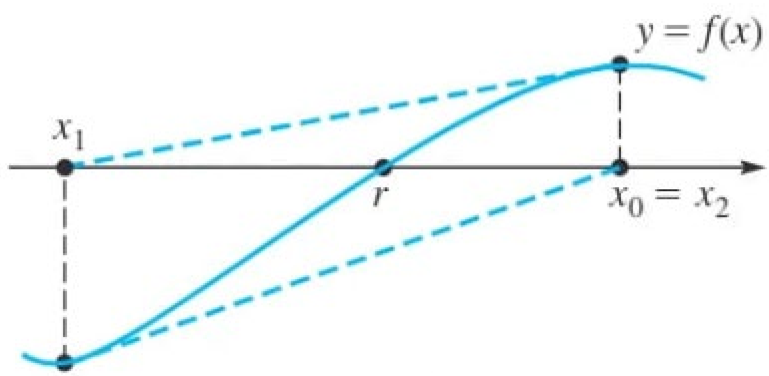
Failure of Newton's



(a) Runaway



(b) Flat spot



(c) Cycle

Multiplicity

- Quadratic convergence holds only for simple zero, i.e., $f'(r) \neq 0$
- The multiplicity of the zero is the least m s.t.

$$f^{(k)}(r) = 0, \forall k < m$$

- Newton's method converges linearly for a multiple zero.
- Modified Newton's method with multiplicity m

$$x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)}$$

Nonlinear equations

$$\begin{cases} f_1(x_1, x_2, \ldots, x_N) &= 0 \\ f_2(x_1, x_2, \ldots, x_N) &= 0 \\ &\vdots \\ f_N(x_1, x_2, \ldots, x_N) &= 0 \end{cases} \longrightarrow$$

$$\mathbf{F}(\mathbf{X}) = \mathbf{0}$$

$$\mathbf{F} = [f_1, f_2, \ldots, f_N]^T$$
$$\mathbf{X} = [x_1, x_2, \ldots, x_N]^T$$

Newton's method

$$\mathbf{X}^{(k+1)} = \mathbf{X}^{(k)} - \underbrace{\left[\mathbf{F}'(\mathbf{X}^{(k)})\right]}_{\text{Jacobian matrix}}{}^{-1}\mathbf{F}(\mathbf{X}^{(k)})$$

An example

$$\begin{cases} x + y + z = 3 \\ x^2 + y^2 + z^2 = 5 \\ e^x + xy - xz = 1 \end{cases}$$

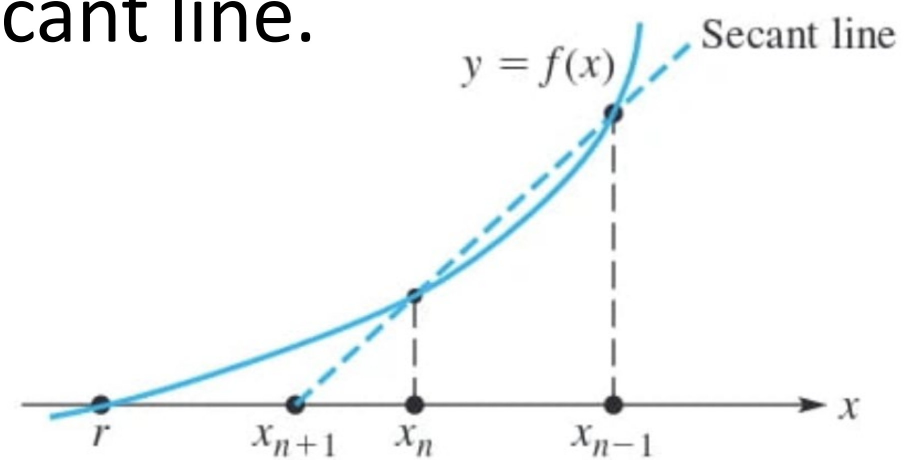
$$\mathbf{F} = \begin{bmatrix} x_1 + x_2 + x_3 - 3 \\ x_1^2 + x_2^2 + x_3^2 - 5 \\ e^{x_1} + x_1x_2 - x_1x_3 - 1 \end{bmatrix}$$

$$\mathbf{J} = \begin{bmatrix} 1 & 1 & 1 \\ 2x_1 & 2x_2 & 2x_3 \\ e^{x_1} + x_2 - x_3 & x_1 & -x_1 \end{bmatrix}$$

Secant method

Overview

- Bisection method requires two points with opposite signs.
- Newton method's drawback lies in the derivative calculation.
- Secant method approximates the function derivative by secant line.



Remarks

- Newton’s method could be called tangent method.
- Secant method requires two initial points, but no need to have opposite signs as bisection.
- Overflow may occur if demon is closer to 0.
- Superlinear convergence!