Artificial Intelligence

CS4365 --- Fall 2022

Constraint Satisfication Problem

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Standard Search Problem

- Standard Search Problems:
  - State is a “black box”: arbitrary data structure
  - Goal test can be any function over states
  - Successor function can also be anything

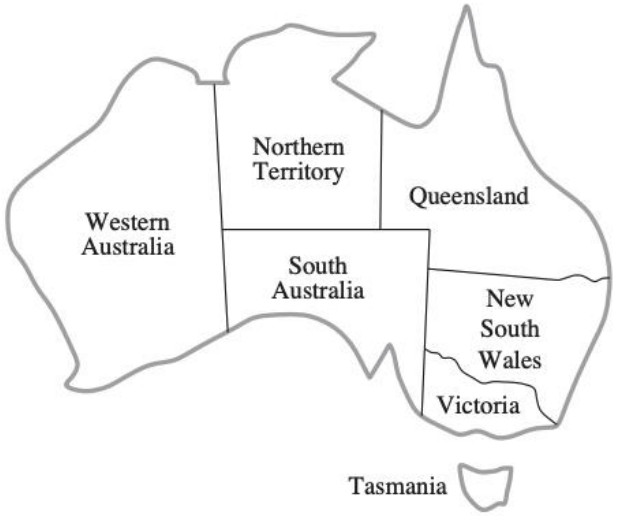
Constraint Satisfication Problems (CSP)

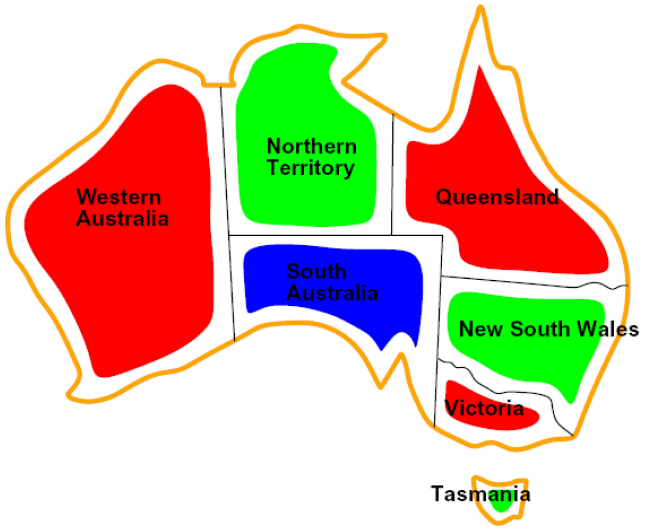
- Constraint satisfaction problems (CSPs):
  - A special subset of search problems
  - State is defined by variable X_i with values from a domain D (sometimes D depends on i)
  - Goal test is a set of constraints specifying allowable combinations of values for subsets of variables

- Why CSPs?
  - Allows useful general-purpose algorithms with more power than standard search algorithms

Example: Map-Coloring Problem

- Color each region either red, green or blue in such a way that no neighboring regions have the same color





  - Variables: different regions
  - Domains: colors

Constraint Satisfication Problems (CSP)

- A powerful representation for (discrete) search problems.

- A Constraint Satisfication Problem (CSP) is defined by:
  - X is a set of n variables X_1, X_2, ..., X_n, each defined by a finite domain D_1, D_2, ..., D_n of possble values
  - C is a set of constraints C_1, C_2, ..., C_m. Each C_i involves some subset of the variables; specifices the allowable combinations of values for that subset.

A solution is an assigment of values to the variables that satisfies all constraints.

Cyptarithmetic as a CSP

- Variables:

  F O U R T W C_1 C_2 C_3

- Domains:

  {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}

- Constraints:

  $O + O = R + 10 * C_1$

  $C_1 + W + W = U + 10 * C_2$

  $C_2 + T + T = O + 10 * C_3$

  $C_3 = F$

  alldiff(F, O, U, R, T, W)

```
   TWO
+  TWO
------
  FOUR
```

Cyptarithmetic Constraint Graph

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  F O U R T W C_1 C_2 C_3

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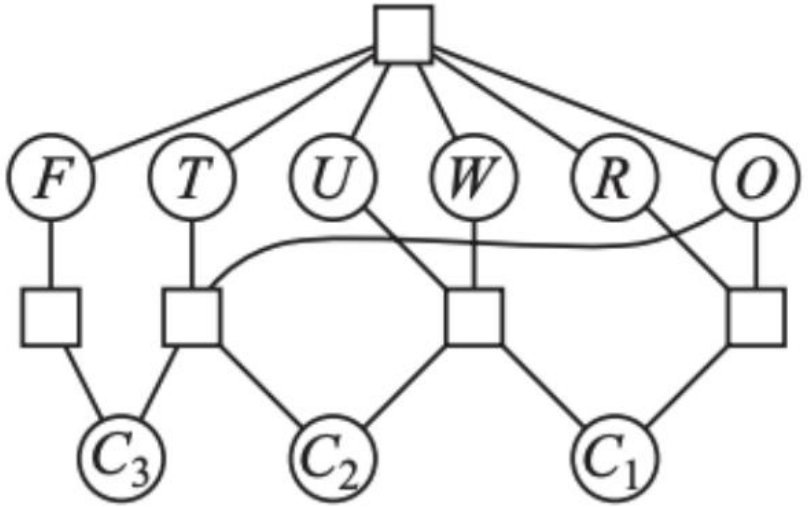


Map-Coloring Problem

- Variables:

  WA, NT, Q, NSW, V, SA, T

- Domains:

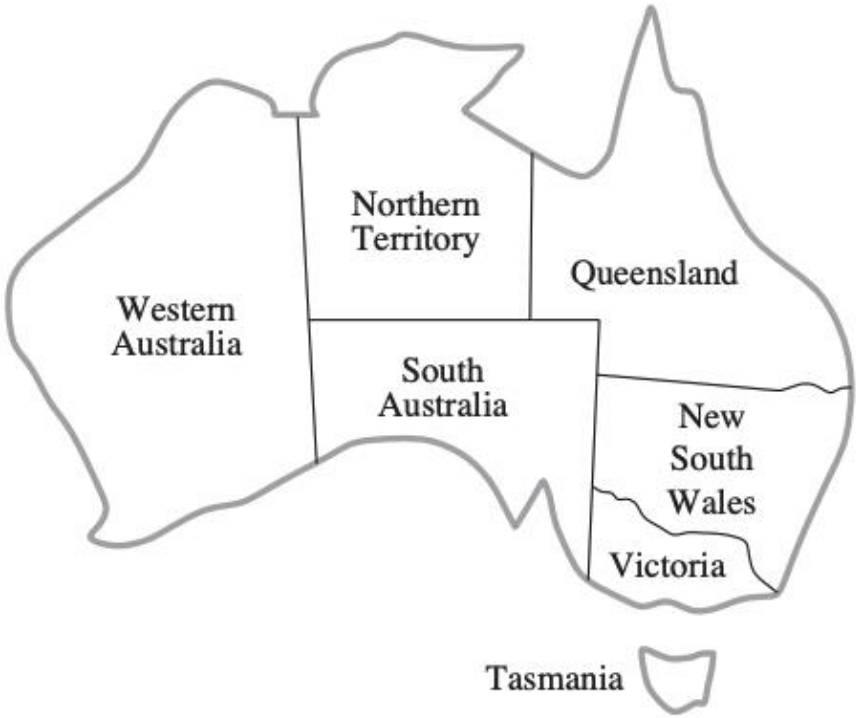
  {red, green, blue}

- Constraints:

  Adjacent regions must have different colors

  Implicit: WA ≠ NT

  Explicit: (WA, NT) ∈ {(red, green), {red, blue}, ...}



Constraint Graph

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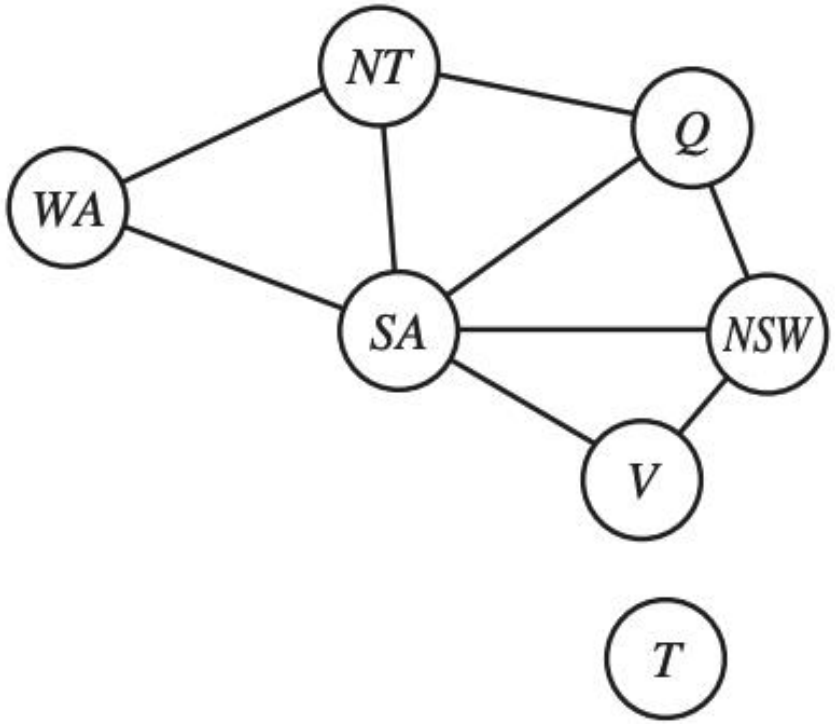

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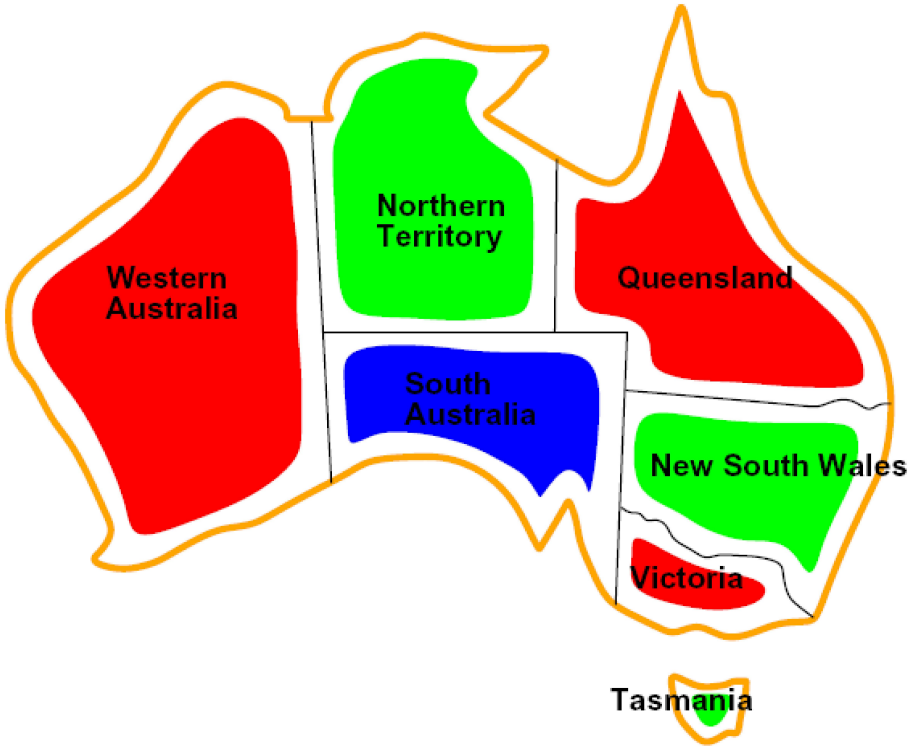


Varieties of CSPs

- Discrete variables
  - Finite domains: each variable can be only assigned to finite number of values.
  - Infinite domains: each variable can be assigned to infinite number of values, e.g., set of integers
- Continuous variables
  - Linear programming

$$
\begin{aligned}
\text{Minimize} \quad & z = 0.6x_1 + 0.35x_2 \\
\text{subject to:} \quad & \\
& 5x_1 + 7x_2 \geq 8 \\
& 4x_1 + 2x_2 \geq 15 \\
& 2x_1 + x_2 \geq 3 \\
& x_1 \geq 0, x_2 \geq 0.
\end{aligned}
$$

Varieties of Constraints

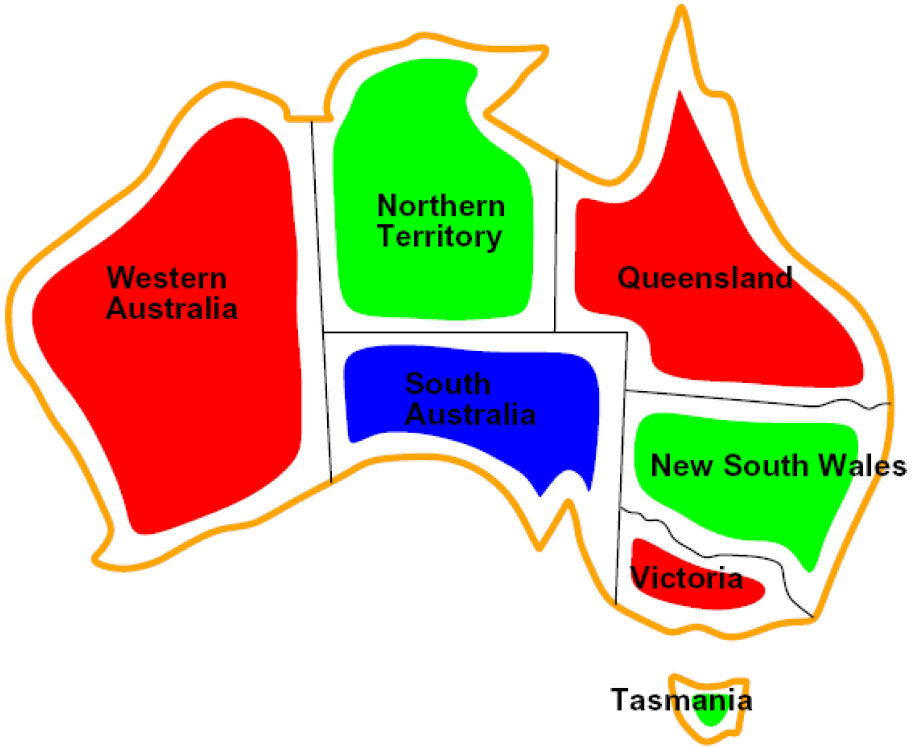
- Unary constraints involve a single variable (equivalent to reducing domains), e.g.:
  - SA ≠ Green
- Binary constraints involve pairs of variables, e.g.:
  - SA ≠ WA
- Higher-order constraints involve 3 or more variables
- Preferences (soft constraints):
  - E.g., red is better than green

Real-World CSPs

- Assignment problems
  - e.g., who teaches what class
- Timetabling problems
  - e.g., which class is offered when and where?
- Transportation scheduling
- Factory scheduling

Constraint Satisfication Problems (CSP)

- For a given CSP the problem is one of the following:
  - Find all solutions
  - Find one solution
  - The optimal solution given an objective function
  - Determine if a solution exists



How to View a CSP as a Search Problem?

- Initial State -- state in which all the variables are unassigned.
- Successor function -- assign a value to a variable from a set
- Goal test -- check if all the variables are assigned and all the constraints are satsified. We call the assignment is complete.
- Path cost -- assumes constant cost for each step

Backtracking Search

- Depth-first search for CPSs with single-variable assigments is called backtracking search

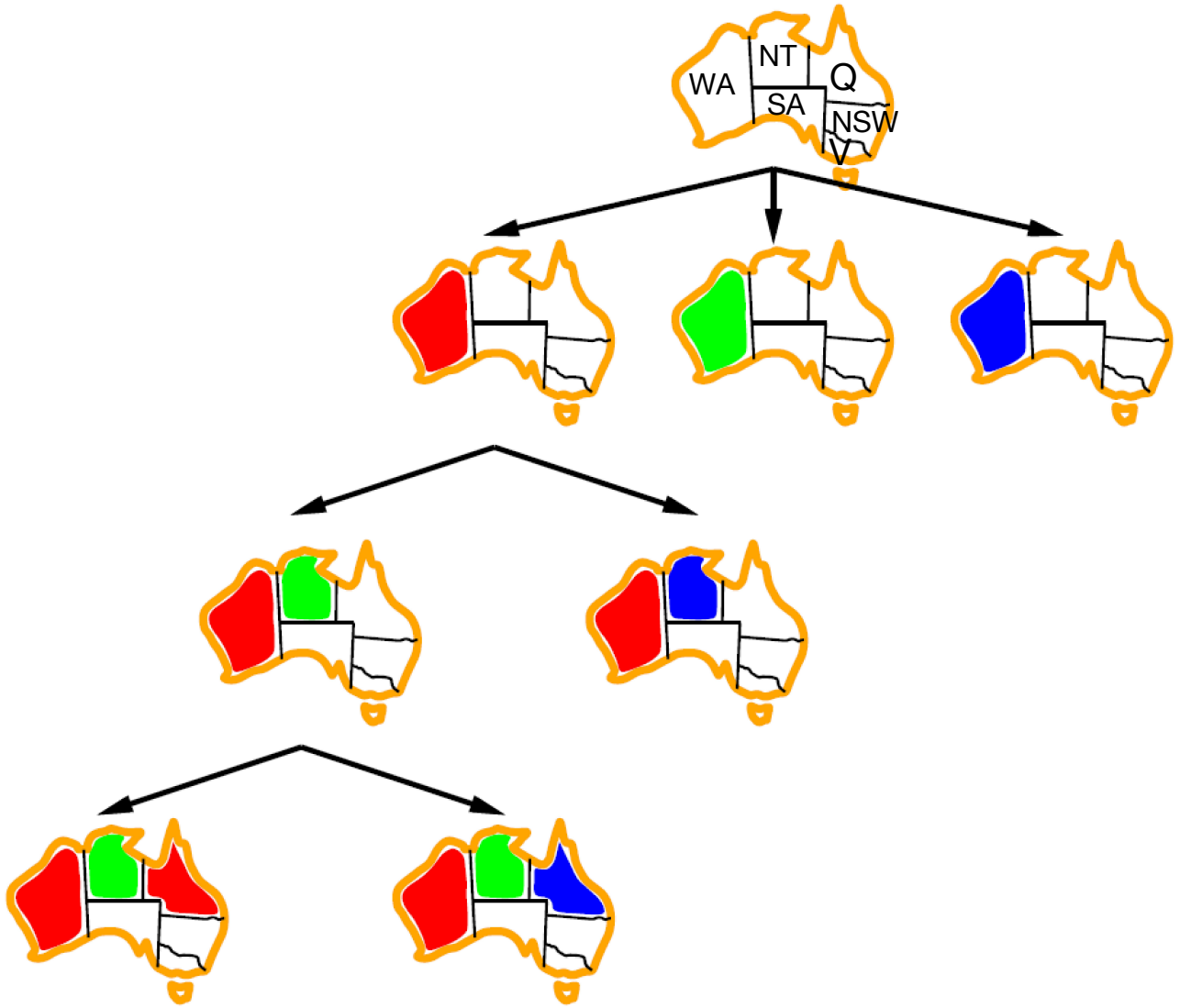
  - Backtracks when a variable has no legal values left to assign

Backtracking search is the basic uninformed algorithm for CSPs.

The Backtracking Search Algorithm

```
Backtrack(assignment, csp):
    If assignment is complete then return assignment
    var ← select one unassigned variable
    for each value in the domain of var do
        if value is consistent with assignment then
            add {var = value} to assignment
            result ← Backtrack(assignment, csp)
            if result ≠ failure then
                return result
            remove {var = value} from assignment
    return failure
```

Backtracking Example



The Backtracking Search Algorithm

```
Backtrack(assignment, csp):
    If assignment is complete then return assignment
    var ← select one unassigned variable  ← which one to select?
    for each value in the domain of var do ← which one to use?
        if value is consistent with assignment then
            add {var = value} to assignment     ← detect failure early
            result ← Backtrack(assignment, csp)
            if result ≠ failure then
                return result
            remove {var = value} from assignment
    return failure
```

Improving Backtracking Efficiency

General-purpose methods can give huge gains in speed:

1. Which variable should be assigned next?
   - Choose the most constrained variable, and break ties by choosing the most constraining variable

2. In what order should its values be tried?
   - Choose the least constraining value

3. Can we detect inevitable failure early?
   - Forward checking, constraint propagation

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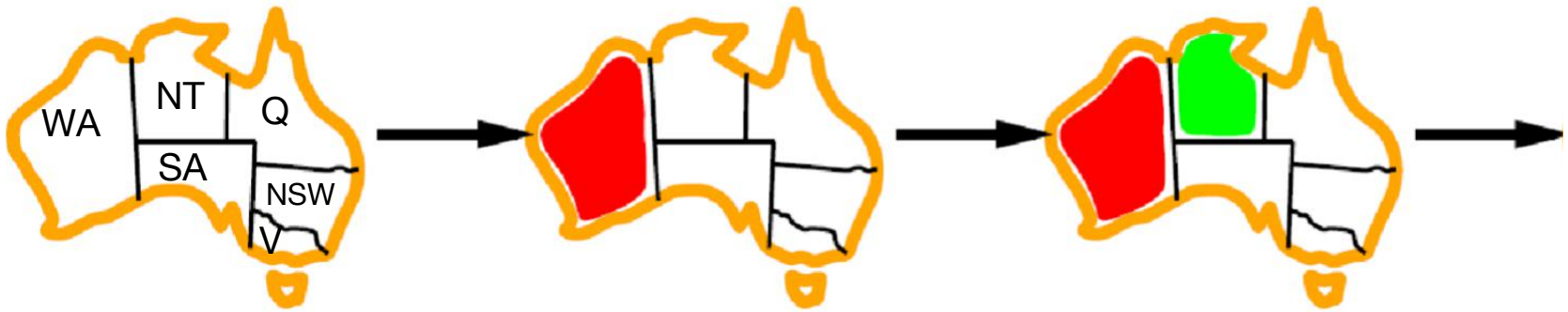
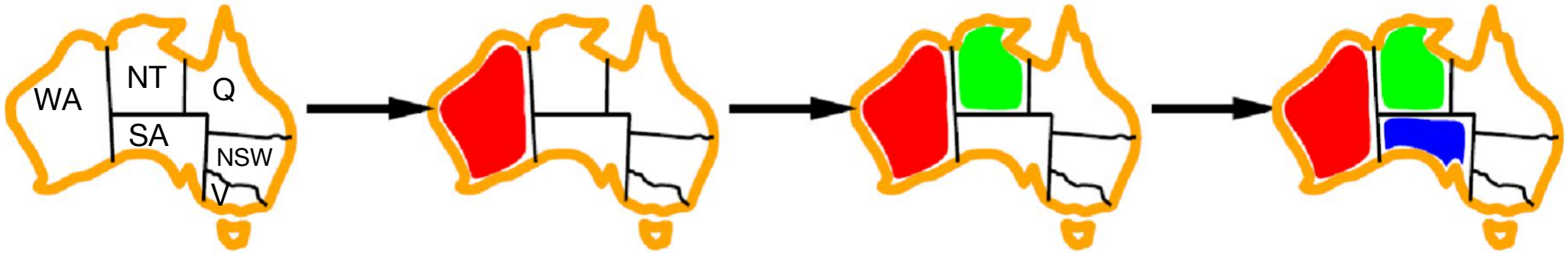
Variable Selection Heuristic 1: Most Constrained Variable



Variable Selection Heuristic 1: Most Constrained Variable



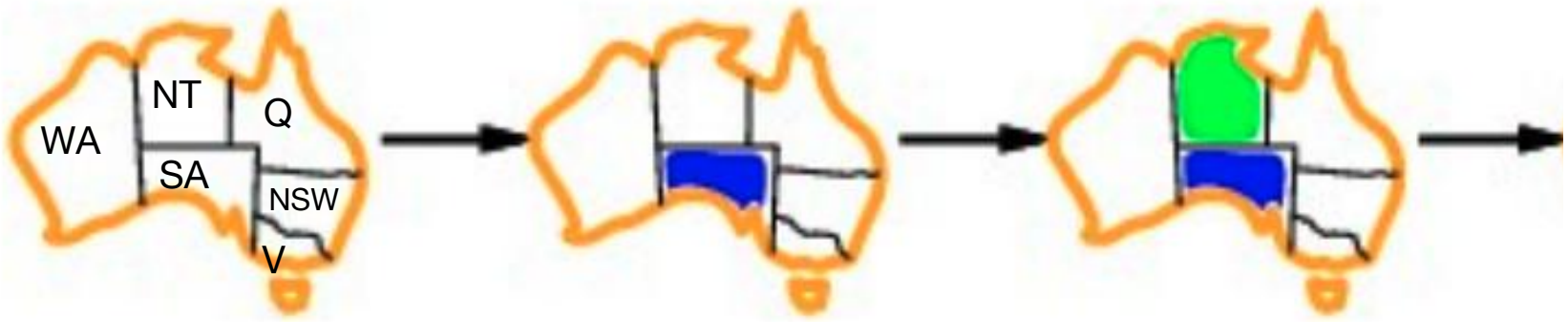
Most constrained variable:

Choose the variable with the fewest legal values

a.k.a. mininum remaining values (MRV) heuristic

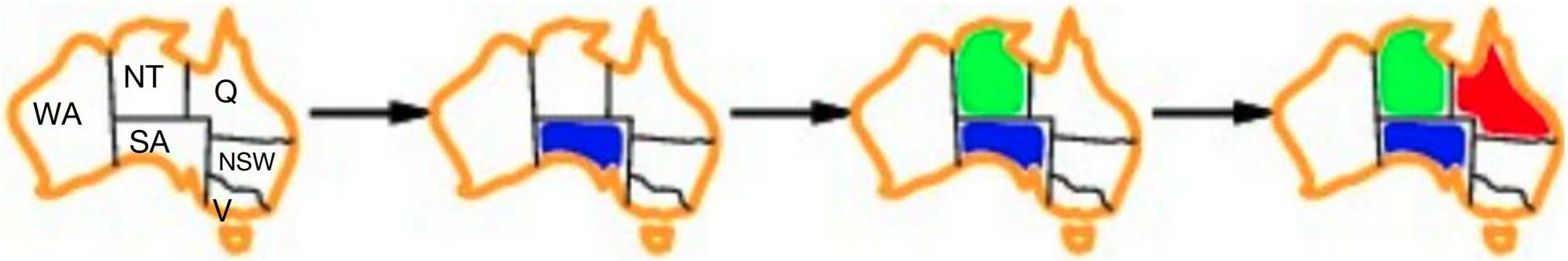
- Why min rather than max?
- “Fail-fast” ordering
- 1000× faster or more

Variable Selection Heuristic 2: Most Constraining Variable



- Tie-breaker among most constrained variables

Variable Selection Heuristic 2: Most Constraining Variable



- Tie-breaker among most constrained variables

Most constraining variable:

choose the variable with the most constraints on remaining variables

Improving Backtracking Efficiency

General-purpose methods can give huge gains in speed:

1. Which variable should be assigned next?
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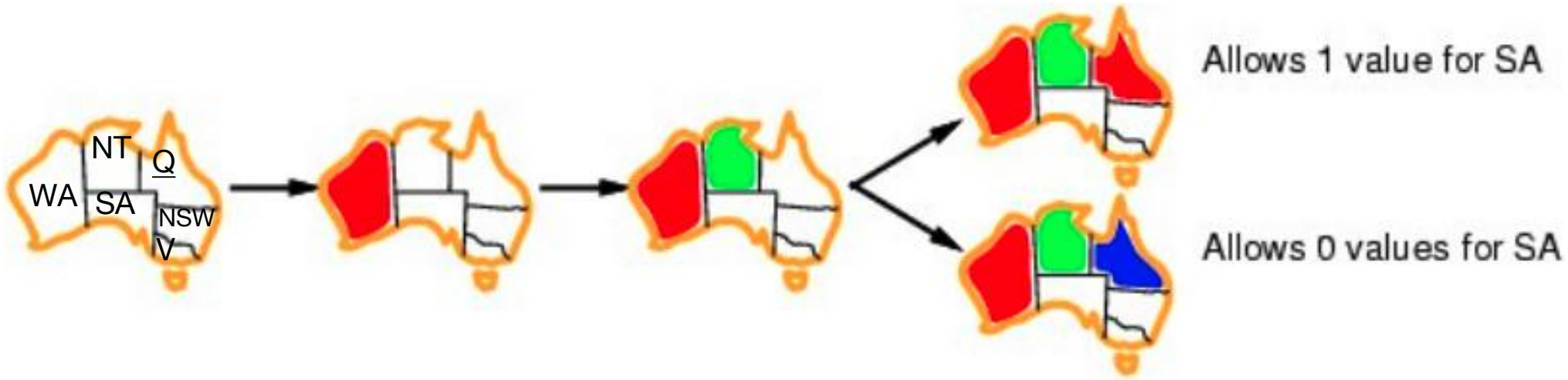
Value Select Heuristic: Least Constraining Value

Value Select Heuristic: Least Constraining Value



- Given a variable, choose the least constraining value:

  the one that rules out the fewest values in the remaining variables

- Why least rather than most?

Variable and Value Ordering Heuristics

- The heuristics should be applied whenever a variable/value needs be chosen during the search process

- They should not be applied once and for all at the begining of the search process

Improving Backtracking Efficiency

General-purpose methods can give huge gains in speed:

1. Which variable should be assigned next?
   - Choose the most constrained variable, and break ties by choosing the most constraining variable

2. In what order should its values be tried?
   - Choose the least constraining value

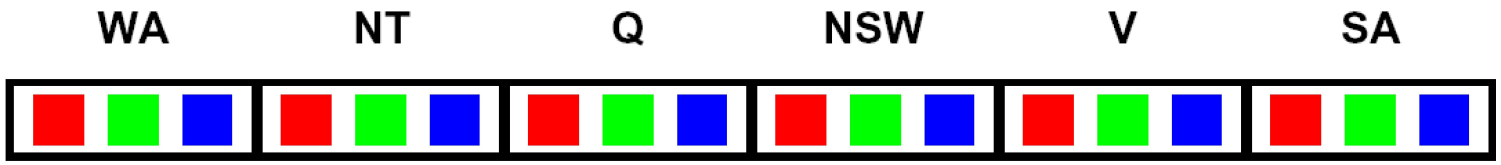
3. Can we detect inevitable failure early?
   - Forward checking, constraint propagation

Consistency-Enforcing Procedure 1: Forward Checking

- Idea:

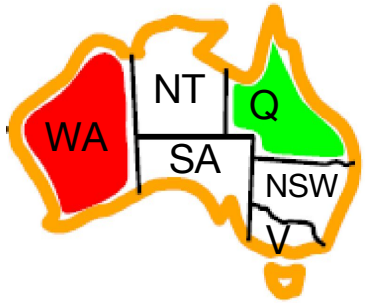
  Keep track of remaining legal values for unsigned variables

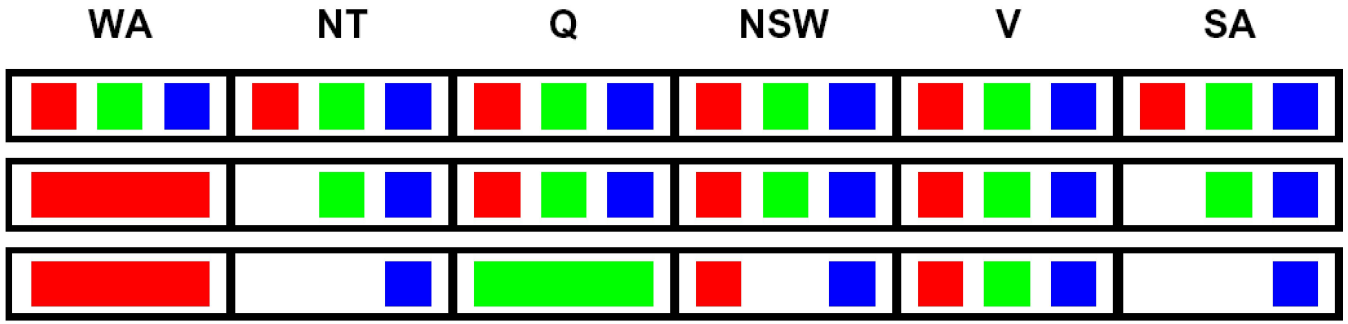
  Terminate search when any variable has no legal values



Consistency-Enforcing Procedure 2: Constraint Propagation

- Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:





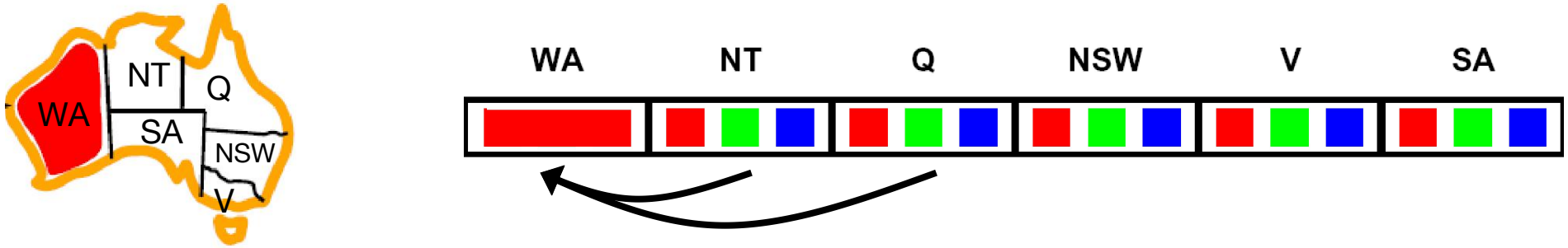
- NT and SA cannot both be blue!
- Constraint propagation repeatedly enforces constrains locally.

Consistency-Enforcing Procedure 2: Constraint Propagation

- Using the constraints to reduce the number of legal values for a variable
- Constraint propagation may be intertwined with search, or it may be done as a preprocessing step, before search starts
- Sometimes this preprocessing can solve the whole problem, so no search is required at all!
- Local consistency

Arc Consistency

- An arc X $\rightarrow$ Y is consistent iff for every x in the tail there is some y in the head which could be assigned without violating a constraint



- Forward checking: Enforcing consistency of arcs pointing to each new assignment

Arc Consistency

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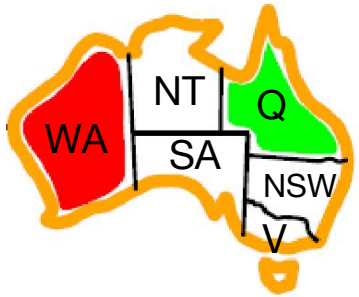
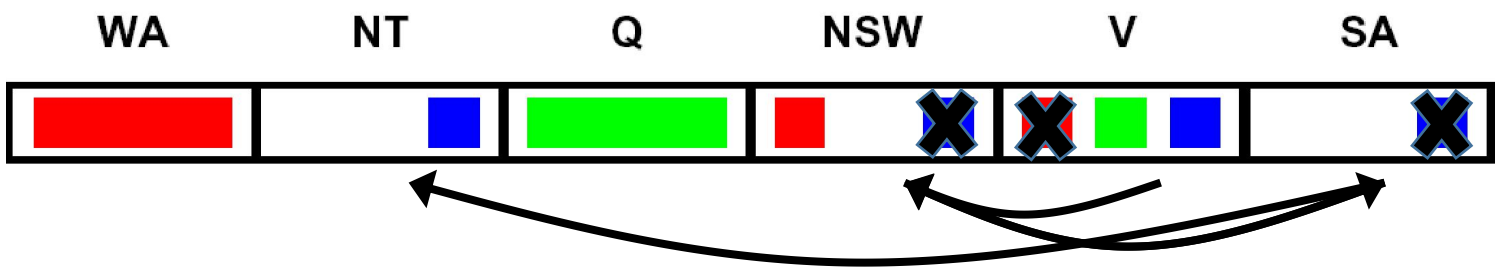




- Important: If X loses a value, neighbors of X need to be rechecked!
- Arc consistency detects failure earlier than forward checking
- Can be run as a preprocessor or after each assignment

Arc Consistency

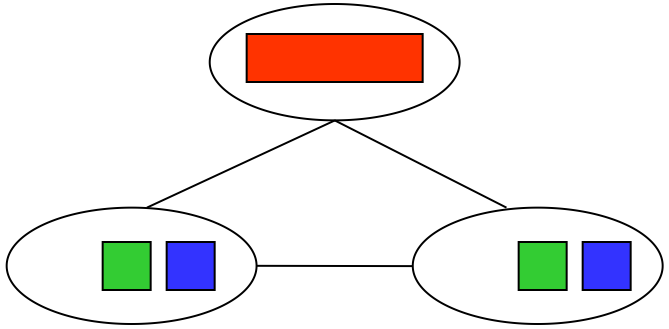
```
function AC-3( csp) returns the CSP, possibly with reduced domains
  inputs: csp, a binary CSP with variables {X_1, X_2, ..., X_n}
  local variables: queue, a queue of arcs, initially all the arcs in csp

  while queue is not empty do
    (X_i, X_j) ← REMOVE-FIRST(queue)
    if REMOVE-INCONSISTENT-VALUES(X_i, X_j) then
      for each X_k in NEIGHBORS[X_i] do
        add (X_k, X_i) to queue
```

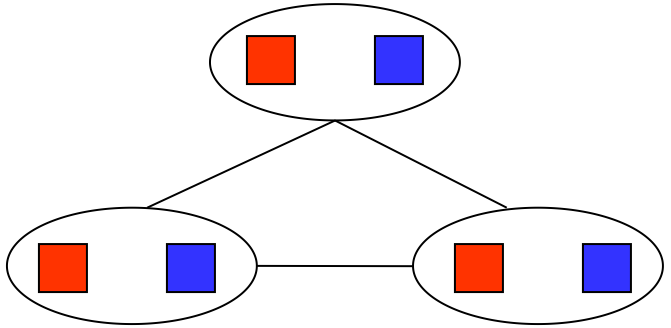
```
function REMOVE-INCONSISTENT-VALUES( X_i, X_j) returns true iff succeeds
  removed ← false
  for each x in DOMAIN[X_i] do
    if no value y in DOMAIN[X_j] allows (x,y) to satisfy the constraint X_i ↔ X_j
      then delete x from DOMAIN[X_i]; removed ← true
  return removed
```

Runtime: $O(n^2d^3)$ n: # of variables d: domain size

Limitations of Arc Consistency

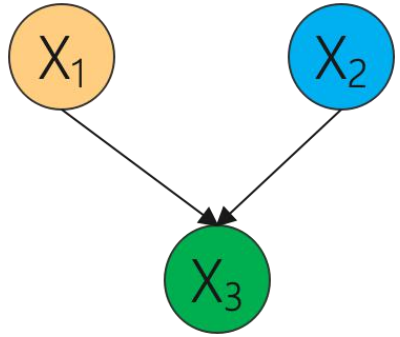
- After enforcing arc consistency:
  - Can have one solution left
  - Can have multiple solutions left
  - Can have no solutions left (and not know it)





K-Consistency

- Increasing degrees of consistency
  - 1-Consistency (Node Consistency): Each single node's domain has a value which meets that node's unary constraints
  - 2-Consistency (Arc Consistency): For each pair of nodes, any consistent assignment to one can be extended to the other
  - K-Consistency: For each k nodes, any consistent assignment to k-1 can be extended to the k^{th} node.



Consistency-Enforcing Procedures

- Forward checking
- Constraint propagation
- Variable/value ordering heuristics

- They can used in combination to improve search process.

General Purpose CSP Algorithm

1. If all values have been successfully assigned then stop, else go on to 2.
2. Apply the consistency enforcing procedure (e.g. forward checking if feeling computationally mean, or constraint propagation if extravagant.)
3. If a deadend is detected then backtrack (e.g., DFS-type backtrack.). Else go on to step 4.
4. Select the next variable to be assigned (using your variable ordering heuristic)
5. Select a promising value (using your value ordering heuristic)
6. Create a new search state. Cache the info you need for backtracking. And go back to 1.

Local Search for CSPs

- Hill-climbing, simulated annealing typically work with "complete" states, i.e., all variables assigned
- To apply to CSPs:

  allow states with unsatisfied constraints operators reassign variable values
- Variable selection: randomly select any conflicted variable

- Value selection by min-conflicts heuristic:

  Choose value that violates the fewest constraints

  i.e., hill-climb with h(n) = total number of violated constraints

Example: 4-Queens

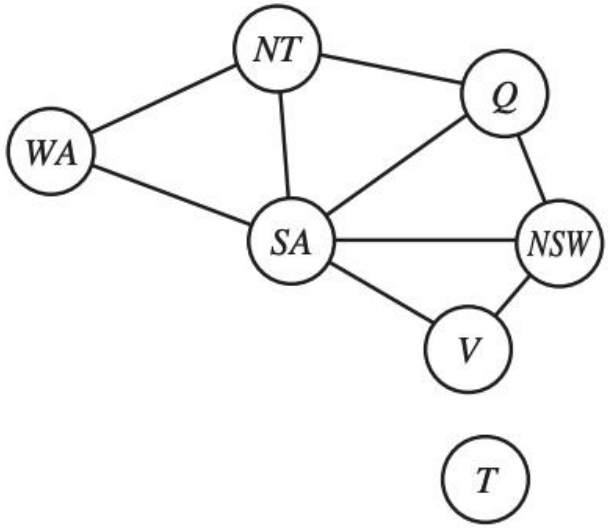
- States: 4 queens in 4 columns (256 states)
- Actions: move queen in column
- Goal test: no attacks
- Evaluation: h(n) = number of attacks

- Give random inital state, can solve n-queens in almost constant time for arbirary n with high probability (e.g., n = 10,000,000)

Problem Encoding

- How can the problem structure help to find a solution quickly?
- Decomposition of the original problem:

$$CSP = \cup_i CSP_i$$



- Solutions can be combined to solve the original problem

$$S = \cup_i S_i$$

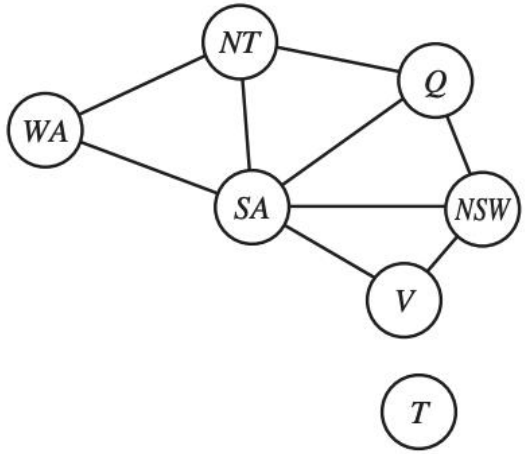
Problem Encoding

- Extreme case: independent subproblems
  - Example: Tasmania and mainland do not interact

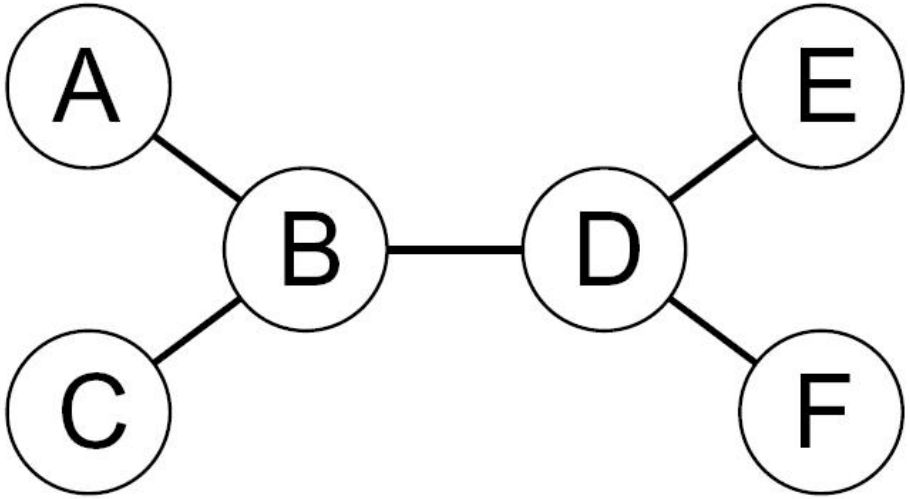
- Independent subproblems are identifiable as connected components of constraint graph

- Suppose a graph of n variables can be broken into subproblems of only c variables:

  - Worst-case solution cost is $O((n/c)(d^c))$, linear in n
  - E.g., n = 80, d = 2, c =20
  - 2^{80} = 4 billion years at 10 million nodes/sec
  - $(4)(2^{20})$ = 0.4 seconds at 10 million nodes/sec



Tree-Structured CSPs

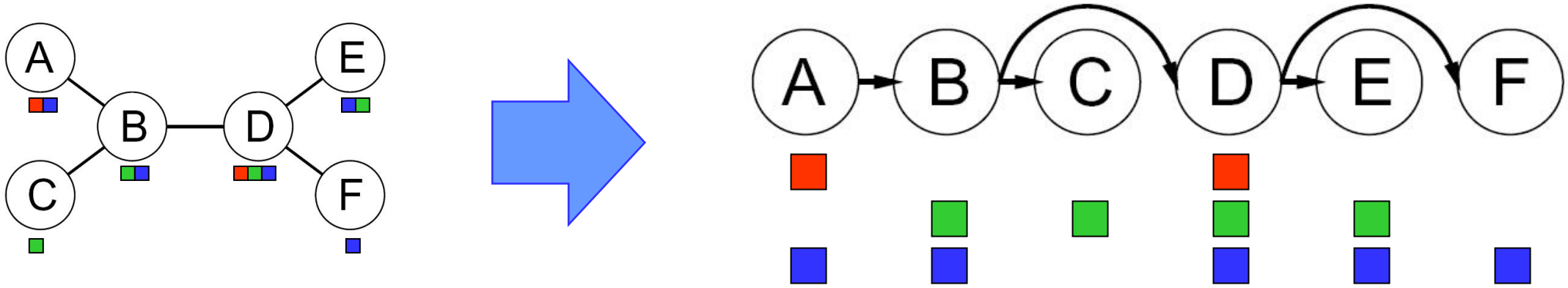


- Theorem: if the constraint graph has no loops, the CSP can be solved in $O(nd^2)$ time

- Compare to general CSPs, where worst-case time is $O(d^n)$

Tree-Structured CSPs

- Algorithm for tree-structured CSPs:
  - Order: Choose a root variable, order variables so that parents precede children



- Remove backward: For i = n : 2, apply RemoveInconsistent(Parent(X_i), X_i)
- Assign forward: For i = 1 : n, assign X_i consistently with Parent(X_i)

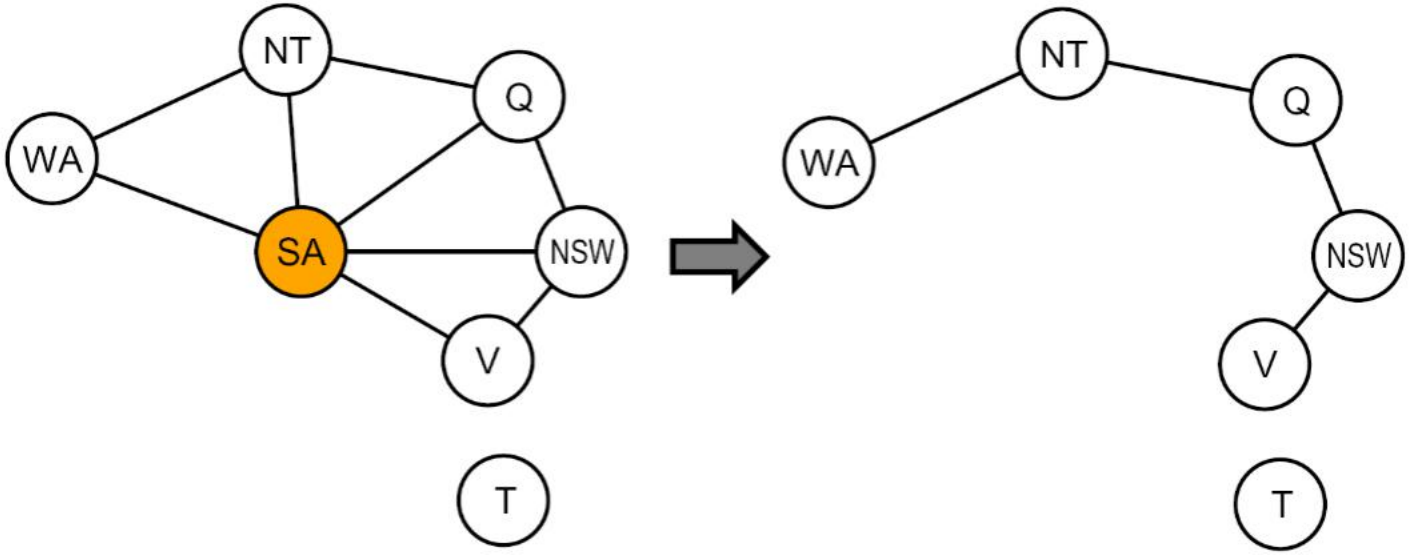
General CSPs

- Removing nodes:
  - cutset conditioning: remainining variables form a tree

- Collapsing nodes
  - Tree decomposition: construct a set of connected subproblems.

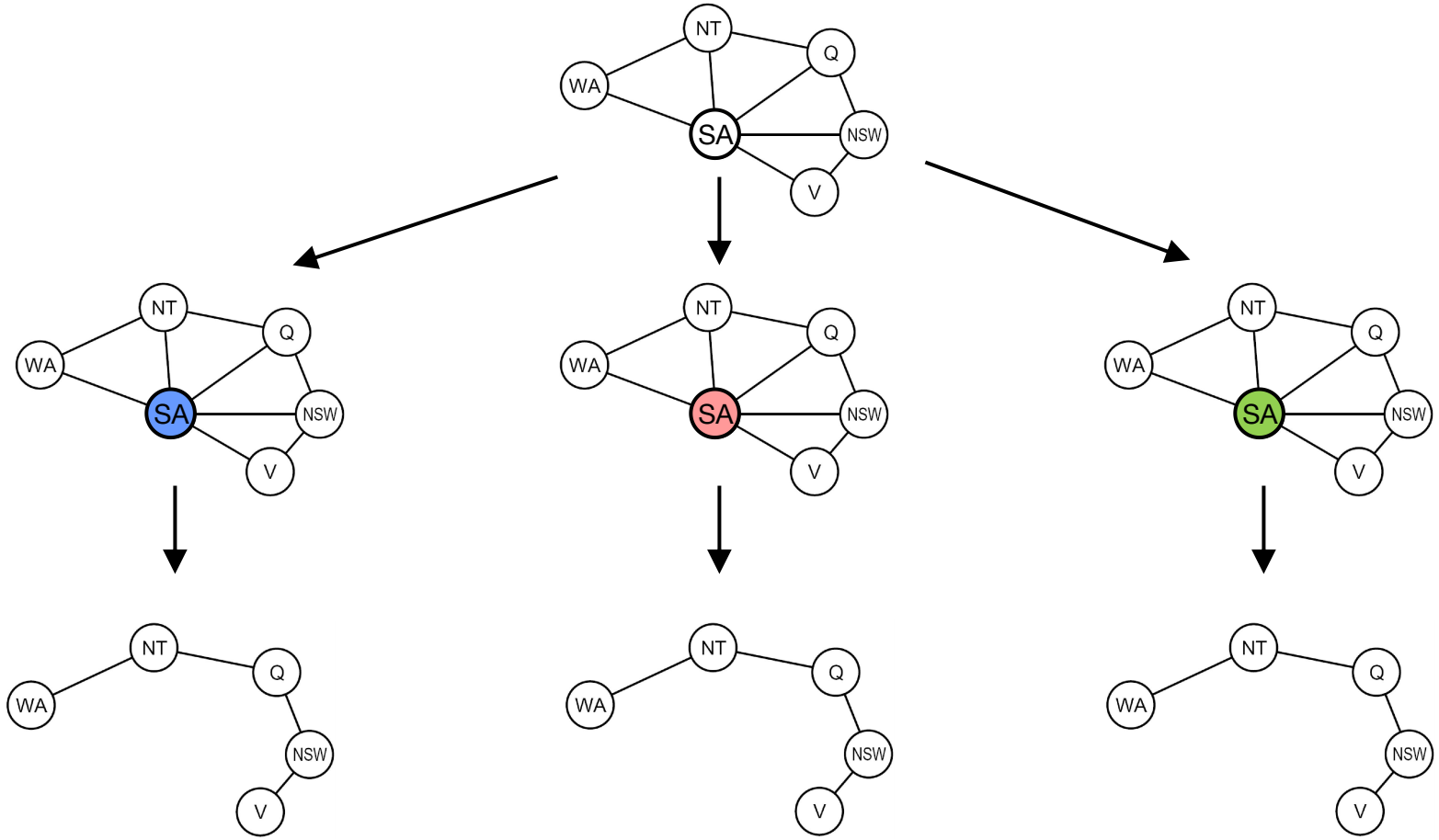
Nearly Tree-Structured CSPs

- Conditioning: instantiate a variable, prune its neighbors' domains



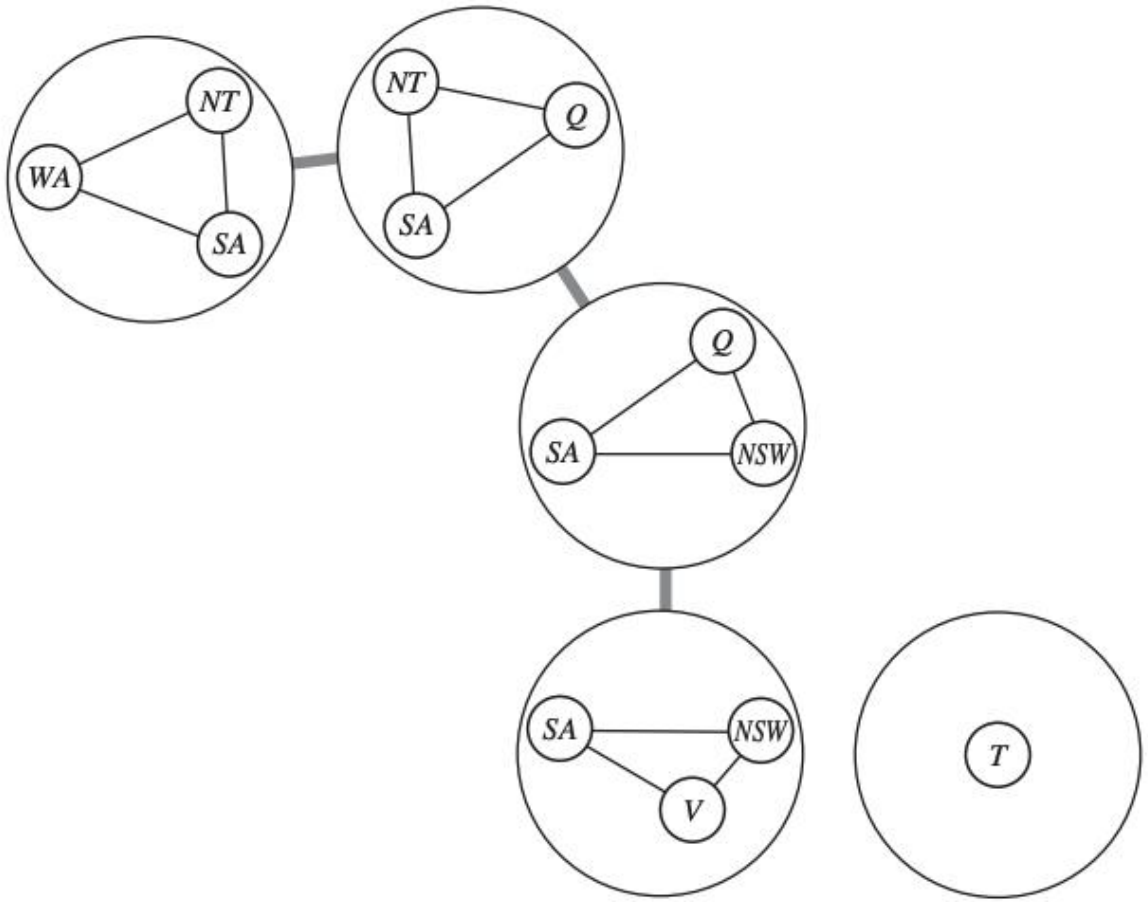
Nearly Tree-Structured CSPs

- Cutset conditioning:
  - instantiate (in all ways) a set of variables S such that the remaining constraint graph is a tree
  - Solve the remaining constraint graph and return it with the assignment of S.
  - If the cutset has size c, then the run time is $O(d^c \cdot (n-c)d^2)$



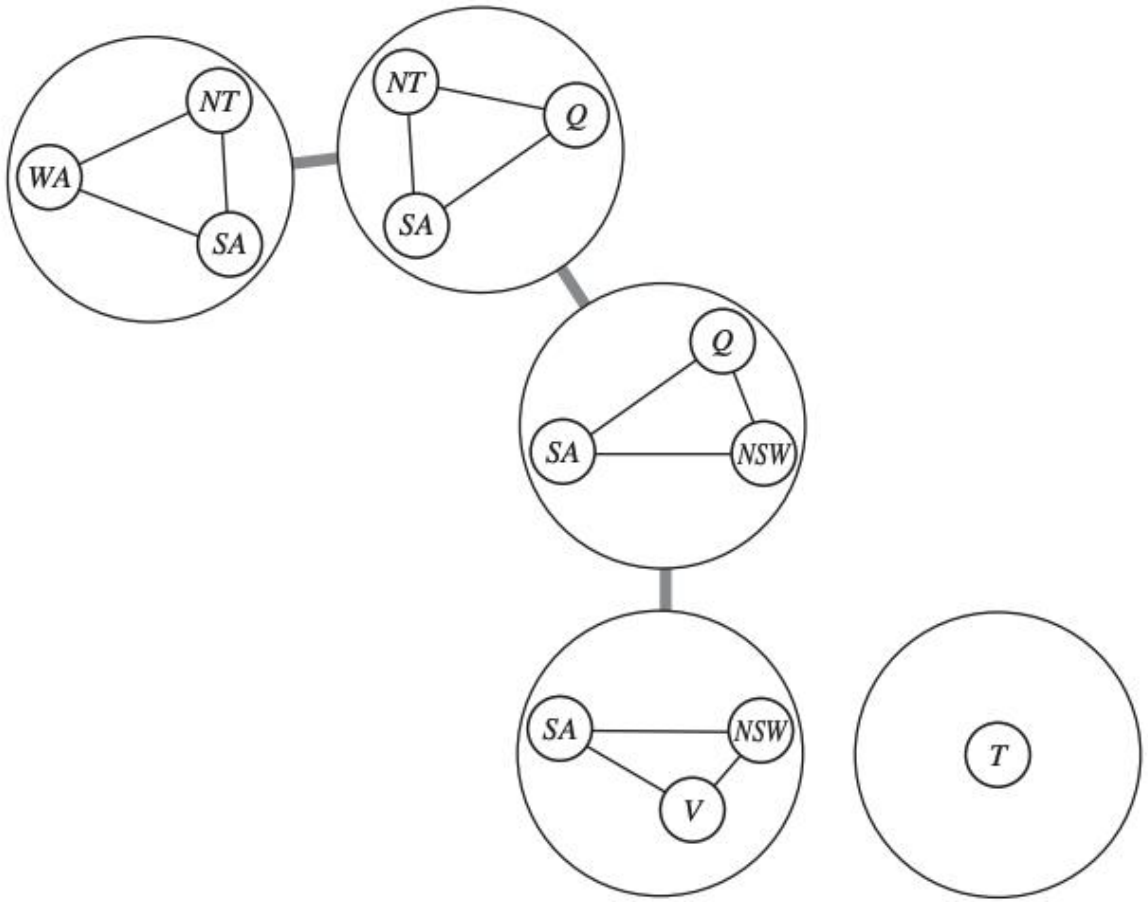
Tree Decompositon

- Every variable must appear in at least one subproblem
- If two variables are connected by a constraint in the original problem, they must appear together (along with the constraint) in at least one of the subproblems.
- If a variable occurs in two subproblems in the tree, it must appear in every subproblem on the path that connects the two



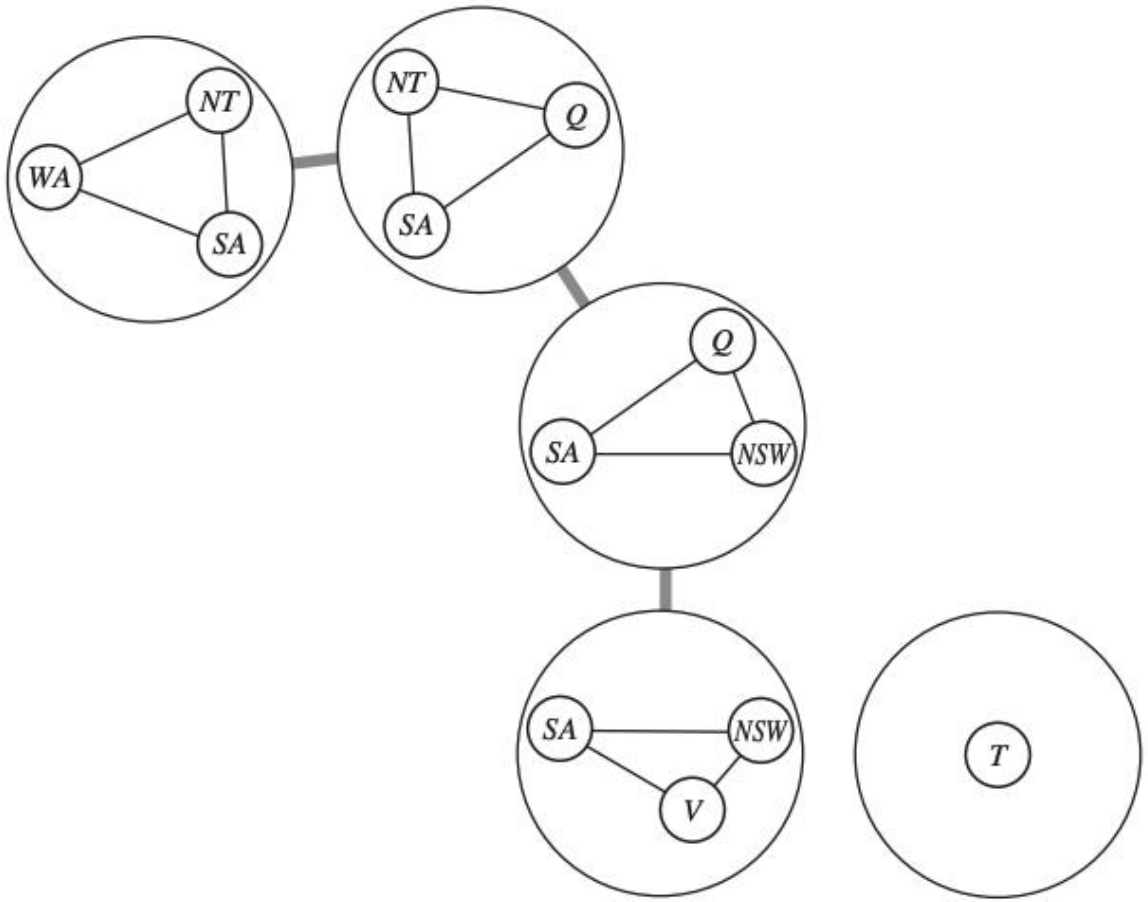
Tree Width

- The subproblem should be as small as possible..
- Tree width w of a tree decomposition is the size of largest sub-problem minus 1
- Tree width of a graph is minimal tree width over all possible tree decompositions (hard to find)



Tree Decompositon

- Solve for all solutions of each subproblem.
- Use tree-structured algorithm, treating the subproblem solutions as variables for those subproblems.
- $O(nd^{w+1})$ where w is the tree width of the graph



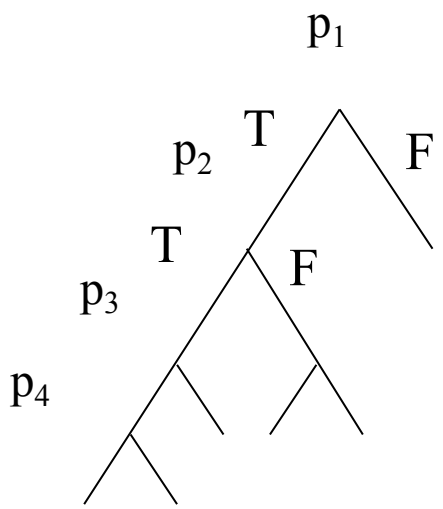
Application: Satisfiability

- 3SAT:
  - find assignments s.t. (p1 ∨ p3 ∨ p4) ∧ (¬p1 ∨ p2 ∨ ¬p3) ∧ … is satisfied

- DFS, GSAT, random walk SAT, ...

Davis-Putnam-Logemann-Loveland (DPLL) tree search algorithm

$$(p1 \vee p3 \vee p4) \wedge (\neg p1 \vee p2 \vee \neg p3) \wedge \ldots$$



- Backtrack when some clause becomes empty

- **Unit propagation** (for variable & value ordering):
  - if some clause only has one literal left, assign that variable the value that satisfies the clause (never need to check the other branch)

Variants of CSPs

- Dynamic CSPs:
  - The constraints are evolved.
  - Can be considered as a sequence of static CSPs.
- Flexible CSPs:
  - Not all constraints need to be satisfied.
- Distributed CSPs:
  - The variables are distributed among multiple agents.

Summary

- CSPs are a special kind of problem:
  - states defined by values of a fixed set of variables
  - goal test defined by constraints on variable values
- Backtracking = DFS with one variable assigned per node
- Variable ordering and value selection heuristics help significantly
- Forward checking prevents assignments that guarantee later failure
- Constraint propagation (e.g., arc consistency) does additional work to constrain values and detect inconsistencies
- Iterative min-conflicts is usually effective in practice