Artificial Intelligence

CS4365 --- Fall 2022

Games with Hidden Information

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Basic Ingredients of Games

- A set of participants
- Each play has a set of options for how to behave: strategies.
- For each choice of strategies, each player receives a payoff that can depend on the strategies selected by everyone

Types of Games

	Deterministic	Chance
Perfect Information	Chess, Checkers, Go	Backgammon
Impefect Information	rock-paper-scissors	Bridge, Poker

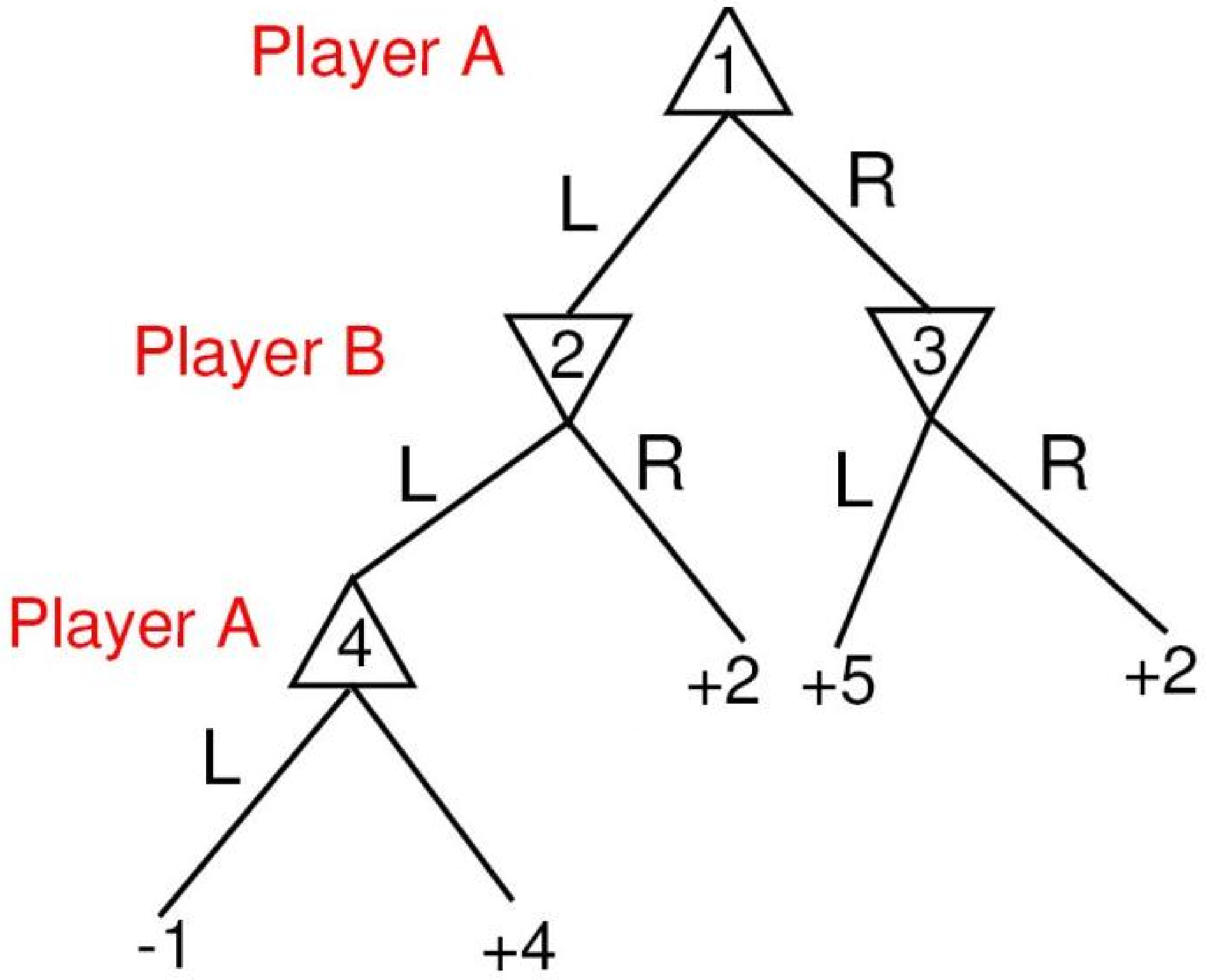
Types of Games

- **Assumptions so far**
  - **Two-player game**: Players A and B.
  - **Perfect information**: Both players see all the states and decisions. Each decision is made sequentially.
  - **Zero-sum**: Player's A gain is exactly equal to player B's loss
- We are going to eliminate these constraints. We will eliminate first the assumption of "perfect information" leading to far more realistic models.
  - Some more game-theoretic definitions → Matrix games
  - Minimax results for perfect information games
  - Minimax results for hidden information games

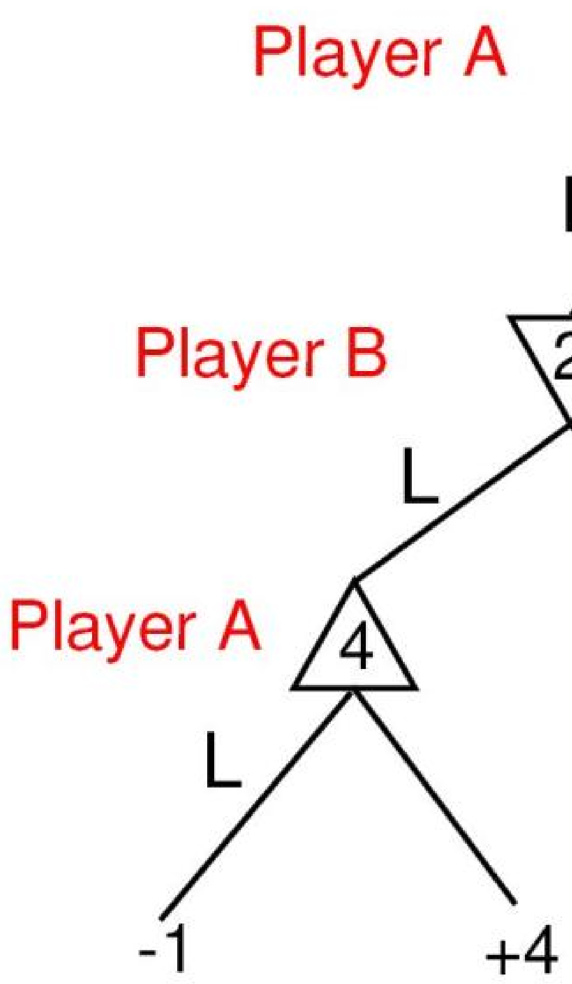
The heart of the problem

- In a 1-agent setting, agent's expected utility maximizing strategy is well-defined
- But in a multiagent system, the outcome may depend on others' strategies also

Search Tree



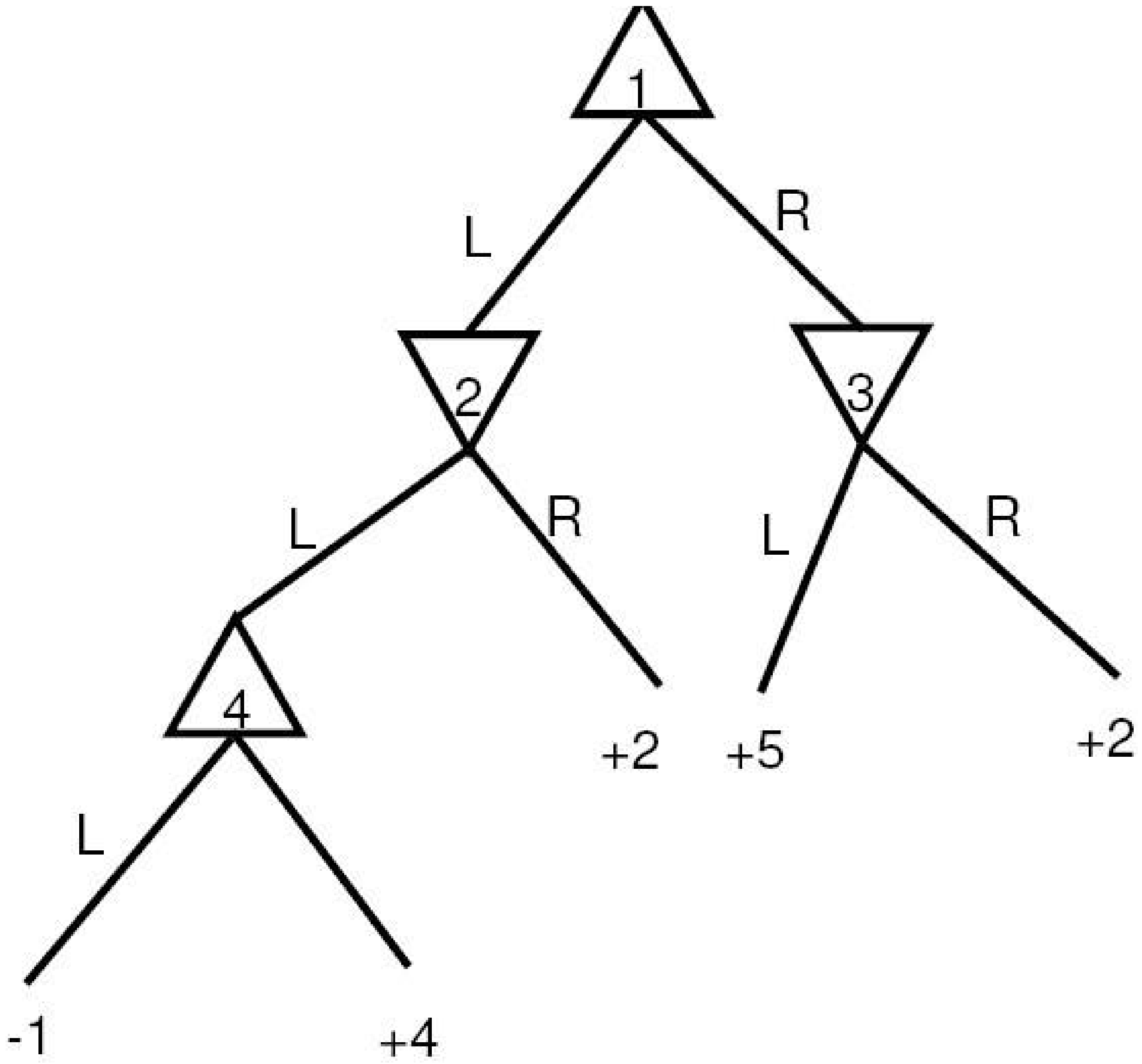
Search Tree



Extensive form of game: Represent the game by a tree

Pure Strategy

A pure strategy for a player defines the move that the player would make for every possible state that the player would see.



Pure Strategy

Pure strategies for A:

Strategy I: (1L,4L)

Strategy II: (1L,4R)

Strategy III: (1R,4L)

Strategy IV: (1R,4R)

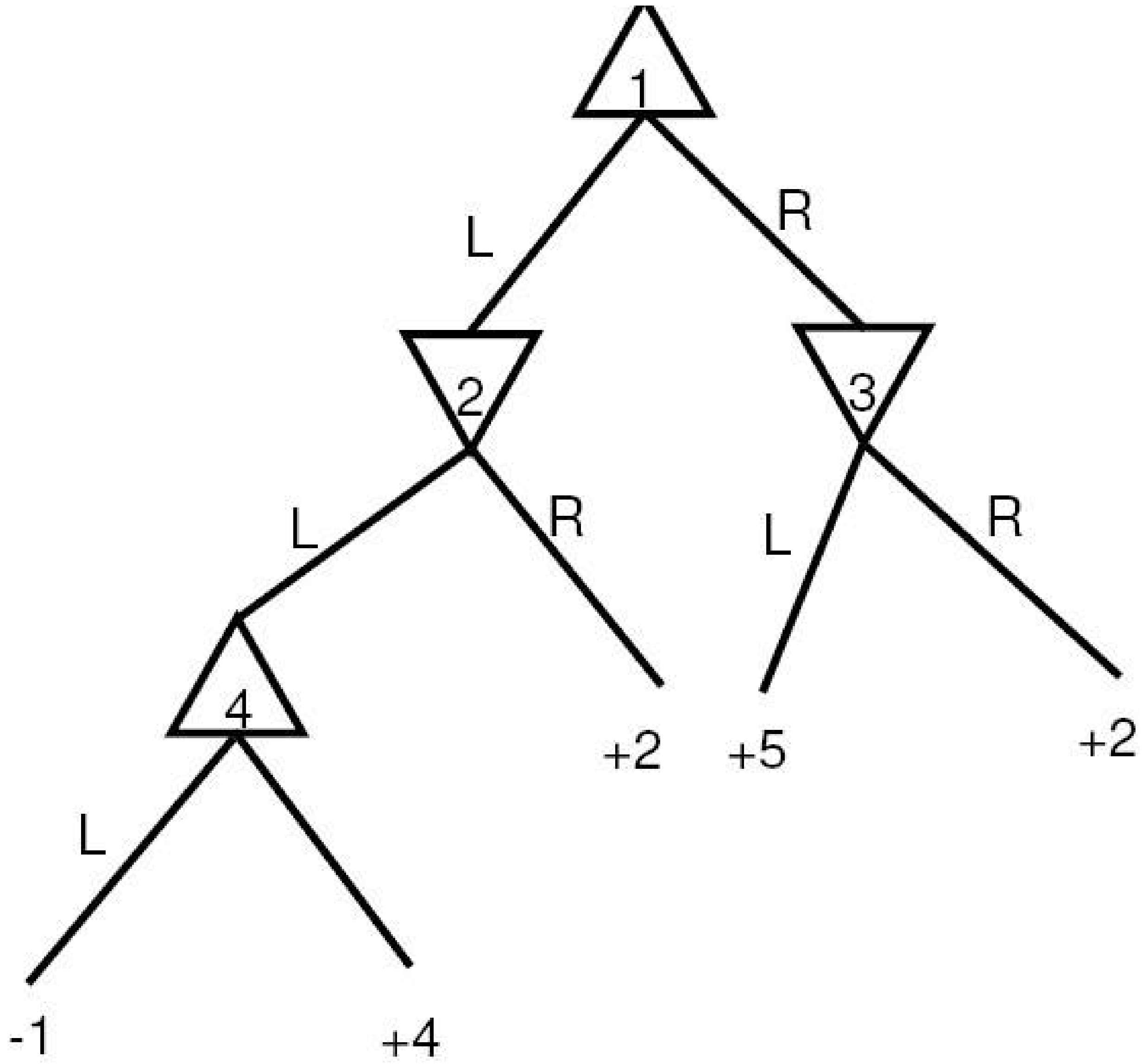
Pure strategies for B:

Strategy I: (2L,3L)

Strategy II: (2L,3R)

Strategy III: (2R,3L)

Strategy IV: (2R,3R)



Matrix Form of Games

Pure strategies for A:

Strategy I: (1L,4L)

Strategy II: (1L,4R)

Strategy III: (1R,4L)

Strategy IV: (1R,4R)

Pure strategies for B:

Strategy I: (2L,3L)

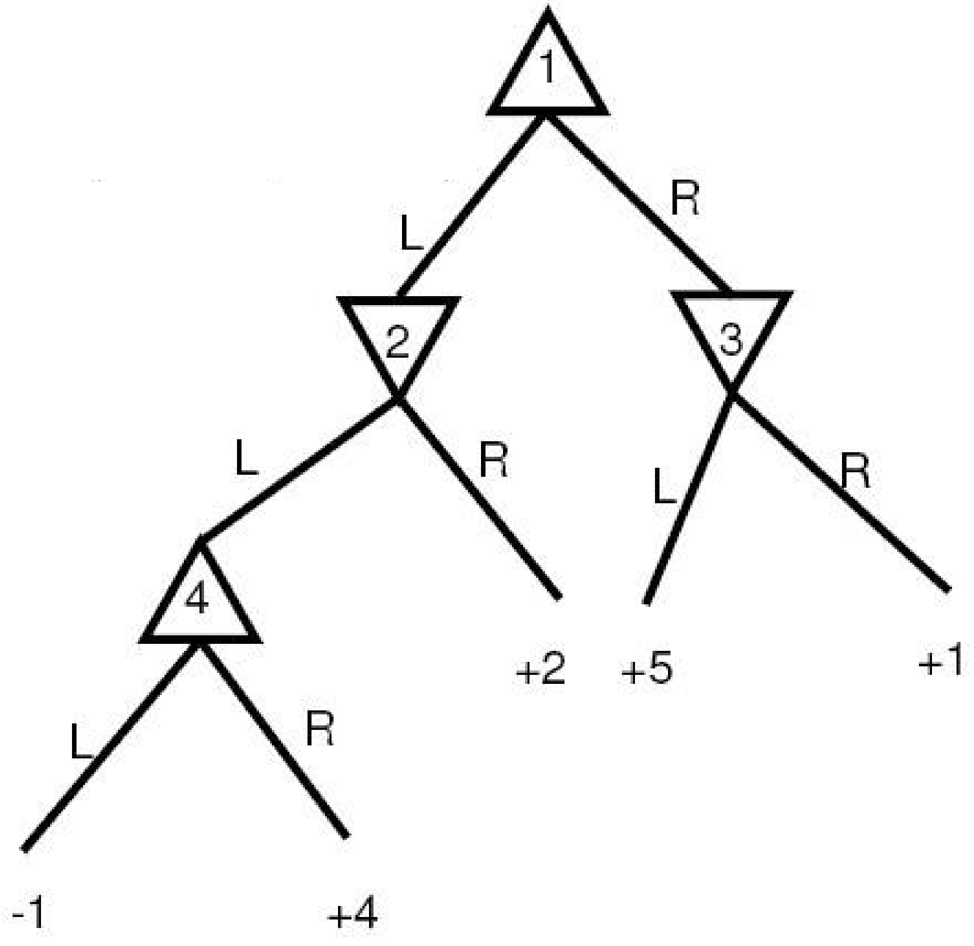
Strategy II: (2L,3R)

Strategy III: (2R,3L)

Strategy IV: (2R,3R)

B

A	I	II	III	IV
I	-1	-1	+2	+2
II	+4	+4	+2	+2
III	+5	+1	+5	+1
IV	+5	+1	+5	+1



Minimax → Matrix Version

B

		I	II	III	IV
-1	I	-1	-1	+2	+2
$\boxed{+2}$	II	+4	+4	+2	+2
+1	III	+5	+1	+5	+1
+1	IV	+5	+1	+5	+1

A

Max value of all the rows

Min value across each row

$$\underset{Rows\ i}{Max}\ \underset{Columns\ j}{Min}\ \mathbf{M}(i,j)$$

Minimax → Matrix Version

B

	I	II	III	IV
I	-1	-1	+2	+2
II	+4	+4	+2	+2
III	+5	+1	+5	+1
IV	+5	+1	+5	+1
	+5	+4	+5	$\boxed{+2}$

A

Max value across each column

Min of all the columns

$$\underset{Columns\ j}{Min}\ \underset{Rows\ i}{Max}\ \mathbf{M}(i,j)$$

Note that we find the same value and same strategies in both cases. Is that always the case?

Max value =
game value = +2

		I	II	III	IV
-1	I	-1	-1	+2	+2
+2	II	+4	+4	+2	$\boxed{+2}$
+1	III	+5	+1	+5	+1
+1	IV	+5	+1	+5	+1

Min value across each row

$$\underset{Rows\ i}{Max}\ \underset{Columns\ j}{Min}\ \mathbf{M}(i,j)$$

Max value across each column

	I	II	III	IV
I	-1	-1	+2	+2
II	+4	+4	+2	$\boxed{+2}$
III	+5	+1	+5	+1
IV	+5	+1	+5	+1
	+5	+4	+5	+2

Min value =
game value = +2

$$\underset{Columns\ j}{Min}\ \underset{Rows\ i}{Max}\ \mathbf{M}(i,j)$$

Minimax vs. Maximin

- **Fundamental Theorem I** (von Neumann):
  - For a two-player, zero-sum game with perfect information:
    - There always exists an optimal pure strategy for each player
    - Minimax = Maximin
- Note: This is a game-theoretic formalization of the minimax search algorithm.

Another (Simple) Game

- The two Players A and B each have a coin

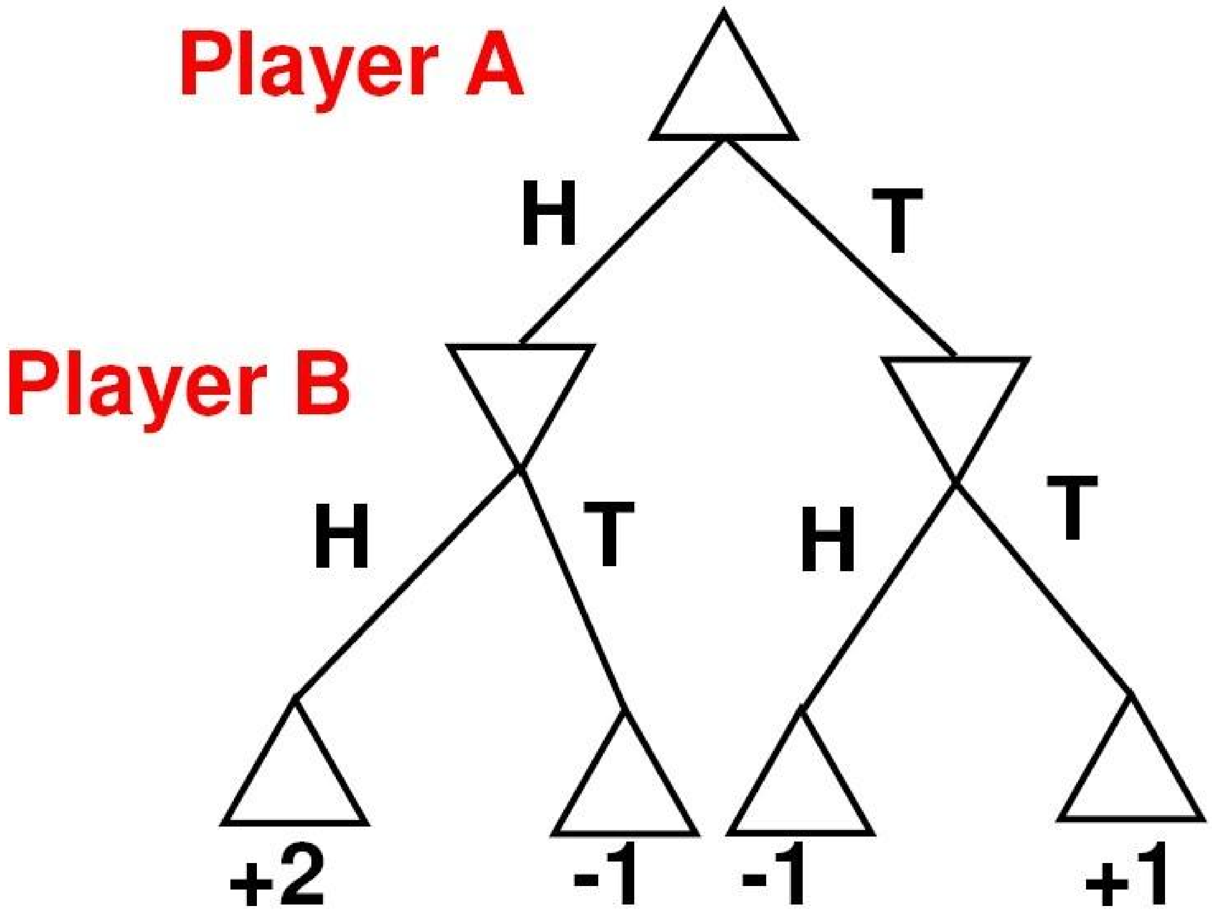
- They show each other their coin, choosing to show either head or tail

- If they both choose head Player B pays Player A \$2
- If they both choose tail Player B pays Player A \$1
- If they choose different sides Player A pays Player B \$1

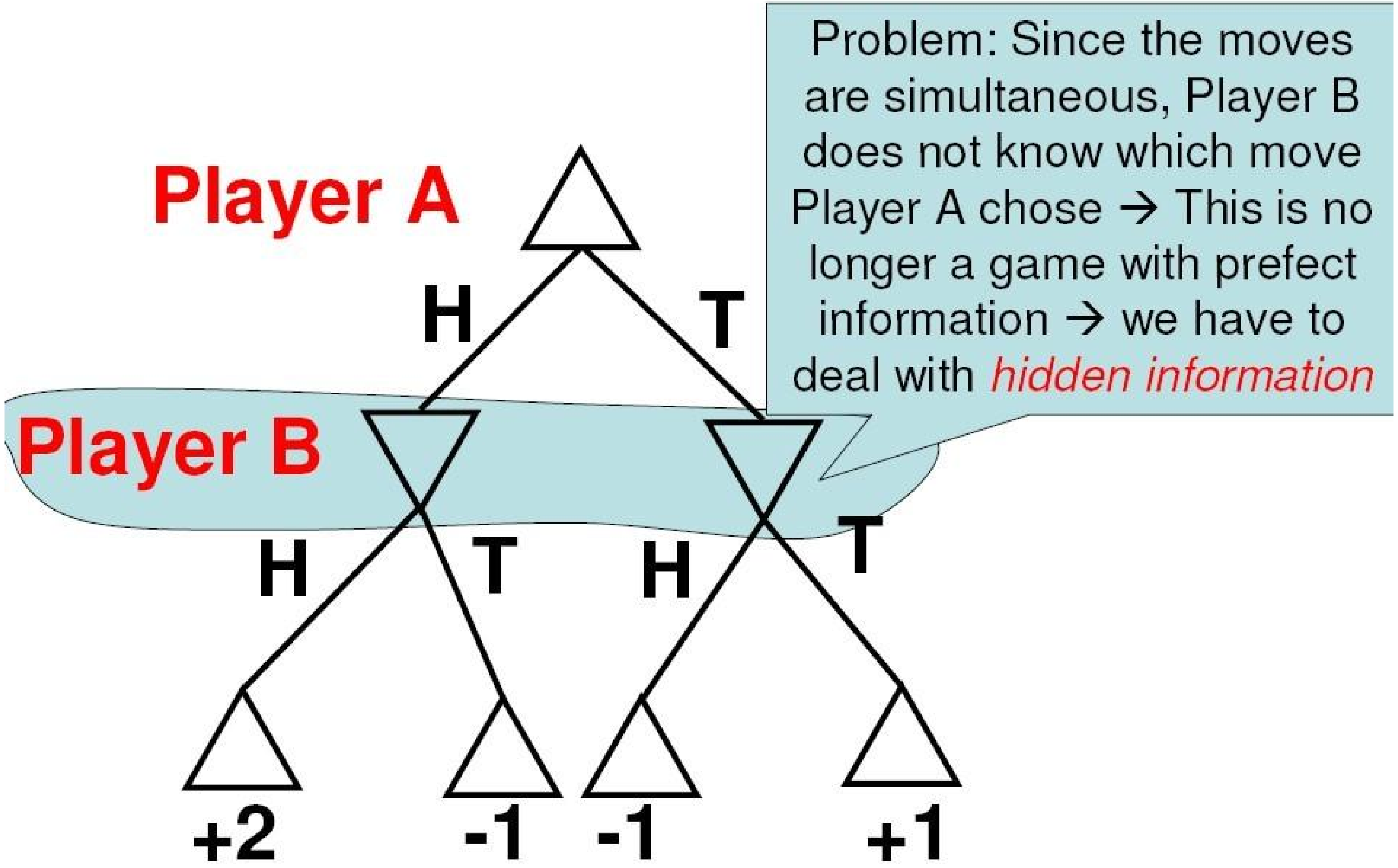
Side Note About All Toy Examples

- This kind of toy example models a large number of real-life situations.
- For example: Player A is a business owner (e.g., a restaurant, a plant..) and Player B is an inspector. The inspector picks a day to conduct the inspection; the owner picks a day to hide the bad stuff. Player B wins if the days are different; Player A wins if the days are the same.
- This class of problems can be reduced to the “coin game”(with different payoff distributions, perhaps).

Extensive Form



Extensive Form



Matrix Normal Form

Player B

Player A

	H	T
H	+2	-1
T	-1	+1

- It is no longer the case that maximin = minimax (easy to verify: -1 vs. +1)
- Therefore: It appears that there is **no pure strategy solution**
- In fact, in general, none of the pure strategies are solutions to a **zero-sum game with hidden information**

Why No Pure Strategy Solutions?

- If Player A considers move H, he has to assume that Player B will choose the worst-case move (for A), which is move T

- Therefore Player A should try move T instead, but then he has to assume that Player B will choose the worst-case move (for A), which is move H.

- Therefore Player A should consider move H, but then he has to assume that Player B will choose the worst-case move (for A), which is move T……….

		Player B	
		H	T
Player A	H	+2	-1
	T	-1	+1

	H	T
H	+2	-1
T	-1	+2

Using Random Strategies

- Suppose that, instead of choosing a fixed pure strategy, Player A chooses randomly strategy H with probability p, and strategy T with probability 1-p.
- If Player B chooses move H, the expected payoff for Player A is:

$$\boxed{p \times (+2) + (1-p) \times (-1) = 3p - 1}$$

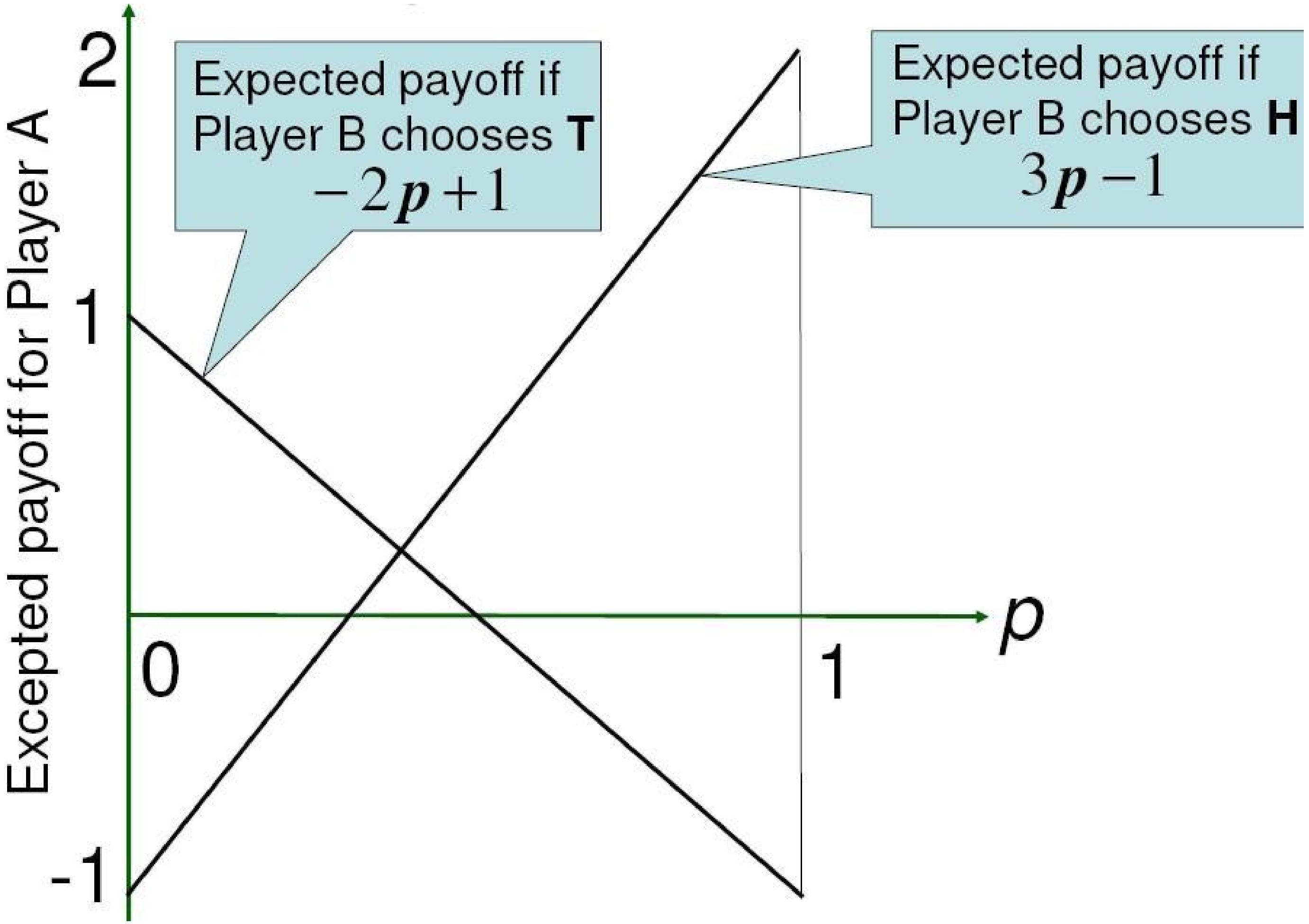
- If Player B chooses move **T**, the *expected* payoff for Player A is:

$$\boxed{p \times (-1) + (1-p) \times (+1) = -2p + 1}$$

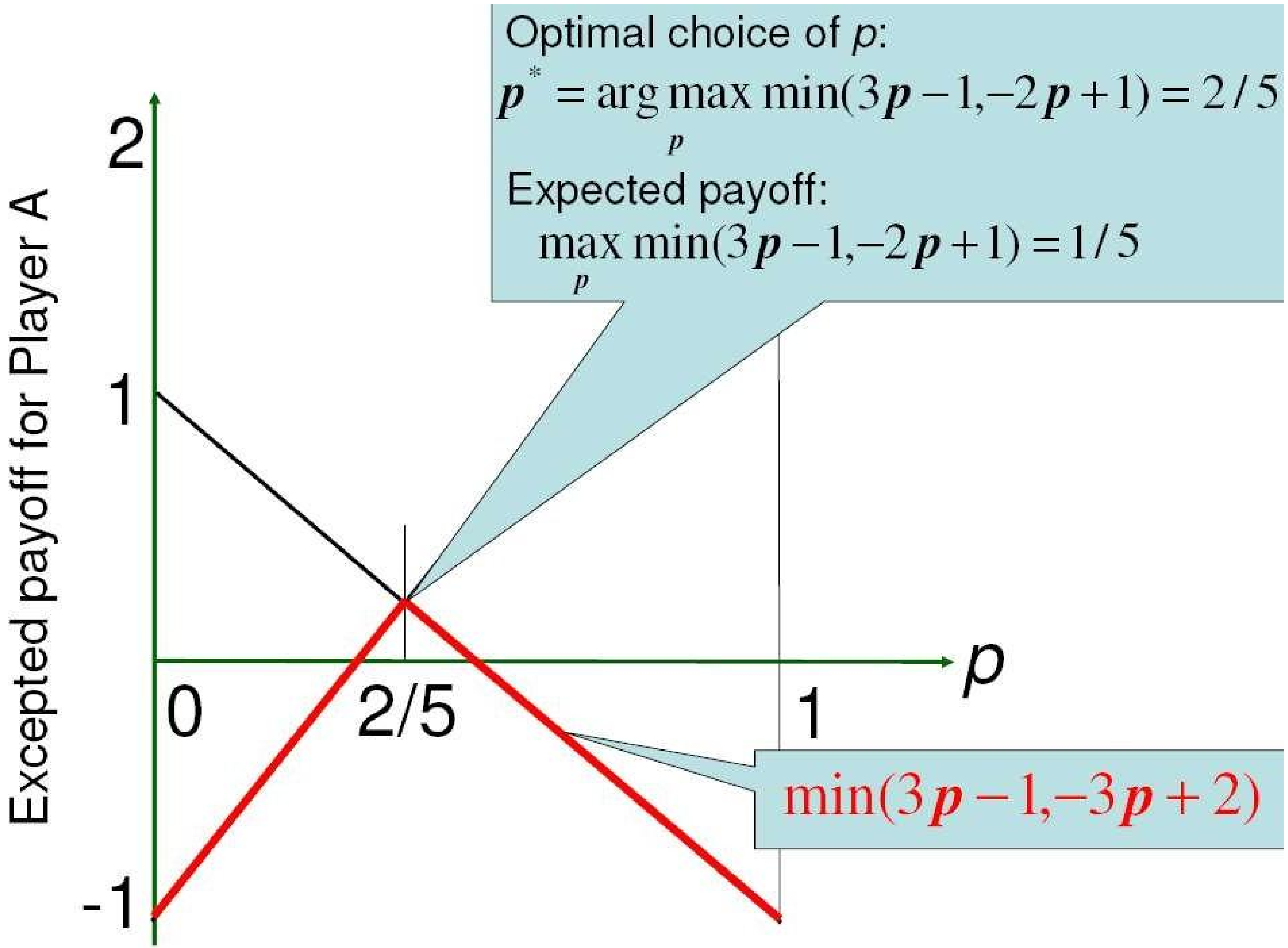
- So, the worst case is when Player B chooses a strategy that *minimizes* the payoff between the 2 cases: $\boxed{\min(3p-1, -2p+1)}$
- Player A should then adjust the probability *p* so that its payoff is *maximized* (note the similarity with the standard maximin procedure):

$$\boxed{\max_p \min(3p-1, -2p+1)}$$

Graphical Solution



Graphical Solution



Mixed Strategies

- It is no longer possible to find an optimal pure strategy for Player A.
- We need to change the problem a bit: We assume that Player A chooses a pure strategy randomly at the beginning of the game.
- In that scenario, Player A selects one pure strategy probability p and the other one with probability 1-p.
- This strategy of choosing pure strategies randomly is called a mixed strategy for Player A and is entirely defined by the probability p.
- The result that we derived for the simple example holds for general games. It yields a procedure for finding the optimal mixed strategy for zero-sum games.

Minimax with Mixed Strategies

- Theorem II (von Neumann):
  - For a two-player, zero-sum game with hidden information, there always exists an optimal mixed strategy with value

$$\boxed{\max_{p} \min(\boldsymbol{p}\times\boldsymbol{m}_{11} + (1-\boldsymbol{p})\times\boldsymbol{m}_{21}, \boldsymbol{p}\times\boldsymbol{m}_{12} + (1-\boldsymbol{p})\times\boldsymbol{m}_{22})}$$

  - Where the *matrix form of the game is:*

m_{11}	m_{12}
m_{21}	m_{22}

  - *Note: This is a direct generalization of the minimax result to mixed strategies.*

Minimax with Mixed Strategies

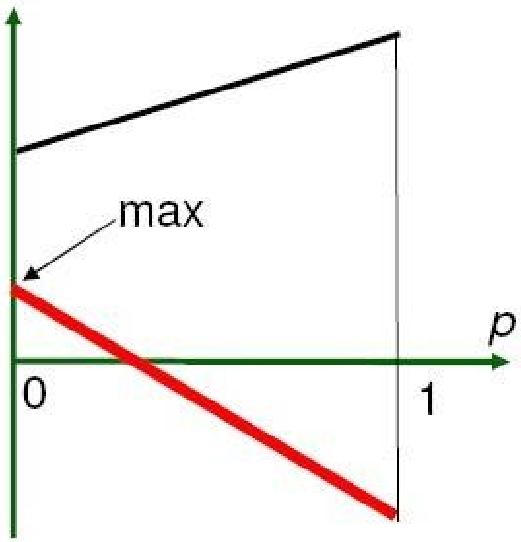
- Theorem II (von Neumann):
  - For a two-player, zero-sum game with hidden information:
  - There always exists an optimal mixed strategy with value

  - In addition, just like for games with perfect information, it does not matter in which order we look at the players, minimax is the same as maximin

$$\max_{p} \min(p \times m_{11} + (1-p) \times m_{21}, p \times m_{12} + (1-p) \times m_{22}) =$$

$$\min_{q} \max(q \times m_{11} + (1-q) \times m_{12}, q \times m_{21} + (1-q) \times m_{22})$$

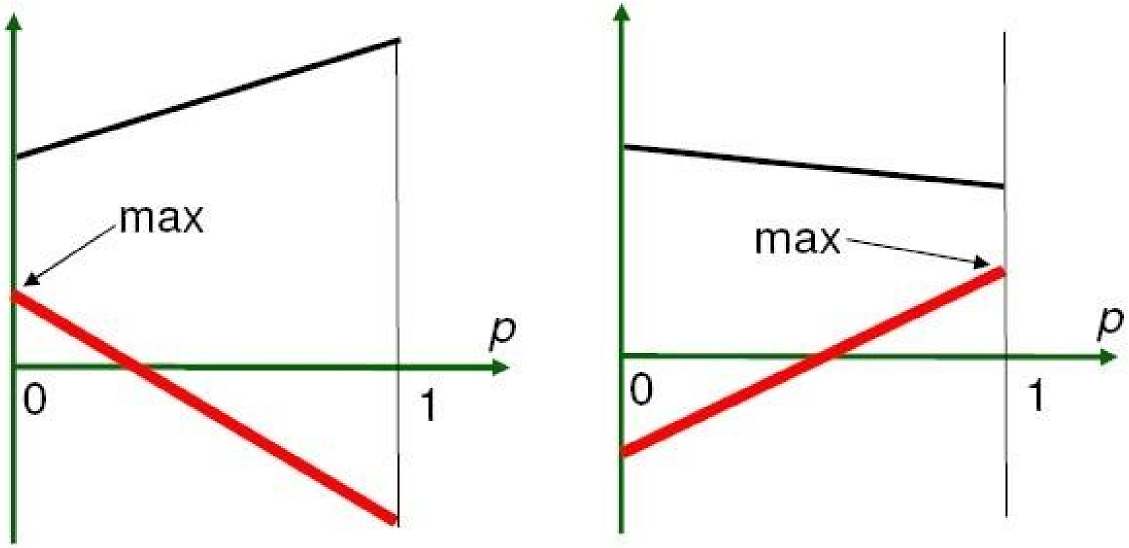
Recipe for 2 x 2 Games



$$\min(p \times m_{11} + (1-p) \times m_{21}, p \times m_{12} + (1-p) \times m_{22})$$

- Since the two functions of *p* are linear, the maximum is attained either for:
  - $p = 0$

Recipe for 2 x 2 Games



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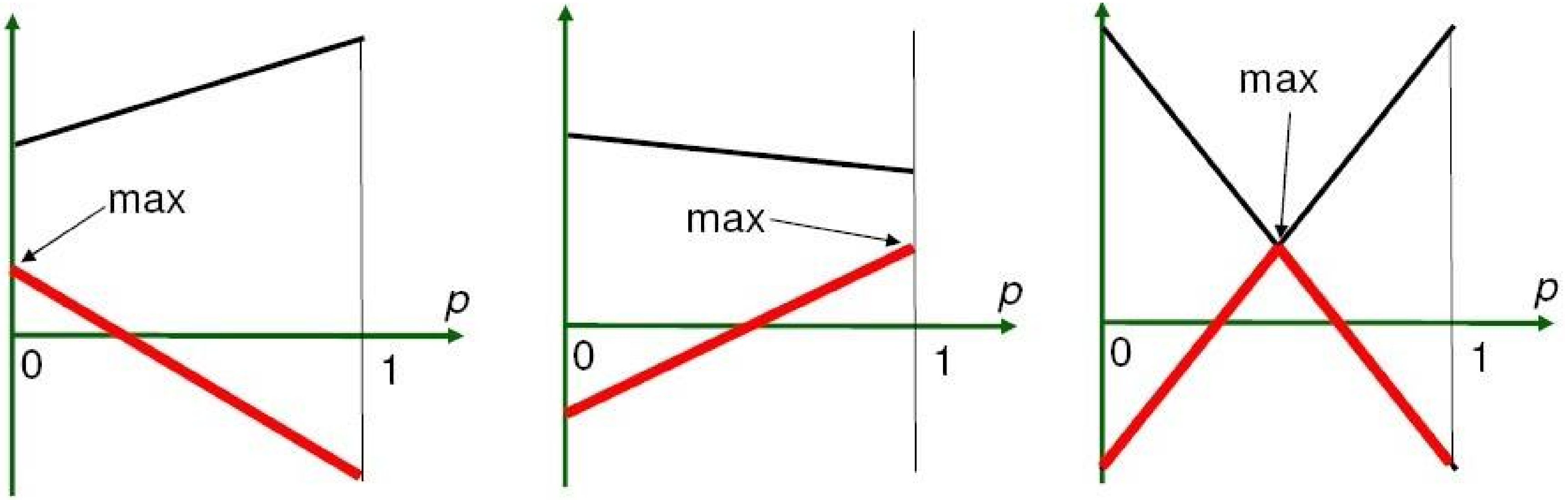
  - $p = 1$

Recipe for 2 x 2 Games



$$\min(p \times m_{11} + (1-p) \times m_{21}, p \times m_{12} + (1-p) \times m_{22})$$

- Since the two functions of *p* are linear, the maximum is attained either for:
  - $p = 0$
  - $p = 1$
  - The intersection of the two lines, if it occurs for *p* between 0 and 1

General Case: N x M Games

- We have illustrated the problem on 2x2 games (2 strategies for each of Player A and Player B)
- The result generalizes to NxM games, although it is more difficult to compute
- A mixed strategy is a vector of probabilities (summing to 1!) $p = (p_1,..,p_N)$. p_i is the probability with which strategy i will be chosen by Player A.
- The optimal strategy is found by solving the problem:

$$\max_{\mathbf{p}} \min_{j} \sum_{i} p_i m_{ij}$$

$$\sum_{i} p_i = 1$$

This is solved by using "Linear Programming"

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$$\sum_i p_i = 1$$

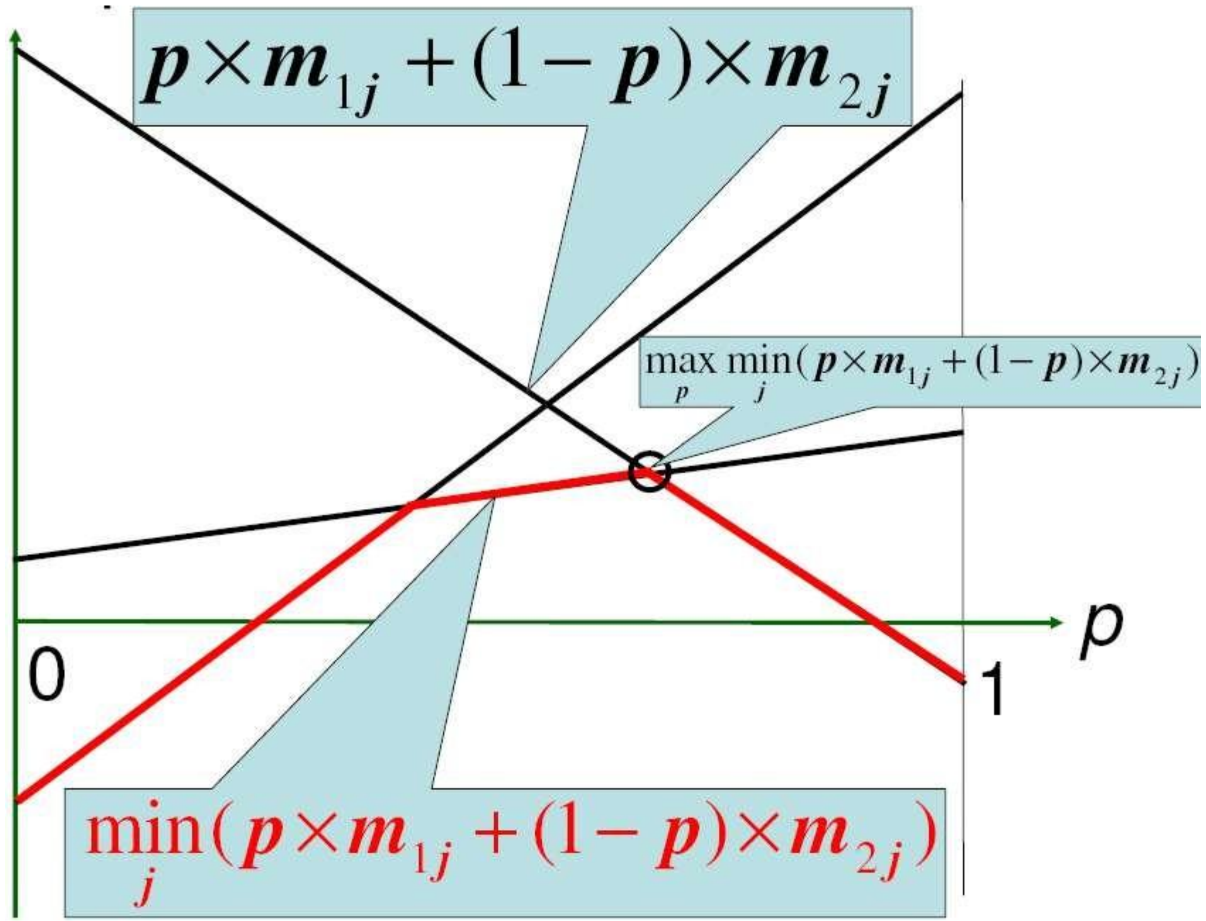
This is solved by using "Linear Programming"

$$\boxed{\sum_i p_i m_{ij}}$$

Expected payoff for Player A if Player B chooses pure strategy number j and Player A chooses pure strategy i with prob. p_i

Graphical Illustration: 2 x M Game

- Each strategy of the opponent corresponds to one straight line
- The opponent aims to minimize the payoff
- The player aims to maximize the minimum payoff



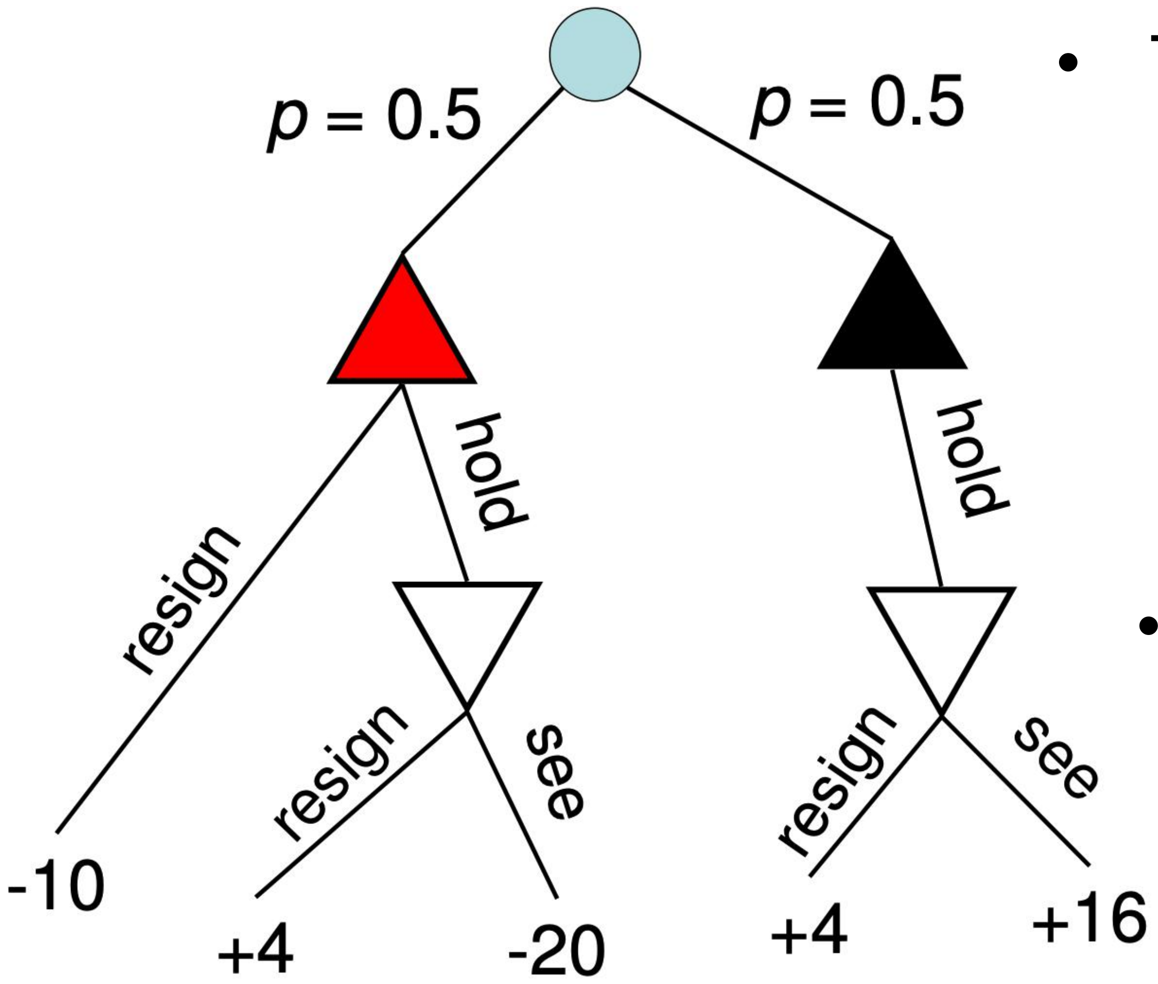
Discussion

- The criterion for selecting the optimal mixed strategy is the average payoff that Player A would receive over many runs of the game.
- It may seem strange to use random choices of pure strategies as “mixed” strategies and to search for optimal mixed strategies.
- In fact, it formalizes what happens in common situations. For example, in poker, if Player A follows a single pure strategy (taking the same action every time a particular configuration of cards is dealt), Player B can guess and respond to that strategy and lower Player A’s payoffs.
- The right thing to do is for Player A to change randomly the way each configuration is handled, according to some policy. A good player would use a good policy.

Another Example: Poker

- Players A and B play with two types of cards: Red and Black
- Player A is dealt one card at random (50% probability of being Red)
- If the card is red, Player A may *resign* and loses $10
- Or Player A may *hold*
  - B may then *resign* A wins $4
  - B may *see*
    - A loses $20 if the card is Red
    - A wins $16 otherwise

Another Example: Poker



- The game is non-deterministic

- Hidden information: Player B cannot know which of these 2 states it's in

Another Example: Poker

Player B

Player A	Resign	See
Resign		
Hold		

- Generate the matrix form of the game (be careful: It’s not a deterministic game)
- Find the expected payoff for Player A
- Find the optimal mixed strategy

Summary

- Matrix form of games
- Minimax procedure and theorem for games with perfect information   Always a *pure strategy* solution
- Minimax procedure and theorem for games with imperfect information   Always a *mixed strategy* solution
- Procedure for solving 2x2 games with hidden information
- Understanding of how the problem is formalized for NxM games (actually solving them requires linear programming tools which will not be covered here)
- Important: These results apply only to *zero-sum games*. This is still a severe restriction as most realistic decision- making problems cannot be modeled as *zero sum games.*