Adaptive Simpson's Rule Example

Consider the integral

$$\int_1^3 e^{2x} \sin 3x \; dx,$$

and the error tolerance $\epsilon = 0.2$. We apply a few steps of the adaptive simpson's rule method.

Let

$$f(x) = e^{2x} \sin 3x,$$

the integrand.

Step 1: Approximation at this step is $S(1,2) + S(2,3)$. We first check the error of this. This is through the formula

$$\frac{1}{10}\left|S(a,b) - S(a,\frac{a+b}{2}) - S(\frac{a+b}{2},b)\right|.$$

Note we use $\frac{1}{10}$ rather than $\frac{1}{15}$ to be safe. So calculating,

$$\begin{aligned} S(1,3) &= 35.42697658812284 \\ S(1,2) &= -15.45828245392933 \\ S(2,3) &= 117.9751755250024, \end{aligned}$$

and thus our approximation of error is

$$|f(x) - S(1,2) - S(2,3)| \approx 6.70899164829503.$$

This error is not yet acceptable since it is greater than $\epsilon = 0.2$, so we add midpoints and continue to the next step.

Step 2: Approximation at this step is $S(1,1.5) + S(1.5,2)$ for the integral in $[1,2]$ and $S(2,2.5) + S(2.5,3)$ for the integral in $[2,3]$. Thus the total approximation is the sum of these for the integral in $[1,3]$. Calculating,

$$\begin{aligned} S(1,2) &= -15.45828245392933 \\ S(1,1.5) &= -3.87030357255464 \\ S(1.5,2) &= -12.38881686458909 \end{aligned}$$

and

$$\begin{aligned} S(1,3) &= 117.9751755250024 \\ S(1,2) &= 23.83355636842984 \\ S(2,3) &= 100.7072692285579, \end{aligned}$$

leading to the error approximations of 0.08008379832144 for the integral in $[1,2]$ and -0.65656500719853 for the integral in $[2,3]$. This error in $[1,2]$ is acceptable since it is less than $\frac{\epsilon}{2} = 0.1$. The error in $[2,3]$, however, is not and we add midpoints and continue with it to the next step.

Step 3: Approximation at this step is $S(2,2.25) + S(2.25,2.5)$ for the integral in $[2,2.5]$ and $S(2.5,2.75) + S(2.75,3)$ for the integral in $[2.5,3]$. Calculating,

$$\begin{aligned} S(1,2) &= 23.83355636842984 \\ S(1,1.5) &= 2.12361566688147 \\ S(1.5,2) &= 21.85661747203629 \end{aligned}$$

and

$$\begin{aligned} S(1,2) &= 100.7072692285579 \\ S(1,1.5) &= 46.96091888208836 \\ S(1.5,2) &= 53.89025695750476, \end{aligned}$$

leading to the error approximations of 0.01466767704879 for the integral in $[2,2.5]$ and 0.01439066110352 for the integral in $[2.5,3]$. Both errors are acceptable since they are less than $\frac{\epsilon}{4} = 0.5$. Thus we stop at this step.

The final approximation for our problem is the sum

$$S(1,1.5) + S(1.5,2) + S(2,2.25) + S(2.25,2.5) + S(2.5,2.75) + S(2.75,3).$$

The first two terms sum up to -16.25912043714373, which approximates the integral in $[1,2]$. The next two sum up to 23.98023313891776, which approximates the integral in $[2,2.5]$. The last two sum up to 100.8511758395931, which approximates the integral in $[2.5,3]$. Thus, in total, the sum 108.5722885413671 approximates the whole integral with an approximate error less than $\epsilon = 0.2$.

If the error tolerance is small, this process should be programmed up.