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Exercises

Section 5.2: 1, 2, 4, 9, 17(c,d)

1. What is the numerical value of the composite trapezoid rule applied to the reciprocal function f (x) = x^{-1} using the points 1, 4/ 3 , and 2?

Answer:

function f (x) = x^{-1}

f(1) = 1; f(4/3) = ¾; f(2) = ½

$$\int_1^2 f(x)\,dx = \int_1^2 \frac{1}{x}dx$$

Trapezoid rule at point 1, 4/3

$$\frac{\frac{4}{3}-1}{2}\left[f(1)+f\left(\frac{4}{3}\right)\right] = \frac{1}{6}\left[1+\frac{3}{4}\right] = \frac{7}{24}$$

Trapezoid rule at point 4/3,2

$$\frac{2-\frac{4}{3}}{2}\left[f\left(\frac{4}{3}\right)+f(2)\right] = \frac{2}{6}\left[\frac{3}{4}+\frac{1}{2}\right] = \frac{10}{24}$$

Total:

$$\int_1^2 f(x)\,dx = \int_1^2 \frac{1}{x}dx = \frac{7}{24}+\frac{10}{24} = \frac{17}{24} \approx 0.70833$$

2. Compute an approximate value of $\int_0^1 (x^2+1)^{-1}dx$ by using the composite trapezoid rule with three points. Then compare with the actual value of the integral. Next, determine the error formula and numerically verify an upper bound on it.

Answer:

$$\int_0^1 (x^2+1)^{-1}dx = [\arctan x]_0^1 = \frac{\pi}{4} \approx 0.785398$$

$$f(x) = \frac{1}{x^2+1}$$

f(0) = 1; f(1/2) = 4/5; f(1) = ½

Trapezoid rule at point 0, 1/2

$$\frac{0.5}{2}[f(0.5)-f(0)]=\frac{1}{4}\left[\frac{4}{5}+1\right]=\frac{9}{20}$$

Trapezoid rule at point 1/2 ,1

$$\frac{0.5}{2}[f(1)-f(0.5)]=\frac{1}{4}\left[\frac{1}{2}+\frac{4}{5}\right]=\frac{13}{40}$$

Total

$$\int_0^1 (x^2+1)^{-1}dx=\frac{9}{20}+\frac{13}{40}=\frac{31}{40}\approx 0.775$$

The required error: 0.785398 – 0.775 ~ 0.010398

$$|E_T|\le\frac{K(b-a)^3}{12n^2}\ with\ n=2$$

$$K=|f''(x)|\ maximum\ value\ of\ f(x)\ \ on\ [0,1]$$

$$f(x)=(x^2+1)^{-1}$$
$$f'(x)=-1(x^2+1)^{-2}\cdot 2n=-2x(x^2+1)^{-2}$$
$$f''(x)=-2(x^2+1)^{-2}+4x(x^2+1)^{-3}$$
$$=\frac{6x^2-2}{(x^2+1)^3}$$

|f’’(1)| = 0.5

|f’’(0)|= 2

|f’’(x)| maximum at x=0 belong [0,1]

$$|E_T|\le\frac{2(1-0)^3}{12\cdot 2^2}$$

$$|E_T|\le\frac{2}{48}\approx 0.0416$$

Upper bound is 0.0416

4. Obtain an upper bound on the absolute error when we compute $\int_0^6 \sin x^2\,dx$ by means of the composite trapezoid rule using 101 equally spaced points.

Answer:

Theorem on Precision of Composite Trapezoid Rule

If f'' exists and is continuous on the interval $[a, b]$ and if the composite trapezoid rule T with uniform spacing h is used to estimate the integral $I = \int_a^b f(x)\,dx$, then for some ζ in (a, b),

$$I - T = -\frac{1}{12}(b-a)h^2 f''(\zeta) = \mathcal{O}(h^2)$$

$f(x) = \sin x^2$

$f'(x) = 2x \cos x^2$

$f''(x) = 2 \cos x^2 + 4x^2 \sin x^2$

Triangle inequality:

$|f''(x)| = |2 \cos x^2 + 4x^2 \sin x^2| \le 2|\cos x^2| + 4|x^2 \sin x^2| < 2 + 4(36) = 146$

$$|I-T| = \frac{(6-0)\left(\frac{6-0}{100}\right)^2}{12}|f''(\zeta)| < \frac{18}{10000}(146) = 0.2628$$

Hence, upper bound is 0.2628.

9. Prove that if a function is concave downward, then the trapezoid rule underestimates the integral.

Answer:

$$I - T = \frac{-1}{12}(b-a)\left(\frac{b-a}{n}\right)^2 f''(\zeta)$$

Concave downward, Therefore, the error is positive

$I - T = \frac{-1}{12}(b-a)\left(\frac{b-a}{n}\right)^2 f''(\zeta) \ge 0$. Hence $I \ge T$, then the trapezoid rule underestimates the integral.

17. Consider the integral $I(h) \equiv \int_a^{a+h} f(x)\,dx$. Establish an expression for the error term for each of the following rules:

c. $I(h) \approx h f(a)$ d. $I(h) \approx h f(a) - 1/2\, h^2 f'(a)$

For each, determine the corresponding general rule and error terms for the integral $\int_a^b f(x)\,dx$, where the partition is uniform; that is, $x_i = a + ih$ and $h = (b - a)/n$ for $0 \le i \le n$

Answer:

$I(h) \equiv \int_a^{a+h} f(x)\,dx.$

Consider the series of f(x) with variation t:

$$f(x+t) = f(x) + f'(x)t + \frac{f''(x)}{2}t^2 + \dots$$

c. I(h) ≈ h f (a)

$$\int_a^{a+h} f(x)\,dx = \int_0^h f(a+t)\,dt = \int_0^h \left[f(a) + f'(a)t + \frac{f''(a)}{2}t^2 + \dots \right] dt$$

$$= \left[f(a)t + f'(a)\frac{t^2}{2} + f''(a)\frac{t^3}{6} \right]_0^h = hf(a) + f'(\zeta)\frac{h^2}{2}$$

$$f'(\zeta)\frac{h^2}{2}\ for\ some\ \zeta \in [a, a+h]$$

We have extend the rule to any interval [a,b] we can compute

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n-1}\left[hf(x_i) + f'(\zeta_i)\frac{h^2}{2} \right] = h\sum_{i=0}^{n-1}[f(x_i)] + \frac{h^2}{2}\sum_{i=0}^{n-1} f''(\zeta_i)$$

$$\int_a^b f(x)\,dx \approx h\sum_{i=0}^{n-1}[f(x_i)]$$

Where h (b-a)/n. This has an error term

$$\frac{h^2}{2}\sum_{i=0}^{n-1} f''(\zeta_i) = \frac{(b-a)h}{2n}\sum_{i=0}^{n-1} f'(\zeta_i) = \frac{(b-a)h}{2} f'(\zeta_i)\ (\zeta \in (a,b))$$

d. I(h) ≈ h f (a) – 1/2 h2 f’ (a)

$$\int_a^{a+h} f(x)\,dx = \int_0^h f(a+t)\,dt = \int_0^h \left[f(a) + f'(a)t + \frac{f''(a)}{2}t^2 + \dots \right] dt$$

$$= \left[f(a)t + f'(a)\frac{t^2}{2} + f''(a)\frac{t^3}{6} \right]_0^h = hf(a) + f'(\zeta)\frac{h^2}{2}$$

$$f'(\zeta)\frac{h^2}{2}\ for\ some\ \zeta \in [a, a+h]$$

We have extend the rule to any interval [a,b] we can compute

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n-1}\left[hf(x_i) + f'(\zeta_i)\frac{h^2}{2} + f''(\zeta_i)\frac{h^3}{6} \right]$$

$$= \left(h\sum_{i=0}^{n-1}[f(x_i)] + \frac{h^2}{2}\sum_{i=0}^{n-1} f'(\zeta_i) \right) + \frac{h^3}{6}\sum_{i=0}^{n-1} f''(\zeta_i)$$

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n-1}\left[hf(x_i) + f'(x_i)\frac{h^2}{2}\right]$$

Where h (b-a)/n. This has an error term

$$\frac{h^3}{6}\sum_{i=0}^{n-1} f''(\zeta_i) = \frac{(b-a)h^2}{6}\sum_{i=0}^{n-1} f''(\zeta_i) = \frac{(b-a)h^2}{6} f''(\zeta_i)\, with\ (\zeta \in (a,b))$$

Section 6.1: 1, 2(b,c), 8

1. Compute $\int_0^1 (1 + x^2)^{-1}$ by the basic Simpson's Rule, using the three partition points x = 0, 0.5, and 1. Compare with the true solution.

Answer:

f(x)= $(1 + x^2)^{-1}$

f(0) = 1; f(0.5) = 0.8; f(1)=0.5

$$\int_a^b f(x)\,dx \approx \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$\int_0^1 (1 + x^2)^{-1} dx \approx \frac{(1-0)}{6}\left[f(0) + 4f\left(\frac{1+0}{2}\right) + f(1)\right] = \frac{1}{6}[1 + 4\cdot 0.8 + 0.5] = 0.783$$

True solution:

$$\int_0^1 (1 + x^2)^{-1} dx = \left[\frac{1}{\tan x}\right]_0^1 = 0.785$$

Compare |0.783-0.785|=0.002

2. Consider the integral $\int_0^1 \sin(\pi x^2/2)\ dx$. Suppose that we wish to integrate numerically, with an error of magnitude less than 10^{-3}.

b. Composite Simpson's Rule? c. Composite Simpson's 38 Rule?

Answer:

a.

$$E = I - T = \frac{-(b-a)h^2}{12} f''(\zeta)\ when\ \zeta \in (a,b)$$

$$|E| \le \frac{(b-a)h^2}{12} M\ where\ M = f''(\zeta)\ when\ \zeta \in (a,b)$$

triangle inequality

$$f(x) = \sin\left(\frac{\pi x^2}{2}\right) \ -> f'(x) = \pi x \cos\left(\frac{\pi x^2}{2}\right) -> f''(x) = \pi \cos\left(\frac{\pi x^2}{2}\right) - (\pi x)^2 \sin\left(\frac{\pi x^2}{2}\right)$$

Triangle inequality

$$|f''(x)| = |\pi\cos\left(\frac{\pi x^2}{2}\right) - (\pi x)^2 \sin\left(\frac{\pi x^2}{2}\right)| \le \pi\,|\cos\left(\frac{\pi x^2}{2}\right)| + |(\pi x)^2 \sin\left(\frac{\pi x^2}{2}\right)| \le \pi + \pi^2 \approx 13.01$$

$$|E| \le \frac{(b-a)h^2}{12}M = \frac{1\cdot 13.01.h^2}{12}$$

Given error of magnitude less than 10^{-3}.

$$\frac{13.01.h^2}{12} \le 10^{-3} -> h^2 \le 0.03037$$

b. Composite Simpson's Rule?

$$E = I - T = \frac{-(b-a)h^4}{180} f^{(4)}(\zeta) \ \ when\ \zeta \in (a,b)$$

$$f^{(3)}(x) = -3\pi^2 x \sin\left(\frac{\pi x^2}{2}\right) - \pi^3 x^3 \cos\left(\frac{\pi x^2}{2}\right) -> f^{(4)}(x) = -3\pi^2 \sin\left(\frac{\pi x^2}{2}\right) - 6\pi^3 x^3 \cos\left(\frac{\pi x^2}{2}\right) + \pi^4 x^4 \sin\left(\frac{\pi x^2}{2}\right)$$

triangle inequality

$$|f^{(4)}(x)| = |-3\pi^2 \sin\left(\frac{\pi x^2}{2}\right) - 6\pi^3 x^3 \cos\left(\frac{\pi x^2}{2}\right) + \pi^4 x^4 \sin\left(\frac{\pi x^2}{2}\right)| \le 3\pi^2\,|\sin\left(\frac{\pi x^2}{2}\right)| + 6\pi^3 |x^3 \cos\left(\frac{\pi x^2}{2}\right)| + \pi^4 |x^4 \sin\left(\frac{\pi x^2}{2}\right)|$$

$$\le 3\pi^2 + 6\pi^3 + \pi^4 \approx 313.06$$

$$|E| \le \frac{(b-a)h^4}{180}M = \frac{313.6.h^4}{180}$$

Given error of magnitude less than 10^{-3}.

$$\frac{313.6.h^4}{180} \le 10^{-3} -> h < 0.1549$$

c. Composite Simpson's 38 Rule?

$$E = I - T_{\frac{3}{8}} = \frac{-(b-a)h^4}{80} f^{(4)}(\zeta) \ \ when\ \zeta \in (a,b)$$

$$|E| \le \frac{(b-a)h^4}{80}M = \frac{313.6.h^4}{80}$$

Given error of magnitude less than 10^{-3}.

$$\frac{313.6.h^4}{80} \le 10^{-3} -> h < 0.1264$$

8. A numerical integration scheme that is not as well known is the basic Simpson's 3/ 8 Rule over three subintervals: $\int_a^{a+3h} f(x)\,dx \approx \frac{3h}{8}[f(a) + 3f(a+h) + 3f(a+2h) + f(a+3h)]$ Establish the error term for this rule, and explain why this rule is overshadowed by Simpson's Rule.

Answer:

$$\int_a^{a+3h} f(x)\,dx \approx \frac{3h}{8}[f(a) + 3f(a+h) + 3f(a+2h) + f(a+3h)]$$

Consider:

$$\int_a^b f(x)\,dx = Af(a) + Bf(a+h) + Cf(a+2h) + Df(a+3h)$$

The error term in Simpson's rule can be established by using the Taylor series from Section 1.2:

$$f(a+h) = f + hf' + \frac{1}{2!}h^2 f'' + \frac{1}{3!}h^3 f''' + \frac{1}{4!}h^4 f^{(4)} + \cdots$$

where the functions f, f', f'', ... on the right-hand side are evaluated at a. Now replacing h by $2h$, we have

$$f(a+2h) = f + 2hf' + 2h^2 f'' + \frac{4}{3}h^3 f''' + \frac{2^4}{4!}h^4 f^{(4)} + \cdots$$

$$f(a+h) = f(a) + f'(a)h + f''(a)\frac{h^2}{2} + f'''(a)\frac{h^3}{6} + f^{(4)}(a)\frac{h^4}{24} + \ldots$$

$$f(a+2h) = f(a) + 2f'(a)h + 2f''(a)h^2 + 4f'''(a)\frac{h^3}{3} + 2f^{(4)}(a)\frac{h^4}{3} + \ldots$$

$$f(a+3h) = f(a) + 3f'(a)h + 9f''(a)\frac{h^2}{2} + 9f'''(a)\frac{h^3}{2} + 27f^{(4)}(a)\frac{h^4}{8} + \ldots$$

We have computed the sum:

$$[f(a) + 3f(a+h) + 3f(a+2h) + f(a+3h)] =$$

$$= 8f(a) + 12f'(a)h + 12f''(a)h^2 + 9f'''(a)h^3 + \frac{11}{2}f^{(4)}(a)h^4 + \ldots$$

$$\frac{3h}{8}[f(a) + 3f(a+h) + 3f(a+2h) + f(a+3h)] =$$

$$= 3f(a)h + \frac{9}{2}f'(a)h^2 + \frac{9}{2}f''(a)h^3 + \frac{27}{8}f'''(a)h^4 + \frac{33}{16}f^{(4)}(a)h^5 + \ldots \qquad (1)$$

We also integrate

$$\int_a^{a+3h} f(x)\,dx = \int_0^{3h} f(a+t)\,dt = \int_0^{3h}\left(f(a) + f'(a)t + f''(a)\frac{t^2}{2} + f'''(a)\frac{t^3}{6} + f^{(4)}(a)\frac{t^4}{24} + \ldots\right)dt =$$

$$= \left[f(a)t + f'(a)\frac{t^2}{2} + f''(a)\frac{t^3}{6} + f'''(a)\frac{t^4}{24} + f^{(4)}(a)\frac{t^5}{120} + \ldots\right]_0^{3h} = \qquad (2)$$

$$= 3hf(a) + f'(a)\frac{9h^2}{2} + f''(a)\frac{9h^3}{2} + f'''(a)\frac{27h^4}{8} + f^{(4)}(a)\frac{81h^5}{40} + \ldots$$

From (1):

$3f(a)h + \frac{9}{2}f'(a)h^2 + \frac{9}{2}f''(a)h^3 + \frac{27}{8}f'''(a)h^4 = \frac{3h}{8}\,[f(a) + 3f(a+h) + 3f(a+2h) + f(a+3h)] - \left[\frac{33}{16}f^{(4)}(a)h^5 + \dots\right]$

From (2):

$$\int_a^{a+3h} f(x)\,dx = \left[3hf(a) + f'(a)\frac{9h^2}{2} + f''(a)\frac{9h^3}{2} + f'''(a)\frac{27h^4}{8}\right] + f^{(4)}(a)\frac{81h^5}{40} + \dots$$

$$= 3h/8\ [f(a) + 3f(a+h) + 3f(a+2h) + f(a+3h)] - \left[\frac{33}{16}f^{(4)}(a)h^5 + \dots\right] + f^{(4)}(a)\frac{81h^5}{40} + \dots =$$

$$= \frac{3h}{8}\ [f(a) + 3f(a+h) + 3f(a+2h) + f(a+3h)] - \left[\frac{3}{80}f^{(4)}(\zeta)h^5 + \dots\right]$$

Hence error is $-\left[\frac{3}{80}f^{(4)}(\zeta)h^5 + \dots\right]$ *with* $\zeta \in [a, a+3h]$. The Simpson's rule is preferred since it relies only on three function evaluations instead of 4. Also, the evaluation in the 3/8 error remains in the same order.

Section 6.2: 1, 2b, 5, 8, 9, 11

1. A Gaussian quadrature rule for the interval [−1, 1] can be used on the interval [a, b] by applying a suitable linear transformation. Approximate

$$\int_0^2 e^{-x^2}dx$$

using the transformed rule from Table 6.1 with n = 1.

TABLE 6.1 Gaussian Quadrature Nodes and Weights

n	Nodes x_i	Weights A_i
1	$-\sqrt{\frac{1}{3}}$	1
	$+\sqrt{\frac{1}{3}}$	1
2	$-\sqrt{\frac{3}{5}}$	$\frac{5}{9}$
	0	$\frac{8}{9}$
	$+\sqrt{\frac{3}{5}}$	$\frac{5}{9}$
3	$-\sqrt{\frac{1}{7}(3-4\sqrt{0.3})}$	$\frac{1}{2}+\frac{1}{12}\sqrt{\frac{10}{3}}$
	$-\sqrt{\frac{1}{7}(3+4\sqrt{0.3})}$	$\frac{1}{2}-\frac{1}{12}\sqrt{\frac{10}{3}}$
	$+\sqrt{\frac{1}{7}(3-4\sqrt{0.3})}$	$\frac{1}{2}+\frac{1}{12}\sqrt{\frac{10}{3}}$
	$+\sqrt{\frac{1}{7}(3+4\sqrt{0.3})}$	$\frac{1}{2}-\frac{1}{12}\sqrt{\frac{10}{3}}$
4	$-\sqrt{\frac{1}{9}\left(5-2\sqrt{\frac{10}{7}}\right)}$	$0.3\left(\frac{-0.7+5\sqrt{0.7}}{-2+5\sqrt{0.7}}\right)$
	$-\sqrt{\frac{1}{9}\left(5+2\sqrt{\frac{10}{7}}\right)}$	$0.3\left(\frac{0.7+5\sqrt{0.7})}{2+5\sqrt{0.7}}\right)$
	0	$\frac{128}{225}$
	$+\sqrt{\frac{1}{9}\left(5-2\sqrt{\frac{10}{7}}\right)}$	$0.3\left(\frac{-0.7+5\sqrt{0.7}}{-2+5\sqrt{0.7}}\right)$
	$+\sqrt{\frac{1}{9}\left(5+2\sqrt{\frac{10}{7}}\right)}$	$0.3\left(\frac{0.7+5\sqrt{0.7}}{2+5\sqrt{0.7}}\right)$

Answer:

$\int_0^2 e^{-x^2}dx$ Let x = y +1 -> dx = dy-> $\int_{-1}^1 e^{-(y+1)^2}dy$

$$\int_{-1}^1 e^{-(y+1)^2}dy \approx 1f\left(-\sqrt{\frac{1}{3}}\right)+1f\left(\sqrt{\frac{1}{3}}\right)=e^{-\left(1-\sqrt{\frac{1}{3}}\right)^2}+e^{\left(1+\sqrt{\frac{1}{3}}\right)^2}\approx 0.91949$$

2. Using Table 6.1, show directly that the Gaussian quadrature rule is exact for the polynomials 1, x, x^2,..., x^{2n+1} when

b. n = 3

Answer:

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n} A_i f(x_i)$$

Here, $A_i = \int_a^b l_i f(x)\,dx$ and $l_i(x) = \prod_{j=0;\,j\neq 1}^{n} \left(\frac{x-x_j}{x_i-x_j}\right)$

For n =3, nodes are $-\sqrt{\frac{1}{7}\left(3-4\sqrt{0.3}\right)}, -\sqrt{\frac{1}{7}\left(3-4\sqrt{0.3}\right)},$

$\sqrt{\frac{1}{7}\left(3-4\sqrt{0.3}\right)}, \sqrt{\frac{1}{7}\left(3-4\sqrt{0.3}\right)}$ corresponding weights $\frac{1}{2}+\frac{1}{12}\sqrt{\frac{10}{3}}, \frac{1}{2}-\frac{1}{12}\sqrt{\frac{10}{3}},$ $\frac{1}{2}+\frac{1}{12}\sqrt{\frac{10}{3}}, \frac{1}{2}-\frac{1}{12}\sqrt{\frac{10}{3}}$

f(x) = $1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7$

$$\int_a^b f(x)\,dx = f\left(-\sqrt{\frac{1}{7}\left(3-4\sqrt{0.3}\right)}\right)+f\left(-\sqrt{\frac{1}{7}\left(3-4\sqrt{0.3}\right)}\right)+f\left(-\sqrt{\frac{1}{7}\left(3-4\sqrt{0.3}\right)}\right)+f\left(-\sqrt{\frac{1}{7}\left(3-4\sqrt{0.3}\right)}\right)$$

$$\sum_{i=0}^{n} A_i f(x_i) = A_1 f(x_1) + A_2 f(x_2) + A_3 f(x_3) + A_4 f(x_4)$$

$$= \left(\frac{1}{2}+\frac{1}{12}\sqrt{\frac{10}{3}}\right) f\left(-\sqrt{\frac{1}{7}\left(3-4\sqrt{0.3}\right)}\right) + \\ + \left(\frac{1}{2}-\frac{1}{12}\sqrt{\frac{10}{3}}\right) f\left(-\sqrt{\frac{1}{7}\left(3-4\sqrt{0.3}\right)}\right) \\ + \left(\frac{1}{2}+\frac{1}{12}\sqrt{\frac{10}{3}}\right) f\left(\sqrt{\frac{1}{7}\left(3-4\sqrt{0.3}\right)}\right) \\ + \left(\frac{1}{2}-\frac{1}{12}\sqrt{\frac{10}{3}}\right) f\left(\sqrt{\frac{1}{7}\left(3-4\sqrt{0.3}\right)}\right)$$

Therefore, $\int_a^b f(x)\,dx \approx \sum_{i=0}^{n} A_i f(x_i)$ Hence quadratic rule is n = 3.

5. Construct a rule of the form $\int_{-1}^{1} f(x)\,dx \approx \alpha f\left(\frac{-1}{2}\right) + \beta f(0) + \gamma f\left(\frac{1}{2}\right)$ that is exact for all polynomials of degree ≤ 2; that is, determine values for α, β, and γ . Hint: Make the relation exact for 1, x, and x^2 and find a solution of the resulting equations. If it is exact for these polynomials, it is exact for all polynomials of degree ≤ 210.

Answer:

$$\int_{-1}^{1} f(x)\,dx \approx \alpha f\left(\frac{-1}{2}\right) + \beta f(0) + \gamma f\left(\frac{1}{2}\right)$$

Make the relation exact for 1, x, and x^2

$$\int_{-1}^{1} dx = 2 = \alpha + \beta + \gamma$$

$$\int_{-1}^{1} x\,dx = 0 = \frac{-1}{2}\alpha + \frac{1}{2}\gamma$$

$$\int_{-1}^{1} x^2\,dx = \frac{2}{3} = \frac{1}{4}\alpha + \frac{1}{4}\gamma$$

Then $\alpha = \gamma = \frac{4}{3}; \beta = -\frac{2}{3}$

$$\int_{-1}^{1} f(x)\,dx \approx \frac{4f\left(\frac{-1}{2}\right)}{3} + \frac{2f(0)}{3} + \frac{4f\left(\frac{1}{2}\right)}{3}$$

8. Derive a formula of the $\int_{a}^{b} f(x)\,dx \approx w_0 f(a) + w_1 f(b) + w_2 f'(a) + w_3 f'(b)$ that is exact for polynomials of the highest degree possible.11

Answer:

$$\int_{a}^{b} f(x)\,dx \approx w_0 f(a) + w_1 f(b) + w_2 f'(a) + w_3 f'(b)$$

Let use a = 0; b =1

$$\int_{0}^{1} f(x)\,dx \approx w_0 f(a) + w_1 f(b) + w_2 f'(a) + w_3 f'(b)$$

Make the relation exact for 1, x, x^2, x^3

$$\int_{0}^{1} dx = 1 \approx w_0 + w_1$$

$$\int_{0}^{1} x\,dx = \frac{1}{2} \approx w_1 + w_2 + w_3$$

$$\int_{0}^{1} x^2\,dx = \frac{1}{3} \approx w_1 + 2w_3$$

$$\int_{0}^{1} x^3\,dx = \frac{1}{4} \approx w_1 + 3w_3$$

$$w_0 = w_1 = \frac{1}{2}; w_2 = \frac{1}{12}; w_3 = -\frac{1}{12}$$

$$\int_0^1 f(x)\,dx \approx \frac{1f(a)}{2}+\frac{1f(b)}{2}+\frac{1f'(a)}{12}-\frac{1f'(b)}{12}$$

Linear map: y = a + (b-a)x -> dx = (b-a)dy to exchange [0,1] and [a,b]

$$\int_a^b f(x)\,dx \approx (b-a)\left[\frac{1f(a)}{2}+\frac{1f(b)}{2}+\frac{1f'(a)}{12}-\frac{1f'(b)}{12}\right]$$

9**. Derive the Gaussian quadrature rule of the form**

$$\int_{-1}^{1} f(x)x^2\,dx \approx af(-\alpha)+bf(0)+cf(\alpha)$$

that is exact for all polynomials of as high a degree as possible; that is, determine α, a, b, and c.

Answer:

$$\int_{-1}^{1} f(x)x^2\,dx \approx af(-\alpha)+bf(0)+cf(\alpha)$$

Make the relation exact for 1, x, x^2, x^3, x^4

$$\int_{-1}^{1} x^2\,dx \approx \frac{2}{3}= a+b+c$$

$$\int_{-1}^{1} x^3\,dx \approx 0= -a\alpha+c\alpha$$

$$\int_{-1}^{1} x^4\,dx \approx \frac{2}{5}= a\alpha^2+c\alpha^2$$

$$\int_{-1}^{1} x^5\,dx \approx 0= -a\alpha^3+c\alpha^3$$

$$\int_{-1}^{1} x^6\,dx \approx \frac{2}{7}= a\alpha^4+c\alpha^4$$

$$\alpha^2=\frac{5}{7}\ then\ \alpha=\sqrt{\frac{5}{7}};\ a=\frac{2}{25};\ b=\frac{8}{75};\ c=\frac{7}{25}$$

$$\int_{-1}^{1} f(x)x^2\,dx \approx \frac{7f\left(-\sqrt{\frac{5}{7}}\right)}{25}+\frac{8f(0)}{75}+\frac{7f\left(\sqrt{\frac{5}{7}}\right)}{25}$$

11. Derive a numerical integration formula of the form

$$\int_{x_{n-1}}^{x_{n+1}} f(x)\,dx \approx Af(x_n)+Bf'(x_{n-1})+Cf''(x_{n+1})$$

for uniformly spaced points xn−1, xn, and xn+1 with spacing h. The formula should be exact for polynomials of as high a degree as possible. Hint: Consider

$$\int_{-h}^{h} f(x)\,dx \approx Af(0)+Bf'(-h)+Cf''(h)$$

Answer:

$$\int_{-h}^{h} f(x)\,dx \approx Af(0) + Bf'(-h) + Cf''(h)$$

Let use h = 1

$$\int_{-1}^{1} f(x)\,dx \approx Af(0) + Bf'(-h) + Cf''(h)$$

Make the relation exact for 1, x, x^2

$$\int_{-1}^{1} dx = 2 \approx A$$

$$\int_{-1}^{1} xdx = 0 \approx B$$

$$\int_{-1}^{1} x^2\,dx = \frac{2}{3} \approx -2A + 2C$$

$$Then\ A = 2;\ B = 0;\ C = \frac{1}{3}$$

$$\int_{-1}^{1} f(x)\,dx \approx 2f(0) + \frac{1f''(h)}{3}$$

Linear map: y= x_n + hx -> dy = h dx to exchange [-1,1] to [x_{n-1},x_{n+1}]

$$\int_{x_{n-1}}^{x_{n+1}} f(x)\,dx \approx h\left[2f(x_n) + \frac{h^2 f''(x_{n+1})}{3}\right]$$

Computing Exercises

Section 5.2: 1, 2, 5 (c)

1. Write

real function Trapezoid Uniform(f, a, b, n) to calculate $\int_a^b f(x)\,dx$ using the composite trapezoid rule with n equal subintervals.

Answer:

```
% Trapezoid_uniform
function num = Trapezoid_Uniform(f,a,b,n)

h = (b-a)/n;

x = [a+h:h:b-h];

num = h/2*(2*sum(feval(f,x))+feval(f,a)+feval(f,b));
```

```
end
```

2. (Continuation) Test the code written in the preceding computer problem on the following functions. In each case, compare with the correct answer.

[a]**a.** $\int_0^{\pi} \sin x\, dx$ [a]**b.** $\int_0^1 e^x\, dx$ [a]**c.** $\int_0^1 \arctan x\, dx$

Answer:

Code:

```
clc;
f= @(x) (sin(x));
g= @(x) (exp(x));
p= @(x) (atan(x));

i1 = Trapezoid_Uniform(f,0,pi,2);
fprintf('sin(x) = %f\n',i1);

i2 = Trapezoid_Uniform(g,0,1,2);
fprintf('exp(x) = %f\n',i2);

i3 = Trapezoid_Uniform(p,0,1,2);
fprintf('arctan(x) = %f\n',i3);

% Trapezoid_uniform
function num = Trapezoid_Uniform(f,a,b,n)

h = (b-a)/n;

x = [a+h:h:b-h];

num = h/2*(2*sum(feval(f,x))+feval(f,a)+feval(f,b));
end
```

Sample Screenshot:

```
sin(x) = 1.570796
exp(x) = 1.753931
arctan(x) = 0.428173
>>
```

5. Compute these integrals by using small and large values for the lower and upper limits and applying a numerical method. Then compute them by first making the indicated change of variable.

c. $\int_0^{\infty} \sin x^2\, dx = \frac{1}{2} * \sqrt{\frac{\pi}{2}}$, using x = tan t (Fresnel sine integral)

Answer:

Code:

```
clc;

syms x;
fun = (sin(x))^2;
int(fun,x,0,inf)


syms x;
fun = sin(tan(x))^2*(sec(x))^2;
int(fun,x,0,pi/2)
```

Screenshot:

```
ans =

Inf


ans =

Inf
```

Section 6.1: 1

1. Find approximate values for the two integrals

$$4\int_0^1 \frac{dx}{1+x^2} \qquad 8\int_0^{\frac{1}{\sqrt{2}}} \left(\sqrt{1-x^2}-x\right) dx$$

Use recursive function Simpson with ε = 1 /2×10^{-5} and level max = 4. Sketch the curves of the integrand f (x) in each case, and show how Simpson partitions the intervals. You may want to print the intervals at which new values are added to Simpson result in function Simpson and also to print values of f (x) over the entire interval [a, b] in order to sketch the curves.

Answer:

Code:

```
clc;
fprintf('1.\n');
f =  @(x) 4./(1 +  x^2);
sLevel_son(f,0,1,10);

fprintf('2.\n');
g =  @(x) 8*(sqrt(1 - x^2) - x);
sLevel_son(g,0,0.5,10);

function sLevel_son(func,a,b,total) %Function
```

```
%Initialize value
y= 0;
Flag1= 0;
Flag2= 0;
Error_estimate= 0;
maxLevel= 4;
saveFu1= zeros(maxLevel,3);
saveFu2= zeros(maxLevel,3);
result= zeros(maxLevel);
total2= total +  10*eps;
total1= total2*15/(b - a);
index1= a:(b - a)/4:b;

%Loop
for index2= 1:5
f(index2)= feval(func,index1(index2));
end

%Initialize value
inter= 5;
level1= 1;

%Loop
while level1>0
%Save the right half sub interval information
    for index3= 1:3
        saveFu1(level1,index3)= f(index3 +  2);            %Fix saveFuntion_1
        saveFu2(level1,index3)= index1(index3 +  2);      %Fix saveFuntion_2

    end

%Find h
h= (index1(5) - index1(1))/4;

%Find
result(level1)= (h/3)*(f(3) +  4*f(4) +  f(5));

%Check
if Flag2<1
    sLevel_1= 2*(h/3)*(f(1) +  4*f(3) +  f(5));
end
sLevel_l= (h/3)*(f(1) +  4*f(2) +  f(3));
sLevel_2= sLevel_l +  result(level1);
diff1= abs(sLevel_1 - sLevel_2);

%Check
if diff1<= total1*4*h
level1= level1 - 1;
Flag2= 0;
y= y +  sLevel_2;
Error_estimate= Error_estimate +  diff1/15;

    %Check
    if level1<1
```

```
        %Display
        fprintf('Number of intervals: %g. \n',inter)
        fprintf('The computed value of the integral: %g. \n',y)
        fprintf('Predicted error: %g. \n',Error_estimate)
    return
    end
    %Loop
    for index2= 1:3
        index5= 2*index2 - 1;
        f(index5)= saveFu1(level1,index2);
        index1(index5)= saveFu2(level1,index2);
    end
    %Otherwise
else
    level1= level1 +  1;
    sLevel_1= sLevel_l;

        %Check
        if level1 <=  maxLevel
            f(5)= f(3);
            f(3)= f(2);
            index1(5)= index1(3);index1(3)= index1(2);
            Flag2= 1;
            %Otherwise
        else
            Flag1= Flag1 +  1;
            level1= level1 - 1;
            y= y +  sLevel_2;
            Flag2= 0;
            Error_estimate= Error_estimate +  diff1/15;   %Fix Error_estimate
        end
end

        %Loop
        for index3= 1:2
            %Update
            index6= 2*index3;
            index1(index6)= .5*(index1(index6 +  1) +  index1(index6 - 1));
            f(index6)= feval(func,index1(index6));
        end
    inter= inter +  2;

    end
end
```

Screenshot:

```
1.
Number of intervals: 5.
The computed value of the integral: 3.14157.
Predicted error: 0.00054902.
2.
Number of intervals: 5.
The computed value of the integral: 2.82641.
Predicted error: 2.72397e-05.
```

Section 6.2: 1, 3, 6

1. Write a program to evaluate an integral $\int_a^b f(x)\,dx$ using Formula (5).

Answer:

```
clc;

function result = Gauss(f,a,b)
% Gauss - 3 Point
x3=[-sqrt(3/5), 0, sqrt(3/5)];
weight3 = [5/9, 8/9, 5/9];

tempx3 = (b-a)/2*weight3;
temp_weight3 = (b-a)/2*x3 + (b+a)/2;

temp = f(tempx3);
result = double(sum(temp_weight3.*temp));
end
```

3. (Continuation) Compute $\int_0^1 x^{-1}\sin x\,dx$ by the Gaussian Formula (5) suitably modified.

Answer:

```
clc;

format long;

a = 0;
b = 1;
f =  @(x) (x.^(-1)).*sin(x); % Any function
fprintf('Result: %f. \n',Gauss(f,0,1))

function result = Gauss(f,a,b)
% Gauss - 3 Point
x3=[-sqrt(3/5), 0, sqrt(3/5)];
weight3 = [5/9, 8/9, 5/9];

tempx3 = (b-a)/2*weight3;
temp_weight3 = (b-a)/2*x3 + (b+a)/2;
```

```
temp = f(tempx3);
result = double(sum(temp_weight3.*temp));
end
```

Screenshot:

```
Result: 1.470890.
```

6. Apply and compare the composite rules for Trapezoid, Midpoint, Two-Point Gaussian, and Simpson's 1/ 3 Rule for approximating the integral $\int_0^{2\pi} e^{-x} \cos x \, dx \approx 0.49906\,62786\,34$ using 32 applications of each basic rule.

Answer:

Code:

```
clc;
f =  @(x) exp(-x).*cos(x);

a = 0;
b = 2*pi;
n = 32; %number of intervals
h = (b-a)/n; %width of interval
x = linspace(a, b, n);%x of interval
x_midpts = linspace(a+h/2, b-h/2, n);%midpoints of interval

result_trapz = h*sum(f(x));              % Trapezoid rule
result_trapz_Midpoint = h*sum(f(x_midpts));      % Trapezoid Midpoint rule
result_simpson= h.*sum((1/6)*(f(x_midpts)+f(x_midpts+h)+4.*f((2*x_midpts+h)/2)));   %
Simpson's 1/3
result_gauss= h*sum((1/2)*(f(x_midpts)+f(x_midpts+h)));                             %
Two-Point Gaussian Quadrature

fprintf('Approximate integral using Trapezoid rule: %f. \n',result_trapz);
fprintf('Approximate integral using Trapezoid Midpoint rule: %f.
\n',result_trapz_Midpoint);
fprintf('Approximate integral using Simpson 1/3 rule: %f. \n',result_simpson);
fprintf('Approximate integral using Two-Point Gaussian Quadrature  rule: %f.
\n',result_gauss);
```

Screenshot:

```
Approximate integral using Trapezoid rule: 0.585143.
Approximate integral using Trapezoid Midpoint rule: 0.497459.
Approximate integral using Simpson 1/3 rule: 0.405877.
Approximate integral using Two-Point Gaussian Quadrature  rule: 0.409059.
>>
```