For the rest of the semester

- 11/28: eigenvalue
- 11/30: SVD
- 12/5: SOR (not covered in final)
- 12/7: hw6 due and final review (going over some homework problems)

- 12/12 Final exam 2-4pm (same classroom)
  - Scientific calculator
  - Single A4 page

Today's Agenda

- Eigenvalues and eigenvectors
- Matrix/eigenvalue properties
- Power method to find the largest eigenvalue in magnitude (not covered in Final)

Overview

- When $Ax = \lambda x$ is valid with $x \neq 0$, we say that λ is an eigenvalue of A and x is an accompanying eigenvector.
- Interpretation: Ax is scalar multiple of x.
- If λ is an eigenvalue of A, the set (subspace) $\{x \in R^n | Ax = \lambda x\}$ is called eigenspace.
- Eigenvalues and eigenvectors can take complex values.

How to find eigenvalues?

- Suppose A is a square matrix of size $n \times n$.
- It follows from $Ax = \lambda x$ that $(A - \lambda I)x = 0$.
- Then $A - \lambda I$ is singular, i.e., $|A - \lambda I| = 0$.
- Define characteristic polynomial of A
$$p(\lambda) = |A - \lambda I|$$
- It is a polynomial of degree n and must have n roots (complex zeros and repeated zeros with multiplicity).
- Root finding to find eigenvalues is a direct method.

Matlab

```
A =

     1     3    -7
    -3     4     1
     2    -5     3
```

[V,D]=eig(A);

```
V =

   0.7267 + 0.0000i   0.7267 + 0.0000i   0.7916 + 0.0000i
  -0.0680 + 0.4533i  -0.0680 - 0.4533i   0.5137 + 0.0000i
  -0.3395 - 0.3829i  -0.3395 + 0.3829i   0.3308 + 0.0000i


D =

   3.9893 + 5.5601i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   3.9893 - 5.5601i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0214 + 0.0000i
```

Properties

Matrix properties

- Symmetric: $A = A^T$
- A complex matrix is Hermitian if $A = A^*$ (conjugate transpose).
- Positive definite if $x^T A x > 0, \forall x \neq 0$.

- Two matrices A and B are similar if there exists a nonsingular matrix P such that $B = PAP^{-1}$.
- Similar matrices have the same eigenvalues.

Matrix properties (cont'd)

- Every square real matrix is similar to a triangular matrix.
- Matrices A and B are unitarily similar if there exists a unitary matrix U s.t. $B = U^* A U$.
- Schur's Theorem: Every square matrix is unitarily similar to a triangular matrix.
- Every square Hermitian matrix is unitarily similar to a diagonal matrix.

Eigenvalue properties

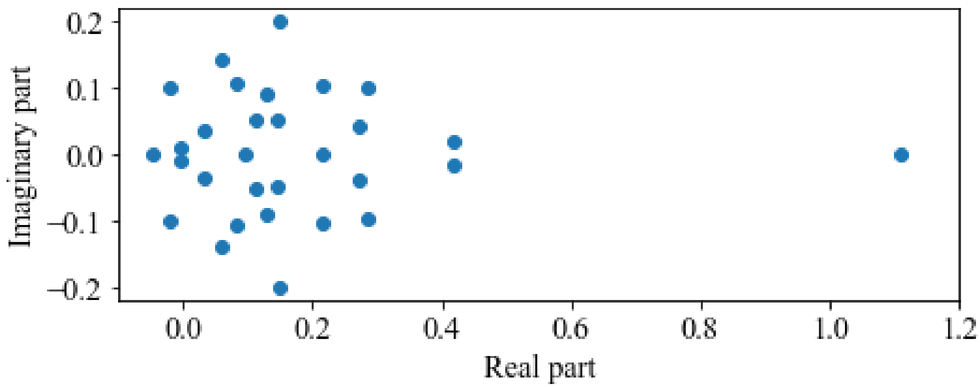
Matrix Eigenvalue Properties

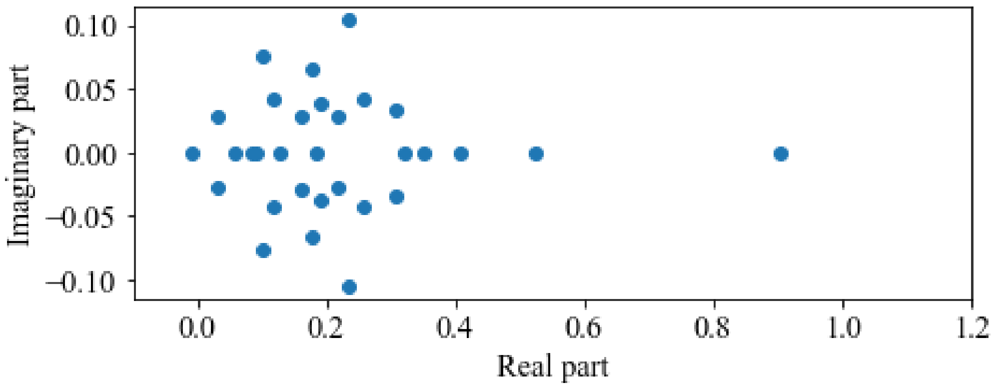
The following statements are true for any square matrix $\boldsymbol{A}$:

1. If λ is an eigenvalue of $\boldsymbol{A}$, then $p(\lambda)$ is an eigenvalue of $p(\boldsymbol{A})$, for any polynomial p. In particular, λ^k is an eigenvalue of $\boldsymbol{A}^k$.
2. If $\boldsymbol{A}$ is invertible and λ is an eigenvalue of $\boldsymbol{A}$, then $p(1/\lambda)$ is an eigenvalue of $p(\boldsymbol{A}^{-1})$, for any polynomial p. In particular, λ^{-1} is an eigenvalue of $\boldsymbol{A}^{-1}$.
3. If $\boldsymbol{A}$ is real and symmetric, then its eigenvalues are real.
4. If $\boldsymbol{A}$ is complex and Hermitian, then its eigenvalues are real.
5. If $\boldsymbol{A}$ is Hermitian and positive definite, then its eigenvalues are positive.
6. If $\boldsymbol{P}$ is invertible, then $\boldsymbol{A}$ and $\boldsymbol{PAP}^{-1}$ have the same characteristic polynomial (and the same eigenvalues).

Localization of eigenvalues

- Suppose $x_{k+1} = Ax_k$ for $k = 1, 2, \cdots$
- What do eigenvalue tell us?





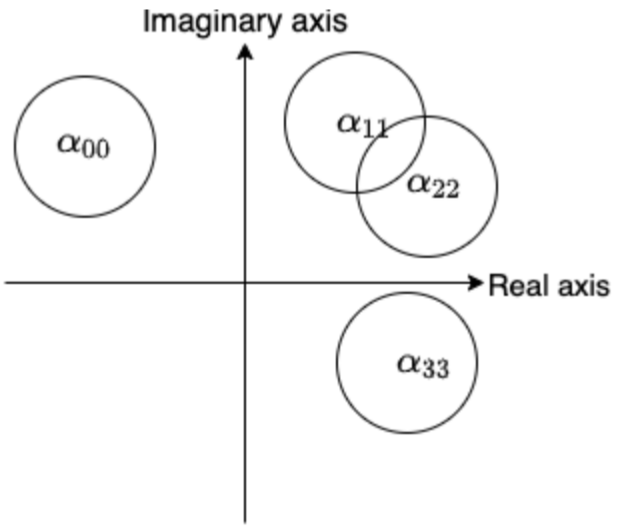
Gershgorin's Theroem

Gershgorin's Theorem

All eigenvalues of an $n \times n$ matrix $\mathbf{A} = (a_{ii})$ are contained in the union of the n discs $C_i = C_i(a_{ii}, r_i)$ in the complex plane with center a_{ii} and radii r_i given by the sum of the magnitudes of the off-diagonal entries in the ith row.

$$A = \begin{bmatrix} \alpha_{00} & \alpha_{01} & \alpha_{02} & \alpha_{03} \\ \alpha_{10} & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{20} & \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{30} & \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix}$$

$$\begin{aligned} \rho_0 &= |\alpha_{01}| + |\alpha_{02}| + |\alpha_{03}| \\ \rho_1 &= |\alpha_{10}| + |\alpha_{12}| + |\alpha_{13}| \\ \rho_2 &= |\alpha_{20}| + |\alpha_{21}| + |\alpha_{23}| \\ \rho_3 &= |\alpha_{30}| + |\alpha_{31}| + |\alpha_{32}| \end{aligned}$$



Proof sketch

- Suppose x is an eigenvector for λ
- Let m be the index of the largest magnitude of x
- Scale x such that $|\xi_m| = 1$, and $|\xi_i| \leq 1$
- x is eigenvector, so $(\lambda - a_{mm})\xi_m = -\sum_{j=1}^{n} a_{mj}\xi_j,$

$$\Rightarrow \quad |\lambda - a_{mm}| \leq \sum_{\substack{j=1 \\ j\neq m}}^{n} |a_{mj}||\xi_j| \leq \sum_{\substack{j=1 \\ j\neq m}}^{n} |a_{mj}| = \rho_m.$$

Usage of the theorem

- If A is a strictly diagonally dominant matrix, then it is non-singular.

- Spectral radius is upper bounded by the matrix inf norm

$$\rho(A) \leq ||A||_\infty$$

- Recall GS and GJ converge if and only if $\rho(A) < 1$.

Power method

Not covered in Final

Rationale

- WLOG assume that eigenvalues of $A \in C^{n\times n}$ are ordered by $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$, and eigenpairs are given by (λ_j, x_j).
- Assume A is diagonalizable for easy analysis
- Initially $q^{(0)} = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$ (assume $a_1 \neq 0$)
- Consider $q^{(k)} = Aq^{(k-1)} = A^k q^{(0)}$

$$= a_1 \lambda_1^k x_1 + a_2 \lambda_2^k x_2 + \cdots + a_n \lambda_n^k x_n$$

As $k \to \infty$, one has $\frac{q^{(k)}}{\lambda_1^k} \to a_1 x_1$

But don't know λ_1
Need to normalize q

Power method/iteration

- λ_1 is called dominant eigenvalue if $|\lambda_1| > |\lambda_2|$, thus having a unique eigenvector x_1.
- Power iteration goes by

Inexpensive
Sparse friendly

$$\boxed{\begin{aligned} z^{(k)} &= Aq^{(k-1)} \\ q^{(k)} &= z^{(k)} / \| z^{(k)} \|_2 \\ \lambda^{(k)} &= [q^{(k)}]^H A q^{(k)} \end{aligned}} \quad \begin{aligned} &\to x_1. \\ &\to \lambda_1 \end{aligned}$$

- Stopping condition: $\dfrac{||z^{(k)} - \lambda^{(k)} q^{(k)}||_2}{||A||_F} < \epsilon$

Convergence

- Recall that $$A^k q^{(0)} = a_1 \lambda_1^k \left(x_1 + \sum_{j=2}^{n} \frac{a_j}{a_1} \left(\frac{\lambda_j}{\lambda_1} \right)^k x_j \right)$$
- As $q^{(k)} \in span\left(A^k q^{(0)}\right)$, we have

$$\text{dist}\left(\text{span}\{q^{(k)}\}, \text{span}\{x_1\}\right) = O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right) \qquad |\lambda_1 - \lambda^{(k)}| = O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right)$$

- The convergence rate depends on $r = \left|\frac{\lambda_2}{\lambda_1}\right|$.

Inverse power method

- Suppose A is non-singular
- Apply the power method to A^{-1}
- Its eigenvalues are λ_j^{-1}
- The inverse power method finds the eigenvector for the smallest eigenvalue (in magnitude).