Part 1:

Q.1: Probability I

$P(A) \geq 0$ for all event $A \subset S$

$P(S) = 1$

If A and B are disjoint events, then $P(A \cup B) = P(A) + P(B)$

Prove:

1) $P(\sim A) = 1 - P(A)$ where $\sim A$ is complement of A

Answer: Sample space: $S = A \cup \sim A$

Note: $A \cap \sim A \neq \emptyset$. So A and $\sim A$ are mutually exclusive.

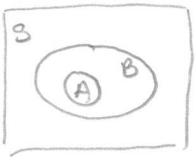
$P(S) = P(A \cup \sim A)$

$P(A) + P(\sim A) + \emptyset$ (axiom 3)

$\Rightarrow P(A) + P(\sim A) = 1$ (when $P(S) = 1$)

$\Rightarrow P(\sim A) = 1 - P(A)$ (Proved)

2) If $A \subset B$, then $P(A) \leq P(B)$



•) Like the picture we can consider when $A \subset B$

$P(B) = P(A \cap B) + P(\sim A \cap B)$

$= P(A) + P(\sim A \cap B)$

Due to: $P(\sim A \cap B) \geq 0$

So: $P(B) \geq P(A)$

or $P(A) \leq P(B)$ (Proved)

0.2. Probability II

$P(C) = 0{,}008$ is Probability that the patient has coronavirus.

$P(\neg C) = 1 - P(C) = 0.992$ is probability that the patient does not.

$P(P \mid C) = 0.98$. is Probability correst positive result of the cases in which disease is actually present.

$P(N \mid C) = 1 - P(P \mid C) = 0{,}02$. is probability not correst positive. result of the cases in which disease is actually present.

$P(N \mid \neg C) = 0.97$ is Probability correst Negative result of the case in which disease. is Not present.

$P(P \mid \neg C) = 1 - P(N \mid \neg C) = 0{,}03$ is Probability Not correct Negative Result of the case in which disease is Not present

•) $P(C \mid P) = \dfrac{P(P \mid C) \cdot P(C)}{P(P)}$

•) $P(P) = P(P \mid C) \cdot P(C) + P(P \mid \neg C) \cdot P(\neg C)$

$= 0.98 \cdot 0.008 + 0.03 \cdot 0.992 = 0{,}0376.$

•) $P(C \mid P) = \dfrac{P(P \mid C) \cdot P(C)}{P(P)} = \dfrac{0.98 \cdot 0{,}008}{0.0376} = 0{,}2085 \approx 21\%$

•) Even though the patient test positive, the probability that the patient has coronavirus is 21

•) The answer maybe Lies when pior probability of having the Coronaoirus is low.

•) Bayes rule let us Know the pior is low, we need to have many more evident to convince that patient has coronavirus.

0.3 Probability III.

$P(A) = 0.2$ → $P(\neg A) = 1 - P(A) = 0.8$

$P(B) = 0.6$ → $P(\neg B) = 1 - P(B) = 0.4$

$P(B|A) = 0.9$

Compute: $P(\neg B|\neg A)$

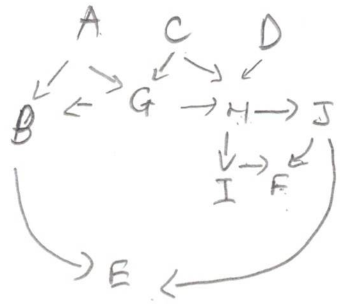
Answer:

$$P(B|A) = \frac{P(B\cap A)}{P(A)} \Rightarrow P(B\cap A) = P(B|A).P(A) = 0.9 \ast 0.2 = 0.18$$

$$P(B|\sim A) = \frac{P(B\cap \sim A)}{P(\sim A)} = \frac{P(B) - P(B\cap A)}{P(\sim A)} = \frac{0.6 - 0.18}{0.8} = 0.525.$$

$P(B|\sim A) = 0.525$

4. Conditional Independence.



$X \perp\!\!\!\perp Z \mid Y$. mean that X is conditionally indepent of Z given that the value of y is known.

a) $A \perp\!\!\!\perp D \mid \{\}$. guaranteed to be true.

No active path found.

b) $A \perp\!\!\!\perp D \mid E$. Not guaranteed to be true.

active path - A → B → E (Observed) ← J ← H ← D

c) $A \perp\!\!\!\perp C \mid F$. Not guaranteed to be true.

active path: A → G → H → J → F (Observed) ← I ← H ← C.

d) $C \perp\!\!\!\perp F \mid \{E, H\}$. Not guaranteed to be true.

active path: C → G → B → E (observed) ← J → F

e) $G \perp\!\!\!\perp F \mid \{B, H\}$. guaranteed to be true.

N0 active path found.

f) $G \perp\!\!\!\perp F \mid \{E, H\}$. Not guaranteed to be true.

active path: G → B → E (Observred) ← J → F

g) $G \perp\!\!\!\perp F \mid \{E, H, J\}$. guarantee' to be true.

No active path found.

0.5. Bayesian Networks

a) $P(f|a) = \dfrac{P(f,a)}{P(a)}$

- $P(f,a) = P(f,a,b) + P(f,a,\neg b) = P(f,a,b,c) + P(f,a,b,\neg c) + P(f,a,\neg b,c) + P(f,a,\neg b,\neg c)$
- $P(f,a,b,c) = P(f|c) \cdot P(c|a,b) \cdot P(a) . P(b)$

$= 0.3 \cdot 0.9 \cdot 0.7 . 0.8 = 0,1512$

$P(f,a,\neg b,c) = P(f|c) . P(c|a,\neg b) . P(a) . P(\neg b)$

$= 0.3 . 0.7 . 0,7 . (1-0.8) = 0.0294$

$P(f,a,b,\neg c) = P(f|\neg c) \cdot P(\neg c|a,b) . P(a) . P(b)$

$= 0.8 . (1-0.9) . 0.7 . 0.8 = 0.0448$

$P(f,a,\neg b,\neg c) = P(f|\neg c) . P(\neg c|a,\neg b) . P(a) . P(\neg b)$

$= 0.8 . (1-0.7) . 0.7 . (1-0.8) = 0,0336$

- $P(f,a) = 0,259$
- $P(f|a) = \dfrac{P(f,a)}{P(a)} = \dfrac{0.259}{0,7} = 0.37$

b) $P(f|\neg c) = 0.8$

c) $P(f|a,b) = \dfrac{P(f,a,b)}{P(a,b)}$

$P(a,b) = P(a) . P(b) = 0.56$

$P(f,a,b) = P(f,a,b,c) + P(f,a,b,\neg c) = 0,1512 + 0.0448 = 0,196$

$P(f|a,b) = \dfrac{P(f,a,b)}{P(a,b)} = 0.35$

d) $P(f|a,b,c) = \dfrac{P(f,a,b,c)}{P(a,b,c)} = \dfrac{P(f|c) \cdot P(c|a,b) \cdot P(a) . P(b)}{P(c|a,b) \cdot P(a) \cdot P(b)} = P(f|c)$

$= 0.3$

e) $P(a|f) = \dfrac{P(a,f)}{P(f)}$

$P(a,f) = P(f,a) = 0.259.$

$P(f) = P(f|-c) . P(c) + P(f|\neg c) \cdot P(\neg c)$

$P(\sim c) = P(\sim c|b) . P(b) + P(\sim c|\sim b) \cdot P(\sim b)$

$P(\sim c|b) = P(\sim c|a,b) \cdot P(a) \cdot P(b) + P(\sim c|\sim a,b) \cdot P(\sim a) \cdot P(b)$

$= (1-0.9) \cdot 0.7 \cdot 0.8 + (1-0{,}6) \cdot (1-0.7) \cdot 0.8$

$= 0{,}152$

$P(\sim c|\sim b) = P(\sim c|a,\sim b) \cdot P(a) \cdot P(\sim b) + P(\sim c|\sim a,\sim b) \cdot P(\sim a) \cdot P(\sim b)$

$= (1-0.7) \cdot 0.7 \cdot (1-0.8) + (1-0.1) \cdot (1-0{,}7) \cdot (1-0.8)$

$= 0{,}096$

$P(\sim c) = 0{,}1408 \Rightarrow P(c) = 0{,}8592$

$P(f) = 0{,}3704$

$$P(a|f) = \frac{0{,}259}{0.3704} = 0{,}699$$

f) $$P(c|g,h) = \frac{P(c,g,h)}{P(g,h)}$$

$P(g,h) = P(g,h,d,f) + P(g,h,d,\neg f) + P(g,h,\neg d,f) + P(g,h,\neg d,\neg f)$

$P(g,h,d,f) = P(g|d) \cdot P(h|f) \cdot P(d) \cdot P(f)$

$P(d) = P(d|c)\, P(c) + P(d|\neg c)\, P(\neg c)$

$= 0.8 \cdot 0.8592 + 0.6 \cdot 0{,}1408 = 0{,}77184$

$P(g,h,d,f) = 0{,}75 \cdot 0{,}65 \cdot 0{,}77184 \cdot 0{,}3704 = 0{,}14$

$P(g,h,d,\neg f) = P(g|d) \cdot P(h|\neg f) \cdot P(d) \cdot P(\neg f)$

$= 0{,}75 \cdot 0{,}25 \cdot 0{,}77184 \cdot (1-0{,}3704) = 0{,}092$

$P(g,h,\neg d,f) = P(g|\neg d) \cdot P(h|f) \cdot P(\neg d) \cdot P(f)$

$= 0{,}15 \cdot 0{,}65 \cdot (1-0{,}77184) \cdot 0{,}3704 = 0{,}008$

$P(g,h,\neg d,\neg f) = P(g|\neg d) \cdot P(h|\neg f) \cdot P(\neg d) \cdot P(\neg f)$

$= 0{,}15 \cdot 0{,}25 \cdot (1-0{,}77184)(1-0{,}3704) = 0{,}005$

$P(g,h) = 0{,}245$

$$P(c,g,h) = P(c,g,h,d,f) + P(c,g,h,d,\neg f) + P(c,g,h,\neg d,f) + P(c,g,h,\neg d,\neg f) =$$

$$P(c,g,h,d,f) = P(g|d) \cdot P(h|f) \cdot P(d|c) \cdot P(f|c) \cdot P(c)$$
$$= 0{,}75 \cdot 0{,}65 \cdot 0{,}8 \cdot 0{,}3 \cdot 0{,}8592 = 0{,}1$$

$$P(c,g,h,d,\neg f) = P(g|d) \cdot P(h|\neg f) \cdot P(d|c) \cdot P(f|c) \cdot P(c)$$
$$= 0{,}75 \cdot 0{,}25 \cdot 0{,}8 \cdot 0{,}3 \cdot 0{,}8592 = 0{,}038664$$

$$P(c,g,h,\neg d,f) = P(g|\neg d) \cdot P(h|f) \cdot P(\neg d|c) \cdot P(f|c) \cdot P(c)$$
$$= 0{,}15 \cdot 0{,}65 \cdot (1-0{,}8) \cdot 0{,}3 \cdot 0{,}8592 = 0{,}005$$

$$P(c,g,h,\neg d,\neg f) = P(g|\neg d) \cdot P(h|\neg f) \cdot P(\neg d|c) \cdot P(f|c) \cdot P(c)$$
$$= 0{,}13 \cdot 0{,}25 \cdot 0{,}2 \cdot 0{,}3 \cdot 0{,}8592 = 0{,}002$$

$$P(c,g.h) = 0{,}146$$

$$P(c|g;h) = \frac{P(c,g,h)}{P(g,h)} = \frac{0{,}146}{0{,}245} = 0{,}594$$