Today's agenda

Section 4.1: polynomial interpolation

- Newton form

Section 4.2: error analysis

Recall the example

x	0	1	-1	2	-2
y	-5	-3	-15	39	-9

- $p_0(x) = -5$
- $p_1(x) = -5 + 2x$
- $p_2(x) = -5 + 2x - 4x(x-1)$
- $p_3(x) = -5 + 2x - 4x(x-1) + 8x(x-1)(x+1)$

$$p(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \cdots + a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1})$$

Nested form

$$p_4(x) = -5 + x(2 + (x-1)(-4 + (x+1)(8 + (x-2)3)))$$

Summary

$$p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

- Concise notation

$$p_n(x) = \sum_{i=0}^{n} a_i \prod_{j=0}^{i-1}(x - x_j) \text{ with } \prod_{j=0}^{-1}(x - x_j) = 1$$

- Nested form

$$p(x) = a_0 + (x - x_0)(a_1 + (x - x_1)(a_2 + \cdots + (x - x_{n-1})a_n))\cdots))$$

Finding coefficients $\{a_j\}_{j=0}^{n}$

- The nodes are assumed to be distinct, but no assumption is made about their position on the real line.
- Newton form

$$p_n(x) = p_{n-1}(x) + a_n(x - x_0)\cdots(x - x_{n-1})$$

- p_n is obtained by p_{n-1} by adding one more term.

$$\begin{cases} f(x_0) = a_0 \\ f(x_1) = a_0 + a_1(x_1 - x_0) \\ f(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) \\ \qquad \text{etc.} \end{cases}$$

Divided difference

- a_0 depends on $f(x_0)$
- a_1 depends on $f(x_0), f(x_1)$, ...
- In general, a_k uniquely depends on the value of f at nodes $x_0, x_1, \cdots, x_k$
- Notation: $a_k = f[x_0, x_1, \cdots, x_k]$.
- The quantity $f[x_0, x_1, \cdots, x_k]$ is called divided difference of order k for f.
- Newton form of the interpolating polynomial

$$p_n(x) = \sum_{i=0}^{n} \left\{ f[x_0, x_1, \ldots, x_i] \prod_{j=0}^{i-1} (x - x_j) \right\}$$

Divided difference (cont'd)

$$f(x_0) = a_0$$
$$f(x_1) = a_0 + a_1(x_1 - x_0)$$
$$f(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

$$a_0 = f(x_0)$$

$$a_1 = \frac{f(x_1) - a_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$a_2 = \frac{f(x_2) - a_0 - a_1(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)} = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Recursive property

The divided differences obey the formula

$$f[x_0, x_1, \ldots, x_k] = \frac{f[x_1, x_2, \ldots, x_k] - f[x_0, x_1, \ldots, x_{k-1}]}{x_k - x_0}$$

The divided difference $f[x_0, x_1, \ldots, x_k]$ is invariant under all permutations of the arguments $x_0, x_1, \ldots, x_k$.

x	$f[\,]$	$f[\,,\,]$	$f[\,,\,,\,]$	$f[\,,\,,\,,\,]$
x_0	$\boxed{f[x_0]}$			
		$\boxed{f[x_0, x_1]}$		
x_1	$f[x_1]$		$\boxed{f[x_0, x_1, x_2]}$	
		$f[x_1, x_2]$		$\boxed{f[x_0, x_1, x_2, x_3]}$
x_2	$f[x_2]$		$f[x_1, x_2, x_3]$	
		$f[x_2, x_3]$		
x_3	$f[x_3]$			

Example

Let $f(x) = x^3 + 2x^2 + x + 1$.

a) Find the polynomial of degree 4 that interpolates the values of f at $\pm 2, \pm 1, 0$.

b) Find the polynomial of degree 2 that interpolates the values of f at $\pm 1, 0$.

$$p_4(x) = -1 + 2(x+2) - (x+2)(x+1) + (x+2)(x+1)x$$

$$p_2(x) = 1 + 2(x+1)x$$

Theorems on Interpolation Errors

First Interpolation Error Theorem

Access the interpolation errors by means of a formula that involves higher-order derivative.

First Interpolation Error Theorem

If p is the polynomial of degree at most n that interpolates f at the $n+1$ distinct nodes $x_0, x_1, \ldots, x_n$ belonging to an interval $[a, b]$ and if $f^{(n+1)}$ is continuous, then for each x in $[a, b]$, there is a ξ in (a, b) for which

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i=0}^{n} (x - x_i) \qquad (2)$$

The maximum error of a linear interpolation is bounded by $\frac{1}{8} h^2 M$, where $h = x_1 - x_0, M = \max_{x_0 \le x \le x_1} |f''(x)|$