Today’s agenda

- Adaptive Simpson’s rule (not covered in exams)
- Gaussian quadrature formulas

Simpson's Rules

- Simpson's 1/3 rule

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$\boxed{-\frac{h^4}{180}(b-a)f^{(4)}(\xi)}$$

- Simpson's 3/8 rule

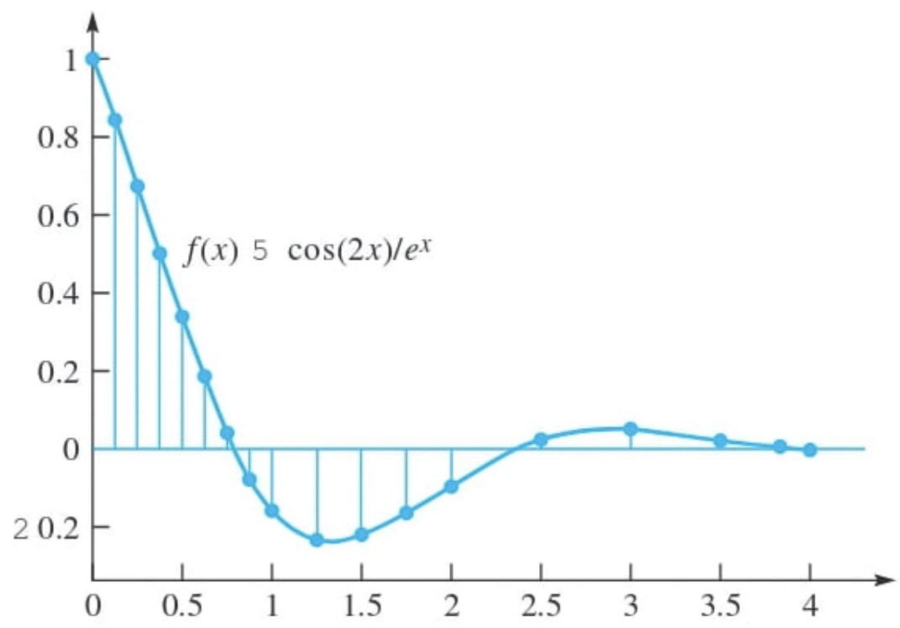
$$\begin{aligned}\int_a^b f(x)\,dx &\approx \frac{3h}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] \\ &= \frac{(b-a)}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right]\end{aligned}$$

$$\boxed{-\frac{h^4}{80}(b-a)f^{(4)}(\xi)}$$

- The 3/8 rule is about twice as accurate as the 1/3 one, but uses one more function value. Same order of accuracy though.

Adaptive Procedure

- The partitioning of the interval is automatically determined.
- We divide the interval into two subintervals and then decide whether each of them is to be divided into more subintervals.



Adaptive (cont'd)

$$I \equiv \int_a^b f(x)\,dx = S(a,b) + E(a,b)$$

$$S(a,b) = \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$E(a,b) = -\frac{1}{90}\left(\frac{b-a}{2}\right)^5 f^{(4)}(a) + \cdots$$

Example

Consider the integral $\int_1^3 e^{2x} \sin 3x \, dx$ and error tolerance $\epsilon = 0.2$. Apply a few steps of adaptive Simpson's rule.

$$\begin{aligned} S(1,3) &= 35.42697658812284 \\ S(1,2) &= -15.45828245392933 \\ S(2,3) &= 117.9751755250024, \end{aligned}$$

$$\frac{|S_1 - S_2|}{15} = 4.47 > \epsilon$$

Gaussian Quadrature Formula

Overview

- Most numerical integration formulas conform to $$\int_a^b f(x)\,dx \approx A_0 f(x_0) + A_1 f(x_1) + \cdots + A_n f(x_n)$$

  with the nodes x_j and the weights A_j.

- Recall Lagrange interpolation formula:

$$p(x) = \sum_{i=0}^{n} f(x_i)\,\ell_i(x) \quad \text{where} \quad \ell_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \left(\frac{x - x_j}{x_i - x_j} \right)$$

Simpson's Rule

Lagrange quadratic polynomial through $(0, f(0)), (h, f(h))$ and $(2h, f(2h))$:

$$p(x) = \frac{1}{2h^2}(x-h)(x-2h)f(0) - \frac{1}{h^2}x(x-2h)f(h) + \frac{1}{2h^2}x(x-h)f(2h)$$

$$\int_0^{2h} f(x)\,dx \approx \int_0^{2h} p(x)\,dx = \frac{h}{3}[f(0) + 4f(h) + f(2h)]$$

Example

Determine the quadrature formula when the interval is [-2,2] and the nodes are -1, 0, and 1.

$$\int_{-2}^{2} f(x)\,dx \approx \frac{8}{3}f(-1) - \frac{4}{3}f(0) + \frac{8}{3}f(1)$$

The formula gives exact values for $1, x, x^2$.

Thus it provides correct values for all quadratic polynomials.