Today's agenda

HW2 has been extended to next Monday (9/26).
Feedback on HW1 will be given on Wed.

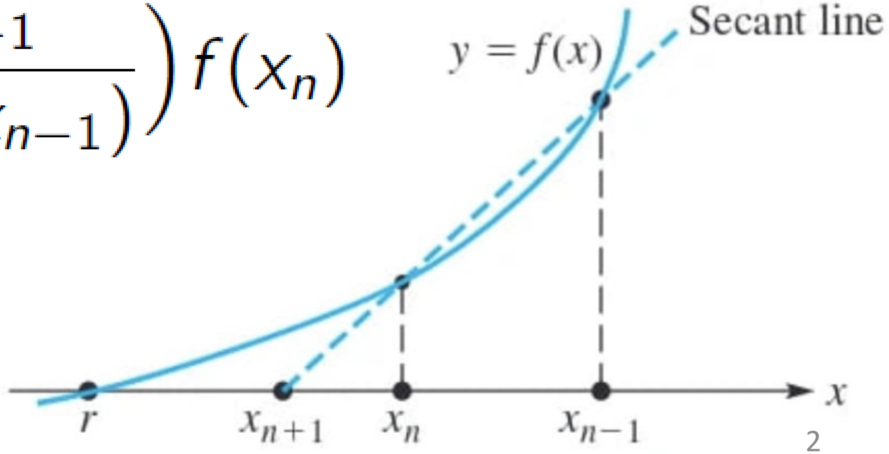
- Section 3.3: secant method
- Section 4.1: polynomial interpolation

Recap on Secant

Secant method approximates the function derivative by secant line.

$$f'(x_n) \approx \frac{f(x_{n-1}) - f(x_n)}{x_{n-1} - x_n}$$

$$x_{n+1} = x_n - \left(\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right) f(x_n)$$



Remarks

- Secant method requires two previous elements of the sequence.
- $f(x_{n-1})$ will be converging to 0
  - $f(x_{n-1}) - f(x_n)$ will be converging to 0
- If $f(x_{n-1}), f(x_n)$ have the same sign
  - Loss of significance
- So, a proper stopping condition

$$|f(x_n) - f(x_{n-1})| \leq \delta |f(x_n)|$$

Convergence analysis

The secant method has Superlinear convergence.

Comparison

Methods	Pros	Cons
Bisection	Only need continuous function	Requires two points with opposite signs
		slow
Newton's	fast	Only near root
		Requires f'
Secant	Relatively fast	Requires two points

Fixed-point iteration

- A value of x s.t. $x = g(x)$ is a fixed point of g as x is unchanged when g is applied to it.
- Fixed-point iteration

$$x_{n+1} = g(x_n)$$

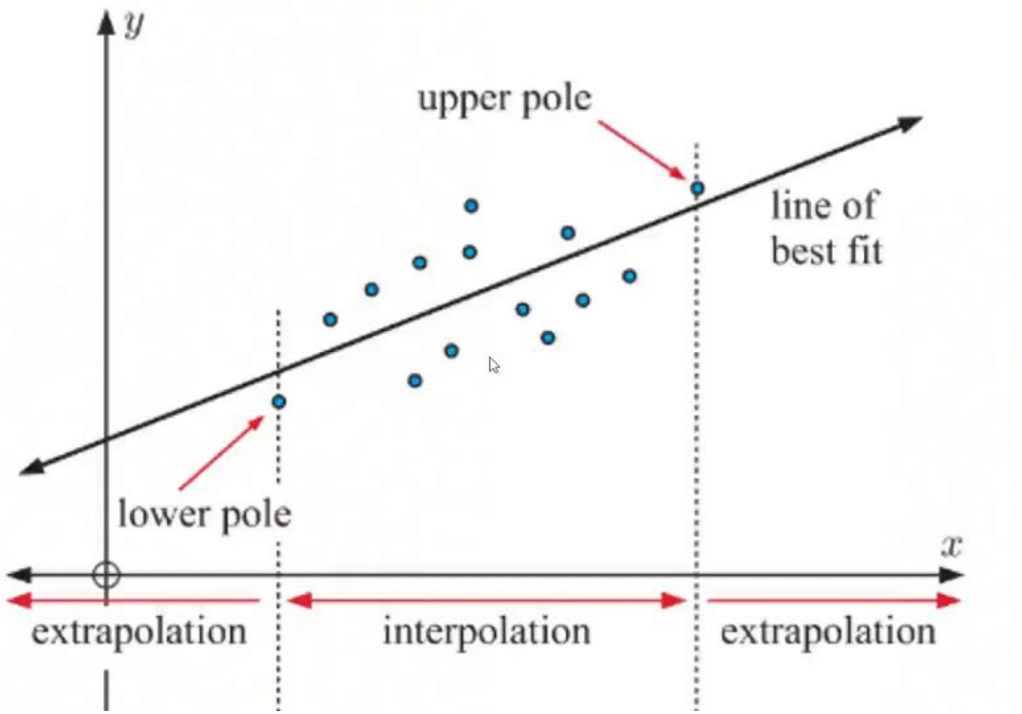
- Locally convergent if $x^* = g(x^*), |g'| < 1$.
- Given $f(x) = 0$, there may be many equivalent fixed-point problem $x = g(x)$ with different functions, some better than others.

Polynomial Interpolation

Interpolation / Extrapolation

In between the points = reliable

Outside the points = unreliable



Interpolation

x	x_0	x_1	$\cdots$	x_n
y	y_0	y_1	$\cdots$	y_n

1. Given a set of pairs (x_i, y_i), find a simple function that reproduces the given points?
2. What if given table is contaminated by errors?
3. Suppose a function is given but expensive to evaluate, we want to find a reasonable approximation.

Polynomial interpolation

x	x_0	x_1	$\cdots$	x_n
y	y_0	y_1	$\cdots$	y_n

- Suppose we have $n+1$ distinct points.
- We want to find a polynomial function that passes through all points: $p(x_i) = y_i$.
- We call p interpolates the table.
- The points x_i are called nodes.

Linear interpolation

- A straight line can be passed through 2 points, a linear function p can be defined by

$$p(x) = \left(\frac{x - x_1}{x_0 - x_1}\right) y_0 + \left(\frac{x - x_0}{x_1 - x_0}\right) y_1$$

$$= y_0 + \left(\frac{y_1 - y_0}{x_1 - x_0}\right)(x - x_0)$$

- It can be verified that $p(x_0) = y_0, p(x_1) = y_1$.
- This polynomial is called linear interpolation.

Lagrange form

- We wish to interpolate at $x_0, x_1, \cdots, x_n$.
- We define a set of cardinal polynomials:

$$\ell_i(x_j) = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

- The Lagrange form of the interpolation polynomial is given by

$$p_n(x) = \sum_{i=0}^{n} \ell_i(x) f(x_i)$$

Lagrange form (cont'd)

$$\ell_i(x) = \left(\frac{x - x_0}{x_i - x_0}\right)\left(\frac{x - x_1}{x_i - x_1}\right)\cdots\left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)\left(\frac{x - x_{i+1}}{x_i - x_{i+1}}\right)\cdots\left(\frac{x - x_n}{x_i - x_n}\right)$$

- In short, $\ell_i(x) = \prod_{\substack{j \neq i \\ j=0}}^{n} \left(\frac{x - x_j}{x_i - x_j}\right) \qquad (0 \le i \le n)$
- Each is a polynomial of degree n and $l_i(x_i) = 1$.
- The Lagrange interpolation proves the existence of an interpolating polynomial for any table of values.

If points $x_0, x_1, \ldots, x_n$ are distinct, then for arbitrary real values $y_0, y_1, \ldots, y_n$, there is a unique polynomial p of degree at most n such that $p(x_i) = y_i$ for $0 \leqq i \leqq n$.

Example

x	$\frac{1}{3}$	$\frac{1}{4}$	1
$f(x)$	2	-1	7

$$\ell_0(x) = \frac{\left(x - \frac{1}{4}\right)(x - 1)}{\left(\frac{1}{3} - \frac{1}{4}\right)\left(\frac{1}{3} - 1\right)} = -18\left(x - \frac{1}{4}\right)(x - 1)$$

$$\ell_1(x) = \frac{\left(x - \frac{1}{3}\right)(x - 1)}{\left(\frac{1}{4} - \frac{1}{3}\right)\left(\frac{1}{4} - 1\right)} = 16\left(x - \frac{1}{3}\right)(x - 1)$$

$$\ell_2(x) = \frac{\left(x - \frac{1}{3}\right)\left(x - \frac{1}{4}\right)}{\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right)} = 2\left(x - \frac{1}{3}\right)\left(x - \frac{1}{4}\right)$$

$$p_2(x) = -36\left(x - \frac{1}{4}\right)(x - 1) - 16\left(x - \frac{1}{3}\right)(x - 1) + 14\left(x - \frac{1}{3}\right)\left(x - \frac{1}{4}\right)$$

Vandermonde Matrix

x	x_0	x_1	$\cdots$	x_n
y	y_0	y_1	$\cdots$	y_n

$$p_n(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$$

$$\boxed{\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix}} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Vandermond matrix