CS 4365 Artificial Intelligence

Assignment 4: Probabilistic Reasoning

Due: Thursday, Dec 1st

Instructions:

1. Your solution to this assignment must be submitted via eLearning.
2. For the written problems, submit your solution as a **single PDF** file.
   - Only use **blue or black pen** (black is preferred). Scan your PDF using a scanner and upload it. Make sure your final PDF is **legible**. Regrades due to non-compliance will receive a 30% score penalty.
   - Verify that both your answers and procedure are **correct, ordered, clean, and self-explanatory** before writing. Please ask yourself the following questions before submitting:
     - Are my answers and procedure legible?
     - Are my answers and procedure in the same order as they were presented in the assignment? Do they follow the specified notation?
     - Are there any corrections or scratched out parts that reflect negatively on my work?
     - Can my work be easily understood by someone else? Did I properly define variables or functions that I am using? Can the different steps of my development of a problem be easily identified, followed, and understood by someone else? Are there any gaps in my development of the problem that need any sort of justification (be it calculations or a written explanation)? Is it clear how I arrived to each and every result in my procedure and final answers? Could someone describe my submission as messy?
3. **You may work individually or in a group of two**. Only one submission should be made per group. If you work in a group, **make sure to indicate both group members when submitting through eLearning**.
4. **IMPORTANT**: As long as you follow these guidelines, your submission should be in good shape; if not, we reserve the right to penalize answers and/or submissions as we see fit.

Part I

0.1 Probability I (16 points)

Let S be the sample space, consider the following three axioms of probability:

$P(A) \geq 0$ for all event $A \subset S$,

$P(S) = 1$

If A and B are disjoint events, then $P(A \cup B) = P(A) + P(B)$

Using these axioms, prove that:

1. (8 pts) $P(\sim A) = 1 - P(A)$, where $\sim A$ is complement of A.

   Answer: we have $\sim A \cup A = S$. Since $\sim A$ and A are disjoint, we have $P(A) + P(\sim A) = P(A \cup \sim A) = P(S) = 1$, so $P(\sim A) = 1 - P(A)$.

2. (8 pts) If $A \subset B$, then $P(A) \leq P(B)$

   Answer: if $A \subset B$, then $A \cap B = A$. We also have $P(B \cap \sim A) = P(B) - P(A \cap B)$, so we have

$$0 \leq P(B \cap \sim A) = P(B) - P(A) \tag{1}$$

0.2 Probability II (14 points)

Consider a medical diagnosis problem in which there are two alternative hypotheses: (1) that the patient has coronavirus, and (2) that the patient does not. The available data is from a particular laboratory test with two possible outcomes: positive and negative. We have prior knowledge that the fraction of the entire population of people who have this disease is 0.008. Furthermore, the lab test is only an imperfect indicator of the disease. The test returns a correct positive result in only 98% of the cases in which the disease is actually present and a correct negative result in only 97% of the cases in which the disease is not present. In other cases, the test returns the opposite result. Suppose we now observe a new patient for whom the lab test returns a positive result. Should we diagnose the patient as having coronavirus or not? Explain your answer by showing the relevant calculations.

Answer: We have $P(disease) = 0.008$, $P(positive|disease) = 0.98$, $P(negative|no\ disease) = 0.97$. We need to compute $P(disease|positive)$.

$$\begin{aligned}
P(disease|positive) &= \frac{P(disease, positive)}{P(positive)} && (2)\\
&= \frac{P(positive|disease)P(disease)}{P(positive)} && (3)\\
&= \frac{P(positive|disease)P(disease)}{P(positive|disease)P(disease) + P(positive|no\ disease)P(no\ disease)} && (4)\\
&= \frac{0.98 * 0.008}{0.98 * 0.008 + (1 - 0.008) * (1 - 0.97)} && (5)\\
&= \frac{0.00784}{0.00784 + 0.02976} && (6)\\
&= 0.2085 && (7)
\end{aligned}$$

We shouldn't diagnose the patient as having coronavirus.

0.3 Probability III (12 points)

There are two events, A and B. You have the following information about them: $P(A) = 0.2$, $P(B) = 0.6$, and $P(B|A) = 0.9$. Compute $P(B|\sim A)$. Show your work.

Answer: $P(\sim A) = 1 - P(A) = 0.8$. From $P(B|A) = 0.9$ we have,

$$\begin{aligned} P(B|A) &= \frac{P(B,A)}{P(A)} && (8) \\ &= \frac{P(B,A)}{0.2} = 0.9 && (9) \\ & && (10) \end{aligned}$$

so, we have $P(B,A) = 0.18$. So,

$$\begin{aligned} P(B|\sim A) &= \frac{P(B,\sim A)}{P(\sim A)} && (11) \\ &= \frac{P(B) - P(B,A)}{P(\sim A)} && (12) \\ &= \frac{0.6 - 0.18}{0.8} = 0.525 && (13) \end{aligned}$$

0.4 Conditional Independence (28 points)

Consider a Bayes net represented by ten variables, A, B, C, D, E, F, G, H, I, and J. We know the following things about these variables.

- A, C, D do not depend on other variables.
- G depends only on A and C.
- H depends only on C, D, and G
- B depends only on A and G
- J depends only on H.
- I depends only on H.
- E depends only on B and J.
- F depends only on I and J.

Use d-separation to determine whether the following conditional independencies are implied by this Bayes net. The notation $X \perp\!\!\!\perp Z \mid Y$ means that X is conditionally independent of Z given that the value of Y is known.

(a) $A \perp\!\!\!\perp D \mid \{\}$ Yes

(b) $A \perp\!\!\!\perp D \mid E$ No

(c) $A \perp\!\!\!\perp C \mid F$ No

(d) $C \perp\!\!\!\perp F \mid \{E, H\}$ No

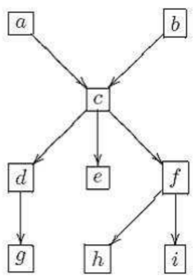
(e) $G \perp\!\!\!\perp F \mid \{B, H\}$ Yes

(f) $G \perp\!\!\!\perp F \mid \{E, H\}$ No

(g) $G \perp\!\!\!\perp F \mid \{E, H, J\}$ Yes

0.5 Bayesian Networks (30 points)

Consider the following Bayesian network:



at node a:

Prob(a)
0.7

at node b:

Prob(b)
0.8

at node c:

a	b	Prob(c)
0	0	0.1
0	1	0.6
1	0	0.7
1	1	0.9

at node d:

c	Prob(d)
0	0.6
1	0.8

at node e:

c	Prob(e)
0	0.1
1	0.7

at node f:

c	Prob(f)
0	0.8
1	0.3

at node g:

d	Prob(g)
0	0.15
1	0.75

at node h:

f	Prob(h)
0	0.25
1	0.65

at node i:

f	Prob(i)
0	0.35
1	0.55

(a) (5 pts) Compute $P(f|a)$. Show your work.

Answer:

$$\begin{aligned}
P(f|a) &= \frac{P(f,a)}{P(a)} && (14)\\
&= \frac{\sum_{b',c'} P(a,b',c',f)}{P(a)} && (15)\\
&= \frac{\sum_{b',c'} P(f|c')P(c'|a,b')P(a)P(b')}{P(a)} && (16)\\
&= \sum_{b'} P(b') \sum_{c'} P(f|c')P(c'|a,b') && (17)\\
&= 0.8 * (0.3 * 0.9 + 0.8 * 0.1) + 0.2 * (0.3 * 0.7 + 0.8 * 0.3) = 0.37 && (18)
\end{aligned}$$

(b) (5 pts) Compute $P(f| \sim c)$. Show your work.

Answer: 0.8

(c) (5 pts) Compute $P(f|a,b)$. Show your work.

Answer:

$$
\begin{aligned}
P(f|a,b) &= \frac{P(f,a,b)}{P(a,b)} && (19)\\
&= \frac{\sum_{c'} P(f,a,b,c')}{P(a)(b)} && (20)\\
&= \frac{\sum_{c'} P(f|c')P(c'|a,b)P(a)P(b)}{P(a)P(b)} && (21)\\
&= \sum_{c'} P(f|c')P(c'|a,b) && (22)\\
&= 0.3 * 0.9 + 0.8 * 0.1 = 0.35 && (23)
\end{aligned}
$$

(d) (5 pts) Compute $P(f|a,b,c)$. Show your work.

Answer: $P(f|a,b,c) = P(f|c) = 0.3$

(e) (5 pts) Compute $P(a|f)$. Show your work.

Answer:

$$
\begin{aligned}
P(a|f) &= \frac{P(a,f)}{P(f)} && (24)\\
&= \frac{\sum_{b',c'} P(a,b',c',f)}{\sum_{a',b',c'} P(a',b',c',f)} && (25)\\
&= \frac{\sum_{b',c'} P(f|c')P(c'|a,b')P(a)P(b')}{\sum_{a',b',c'} P(f|c')P(c'|a',b')P(a')P(b')} && (26)\\
&= \frac{\sum_{b'} P(b') \sum_{c'} P(f|c')P(c'|a,b')P(a)}{\sum_{a',b'} \sum_{c'} P(f|c')P(c'|a',b')P(a')P(b')} && (27)\\
&= \frac{0.259}{0.424} = 0.6108 && (28)
\end{aligned}
$$

(f) (5 pts) Compute $P(c|g,h)$. Show your work.

Answer:

$$
\begin{aligned}
P(c) &= \sum_{a',b'} P(c,a',b') && (29)\\
&= \sum_{a',b'} P(c|a',b')P(a')P(b') && (30)\\
&= \sum_{a',b'} P(c|a',b')P(a')P(b') && (31)\\
&= 0.7 * 0.8 * 0.9 + 0.3 * 0.8 * 0.6 + 0.3 * 0.2 * 0.1 + 0.7 * 0.2 * 0.7 = 0.752 && (32)
\end{aligned}
$$

$$
\begin{aligned}
P(c|g,h) &= \frac{P(c,g,h)}{P(g,h)} && (33)\\
&= \frac{\sum_{d',f'} P(c,g,h,d',f')}{\sum_{d',f',c'} P(c',g,h,d',f')} && (34)\\
&= \frac{\sum_{d',f'} P(c)P(d'|c)P(g|d')P(f'|c)P(h|f')}{\sum_{d',f',c'} P(c')P(d'|c')P(g|d')P(f'|c')P(h|f')} && (35)\\
&= \frac{0.752*((0.8*0.75+0.2*0.15)*(0.3*0.65+0.7*0.25))}{0.752*((0.8*0.75+0.2*0.15)*(0.3*0.65+0.7*0.25))+0.248*((0.6*0.75+0.4*0.15)*(0.8*0.65+0.2*0.25))} && (36)\\
&= \frac{0.1752}{0.1752+0.072} = 0.7087 && (37)
\end{aligned}
$$