Gaussian Quadrature Formulas

Overview

- Most numerical integration formulas conform to $$\int_a^b f(x)\,dx \approx A_0 f(x_0) + A_1 f(x_1) + \cdots + A_n f(x_n)$$

  with the nodes x_j and the weights A_j.

- Recall Lagrange interpolation formula:

$$p(x) = \sum_{i=0}^{n} f(x_i)\,\ell_i(x) \quad \text{where} \quad \ell_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \left(\frac{x - x_j}{x_i - x_j}\right)$$

Simpson's Rule

Lagrange quadratic polynomial through $(0, f(0)), (h, f(h))$ and $(2h, f(2h))$:

$$p(x) = \frac{1}{2h^2}(x-h)(x-2h)f(0) - \frac{1}{h^2}x(x-2h)f(h) + \frac{1}{2h^2}x(x-h)f(2h)$$

$$\int_0^{2h} f(x)\,dx \approx \int_0^{2h} p(x)\,dx = \frac{h}{3}[f(0) + 4f(h) + f(2h)]$$

Change of interval

- Integration rules are usually derived on an interval such as [0,1] or [-1,1].
- Often we want to use these rules over a different intervals.
- We can do so by a linear change of variables.

$$\int_a^b f(x)\,dx = \left(\frac{b-a}{d-c}\right)\int_c^d f(\lambda(t))\,dt$$

$$\lambda(t) = \left(\frac{b-a}{d-c}\right)t + \left(\frac{ad-bc}{d-c}\right)$$

Quadrature Rule

- Recall $$\int_a^b f(x)\,dx \approx A_0 f(x_0) + A_1 f(x_1) + \cdots + A_n f(x_n)$$

- This formula is exact for all polynomials of degree n with arbitrary $n+1$ nodes in (a, b).

- Gaussian Quadrature Theorem states that with a set of carefully chosen nodes, the formula is exact for polynomials of degree $2n+1$.

Theorem

Gaussian Quadrature Theorem

Let q be a nontrivial polynomial of degree $n+1$ such that

$$\int_a^b x^k q(x)\,dx = 0 \qquad (0 \leqq k \leqq n)$$

Let $x_0, x_1, \ldots, x_n$ be the zeros of q. Then the formula

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n} A_i f(x_i) \quad \text{where} \quad A_i = \int_a^b \ell_i(x)\,dx \tag{6}$$

with these x_i's as nodes will be exact for all polynomials of degree at most $2n+1$. Furthermore, the nodes lie in the open interval (a, b).

$$\int_a^b f(x)\,dx = \int_a^b p(x)q(x)\,dx + \int_a^b r(x)\,dx = \int_a^b r(x)\,dx$$

$$= \sum_{i=0}^{n} A_i r(x_i) = \sum_{i=0}^{n} A_i f(x_i)$$

Example

- Determine the Gaussian quadrature formula with three nodes/weights for the integral

$$\int_{-1}^{1} f(x)dx$$

- Use this formula to approximate the integral

$$\int_{0}^{1} e^{-x^2}dx$$

Legendre polynomials

- Aim at efficient methods for generating the special polynomials whose roots are used as nodes.
- Consider $\int_{-1}^{1} f(x)dx$ and standardize q_n such that $q_n(1) = 1 \rightarrow$ Legendre polynomials
- The first few:

$$q_0(x) = 1$$

$$q_1(x) = x$$

$$q_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$$

- Recurrence relation:

$$q_n(x) = \left(\frac{2n-1}{n}\right)xq_{n-1}(x) - \left(\frac{n-1}{n}\right)q_{n-2}(x)$$

Gaussian Quadrature formulas

n	Nodes x_i	Weights A_i
3	$-\sqrt{\frac{1}{7}\left(3-4\sqrt{0.3}\right)}$	$\frac{1}{2}+\frac{1}{12}\sqrt{\frac{10}{3}}$
	$-\sqrt{\frac{1}{7}\left(3+4\sqrt{0.3}\right)}$	$\frac{1}{2}-\frac{1}{12}\sqrt{\frac{10}{3}}$
	$+\sqrt{\frac{1}{7}\left(3-4\sqrt{0.3}\right)}$	$\frac{1}{2}+\frac{1}{12}\sqrt{\frac{10}{3}}$
	$+\sqrt{\frac{1}{7}\left(3+4\sqrt{0.3}\right)}$	$\frac{1}{2}-\frac{1}{12}\sqrt{\frac{10}{3}}$

Remarks

- Most nodes/weights are irrational numbers, so they are not used in computations by hand.
- In programming, Gaussian quadrature formulas usually give greater accuracy with fewer function evaluations.
- The choice of quadrature formula depends applications.