Today's agenda

Section 4.1: polynomial interpolation

- Lagrange form
- Vandermonde matrix
- Newton form

My office hour on Wed will be changed to 1:30-2:30.

Lagrange form

$$p(x) = \left(\frac{x - x_1}{x_0 - x_1}\right) y_0 + \left(\frac{x - x_0}{x_1 - x_0}\right) y_1$$

x	$\frac{1}{3}$	$\frac{1}{4}$	1
$f(x)$	2	-1	7

$$\ell_0(x) = \frac{\left(x - \frac{1}{4}\right)(x-1)}{\left(\frac{1}{3} - \frac{1}{4}\right)\left(\frac{1}{3} - 1\right)} = -18\left(x - \frac{1}{4}\right)(x-1)$$

$$\ell_1(x) = \frac{\left(x - \frac{1}{3}\right)(x-1)}{\left(\frac{1}{4} - \frac{1}{3}\right)\left(\frac{1}{4} - 1\right)} = 16\left(x - \frac{1}{3}\right)(x-1)$$

$$\ell_2(x) = \frac{\left(x - \frac{1}{3}\right)\left(x - \frac{1}{4}\right)}{\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right)} = 2\left(x - \frac{1}{3}\right)\left(x - \frac{1}{4}\right)$$

$$p_2(x) = -36\left(x - \frac{1}{4}\right)(x-1) - 16\left(x - \frac{1}{3}\right)(x-1) + 14\left(x - \frac{1}{3}\right)\left(x - \frac{1}{4}\right)$$

Lagrange form (cont'd)

$$\ell_i(x) = \left(\frac{x - x_0}{x_i - x_0}\right)\left(\frac{x - x_1}{x_i - x_1}\right)\cdots\left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)\left(\frac{x - x_{i+1}}{x_i - x_{i+1}}\right)\cdots\left(\frac{x - x_n}{x_i - x_n}\right)$$

- In short, $\ell_i(x) = \prod_{\substack{j \neq i \\ j=0}}^{n} \left(\frac{x - x_j}{x_i - x_j}\right) \qquad (0 \leq i \leq n)$
- The Lagrange form of the interpolation polynomial is given by

$$p_n(x) = \sum_{i=0}^{n} \ell_i(x) f(x_i)$$

- The Lagrange interpolation proves the existence of an interpolating polynomial for any table of values.

If points $x_0, x_1, \ldots, x_n$ are distinct, then for arbitrary real values $y_0, y_1, \ldots, y_n$, there is a unique polynomial p of degree at most n such that $p(x_i) = y_i$ for $0 \leq i \leq n$.

Vandermonde Matrix

x	x_0	x_1	$\cdots$	x_n
y	y_0	y_1	$\cdots$	y_n

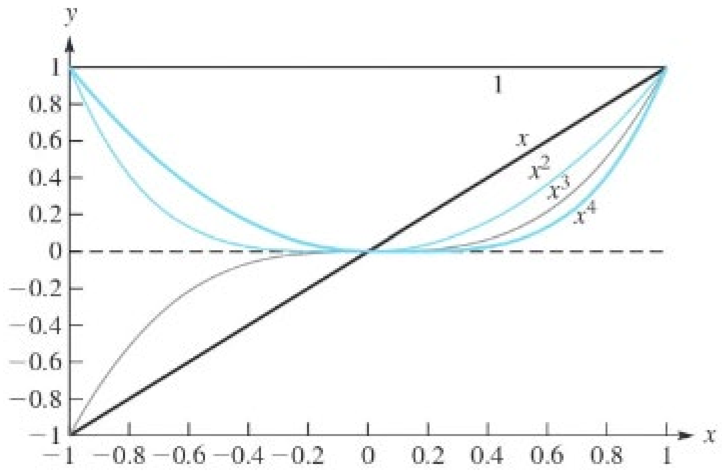
$$p_n(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$$

$$\boxed{\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix}} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Vandermond matrix

Vandermonde (cont'd)

- Vandermonde matrix is nonsingular for distinct nodes.
- It is ill-conditioned as n increases.
- For large n, the monomials are less distinguishable from one another.

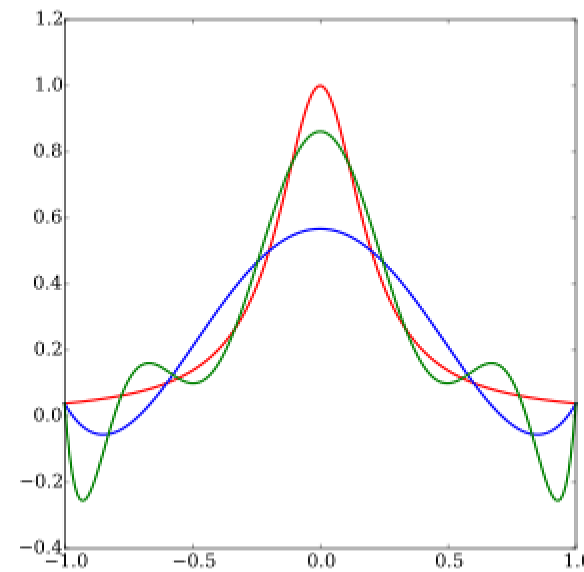


High-degree polynomials often oscillate widely and are highly sensitive to small changes in the data.

Runge's phenomenon

- A polynomial of degree n has n roots.
- If all roots are real, the curve crosses the x-axis n times.
- These many turns result in wide oscillations.
- A specific example is provided by Runge function

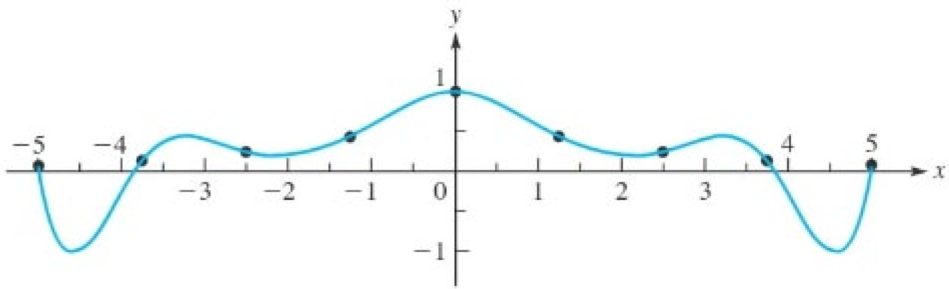
$$f(x) = (1 + 25x^2)^{-1}$$



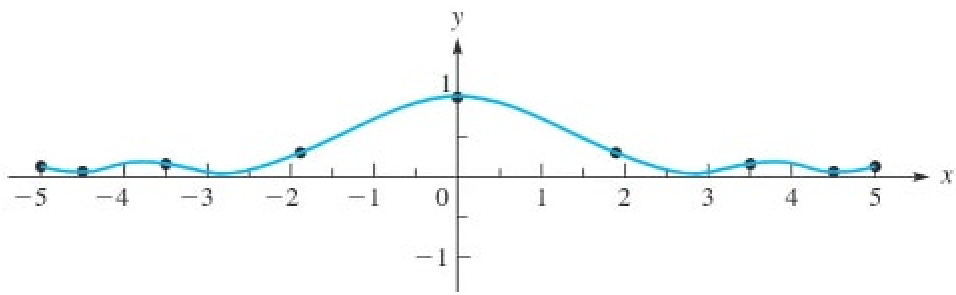
$$\lim_{n\to\infty}\left(\max_{-1\le x\le 1}|f(x) - P_n(x)|\right) = \infty.$$

Chebyshev nodes

$$f(x) = (1 + x^2)^{-1}$$



Equally spaced nodes



Chebyshev nodes

$$x_i = \cos\left[\left(\frac{2i+1}{2n+2}\right)\pi\right]$$

Extension

Consider a linear combination:

$$f(x) \approx c_0\varphi_0(x) + c_1\varphi_1(x) + c_2\varphi_2(x) + \cdots + c_n\varphi_n(x)$$

- $\{\varphi_0, \varphi_1, \cdots, \varphi_n\}$ are called basis functions.
- The coefficients $\{c_j\}$ are to be determined.
- Monomials $\{1, x, x^2, x^3, \dots\}$ are the simplest basis function.

Chebyshev polynomial

$$T_0(x) = 1$$

$$T_1(x) = x$$

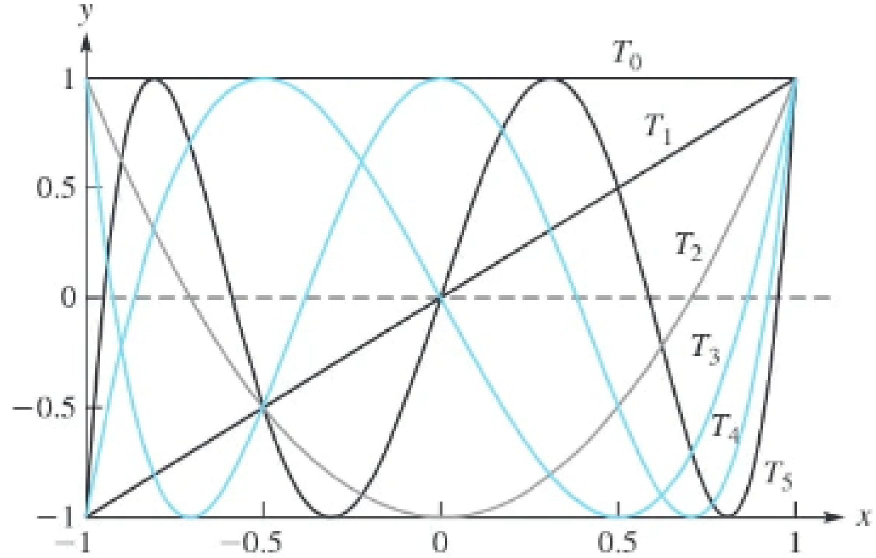
$$T_2(x) = 2x^2 - 1$$

$$T_3(x) = 4x^3 - 3x$$

$$T_4(x) = 8x^4 - 8x^2 + 1$$

$$T_5(x) = 16x^5 - 20x^3 + 5x$$

$$\begin{cases} T_0(x) = 1, \qquad T_1(x) = x \\ T_i(x) = 2xT_{i-1}(x) - T_{i-2}(x) \end{cases}$$



Chebyshev (cont'd)

- Cheb poly are usually defined on [-1,1] and can be used in any interval with changes of variable.
- Cheb nodes are roots of Cheb poly.
- Extreme points of Cheb polynomials are equal in magnitude and alternative in sign.
- This property tends to distribute the error uniformly.
- The maximum error over the interval of interpolation can be minimized.

Newton form

One drawback of Lagrange and Vandermonde is that adding one data point completely revamps the entire process.

Motivating example

- Recall

x	$\frac{1}{3}$	$\frac{1}{4}$	1
$f(x)$	2	-1	7

$$p_2(x) = -36\left(x-\frac{1}{4}\right)(x-1) - 16\left(x-\frac{1}{3}\right)(x-1) + 14\left(x-\frac{1}{3}\right)\left(x-\frac{1}{4}\right)$$

- It can be simplified to $p_2(x) = -\frac{79}{6} + \frac{349}{6}x - 38x^2$

- We will learn that this polynomial can be written in the nested Newton form:

$$p_2(x) = 2 + \left(x-\frac{1}{3}\right)\left[36 + \left(x-\frac{1}{4}\right)(-38)\right]$$

Remark

- The resulting polynomial is the Newton form of the interpolating polynomial.
- Newton form involves fewer operations.
- Newton and Lagrange forms are the same polynomial.
- Newton is better off to accommodate additional data points.

Example

x	0	1	-1	2	-2
y	-5	-3	-15	39	-9

$$p_4(x) = -5 + x(2 + (x-1)(-4 + (x+1)(8 + (x-2)3)))$$

Nested form

$$
\begin{aligned}
p(x) = &a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots \\
&+ a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1})
\end{aligned}
$$

$$
\begin{aligned}
p(x) &= a_0 + (x - x_0)(a_1 + (x - x_1)(a_2 + \cdots + (x - x_{n-1})a_n))\cdots)) \\
&= (\cdots((a_n(x - x_{n-1}) + a_{n-1})(x - x_{n-2}) + a_{n-2})\cdots)(x - x_0) + a_0
\end{aligned}
$$