Today's Agenda

- Matrix factorizations
  - LU
  - LDL^T
  - Cholesky factorization LL^T
  - SVD (later)

- Matrix inversion: A^{-1}

LU factorization

- Unit lower triangular

$$\mathbf{L} = \begin{bmatrix} 1 & & & & \\ \ell_{21} & 1 & & & \\ \ell_{31} & \ell_{32} & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ \ell_{n1} & \ell_{n2} & \ell_{n3} & \cdots & 1 \end{bmatrix}$$

$$\mathbf{A} = \mathbf{LU}$$

- Upper triangular

$$\mathbf{U} = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ & u_{22} & u_{23} & \cdots & u_{2n} \\ & & u_{33} & \cdots & u_{3n} \\ & & & \ddots & \vdots \\ & & & & u_{nn} \end{bmatrix}$$

Applications of SVD

- Condition number
- Matrix inverse
- Low-rank approximation

Matrix rep.

- Recall forward elimination: $\boldsymbol{Ax} = \boldsymbol{b}$

$$\boxed{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ 26 \\ -19 \\ -34 \end{bmatrix}$$

- Question: can we find a matrix **M** such that

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & -12 & 8 & 1 \\ 0 & 2 & 3 & -14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -27 \\ -18 \end{bmatrix}$$

$$\boldsymbol{MAx} = \boldsymbol{Mb}$$

Remarks

$$\mathbf{M}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

- The diagonal are all 1's.
- The only other nonzero elements are in the first column.
- These numbers are the negative of the multipliers.
- It is a lower triangular matrix.
- When multiplying on the left of a matrix $\boldsymbol{A}$, the rows of $\boldsymbol{M}_1$ sweep across the columns of $\boldsymbol{A}$.

Naïve Gaussian elimination

$$\mathbf{M}_2 = \boxed{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & 1 \end{bmatrix}} \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & -12 & 8 & 1 \\ 0 & 2 & 3 & -14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -27 \\ -18 \end{bmatrix}$$

$$\mathbf{M}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix}$$

$$\mathbf{M}_3\mathbf{M}_2\mathbf{M}_1\mathbf{A}\mathbf{x} = \mathbf{M}_3\mathbf{M}_2\mathbf{M}_1\mathbf{b}$$

$$\mathbf{M}\mathbf{A} = \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{bmatrix} \equiv \mathbf{U}$$

LU factorization

$$\mathbf{MA} = \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{bmatrix} \equiv \mathbf{U}$$

$$\begin{aligned} \mathbf{MA} &= \mathbf{U} \\ \mathbf{A} &= \mathbf{M}^{-1}\mathbf{U} \\ &= \mathbf{M}_1^{-1}\mathbf{M}_2^{-1}\mathbf{M}_3^{-1}\mathbf{U} \\ &= \mathbf{LU} \end{aligned}$$

- The inverse of each M is obtained by changing the signs of the negative multiplier entries.

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & -\frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

Generally

- To subtract λ times row p from row q, we apply $\boldsymbol{M}_{pq}$ to the identity matrix

$$m_{ij} = \begin{cases} 1, & \text{if } i = j \\ -\lambda, & \text{if } i = q \text{ and } j = p \\ 0, & \text{otherwise} \end{cases}$$

- Solving linear system by LU factorization

$$\mathbf{Ax} = \mathbf{b} \qquad \mathbf{Lz} = \mathbf{b}$$

$$\mathbf{LUx} = \mathbf{b} \qquad \mathbf{Ux} = \mathbf{z}$$

- It is useful for problems that involve the same matrix A but different right-hand vectors.

What about pivoting?

- Interchanging the rows can be accomplished by multiplying on the left by a permutation matrix

"Column the 1 Goes"

$$\begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{array}{c} \Rightarrow 3 \\ \Rightarrow 1 \\ \Rightarrow 2 \end{array} \quad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{31} & a_{32} & a_{33} \\ a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

- The matrix PA is A with its row rearranged.
- P is invertible and its inverse is itself.

Pivoting (cont'd)

- The permutation matrix is obtained by the indexing ℓ.
- Suppose we have LU for PA

$$\mathsf{PAx} = \mathsf{Pb} \qquad \mathsf{Ly} = \mathsf{Pb}$$

$$\mathsf{LUx} = \mathsf{Pb} \qquad \mathsf{Ux} = \mathsf{y}$$

LDL^T

For a symmetric matrix A

Derivation

- Suppose a symmetric matrix A has an ordinary LU factorization.
- Then $\mathbf{LU} = \mathbf{A} = \mathbf{A}^T = (\mathbf{LU})^T = \mathbf{U}^T\mathbf{L}^T$
- Unit lower triangular matrix is invertible, so

  $\mathbf{U} = \mathbf{L}^{-1}\mathbf{U}^T\mathbf{L}^T$ then $\mathbf{U}(\mathbf{L}^T)^{-1} = \mathbf{L}^{-1}\mathbf{U}^T$
- LHS is upper and RHS is lower $\Rightarrow$ diagonal.
- Therefore, $\mathbf{U} = \mathbf{D}\mathbf{L}^T$ and $\mathbf{A} = \mathbf{LU} = \mathbf{L}\mathbf{D}\mathbf{L}^T$.

Cholesky Factorization

For a symmetric and positive definite (SPD) matrix

Derivation

- SPD means $\boldsymbol{A} = \boldsymbol{A}^T, x^T \boldsymbol{A} x > 0\ \forall x \neq 0$.
- By symmetry, $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{D}\boldsymbol{L}^T$.
- $\boldsymbol{D}$ is positive definite: all diagonal elements >0.
- We have $\boldsymbol{A} = \tilde{\boldsymbol{L}}\tilde{\boldsymbol{L}}^T, \tilde{\boldsymbol{L}} \coloneqq \boldsymbol{L}\boldsymbol{D}^{1/2}$.
- In fact, the Cholesky decomposition is unique.

Computing A^{-1}

Overview

- It may be necessary to compute the matrix inverse that satisfies $\mathbf{AX} = \mathbf{I}$
- If $x^{(j)}$ denotes the jth column of **X** and $I^{(j)}$ denotes the jth column of **I**

$$\mathbf{A}[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(n)}] = [\mathbf{I}^{(1)}, \mathbf{I}^{(2)}, \dots, \mathbf{I}^{(n)}]$$

- Then $\mathbf{A}^{-1} = [\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(n)}]$

Singular Value Decomposition

SVD

Overview

- It can be applied to any matrix, whether square or not.
- For a matrix **A** with any shape, $\boldsymbol{A}^T\boldsymbol{A}$ is symmetric and semi-positive definite.
- Prove all eigenvalues of $\boldsymbol{A}^T\boldsymbol{A}$ are nonnegative.
- SVD: