Today's Agenda

- Condition number
- Pivoting
- HW5 is extended to Wed (11/16)

Vector induced matrix norms

- L1 norm $\|A\|_1 = \max_{\|\mathbf{x}\|_1=1} \|A\mathbf{x}\|_1 = \max_{1\le j\le n} \sum_{i=1}^{m} |a_{ij}|.$
  - Maximum column sum
- L infinity norm $\|A\|_\infty = \max_{\|\mathbf{x}\|_\infty=1} \|A\mathbf{x}\|_\infty = \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}|.$
  - Maximum row sum
- Important inequality

$$||Ax||_v \leq ||A||_M \cdot ||x||_v$$

Condition number

- The condition number is defined by

$$\kappa(A) = ||A|| \cdot ||A^{-1}|| = \frac{\sigma_{max}(A)}{\sigma_{min}(A)}$$

- It indicates how close A is to being numerically singular.

- If $\kappa(A)$ is large, A is ill-conditioned; no expectation of a true solution or even close to it.

Pivoting

Naïve Gaussian can fail

- Gaussian elimination would fail if $a_{11} = 0$.

$$\begin{cases} 0x_1 & + & x_2 & = & 1 \\ x_1 & + & x_2 & = & 2 \end{cases} \qquad \begin{cases} \varepsilon x_1 & + & x_2 & = & 1 \\ x_1 & + & x_2 & = & 2 \end{cases}$$

- How about this for a small number $\epsilon \neq 0$?

$$\begin{cases} \varepsilon x_1 + & x_2 & = & 1 \\ & (1-\varepsilon^{-1})x_2 & = & 2-\varepsilon^{-1} \end{cases}$$

On 8-digit decimal computer

- Consider $\epsilon = 10^{-9} \Rightarrow \epsilon^{-1} = 10^{9}$.
- To compute $2 - \epsilon^{-1}$, the computer must interpret the numbers as

$$\begin{aligned} \varepsilon^{-1} = 10^9 &= 0.10000\,000 \times 10^{10} &= 0.10000\,00000\,00000\,0 \times 10^{10} \\ 2 &= 0.20000\,000 \times 10^{1} &= 0.00000\,00002\,00000\,0 \times 10^{10} \end{aligned}$$

- Thus $2 - \epsilon^{-1}$ is rounded to ϵ^{-1}.

Remedy

$$\begin{cases} 0x_1 + x_2 = 1 \\ x_1 + x_2 = 2 \end{cases} \qquad \begin{cases} \varepsilon x_1 + x_2 = 1 \\ x_1 + x_2 = 2 \end{cases}$$

Switch the two rows

$$\begin{cases} x_1 + x_2 = 2 \\ 0x_1 + x_2 = 1 \end{cases} \qquad \boxed{\begin{cases} x_1 + x_2 = 2 \\ \varepsilon x_1 + x_2 = 1 \end{cases}}$$

Necessary to switch

- Given

$$\boxed{\begin{cases} x_1 + x_2 = 2 \\ \varepsilon x_1 + x_2 = 1 \end{cases}}$$

- After elimination

$$\begin{cases} x_1 + x_2 = 2 \\ (1-\varepsilon)x_2 = 1 - 2\varepsilon \end{cases}$$

- Solution

$$x_2 = 1 - 2\varepsilon / 1 - \varepsilon \approx 1$$

$$x_1 = 2 - x_2 \approx 1$$

- What if

$$\begin{cases} x_1 + \varepsilon^{-1} x_2 = \varepsilon^{-1} \\ x_1 + x_2 = 2 \end{cases}$$

- After elimination

$$\begin{cases} x_1 + \varepsilon^{-1} x_2 = \varepsilon^{-1} \\ (1-\varepsilon^{-1})x_2 = 2 - \varepsilon^{-1} \end{cases}$$

- Solution

$$x_2 = (2 - \varepsilon^{-1}) / (1 - \varepsilon^{-1}) \approx 1$$

$$x_1 = \varepsilon^{-1} - \varepsilon^{-1} x_2 \approx 0$$

Pivoting types

- Complete pivoting: search over all entries in the submatrices for the largest entry in absolute value and then interchanges rows and columns to move it into the pivot position.

- Partial pivoting: search just the first column in the submatrix at each stage.

- Scaled partial pivoting: introduce a scale factor

$$s_i = \max_{1 \leq j \leq n} |a_{ij}| \qquad (1 \leq i \leq n)$$

select the equation for which $a_{i,1}/s_i$ is greatest.

Scaled partial pivoting

- Compute a scale factor for each equation

$$s_i = \max_{1 \leqq j \leqq n} |a_{ij}| \qquad (1 \leq i \leq n)$$

- We use the equation for which the ratio $|a_{i,1}|/s_i$ is largest as the pivot equation. Let l_1 be the first index for which this ratio is largest.
- Create 0's except for the pivot equation
- Need to keep track of the indices.

Index vector

- At beginning, define

$$\vec{l} := [l_1, l_2, \cdots, l_n] = [1, 2, \cdots, n]$$

- Suppose j to be the first index of the largest ratio
- Now interchange l_j with l_1.
- Only entries in $\vec{l}$ are being interchanged, not the equations.
  – Avoid unnecessary process of moving equations around in the computer memory.

Second step

- We scan the ratios $\{|a_{l_i,2}|/s_{l_i}\}$ for $i = 2, \cdots, n$
- Let j to be the first index of the greatest ratio, interchange l_j with l_2.
- Then multiplier $a_{l_i,2}/a_{l_2,2}$ times equation l_2 are subtracted from equation l_i for $i = 3, \cdots, n$.
- At step k, select j be the first index of the largest of the ratios $\{|a_{l_i,k}|/s_{l_i}\}$ for $i = k, \cdots, n$ and interchange l_j with l_k.
- Then multiplier $a_{l_i,k}/a_{l_k,k}$ times equation l_k are subtracted from equation l_i for $i = k+1, \cdots, n$.

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Example

$$\begin{bmatrix} 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \\ 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -19 \\ -34 \\ 16 \\ 26 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -12 & 8 & 1 \\ 0 & 2 & 3 & -14 \\ 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -27 \\ -18 \\ 16 \\ -6 \end{bmatrix}$$

Example (cont'd)

$$\begin{bmatrix} 0 & -12 & 8 & 1 \\ 0 & 2 & 3 & -14 \\ 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -27 \\ -18 \\ 16 \\ -6 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -12 & 8 & 1 \\ 0 & 0 & 13/3 & -83/6 \\ 6 & -2 & 2 & 4 \\ 0 & 0 & -2/3 & 5/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -27 \\ -45/2 \\ 16 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -12 & 8 & 1 \\ 0 & 0 & 13/3 & -83/6 \\ 6 & -2 & 2 & 4 \\ 0 & 0 & 0 & -6/13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -27 \\ -45/2 \\ 16 \\ -6/13 \end{bmatrix}$$