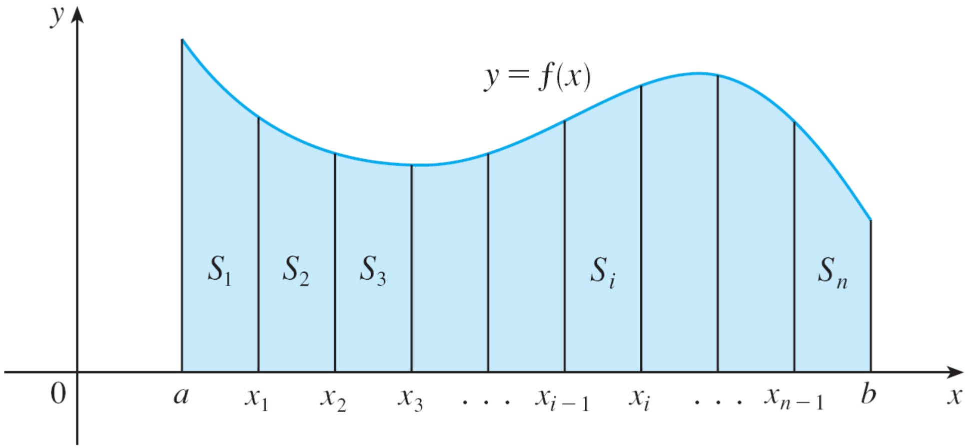
Today's agenda

- Calculus review
- Numerical integration
  - trapezoid rule
  - Simpson's rules
  - Gaussian Quadrature formulas

- Midterm exam is scheduled on Oct 19 (Wed in class)
  - Cut off materials to Simpson's rules (closed interval)

Recall Riemann sum

- We subdivide S into n strips $S_1, S_2, \ldots, S_n$ of equal width.



$x_1 = a + \Delta x,$

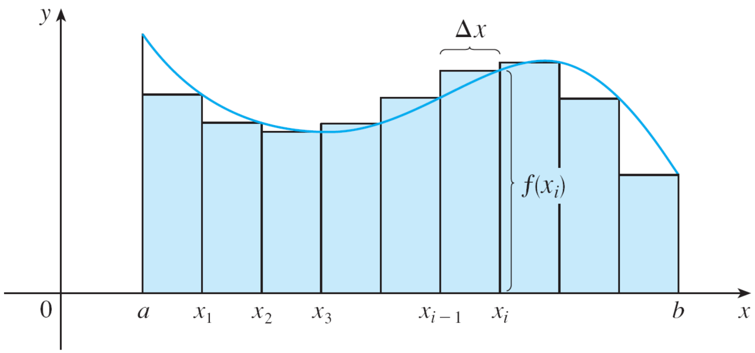
$x_2 = a + 2\,\Delta x,$

$x_3 = a + 3\,\Delta x,$

$\vdots$

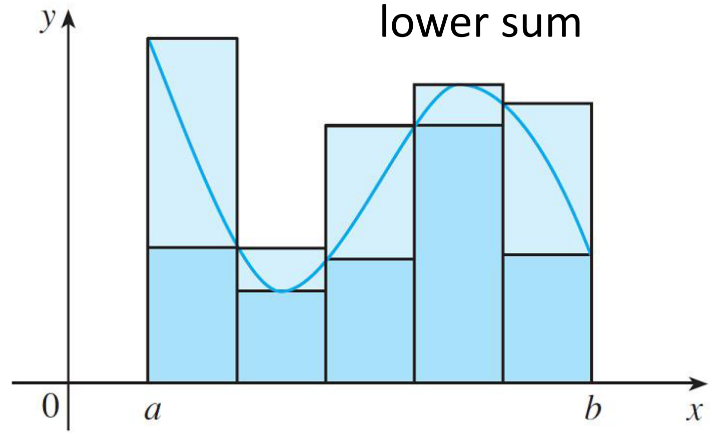
- The width of the interval $[a, b]$ is $b - a$, so the width of each of the n strips is $\Delta x = \dfrac{b-a}{n}$

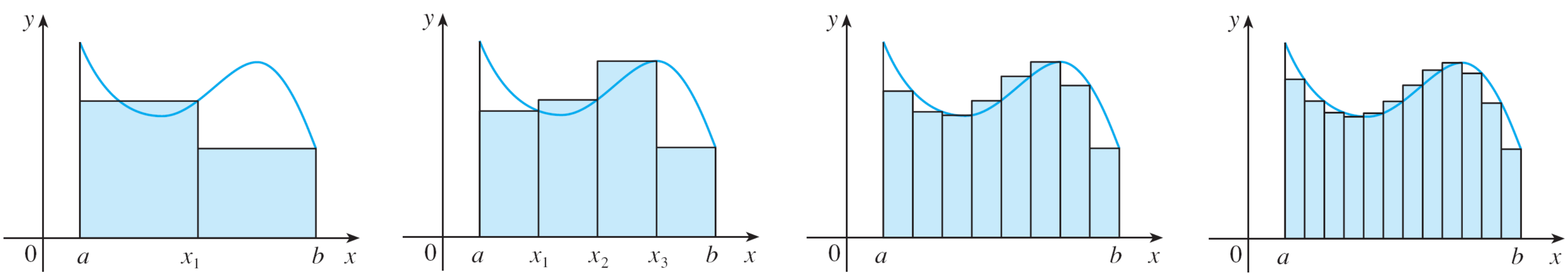
Sample points



$$R_n = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x$$

Upper sum and lower sum





(a) $n = 2$ (b) $n = 4$ (c) $n = 8$ (d) $n = 12$

Fundamental Theory of Calc.

The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt \qquad a \leqslant x \leqslant b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

The Fundamental Theorem of Calculus Suppose f is continuous on $[a, b]$.

1. If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.
2. $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

Definite v.s. Indefinite Integrals

- You should distinguish carefully between definite and indefinite integrals:

❑A definite integral $\int_a^b f(t)dt$ is a *number*

❑an indefinite integral $\int_a^x f(t)dt$ is a *function* (or family of functions).

- The connection between them is given by Part 2 of the Fundamental Theorem: If *f* is continuous on [*a*, *b*], then $\int_a^b f(x)\,dx = \int f(x)\,dx\Big]_a^b$

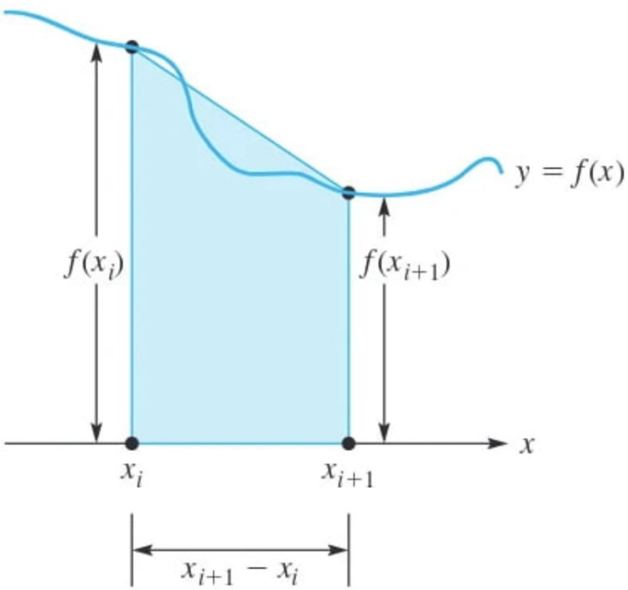
Numerical integration

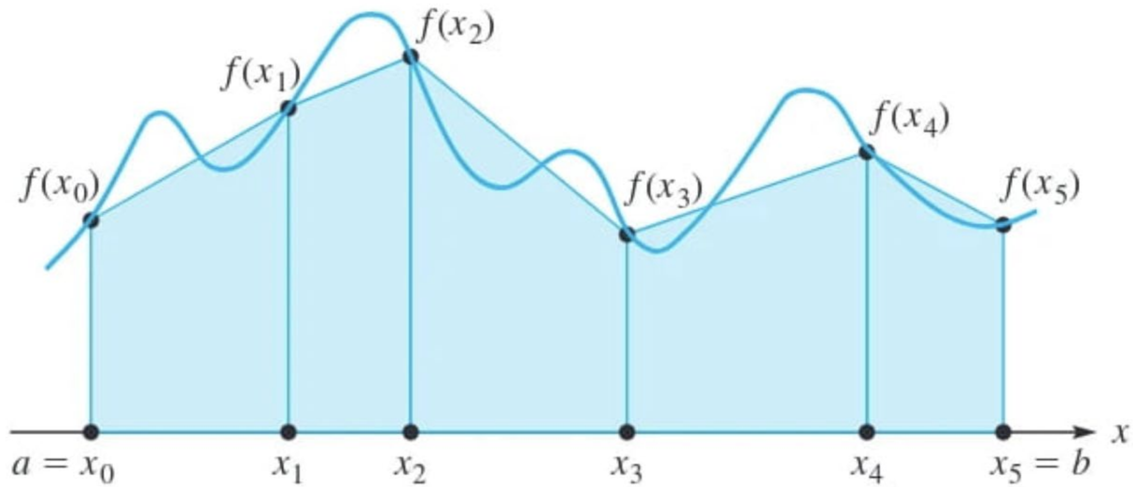
Trapezoid rule

Trapezoid rule

Trapezoid area = base times the average height

$$\int_{x_i}^{x_{i+1}} f(x)\,dx \approx \frac{1}{2}(x_{i+1} - x_i)[f(x_i) + f(x_{i+1})]$$





composite trapezoid rule:

$$\int_a^b f(x)\,dx \approx \frac{1}{2}\sum_{i=0}^{n-1}(x_{i+1} - x_i)[f(x_i) + f(x_{i+1})]$$

Uniform sampling

- Recall that $$\int_a^b f(x)\,dx \approx \frac{1}{2}\sum_{i=0}^{n-1}(x_{i+1}-x_i)[f(x_i)+f(x_{i+1})]$$
- Uniform sampling means points x_i are equally spaced: $x_i = a + ih, h = \dfrac{b-a}{n}$
- The formula becomes

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\sum_{i=0}^{n-1}[f(x_i)+f(x_{i+1})]$$

$$= h\left\{\frac{1}{2}[f(x_0)+f(x_n)] + \sum_{i=1}^{n-1} f(x_i)\right\}$$

Example

Write the pseudocode for Trapezoid rule and apply to $\int_0^1 e^{-x^2}\,dx$ when $n = 60$.

- The computer output is 0.746807.
- This integral is related to the error function

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$$

$$\int_0^1 e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}\,\mathrm{erf}(1) \approx 0.74682\,41330$$

Error analysis

Theorem on Precision of Composite Trapezoid Rule

If f'' exists and is continuous on the interval $[a, b]$ and if the composite trapezoid rule T with uniform spacing h is used to estimate the integral $I = \int_a^b f(x)\,dx$, then for some ζ in (a, b),

$$I - T = -\frac{1}{12}(b-a)h^2 f''(\zeta) = \mathcal{O}(h^2)$$

First Interpolation Error Theorem

If p is the polynomial of degree at most n that interpolates f at the $n+1$ distinct nodes $x_0, x_1, \ldots, x_n$ belonging to an interval $[a, b]$ and if $f^{(n+1)}$ is continuous, then for each x in $[a, b]$, there is a ξ in (a, b) for which

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i=0}^{n} (x - x_i) \tag{2}$$

Error analysis (cont'd)

The Mean Value Theorem for Integrals If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$f(c) = f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

that is,

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

$$\begin{aligned}\int_a^b f(x)\,dx &= (b-a)\int_0^1 f[a + t(b-a)]\,dt \\ &= (b-a)\int_0^1 g(t)\,dt \\ &= (b-a)\left\{\frac{1}{2}[g(0)+g(1)] - \frac{1}{12}g''(\zeta)\right\} \\ &= \frac{b-a}{2}[f(a)+f(b)] - \frac{(b-a)^3}{12}f''(\xi)\end{aligned}$$

$$\begin{aligned}\int_a^b f(x)\,dx &= \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} f(x)\,dx \\ &= \frac{h}{2}\sum_{i=0}^{n-1}[f(x_i)+f(x_{i+1})] - \frac{h^3}{12}\sum_{i=0}^{n-1} f''(\xi_i)\end{aligned}$$

$$-\frac{h^3}{12}\sum_{i=0}^{n-1} f''(\xi_i) = -\frac{b-a}{12}h^2\left[\frac{1}{n}\sum_{i=0}^{n-1} f''(\xi_i)\right] = -\frac{b-a}{12}h^2 f''(\zeta) = \mathcal{O}(h^2)$$

Example

If the trapezoid rule is to be used to compute

$$\int_0^1 e^{-x^2}\,dx$$

With an error of at most $\frac{1}{2} \times 10^4$, how many points should be used?