Midterm topics

- Computer arithmetic
- Root finding
- Interpolation
- Integration

Not covered topics

-Splines

-Adaptive Simpson

-Quadrature

Midterm exam

- 5 major problems with subproblems
- Some conceptual questions (T/F and multiple choice)
- No pseudo code

Computer Arithmetic

Concepts

- Absolute error v.s. Relative error
- Taylor series and Taylor Theorem
- Deflating $p(x) = (x - r)q(x) + p(r)$ (synthetic division)

- Floating point representation
- Machine number
- Loss of significance

Float-point rep.

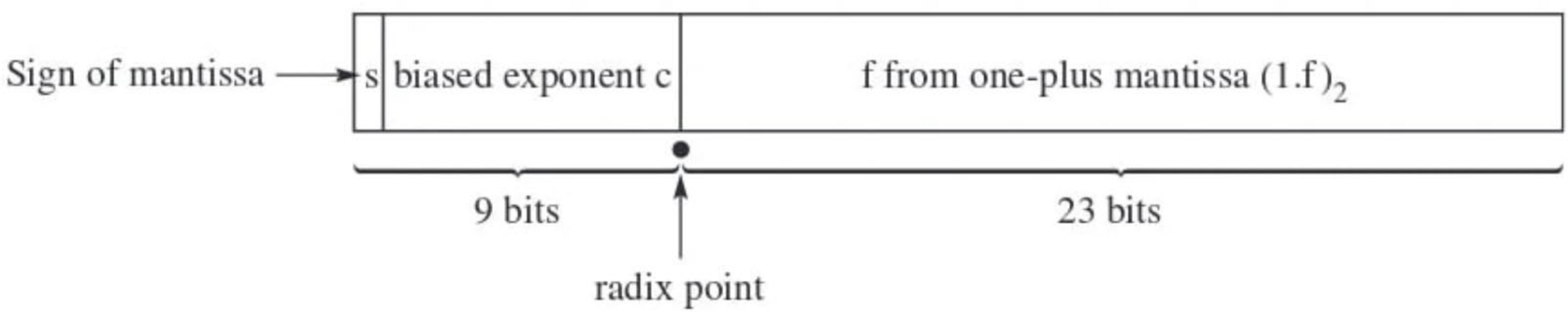
- If $x \neq 0$, it can be written as

$$x = \pm q \times 2^{m} \left(\frac{1}{2} \leq q < 1\right)$$

- The mantissa would be expressed a sequence of binary values (0 or 1)

$$q = (0.b_1 b_2 b_3 \cdots)_2$$

- $b_1 \neq 0 \rightarrow b_1 = 1 \rightarrow q \geq \frac{1}{2}.$



Loss of significance

- It occurs in the subtraction of two nearly equal numbers, which produces a result much smaller than either one.

- Theorem

Let x and y be normalized floating-point machine numbers, where $x > y > 0$. If $2^{-p} \leqq 1 - (y/x) \leqq 2^{-q}$ for some positive integers p and q, then at most p and at least q significant binary bits are lost in the subtraction $x - y$.

- How to avoid?
  - Double precision
  - Taylor series
  - Rationalization
  - Trigonometric identities
  - Logarithmic properties
  - Range reduction

Root finding

Bisection

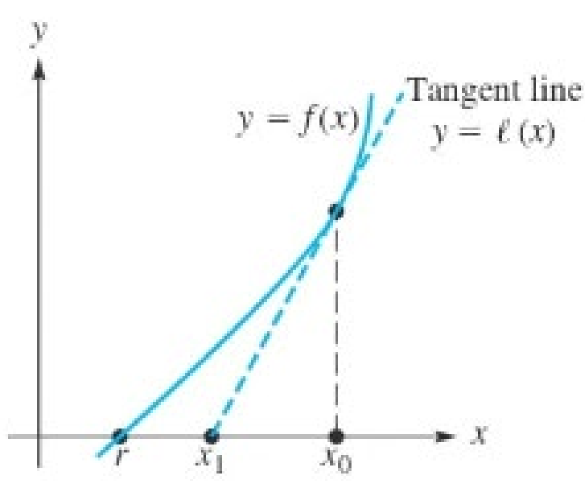
Newton's

Secant

Algorithms

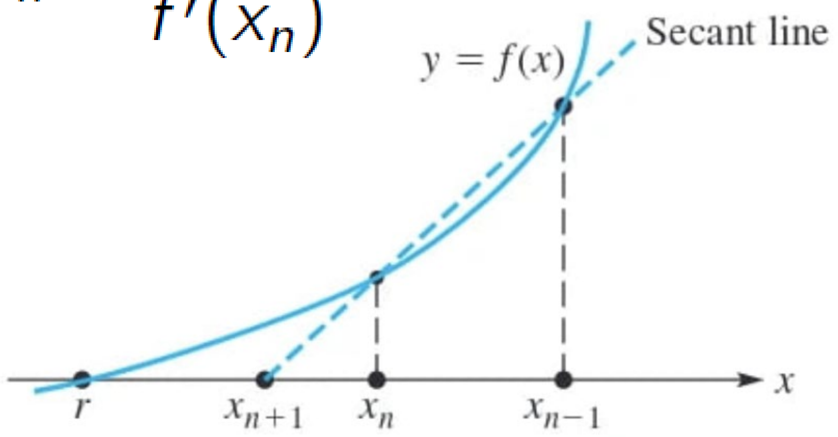
- Bisection method
- Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$



- Secant method

$$x_{n+1} = x_n - \left(\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right) f(x_n)$$



Convergence analysis

If the bisection algorithm is applied to a continuous function f on an interval $[a, b]$, where $f(a)f(b) < 0$, then, after n steps, an approximate root will have been computed with error at most $(b - a)/2^{n+1}$.

Newton's Method Theorem

If f, f', and f'' are continuous in a neighborhood of a root r of f and if $f'(r) \neq 0$, then there is a positive δ with the following property: If the initial point in Newton's method satisfies $|r - x_0| \leq \delta$, then all subsequent points x_n satisfy the same inequality, converge to r, and do so quadratically; that is,

$$|r - x_{n+1}| \leq c(\delta)|r - x_n|^2$$

Fixed-point iteration

$$x_{n+1} = g(x_n)$$

Locally convergent if $x^* = g(x^*)$, $|g'| < 1$.

Convergence rate

- Linear convergence: $C \in [0,1)$
$$|x_{n+1} - x^*| \le C|x_n - x^*|$$

Bisection is not linear

- Superlinear convergence: $\alpha \in (1,2)$
$$|x_{n+1} - x^*| \le C|x_n - x^*|^\alpha$$

Secant

- Quadratic convergence
$$|x_{n+1} - x^*| \le C|x_n - x^*|^2$$

Newton’s

Interpolation

Vandermonde matrix

Lagrange form

Divided difference

Vandermonde Matrix

x	x_0	x_1	$\cdots$	x_n
y	y_0	y_1	$\cdots$	y_n

$$p_n(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$$

$$\boxed{\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix}} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Can be used to prove the existence and uniqueness!

Vandermond matrix

Lagrange Form

The Lagrange form of the interpolation polynomial is given by

$$p_n(x) = \sum_{i=0}^{n} \ell_i(x) f(x_i)$$

$$\ell_i(x) = \left(\frac{x - x_0}{x_i - x_0}\right)\left(\frac{x - x_1}{x_i - x_1}\right)\cdots\left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)\left(\frac{x - x_{i+1}}{x_i - x_{i+1}}\right)\cdots\left(\frac{x - x_n}{x_i - x_n}\right)$$

Example

x	$\frac{1}{3}$	$\frac{1}{4}$	1
$f(x)$	2	-1	7

$$\ell_0(x) = \frac{\left(x - \frac{1}{4}\right)(x-1)}{\left(\frac{1}{3} - \frac{1}{4}\right)\left(\frac{1}{3} - 1\right)} = -18\left(x - \frac{1}{4}\right)(x-1)$$

$$\ell_1(x) = \frac{\left(x - \frac{1}{3}\right)(x-1)}{\left(\frac{1}{4} - \frac{1}{3}\right)\left(\frac{1}{4} - 1\right)} = 16\left(x - \frac{1}{3}\right)(x-1)$$

$$\ell_2(x) = \frac{\left(x - \frac{1}{3}\right)\left(x - \frac{1}{4}\right)}{\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right)} = 2\left(x - \frac{1}{3}\right)\left(x - \frac{1}{4}\right)$$

$$p_2(x) = -36\left(x - \frac{1}{4}\right)(x-1) - 16\left(x - \frac{1}{3}\right)(x-1) + 14\left(x - \frac{1}{3}\right)\left(x - \frac{1}{4}\right)$$

Divided Difference

- Concise notation

$$p_n(x) = \sum_{i=0}^{n} a_i \prod_{j=0}^{i-1} (x - x_j)$$

with $\prod_{j=0}^{-1} (x - x_j) = 1$

- Nested form

$$p(x) = a_0 + (x - x_0)(a_1 + (x - x_1)(a_2 + \cdots + (x - x_{n-1})a_n))\cdots))$$

x	$f[\,]$	$f[\,,\,]$	$f[\,,\,,\,]$	$f[\,,\,,\,,\,]$
x_0	$\boxed{f[x_0]}$			
		$\boxed{f[x_0, x_1]}$		
x_1	$f[x_1]$		$\boxed{f[x_0, x_1, x_2]}$	
		$f[x_1, x_2]$		$\boxed{f[x_0, x_1, x_2, x_3]}$
x_2	$f[x_2]$		$f[x_1, x_2, x_3]$	
		$f[x_2, x_3]$		
x_3	$f[x_3]$			

Example

Let $f(x) = x^3 + 2x^2 + x + 1.$

a) Find the polynomial of degree 4 that interpolates the values of f at $\pm 2, \pm 1, 0.$

b) Find the polynomial of degree 2 that interpolates the values of f at $\pm 1, 0.$

$$p_4(x) = -1 + 2(x+2) - (x+2)(x+1) + (x+2)(x+1)x$$

$$p_2(x) = 1 + 2(x+1)x$$

Interpolation errors

First Interpolation Error Theorem

If p is the polynomial of degree at most n that interpolates f at the $n+1$ distinct nodes $x_0, x_1, \ldots, x_n$ belonging to an interval $[a, b]$ and if $f^{(n+1)}$ is continuous, then for each x in $[a, b]$, there is a ξ in (a, b) for which

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i=0}^{n} (x - x_i) \qquad (2)$$

Second Interpolation Error Theorem

Let f be a function such that $f^{(n+1)}$ is continuous on $[a, b]$ and satisfies $|f^{(n+1)}(x)| \leq M$. Let p be the polynomial of degree $\leq n$ that interpolates f at $n+1$ equally spaced nodes in $[a, b]$, including the endpoints. Then on $[a, b]$,

$$|f(x) - p(x)| \leq \frac{1}{4(n+1)} M h^{n+1} \qquad (6)$$

where $h = (b - a)/n$ is the spacing between nodes.

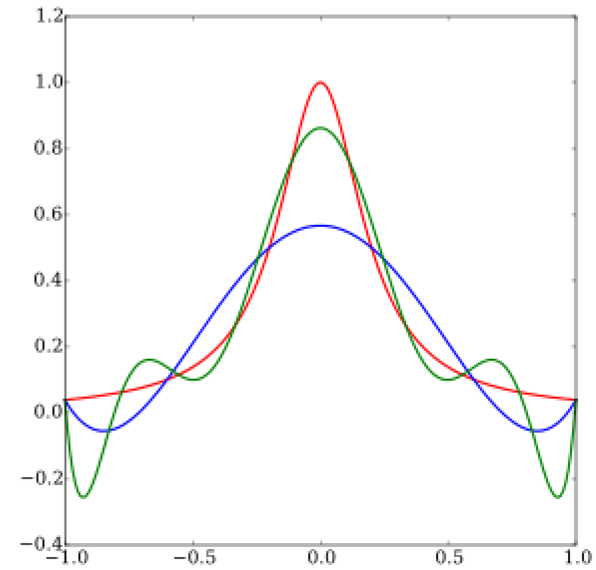
Runge's phenomenon

- A polynomial of degree n has n roots.

  If all roots are real, the curve crosses the x-axis n times.
- These many turns result in wide oscillations.
- A specific example is provided by Runge function

$$f(x) = (1 + 25x^2)^{-1}$$

- Replace the equispaced nodes by Chebyshev nodes



$$\lim_{n\to\infty}\left(\max_{-1\leq x\leq 1}|f(x) - P_n(x)|\right) = \infty.$$

Integration

Trapezoid

Simpson's

Formula

- Trapezoid

$$\int_a^b f(x)\,dx \approx \frac{1}{2}(b-a)[f(a)+f(b)]$$

- Simpson's 1/3 Rule

$$\int_a^b f(x)\,dx \approx \frac{1}{6}(b-a)\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$

- No need to memorize 3/8 rule

Error analysis

Theorem on Precision of Composite Trapezoid Rule

If f'' exists and is continuous on the interval $[a, b]$ and if the composite trapezoid rule T with uniform spacing h is used to estimate the integral $I = \int_a^b f(x)\,dx$, then for some ζ in (a, b),

$$I - T = \boxed{-\frac{1}{12}(b-a)h^2 f''(\zeta)} = \mathcal{O}(h^2)$$

Simpson's 1/3 Rule: $-\frac{1}{180}(b-a)h^4 f^{(4)}(\xi)$

Simpson's 3/8 rule has the same order but lower in accuracy.