Today's agenda

- Algorithm/pseudocode of Newton form
- Error analysis
- Spline function (not required in exams)

Divided difference

The divided differences obey the formula

$$f[x_0, x_1, \ldots, x_k] = \frac{f[x_1, x_2, \ldots, x_k] - f[x_0, x_1, \ldots, x_{k-1}]}{x_k - x_0}$$

$$f[x_i, x_{i+1}, \ldots, x_{j-1}, x_j] = \frac{f[x_{i+1}, x_{i+2}, \ldots, x_j] - f[x_i, x_{i+1}, \ldots, x_{j-1}]}{x_j - x_i}$$

x	$f[\,]$	$f[\,,\,]$	$f[\,,\,,\,]$	$f[\,,\,,\,,\,]$
x_0	$\boxed{f[x_0]}$			
		$\boxed{f[x_0, x_1]}$		
x_1	$f[x_1]$		$\boxed{f[x_0, x_1, x_2]}$	
		$f[x_1, x_2]$		$\boxed{f[x_0, x_1, x_2, x_3]}$
x_2	$f[x_2]$		$f[x_1, x_2, x_3]$	
		$f[x_2, x_3]$		
x_3	$f[x_3]$			

Pseudocode

Input: $x_{0:n}, f(x_{0:n})$

$$f[x_i, x_{i+1}, \ldots, x_{j-1}, x_j] = \frac{f[x_{i+1}, x_{i+2}, \ldots, x_j] - f[x_i, x_{i+1}, \ldots, x_{j-1}]}{x_j - x_i}$$

Output: $a_{ij} = f\left[x_i, x_{i+1}, \cdots, x_j\right]$

$$a_i = f[x_i, x_{i+1}, \cdots, x_j]$$

Evaluate the polynomial

Input: $x_{0:n}, f(x_{0:n}), a_i$, t

$$p(x) = a_0 + (x - x_0)(a_1 + (x - x_1)(a_2 + \cdots + (x - x_{n-1})a_n))\cdots))$$

output: $f(t)$

Theorems on Interpolation Errors

First Interpolation Error Theorem

Access the interpolation errors by means of a formula that involves higher-order derivative.

First Interpolation Error Theorem

If p is the polynomial of degree at most n that interpolates f at the $n+1$ distinct nodes $x_0, x_1, \ldots, x_n$ belonging to an interval $[a, b]$ and if $f^{(n+1)}$ is continuous, then for each x in $[a, b]$, there is a ξ in (a, b) for which

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i=0}^{n} (x - x_i) \tag{2}$$

The maximum error of a linear interpolation is bounded by $\frac{1}{8} h^2 M$, where $h = x_1 - x_0, M = \max_{x_0 \le x \le x_1} |f''(x)|$

Second Interpolation Theorem

Special case for equally spaced nodes

Second Interpolation Error Theorem

Let f be a function such that $f^{(n+1)}$ is continuous on $[a, b]$ and satisfies $|f^{(n+1)}(x)| \leqq M$. Let p be the polynomial of degree $\leqq n$ that interpolates f at $n+1$ equally spaced nodes in $[a, b]$, including the endpoints. Then on $[a, b]$,

$$|f(x) - p(x)| \leqq \frac{1}{4(n+1)} M h^{n+1} \tag{6}$$

where $h = (b-a)/n$ is the spacing between nodes.

Upper Bound Lemma

Suppose that $x_i = a + ih$ for $i = 0, 1, \ldots, n$ and that $h = (b-a)/n$. Then for any $x \in [a, b]$

$$\prod_{i=0}^{n} |x - x_i| \leqq \frac{1}{4} h^{n+1} n! \tag{4}$$

Theorems on Newton Form

Third Interpolation Error Theorem

If p is the polynomial of degree n that interpolates the function f at nodes $x_0, x_1, \ldots, x_n$, then for any x that is not a node,

$$f(x) - p(x) = f[x_0, x_1, \ldots, x_n, x] \prod_{i=0}^{n} (x - x_i)$$

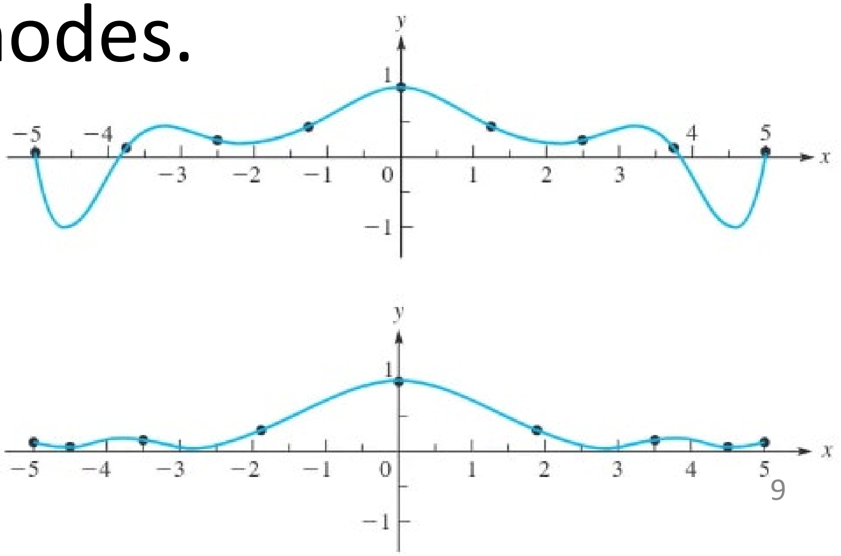
Divided Differences and Derivatives

If $f^{(n)}$ is continuous on $[a, b]$ and if $x_0, x_1, \ldots, x_n$ are any $n + 1$ distinct points in $[a, b]$, then for some ξ in (a, b),

$$f[x_0, x_1, \ldots, x_n] = \frac{1}{n!} f^{(n)}(\xi)$$

Remarks

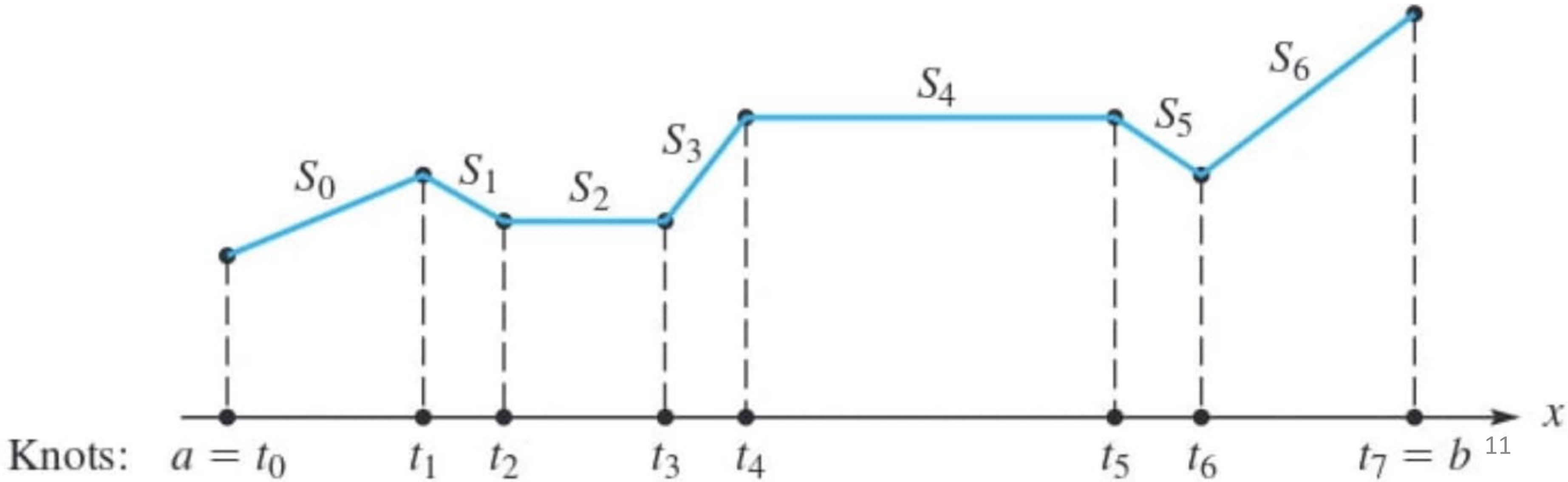
- These theorems give loop upper bounds on the interpolation errors.
- When the order n is small, one can find tighter upper bounds.
- Uniform nodes often result in larger errors compared to Chebyshev nodes.



Chapter 6. Spline Functions

Overview

- A spline function is a function that consists of polynomial pieces joined together with smoothness.
- A simple example is polygonal function (or spline of degree 1).
- The points $t_0, t_1, \cdots, t_n$ are termed knots.



First-degree spline

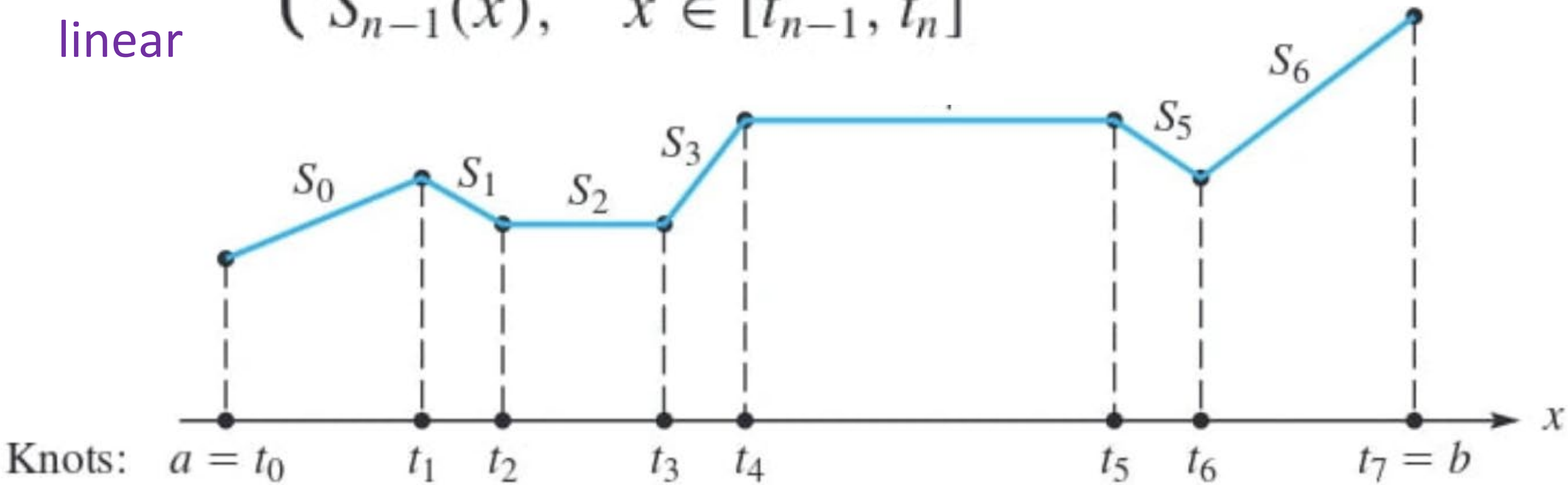
Piece-wise defined function

Linear polynomial

$$S(x) = \begin{cases} S_0(x), & x \in [t_0, t_1] \\ S_1(x), & x \in [t_1, t_2] \\ \vdots & \vdots \\ S_{n-1}(x), & x \in [t_{n-1}, t_n] \end{cases}$$

$$S_i(x) = a_i x + b_i$$

Piecewise linear



Spline Definition

Spline of Degree 1

A function S is called a **spline of degree 1** if:

1. The domain of S is an interval $[a, b]$.
2. S is continuous on $[a, b]$.
3. There is a partitioning of the interval $a = t_0 < t_1 < \cdots < t_n = b$ such that S is a linear polynomial on each subinterval $[t_i, t_{i+1}]$.

Spline of Degree 2

A function Q is called a **spline of degree 2** if:

1. The domain of Q is an interval $[a, b]$.
2. Q and Q' are continuous on $[a, b]$.
3. There are points t_i (called **knots**) such that $a = t_0 < t_1 < \cdots < t_n = b$ and Q is a polynomial of degree at most 2 on each subinterval $[t_i, t_{i+1}]$.

Example

$$S(x) = \begin{cases} x, & x \in [-1, 0] \\ 1 - x, & x \in (0, 1) \\ 2x - 2, & x \in [1, 2] \end{cases}$$

$$Q(x) = \begin{cases} x^2 & (-10 \leqq x \leqq 0) \\ -x^2 & (0 \leqq x \leqq 1) \\ 1 - 2x & (1 \leqq x \leqq 20) \end{cases}$$

Cubic splines

- Linear/quadratic splines are not smooth.
- Most popular splines are order 3, termed cubic splines.
- As S, S', S'' are continuous, the graph of the function will appear smooth to the eye.
- Natural cubic spline $S''(t_0) = S''(t_n) = 0$
- Example

x	-1	0	1
y	1	2	-1

Example solution

$$S(x) = \begin{cases} S_0(s) = ax^3 + bx^2 + cx + d & x \in [-1, 0] \\ S_1(s) = ex^3 + fx^2 + gx + h & x \in [0, 1] \end{cases}$$

$$S'(x) = \begin{cases} S_0'(x) = 3ax^2 + 2bx + c \\ S_1'(x) = 3ex^2 + 2fx + g \end{cases}$$

$$S''(x) = \begin{cases} S_0''(x) = 6ax + 2b \\ S_1''(s) = 6ex + 2f \end{cases}$$