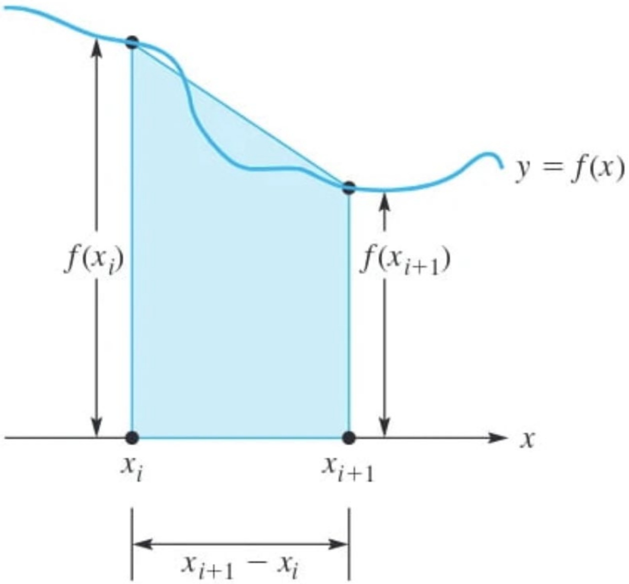
Simpson’s Rules

Recall Trapezoid Rule

Trapezoid area = base times the average height

$$\int_{x_i}^{x_{i+1}} f(x)\,dx \approx \frac{1}{2}(x_{i+1}-x_i)[f(x_i)+f(x_{i+1})]$$



$$\int_a^b f(x)\,dx \approx Af(a) + Bf(b)$$

Goal: find A, B such that the resulting integration is exact for any linear functions.

$$\int_a^b f(x)\,dx \approx \frac{1}{2}(b-a)[f(a)+f(b)]$$

Simpson's Rule

Consider a quadratic polynomial passing through three points $f(a), f\left(\frac{a+b}{2}\right)$, and $f(b)$

$$\int_a^b f(x)\,dx \approx Af(a) + Bf\left(\frac{a+b}{2}\right) + Cf(b)$$

Goal: find A, B, C such that the formula integrates correctly $1, x, x^2$.

Simpson (cont'd)

- Start by $\int_{-1}^{1} f(x)\,dx \approx Af(-1) + Bf(0) + Cf(1)$

- The solution is $A = \frac{1}{3}$, $B = \frac{4}{3}$, and $C = \frac{1}{3}$.
- Using a linear mapping

$$x = \frac{1}{2}(b-a)t + \frac{1}{2}(a+b)$$

Basic Simpson's rule

$$\int_a^b f(x)\,dx \approx \frac{1}{6}(b-a)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Example

Find approximate value for the integral $\int_0^1 e^{-x^2}\,dx$

Using basic trapezoid rule and the basic Simpson rule.

$$\int_0^1 e^{-x^2}\,dx \approx \frac{1}{2}\left[e^0 + e^{-1}\right] \approx 0.5[1 + 0.36788] = 0.68394$$

$$\int_0^1 e^{-x^2}\,dx \approx \frac{1}{6}\left[e^0 + 4e^{-0.25} + e^{-1}\right] = 0.7472$$

$$\int_0^1 e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}\,\mathrm{erf}(1) \approx 0.7468241330$$

Uniform spacing

- Consider partition points $a, a+h, a+2h$ in the basic Simpson's Rule

$$\int_a^{a+2h} f(x)\,dx \approx \frac{h}{3}[f(a) + 4f(a+h) + f(a+2h)]$$

- It computes the integral of a quadratic polynomial over an interval of length $2h$ using 3 points: two endpoints and the middle point.

- Composite Simpson's Rule (n is an even number)

$$\int_a^b f(x)\,dx = \sum_{i=1}^{n/2} \int_{a+2(i-1)h}^{a+2ih} f(x)\,dx$$

Same complexity as Trapezoid's!

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left\{[f(a) + f(b)] + 4\sum_{i=1}^{n/2} f[a + (2i-1)h] + 2\sum_{i=1}^{(n-2)/2} f(a + 2ih)\right\}$$

Error analysis

Theorem on Precision of Composite Trapezoid Rule

If f'' exists and is continuous on the interval $[a, b]$ and if the composite trapezoid rule T with uniform spacing h is used to estimate the integral $I = \int_a^b f(x)\,dx$, then for some ζ in (a, b),

$$I - T = \boxed{-\frac{1}{12}(b-a)h^2 f''(\zeta)} = \mathcal{O}(h^2)$$

Simpson's Rule: $-\frac{1}{180}(b-a)h^4 f^{(4)}(\xi)$

Proof

$$\int_a^{a+2h} f(x)\,dx \approx \frac{h}{3}[f(a) + 4f(a+h) + f(a+2h)]$$

$$f(a+h) = f + hf' + \frac{1}{2!}h^2 f'' + \frac{1}{3!}h^3 f''' + \frac{1}{4!}h^4 f^{(4)} + \cdots$$

$$f(a+2h) = f + 2hf' + 2h^2 f'' + \frac{4}{3}h^3 f''' + \frac{2^4}{4!}h^4 f^{(4)} + \cdots$$

$$f(a)+4f(a+h)+f(a+2h) = 6f+6hf'+4h^2f''+2h^3f'''+\frac{20}{4!}h^4f^{(4)}+\cdots$$

Proof (cont'd)

$$F(a+2h) = F(a) + 2hF'(a) + 2h^2F''(a) + \frac{4}{3}h^3F'''(a) + \frac{2}{3}h^4F^{(4)}(a) + \frac{2^5}{5!}h^5F^{(5)}(a) + \cdots$$

$$\int_a^{a+2h} f(x)\,dx = \frac{h}{3}[f(a) + 4f(a+h) + f(a+2h)] - \frac{h^5}{90}f^{(4)} - \cdots$$

$$\int_a^b f(x)\,dx \approx \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

with error term

$$-\frac{1}{90}\left(\frac{b-a}{2}\right)^5 f^{(4)}(\xi)$$

for some ξ in (a, b).

Simpson's Rules

- Simpson's 1/3 rule

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] \qquad -\frac{h^4}{180}(b-a)f^{(4)}(\xi)$$

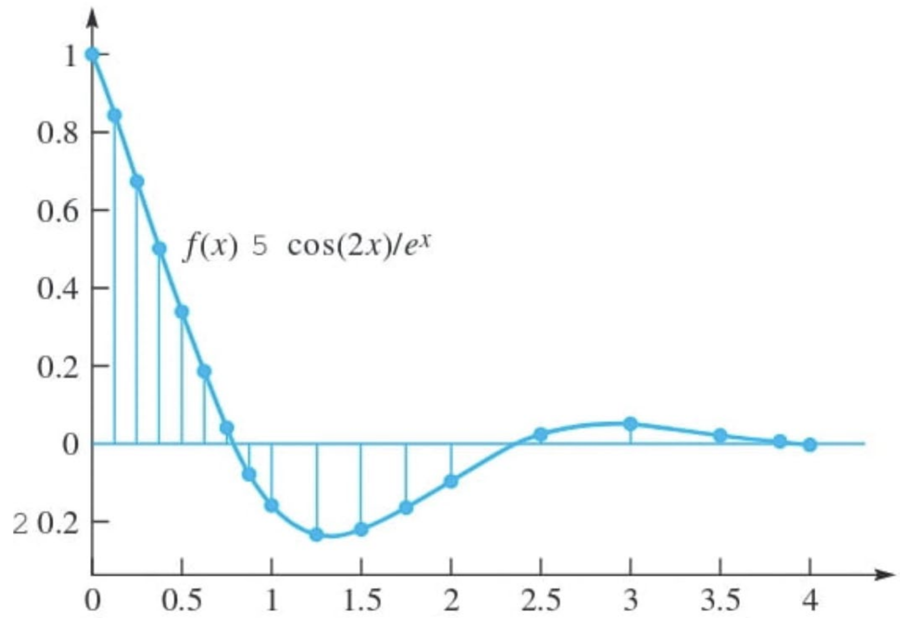
- Simpson's 3/8 rule

$$\int_a^b f(x)\,dx \approx \frac{3h}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right]$$
$$= \frac{(b-a)}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] \qquad -\frac{h^4}{80}(b-a)f^{(4)}(\xi)$$

- The 3/8 rule is about twice as accurate as the 1/3 one, but uses one more function value. Same order of accuracy though.

Adaptive Procedure

- The partitioning of the interval is automatically determined.
- We divide the interval into two subintervals and then decide whether each of them is to be divided into more subintervals.



Adaptive (cont'd)

$$I \equiv \int_a^b f(x)\,dx = S(a,b) + E(a,b)$$

$$S(a,b) = \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$E(a,b) = -\frac{1}{90}\left(\frac{b-a}{2}\right)^5 f^{(4)}(a) + \cdots$$