Nam Nguyen

npn190000

Exercises

Section 8.1: 1(b), 2, 9, 10, 19, 20

1. Using naive Gaussian elimination, factor the following matrices in the form A = LU, where L is a unit lower triangular matrix and U is an upper triangular matrix.

b. $A = \begin{bmatrix} 1 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 3 & -1 \\ 3 & -3 & 0 & 6 \\ 0 & 2 & 4 & -6 \end{bmatrix}$

Answer:

$$A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 3 & -1 \\ 3 & -3 & 0 & 6 \\ 0 & 2 & 4 & -6 \end{bmatrix}$$

$u_{11} = 1;\ u_{12} = 0;\ u_{13} = \frac{1}{3}; u_{14} = 0$

$l_{21} u_{11} = 0 -> l_{21} = 0$

$l_{21} u_{12} + u_{22} = 1 -> u_{22} = 1$

$l_{21} u_{13} + u_{23} = 3 - > u_{23} = 3$

$l_{21} u_{14} + u_{24} = -1 -> u_{24} = -1$

$l_{31} u_{11} = 3 -> l_{31} = 3$

$l_{31} u_{12} + l_{32} u_{22} = -3 -> l_{32} = -3$

$l_{31} u_{13} + l_{32} u_{23} + u_{33} = 0 -> u_{33} = 8$

$l_{31} u_{14} + l_{32} u_{24} + u_{44} = 6 -> u_{34} = 3$

$l_{41} = 0$

$l_{41} u_{12} + l_{42} u_{22} = 2 -> l_{42} = 2$

$l_{41} u_{13} + l_{42} u_{23} + l_{43} u_{33} = 4 -> l_{43} = -\frac{1}{4}$

$l_{41} u_{14} + l_{42} u_{24} + l_{43} u_{34} + u_{44} = -6 -> l_{44} = -\frac{13}{4}$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & -3 & 1 & 0 \\ 0 & 2 & -\frac{1}{4} & 1 \end{bmatrix}; U = \begin{bmatrix} 1 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 8 & 3 \\ 0 & 0 & 0 & -\frac{13}{4} \end{bmatrix}$$

2. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 0 & 0 \\ 0 & 9 & 4 & 0 \\ 5 & 0 & 8 & 10 \end{bmatrix}$$

a. Determine a unit lower triangular matrix M and an upper triangular matrix U such that MA = U.

b. Determine a unit lower triangular matrix L and an upper triangular matrix U such that A = LU. Show that M L = I so that L = M^{-1} .

Answer:

a. MA =U

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 0 & 0 \\ 0 & 9 & 4 & 0 \\ 5 & 0 & 8 & 10 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -5 & 0 & 0 & 1 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix}$$

$$M = E_3 E_2 E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ -5 & 6 & -2 & 1 \end{bmatrix}$$

$$U = M \cdot A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

b. A= LU with L = M^{-1}

$$A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 5 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$ML = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ -5 & 6 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 5 & 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

9. Consider

$$\boldsymbol{A} = \begin{bmatrix} 2 & -1 & 2 \\ 2 & -3 & 3 \\ 6 & -1 & 8 \end{bmatrix}$$

a. Find the matrix factorization A = **LDU’** , where L is unit lower triangular, D is diagonal, and U’ is unit upper triangular.

b. Use this decomposition of A to solve Ax = b, where b = $[-2, -5, 0]^T$.

Answer:

a.

$$A = \begin{bmatrix} 2 & -1 & 2 \\ 2 & -3 & 3 \\ 6 & -1 & 8 \end{bmatrix}$$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & -1 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

$$A = LDU'$$

$$U = DU' = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & 1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

b.

$A = LDU' \ ; Ax = b$

$LDU'x = b$

$Process: Lz = b$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}=\begin{bmatrix} -2 \\ -5 \\ 0 \end{bmatrix}$$

$Then\ z_1 = -2;\ z_2 = -3; z_3 = 3$

$Process: Dy = z$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}=\begin{bmatrix} -2 \\ -3 \\ 3 \end{bmatrix}$$

$Then\ \ y_1 = -1;\ y_2 = \frac{3}{2}; y_3 = 1$

$Process: U'x = y$

$$\begin{bmatrix} 1 & \frac{-1}{2} & 1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} -1 \\ \frac{3}{2} \\ 1 \end{bmatrix}$$

$Then\ \ x_1 = -1;\ x_2 = 2; x_3 = 1$

10. Repeat the preceding problem for

$$\boldsymbol{A} = \begin{bmatrix} -2 & 1 & -2 \\ -4 & 3 & -3 \\ 2 & 2 & 4 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix}$$

Answer:

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$U = DU' = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$LDU'x = b$

$Process: Lz = b$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix}$$

$Then\ z_1 = 1 ;\ z_2 = 2 ;\ z_3 = -1$

$Process: Dy = z$

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$Then\ y_1 = -\frac{1}{2};\ y_2 = 2 ;\ y_3 = 1$

$Process: U'x = y$

$$\begin{bmatrix} 1 & -\frac{1}{2} & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 2 \\ 1 \end{bmatrix}$$

$Then\ x_1 = -1;\ x_2 = 1 ;\ x_3 = 1$

19. a. Prove that the product of two lower triangular matrices is lower triangular.

b. Prove that the product of two unit lower triangular matrices is unit lower triangular.

c. Prove that the inverse of a unit lower triangular matrix is unit lower triangular.

d. By using the transpose operation, prove that all of the preceding results are true for upper triangular matrices.

Answer:

a. Prove that the product of two lower triangular matrices is lower triangular.

Matrix A , matrix b, they have a_{ij} and b_{ij} entries, whenever j > i $a_{ij} = 0$ and $b_{ij} = 0$;, Let C = A*B

$$C_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Whenever $j > i$, we have:

$$C_{ij} = \sum_{k=1}^{i} a_{ik} b_{kj} + \sum_{k=i+1}^{n} a_{ik} b_{kj} = \sum_{k=1}^{i} a_{ik} \cdot 0 + \sum_{k=i+1}^{n} 0 \cdot b_{kj} = 0$$

Then matrix C is a low triangle matrix(Proved)

b. Prove that the product of two unit lower triangular matrices is unit lower triangular .

Matrix A , matrix b, they have entries, $a_{ii} = 1$ and $b_{ii} = 1$;, Let C = A*B

$$C_{ii} = \sum_{k=1}^{n} a_{ik} b_{ki}.$$

$$C_{ii} = \sum_{k=1}^{i-1} a_{ik} b_{ki} + a_{ii} b_{ii} + \sum_{k=i+1}^{n} a_{ik} b_{ki} = \sum_{k=1}^{i} a_{ik} \cdot 0 + 1 + \sum_{k=i+1}^{n} 0 \cdot b_{ki} = 1$$

Then matrix C is a unit lower triangular(Proved)

c. Prove that the inverse of a unit lower triangular. triangular matrix is unit lower

Matrix A is lower triangular matrix, and X, such AX = I. Consider column of X; $AX^{(j)} = I^{(j)}$. Vector $I_i^{(j)}$ *is* 0 except When i = j $I_i^{(j)}$ *is* 1

Since A is lower triangular matrix. $AX^{(j)} = I^{(j)}$. Then

$$X_k^{(j)} = 0 \text{ when } k < j$$

$$\text{When } k = j$$

$$X_k^{(j)} = \frac{I_i^{(j)} - 0}{a_{jj}} = \frac{1}{1} = 1$$

$$\text{Then } X \begin{bmatrix} 1 & 0 & 0 & \cdots 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & \cdots & \\ & & & 1 \end{bmatrix}$$

d. By using the transpose operation, prove that all of the preceding results are true for upper triangular matrices.

Upper triangular matrices A and B , C = AB also be upper triangular since $C^T = A^T B^T$ is lower triangular since the transpose matrices A^T and B^T are lower triangular, and their product C^T is once again lower triangular. Then its transpose C is upper triangular. Also, given unit upper triangular matrix A, its inverse A^{-1} is also unit upper triangular.

$$(A^{-1}A)^T = I^T = I$$

$$\text{Then } A^T (A^{-1})^T = (A^{-1}A)^T$$

$$\text{Inverse } A^T = (A^{-1})^T. \text{ So } (A^T)^{-1} = (A^{-1})^T$$

A^T is unit lover triangular and its invers $(A^T)^{-1}$ and $\left(A^{-1}\right)^T$ is lower triangular, Then A^{-1} is upper triangular. (proved)

20. Let L be lower triangular, U be upper triangular, and D be diagonal.

a. If **L** and **U** are both unit triangular and **LDU** is diagonal, does it follow that L and U are diagonal?

b. If **LDU** is nonsingular and diagonal, does it follow that L and U are diagonal?

c. If L and U are both unit triangular and if **LDU** is diagonal, does it follow that **L = U = I**?

Answer:

a. Consider D =0, then LDU = 0 regardless of what form L and U assume. L and U are not necessarily diagonal in this case.

b. If LDU is invertible, then D can be invertible since $(LDU)^{-1} = U^{-1} D^{-1} L^{-1}$ and both L and U are invertible when they are unit triangular form and have nonzero determinant. Hence LDU to be invertible, must D mean its has no nonzero diagonal terms.

The product of a diagonal matrix with an unit upper triangular matrix is an upper triangular matrix:

DU = U', since the diagonal matrix is just a row scaling matrix. We also know U' is invertible when D has nonzero diagonal entries, U' does not either. So U' has a nonzero determinant and is invertible.

Consider: LDU = D', we have $L = D'U'^{-1}$. The inverse of an upper triangular matrix is a upper triangular, and also that the product of two upper triangular matrices is also upper triangular.

When U'^{-1} and D' are upper triangular, Hence form of L is simultaneously upper triangular and lower triangular is for it to be diagonal.

From defining the lower triangular matrix L' = LD, since the diagonal matrix is just a row scaling matrix. We also know L' is invertible when D has nonzero diagonal entries, L' does not either. So L' has a nonzero determinant and is invertible.

Consider: LDU = D', we have $U = L'^{-1}U'$. The inverse of an upper triangular matrix is a lower triangular, and also that the product of two lower triangular matrices is also lower triangular.

When L'^{-1} and D' are upper triangular, Hence form of U is simultaneously upper triangular and lower triangular is for it to be diagonal.

Hence, If LDU is nonsingular and diagonal, does it follow that L and U are diagonal.

c. Consider D =0, then LDU = 0 regardless of what form L and U assume.

If L and U are unit triangular and If LDU = I, does it follow that L = U = I?

Consider, if LDU is invertible, then both L and U are diagonal (from proved question b above). They are both unit triangular, it follow that L = U = I

Section 8.2: 4, 7, 8, 11(a,c)

4. (Multiple choice) From a vector norm, we can create a subordinate matrix norm. Which relation is satisfied by every subordinate matrix norm?

a. ||Ax|| ||A|| ||x||

b. ||I|| = 1

c. ||AB|| ||A|| ||B||

d. ||A+ B|| ||A||+||B||

e. None of these.

Answer: b. ||I|| = 1

7. (Multiple choice) A sufficient condition for the Jacobi method to converge for the linear system Ax = b.

a. A − I is diagonally dominant.

b. A is diagonally dominant.

c. G is nonsingular.

d. The spectral radius of G is less than 1.

e. None of these

Answer: b. A is diagonally dominant

8. (Multiple choice)

A sufficient condition for the Gauss-Seidel method to work on the linear system **Ax = b**.

a. A is diagonally dominant.

b. A − I is diagonally dominant.

c. The spectral radius of A is less than 1.

d. G is nonsingular.

e. None of these

Answer: a. A is diagonally dominant

11. Determine the condition numbers $\kappa(A)$ of these matrices:

a. $\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$

c. $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Answer:

a.

$$A^TA = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 5 & -4 & 1 \\ -4 & 6 & -4 \\ 1 & -4 & 5 \end{bmatrix}$$

$$\begin{bmatrix} (5-\lambda) & -4 & 1 \\ -4 & (6-\lambda) & -4 \\ 1 & -4 & (5-\lambda) \end{bmatrix} = 0$$

Solve the characteristic polynomial for eigenvalues

$$(5-\lambda)(6-\lambda)(5-\lambda) + 32 - (6-\lambda) - 32(5-\lambda) = 0$$

$$Then: \lambda^3 - 16\lambda^2 + 52\lambda - 16 = 0$$

$$Then: (\lambda - 4)\left(\lambda^2 - 12\lambda + 4\right) = 0$$

$$Then: \lambda = 4; \lambda = 6 \pm 4\sqrt{2}$$

$$k(A)\sqrt{\frac{6+4\sqrt{2}}{6-4\sqrt{2}}} = \frac{6+4\sqrt{2}}{2} \approx 5.83$$

c. Its singular values. $k(A) = \dfrac{3}{1} = 3$

Section 8.3: 8, 9, 10, 11, 12, 13, 14, 15

8. (Multiple choice) Let A be an n ×n invertible (nonsingular) matrix. Let x be a nonzero vector. Suppose that **Ax = λx**. Which equation does not follow from these hypotheses?

a. $A^k x = \lambda^k x$

b. $\lambda^{-k} x = (A^{-1})^k x$ for $k \geq 0$

c. $p(A)x = p(\lambda)x$ for any polynomial p

d. $A^k x = (1 - \lambda)^k x$

e. None of these.

Answer: d. Ak x = (1 − λ)k x

9. (Multiple choice) For what values of s will the matrix $I - svv^*$ be unitary, where v is a column vector of unit length?

a. 0, 1

b. 0, 2

c. 1, 2

d. 0, $\sqrt{2}$

e. None of these.

Answer: b. 0, 2

10. (Multiple choice) Let **U and V** be unitary n × n matrices, possibly complex. Which conclusion is not justified?

a. U + V is unitary.

b. U^* is unitary.

c. UV is unitary.

d. $U - vv^*$ is unitary when $||v|| = \sqrt{2}$ and v is a column vector.

e. None of these.

Answer: a. U + V is unitary

11. (Multiple choice) Which assertion is true?

a. Every n × n matrix has n distinct (different) eigenvalues.

b. The eigenvalues of a real matrix are real.

c. If U is a unitary matrix, then $U^* = U^T$

d. A square matrix and its transpose have the same eigenvalues.

e. None of these.

Answer: d. A square matrix and its transpose have the same eigenvalues

12. (Multiple choice) Consider the symmetric matrix

$$A = \begin{bmatrix} 1 & 3 & 4 & -1 \\ 3 & 7 & -6 & 1 \\ 4 & -6 & 3 & 0 \\ -1 & 1 & 0 & 5 \end{bmatrix}$$

What is the smallest interval derived from Gershgorin's Theorem such that all eigenvalues of the matrix A lie in that interval?

a. [−7, 9]

b. [−7, 13]

c. [3, 7]

d. [−3, 17]

e. None of these.

Answer: e. None of these

13. (True or false) Gershgorin's Theorem asserts that every eigenvalue λ of an n ×n matrix A must satisfy one of these inequalities:

$$|\lambda - a_{ii}| \leqq \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \quad \text{for} \quad 1 \leqq i \leqq n.$$

Answer: True

14. (True or false) A consequence of Schur's Theorem is that every square matrix A can be factored as A = PTP^{-1} , where P is a nonsingular matrix and T is upper triangular.

Answer: True

15. (True or false) A consequence of Schur's Theorem is that every (real) symmetric matrix **A** can be factored in the form **A = PDP**$^{-1}$, where **P** is unitary and **D** is diagonal.

Answer: True

Computing Exercises

Section 8.1: 4, 8

4. Write and test a procedure for determining A^{-1} for a given square matrix A of order n. Your procedure should use procedures Gauss and Solve.

Answer:

Source code:

```
clc
M = [0 2 3 2;
    3 0 4 5;
    0 5 0 2;
    0 1 2 3];
A = M;

Ni = length(A(1,:));
Nj = length(A(:,1));


Position_x = 1:Ni;
Position_y = 1:Nj;

for index = 1: Ni-1
    pct =index; qdt = index;
    pivot_checking = 0;
    for i =index : Ni
        for j = index:Ni
            tmp = abs(A(uint8(Position_x(i)),uint8(Position_y(j))));
            if (tmp > pivot_checking)
                pivot_checking = tmp; pct = i; qdt = j;
            end
        end
    end
    if pivot_checking == 0
        fprintf("Pivot is zero, Can not inverse \n")
    end

    Position_x([index pct]) = Position_x([pct index]);
    Position_y([index qdt]) = Position_y([qdt index]);

    for i = index+1:Ni
        if A(Position_x(i),Position_y(index)) ~= 0
            mult =
A(Position_x(i),Position_y(index))/A(Position_x(index),Position_y(index));
            A(Position_x(i),Position_y(index)) = mult;
            for j = index+1:Ni
                A(Position_x(i),Position_y(j)) =  A(Position_x(i),Position_y(j)) -
mult* A(Position_x(index),Position_y(j));
            end
        end
    end
end

I = eye(size(A));
for index=1:Ni
    for i = 2:Ni
        for j =1:i-1
            I(Position_x(i),index) = I(Position_x(i),index)-
A(Position_x(i),Position_y(j))*I(Position_x(j),index);
        end
    end
end
```

```
Invert = zeros(Ni,Nj);
for index=1:Ni
    for i = Ni:-1:1
        for j =i +1:Ni
            I(Position_x(i),index) = I(Position_x(i),index)-
A(Position_x(i),Position_y(j))*Invert(Position_x(j),index);
        end
        Invert(Position_y(i),index) =
I(Position_x(i),index)/A(Position_x(i),Position_y(i));
    end
end

fprintf("\nOriginal Matrix:\n")
fprintf("%f %f %f %f \n",M);
fprintf("\nInverse Matrix:\n")
fprintf("%f %f %f %f \n",Invert);
```

Result:

```
Original Matrix:
0.000000 3.000000 0.000000 0.000000
2.000000 0.000000 5.000000 1.000000
3.000000 4.000000 0.000000 2.000000
2.000000 5.000000 2.000000 3.000000

Inverse Matrix:
-0.024691 -0.007901 0.490196 0.019753
0.333333 0.026667 -0.117647 -0.066667
0.135802 0.243457 -0.196078 -0.108642
-0.629630 -0.201481 0.000000 0.503704
```

8. Investigate the numerical difficulties in inverting the following matrix:

$$\boldsymbol{A} = \begin{bmatrix} -0.0001 & 5.096 & 5.101 & 1.853 \\ 0. & 3.737 & 3.740 & 3.392 \\ 0. & 0. & 0.006 & 5.254 \\ 0. & 0. & 0. & 4.567 \end{bmatrix}$$

Answer:

Code:

```
A = [-0.0001, 5.096 , 5.101 , 1.853;
    0.0 , 3.737 , 3.740 , 3.392;
    0.0 , 0.0 , 0.006 , 5.254;
    0.0 , 0.0 , 0.0 , 4.567];
B = inv(A)
```

Result:

Command Window

```
  1.0e+04 *

  -1.0000    1.3637    0.1515   -0.7814
        0    0.0000   -0.0167    0.0192
        0         0    0.0167   -0.0192
        0         0         0    0.0000
```

Section 8.2: 3, 4

3. Using the Jacobi, Gauss-Seidel, and the SOR (ω = 1.4) iterative methods, write and run code to solve the following linear system to four decimal places of accuracy:

$$\begin{bmatrix} 7 & 3 & -1 & 2 \\ 3 & 8 & 1 & -4 \\ -1 & 1 & 4 & -1 \\ 2 & -4 & -1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix}$$

Compare the number of iterations in each case. Hint: Here, the exact solution is $x = (-1, 1, -1, 1)^T$

Answer:

Code for Using the Jacobi:

```
clc;

a = [7 3 -1 2;
    3 8 1 -4;
    -1 1 4 -1;
    2 -4 -1 6];
b = [-1 0 -3 1];

l_0 =[0 0 0 0]';
T = 0.000001;

y = jacobi(a,b,l_0,T);
fprintf ("Result Using the Jacobi y = %f %f %f %f \n",y)

function x1 = jacobi(a,b,l_0,T)

n = length(b);
% Find value of x
for j = 1 : n
    x(j) = ((b(j) - a(j,[1:j-1,j+1:n]) * l_0([1:j-1,j+1:n])) / a(j,j));

end

    x1 = x';
    k = 1;

    while norm(x1-l_0,1) > T
```

```
        for j = 1 : n
            temp(j) = ((b(j) - a(j,[1:j-1,j+1:n]) * x1([1:j-1,j+1:n])) / a(j,j));
        end

        l_0 = x1;
        x1 = temp';
        k = k + 1;
    end

    fprintf ("k = %f \n",k)
    x = x1';
end
```

```
Result:
k = 100.000000
Result Using the Jacobi y = -0.999998 0.999998 -0.999999 0.999998
``
```

Code Gauss-Seidel,:

```
clc;

a = [7 3 -1 2;
    3 8 1 -4;
    -1 1 4 -1;
    2 -4 -1 6];
b = [-1 0 -3 1];
l_0=[0 0 0 0]';
T=1e-5;
y = GS(a,b,l_0,T);
fprintf("Result Using the Gauss-Seidel y = %f %f %f %f \n",y)


%Display iteration


function x1 = GS(a,b,l_0,T)
    n=size(l_0,1);
    error=Inf;

    % Assign values

    k=0;

    while error>T

        l_current=l_0;
        for i=1:n
            sgma=0;

            for j=1:i-1
                sgma=sgma+a(i,j)*l_0(j);
            end

            for j=i+1:n
```

```
                sgma=sgma+a(i,j)*l_current(j);
            end
            l_0(i)=(1/a(i,i))*(b(i)-sgma);
        end

        k=k+1;
        error=norm(l_current-l_0);
        x1 = l_0;


    end
    fprintf("k=%f \n",k)
end
```

Result:

Command Window

```
k=43.000000
Result Using the Gauss-Seidel y = -0.999985 0.999984 -0.999997 0.999985
``
```

Code SOR (ω = 1.4)

```
clc;

a = [7 3 -1 2;
    3 8 1 -4;
    -1 1 4 -1;
    2 -4 -1 6];
b = [-1 0 -3 1];
l_0=[0 0 0 0]';
T=1e-5;
y = SOR(a,b,l_0,T);
fprintf("Result Using the SOR y = %f %f %f %f \n",y)


function x1 = SOR(a,b,l_0,T)
    lambda=1.4;
    n=length(l_0);
    x=l_0;
    error=1;
    k = 0;

    while (error>T)
        l_current=x;
            for i=1:n
                I = [1:i-1 i+1:n];
                x(i) = (1-lambda)*x(i)+lambda/a(i,i)*( b(i)-a(i,I)*x(I) );
            end
        error = norm(x-l_current)/norm(x);
        k = k+1;
    end
```

```
    x1=x;
    fprintf("k=%f \n",k);
end
```

Result:

```
Command Window
k=14.000000
Result Using the SOR y = -0.999998 0.999999 -1.000001 0.999997
>>
```

4. (Continuation) Solve the system using the SOR iterative method with values of ω = 1(0.1)2. Plot the number of iterations for convergence versus the values of ω. Which value of ω results in the fastest convergence?

Answer:

Code:

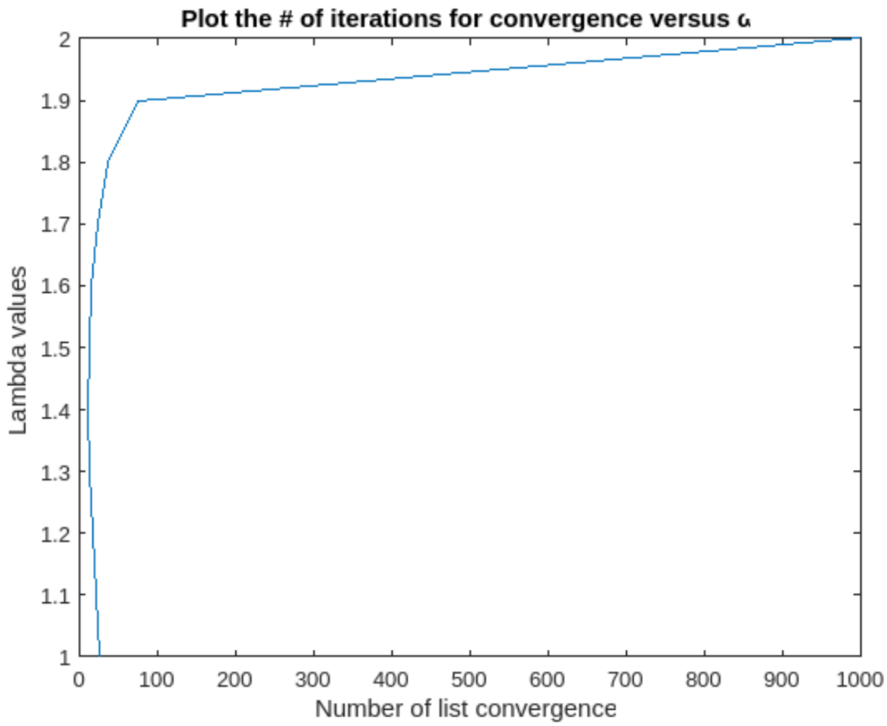
```
clear all;
clc;
a = [7 3 -1 2;
    3 8 1 -4;
    -1 1 4 -1;
    2 -4 -1 6];
b = [-1 0 -3 1]';
l_0=[0 0 0 0]';
T=1e-3;
list = [];
i=1;
MaxCon = 0;
current_w = 0;

for lambda = 1:0.1:2
  list(i) = SOR( a,b, l_0, lambda, T );

  % Finding fast convergence
  if MaxCon < list(i)
    MaxCon = list(i);
    current_lambda = lambda;
  end

  i= i+1;
end
fprintf("Fastest convergence at value of Lambda  = %d", current_lambda);
lambda = 1:0.1:2;
% Plot as follows
plot(list,lambda);
title("Plot the # of iterations for convergence versus ω");

xlabel("Number of list convergence ");
```

```
ylabel("Lambda values ");


function [list] = SOR(a, b,l_0, lambda, T)
    list = 0;
    norm_current = norm(b);

    if (norm_current == 0.0)
        norm_current = 1.0;
    end

    temp = b - a * l_0;
    err = norm(temp) / norm_current;
    if (err < T)
        return
    end

    b = lambda * b;
    M =  lambda * tril(a, -1) + diag(diag(a));
    N = -lambda * triu(a,  1) + (1.0 - lambda) * diag(diag(a));

    for list = 1 : (1/T)
        x_1 = l_0;
        l_0 = M \ (N * l_0 + b);     % adjust the aproximation
        err = norm(x_1 - l_0, 1);   % compute error
        if (err <= T)
            break
        end
    end

end
```

Result:

Command Window

```
Fastest convergence at value of Lambda  = 2
```



Section 8.3: 1, 10, 12

1. Use Matlab, Maple, Mathematica, or other computer programs available to you to compute the eigenvalues and eigenvectors of these matrices:

a. $A = \begin{bmatrix} 1 & 7 \\ 2 & -5 \end{bmatrix}$

b. $\begin{bmatrix} 4 & -7 & 3 & 2 & 3 \\ 1 & 6 & 11 & -1 & 2 \\ 5 & -5 & -2 & -4 & 1 \\ 9 & -3 & 1 & 6 & 5 \\ 3 & 2 & 5 & -5 & 1 \end{bmatrix}$

Answer:

Code:

```
clc;
A=[1 7;2 -5];

fprintf("Matrix A \n");
eigs(A)

B=[4 -7 3 2 3;1 6 11 -1 2;5 -5 -2 -4 1;9 -3 1 6 5;3 2 5 -5 1];
fprintf("Matrix B \n");

eigs(B)
```

Result:

Command Window

```
Matrix A

ans =

   -6.7958
    2.7958

Matrix B

ans =

   5.2665 + 8.8827i
   5.2665 - 8.8827i
   9.6921 + 0.0000i
  -5.2339 + 0.0000i
   0.0087 + 0.0000i

>>
```

10. Using mathematical software such as Matlab, Maple, or Mathematica, compute the singular value decomposition of these matrices, and verify that each result satisfies the equation $\mathbf{A} = \mathbf{UDV}^T$:

a. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 3 & -2 \\ 2 & 7 & 5 \\ -2 & -3 & 4 \\ 5 & -3 & -2 \end{bmatrix}$

Create the diagonal matrix $\mathbf{D} = \mathbf{U}^T\mathbf{AV}$ to check the results (always recommended). One can see the effects of roundoff errors in these calculations, for the off-diagonal elements in **D** are theoretically zero.

Answer:

a.

Code:

```
clc;
 A = [1 1; 0 1;1 0];
%A = [1, 3,-2, 2, 7, 5, -2, -3, 4, 5, -3, -2];
[U, S, V] = svd(A)
U*S*V'
```

Result:

```
U =

   -0.8165   -0.0000   -0.5774
   -0.4082   -0.7071    0.5774
   -0.4082    0.7071    0.5774


S =

    1.7321         0
         0    1.0000
         0         0


V =

   -0.7071    0.7071
   -0.7071   -0.7071


ans =

    1.0000    1.0000
    0.0000    1.0000
    1.0000   -0.0000
```

b.

Code:

```
clc;
 %A = [1 1; 0 1;1 0];
A = [1  3 -2; 2 7 5; -2 -3 4; 5 -3 -2];
[U, S, V] = svd(A)
U*S*V'
```

Result:

```
U =

   -0.1773   -0.4339    0.3176   -0.8243
   -0.9125   -0.0655   -0.3961    0.0782
    0.0778    0.7216   -0.4080   -0.5538
    0.3603   -0.5354   -0.7588   -0.0880
```

```
S =

    9.4469         0         0
         0    6.8963         0
         0         0    4.7115
         0         0         0


V =

   -0.0377   -0.6794   -0.7328
   -0.8716   -0.3363    0.3567
   -0.4888    0.6522   -0.5795


ans =

    1.0000    3.0000   -2.0000
    2.0000    7.0000    5.0000
   -2.0000   -3.0000    4.0000
    5.0000   -3.0000   -2.0000
```

12. Find the singular value decomposition of these matrices:

a. $\begin{bmatrix} 2 & 1 & -2 \end{bmatrix}$ **b.** $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ **c.** $\begin{bmatrix} -\frac{5}{2}+3\sqrt{3} & \frac{5}{2}\sqrt{3}+3 \end{bmatrix}$

d. $\begin{bmatrix} 2 & 2 & 2 & 2 \\ \frac{17}{10} & \frac{1}{10} & -\frac{17}{10} & -\frac{1}{10} \\ \frac{3}{5} & \frac{9}{5} & -\frac{3}{5} & -\frac{9}{5} \end{bmatrix}$ **e.** $\begin{bmatrix} \frac{7}{2}-\frac{13}{6}\sqrt{6} & \frac{7}{2}+\frac{13}{6}\sqrt{6} \\ -\frac{7}{2}-\frac{13}{6}\sqrt{6} & -\frac{7}{2}+\frac{13}{6}\sqrt{6} \\ -\frac{13}{6}\sqrt{6} & \frac{13}{6}\sqrt{6} \end{bmatrix}$

Answer:

Code:

```
clc;
a = [2, 1, 2];

b = [3; 4];

c = [(-5/2)+3*(3)^(1/2) (5/2)*(3^(1/2))+3];

d = [2, 2, 2, 2; 17/10, 1/10, -17/10, -1/10; 3/5, 9/5, -3/5, -9/5];
```

```
e = [(7/2)-((13/6)*6^(1/2)), (7/2)+((13/6)*6^(1/2)); (-7/2)-((13/6)*6^(1/2)), (-
7/2)+((13/6)*6^(1/2)); -((13/6)*6^(1/2)), ((13/6)*6^(1/2))] ;

fprintf("SVD of a is:\n ")
disp(svd(a))
[U, S, V] = svd(a)

fprintf("SVD of b is:\n ")
disp(svd(b))
[U, S, V] = svd(b)

fprintf("SVD of c is:\n ")
disp(svd(c))
[U, S, V] = svd(c)

fprintf("SVD of d is:\n ")
disp(svd(d))
[U, S, V] = svd(d)

fprintf("SVD of e is:\n ")
disp(svd(e))
[U, S, V] = svd(e)
```

```
Result:
SVD of a is:
      3


U =

     1


S =

     3     0     0


V =

    0.6667   -0.3333   -0.6667
    0.3333    0.9333   -0.1333
    0.6667   -0.1333    0.7333

SVD of b is:
      5


U =

    0.6000   -0.8000
    0.8000    0.6000
```

```
S =

     5
     0


V =

     1

SVD of c is:
     7.8102


U =

     1


S =

    7.8102         0


V =

    0.3452   -0.9385
    0.9385    0.3452

SVD of d is:
     4.0000
    3.0000
    2.0000


U =

   -1.0000         0         0
         0   -0.6000   -0.8000
         0   -0.8000    0.6000


S =

     4     0     0     0
     0     3     0     0
     0     0     2     0


V =

   -0.5000   -0.5000   -0.5000   -0.5000
   -0.5000   -0.5000    0.5000    0.5000
   -0.5000    0.5000    0.5000   -0.5000
```

```
   -0.5000    0.5000   -0.5000    0.5000

SVD of e is:
    13.0000
    7.0000


U =

   -0.5774    0.7071   -0.4082
   -0.5774   -0.7071   -0.4082
   -0.5774   -0.0000    0.8165


S =

   13.0000         0
         0    7.0000
         0         0


V =

    0.7071    0.7071
   -0.7071    0.7071
```