Today's Agenda

Structured linear system

– Tridiagonal

– Diagonally dominant

– Banded

Tridiagonal system

Tridiagonal system

- A tridiagonal matrix is characterized by
$$a_{ij} = 0 \text{ if } |\mathrm{i} - \mathrm{j}| \geq 2.$$

- Memory friendly: $n \times n$ tridiagonal matrix only requires $3n - 2$ memory locations.

- Naïve Gaussian elimination is used; no need for pivoting!

- Similarly, there is a penta-diagonal system
$$a_{ij} = 0 \text{ if } |\mathrm{i} - \mathrm{j}| \geq 3.$$

Tridiagonal system

$$
\begin{bmatrix}
d_1 & c_1 & & & & & & \\
a_1 & d_2 & c_2 & & & & & \\
 & a_2 & d_3 & c_3 & & & & \\
 & & \ddots & \ddots & \ddots & & & \\
 & & & a_{i-1} & d_i & c_i & & \\
 & & & & \ddots & \ddots & \ddots & \\
 & & & & & a_{n-2} & d_{n-1} & c_{n-1} \\
 & & & & & & a_{n-1} & d_n
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_i \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_i \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix}
$$

$$
\begin{cases}
d_2 \leftarrow d_2 - (a_1/d_1)\, c_1 \\
b_2 \leftarrow b_2 - (a_1/d_1)\, b_1
\end{cases}
$$

$$
\begin{cases}
d_i \leftarrow d_i - (a_{i-1}/d_{i-1})\, c_{i-1} \\
b_i \leftarrow b_i - (a_{i-1}/d_{i-1})\, b_{i-1}
\end{cases}
\qquad (2 \le i \le n)
$$

After forward elimination

- We get an upper-triagonal matrix

$$\begin{bmatrix} d_1 & c_1 & & & & & & \\ & d_2 & c_2 & & & & & \\ & & d_3 & c_3 & & & & \\ & & & \ddots & \ddots & & & \\ & & & & d_i & c_i & & \\ & & & & & \ddots & \ddots & \\ & & & & & & d_{n-1} & c_{n-1} \\ & & & & & & & d_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_i \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_i \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix}$$

- Note that b, d are changed, but c's are the same.
- Back substitution: $x_n \leftarrow b_n / d_n$

$$x_i \leftarrow (b_i - c_i x_{i+1}) / d_i \qquad (i = n-1, n-2, \ldots, 1)$$

Pseudocode

Function x = Tri(a, d, c, b)

$$\begin{cases} d_i \leftarrow d_i - \left(a_{i-1}/d_{i-1}\right) c_{i-1} & \\ b_i \leftarrow b_i - \left(a_{i-1}/d_{i-1}\right) b_{i-1} & (2 \leq i \leq n) \end{cases}$$

$$x_n \leftarrow b_n/d_n$$

$$x_i \leftarrow \left(b_i - c_i x_{i+1}\right)/d_i \qquad (i = n-1, n-2, \ldots, 1)$$

Strictly diagonal dominance

Strictly diagonal matrix

- Since Tri does not involve pivoting, will it fail?
- It is possible to encounter division by zero.
- If the matrix is diagonally dominant, then Tri will succeed.

A general matrix $A = (a_{ij})_{n\times n}$ is **strictly diagonally dominant** if

$$|a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \qquad (1 \leqq i \leqq n)$$

Strictly dominant tridiagonal

For a tridiagonal matrix, strict diagonal dominance means

$$
\begin{bmatrix}
d_1 & c_1 & & & & & & \\
a_1 & d_2 & c_2 & & & & & \\
 & a_2 & d_3 & c_3 & & & & \\
 & & \ddots & \ddots & \ddots & & & \\
 & & & a_{i-1} & d_i & c_i & & \\
 & & & & \ddots & \ddots & \ddots & \\
 & & & & & a_{n-2} & d_{n-1} & c_{n-1} \\
 & & & & & & a_{n-1} & d_n
\end{bmatrix}
\Rightarrow
\begin{gathered}
|d_i| > |a_{i-1}| + |c_i| \\
(1 \le i \le n)
\end{gathered}
$$

No need for pivoting

We'll verify that Tri preserves strictly diagonal dominance.

$$\begin{cases} \widehat{d}_1 = d_1 \\ \widehat{d}_i = d_i - \left(a_{i-1}/\widehat{d}_{i-1}\right)c_{i-1} \qquad (2 \le i \le n) \end{cases}$$

If $|d_i| > |a_{i-1}| + |c_i| \qquad (1 \le i \le n)$

then $|\widehat{d}_i| > |c_i|$.

Case study

Symmetric tridiagonal system

$$
\begin{bmatrix}
d_1 & c_1 & & & & & & \\
c_1 & d_2 & c_2 & & & & & \\
 & c_2 & d_3 & c_3 & & & & \\
 & & \ddots & \ddots & \ddots & & & \\
 & & & c_{i-1} & d_i & c_i & & \\
 & & & & \ddots & \ddots & \ddots & \\
 & & & & & c_{n-2} & d_{n-1} & c_{n-1} \\
 & & & & & & c_{n-1} & d_n
\end{bmatrix}
\begin{bmatrix}
x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_i \\ \vdots \\ x_{n-1} \\ x_n
\end{bmatrix}
=
\begin{bmatrix}
b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_i \\ \vdots \\ b_{n-1} \\ b_n
\end{bmatrix}
$$

Penta-diagonal system

$$
\begin{bmatrix}
d_1 & c_1 & f_1 & & & & & & & \\
a_1 & d_2 & c_2 & f_2 & & & & & & \\
e_1 & a_2 & d_3 & c_3 & f_3 & & & & & \\
 & e_2 & a_3 & d_4 & c_4 & f_4 & & & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\
 & & & e_{i-2} & a_{i-1} & d_i & c_i & f_i & & \\
 & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & & & & e_{n-4} & a_{n-3} & d_{n-2} & c_{n-2} & f_{n-2} \\
 & & & & & & e_{n-3} & a_{n-2} & d_{n-1} & c_{n-1} \\
 & & & & & & & e_{n-2} & a_{n-1} & d_n
\end{bmatrix}
\begin{bmatrix}
x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_i \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n
\end{bmatrix}
=
\begin{bmatrix}
b_1 \\ b_2 \\ b_3 \\ b_4 \\ \vdots \\ b_i \\ \vdots \\ b_{n-2} \\ b_{n-1} \\ b_n
\end{bmatrix}
$$

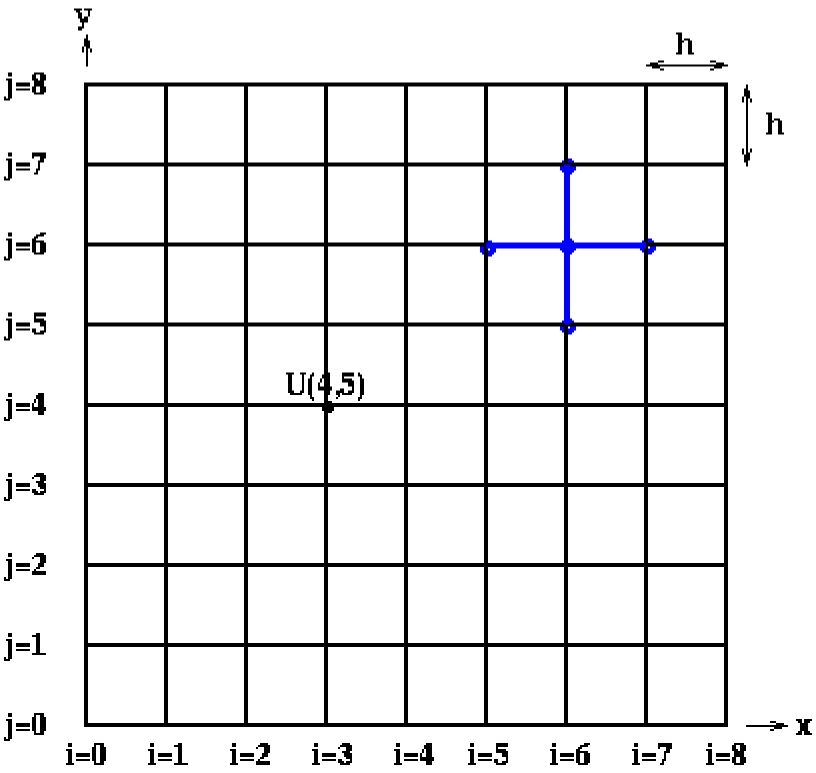
Banded system

- Poisson equation

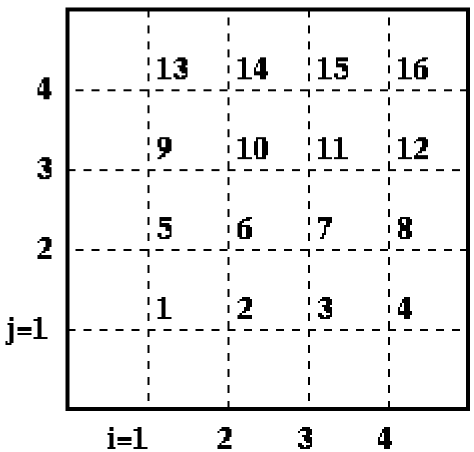
```
d^2 u(x,y)   d^2 u(x,y)
---------- + ----------  = f(x,y)
  d x^2         d y^2

u(x,y) = 0 if (x,y) is on the boundary of Omega
```

- Discretization

4*U(i,j) – U(i–1,j) – U(i+1,j) – U(i,j–1) – U(i,j+1) = b(i,j)



Small-scale example



$$
\left[\begin{array}{cccc|cccc|cccc|cccc}
4 & -1 & & & -1 & & & & & & & & & & & \\
-1 & 4 & -1 & & & -1 & & & & & & & & & & \\
 & -1 & 4 & -1 & & & -1 & & & & & & & & & \\
 & & -1 & 4 & & & & -1 & & & & & & & & \\
\hline
-1 & & & & 4 & -1 & & & -1 & & & & & & & \\
 & -1 & & & -1 & 4 & -1 & & & -1 & & & & & & \\
 & & -1 & & & -1 & 4 & -1 & & & -1 & & & & & \\
 & & & -1 & & & -1 & 4 & & & & -1 & & & & \\
\hline
 & & & & -1 & & & & 4 & -1 & & & -1 & & & \\
 & & & & & -1 & & & -1 & 4 & -1 & & & -1 & & \\
 & & & & & & -1 & & & -1 & 4 & -1 & & & -1 & \\
 & & & & & & & -1 & & & -1 & 4 & & & & -1 \\
\hline
 & & & & & & & & -1 & & & & 4 & -1 & & \\
 & & & & & & & & & -1 & & & -1 & 4 & -1 & \\
 & & & & & & & & & & -1 & & & -1 & 4 & -1 \\
 & & & & & & & & & & & -1 & & & -1 & 4
\end{array}\right]
*
\left[\begin{array}{c}
U(1,1)\\ U(2,1)\\ U(3,1)\\ U(4,1)\\ U(1,2)\\ U(2,2)\\ U(3,2)\\ U(4,2)\\ U(1,3)\\ U(2,3)\\ U(3,3)\\ U(4,3)\\ U(1,4)\\ U(2,4)\\ U(3,4)\\ U(4,4)
\end{array}\right]
=
\left[\begin{array}{c}
b(1,1)\\ b(2,1)\\ b(3,1)\\ b(4,1)\\ b(1,2)\\ b(2,2)\\ b(3,2)\\ b(4,2)\\ b(1,3)\\ b(2,3)\\ b(3,3)\\ b(4,3)\\ b(1,4)\\ b(2,4)\\ b(3,4)\\ b(4,4)
\end{array}\right]
$$