Agenda

9	10/17	Midterm review	10/19	**Midterm**	
10	10/24	Gaussian elimination (2.1)	10/26	Pivoting (2.2)	**HW4**
11	10/31	Pivoting (2.2)	11/2	Structured system (2.3)	
12	11/7	Factorization (8.1)	11/9		
13	11/14	Iterative (8.4)	11/16		**HW5**
14	11/21	**Fall break**	11/23	**Fall break**	
15	11/28	Eigenvalue (8.2)	11/30	Power method (8.3)	
16	12/5	Topic TBD	12/8	Review	**HW6**
	12/12	**2:00-4:45pm Final Exam**			

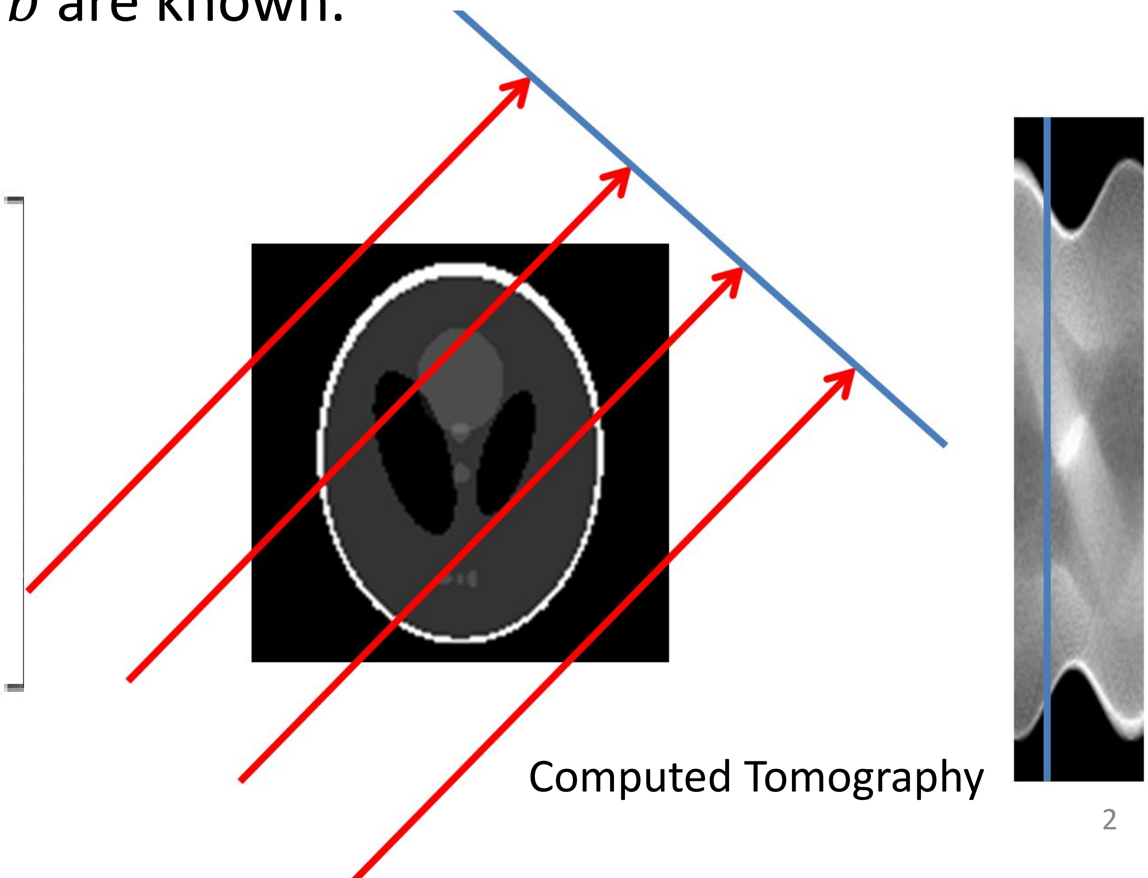
Overview

Solving an algebraic linear system

$$Ax = b$$

for the unknown vector x, when the coefficient matrix A and the right-hand side vector b are known.

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix}$$



Computed Tomography

Overview (cont'd)

- The system may or may hot have a solution, and if it has a solution, it may or may not be unique.
- Here we assume A is a square, invertible (nonsingular) matrix, so there exists a unique solution.
- In pure math, we get the solution
$$x^* = A^{-1}b$$
- But it is advisable to solve the system directly rather than explicitly computing the inverse.
- We will discuss two types of methods
  - Direct approach: Gaussian elimination
  - Iterative approach that generates $x_1, x_2, \cdots \to x^*$

Example

$$\begin{cases} 6x_1 - 2x_2 + 2x_3 + 4x_4 = 16 \\ 12x_1 - 8x_2 + 6x_3 + 10x_4 = 26 \\ 3x_1 - 13x_2 + 9x_3 + 3x_4 = -19 \\ -6x_1 + 4x_2 + x_3 - 18x_4 = -34 \end{cases}$$

$$\begin{cases} 6x_1 - 2x_2 + 2x_3 + 4x_4 = 16 \\ \quad\;\; - 4x_2 + 2x_3 + 2x_4 = -6 \\ \quad\;\; - 12x_2 + 8x_3 + x_4 = -27 \\ \quad\;\; 2x_2 + 3x_3 - 14x_4 = -18 \end{cases}$$

Remarks

- The first equation was not altered, which is called the pivot equation.
- Keep going

$$\begin{cases} 6x_1 - 2x_2 + 2x_3 + 4x_4 = 16 \\ \quad\; - 4x_2 + 2x_3 + 2x_4 = -6 \\ \qquad\qquad\;\; 2x_3 - 5x_4 = -9 \\ \qquad\qquad\qquad\; - 3x_4 = -3 \end{cases}$$

- The above system is said to be in the upper triangular form.
- All the linear systems are equivalent.

Gaussian Elimination

- We just completed the first phase called forward elimination.
- We now proceed with the second phase: back substitution.
- Starting from …

$$
\begin{cases}
6x_1 & - & 2x_2 & + & 2x_3 & + & 4x_4 & = & 16 \\
 & - & 4x_2 & + & 2x_3 & + & 2x_4 & = & -6 \\
 & & & & 2x_3 & - & 5x_4 & = & -9 \\
 & & & & & - & 3x_4 & = & -3
\end{cases}
$$

Generally

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{bmatrix}$$

multiplier

$$2 \le i \le n \qquad \begin{cases} a_{ij} \leftarrow a_{ij} - \left(\dfrac{a_{i1}}{a_{11}}\right) a_{1j} & (1 \leqq j \leqq n) \\ b_i \leftarrow b_i - \left(\dfrac{a_{i1}}{a_{11}}\right) b_1 \end{cases}$$

After the first step

Pivot element

Pivot equation

$$
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
0 & a_{22} & a_{23} & \cdots & a_{2n} \\
0 & a_{23} & a_{33} & \cdots & a_{3n} \\
\vdots & \vdots & \vdots & & \vdots \\
0 & a_{i2} & a_{i3} & \cdots & a_{in} \\
\vdots & \vdots & \vdots & & \vdots \\
0 & a_{n2} & a_{n3} & \cdots & a_{nn}
\end{bmatrix}
\begin{bmatrix}
x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_i \\ \vdots \\ x_n
\end{bmatrix}
=
\begin{bmatrix}
b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_i \\ \vdots \\ b_n
\end{bmatrix}
$$

$3 \leq i \leq n$

$$
\begin{cases}
a_{ij} \leftarrow a_{ij} - \left(\dfrac{a_{i2}}{a_{22}}\right) a_{2j} & (2 \leqq j \leqq n) \\
b_i \leftarrow b_i - \left(\dfrac{a_{i2}}{a_{22}}\right) b_2
\end{cases}
$$

After the kth step

$$
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots & & \cdots & & \cdots & a_{1n} \\
0 & a_{22} & a_{23} & \cdots & & \cdots & & \cdots & a_{2n} \\
0 & 0 & a_{33} & \cdots & & \cdots & & \cdots & a_{3n} \\
\vdots & \vdots & \vdots & \ddots & & & & & \vdots \\
0 & 0 & 0 & \cdots & a_{kk} & \cdots & a_{kj} & \cdots & a_{kn} \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots \\
0 & 0 & 0 & \cdots & a_{ik} & \cdots & a_{ij} & \cdots & a_{in} \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots \\
0 & 0 & 0 & \cdots & a_{nk} & \cdots & a_{nj} & \cdots & a_{nn}
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_k \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_k \\ \vdots \\ b_i \\ \vdots \\ b_n \end{bmatrix}
$$

$$
k+1 \le i \le n \qquad
\begin{cases}
a_{ij} \leftarrow a_{ij} - \left(\dfrac{a_{ik}}{a_{kk}}\right) a_{kj} & (k \leqq j \leqq n) \\[2ex]
b_i \leftarrow b_i - \left(\dfrac{a_{ik}}{a_{kk}}\right) b_k
\end{cases}
$$

Forward elimination

MatLab pseudo-code

Back substitution

$$
\begin{cases}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\
a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\
a_{33}x_3 + \cdots + a_{3n}x_n = b_3 \\
\ddots \cdots \qquad \cdots \qquad \vdots \\
a_{n-1,n-1}x_{n-1} + a_{n-1,n}x_n = b_{n-1} \\
a_{nn}x_n = b_n
\end{cases}
$$

Here coefficients are not the original ones, but are the ones that have been altered by the elimination.

Mathematical analysis

LU factorization

Gaussian elimination transforms a matrix into the product of a unit lower triangular matrix and an upper triangular matrix, i.e.,

$$A = LU$$

Lower Triangular

$$\begin{bmatrix} A00 & A01 & A02 \\ A10 & A11 & A12 \\ A20 & A21 & A22 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ L10 & 1 & 0 \\ L20 & L21 & 1 \end{bmatrix} \begin{bmatrix} U00 & U01 & U02 \\ 0 & U11 & U12 \\ 0 & 0 & U22 \end{bmatrix}$$

Upper Triangular

Pivoting

- Recall $a_{ij} \leftarrow a_{ij} - \left(\frac{a_{ik}}{a_{kk}}\right) a_{kj}$
- We must expect all quantities to be infected with roundoff errors.
- The roundoff error in a_{kj} is multiplied by $\left(\frac{a_{ik}}{a_{kk}}\right)$.
- The small pivot elements would lead to large multipliers and to worse roundoff errors.
- Take-home message: Selecting a large value for pivoting every time (next lecture)

Error analysis

For a linear system $Ax = b$ having the true solution x and a computed solution $\tilde{x}$, we define

- Error vector: $e = \tilde{x} - x$
- Residual vector: $r = A\tilde{x} - b$

For two solutions, how do we evaluate which one is better?

- Look at the residual vector: smaller the better!