Dimension Theory and Kähler Differentials

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Abstract

The Krull dimension of a commutative Noetherian ring is the length of its longest chain of prime ideals. This project covers various ways of calculating the dimension of a local ring, as expounded in Atiyah and MacDonald [AM70], and would go on to tie it in with the transcendence degree of the field of functions on an algebraic variety and with the notion of Kähler differentials, as in [Mat80] and [Har13].

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Chapter 1

Dimension Theory

This chapter closely follows the description of Dimension Theory in [AM70, Ch11]. The equivalence of three different definitions of dimension is establised. Then an account of regular local rings and the transcendence degree of the field of functions on an algebraic variety is given. Extra detail and edits are added when it is appropriate.

1.1 Hilbert Polynomials

Let $A = \oplus_n A_n$ be a Noetherian graded ring and $M = \oplus_n M_n$ a finitely generated graded A-module. A non-zero element of M_n is said to be a homogeneous component of M of degree n. By definition of the direct sum, any element $x \in M$ can be written uniquely as a finite sum $\Sigma_n x_n$, where $x_n \in M_n$ for all $n \geq 0$ and all but a finite number of the x_n are 0. We know from [AM70, 10.7], that A_0 is a Noetherian ring and A is finitely generated as an A_0-algebra. So, viewing A as a graded A-module, we may say A is finitely generated as an A_0-algebra by some homogeneous components $x_1, \dots, x_s$ of degrees $k_1, \dots, k_s$ (all > 0). Again, we can say M is generated by a finite number of homogeneous $m_1, \dots, m_t$ with degrees $r_1, \dots, r_t$. Therefore, every element of M_n is of the form $\Sigma_j f_j(x) m_j$, where $f_j(x)$ is homogeneous of degree $n - r_j$ (0 of $n < r_j$). It follows that M_n is finitely generated as an A_0-module, namely it is generated by all $g_j(x) m_j$ where $g_j(x)$ is a monomial in the x_i of total degree $n - r_j$.

An *additive function* on the class of modules of a given ring is a function λ with values in $\mathbb{Z}$ such that for each exact sequence of modules $0 \to M' \to M \to M'' \to 0$, we have $\lambda(M') - \lambda(M) + \lambda(M'') = 0$.

Example 1.1. A *chain* of submodules of a module M is a sequence of submodules of M such that

$$M = M_0 \supsetneq M_1 \supsetneq \dots \supsetneq M_n = 0$$

The length of this chain is said to be n. A composition series of M is a maximal chain, i.e, a chain in which no more submodules can be added. By [AM70, 6.7], all composition series of a fixed module M have the same length. This number is called *length* of M, and is denoted $l(M)$. This length defined on modules describes an additive function on the class of all modules of a given ring with finite length [AM70, 6.9]

The *Poincare series* of a graded module M is the generating function of $\lambda(M_n)$, i.e., it is the power series

$$P(M, t) = \Sigma_{n=0}^{\infty} \lambda(M_n) t^n \quad \in \quad \mathbb{Z}[[t]]$$

Theorem 1.2 (Hilbert, Serre). *$P(M,T)$ is rational function in t of the form $f(t)/\Pi_{i=1}^{s}(1-t^{k_s})$, where $f(t) \in Z[t]$ and s and k_i correspond to the generators of A.*

Proof. We proceed with induction on s. Start with $s = 0$. We have $A = A_0$, so since M is finitely generated A_0-module, we have $M_n = 0$, for large enough n. Thus $P(M,t)$ is a polynomial in this case.

Now assume $s > 0$. We know multiplication by x_s defines an A-module homomorphism $f_n : M_n \to M_{n+k_s}$ for each $n \geq 0$. Let $K_n = \ker f_n$, $I_n = \text{Im } f_n$ and $L_n = M_{n+k_s}/I_n$. So we have an exact sequence,

$$0 \to K_n \to M_n \to M_{n+k_s} \to L_{n+k_s} \to 0$$

$K = \oplus_n K_n$ is submodule of M so it is a finitely generated A-module. $L = \oplus_n L_n$ is quotient module of M so it is also a finitely generated A-module. Both K and L are annihilated by x_s, hence they are $A_0[x_1, \ldots, x_{s-1}]$-modules. Applying λ to the exact sequence, we get

$$\lambda(K_n) - \lambda(M_n) + \lambda(M_{n+k_s}) - \lambda(L_{n+k_s}) = 0$$

by [AM70, 2.11]. Multiplying by t^{n+k_s}, and summing with respect to n, we get

$$(1 - t^{k_s})P(M,t) = P(L,t) - t^{k_s}P(K,t) + g(t).$$

where $g(t)$ is a residue polynomial we get by balancing the indices for summation. $P(L,t)$ and $P(K,t)$ already have the desired form by assumption as they are $A_0[x_1 \ldots, x_{s-1}]$-modules. So the result follows. $\square$

We denote by $d(M)$ the order of the pole at $P(M,t)$ at $t = 1$, i.e, the largest n such that $P(M,t) = \frac{g(t)}{(1-t)^n}$, where $g(t)$ is a rational function with $g(1) \neq 0$. This number provides a measure of the "size" of M relative to λ. We consider the case when $k_i = 1$ for each i.

Corollary 1.3. *If each $k_i = 1$, then for all large enough n, $\lambda(M_n)$ is a polynomial in n with rational coefficients and degree $d(M) - 1$.*

Proof. We know $\lambda(M_n)$ is the coefficient of t^n in $\frac{f(t)}{(1-t)^s}$. Cancelling powers of $(1-t)$ we may assume $s = d$ and that $f(1) \neq 0$. Suppose $f(t) = \Sigma_{k=0}^{N} a_k t^k$. Let $d(M) = d$. We know

$$(1-t)^{-d} = \Sigma_{k=1}^{\infty} \binom{d+k-1}{d-1} t^k$$

so we extract the coefficient of t^n from the expansion of the product of $f(t)$ and $(1-t)^{-d}$ and we see

$$\lambda(M_n) = \Sigma_{k=0}^{N} a_k \binom{d+n-k-1}{d-1}$$

for all $n \geq N$. Notice the sum on the right-hand side is a polynomial in n (whenever $n \geq N$) with leading term $(\Sigma a_k)n^{d-1}/(d-1)! \neq 0$. $\square$

Remark 1.4. 1. Note that a polynomial $f(x)$ does not need to have integer coefficients to have integer values for each $f(n)$, for example $\frac{1}{2}x(x+1)$. So it is not a problem that the polynomial for $\lambda(M_n)$ has rational coefficients. The polynomial for $\lambda(M_n)$ is usually called the *Hilbert polynomial* of M with respect to λ. Note that this corollary gives us a different way of calculating $d(M)$ when $k_i = 1$ for all i.

2. In the case that $d(M) = 0$, note that the leading term is given by $(\Sigma a_k)n^{-1}/(-1)! \neq 0$. So we adopt the convention that the degree of the zero polynomial is -1.

3. We use this corollary to define this "measure" d on Noetherian local rings.

Proposition 1.5. *If $x \in A_k$ is not a zero divisor in M, then $d(M/xM) = d(M) - 1$.*

Proof. Multiplication by $x \in A_k$ is a module homomorphism for which the kernel is zero. Therefore, $P(K,t) = P(0,t) = 0$ and using the equality for the power series from Theorem 1.2, we have $d(M) - 1 = d(L)$. □

We intend to use the Hilbert, Serre theorem in the case that A_0 is an Artin ring (in particular, a field) and $\lambda(M)$ is the length of the module, which we know is additive.

Example 1.6. Let A_0 be an artin ring and x_i are independent indeterminates. Then A_n is a free A_0-module generated by monomials of degree n, i.e., by $x_1^{m_1} \cdots x_s^{m_s}$, where $\Sigma m_i = n$. There are $\binom{s+n-1}{s-1}$ monomials, so $\lambda(A_n) = \binom{s+n-1}{s-1}$ and hence $P(A,t) = (1-t)^{-d}$.

Proposition 1.7. *Let A be a Noetherian local ring, $\mathfrak{m}$ its maximal ideal, $\mathfrak{q}$ an $\mathfrak{m}$-primary ideal, M a finitely generated A-module, and (M_n) is $\mathfrak{q}$-stable filtration of M. Then*

1. *M/M_n is of finite length, for each $n \geq 0$.*
2. *for all sufficiently large n this length is polynomial $g(n)$ of degree $\leq s$ in n, where s is the least number of generators of $\mathfrak{q}$.*
3. *the degree and leading coefficient of $g(n)$ depend only M and not on the filtration chosen.*

Proof. 1. Let $G(A) = \oplus_n \mathfrak{q}^n/\mathfrak{q}^{n+1}$, $G(M) = \oplus_n M_n/M_{n+1}$. Then $A/\mathfrak{q}$ is Artin ring (it is easy to show there is no other prime ideal $A/\mathfrak{q}$ other than $\mathfrak{m}/\mathfrak{q}$, so dim $A/\mathfrak{q}$ is 0, and we know $A/\mathfrak{q}$ is Noetherian). Also, $G(A)$ is Noetherian and $G(M)$ is finitely generated graded $G(A)$-module. [10.22, AM]. Each $G_n(M) = M_n/M_{n+1}$ is a Noetherian A-module annihilated by $\mathfrak{q}$ because by definition $\mathfrak{q}M_n \subseteq M_{n+1}$. Therefore $G_n(M)$ is Noetherian A/q-module and therefore has finite length since $A/\mathfrak{q}$ is Artin (follows from [6.10, AM]). Now note

$$l_n = l(M/M_n) = \Sigma_{i=1}^n l(M_{i-1}/M_r)$$

can be shown inductively by considering the exact sequence $0 \to M/M_{n-1} \to M/M_n \to M_{n-1}/M_n \to 0$, so it follows that M/M_n is of finite length for each $n \geq 0$.

2. A is Noetherian so every ideal of A is finitely generated. Let $x_1, \ldots, x_s$ generate $\mathfrak{q}$, the images $\bar{x}_i$ in $\mathfrak{q}/\mathfrak{q}^2$ generate $G(A)$ as an $A/\mathfrak{q}$-algebra [10.22 AM], and each $\bar{x}_i$ has degree 1. So by one of our previous propositions, we have $l(M_n/M_{n+1}) = f(n)$, where $f(n)$ is a polynomial in n of degree $\leq s-1$ for all large enough n. Now note that $l_{n+1} - l_n = f(n)$, so l_n is polynomial $g(n)$ of degree $\leq s$ for all large enough n.

3. To prove this we use the following lemma: any two stable $\mathfrak{q}$-filtrations of M have a bounded difference [10.6, AM]. Let (M'_n) be another stable $\mathfrak{q}$-filtration of M and let $g'(n) = l(M/M'_n)$. By the lemma, there exist an integer n_0 such that $M_{n+n_0} \subseteq M'_n$ and $M'_{n+n_0} \subseteq M_n$ for all $n \geq 0$. Therefore it follows that $g(n+n_0) \geq g'(n)$ and $g'(n+n_0) \geq g(n)$. We know g and g' are polynomial for all large enough n, so $g(n)/g'(n)$ tends to 1 as $n \to \infty$. So we know they have the same degree and the same leading coefficient. □

Let $\chi_{\mathfrak{q}}^M$ denote the polynomial corresponding to the filtration $M_n = \mathfrak{q}^n M$, i.e.,

$$\chi_{\mathfrak{q}}^M(n) = l(M/\mathfrak{q}^n M)$$

for all large n. We write $\chi_{\mathfrak{q}}$ for $M = A$ and call it the *characteristic polynomial* of the $\mathfrak{m}$-primary ideal $\mathfrak{q}$.

Corollary 1.8. *For all large n, the length $l(A/\mathfrak{q}^n)$ is a polynomial $\chi_{\mathfrak{q}}$ of degree $\leq s$, where s is the least number of generators of $\mathfrak{q}$.*

Proposition 1.9. *If A, $\mathfrak{m}$, $\mathfrak{q}$ are as above, then* $\deg \chi_{\mathfrak{q}} = \deg \chi_{\mathfrak{m}}$.

Proof. Every ideal contains a power of its radical [7.16, AM], so let $\mathfrak{m} \subseteq \mathfrak{q} \subseteq \mathfrak{m}^r$ for some r. Then we have $\mathfrak{m}^n \subseteq \mathfrak{q}^n \subseteq \mathfrak{m}^{nr}$, which leads to

$$\chi_{\mathfrak{m}}(n) \leq \chi_{\mathfrak{q}}(n) \leq \chi_{\mathfrak{m}}(rn)$$

for all large enough n. We note $\chi_{\mathfrak{m}}(n)/\chi_{\mathfrak{q}}(n)$ tends to 1 as $n \to \infty$ and therefore, the degrees are the same. $\square$

We denote by $d(A)$ the common degree of $\chi_{\mathfrak{q}}$. In the context of Corollary 1.3, we define $d(A) := d(G_{\mathfrak{m}}(A))$. Note that we have expanded our notion of measure from graded noetherian rings to Noetherian local rings.

1.2 Dimension Theory of Noetherian Local rings

In this section we establish the equivalence of three different definition of dimension.

Theorem 1.10 (Dimension Theorem.)**.** *For any Noetherian local ring A, the following are equal:*

1. $\dim A =$ *the Krull dimension of A, i.e., the maximum length of chains of prime ideal in A;*
2. $d(A) =$ *the degree of the characteristic polynomial $\chi_{\mathfrak{m}}(n) = l(A/\mathfrak{m}^n)$;*
3. $\delta(A) =$ *the least number of generators of an m-primary ideal of A.*

A proof of this theorem can be achieved by proving $\delta(A) \geq d(A) \geq \dim A \geq \delta(A)$.

Proposition 1.11. $\delta(A) \geq d(A)$.

Proof. Corollary 1.8 and Proposoition 1.9 give us this result. $\square$

Proposition 1.12. *Let A be a Noetherian local ring, $\mathfrak{m}$ its maximal ideal, and $\mathfrak{q}$ a $\mathfrak{m}$-primary ideal. Let M be a finitely generated A-module, $x \in A$ a non zero-divisor in M and $M' = M/xM$. Then*

$$\deg \chi_{\mathfrak{q}}^{M'} \leq \deg \chi_{\mathfrak{q}}^{M} - 1$$

Proof. Let $N = xM$. Then $N \cong M$ as A-modules since x is a non zero-divisor. Let $N_n = N \cap \mathfrak{q}^n M$ and consider the exact sequence

$$0 \to N/N_n \to M/\mathfrak{q}^n M \to M'/\mathfrak{q}^n M' \to 0$$

Let $g(n) = N/N_n$. Then using Proposition 1.7, we have

$$g(n) - \chi_{\mathfrak{q}}^{M} + \chi_{\mathfrak{q}}^{M'} = 0$$

for all large n. By the Artin-Rees lemma [AM70, 10.9], (N_n) is a stable $\mathfrak{q}$-filtration of N. Since $N \cong M$, Proposition 1.7 implies that $g(n)$ and $\chi_{\mathfrak{q}}^{M}(n)$ have the same leading term. So the result follows. $\square$

Corollary 1.13. *If A is Noetherian local ring, x a non zero divisor in A, then $d(A/(x)) \leq d(A) - 1$.*

Proposition 1.14. $d(A) \geq \dim A$.

Proof. We use induction to prove this. If $d(A) = 0$, then the degree of $\chi_{\mathfrak{m}}$ is 0, which means then $l(A/\mathfrak{m}^n)$ is constant for all large n. This leads to $\mathfrak{m}^n = \mathfrak{m}^{n+1}$ for some n, and by nakayama's lemma we have $\mathfrak{m}^n = 0$. So by [AM70, 8.6], A is an Artin ring and $\dim A = 0$.

Suppose $d > 0$, and let $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_r$ be a chain of prime ideals in A. Let $x \in \mathfrak{p}_1$, $x \notin \mathfrak{p}_0$. Note that x' the image of x in $A' = A/\mathfrak{p}_0$ is not zero. Note that A' is noetherian and an integral domain. Hence, by the previous proposition we have

$$d(A'/(x')) \leq d(A') - 1$$

Also, if $\mathfrak{m}$ is the maximal ideal of A, $\mathfrak{m}'$ the image of $\mathfrak{m}$ in A', then $A'/\mathfrak{m}'^n$ is the homomorphic image of $A/\mathfrak{m}^n$, hence $l(A/\mathfrak{m}^n \geq l(A'/\mathfrak{m}'^n)$. Since n is arbitrary here, we can conclude that $d(A) \geq d(A')$. Consequently

$$d(A'/(x')) \leq d(A) - 1$$

The image of the chain $\mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_r$ in $A'/(x')$ is also a chain of prime ideal. By assumption we know $r - 1 \leq d(A) - 1$, so we have $\dim A = r \leq d(A)$ as required. □

Corollary 1.15. *If A is a Noetherian local ring,* $\dim A$ *is finite.*

Proposition 1.16. *Let A be a Noetherian local ring of dimension d. Then there exists an $\mathfrak{m}$-primary ideal in A generated by d elements. Therefore,* $\dim A \geq \delta(A)$.

Proof. We begin by constructing the set $x_1, \ldots, x_d$ inductively in such a way that every prime ideal containing $(x_1, \ldots, x_i)$ has height $\geq i$ for each i.

Note the condition is trivially true for $i = 0$. Suppose we have $i > 0$ and $x_1, \ldots, x_{i-1}$. We take $\mathfrak{p}_1, \ldots, \mathfrak{p}_\mathfrak{j}$ to be the minimal prime ideals of $(x_1, \ldots, x_{i-1})$ which have height exactly $i - 1$ (these minimal prime ideals may not exist, but this doesn't affect us). We have height $\mathfrak{m} = \dim A = d > i - 1$, so we know $\mathfrak{m} \neq \mathfrak{p}_i$ for any $1 \leq i \leq j$. This implies $\mathfrak{m} \neq \bigcup \mathfrak{p}_i$ by the prime avoidance lemma [AM70, 1.11]. Now proceed by choosing $x_i \in \mathfrak{m}$ such that $x_i \notin \bigcup \mathfrak{p}_i$ and let $\mathfrak{q}$ be a prime containing $(x_1, \ldots, x_i)$. Then $\mathfrak{q}$ contains a minimal prime ideal $\mathfrak{p}$ of $(x_1, \ldots, x_{i-1})$. If $\mathfrak{p} = \mathfrak{p}_i$ for some i, we have $x_i \in \mathfrak{q}$, $x_i \notin \mathfrak{p}_\mathfrak{i}$, hence $\mathfrak{q} \subset \mathfrak{p}$ and therefore, height $\mathfrak{q} \geq i$. If $\mathfrak{p} \neq \mathfrak{p}_i$ for any i, then height $\mathfrak{p} \geq i$ so height $\mathfrak{q} \geq i$. Thus every prime ideal containing $(x_1, \ldots, x_i)$ has height $\geq i$.

Now we show that $(x_1, \ldots, x_i)$ is $\mathfrak{m}$-primary. Take any prime ideal ideal containing $(x_1, \ldots, x_i)$. We know that height $\mathfrak{p} \geq d$. Since A is Noetherian and local, it follows that $\mathfrak{m} = \mathfrak{p}$ because if $\mathfrak{p} \subset \mathfrak{m}$, then height $\mathfrak{p} <$ height $\mathfrak{m} = d$, which is a contradiction. □

So our proof of Dimension Theorem 1.10 is complete.

Example 1.17. if A is the localization of the polynomial ring $k[x_1, \ldots, x_n]$ at the maximal ideal $(x_1, \ldots, x_n)$, then $G_\mathfrak{m}(A)$ is a polynomial ring in n indeterminates and $P(G_\mathfrak{m}(A), t) = (1 - t)^{-n}$ [Example 1.6]. Therefore, from the previous theorem, we $\dim A_\mathfrak{m} = n$.

Corollary 1.18. *If A is noetherian local ring with $\mathfrak{m}$ as its maximal ideal, then* $\dim(\mathfrak{m}/\mathfrak{m}^2) \geq \dim A$.

Proof. If the images of $x_i \in \mathfrak{m}$, $(1 \leq i \leq s)$ in $\mathfrak{m}/\mathfrak{m}^2$ generate $\mathfrak{m}/\mathfrak{m}^2$, then by Nakayama's lemma x_i generate $\mathfrak{m}$. Therefore, by Theorem 1.10, $\dim A =$ height $\mathfrak{m} \leq s = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$. □

Corollary 1.19. *Let A be a Noetherian ring, $x_1, \ldots, x_r \in A$. Then every minimal prime ideal of $(x_1, \ldots, x_r)$ has height $\leq r$.*

Proof. In $A_\mathfrak{p}$, the ideal $(x_1, \ldots, x_r)$ becomes $\mathfrak{p}^e$-primary, hence $r \geq$ height $\mathfrak{p} = \dim A$. □

Corollary 1.20 (Krull's Principal Ideal Theorem)**.** *Let A be a Noetherian ring and x an element in A which is neither zero divisor nor a unit. Then every minimal prime ideal $\mathfrak{p}$ of (x) has height 1.*

Proof. By the previous corollary, we know height $\mathfrak{p} \leq 1$. If height $\mathfrak{p} = 0$, then $\mathfrak{p}$ is minimal prime ideal of the zero ideal. Then from [AM70, 4.7], we know every element of $\mathfrak{p}$ is a zero-divisor. But $x \in \mathfrak{p}$ so we have a contradiction. □

Corollary 1.21. *Let A be a Noetherian local ring, x an element of $\mathfrak{m}$ which is not zero-divisor. Then* $\dim A/(x) = \dim A - 1$

Proof. Let $d = \dim A/(x)$. We know from Theorem 1.10 and Corollary 1.13 that $d \leq \dim A - 1$. For the opposite inequality, we observe that if $x_1, \dots, x_d$ are elements of $\mathfrak{m}$ such that their images in $A/(x)$ generate an $m/(x)$-primary ideal, then $(x, x_1, \dots, x_d)$ is an $\mathfrak{m}$-primary ideal. Using corollary [add no.], we get $d + 1 \geq \dim A$. □

We call $x_1, \dots, x_d$ a *system of parameters* of A when $d = \dim A$ and x_i generate an $\mathfrak{m}$-primary ideal.

Proposition 1.22. *Let $x_1, \dots, x_d$ be a system of parameters for A and let $\mathfrak{q} = (x_1, \dots, x_d)$ be the $\mathfrak{m}$-primary ideal generated by them. Let $f(t_1, \dots, t_d)$ be a homogeneous polynomial of degree s with coefficients in A and assume*

$$f(x_1, \dots, x_d) \in \mathfrak{q}^{s+1}.$$

Then all coeffecients of f are in $\mathfrak{m}$.

Proof. Consider the epimorphism

$$\alpha : A/\mathfrak{q}[t_1, \dots, t_d] \to G_{\mathfrak{q}}(A)$$

which maps t_i to the image $\bar{x_i} = x_i \bmod \mathfrak{q}$. Note that the assumption on f implies that $\bar{f}[t_1, \dots, t_d] = f[t_1, \dots, t_d] \bmod \mathfrak{q}$ is in the kernel of α. Now assume, if possible, that a coefficient of f is a unit. Then $\bar{f}$ is a not zero divisor [AM70, Ch1. Ex3]. So we have

$$d(G_{\mathfrak{q}}(A)) \leq d(A/\mathfrak{q}[t_1, \dots, t_d]/(\bar{f})) = d(A/\mathfrak{q}[t_1, \dots, t_d]) - 1 = d - 1$$

But $d(G_{\mathfrak{q}}(A)) = d$ by Theorem 1.10. So we have a contradiction. □

Corollary 1.23. *If $k \subset A$ is a field isomorphic to the residue field $A/\mathfrak{m}$ and $x_1, \dots, x_d$ is a system of parameters. Then $x_1, \dots, x_d$ are algebraically independent over k.*

Proof. Assume there exist a polynomial f with coefficients in k such that $f(x_1, \dots, x_d) = 0$. If $f \not\equiv 0$, we can write $f = f_s + f'$, where f_s is the homogeneous part of f of degree s and $f_s \not\equiv 0$. We apply the previous proposition to f_s and deduce that it has all coefficients in m. Since f_s has coefficients in k this implies $f_s = 0$. So we have a contradiction and x_i are algebraically independent. □

1.3 Regular Local Rings

Theorem 1.24. *Let A be a Noetherian local ring of dimension d, $\mathfrak{m}$ its maximal ideal, $k = A/\mathfrak{m}$. Then the following are equivalent:*

1. $G_{\mathfrak{m}}(A) \cong k[t_1, \dots, t_d]$
2. $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = d$
3. *m is generated by d elements.*

Proof. 1. $\implies$ 2. Since A is Noetherian, we can take $\mathfrak{m}$ to be finitely generated by $x_1, \dots x_d$. Let $\bar{x_i}$ be the image of x_i, in $\mathfrak{m}/\mathfrak{m}^2$. Then we know $G_{\mathfrak{m}}(A) = (A/\mathfrak{m})[x_1, \dots, x_d] \cong k[t_1, \dots, t_d]$.
2. $\implies$ 3. Follows from Nakayama's Lemma.
3. $\implies$ 1. Let $\mathfrak{m} = (x_1, \dots, x_d)$. Then the map defined in Proposition 1.22 is isomorphism of graded rings. □

A *regular local ring* is one that satisfies the condition above. These rings are also integral domains. This is because of a more general result.

Lemma 1.25. *Let A be a ring, $\mathfrak{a}$ an ideal of A such that $\bigcap_n \mathfrak{a}^n = 0$. Suppose that $G_{\mathfrak{a}}(A)$ is an integral domain. Then A is an integral domain.*

Proof. Let x, y be non-zero elements in A. Since $\bigcap_n \mathfrak{a}^n = 0$, we get the chain $A \supset \mathfrak{a} \supset \mathfrak{a}^2 \supset \ldots \supset 0$. So there exist $r, s \geq 0$ such that $x \in \mathfrak{a}^r$, $x \notin \mathfrak{a}^{r+1}$,$y \in \mathfrak{a}^s$, and $y \notin \mathfrak{a}^{s+1}$. Let $\bar{x}$, $\bar{y}$ be the images of x, y in $\mathfrak{a}^r/\mathfrak{a}^{r+1}$, $\mathfrak{a}^s/\mathfrak{a}^{s+1}$ respectively. Then $\bar{x}, \bar{y} \neq 0$ and so $\bar{x}\bar{y} \neq 0$ in $\mathfrak{a}^{r+s}/\mathfrak{a}^{r+s+1}$ by assumption. This implies $xy \neq 0$. ☐

Proposition 1.26. *A regular local ring is integrally closed.*

Proof. We know from [AM70, 5.13] that a local ring A is integrally closed since $A \cong A_{\mathfrak{m}}$ ☐

Example 1.27. Let $A = k[x_1, \ldots, x_n]$ (k any field, x_i independent indeterminates); let $m = (x_1, \ldots, x_n)$. Then $A_{\mathfrak{m}}$ (the local ring of affine space k^n at the origin) is a regular local ring. This follows from the first definition of regularity since $G_{\mathfrak{m}}(A)$ is a polynomial ring in n variables.

Proposition 1.28. *Let $f \in k[x_1, \ldots, x_n]$ be an irreducible polynomial over an algebraically closed field k. Let $A = k[x_1, \ldots, x_n]/(f)$, and let $\mathfrak{m}$ be the maximal ideal of A corresponding to the point P. Then P is smooth if and only if $A_{\mathfrak{m}}$ is a regular local ring.*

Proof. By Proposition 1.18, we have dim $A_{\mathfrak{m}} = n - 1$. Now

$$\mathfrak{m}/\mathfrak{m}^2 \cong (x_1, \ldots, x_n)/(x_1, \ldots, x_n)^2 + (f)$$

and it has dimension $n - 1$ if and only if $f \notin (x_1, \ldots, x_n)^2$, and therefore, if and only if the variety $f(x) = 0$ is smooth. Therefore, P is smooth, if and only if $A_{\mathfrak{m}}$ is a regular local ring. ☐

1.4 Transcendental Dimension

Let K/k be an arbitrary extension of fields. A subset $S \subset K$ is called *algebraically independent* over k if for all finite sets of elements $a_1, ..., a_n \in S$, no non-zero $f \in k[X_1, ..., X_n]$ satisfies $f(a_1, ..., a_n) = 0$ in K. An algebraically independent subset of $S \subset K$ is called a *transcendental basis* of K/k if K is an algebraic extension of the field $k(S)$. One can show that every field extension has a transcendence basis, and that all transcendence bases have the same cardinality [Mil20, 9.12]. This cardinality is called the *transcendence degree* of the extension and is denoted tr.d K/k.

Let k be an algebraically closed field and let V be an irreducible affine variety over k. We know the coordinate ring is the $A(V)$ is of the form

$$A(V) = k[x_1, \ldots, x_n]/\mathfrak{p}$$

where $\mathfrak{p}$ is a prime ideal. The field of fractions of $A(V)$ is the field of rational functions on V and is denoted by $k(V)$. It is a finitely generated extension of k, so it has a finite transcendence degree. This number is called the *dimension* of V. We know, by Nullstellensatz, each point of V corresponds bijectively to a maximal ideal of $A(V)$. If P has maximal ideal $\mathfrak{m}$, the *local dimension* of P is dim $A(V)_{\mathfrak{m}}$. It turns out that the dimension and the local dimensions of an irreducible affine variety are equal. First we discuss some preliminaries.

Lemma 1.29. *Let $B \subseteq A$ be integral domains with B integrally closed and A integral over B. Let $\mathfrak{m}$ be a maximal ideal of A, and let $\mathfrak{n} = \mathfrak{m} \cap B$. Then $\mathfrak{n}$ is maximal and* $\dim A_{\mathfrak{m}} = \dim B_{\mathfrak{n}}$

Proof. We know $\mathfrak{n}$ is a maximal ideal of B from [AM70, 5.8]. Also, given a strict chain of prime

$$\mathfrak{m} \supset \mathfrak{q}_1 \supset \ldots \supset\subset \mathfrak{q}_0$$

in, using [AM70, 5.9], its intersection with B is strict chain of prime in B. So we have $\dim A_{\mathfrak{m}} \leq \dim B_{\mathfrak{n}}$. Conversely, any strict chain of primes in B can be lifted to a strict chain of primes in A by [AM70, 5.16]. So we have the $A_{\mathfrak{m}} \geq \dim B_{\mathfrak{n}}$ □

Proposition 1.30. *Any algebraically closed field is infinite.*

Proof. A field k is algebraically closed if each non-constant polynomial in $k[x]$ has a root in k. Let k be a finite field and consider the polynomial

$$f(x) = 1 + \prod_{y \in k}(x - y)$$

The coefficients of $f(x)$ lie in the field k, and thus $f(x) \in k[x]$. Of course, $f(x)$ is a non-constant polynomial. Note the for each $y \in k$, $f(y) = 1 \neq 0$. So f has no root in k. So k is not algebraically closed. □

Lemma 1.31 (Nother Normalization Lemma)**.** *Let k be a field and let $A \neq 0$ be a finitely generated k-algebra. Then there exist elements $y_1, \ldots, y_r \in A$ which are algebraically independent over k and such that A is integral over $k[y_1, \ldots, y_r]$.*

Proof. A proof for the case when k is infinite is given. Since we are considering algebraically closed fields, the finite case is not relevant, though still true.

Let $x_1, \ldots, x_n$ generate A as a k-algebra and renumber the x_i so that $x_1, \ldots, x_r$ are algebraically independent over k and each of $x_{r+1}, \ldots, x_n$ is algebraic over $k[x_1, \ldots, x_r]$. Now we proceed by induction on n. If $n = r$ there is nothing to do, so assume $n > r$ and the result is true for $n-1$ generators. The generator x_n is algebraic over $k[x_1, \ldots, x_{n-1}]$, hence there exists a polynomial $f \neq 0$ in n variables such that $f(x_1, \ldots, x_{n-1}, x_n) = 0$. Let F be the homogeneous part of highest degree in f. Since k is infinite, there exist $\lambda_1, \ldots, \lambda_{n-1}$ such that $F(\lambda_1, \ldots, \lambda_{n-1}, 1) \neq 0$. Put $x_i = x_1 -_i x_n$ for $(1 \leq i \leq n-1)$. Show that x_n is integral over the ring $A' = k[x'_i, ..., x'_{n-1}]$, and hence that A is integral over A'. Then apply the inductive hypothesis to A' to complete the proof. □

Theorem 1.32. *Given any irreducible variety V over k the local dimension of V at any point is equal to* $\dim V$.

Proof. Using Nother Normalization, we can find a subset $B = k[y_1, \ldots, y_d]$ of $A(V)$ such that $d = \dim V$ and $A(V)$ is integral over B. We know B is integrally closed so now we apply Lemma 1.29. This reduces the problem to B. Any point on B, i.e an affine space, can be taken as the origin of coordinates. We know from example [1.17] that dim B $= d$, so we are done. □

Corollary 1.33. *For every maximal ideal $\mathfrak{m}$ of $A(V)$, we have*

$$\dim A(V) = \dim A(V)_{\mathfrak{m}}$$

Proof. We have $\dim A(V) = \sup_{\mathfrak{m}} \dim A(V)_{\mathfrak{m}}$ by definition of Krull dimension. But we have shown $\dim A(V)_{\mathfrak{m}} = \dim V$ for all $\mathfrak{m}$ □

1.5 Exercises

Example 1.34. Infinite dimension is possible. Let k be a field and let $A = k[x_1, x_2, \ldots, x_n, \ldots]$ be a polynomial ring over k in a countably infinite set of indeterminates. Let $m_1, m_2, \ldots,$ be an increasing sequence of positive integers such that $m_{i+1} - m_i > m_i - m_{i-1}$ for all $i > 1$. Let $\mathfrak{p}_i = (x_{m_i+1}, x_{m_{i+1}})$, and let S be the complement in A of the union of the ideals $\mathfrak{p}_i$.

Each $\mathfrak{p}_i$ is a prime ideal and therefore the set S is multiplicatively closed [AM70, p38]. The ring $S^{-1}A$ is Noetherian by [AM70, Ch7, Ex9]. Each $S^{-1}\mathfrak{p}_i$ has height equal to $m_{i+1} - m_i$, hence $\dim S^{-1}A = \infty$

Proposition 1.35. *Let A be a Noetherian ring. Then*

$$\dim A[x] = 1 + \dim A,$$

and hence, by induction, on n

$$\dim A[x_1, \ldots, x_n] = n + \dim A.$$

Proof. Let $\mathfrak{p}$ be a prime ideal of height m in A. Then, there exist $a_1, a_2, \ldots, a_m \in \mathfrak{p}$ such that $\mathfrak{p}$ is a minimal prime ideal belonging to $a = (a_1, a_2, \ldots, a_m)$.

By [AM70, Ch4, Ex7], $\mathfrak{p}[x]$ is a minimal prime ideal of $a[x]$ and therefore the height $\mathfrak{p}[x] \leq m$. However, a chain of prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_m = \mathfrak{p}$ induces a chain $\mathfrak{p}_0[x] \subset \mathfrak{p}_1[x] \subset \ldots \subset \mathfrak{p}_m[x] = \mathfrak{p}[x]$ (since A is Noetherian), and therefore the height of $\mathfrak{p}[x]$ is at least m. We deduce that the heights of $\mathfrak{p}$ and $\mathfrak{p}[x]$ are equal. By [AM70, Ch11, Ex6], $\dim A[x] \geq \dim A + 1$ and now we have the opposite inequality too, $\dim A[x] \leq \dim A + 1$. The desired result, $\dim A[x] = \dim A + 1$, follows. So by induction, $\dim A[x_1, \ldots, x_n] = n + \dim A$ follows. $\square$

Chapter 2

Kähler Differentials

This section summarises the sections on Kähler Differentials given in [Har13], [Mat80], [MR89] and [ALA00] to successfully relate Dimension Theory with the notion of Kähler Differentials. Extra details and edits are added when appropriate.

2.1 Definition and Properties

Let A is a k-algebra via a ring homomorphism $f : k \to A$. A k-*derivation*, or a derivation over k is a map $D : A \to M$ such that $D(a+b) = Da + Db$, $D(ab) = aDb + bDa$, and $D \circ f = 0$. The set of all k-derivations of A into M is written as $\mathrm{Der}_k(A, M)$.

Example 2.1. If $A = k[x, y]$ is a polynomial ring, then the partial derivative $\frac{\partial}{\partial x}$ is a $k[y]$-derivative of A to itself.

Proposition 2.2. *Let k be ring and A an k-algebra via homomophism $f : k \to A$. Then there exists an A-module M_0 and a derivation $d \in \mathrm{Der}_k(A, M_0)$ such that for any A-module M and any $D \in \mathrm{Der}_k(A, M)$, there is unique A-module homomorphism $f : M_0 \to M$ such that $D = f \circ d$.*

Proof. Define $\mu : A \otimes_k A \to A$ by $\mu(x \otimes y) = xy$. Note that μ is a homomorphim of k-algebras. Set

$$I = \mathrm{Ker}\ \mu,\ \Omega_{A/k} = I/I^2, \text{ and } B = A \otimes_k A/I^2$$

μ induces $\mu' : B \to A$ with $\mu'(x \otimes y \bmod I^2) = xy$. Define k-algebra homomorphisms $\lambda_i : A \to B$ for $i = 1, 2$ by

$$\lambda_1(a) = a \otimes 1 \bmod I^2 \text{ and } \lambda_2(a) = 1 \otimes a \bmod I^2$$

Observe that the following diagram of k-algebras commutes for both $i = 1, 2$

$$\begin{tikzcd} B \arrow[r, "\mu'"] & A \\ & A \arrow[u, "1_A"'] \arrow[ul, "\lambda_i"] \end{tikzcd}$$

Define $d := \lambda_2 - \lambda_1$. Since $\mu' \circ \lambda_1 = \mu' \circ \lambda_2$, $\mathrm{Im}\ d = \mathrm{Ker}\ \mu' = \Omega_{A/k}$. Therefore, we see that d is map from A to $\Omega_{A/k}$. $\Omega_{A/k}$ can be seen as an A-module via $a \cdot m = \lambda_i(a) \cdot m$, and it is easy to see from here that d is k-derivation from A to $\Omega_{A/k}$.

Now we prove that the pair $(\Omega_{A/k}, d)$ satisfy the requirements of (M_0, d). Given an A-module M, and $D \in \mathrm{Der}_k(A, M)$, define the A-module homomorphism $f : \Omega_{A/k} \to M$ (easy to check that it is one) with $f(x \oplus y) = xDy$. Notice that for any $a \in A$

$$f(da) = f(1 \otimes a \bmod I^2 - a \otimes 1 \bmod I^2) = Da - a \cdot D(1) = Da$$

so $D = f \circ d$. ☐

Remark 2.3. For the reader familiar with the language of categories, the proposition above implies that $M \mapsto \mathrm{Der}_k(A, M)$ represents a representable functor from the category of A-modules $\mathcal{M}_A$ to itself. The representation of this functor is the pair $(\Omega_{A/k}, d)$, and therefore $\mathrm{Der}_k(A, M) \cong \mathrm{Hom}_A(\Omega_{A/k}, M)$. The A-module $\Omega_{A/k}$ is called the *module of Kähler differentials*.

Corollary 2.4. *$\Omega_{A/k}$ is generated as an A-module by $\{dy \mid y \in A\}$.*

Proof. When $x \otimes y \bmod I^2 \in \Omega_{A/k}$, $xy = 0 \bmod I^2$, so $x \otimes y \bmod I^2 = x \otimes 1(1 \otimes x - x \otimes 1) \bmod I^2 = x \cdot dy$. Therefore, given $\omega = \Sigma_i x_i \otimes y_i \in I$, we have $\omega \bmod I^2 = \Sigma_i x_i \cdot dy_i$. ☐

Example 2.5. If A is generated as a k-algebra by a subset $U \subset A$, then $\Omega_{A/k}$ is generated as an A-module by $\{dy \mid y \in U\}$. In particular, if $a \in A$ then there exist $y_i \in U$ and $f \in k[X_1, \ldots, X_n]$ such that $a = f(y_1, \ldots, y_n)$, and the using the definition of d

$$da = \textstyle\sum_{i=1}^n f_i(y_1, \ldots, y_n) dy_i, \text{ where } f_i = \frac{\partial f}{\partial X_i}$$

Hence, if $A = k[x_1, \ldots, x_n]$ is a polynomial ring over k, then $\Omega_{A/k} = Adx_1 \oplus \ldots \oplus Adx_n$ is a free A-module of rank n generated by $dx_1, \ldots, dx_n$, which are linearly independent over A; this follows from the fact that there are $D_i = \frac{\partial}{\partial x_i} \in \mathrm{Der}_k(A, A)$ such that $D_i x_j = \delta_{ij}$.

Example 2.6. Suppose we have a field K of characteristic $p > 0$, a subfield $k \subset K$ such that $K = k(t)$, with $t^p = a \in k$ but $t \notin k$. Then $K \cong k[X]/(X^p - a)$, and since $\frac{\partial (X^p - a)}{\partial x} = 0$, it is easy to see that the ideal $(X^p - a)$ is mapped to it itself under the derivation $\frac{\partial}{\partial x}$ (use the product rule). Therefore, it induces a k-derivation D of K such that $D(f(t)) = 1$.

Suppose further that k is an arbitrary field such that $K^p \subseteq k \subseteq K$ and we are given a p-basis x_i of K/k. The linear independence of dx_i can be established again and therefore, $\Omega_{K/k}$ is a free K-module generated by dx_i.

A transcendental basis S is called a *separating transcendental basis* of K/k if K is a separable algebraic extension of $k(S)$. When a separating transcendental basis of K/k exists, we say K is separably generated over k.

Example 2.7. Let k be a field and K a separably algebraic extension of k. So for any $\alpha \in K$, there is a polynomial $f(X) \in K[X]$ such that $f(\alpha) = 0$ and $f'(\alpha) \neq$. Since $d : K \to \Omega_{K/k}$ is a derivation we have $0 = d(f(\alpha)) = f'(\alpha) d\alpha$, so $d\alpha = 0$. As $\Omega_{K/k}$ is generated by $d\alpha$, $\Omega_{K/k} = 0$

Example 2.8. If the following diagram of ring homomorphisms is commutative,

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \uparrow & & \uparrow \\ k & \longrightarrow & k' \end{array}$$

then there is a natural A-module homomorphism, $\Omega_{A/k} \to \Omega_{A'/k'}$, given by $ad_{A/k}b \mapsto f(a)d_{A'/k'}f(b)$, where f is the homomorphism between A and A'. This homomorphism can be turned into an A'-module homomorphism between $\Omega_{A/k} \otimes_A A' \to \Omega_{A'/k'}$, with $ad_{A/k}b \otimes c \mapsto cf(a)d_{A'/k'}f(b)$. In the case that $A' = A \otimes_k k'$, observe that the A'-module homomorphism is an isomorphism. So we observe that Kähler differentials are compatible with scalar extensions.

Suppose S is a multiplicative subset of A, then we have an isomorphism between $\Omega_{S^{-1}A/k}$ and $S^{-1}\Omega_{A/k}$ given by $d_{S^{-1}A/k}(a/s) \mapsto (1/s)da - (1/s^2)bds$. So we see that Kähler differentials are also compatible with localizations.

Proposition 2.9. *A composition of ring homomorphism $k \xrightarrow{f} A \xrightarrow{g} B$ gives an exact sequence of B-modules*

$$\Omega_{A/k} \otimes_A B \xrightarrow{\alpha} \Omega_{B/k} \xrightarrow{\beta} \Omega_{B/A} \to 0$$

where $\alpha(d_{A/k}a \otimes b) = bd_{B/k}g(a)$ and $\beta(d_{B/k}b) = d_{B/A}b$ for $a \in A$ and $b \in B$.

Proof. First, observe that β is surjective because it maps onto the generators of $\Omega_{B/A}$. Now we want to show $\text{Im } \alpha = \text{Ker } \beta$. Note that the generators of $\Omega_{A/k} \otimes_A B$ are $\{d_{A/k}a \otimes 1 \mid a \in A\}$. The image of these generators under α is $\{d_{B/k}g(a) \mid a \in A\}$. These elements then map to $d_{B/A}g(a)$ which are all zero since $d_{B/A} \circ g = 0$ by definition of a derivation. □

Corollary 2.10. *The map α in the sequence of Kähler differential module given above has a left inverse if and only if for any B-module T, a k-derivation $A \to T$ can be extended to a k-derivation $B \to T$.*

Proof. The map α has a left inverse if and only if the induced map $\alpha' : \text{Hom}_B(\Omega_{B/k}, T) \to \text{Hom}_B(\Omega_{A/k} \otimes_A B, T)$ is surjective for any B-module T. But we have the congruences

$$\text{Hom}_B(\Omega_{A/k} \otimes_A B, T) \cong \text{Hom}_A(\Omega_{A/k},\ \text{Hom}_B(B, T)) \cong \text{Hom}_A(\Omega_{A/k}, T) \cong \text{Der}_k(A, T)$$

We also have the congruence $\text{Der}_k(B, T) \cong \text{Hom}_B(\Omega_{B/k,T})$. So we can translate the surjectivity of α' through these congruences to $\text{Der}_k(B, T) \to \text{Der}_k(A, T)$ □

Note it is clear from the argument given in Proposition 2.9 that if $g : A \to B$ is surjective, then $\Omega_{B/A} = 0$. Also, $B \cong A/\text{ker g}$. This leads to next our proposition.

Proposition 2.11. *Let A be an k-algebra, I be an ideal of A, and $B = A/I$. Then there is a natural exact sequence of B-modules*

$$I/I^2 \xrightarrow{\delta} \Omega_{A/k} \otimes_A B \xrightarrow{\alpha} \Omega_{B/k} \to 0$$

where δ is the B-homomorphism defined by $\delta(x \bmod I^2) = d_{A/k}x \otimes 1$.

Proof. Notice α is clearly surjective. Now to see that $\text{Im } \delta = \text{Ker } \alpha$, we consider the sequence

$$\text{Hom}_B(I/I^2, T) \xleftarrow{\delta'} \text{Hom}_B(\Omega_{A/k} \otimes_A B, T) \xleftarrow{\alpha'} \text{Hom}_B(\Omega_{B/k}, T)$$

where T is an arbitrary B-module. This sequence is exact if and only if $\text{Im } \delta = \text{Ker } \alpha$. [AM70, 2.9]. We know $\text{Hom}_B(\Omega_{A/k} \otimes_A B, T) \cong \text{Hom}_A(\Omega_{A/k}, T)$ from before so we can prove

$$\text{Hom}_B(I/I^2, T) \xleftarrow{\lambda'} \text{Hom}_A(\Omega_{A/k}, T) \xleftarrow{\alpha'} \text{Hom}_B(\Omega_{B/k}, T)$$

where $\lambda(x \bmod I^2) = d_{A/k}x$, is exact. So let $f \in \text{Ker } \lambda'$. then $f \circ \lambda(x \bmod \text{I}^2) = f(d_{A/k}x) = 0$ for every $x \in I$, that is, $f(I) = 0$. But then f can be considered a derivation from $B = A/I$, so $f \in\ \text{Im } \alpha'$. □

Corollary 2.12. *If A is a finitely generated k-algebra or a localization of such an algebra, then $\Omega_{A/k}$ is finitely generated as a A-module.*

Proof. If A is a finitely generated k-module, then A is either a polynomial ring or the quotient of one. Therefore, by the previous Proposition and Example 2.5, we get the exact sequence

$$I/I^2 \longrightarrow \Omega_{F/k} \otimes_F A \longrightarrow \Omega_{A/k} \to 0$$

where $F = \bigoplus_{i=1}^n k$, and we see that $\Omega_{A/k}$ is also finitely generated. We know Kähler differentials are compatible with localizations so if A is the localization of a finitely generated k-algebra, then $\Omega_{A/k}$ is finitely generated again. $\square$

Example 2.13. Let k be a ring, A a k-algebra , and $B = A[X_1, \dots X_n]$. Suppose we have B-module T and a derivation $D \in \mathrm{Der}_k(A, T)$. Then we can extend D to derivation $D' : B \to T$, where D' is obtained by applying D to the coefficients of a polynomial in B. Therefore, the map α from Corollary 2.10 has a left inverse. By the splitting lemma and Example 2.5, we have

$$\Omega_{B/k} \cong (\Omega_{A/k} \otimes_A B) \oplus BdX_1 \oplus \dots \oplus BdX_n$$

Example 2.14. Suppose that $B = k[X_1, \dots, X_n]/(f_1, \dots, f_n) = k[x_1, \dots, x_n]$. Then setting $A = k[X_1, \dots, X_n]$ and using Proposition 2.11, we have

$$\Omega_{B/k} = (\Omega_{A/k} \otimes B)/\Sigma B df_i = F/R$$

where F is the free B-module generated by $dX_1, \dots, dX_n$, and R is the submodule of F generated by df_i.

Example 2.15. Let $B = k[X, Y]/(Y^2 - X^3) = k[x, y]$. Then $\Omega_{B/k}$ is generated by dx and dy module the relation $-3x^2 dx + 2y dy = 0$

2.2 Kähler Differentials and Dimension

Let K be an arbitrary extension of k, and let L be a finitely generated extension of K. We want to compare $\dim_K \Omega_{K/k}$ and $\dim_L \Omega_{L/k}$. Suppose first that $L = K(t)$. Then we have four cases.

1. <u>t is transcendental over K.</u> Then we have $\Omega_{K[t]/k} = (\Omega_{K/k} \otimes_K K[t]) \oplus K[t]dt$ as in Example 2.14. So by localisation $\Omega_{L/k} = (\Omega_{K/k} \otimes_K L) \oplus L dt$, therefore, $\dim_L \Omega_{L/k} = \dim_K \Omega_{K/k} + 1$.

2. <u>t is separably algebraic over K.</u> Let f be the minimal polynomial of t over K. Then $L = K[t] = K[X]/(f)$, $f(t) = 0$, $f'(t) \neq 0$. By Example 2.14, we have $\Omega_{L/k} = ((\Omega_{K/k} \otimes_K L) \oplus L dX)/L\delta f$, where $\delta(f) = df(t) + f'(t)\delta X$. As $f'(t)$ is invertible in L (since L is a field), we have $\Omega_{K/k} \otimes_K L \cong \Omega_{L/k}$. Therefore, we have $\dim_L \Omega_{L/k} = \dim_L \Omega_{K/k} \otimes L = \dim_K \Omega_{K/k}$.

3. <u>$\mathrm{Char}(k) = p, t^p = a \in K, t \notin K, d_{K/k} a = 0$.</u> Then $L = K[t] = K[X]/(X^p - a)$. We have $\delta(X^p - a) = 0$, so $\Omega_{L/k} = (\Omega_{K/k} \otimes_K L) \oplus L dX$ and $\dim_L \Omega_{L/k} = \dim_K \Omega_{K/k} + 1$.

4. <u>$\mathrm{Char}(k) = p, t^p = a \in K, t \notin K, d_{K/k} a \neq 0$.</u> $\delta(X^p - a) \neq 0$ so we get $\dim_L \Omega_{L/k} = \dim_K \Omega_{K/k}$ again as in 2.

Theorem 2.16. *Let K be a extension of a field k, and L a finitely generated extension of K.*

1. *Then* $\dim_L \Omega_{L/k} \geq \dim_K \Omega_{K/k} +$ tr.d. K/k.
2. *The equality holds if* L *is separably generated over* K
3. *If* L *is finitely generated extension of* k*, then* $\dim_L \Omega_{L/k} \geq$ tr.d. L/k*, where equality holds iff* L *is separably generated over* k*. In particular,* $\Omega_{L/k} = 0$*, iff* L *is separably algebraic over* k.

Proof. Any finitely generated extension is obtained by repeating extension listed above. So 1. and 2. are clear now. For 3., the inequality is a special of 1. ($K = k$). Suppose $\Omega_{L/k} = 0$, that is $\dim_L \Omega_{L/k} = 0$. Then $\dim_K \Omega_{K/k} = 0$ for any $k \subseteq K \subseteq L$. Therefore cases 1, 3 and 4 are not possible for L and K. This means L is separably algebraic over k.

Suppose now that $\dim_L =$ tr.d. $L/k = r$. Let $x_1, \ldots, x_r \in L$ be such that $dx_1, \ldots, dx_r$ is basis of $\Omega_{L/k}$ over L. Then we have $\Omega_{L/k(x_1,\ldots,x_r)} = 0$ by Proposition 2.8. So L is separably algebraic over k. Since $r =$ tr.d. L/k, the elements $x_1, \ldots, x_r$ must form a transcendence basis of L over k. $\square$

Then $\dim_K \Omega_{K/k} \geq$ tr.d. K/k*, and equality holds if and only if* K *is separably generated over* k*. (Here* $\dim_K$ *denotes the dimension as a* K*-vector space.)*

Proposition 2.17. *Let* A *be a local ring which contains a field* k *isomorphic to its residue field* A/m*. Then the map* $\delta : m/m^2 \to \Omega_{A/k} \otimes_A k$ *of Proposition 2.11 is an isomorphism.*

Proof. Reinterpreting 2.11 in this context we get the following sequence

$$\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\delta} \Omega_{A/k} \otimes_A k \xrightarrow{\alpha} \Omega_{(A/\mathfrak{m})/k} = \Omega_{k/k} = 0$$

So we know δ is surjective. To prove injectivity, we consider the map $\delta' : \mathrm{Hom}_A(\Omega_{A/k} \otimes A, k) \to \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$ and prove it is surjective. We know from before that $\mathrm{Hom}_k(\Omega_{A/k} \otimes_A k) \cong \mathrm{Der}_k(A, k)$. The image of a k-derivation of B, D under δ' is given by its restriction to $\mathfrak{m}$, and noting that $D(\mathfrak{m}^2) = 0$.

Let $h \in \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$. Given $a \in A$, there is unique representation of a as $a_0 + m$, $a_0 \in k$, $m \in \mathfrak{m}$. Define $D(a) = h(\overline{m})$, where $\overline{m} \in \mathfrak{m}/\mathfrak{m}^2$ is the image of $\overline{m}$. Now we show that D is k-derivation of A. Let $x, y \in A$, and write $x = x_0 + m_x$ and $y = y_0 + m_y$ for unique $x_0, y_0 \in A$ and $m_x, m_y \in \mathfrak{m}$. Then,

$$D(x + y) = h(\overline{m_x} + \overline{m_y}) = h(\overline{m_x}) + h(\overline{m_y}) = D(x) + D(y)$$

$$\begin{aligned} D(xy) &= D(x_0 y_0 + x_0 m_y + y_0 m_x + m_x m_y) = h(\overline{x_0 m_y} + \overline{y_0 m_x} + \overline{m_x m_y}) \\ &= x_0 h(\overline{x_0 m_y}) + y_0 h(\overline{m_x}) = x h(\overline{m_y}) + y h(\overline{m_x}) = x D(y) + y D(x) \end{aligned}$$

Therefore, we see that D is derivation such that $\delta'(D) = h$. $\square$

Theorem 2.18. *Let* A *be a local ring containing a field* k *isomorphic to its residue field. Assume furthermore that* k *is perfect, and that* A *is a localization of a finitely generated* k*-algebra. Then* $\Omega_{A/k}$ *is a free* A*-module of rank equal to* $\dim A$ *if and only if* A *is a regular local ring.*

Proof. ($\Rightarrow$) Assume $\Omega_{A/k}$ is free module of rank $\dim A$. So $\dim_k \Omega_{A/k} \otimes_A k = \dim_A \Omega_{A/k} = \dim A$. Then by the previous proposition $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim A$, which means A is a regular local ring.

($\Leftarrow$) Let A be a regular local ring. So we have by $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim A$, and then by Theorem 2.16, we have $\dim_k \Omega_{A/k} \otimes_A k = d$. If K is the quotient field of A, i.e., $K = S^{-1}A$, where $S = A - 0$, we have the general isomorphism $S^{-1}M \cong M \otimes_A S^{-1}A$ for any A-module M. We get $\Omega_{A/k} \otimes_A K = \Omega_{K/k}$ with the application of Example 2.8. Since k is perfect, K/k must be a separably generated extension [ZS19, 1, Ch. II, Theorem 31, p. 105]. So we apply Proposition 2.9 and Theorem 1.32 and we see that tr.d $K/k = \dim_K \Omega_{K/k} = \dim A$. We know by Corollary 2.12 that $\Omega_{A/k}$ is finitely generated module. So we apply the following lemma to complete our proof. $\square$

Lemma 2.19. *Let A be a noetherian local domain, with residue field k and quotient field K. If M is finitely generate A-module such that* $\dim_k M \otimes_A k = \dim_K M \otimes_A K = r$*, the M is free A-module of rank r.*

Proof. Note first there is natural isomorphism between $M \otimes k$ and $M/\mathfrak{m}M$, where $\mathfrak{m}$ is the maximal ideal of A. Then through [AM70, 2.8], $\dim_k M \otimes_A k = r$ implies M can be generated by r elements. Therefore, it is possible to construct a surjective map $\varphi : A^r \to M$. Let $R = \ker \varphi$. Then we have an exact sequence

$$0 \to R \to A^r \to M \to 0$$

Then we can tensor this sequence with K, to get

$$0 \to R \otimes K \to A^r \otimes K \cong K^r \to M \otimes K \to 0.$$

Since $\dim_K M \otimes_A K = r$, $\dim_K R \otimes_A K = 0$. But R is torsion free so $R = 0$, which turns φ into an isomorphism. □

Corollary 2.20. *Let X be a variety over an algebraically closed field k and $k[X]$ its coordinate ring. Let $p \in X$ and $\mathfrak{m}_p$ its corresponding ideal in $k[X]$. Then $\Omega_{k[x]/k}$ is $k[X]$-module of rank* $\dim X$ *and the localisation of $\Omega_{k[X]/k}$ at $\mathfrak{m}_p$ is a free $k[X]_{\mathfrak{m}_p}$-module if and only if X is smooth.*

Proof. Follows from Theorem 2.18 and Proposition 1.28. □

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