SMFD Assignment-3

Naman Mohan Singh

17th June

Q3. Show that for standard Brownian motion, $\mathbb{E}[W_s W_t] = \min(s,t)$ for $s, t \geq 0$

To prove this, I considered the fact that Brownian motion $\{W_t\}$ has stationary independent increments and is Gaussian with mean zero. Suppose without loss of generality that $0 \leq s \leq t$. Then:

$$W_t = W_s + (W_t - W_s)$$

Taking the expectation of the product:

$$\mathbb{E}[W_s W_t] = \mathbb{E}\left[W_s(W_s + (W_t - W_s))\right] = \mathbb{E}[W_s^2] + \mathbb{E}[W_s(W_t - W_s)]$$

Now since W_s and $(W_t - W_s)$ are independent (from the definition of Brownian motion), and $\mathbb{E}[W_t - W_s] = 0$, the second term becomes:

$$\mathbb{E}[W_s(W_t - W_s)] = \mathbb{E}[W_s] \cdot \mathbb{E}[W_t - W_s] = 0$$

So we are left with:

$$\mathbb{E}[W_s W_t] = \mathbb{E}[W_s^2] = \mathrm{Var}(W_s) = s = \min(s,t)$$

Hence proved.

Q4. Show that $W_t - W_s \sim \mathcal{N}(0, t-s)$ and that Brownian motion has independent increments

Let $0 \leq s < t$. From the properties of Brownian motion:

$$W_t - W_s \sim \mathcal{N}(0, t-s)$$

This is because Brownian increments are normally distributed with mean 0 and variance equal to the length of the interval. Formally:

$$\mathbb{E}[W_t - W_s] = \mathbb{E}[W_t] - \mathbb{E}[W_s] = 0 - 0 = 0$$

$$\mathrm{Var}(W_t - W_s) = \mathrm{Var}(W_t) + \mathrm{Var}(W_s) - 2\mathrm{Cov}(W_t, W_s) = t + s - 2s = t - s$$

So, $W_t - W_s \sim \mathcal{N}(0, t - s)$.

Next, to show independence of increments: consider $X = W_s$ and $Y = W_t - W_s$. These are jointly Gaussian and:

$$\mathrm{Cov}(X, Y) = \mathrm{Cov}(W_s, W_t - W_s) = \mathrm{Cov}(W_s, W_t) - \mathrm{Cov}(W_s, W_s) = s - s = 0$$

Since they are jointly normal with zero covariance, they are independent. Hence, Brownian motion has independent increments.

Q5. Show that $\mathbb{E}[W_t \mid \mathcal{F}_s] = W_s$ for $0 \leq s \leq t$ and conclude that $\{W_t\}$ is a martingale

We want to compute the conditional expectation of W_t given the filtration $\mathcal{F}_s$, which represents all information up to time s. Using the decomposition:

$$W_t = W_s + (W_t - W_s)$$

Taking conditional expectation:

$$\mathbb{E}[W_t \mid \mathcal{F}_s] = \mathbb{E}[W_s + (W_t - W_s) \mid \mathcal{F}_s]$$

Now:

- W_s is known at time s, so $\mathbb{E}[W_s \mid \mathcal{F}_s] = W_s$
- $W_t - W_s$ is independent of $\mathcal{F}_s$, so $\mathbb{E}[W_t - W_s \mid \mathcal{F}_s] = \mathbb{E}[W_t - W_s] = 0$

Hence:

$$\mathbb{E}[W_t \mid \mathcal{F}_s] = W_s$$

This satisfies the definition of a martingale: future expected value equals current value given past history. Therefore, $\{W_t\}$ is a martingale with respect to its natural filtration $\mathcal{F}_t$.