SMFD

Assignment - 2.1

[NAMAN - Q1-2
ASHISH → Q3-4
SHUBHAM → Q5]

(1) state space ∈ {1, 2, 3, 4}

(a)
$$Q = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0.5 & 0.5 & 0 & 0 \\ 2 & 0.25 & 0.75 & 0 & 0 \\ 3 & 0 & 0 & 0.25 & 0.25 \\ 4 & 0 & 0 & 0.75 & 0.25 \end{array}$$

(b) Recurrent states → ~~1, 2, 3, 4 (all)~~ None

Transient states → None

(c) $\pi = \pi Q$, π → stationary Distribution

$[a_1, a_2, a_3, a_4] = [a_1, a_2, a_3, a_4] [Q]$

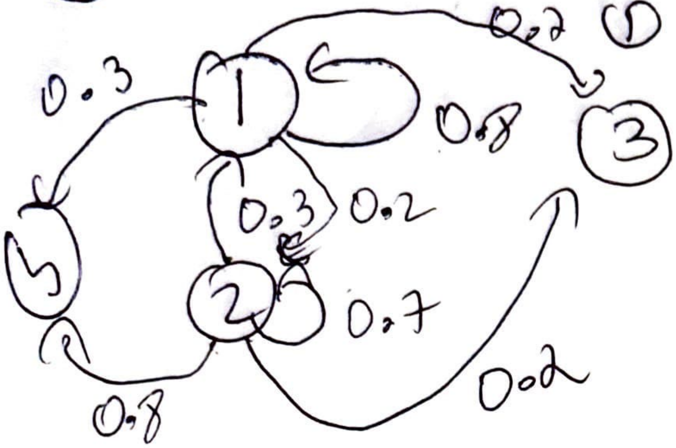
$a_1, a_2, a_3, a_4 = 0.5a_1 + 0.25a_2,\ 0.5a_1 + 0.75a_2,$
$0.25a_3 + 0.75a_4,\ 0.75a_3 + 0.25a_4$ (1)

Solving eq^n (1) & $a_1 + a_2 + a_3 + a_4 = 1$

We get ⇒ $\pi_1 = (0.5, 0.5, 0, 0)$

$\pi_2 = (0, 0, 0.5, 0.5)$

② States ⇒ Team wins ①, Team loses ②, Dinner ③, No Dinner ④



We can also take only ① & ② as states too!

Transition Matrix $\Rightarrow T = \begin{array}{c|cc} & 1 & 2 \\ 1 & 0.8 & 0.2 \\ 2 & 0.3 & 0.7 \end{array}$

Let π_W & π_L be longrun prob. of (1) & (2) resp.

$\Rightarrow \pi_W = 0.8\pi_W + 0.3\pi_L$ $\pi_W + \pi_L = 1$

$\pi_L = 0.2\pi_W + 0.7\pi_L$

$\Rightarrow \boxed{\pi_W = 3/5}$ (a) $\pi_L = 2/5$

Similarly for Dinner, $P(D) = \pi_W \times 0.7 + \pi_L \times 0.2$

$= \frac{3}{5} \times 0.7 + \frac{2}{5} \times 0.2$

$= \boxed{0.5}$ (b)

Now,

E[games until dinner]
games

= E(geometric distribution) → used formula from net!

$= \frac{1}{p} = 2$ games!!

③ **Cat** $\Rightarrow c_1, c_2 \Rightarrow T_C = \begin{bmatrix} 0.2 & 0.8 \\ 0.8 & 0.2 \end{bmatrix}$

(a) π_{c_1} & $\pi_{c_2} \Rightarrow \pi_{c_1} + \pi_{c_2} = 1$ — ①

$0.2\pi_{c_1} + 0.8\pi_{c_2} = \pi_{c_1}$ ②

Solve $\Rightarrow \pi_c = \left[\frac{1}{2}, \frac{1}{2}\right]$

Mouse $m_1, m_2 \Rightarrow M_C = \begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{bmatrix}$

(b) π_{m_1} & $\pi_{m_2} \Rightarrow \pi_{m_1} + \pi_{m_2} = 1$ — ①

$0.7\pi_{m_1} + 0.3\pi_{m_2} = \pi_{m_1}$ — ②

$0.6\pi_{m_1} + 0.4\pi_{m_2} = \pi_{m_2}$ — ③

$\Rightarrow \pi_m = \left[\frac{2}{3}, \frac{1}{3}\right]$

(c)

Z	Cat	Mouse
0	1	1
1	1	2
2	2	1
3	2	2

So, $Z_n = (C_n, M_n)$

$\quad\quad\quad\quad\ \ \downarrow \quad \downarrow$

$\quad\quad\quad\quad\ MC \quad MC$

So pair C_n, M_n is MC.

So, Z_n is also markov chain!

Ans 4 there are total 64 states.
Irreducible & aperiodic therefore must have a unique stationary distribution.

Let the stationary probability of being at square s be $\pi(s)$. then $\sum_{s} \pi(s) = 1$

stationary distribution satisfies

$$\pi(s) = \sum_{s'} \pi(s') \cdot P(s' \to s)$$

we can categories 64 squares into

(A) corners - total 4 — 3 possible moves
(B) edges (non-corners) - total 24 — 5 possible moves
(C) interior square - total 36 — 8 possible moves

Here the stationary distribution is propotional to incoming moves.

let π_A: Prob for a corner
π_B: Prob for edge
π_C: Prob for interior square

$$\pi_A : \pi_B : \pi_C = 3:5:8$$

Total Probability $4\pi_A + 24\pi_B + 36\pi_C = 1$

$$4(3x) + 24(5x) + 36(8x) \Rightarrow x = \frac{1}{420}$$

π_A (corners) $= \frac{3}{420} = \frac{1}{140}$

π_B (edge) $= 5x = \frac{1}{84}$

π_C (interior) $= 8x = \frac{2}{105}$

Ans 5 tick price = 0.01 Rs, it changes every 5 seconds

- transition probabilities
  - Move up: 0.1
  - stay the same: 0.85
  - move down: 0.05

starting price = 120.00 Rs

time from 10:00 AM to 3:00 PM = 5 hours = 18000 seconds

- total number of transitions = $\frac{18000}{5}$ = 3600 time steps

(a) A state in Markov chain is recurrent if the process returns to it infinitely often with probability 1.

we will represent price states as integer ticks (120.00 as 12000, 120.01 as 12001, etc...)

$P(X_{n+1} = X_n + 1) = 0.1$

$P(X_{n+1} = X_n) = 0.85$

$P(X_{n+1} = X_n - 1) = 0.05$

$$\mathbb{E}[X_{n+1} - X_n] = 1\cdot(0.1) + 0\cdot(0.85) + (-1)\cdot(0.05) = 0.05$$

the process tends to move upwards over time., it may never return to previously visited state.

. Hence the stock price is not recurrent.

(b) A stationary distributions π of a Markov chain satisfies.

$$\pi = \pi(P)$$
$$\sum_i \pi(i) = 1$$

- the state space is infinite in one direction
- the drift is positive, meaning the stock is more li[kely] to go up. Due to this asymmetry the chain does not stabilize.

so expected return time to any state may not be finite. so is not positive recurrent and so does not admit a stationary distribution.

c) strike price $K = 125$ Rs
to get a pay off of 5 Rs ($S_T = 130$)
let's say option expires at 1:00 PM
we have total 3 hours = 10800 seconds.
steps $= \frac{10800}{5} = 2160$ time steps.

move up one tick (0.01 Rs) with prob 0.1
stay same with prob 0.85
Move down one tick (-0.01 Rs) with prob 0.05

we need to find the probability that the stock will reaches 130 Rs (or 13000 ticks) before 1:00 PM (within 2160 steps) starting from 12000 ticks

```python
import random

def simulate_once():
    price = 12000  # in ticks
    target = 13000
    steps = 2160

    for _ in range(steps):
        r = random.random()
        if r < 0.1:
            price += 1
        elif r < 0.15:
            price -= 1
        # else stays the same

        if price >= target:
            return 1
    return 0


success_count = 0
trials = 100000
for _ in range(trials):
    success_count += simulate_once()

print(f"Estimated probability of reaching ₹130: {success_count / trials:.4f}")
```

Ans $\rightarrow$ 0