Stamatics

Stochastic Modelling of Financial Derivatives

Assignment - 1.1

Prob : 1–4 ⇒ Naman
and (12)

Prob : 5 - 8 ⇒ Ashish

Prob : 9 - 11 ⇒ Shubham

SMFD

Ass - 1.1 - NAMAN - 230678

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① 1 → envelope in correct letter → Event A

A^C → No letter in correct envelope!

⇒ Out of n! combinations, D_n i.e. the derangement of (n) items, can be used, $D_n = n! - {}^nC_1(n-1)! + {}^nC_2(n-2)! - \dots$

So, $P(A^C) = \frac{D_n}{n!}$, $P(A) = 1 - P(A^C)$

$$\boxed{P(A) = 1 - \frac{D_n}{n!}}$$

For $n=50$,
$$P(A) = 1 - \frac{D_{50}}{50!} = \boxed{0.632}$$

② Brainsteller Problem → Cars and Donkey

→ If I switch → I win

→ If I switch → I win

→ If I switch → I Loose

→ $\boxed{2/3}$ → 2 out of 3 times, i am winning on switching!!

③ (a) $P\left(\frac{A\cap B}{C}\right) = P\left(\frac{A}{B\cap C}\right)\cdot P\left(\frac{B}{C}\right) \rightarrow \boxed{\text{True}}$

$$= \frac{P(A\cap B\cap C)}{\cancel{P(B\cap C)}} \cdot \frac{\cancel{P(B\cap C)}}{P C} = P(A\cap B/C)$$

(b) $P\left(\frac{A\cap B}{C}\right) = P(A/C)\cdot P(B/C) \rightarrow \boxed{\text{False}}$

$(A, B \text{ ind} \Rightarrow P(A\cap B) = P(A)\cdot P(B))$

Also, $P(A\cap B\cap C) = P(A\cap C)\cdot P(B\cap C)$

so, $P\left(\frac{A\cap B}{C}\right) = \frac{P(A\cap B\cap C)}{P(C)} = \frac{P(A\cap C)\cdot P(B\cap C)}{P(C)}$

So, <u>not true!</u> $= P(A/C)\cdot P(B\cap C)$

(c) Note: $P(A|B) = P(A|B\cap D)\cdot P(D/B)$
$+ P(D^c/B)\cdot P(A/D^c\cap B)$

Similarly we write $P(A/B^c)$

and using the result we get $P(A/B^c) > P(A/B)$

$\Rightarrow \boxed{\text{False}}$

④ (a) $\sum x P(X=x) < \infty$, Let $P(X=n) = \frac{k}{n^3}$,

But $\sum x^2 P(X=x) \to \infty$

$\sum n \frac{k}{n^3} = \sum \frac{1}{n^2} \to$ finite

But $\sum n^2\cdot \frac{k}{n^3} = \sum \frac{1}{n} \to$ Infinite!

(b) $\int_{-\infty}^{\infty} x P(X=x) \to$ Finite, But $\int_{\infty}^{\infty} x^2 P(X=x) \to \infty$

(5) N - identical lotteries → prizes 1 to N

we draw (n) with replacement, $M = \max(l_1, \ldots l_n)$

$E(M)$? ⇒ This is a Uniform Distribution,

$$\Rightarrow E(X) = \sum_{k=1}^{N} k \, P(X = k)$$

$$P(X = k) = P(X \le k) - P(X \le k-1)$$

$$= \left(\frac{k}{N}\right)^n - \left(\frac{k-1}{N}\right)^n, \text{ (as uniform distribution)}$$

So $\boxed{E(M) = \sum_{k=1}^{N} k \left[\left(\frac{k}{N}\right)^n - \left(\frac{k-1}{N}\right)^n \right]}$

Ans 6 we need to choose two points X & Y independently on random & need to compute $P(|X-Y|) < 1/3$

d: length of line segment

we will take d=1 here (it will not affect probability)

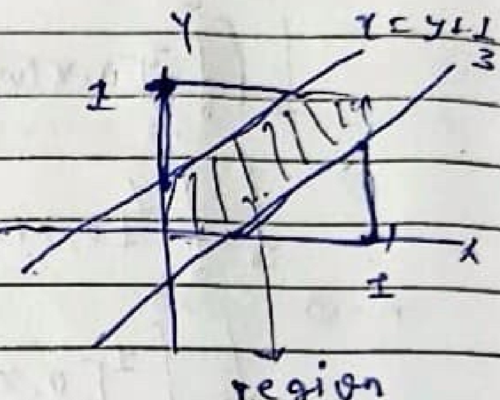
X can take values in [0,1]

Y can take values in [0,1]

total area = 1

area such that $|X-Y| < 1/3$

$\boxed{x < y+\frac{1}{3}}$ & $\boxed{x+\frac{1}{3} \geq y}$



area of region $= 1 - 2\left(\frac{2}{3}\right)\left(\frac{2}{3}\right) \times \frac{1}{2}$

$= 5/9$

$\boxed{P\left(|X-Y| < \frac{1}{3}\right) = 5/9}$ Ans

Ans 7 (a) n+1 total people

the $P_0, P_1 -- P_n$ be the persons &

→ rumor starts from P_0.

→ we want to compute the probability that in r steps no one tells the rumor back to P_0.

→ At each step a person can choose n people excluding the P_0 to tell the rumor

so Probability at that step $= \frac{n}{n+1}$

the choice is uniform & independent at each step.

so the probability that none of r person is P_0

is $= \left(\frac{n}{n+1}\right)^r$

Ans 7

(b) so Initially 1 or one person knows the rumor
after 1st step 2 people and so on
∴ at r step r+1 different people must know
the rumor.

so at any step let's say k, rumor is told to a new
person among n+1−k people

Probability = $\frac{n+1-k}{n}$

Total probability ⇒ $P = \prod_{k=1}^{r} \frac{n+1-k}{n}$

$= \frac{n(n-1)\cdots(n-r+1)}{n^r} = \frac{n!}{(n-r)!\, n^r}$

$\Rightarrow \frac{n!}{(n-r)!\, n^r}$ Ans

Ans 8 $P(A_i^c) = 1 - P(A_i)$

since the events are independent, their complement
are also independent.

for independent events $P\left(\bigcap_{i=1}^{n} A_i^c\right) = \prod_{i=1}^{n} P(A_i^c)$

$= \prod_{i=1}^{n} (1 - P(A_i))$

now using inequality $\boxed{1-x \leq e^{-x}}$ for all $x \in [0,1]$

so $\prod_{i=1}^{n} (1 - P(A_i)) \leq \prod_{i=1}^{n} e^{-P(A_i)} = e^{-\sum_{i=1}^{n} P(A_i)}$

Hence $\boxed{P\left(\bigcap_{i=1}^{n} A_i^c\right) \leq e^{-P(A_1)-P(A_2)\cdots-P(A_n)}}$

Ans 9 let F & G be two distribution functions on $\mathbb{R}$, so the convolution will be

$$(F \times G)(x) = \int_{-\infty}^{\infty} F(x-y)\, dG(y)$$

for a function to be a distribution function

1. It should be non decreasing

since F is non decreasing & G is a probability measure, convolution of F & G gives a non-decreasing function

2. (i) $\lim_{x \to -\infty} H(x) = 0$

as $x \to -\infty$ $F(x-y) \to 0$ so $(F \times G)(x) \to 0$

(ii) $\lim_{x \to \infty} H(x) = 1$

at $x \to \infty$ $F(x-y) \to 1$ so $(F \times G)(x) \to \int dG(y)$
$= 1$

3. Right continuity:-

F is right continous, & convolution with a measure preserves this property.

so $F \times G$ satisfies all properties of a distributive function.

Ans 10 X non negative random variable

$X : \Omega \to [0, \infty) \quad F(x) = \mathbb{P}\{X \le x\}$

~~we know that~~ $\mathbb{E}[X] = \int_{\Omega} X(\omega)\, d\mathbb{P}(\omega)$

$$= \int_{\Omega} \left(\int_0^{X(\omega)} 1\, dx \right) d\mathbb{P}(\omega)$$

$$= \int_{\Omega} \left(\int_0^{\infty} \mathbb{1}_{[0, X(\omega))}(x)\, dx \right) d\mathbb{P}(\omega)$$

$$\Rightarrow \int_{\Omega} \int_0^{\infty} \mathbb{1}_{[0, X(\omega))}(x)\, dx\, d\mathbb{P}(\omega) = \mathbb{E}[X]$$

now

$$\int_{\Omega} \int_0^{\infty} \mathbb{1}_{[0, X(\omega))}(x)\, dx\, d\mathbb{P}(\omega) = \int_0^{\infty} \int_{\Omega} \mathbb{1}_{[0, X(\omega))}(x)\, d\mathbb{P}(\omega)\, dx$$

(fubini's theorem).

$$= \int_0^{\infty} P(X > x)\, dx = \int_0^{\infty} (1 - F(x))\, dx$$

Hence $\mathbb{E}[X] = \int_0^{\infty} (1 - F(x))\, dx$

$-\frac{1}{2\sigma^2}(x^2+\mu^2$ [illegible]$)$

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u be a fixed number in $\mathbb{R}$, $\varphi(x) = e^{ux}$

X be a normal random variable μ (mean) $= \mathbb{E}X$

& S.D(σ) $= [\mathbb{E}(X-\mu)^2]^{1/2}$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

(i) $\mathbb{E}[e^{uX}]$ is moment generating function of a normal distribution.

$$M_X(u) = \int_{-\infty}^{\infty} e^{ux} \cdot \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$e^{ux - \frac{(x-\mu)^2}{2\sigma^2}} = e^{-\frac{1}{2\sigma^2}(x-\mu-u\sigma^2)^2 + \mu u + \frac{1}{2}u^2\sigma^2}$$

$$M_x(u) = e^{\mu u + \frac{1}{2}u^2\sigma^2} \cdot \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu-u\sigma^2)^2} dx}_{\text{normal distribution with shifted mean}}$$

so integral will be 1

$$\boxed{\mathbb{E}[e^{uX}] = e^{\mu u + \frac{1}{2}u^2\sigma^2}}$$

(ii) $\mathbb{E}[\varphi(X)] \geq \varphi(\mathbb{E}X)$

$\varphi(x) = e^{ux} \Rightarrow \mathbb{E}[\varphi(X)] = \mathbb{E}[e^{uX}] = e^{\mu u + \frac{1}{2}u^2\sigma^2}$

$\varphi(\mathbb{E}[X]) = e^{u\mu}$ $\quad \because \mathbb{E}[X] = \mu$

now we can see $\boxed{e^{\mu u + \frac{1}{2}u^2\sigma^2} \geq e^{u\mu}}$ $\because \sigma^2 \geq 0$

Hence $\boxed{\mathbb{E}[e^{uX}] \geq e^{u\mathbb{E}[X]}}$

this proves Jensen's inequality.

```
int dp[101][101];
int mod=1e9+7;
int findpaths(int i, int j){
    if(i==0 && j==0)return 1;
    if(dp[i][j]!=-1)return dp[i][j]%mod; //storing the answers
    int ans=0;                           // using dynamic programming
    if(i==j){ans=findpaths(i,j-1)%mod;}
    else if(j==0 && i!=0){
        ans=findpaths(i-1,j)%mod;
    }
    else{
        ans=findpaths(i-1,j)%mod+findpaths(i,j-1)%mod;
    }
    return dp[i][j]=ans%mod;
}

void solve(){
memset(dp,-1,sizeof(dp));
int n;
cin>>n;
cout<<findpaths(n,n);
}
```

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