4 Show that $L \vee M$ follows from

$$P \wedge Q \wedge R \qquad (Q \rightleftarrows R) \rightarrow (L \vee M)$$

5 Show without constructing truth tables that the following statements cannot all be true simultaneously.

(a) $P \rightleftarrows Q \quad Q \rightarrow R \quad \neg R \vee S \quad \neg P \rightarrow S \quad \neg S$

(b) $R \vee M \quad \neg R \vee S \quad \neg M \quad \neg S$

1-4.2 Rules of Inference

We now describe the process of derivation by which one demonstrates that a particular formula is a valid consequence of a given set of premises. Before we do this, we give two rules of inference which are called rules **P** and **T**.

Rule P: A premise may be introduced at any point in the derivation.
Rule T: A formula S may be introduced in a derivation if S is tautologically implied by any one or more of the preceding formulas in the derivation.

Before we proceed with the actual process of derivation, we list some important implications and equivalences that will be referred to frequently.

Not all the implications and equivalences listed in Tables 1-4.2 and 1-4.3 respectively are independent of one another. One could start with only a minimum number of them and derive the others by using the above rules of inference. Such an axiomatic approach will not be followed. We list here most of the important implications and equivalences and show how some of them are used in a derivation. Those which are used more often than the others are given special names because of their importance.

EXAMPLE 1 Demonstrate that R is a valid inference from the premises $P \rightarrow Q$, $Q \rightarrow R$, and P.

SOLUTION

{1}	(1)	$P \rightarrow Q$	Rule **P**
{2}	(2)	P	Rule **P**
{1, 2}	(3)	Q	Rule **T**, (1), (2), and I_{11} (modus ponens)
{4}	(4)	$Q \rightarrow R$	Rule **P**
{1, 2, 4}	(5)	R	Rule **T**, (3), (4), and I_{11}

The second column of numbers designates the formula as well as the line of derivation in which it occurs. The set of numbers in braces (the first column) for each line shows the premises on which the formula in the line depends. On the right, **P** or **T** represents the rule of inference, followed by a comment showing from which formulas and tautology that particular formula has been obtained. For example, if we follow this notation, the third line shows that the formula in this line is numbered (3) and has been obtained from premises in (1) and (2). The comment on the right says that the formula Q has been introduced using rule **T** and also indicates the details of the application of rule **T**. ////

Table 1-4.2 IMPLICATIONS

I_1	$P \wedge Q \Rightarrow P$	(simplification)
I_2	$P \wedge Q \Rightarrow Q$	
I_3	$P \Rightarrow P \vee Q$	(addition)
I_4	$Q \Rightarrow P \vee Q$	
I_5	$\neg P \Rightarrow P \rightarrow Q$	
I_6	$Q \Rightarrow P \rightarrow Q$	
I_7	$\neg(P \rightarrow Q) \Rightarrow P$	
I_8	$\neg(P \rightarrow Q) \Rightarrow \neg Q$	
I_9	$P, Q \Rightarrow P \wedge Q$	
I_{10}	$\neg P, P \vee Q \Rightarrow Q$	(disjunctive syllogism)
I_{11}	$P, P \rightarrow Q \Rightarrow Q$	(modus ponens)
I_{12}	$\neg Q, P \rightarrow Q \Rightarrow \neg P$	(modus tollens)
I_{13}	$P \rightarrow Q, Q \rightarrow R \Rightarrow P \rightarrow R$	(hypothetical syllogism)
I_{14}	$P \vee Q, P \rightarrow R, Q \rightarrow R \Rightarrow R$	(dilemma)

Table 1-4.3 EQUIVALENCES

E_1	$\neg\neg P \Leftrightarrow P$	(double negation)
E_2	$P \wedge Q \Leftrightarrow Q \wedge P$	(commutative laws)
E_3	$P \vee Q \Leftrightarrow Q \vee P$	
E_4	$(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$	(associative laws)
E_5	$(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$	
E_6	$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$	(distributive laws)
E_7	$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$	
E_8	$\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$	(De Morgan's laws)
E_9	$\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$	
E_{10}	$P \vee P \Leftrightarrow P$	
E_{11}	$P \wedge P \Leftrightarrow P$	
E_{12}	$R \vee (P \wedge \neg P) \Leftrightarrow R$	
E_{13}	$R \wedge (P \vee \neg P) \Leftrightarrow R$	
E_{14}	$R \vee (P \vee \neg P) \Leftrightarrow \mathbf{T}$	
E_{15}	$R \wedge (P \wedge \neg P) \Leftrightarrow \mathbf{F}$	
E_{16}	$P \rightarrow Q \Leftrightarrow \neg P \vee Q$	
E_{17}	$\neg(P \rightarrow Q) \Leftrightarrow P \wedge \neg Q$	
E_{18}	$P \rightarrow Q \Leftrightarrow \neg Q \rightarrow \neg P$	
E_{19}	$P \rightarrow (Q \rightarrow R) \Leftrightarrow (P \wedge Q) \rightarrow R$	
E_{20}	$\neg(P \rightleftarrows Q) \Leftrightarrow P \rightleftarrows \neg Q$	
E_{21}	$P \rightleftarrows Q \Leftrightarrow (P \rightarrow Q) \wedge (Q \rightarrow P)$	
E_{22}	$(P \rightleftarrows Q) \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge \neg Q)$	

EXAMPLE 2 Show that $R \vee S$ follows logically from the premises $C \vee D$, $(C \vee D) \rightarrow \neg H$, $\neg H \rightarrow (A \wedge \neg B)$, and $(A \wedge \neg B) \rightarrow (R \vee S)$.

SOLUTION

{1}	(1)	$(C \vee D) \rightarrow \neg H$	P
{2}	(2)	$\neg H \rightarrow (A \wedge \neg B)$	P
{1, 2}	(3)	$(C \vee D) \rightarrow (A \wedge \neg B)$	T, (1), (2), and I_{13}
{4}	(4)	$(A \wedge \neg B) \rightarrow (R \vee S)$	P
{1, 2, 4}	(5)	$(C \vee D) \rightarrow (R \vee S)$	T, (3), (4), and I_{13}
{6}	(6)	$C \vee D$	P
{1, 2, 4, 6}	(7)	$R \vee S$	T, (5), (6), and I_{11}

The two tautologies frequently used in the above derivations are I_{13}, known as hypothetical syllogism, and I_{11}, known as modus ponens. ////

EXAMPLE 3 Show that $S \vee R$ is tautologically implied by $(P \vee Q) \wedge (P \to R) \wedge (Q \to S)$.

SOLUTION

$\{1\}$	(1)	$P \vee Q$	P
$\{1\}$	(2)	$\neg P \to Q$	T, (1), E_1, and E_{16}
$\{3\}$	(3)	$Q \to S$	P
$\{1, 3\}$	(4)	$\neg P \to S$	T, (2), (3), and I_{13}
$\{1, 3\}$	(5)	$\neg S \to P$	T, (4), E_{18}, and E_1
$\{6\}$	(6)	$P \to R$	P
$\{1, 3, 6\}$	(7)	$\neg S \to R$	T, (5), (6), and I_{13}
$\{1, 3, 6\}$	(8)	$S \vee R$	T, (7), E_{16}, and E_1 ////

EXAMPLE 4 Show that $R \wedge (P \vee Q)$ is a valid conclusion from the premises $P \vee Q$, $Q \to R$, $P \to M$, and $\neg M$.

SOLUTION

$\{1\}$	(1)	$P \to M$	P
$\{2\}$	(2)	$\neg M$	P
$\{1, 2\}$	(3)	$\neg P$	T, (1), (2), and I_{12}
$\{4\}$	(4)	$P \vee Q$	P
$\{1, 2, 4\}$	(5)	Q	T, (3), (4), and I_{10}
$\{6\}$	(6)	$Q \to R$	P
$\{1, 2, 4, 6\}$	(7)	R	T, (5), (6), and I_{11}
$\{1, 2, 4, 6\}$	(8)	$R \wedge (P \vee Q)$	T, (4), (7), and I_9 ////

EXAMPLE 5 Show I_{12}: $\neg Q, P \to Q \Rightarrow \neg P$.

SOLUTION

$\{1\}$	(1)	$P \to Q$	P
$\{1\}$	(2)	$\neg Q \to \neg P$	T, (1), and E_{18}
$\{3\}$	(3)	$\neg Q$	P
$\{1, 3\}$	(4)	$\neg P$	T, (2), (3), and I_{11} ////

We shall now introduce a third inference rule, known as rule CP or rule of conditional proof.

Rule CP ***If we can derive S from R and a set of premises, then we can derive $R \to S$ from the set of premises alone.***

Rule CP is not new for our purpose here because it follows from the equivalence E_{19} which states that

$$(P \wedge R) \rightarrow S \Leftrightarrow P \rightarrow (R \rightarrow S)$$

Let P denote the conjunction of the set of premises and let R be any formula. The above equivalence states that if R is included as an additional premise and S is derived from $P \wedge R$, then $R \rightarrow S$ can be derived from the premises P alone.

Rule CP is also called the *deduction theorem* and is generally used if the conclusion is of the form $R \rightarrow S$. In such cases, R is taken as an additional premise and S is derived from the given premises and R.

EXAMPLE 6 Show that $R \rightarrow S$ can be derived from the premises $P \rightarrow (Q \rightarrow S)$, $\neg R \vee P$, and Q.

SOLUTION Instead of deriving $R \rightarrow S$, we shall include R as an additional premise and show S first.

{1}	(1)	$\neg R \vee P$	P
{2}	(2)	R	P (assumed premise)
{1, 2}	(3)	P	T, (1), (2), and I_{10}
{4}	(4)	$P \rightarrow (Q \rightarrow S)$	P
{1, 2, 4}	(5)	$Q \rightarrow S$	T, (3), (4), and I_{11}
{6}	(6)	Q	P
{1, 2, 4, 6}	(7)	S	T, (5), (6), and I_{11}
{1, 4, 6}	(8)	$R \rightarrow S$	CP

////

These examples show that a derivation consists of a sequence of formulas, each formula in the sequence being either a premise or tautologically implied by formulas appearing before.

In Sec. 1-3.1 we discussed the decision problem in terms of determining whether a given formula is a tautology. We can extend this notion to the determination of validity of arguments. Accordingly, if one can determine in a finite number of steps whether an argument is valid, then the decision problem for validity is solvable.

One solution to the decision problem for validity is provided by the truth table method. Use of this method is often not practical. The method of derivation just discussed provides only a partial solution to the decision problem, because if an argument is valid, then it is possible to show by this method that the argument is valid. On the other hand, if an argument is not valid, then it is very difficult to decide, after a finite number of steps, that this is the case. There are other methods of derivation which do allow one to determine, after a finite number of steps, whether an argument is or is not valid. One such method is described in Sec. 1-4.4, and its computer implementation is given later in Sec. 2-7.

We shall now give some examples of derivation involving statements in

... Theory of Inference for The Predicate Calculus

The method of derivation involving predicate formulas uses the rules of inference given for the statement calculus and also certain additional rules which are required to deal with the formulas involving quantifiers. The rules **P** and **T**, regarding the introduction of a premise at any stage of derivation and the introduction of any formula which follows logically from the formulas already introduced, remain the same. If the conclusion is given in the form of a conditional, we shall also use the rule of conditional proof called **CP**. Occasionally, we may use the indirect method of proof in introducing the negation of the conclusion as an additional premise in order to arrive at a contradiction.

The equivalences and implications of the statement calculus can be used in the process of derivation as before, except that the formulas involved are generalized to predicates. But these formulas do not have any quantifiers in them, while some of the premises or the conclusion may be quantified. In order to use the equivalences and implications, we need some rules on how to eliminate quantifiers during the course of derivation. This elimination is done by *rules of specification* called rules **US** and **ES**. Once the quantifiers are eliminated, the derivation proceeds as in the case of the statement calculus, and the conclusion is reached. It may happen that the desired conclusion is quantified. In this case, we need *rules of generalization* called rules **UG** and **EG**, which can be used to attach a quantifier.

The rules of generalization and specification follow. Here $A(x)$ is used to denote a formula with a free occurrence of x. $A(y)$ denotes a formula obtained by the substitution of y for x in $A(x)$. Recall that for such a substitution $A(x)$ must be free for y.

Rule **US** (Universal Specification) From $(x)A(x)$ one can conclude $A(y)$.

Rule **ES** (Existential Specification) From $(\exists x)A(x)$ one can conclude $A(y)$ provided that y is not free in any given premise and also not free in any prior step of the derivation. These requirements can easily be met by choosing a new variable each time **ES** is used. (The conditions of **ES** are more restrictive than ordinarily required, but they do not affect the possibility of deriving any conclusion.)

Rule **EG** (Existential Generalization) From $A(x)$ one can conclude $(\exists y)A(y)$.

Rule **UG** (Universal Generalization) From $A(x)$ one can conclude $(y)A(y)$ provided that x is not free in any of the given premises and provided that if x is free in a prior step which resulted from use of **ES**, then no variables introduced by that use of **ES** appear free in $A(x)$.

We shall now show, by means of an example, how an invalid conclusion could be arrived at if the second restriction on rule **UG** were not imposed. The other restrictions on **ES** and **UG** are easy to understand.

Let $D(u, v)$: u is divisible by v. Assume that the universe of discourse is $\{5, 7, 10, 11\}$, so that the statement $(\exists u)D(u, 5)$ is true because both $D(5, 5)$

and $D(10, 5)$ are true. On the other hand, $(y)D(y, 5)$ is false because $D(7, 5)$ and $D(11, 5)$ are false. Consider now the following derivation.

$\{1\}$	(1)	$(\exists u)D(u, 5)$	P
$\{1\}$	(2)	$D(x, 5)$	ES, (1)
$\{1\}$	(3)	$(y)D(y, 5)$	UG, (2) (neglecting second restriction)

In step 3 we have obtained from $D(x, 5)$ the conclusion $(y)D(y, 5)$. Obviously x is not free in the premise, and so the first restriction is satisfied. But x is free in step 2 which resulted by use of ES, and that x has been introduced by use of ES and appears free in $D(x, 5)$; hence it cannot be generalized. This is the reason why we obtained a false conclusion from a true premise.

We now give several examples with comments to explain the method of derivation. In the first two examples we use the principles UG and US, but not EG and ES.

EXAMPLE 1 Show that $(x)(H(x) \rightarrow M(x)) \wedge H(s) \Rightarrow M(s)$. Note that this problem is a symbolic translation of a well-known argument known as the "Socrates argument" which is given by:

All men are mortal.

Socrates is a man.

Therefore Socrates is a mortal.

If we denote $H(x)$: x is a man, $M(x)$: x is a mortal, and s: Socrates, we can put the argument in the above form.

SOLUTION

$\{1\}$	(1)	$(x)(H(x) \rightarrow M(x))$	P
$\{1\}$	(2)	$H(s) \rightarrow M(s)$	US, (1)
$\{3\}$	(3)	$H(s)$	P
$\{1, 3\}$	(4)	$M(s)$	T, (2), (3), I_{11}

Note that in step 2 first we remove the universal quantifier. ////

EXAMPLE 2 Show that

$$(x)(P(x) \rightarrow Q(x)) \wedge (x)(Q(x) \rightarrow R(x)) \Rightarrow (x)(P(x) \rightarrow R(x))$$

SOLUTION

$\{1\}$	(1)	$(x)(P(x) \rightarrow Q(x))$	P
$\{1\}$	(2)	$P(y) \rightarrow Q(y)$	US, (1)
$\{3\}$	(3)	$(x)(Q(x) \rightarrow R(x))$	P
$\{3\}$	(4)	$Q(y) \rightarrow R(y)$	US, (3)
$\{1, 3\}$	(5)	$P(y) \rightarrow R(y)$	T, (2), (4), I_{13}
$\{1, 3\}$	(6)	$(x)(P(x) \rightarrow R(x))$	UG, (5)

////

English. We first symbolize the given statements and then use the method of derivation just discussed.

EXAMPLE 7 "If there was a ball game, then traveling was difficult. If they arrived on time, then traveling was not difficult. They arrived on time. Therefore, there was no ball game." Show that these statements constitute a valid argument.

SOLUTION Let

P: There was a ball game.

Q: Traveling was difficult.

R: They arrived on time.

We are required to show that from the premises $P \rightarrow Q$, $R \rightarrow \neg Q$, and R the conclusion $\neg P$ follows. (Complete the rest of the derivation.) ////

EXAMPLE 8 If A works hard, then either B or C will enjoy themselves. If B enjoys himself, then A will not work hard. If D enjoys himself, then C will not. Therefore, if A works hard, D will not enjoy himself.

SOLUTION Let A: A works hard; B: B will enjoy himself; C: C will enjoy himself; D: D will enjoy himself. Show that $A \rightarrow \neg D$ follows from $A \rightarrow B \vee C$, $B \rightarrow \neg A$, and $D \rightarrow \neg C$. Since the conclusion is given in the form of a condition $A \rightarrow \neg D$, include A as an additional premise and show that $\neg D$ follows logically from all the premises including A. Finally, use rule CP to obtain the result. ////

1-4.3 Consistency of Premises and Indirect Method of Proof

A set of formulas $H_1, H_2, \ldots, H_m$ is said to be consistent if their conjunction has the truth value T for some assignment of the truth values to the atomic variables appearing in $H_1, H_2, \ldots, H_m$. If, for every assignment of the truth values to the atomic variables, at least one of the formulas $H_1, H_2, \ldots, H_m$ is false, so that their conjunction is identically false, then the formulas $H_1, H_2, \ldots, H_m$ are called *inconsistent*.

Alternatively, a set of formulas $H_1, H_2, \ldots, H_m$ is inconsistent if their conjunction implies a contradiction, that is,

$$H_1 \wedge H_2 \wedge \cdots \wedge H_m \Rightarrow R \wedge \neg R$$

where R is any formula. Note that $R \wedge \neg R$ is a contradiction, and it is necessary and sufficient for the implication that $H_1 \wedge H_2 \wedge \cdots \wedge H_m$ be a contradiction.

The notion of inconsistency is used in a procedure called *proof by contradiction* or *reductio ad absurdum* or *indirect method of proof*. In order to show that a conclusion C follows logically from the premises $H_1, H_2, \ldots, H_m$, we assume that C is false and consider $\neg C$ as an additional premise. If the new set of premises is inconsistent, so that they imply a contradiction, then the assumption that $\neg C$ is true does not hold simultaneously with $H_1 \wedge H_2 \wedge \cdots \wedge H_m$ being true. Therefore, C is true whenever $H_1 \wedge H_2 \wedge \cdots \wedge H_m$ is true. Thus, C follows logically from the premises $H_1, H_2, \ldots, H_m$.

EXAMPLE 1 Show that $\neg(P \wedge Q)$ follows from $\neg P \wedge \neg Q$.

SOLUTION We introduce $\neg\neg(P \wedge Q)$ as an additional premise and show that this additional premise leads to a contradiction.

{1}	(1)	$\neg\neg(P \wedge Q)$	P (assumed)
{1}	(2)	$P \wedge Q$	T, (1), and E_1
{1}	(3)	P	T, (2), and I_1
{4}	(4)	$\neg P \wedge \neg Q$	P
{4}	(5)	$\neg P$	T, (4), I_1
{1, 4}	(6)	$P \wedge \neg P$	T, (3), (5), I_9 ////

EXAMPLE 2 Show that the following premises are inconsistent.

1 If Jack misses many classes through illness, then he fails high school.
2 If Jack fails high school, then he is uneducated.
3 If Jack reads a lot of books, then he is not uneducated.
4 Jack misses many classes through illness and reads a lot of books.

SOLUTION

E: Jack misses many classes.
S: Jack fails high school.
A: Jack reads a lot of books.
H: Jack is uneducated.

The premises are $E \rightarrow S$, $S \rightarrow H$, $A \rightarrow \neg H$, and $E \wedge A$.

{1}	(1)	$E \rightarrow S$	P
{2}	(2)	$S \rightarrow H$	P
{1, 2}	(3)	$E \rightarrow H$	T, (1), (2), and I_{12}
{4}	(4)	$A \rightarrow \neg H$	P
{4}	(5)	$H \rightarrow \neg A$	T, (4), E_{18}
{1, 2, 4}	(6)	$E \rightarrow \neg A$	T, (3), (5), I_{13}
{1, 2, 4}	(7)	$\neg E \vee \neg A$	T, (6), E_{16}
{1, 2, 4}	(8)	$\neg(E \wedge A)$	T, (7), E_8
{9}	(9)	$E \wedge A$	P
{1, 2, 4, 9}	(10)	$(E \wedge A) \wedge \neg(E \wedge A)$	T, (8), (9), I_9 ////

Proof by contradiction is sometimes convenient. However, it can always be eliminated and replaced by a conditional proof (CP). Observe that

$$P \rightarrow (Q \wedge \neg Q) \Rightarrow \neg P \qquad (1)$$

In the proof by contradiction we show

$$H_1, H_2, \ldots, H_m \Rightarrow C$$

EXAMPLE 3 Show that $(\exists x)M(x)$ follows logically from the premises

$$(x)(H(x) \to M(x)) \quad \text{and} \quad (\exists x)H(x)$$

SOLUTION

$\{1\}$	(1)	$(\exists x)H(x)$	P
$\{1\}$	(2)	$H(y)$	ES, (1)
$\{3\}$	(3)	$(x)(H(x) \to M(x))$	P
$\{3\}$	(4)	$H(y) \to M(y)$	US, (3)
$\{1, 3\}$	(5)	$M(y)$	T, (2), (4), I_{11}
$\{1, 3\}$	(6)	$(\exists x)M(x)$	EG, (5)

Note that in step 2 the variable y is introduced by ES. Therefore a conclusion such as $(x)M(x)$ could not follow from step 5 because it would violate the rules given for UG. ////

EXAMPLE 4 Prove that

$$(\exists x)(P(x) \land Q(x)) \Rightarrow (\exists x)P(x) \land (\exists x)Q(x)$$

SOLUTION

$\{1\}$	(1)	$(\exists x)(P(x) \land Q(x))$	P
$\{1\}$	(2)	$P(y) \land Q(y)$	ES, (1), y fixed
$\{1\}$	(3)	$P(y)$	T, (2), I_1
$\{1\}$	(4)	$Q(y)$	T, (2), I_2
$\{1\}$	(5)	$(\exists x)P(x)$	EG, (3)
$\{1\}$	(6)	$(\exists x)Q(x)$	EG, (4)
$\{1\}$	(7)	$(\exists x)P(x) \land (\exists x)Q(x)$	T, (4), (5), I_9

////

It is instructive to try to prove the converse which does not hold. The derivation is

(1)	$(\exists x)P(x) \land (\exists x)Q(x)$	P
(2)	$(\exists x)P(x)$	T, (1), I_1
(3)	$(\exists x)Q(x)$	T, (1), I_2
(4)	$P(y)$	ES, (2)
(5)	$Q(z)$	ES, (3)

Note that in step 4 y is fixed, and it is no longer possible to use that variable again in step 5.

EXAMPLE 5 Show that from

(a) $(\exists x)(F(x) \land S(x)) \to (y)(M(y) \to W(y))$

(b) $(\exists y)(M(y) \land \neg W(y))$

the conclusion $(x)(F(x) \to \neg S(x))$ follows.

SOLUTION

{1}	(1)	$(\exists y)(M(y) \land \neg W(y))$	P
{1}	(2)	$M(z) \land \neg W(z)$	ES, (1)
{1}	(3)	$\neg(M(z) \to W(z))$	T, (2), E_{17}
{1}	(4)	$(\exists y)\neg(M(y) \to W(y))$	EG, (3)
{1}	(5)	$\neg(y)(M(y) \to W(y))$	E_{26}, (4)
{6}	(6)	$(\exists x)(F(x) \land S(x)) \to (y)(M(y) \to W(y))$	P
{1, 6}	(7)	$\neg(\exists x)(F(x) \land S(x))$	T, (5), (6), I_{12}
{1, 6}	(8)	$(x)\neg(F(x) \land S(x))$	T, (7), E_{25}
{1, 6}	(9)	$\neg(F(x) \land S(x))$	US, (8)
{1, 6}	(10)	$F(x) \to \neg S(x)$	T, (9), E_9, E_{16}, E_{17}
{1, 6}	(11)	$(x)(F(x) \to \neg S(x))$	UG, (10)

EXAMPLE 6 Show that

$$(x)(P(x) \lor Q(x)) \Rightarrow (x)P(x) \lor (\exists x)Q(x)$$

SOLUTION We shall use the indirect method of proof by assuming $\neg((x)P(x) \lor (\exists x)Q(x))$ as an additional premise.

{1}	(1)	$\neg((x)P(x) \lor (\exists x)Q(x))$	P (assumed)
{1}	(2)	$\neg(x)P(x) \land \neg(\exists x)Q(x)$	T, (1), E_9
{1}	(3)	$\neg(x)P(x)$	T, (2), I_1
{1}	(4)	$(\exists x)\neg P(x)$	T, (3), E_{26}
{1}	(5)	$\neg(\exists x)Q(x)$	T, (2), I_2
{1}	(6)	$(x)\neg Q(x)$	T, (5), E_{25}
{1}	(7)	$\neg P(y)$	ES, (4)
{1}	(8)	$\neg Q(y)$	US, (6)
{1}	(9)	$\neg P(y) \land \neg Q(y)$	T, (7), (8), I_9
{1}	(10)	$\neg(P(y) \lor Q(y))$	T, (9), E_9
{11}	(11)	$(x)(P(x) \lor Q(x))$	P
{11}	(12)	$P(y) \lor Q(y)$	US, (11)
{1, 11}	(13)	$\neg(P(y) \lor Q(y)) \land (P(y) \lor Q(y))$	T, (10), (12), I_9, contradiction

////

1-6.5 Formulas Involving More Than One Quantifier

So far we have considered only those formulas in which the universal and existential quantifiers appear singly. We shall now consider cases in which the quanti-