Models of Stochastic Processes

1. Wiener Process

1.1 Non correlated Brownian Motion

A Weiner process, also called Brownian motion, is a model for stochastic processes (for more info: Wikipedia).

For every Wiener process W_t, we can write:

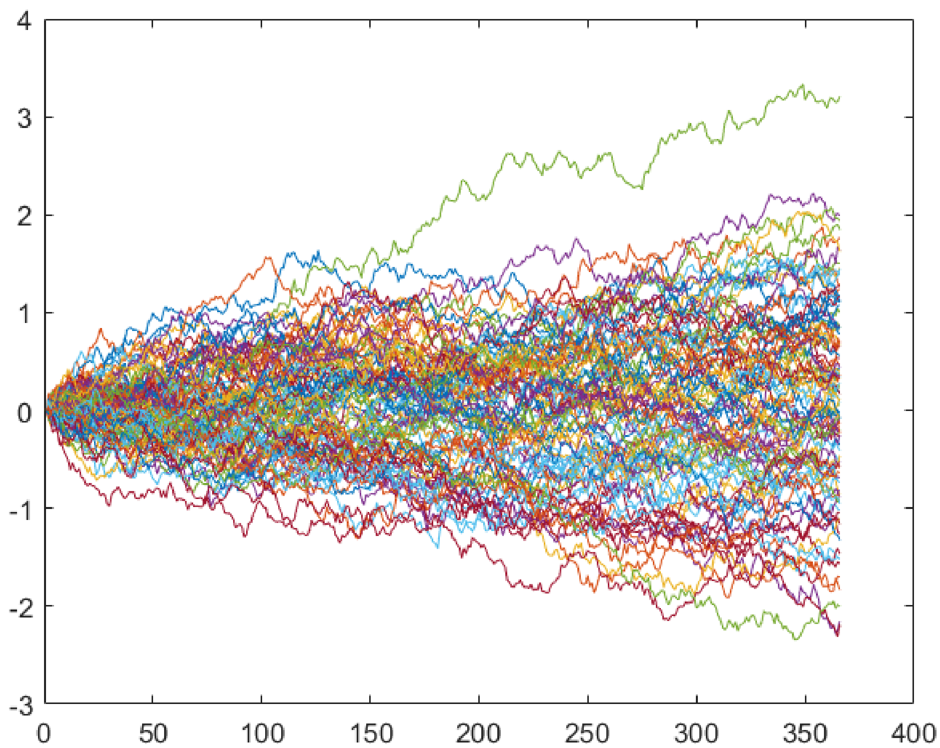
$$\begin{cases} W_{t+1} = W_t + N(0, \Delta t) \\ W_0 = 0 \end{cases}$$

Looking at some simulations of a Weiner process in 4 dimensions, we have:

```
Nb_trj = 10000;
N = 366;
Te = 1/(N-1);
t = 0:Te:1;
dim = 4;
M = zeros(dim,Nb_trj, N);
randM = normrnd(0,sqrt(Te),dim,Nb_trj,N);
for time=2:1:N
   M(:,:,time) = M(:,:,time-1) + randM(:,:,time);
end
```

Plotting the movement in one of four dimensions:

```
figure
for i = 1:100
   temp = M(1,i,:);
   plot(temp(:))
   hold on
end
```



Next, we confirm the mean and varaiance of the process.

```
points = [10 100 300];
m=zeros(dim, 3);
sigma=zeros(dim, 3);
for d = 1:1:4
    for i=1:1:3
        m(d,i) = mean(M(d,:,points(i)));
        sigma(d,i) = std(M(d,:,points(i)));
    end
end
m, sigma
```

```
m = 4×3
   -0.0059   -0.0104   -0.0047
   -0.0025   -0.0103   -0.0040
    0.0003   -0.0006   -0.0060
    0.0016    0.0022   -0.0116
sigma = 4×3
    0.1581    0.5171    0.8974
    0.1588    0.5196    0.9119
    0.1578    0.5140    0.9060
    0.1549    0.5170    0.9063
```

We see that the mean value at the chosen onstants is close to 0, and can be predicted fromt he theory. The standard deviation for each instant $k\Delta t$ should be $\sqrt{k\Delta t}$, since each instance of W_t is independant. TO further confirm independance between the dimensions, we can calculate the cross-correlation between different dimensions.

```
corMatrix_10_12 = corrcoef(M(1,:,10),M(2,:,10));
corMatrix_10_13 = corrcoef(M(1,:,10),M(3,:,10));
corMatrix_10_14 = corrcoef(M(1,:,10),M(4,:,10));
corMatrix_10_23 = corrcoef(M(2,:,10),M(3,:,10));
corMatrix_10_24 = corrcoef(M(2,:,10),M(3,:,10));
corMatrix_10_34 = corrcoef(M(3,:,10),M(4,:,10));

corMatrix_100_12 = corrcoef(M(1,:,100),M(2,:,100));
corMatrix_100_13 = corrcoef(M(1,:,100),M(3,:,100));
corMatrix_100_14 = corrcoef(M(1,:,100),M(4,:,100));
corMatrix_100_23 = corrcoef(M(2,:,100),M(3,:,100));
corMatrix_100_24 = corrcoef(M(2,:,100),M(3,:,100));
corMatrix_100_34 = corrcoef(M(3,:,100),M(4,:,100));

corMatrix_300_12 = corrcoef(M(1,:,300),M(2,:,300));
corMatrix_300_13 = corrcoef(M(1,:,300),M(3,:,300));
corMatrix_300_14 = corrcoef(M(1,:,300),M(4,:,300));
corMatrix_300_23 = corrcoef(M(2,:,300),M(3,:,300));
corMatrix_300_24 = corrcoef(M(2,:,300),M(3,:,300));
corMatrix_300_34 = corrcoef(M(3,:,300),M(4,:,300));

corMatrix_12 = (corMatrix_10_12 + corMatrix_100_12 + corMatrix_300_12)/3;
corMatrix_13 = (corMatrix_10_13 + corMatrix_100_13 + corMatrix_300_13)/3;
corMatrix_14 = (corMatrix_10_14 + corMatrix_100_14 + corMatrix_300_14)/3;
corMatrix_23 = (corMatrix_10_23 + corMatrix_100_23 + corMatrix_300_23)/3;
corMatrix_24 = (corMatrix_10_24 + corMatrix_100_24 + corMatrix_300_24)/3;
corMatrix_34 = (corMatrix_10_34 + corMatrix_100_34 + corMatrix_300_34)/3;

corMatrix_12, corMatrix_13, corMatrix_14, corMatrix_23, corMatrix_24, corMatrix_34
```

```
corMatrix_12 = 2×2
    1.0000   -0.0012
   -0.0012    1.0000
corMatrix_13 = 2×2
    1.0000   -0.0061
   -0.0061    1.0000
corMatrix_14 = 2×2
    1.0000   -0.0042
   -0.0042    1.0000
corMatrix_23 = 2×2
    1.0000    0.0096
    0.0096    1.0000
corMatrix_24 = 2×2
    1.0000    0.0096
    0.0096    1.0000
corMatrix_34 = 2×2
    1.0000    0.0014
    0.0014    1.0000
```

1.2 Correlated Brownian Motion

Let us assume that we want a specific correlation between each of the dimensions of the brownian motion. For the demonstration, let's fix that as:

$$\Sigma = \begin{bmatrix} 1 & 0.2 & 0.8 & 0.5 \\ 0.2 & 1 & 0.3 & 0.2 \\ 0.8 & 0.3 & 1 & 0.9 \\ 0.5 & 0.2 & 0.9 & 1 \end{bmatrix}$$

By Choleski decomposition, we can find a triangular matrix $U\ s.t. \Sigma = U^T U$. To have a Weiner process W_t^{new} whose componants are correlated, we have: $W_t^{\text{new}} = \mathrm{U}\mathrm{W}_t$.

```
CorM = [1 0.2 0.8 0.5; 0.2 1 0.3 0.2; 0.8 0.3 1 0.9; 0.5 0.2 0.9 1];
U = chol(CorM);

M_dash = zeros(dim,Nb_trj, N);
for time=2:1:N
   M_dash(:,:,time) = U'*M(:,:,time);
end
```

```
m_dash=zeros(dim, 3);
sigma_dash=zeros(dim, 3);
for d = 1:1:4
    for i=1:1:3
        m_dash(d,i) = mean(M_dash(d,:,points(i)));
        sigma_dash(d,i) = std(M_dash(d,:,points(i)));
    end
end
m_dash, sigma_dash
```

```
m_dash = 4×3
   -0.0059   -0.0104   -0.0047
   -0.0036   -0.0122   -0.0049
   -0.0049   -0.0101   -0.0078
   -0.0026   -0.0062   -0.0102
sigma_dash = 4×3
    0.1581    0.5171    0.8974
    0.1586    0.5202    0.9102
    0.1576    0.5163    0.8969
    0.1575    0.5153    0.8989
```

```
corMatrix_300_12_dash = corrcoef(M_dash(1,:,300),M_dash(2,:,300))
```

```
corMatrix_300_12_dash = 2×2
    1.0000    0.1910
    0.1910    1.0000
```

```
corMatrix_300_13_dash = corrcoef(M_dash(1,:,300),M_dash(3,:,300))
```

```
corMatrix_300_13_dash = 2×2
    1.0000    0.7935
    0.7935    1.0000
```

```
corMatrix_300_14_dash = corrcoef(M_dash(1,:,300),M_dash(4,:,300))
```

```
corMatrix_300_14_dash = 2×2
    1.0000    0.4869
```

```
    0.4869    1.0000
```

```
corMatrix_300_23_dash = corrcoef(M_dash(2,:,300),M_dash(3,:,300))
```

```
corMatrix_300_23_dash = 2×2
    1.0000    0.3035
    0.3035    1.0000
```

```
corMatrix_300_24_dash = corrcoef(M_dash(2,:,300),M_dash(3,:,300))
```

```
corMatrix_300_24_dash = 2×2
    1.0000    0.3035
    0.3035    1.0000
```

```
corMatrix_300_34_dash = corrcoef(M_dash(3,:,300),M_dash(4,:,300))
```

```
corMatrix_300_34_dash = 2×2
    1.0000    0.8977
    0.8977    1.0000
```

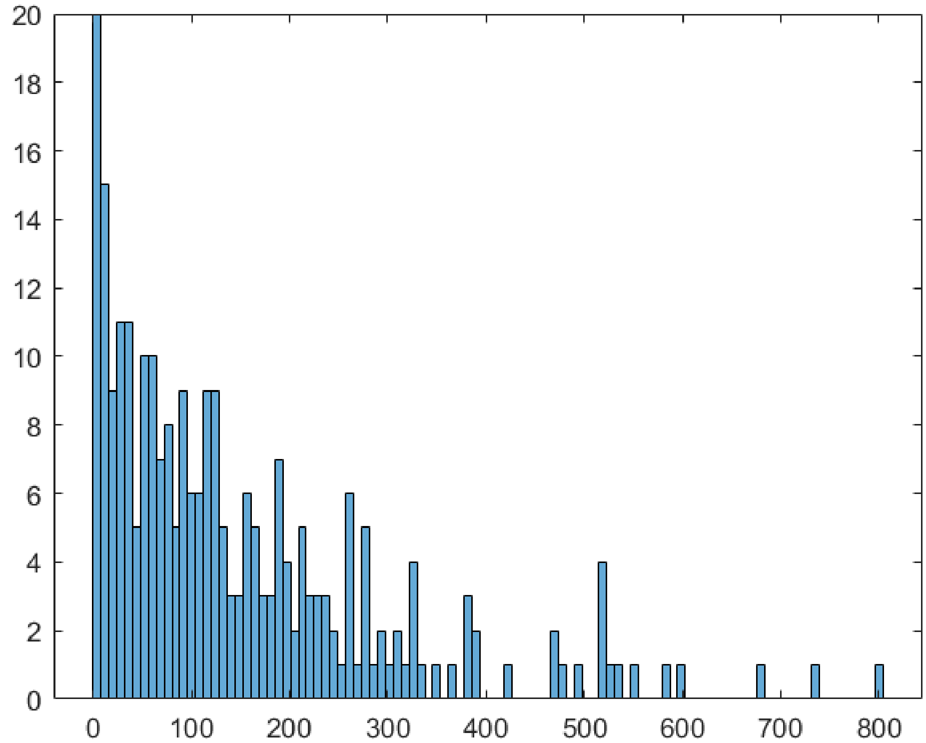
2. Poisson Process

A Poisson process is one where the time interval betweent he arrival of two clients is distributed exponentially

$$\Pr(\tau \geq t) = \exp(-\lambda t) \quad t \geq 0$$

We can emulate the time instants of τ which follows the exponential distribution by using the inverse transform sampling. Given a uniform random variable $u \in [0, 1]$, we have $\tau = -\frac{1}{\lambda}\ln(1-u)$.

```
lambda = 1/140;%consider ecart type
n = round(3600*10*lambda - 1);
tau_exp = gen_exp(n,lambda);
BIN = 100;

figure
histogram(tau_exp, BIN)
```



Estimating the number of insatisfied clients

To calculate the number of insatisfied clients, we assume that we have on average one request every $\frac{1}{\lambda}$ seconds. We can further model the time for treating a single request using a gaussian distribution, with mean of 120 seconds and a standard deviation of 20 seconds. The aim of this section is to calculate the fraction of people for which the waiting time is greater than 10 minutes in a period of 10 hours. For getting better results, we will use Monte Carlo sampling.

```
t_traitement = normrnd(120,20,1,n);
iterations = 1000;
irritated_clients_sum = 0;
for it = 1:1:iterations
  [t_attente, t_fin] = attente(tau_exp, t_traitement);
  irritated_clients_sum  = irritated_clients_sum + numel(t_attente(t_attente > 10*60));
end
irritated_clients = irritated_clients_sum/iterations
```

```
irritated_clients = 8
```

3. Markov Chains

Given $\{Z_n\}$ an IID random variable with values in $\{+1, -1\}$ with the probability distribution

$$\Pr\{Z_n = 1\} = p$$

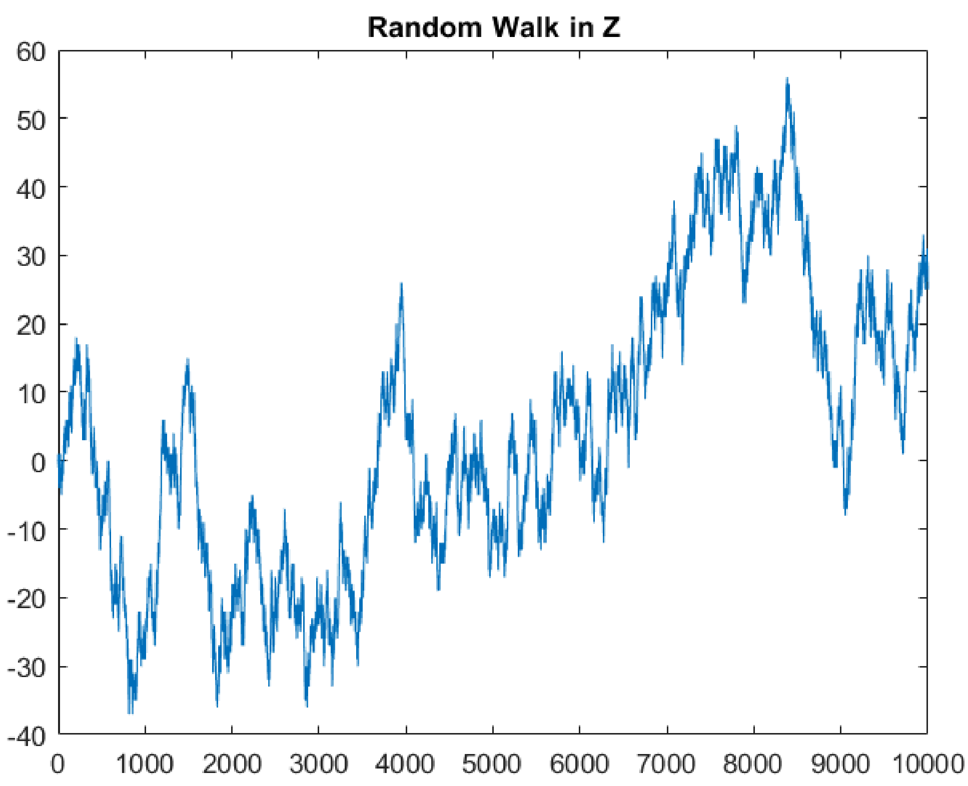
Given another random variable X_0 in (Z) and independant of $\{Z_n\}$. We define the process $\{X_{n+1}\} = X_n + Z_n$. We observe that the value of X_{n+1} is only dependant on the current value X_n and not the preceeding values. This is the property of a Markov Chain. We can aslo show this this Markov chain is homogenous:

$$\Pr\{x_{n+1} = i + z \mid X_n = i\} = \Pr\{Z = z\}$$

Since the distribution of Z_n does not depend on *n*, the Markov chain is homogenous.

We can simulate the Movement of a Markov chain using the code below. We observe that if $p \neq 0.5$, there is either a tendenacy to increase (for $p > 0.5$) or decrease (for $p < 0.5$) if *n* is sufficiently large.

```
n = 10000;
x = zeros(1,n);
p = 0.5;
for i = 2:n
    if rand()<p
        x(i) = x(i-1) + 1;
    else
        x(i) = x(i-1) - 1;
    end
end

figure
plot(x)
title('Random Walk in Z')
```



4. Diffusion

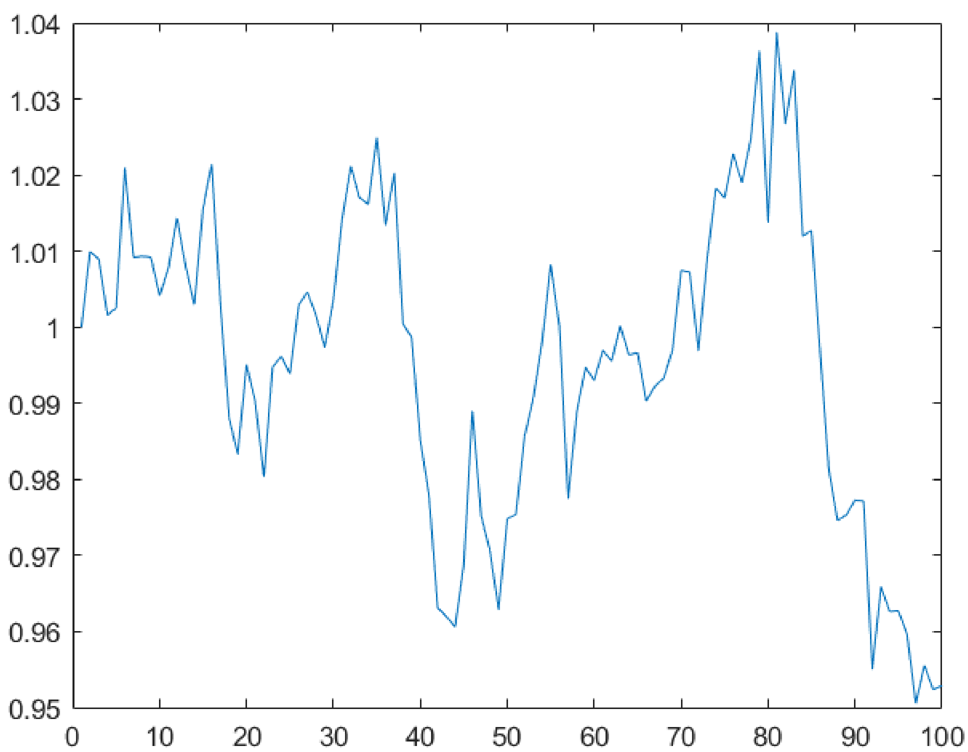
Given a standard monodimensional brownian motion strating from the origin, the process is a solution for the equation:

$$\mathrm{dX}_t = a(b - X_t)\mathrm{dt} + \sigma \mathrm{dW}_t$$

$$\Rightarrow X_{t_n} = \frac{1}{1 + a\delta t} X_{t_{n-1}} + \mathrm{ab}\delta t + \sigma\delta W_t$$

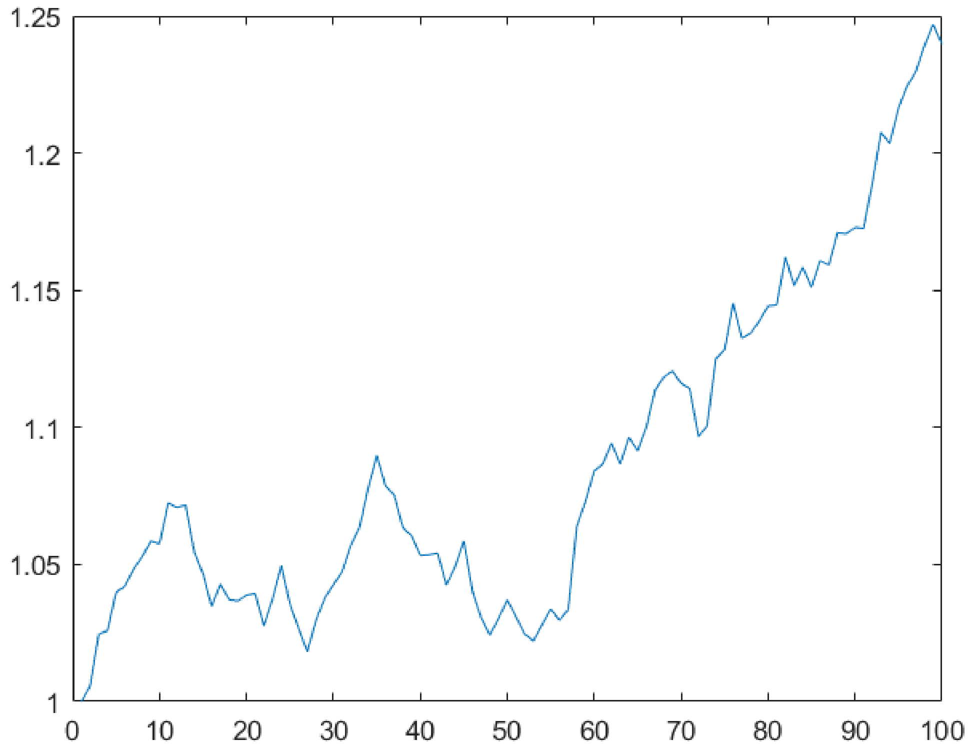
where $\delta W_t \sim N(0, \delta t)$. Below, we show some trajectories using different values of a, b and σ.

```
N = 100;
a = 1;
b = 1;
sig = 1;
dt = 0.01;

x = ones(1, N);
for time=2:1:N
   x(time) = (x(time-1) + a*b*dt + sig*normrnd(0,dt))/(1+a*dt);
end
figure
plot(x)
```



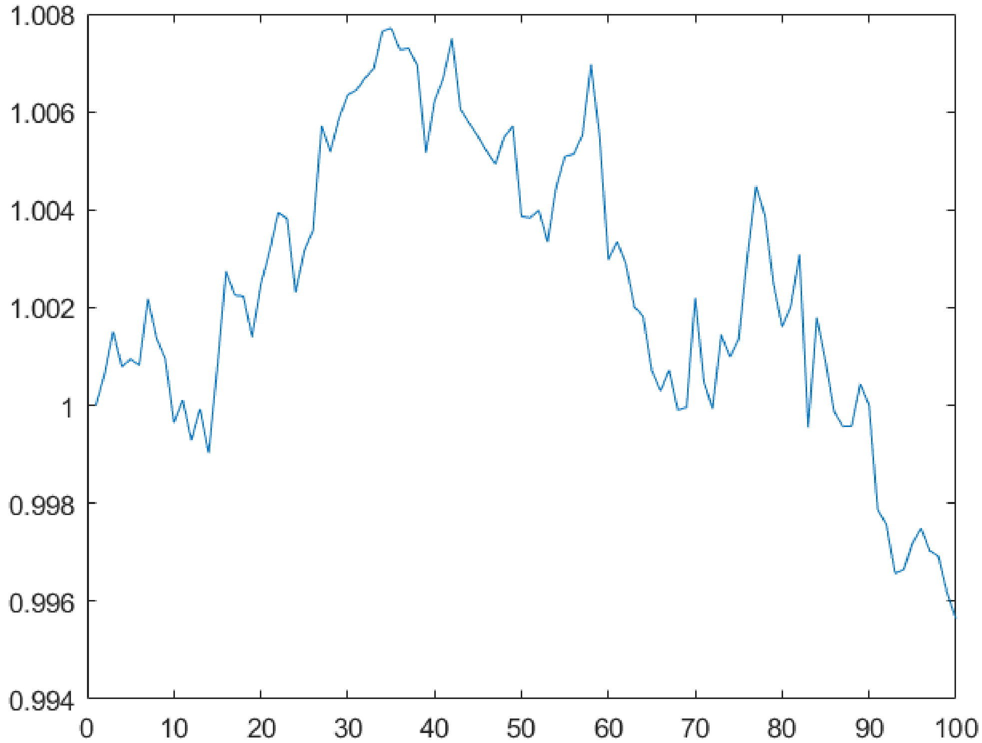

```
a = -1;
b = 1;
```

```
sig = 1;
dt = 0.01;

x = ones(1, N);
for time=2:1:N
   x(time) = (x(time-1) + a*b*dt + sig*normrnd(0,dt))/(1+a*dt);
end
figure
plot(x)
```



Depending on the run, the above graph can either be inscreaing or decreasing.

```
a = -1;
b = 1;
sig = 0.1;
dt = 0.01;

x = ones(1, N);
for time=2:1:N
   x(time) = (x(time-1) + a*b*dt + sig*normrnd(0,dt))/(1+a*dt);
end
figure
plot(x)
```


1.008
1.006
1.004
1.002
1
0.998
0.996
0.994
0
10
20
30
40
50
60
70
80
90
100