Estimation in the Frequency Domain (Discrete Spectrum)

1. The Periodogram

A Periodogram is used to estimate the spectral frequency of a signal. We want to estimate the parameters of the signal:

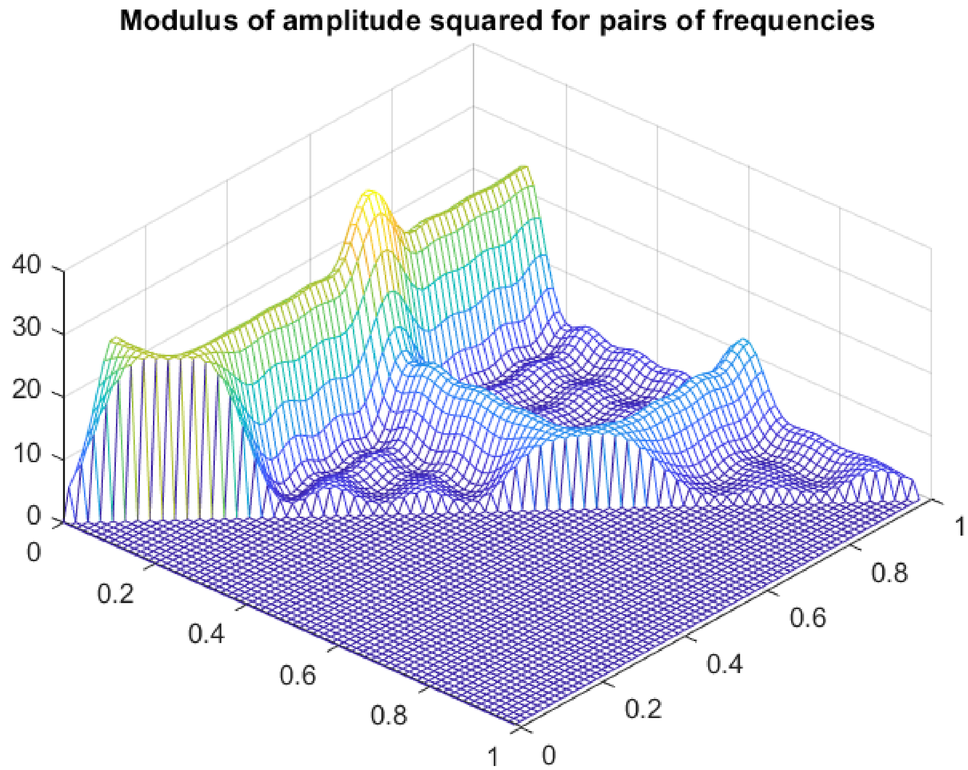
$$x(n) = a_1 e^{j2\pi F_1 n} + a_2 e^{j2\pi F_2 n} + b(n)$$

where $b(n)$ is a white gaussian noise. The code below plots the modulus of the estimated vector magnitudes $\widehat{a_{\text{MC}}} = [\,\widehat{a_1}\widehat{a_2}\,]$ while minimizing the mean squared error over different pairs of fixed frequencies. In other words, it shows the solutions to the following equation for different pairs of F_1 and F_2 fixed:

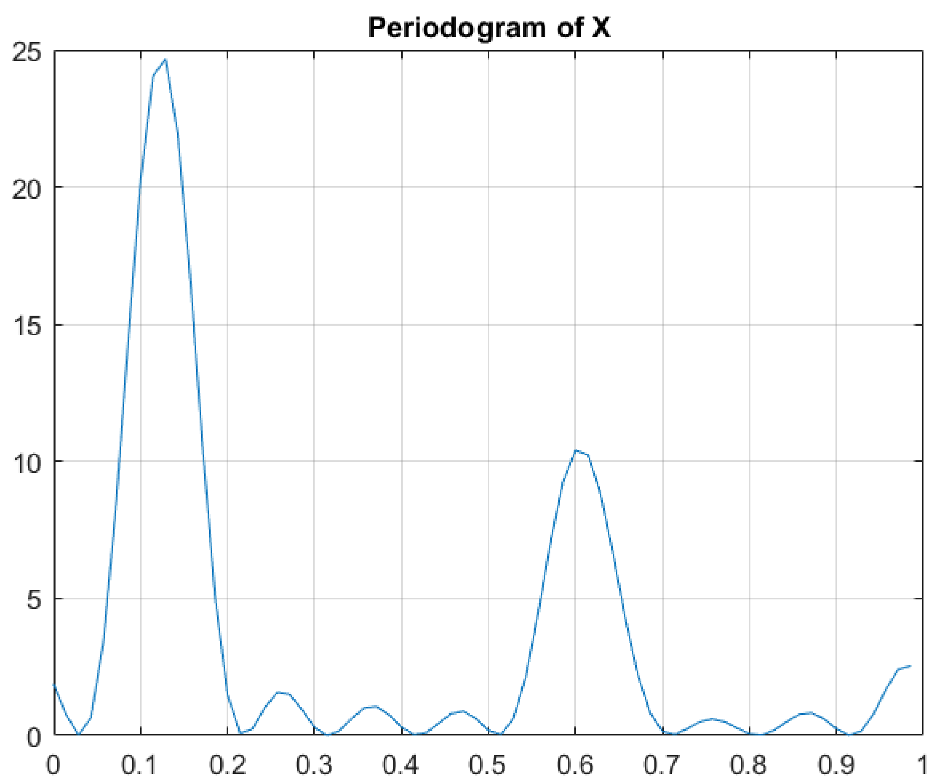
$$\min_{a_1, a_2} \left\| x(n) - \sum_{k=1}^{2} a_k e^{j2\pi F_k n} \right\|^2$$

```
clear all;
a1 = 1.5;   % Want to estiamte
a2 = 1;     % Want to estimate
F1 = 0.12;
F2 = 0.61;
N = 10; temps = (0:N-1)';
s = a1*exp(j*2*pi*F1*temps)+a2*exp(j*2*pi*F2*temps);
SNR = 15;
P = s'*s/N;
sigma = sqrt(P)*(10^(-SNR/20));
x = s+sigma*rand(N, 1);

L=70; f=(0:L-1)/L;
M = exp(j*2*pi*temps*f); % matrice a N lignes et L colonnes
K = zeros(L,L);
for i=1:L
    for j=1:i-1
        E = [M(:,i) M(:,j)];
        K(i, j) = abs(x'*E*pinv(E)*x); %pinv Pseudoinverse
    end
end
mesh(f,f,K); title('Modulus of amplitude squared for pairs of frequencies')
view([41.9 50.7])
```




```
plot(f, abs(fft(x, L)).^2/N); grid; title('Periodogram of X')
```



It is relatively easy to get the global maximum of the two graphs in MATLAB. The function `max()` is sufficient.

To estimate the valeus of $\hat{a} = \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \end{bmatrix}$ and $\hat{F} = \begin{bmatrix} \hat{F}_1 \\ \hat{F}_2 \end{bmatrix}$ using the least squares method shown below:

```
[a_max_cols,rows_ind] = max(K);
[a_max, col_ind] = max(a_max_cols) ;
l=rows_ind(col_ind);
c=col_ind;
f1_mc = f(c);
f2_mc = f(l);
E_mc = [M(:,c) M(:,l)];
a_mc = abs(pinv(E_mc)*x);
f1_mc, f2_mc, a_mc
```

```
f1_mc = 0.1286
f2_mc = 0.6143
a_mc = 2×1
    1.5614
    0.9958
```

Using the periodogram to estimate the same:

```
a_p = [sqrt(23.62*N) sqrt(10.38*N)]/N
```

```
a_p = 1×2
    1.5369    1.0188
```

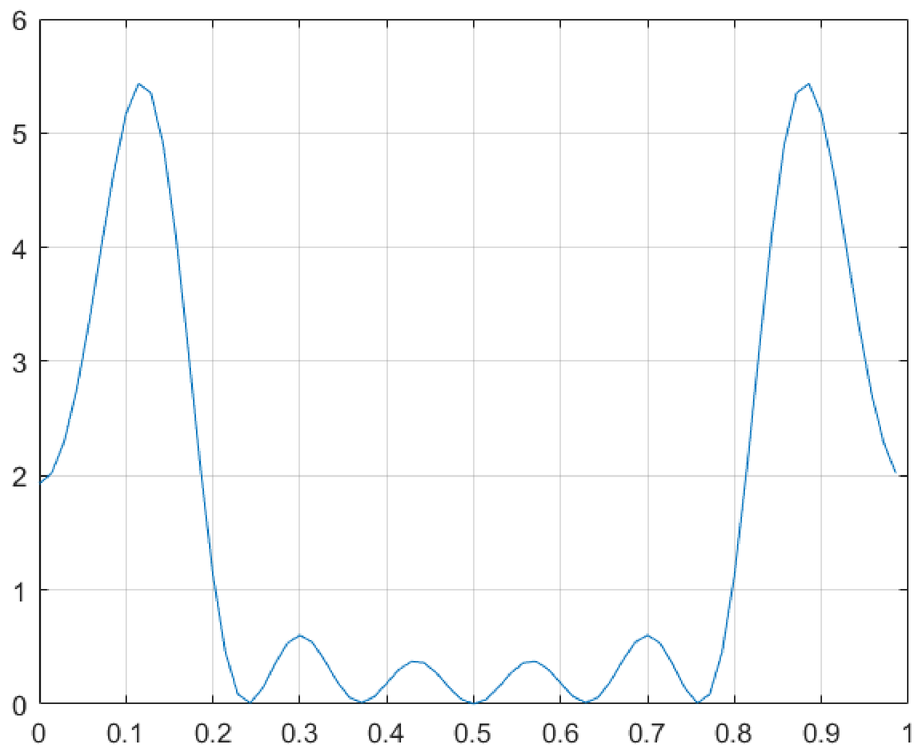
We see that the two estimates are very close to each other, since the resolution of the periodogram is good becasue of zero padding.

The minimal duration *N*of a noisy signal of frequency *F*which justifies using a periodogram is given by the Shanon criteria:

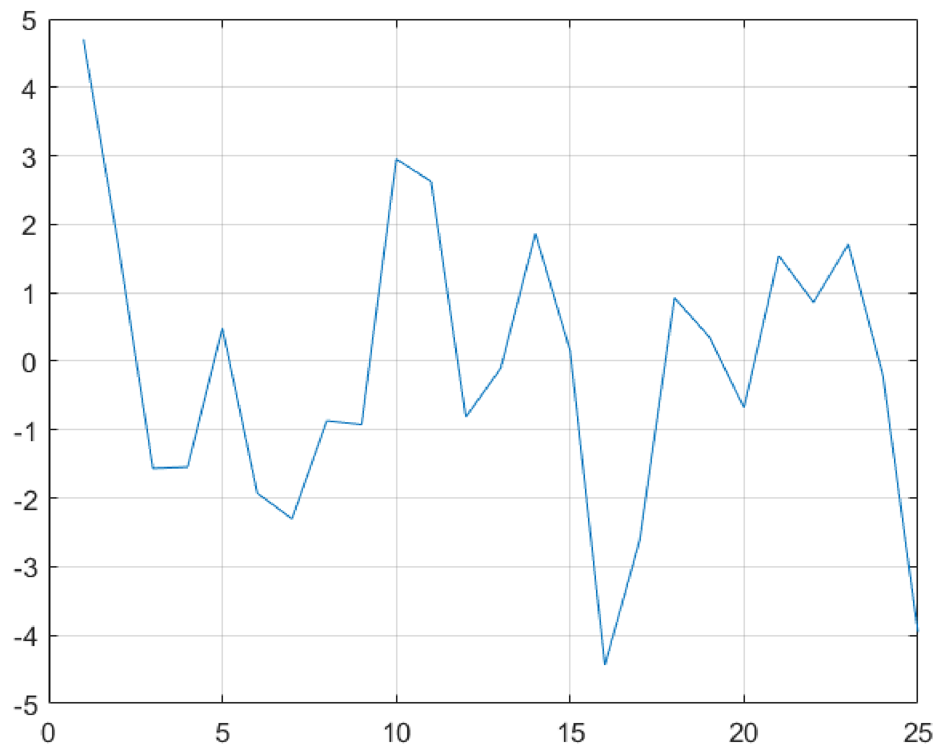
$$\frac{1}{N} > 2F$$

Below, we give an exmaple to show how this criteria can affect the quality of the solution achieved.

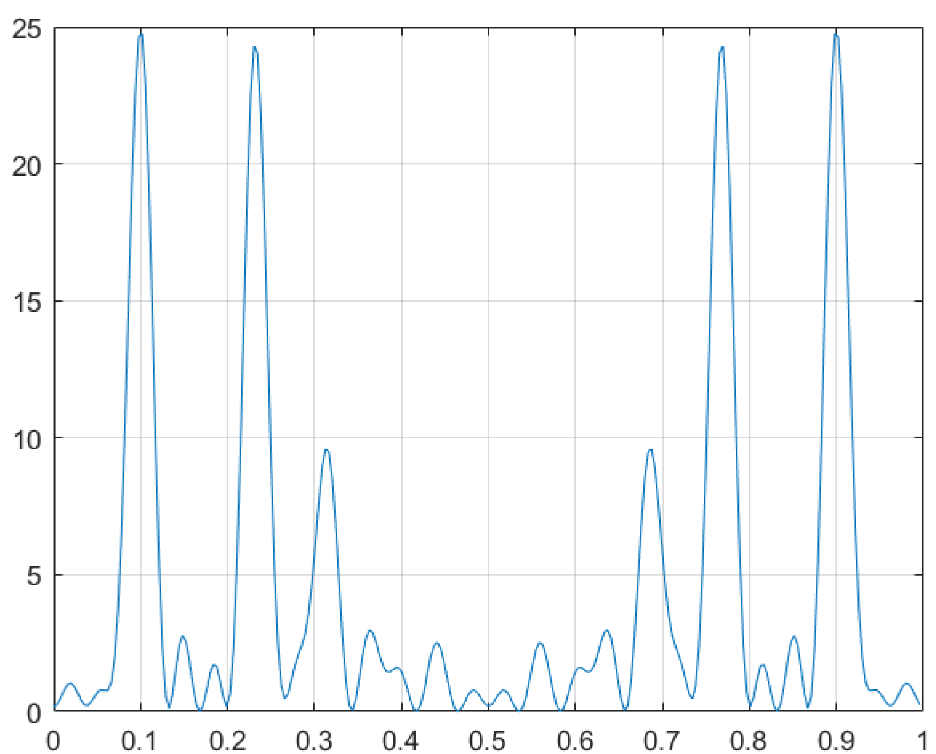
```
F=0.1; a=2; phi=pi/3;
SNR = 30; P = a^2/2;
sigma = sqrt(P)*(10^(-SNR/20));
N = 8;
x = a*cos(2*pi*F*(0:N-1)+phi)+sigma*randn(1, N);
figure; plot(f, abs(fft(x, L)).^2/N); grid;
```




```
a = [2 1.5 1];
F = [0.1; 0.23; 0.3];
N=25;
s = a*cos(2*pi*F*(0:N-1));
x = s+sqrt(0.5)*randn(1,N);
figure
plot(x); axis([0 N -5 5]); grid
```




```
L = 256;
figure; plot((0:L-1)/L, abs(fft(x, L)).^2/N); grid;
```



```
save signal x
```

Below, we load a noisy signal with the harmonics at the frequencies $F = [0.1,\ 0.23, 0.3]$. We attempt to recover these frequencies.

```
load signal.mat
L = 256;
f=(0:L-1)/L;
dF = (0.31-0.23)/2;
periodogramme_temp = abs(fft(x, L)).^2/N;
periodogramme = periodogramme_temp(1:floor(L/2));
P = 3;
pic = zeros(1, P);
indice = zeros(1, P);
frequencies = zeros(1, P);
for i=1:P
    [pic(i), indice(i)] = max(periodogramme);
    periodogramme(indice(i)-L*dF: indice(i)+L*dF)=0;
    frequencies(i) = f(indice(i));
end
```

```
Warning: Integer operands are required for colon operator when used as index.
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```

```
frequencies
```

```
frequencies = 1×3
    0.1016    0.2305    0.3125
```

We see that the frequency estimations are quite accurate.

MUSIC Algorithm: High Resolution method

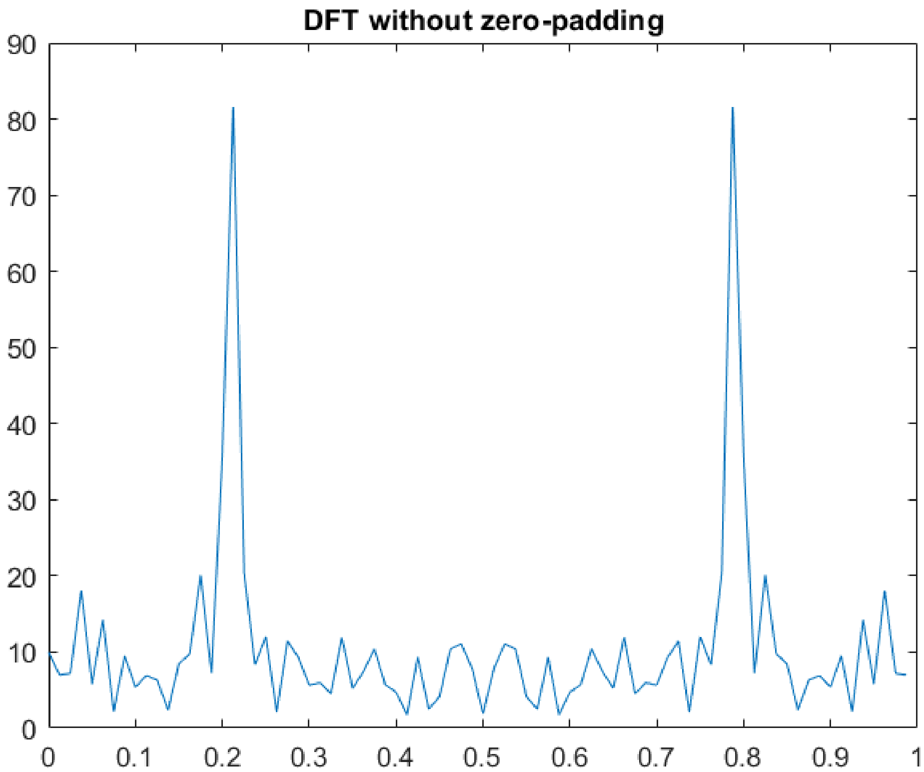
The MUSIC algorithm is used to estimate frequencies which may be very close to each other. Firstly, we will generate a sampling of size *N*of the signal:

$$x(n) = a_1\cos(2\pi F_1 n) + a_2\cos(2\pi F_2 n) + b(n)$$

```
clear all;
a = [1 2]; F = [0.2; 0.21]; N = 80;
s = a*cos(2*pi*F*(0:N-1));
SNR = 5;
P = s*s'/N;
sigma = sqrt(P)*(10^(-SNR/20));
x = s + sigma*randn(1, N);
```

If we calculate the DFT without zero-padding, we get the result:

```
t = (0:N-1)/N;
figure
plot(t, abs(fft(x)))
title('DFT without zero-padding')
```

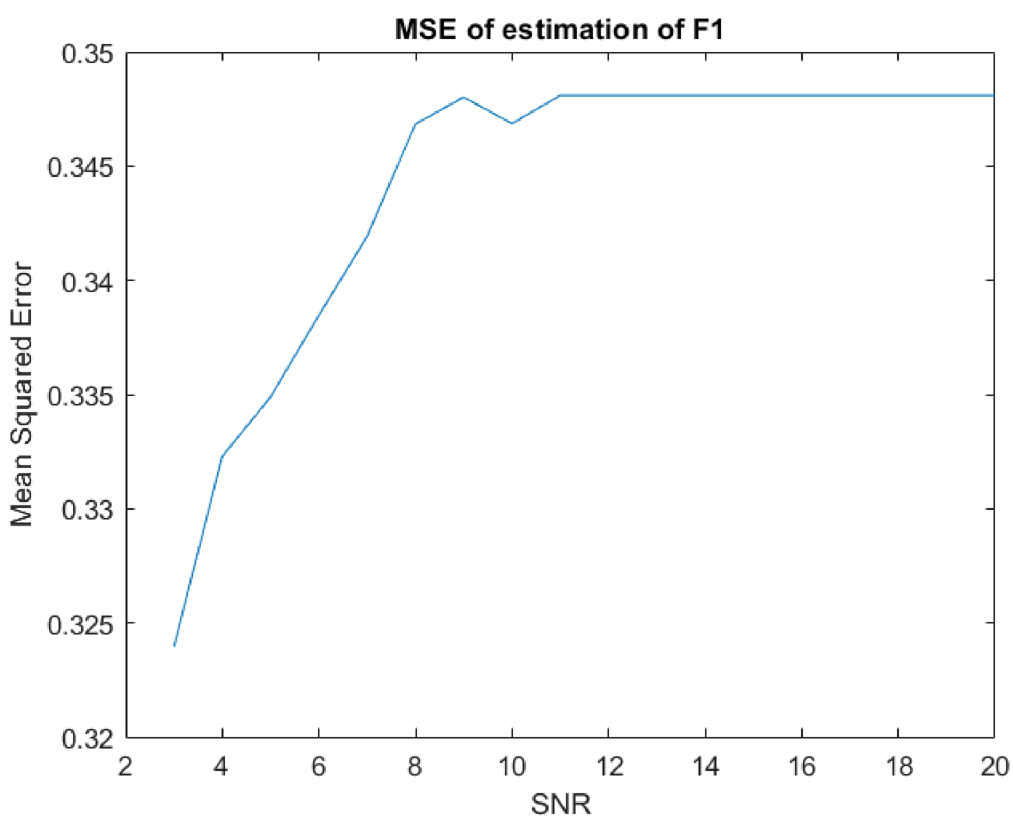


We see that it is not possible to distinguish the two frequencies that are very close to each other, at $F_1 = 0.2$ and $F_2 = 0.21$.

If we now use the MUSIC algorithm (NOTE: CAN TAKE AROUND 5 MINS TO RUN):

```
EQM = zeros(18,1);
M = 40; p = 2;
for SNR = 3:20
    for mc = 1:300
        sigma = sqrt(P)*(10^(-SNR/20));
        x = s + sigma*randn(1, N);

        X = zeros(M, N-M);
        for n=1:N-M
            for j=1:M
                X(j, n) = x(n+j);
            end
        end

        R = zeros(M, M);
        for n=1:N-M
            R = R + (X(:,n)*X(:,n)');
        end
        R = R/(N-M);

        [V, D] = eig(R);    %D(1,1) < ... <D(M, M)
        G = V(:, (1:M-p));
```

```
        n = 200;
        f = (1:n)/n;
        p_music = zeros(n,1);

        for k=1:n
            a = zeros(M,1);
            for j=1:M
                a(j) = exp(2*i*pi*k/n*(j-1));
            end
            p_music(k) = 1/(a'*(G*G')*a);
        end

%         figure
%         plot(f, abs(p_music));
        [Max,Loc] = findpeaks(abs(p_music),'sortstr','descend');
        EQM(SNR-2)=EQM(SNR-2)+((F(1)-Loc(2)/200)^2)/300;
    end
end
figure
plot((3:20), EQM)
title('MSE of estimation of F1')
xlabel('SNR')
ylabel('Mean Squared Error')
```



We observe that the MSE of the F_1 estimate increases with the SNR. It increases till around $\mathrm{SNR} \approx 10$, and then becomes roughly constant.