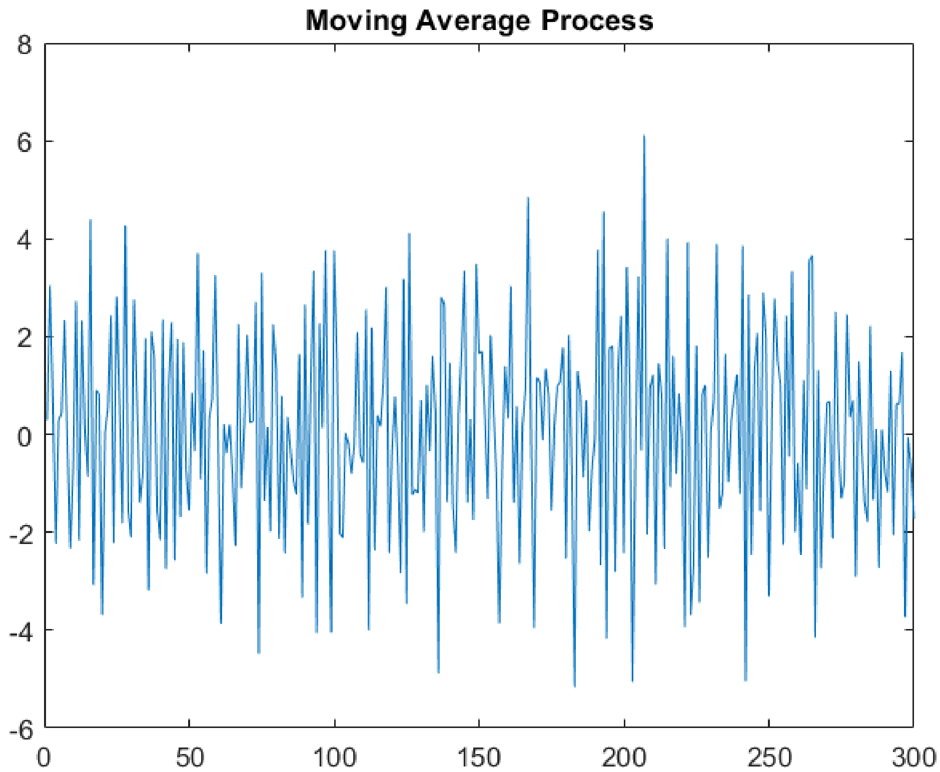
Filtering Random Signals and Time Series Data

1. Moving Average (MA) Process

For more info on a Moving Average process, look at Wikipedia. In the following code, we will simulate samples of a process $X, \mathrm{MA}(2)$ with $b_1 = 1.5,\ b_2 = -1.2$ and $\sigma^2 = 1$. Based on the definition of a Moving Average process, we have:

$$X(n) = 1 + \sum_{k=1}^{2} b_k \epsilon_{n-k}, \qquad \epsilon \sim N(0,1)$$

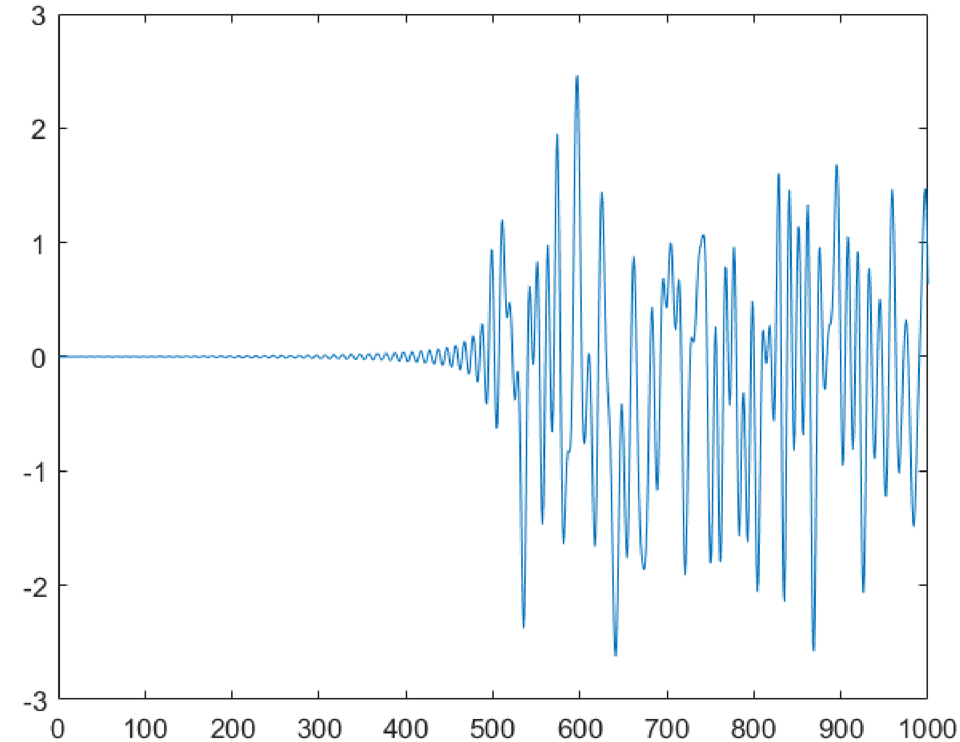
```
N = 300;
b1 = 1.5;
b2 = -1.2;
sig_carr = 1;
X = zeros(N,1);
e = normrnd(0, sig_carr, 2,1);
b = [b1; b2];

for i=1:N
   e(2) = normrnd(0, sig_carr);
   X(i) = normrnd(0, sig_carr) + fliplr(b)'*e;
   e(1) = e(2);
end

figure
plot(X)
title('Moving Average Process')
```



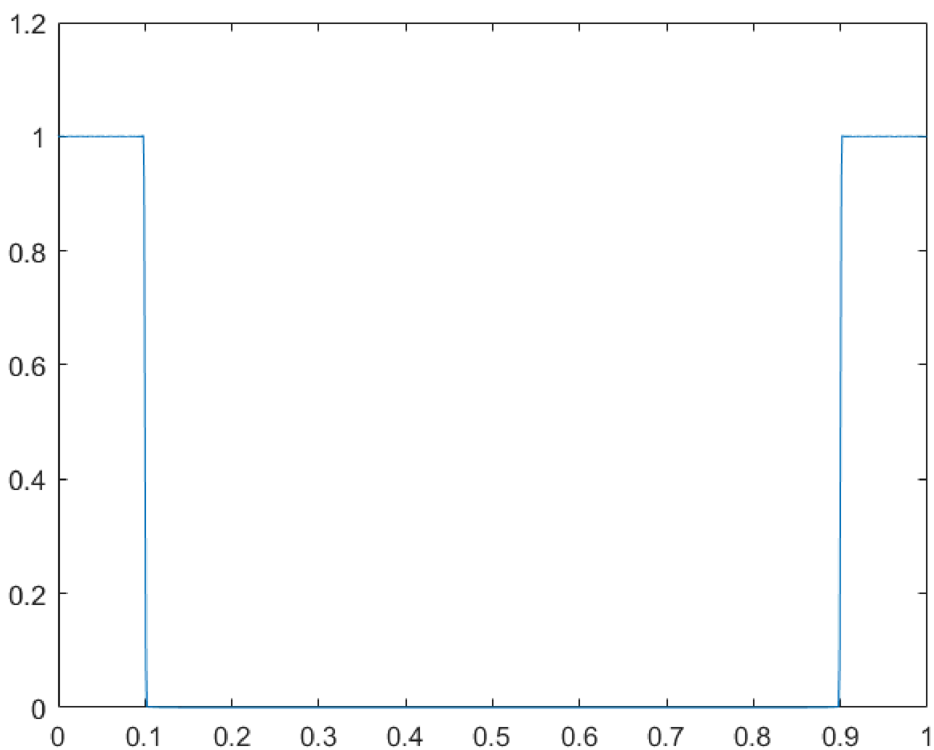
2. Processes with limited frequency band

We will demonstrate a low pass filter *b* with a cutoff frequency of f_c/f_s, where $f_c = 1000\,\mathrm{Hz}$ and the samling frequency is $f_s = 10000\,\mathrm{Hz}$. SInce the spectral density of the process is constant, the TFTD^{-1} is an impulse at $t = 0$, which justifies applying white noise as the input to the filter.

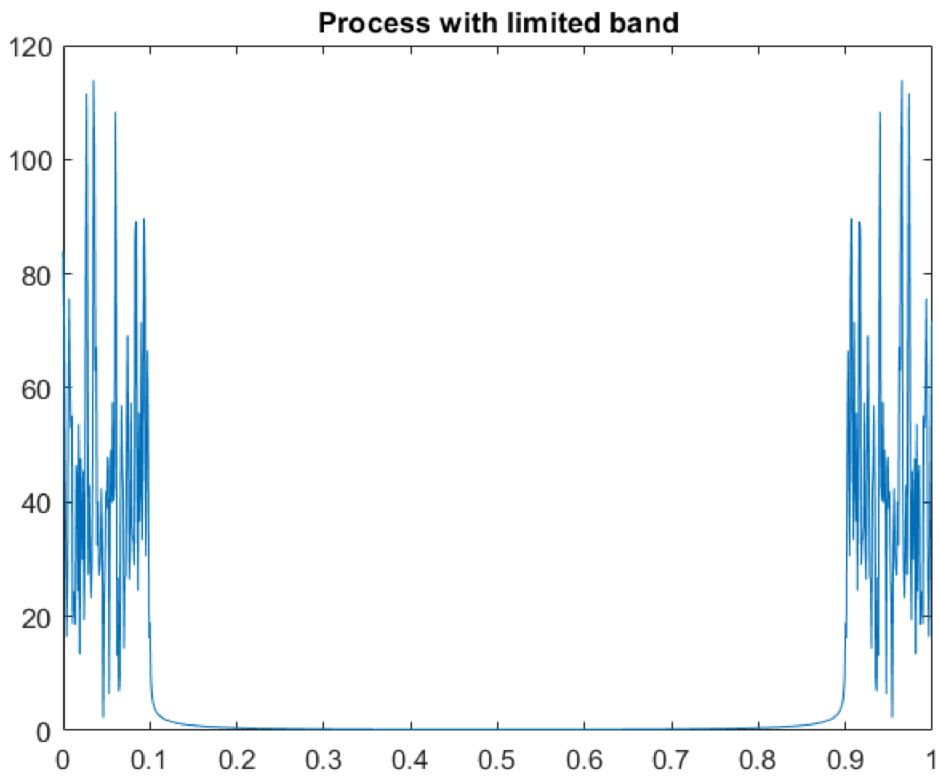
```
fc = 1000;
N = 1000;
fs = 10000;
P = 2;
b = fir1(N-1,2*fc/fs);
a = 1;
%e = normrnd(0, sqrt(sig_carr), N,1);
e = wgn(N, 1, 20*log10(P));
X=filter(b,a,e);
nu = 0:1/N:1-1/N;

figure
plot(X)
```




```
figure
plot(nu, abs(fft(b)))
```



```
figure
Xv = fft(X);
plot(nu, abs(Xv))
title('Process with limited band')
```



3. Noise Smoothening

Given a filter with impulse response:

$$h_{(n)} = \begin{cases} \frac{1}{8} & ,0 \leq n \leq 7 \\ 0 & ,n \geq 8 \end{cases}$$

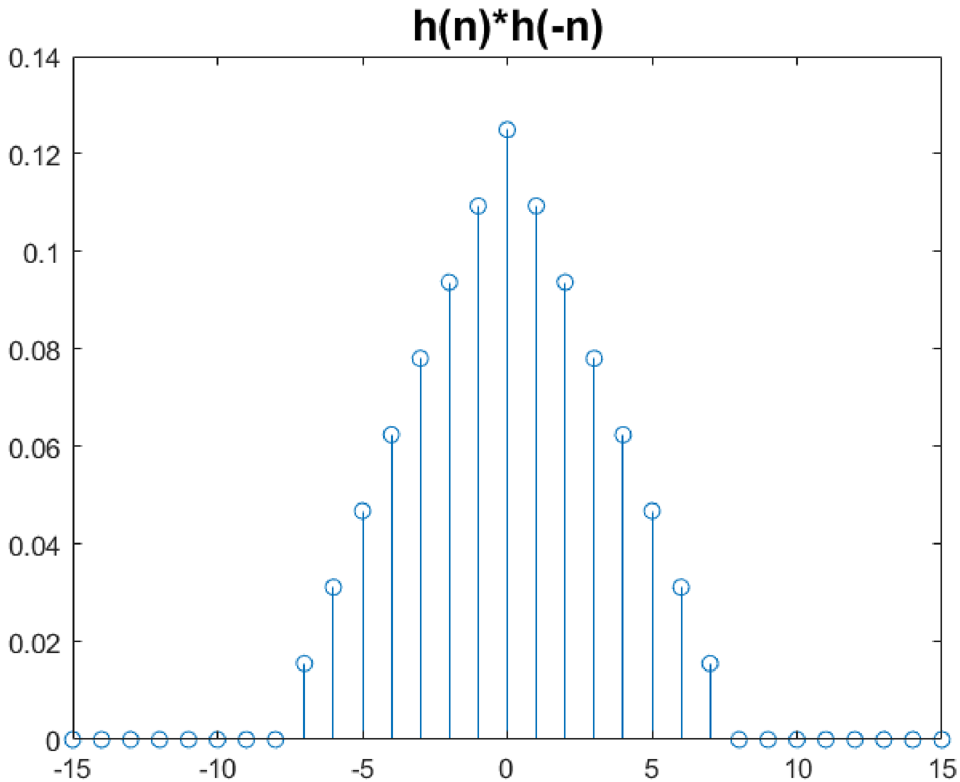
Spectral Density: The spectral density of the process at the output of the filter will be given by the spectral density $s(w)$ of an ARMA process.

$$s(w) = \sigma^2 |H(\mathrm{jw})|^2 = \sigma^2 H(\mathrm{jw}) H^*(\mathrm{jw}) = \sigma^2 \mathrm{FTDT}\{h(n) * h(-n)\}$$

Below, we graphically show the convolution $h(n) * h(-n)$

```
fontsize = 16;

my_h = (1/8) * ones(1,8);
my_h = [ zeros(1,8)  my_h];
my_conv = conv(my_h, flip(my_h));
figure
stem(-15:1:15,my_conv)
```

```
title('h(n)*h(-n)', 'fontsize', fontsize)
```



From the above result, we can write the spectral density by performing the fourier transform in discrete space, where $a = \frac{1}{8}$.

$$s(w) = \sigma^2[8a^2 + 2 \cdot 7a^2\cos(w) + 2 \cdot 5a^2\cos(2w) + 2 \cdot 5a^2\cos(3w) + 2 \cdot 4a^2\cos(4w) + 2 \cdot 3\cos(5w) + \ldots]$$

Autocovariance of filter output: The autocovariance K_X of the filter output is equal to the inverse fourier transform of the spectral density defined previously.

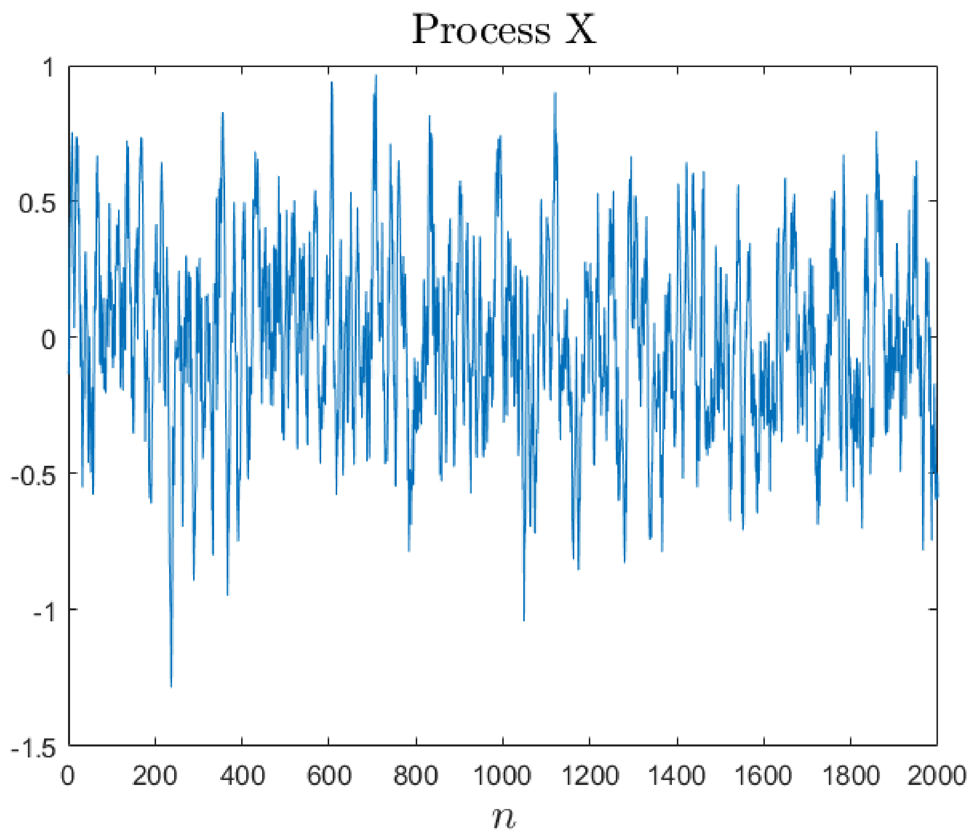
$$K_X(n) = \sigma^2 h(n) * h(-n)$$

Thus, K_X is just the above graph multiplied by σ^2.

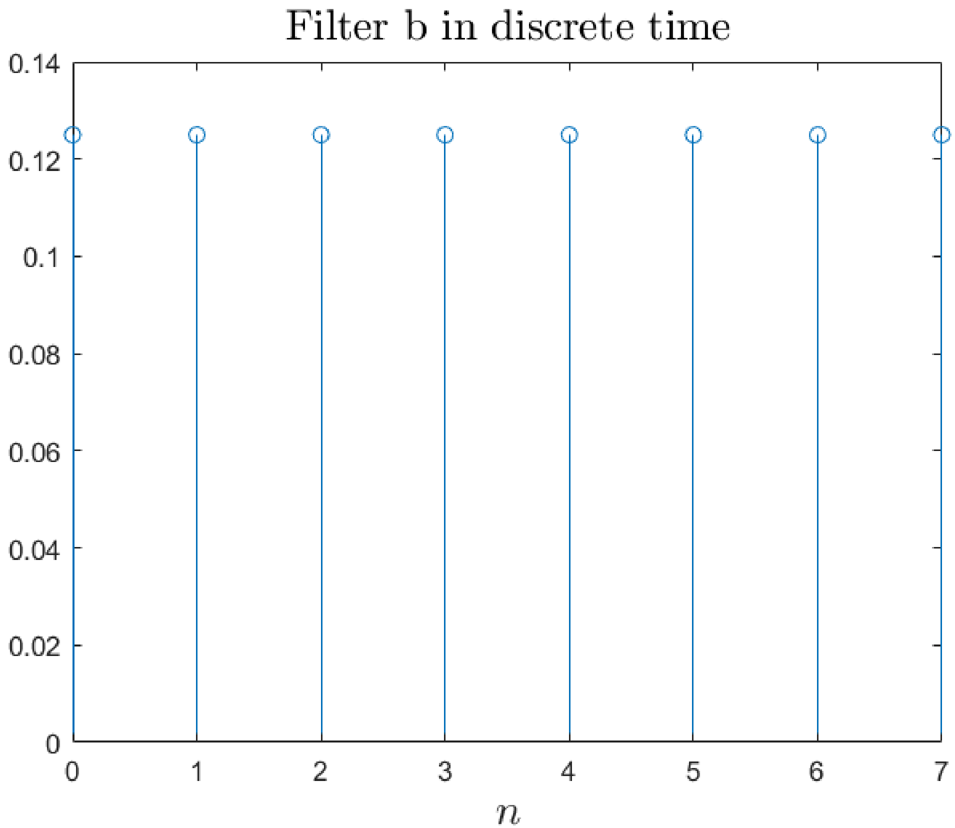
Now, we realize the same results on MATLAB and compare them to the theoretical values.

```
N = 2000;
b = 1/8.*ones(8,1);
a = 1;
P = 1;
e = normrnd(0, 1, N,1);
%e = wgn(N, 1, 20*log10(P));
X=filter(b,a,e);
nu = 0:1/N:1-1/N;

figure
plot(X)
```

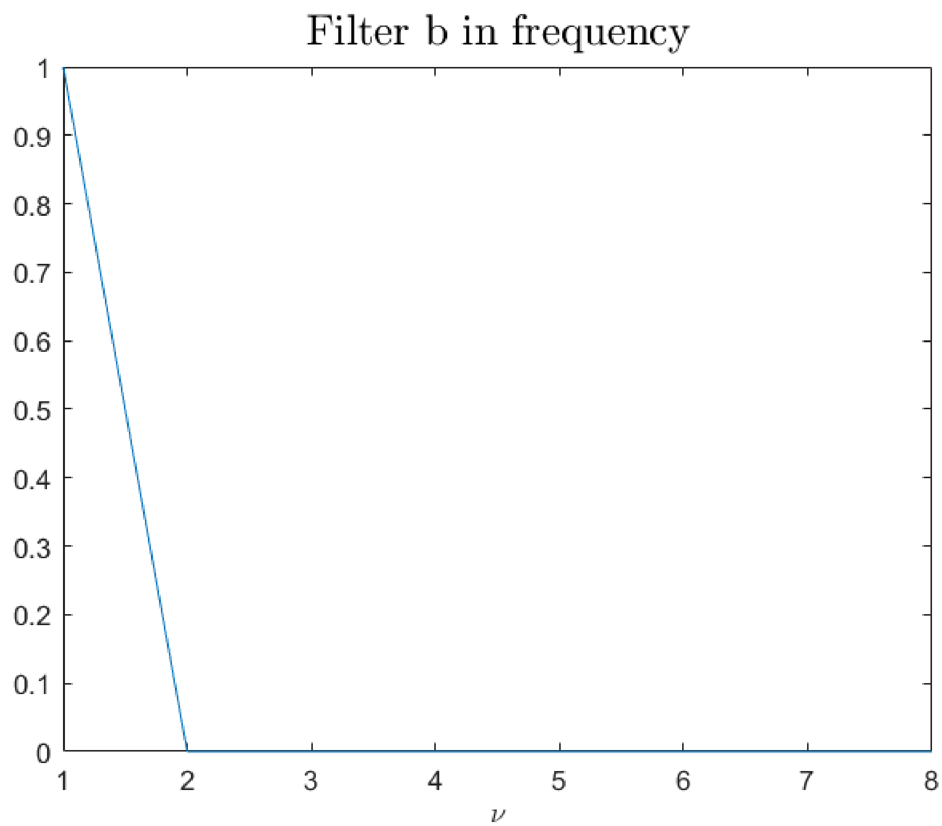
```
title('Process X','interpreter', 'latex','fontsize',fontsize)
xlabel('$n$', 'interpreter', 'latex','fontsize',fontsize)
```



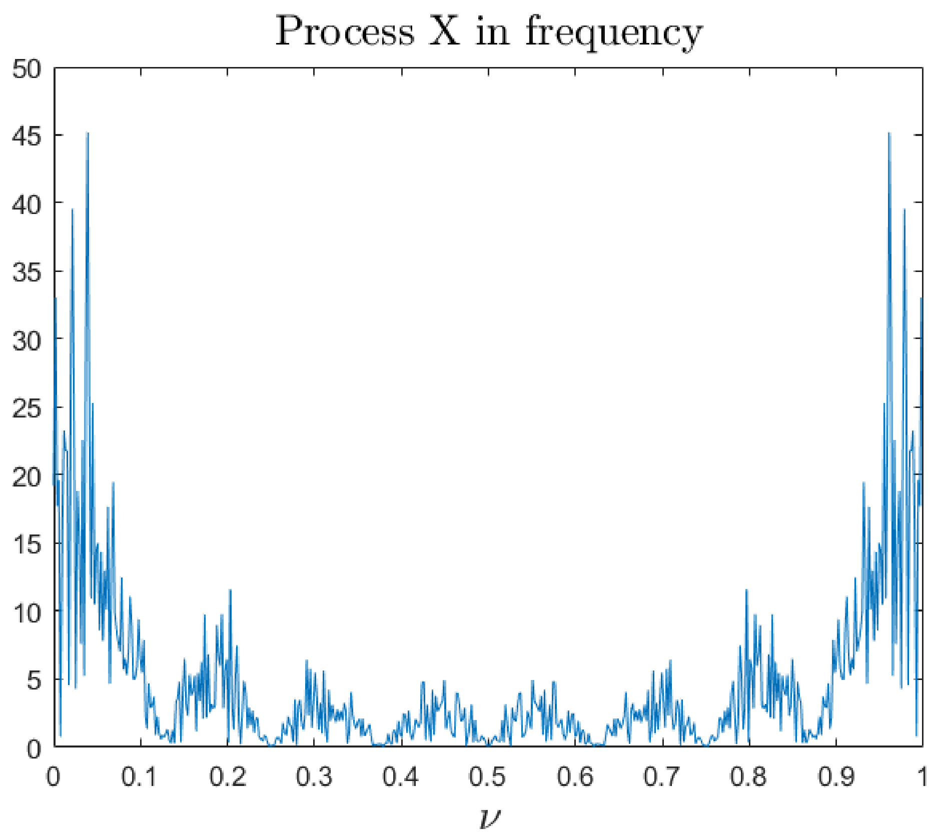

```
figure
stem(0:1:7,b)
title('Filter b in discrete time','interpreter', 'latex','fontsize',fontsize)
xlabel('$n$', 'interpreter', 'latex','fontsize',fontsize)
```



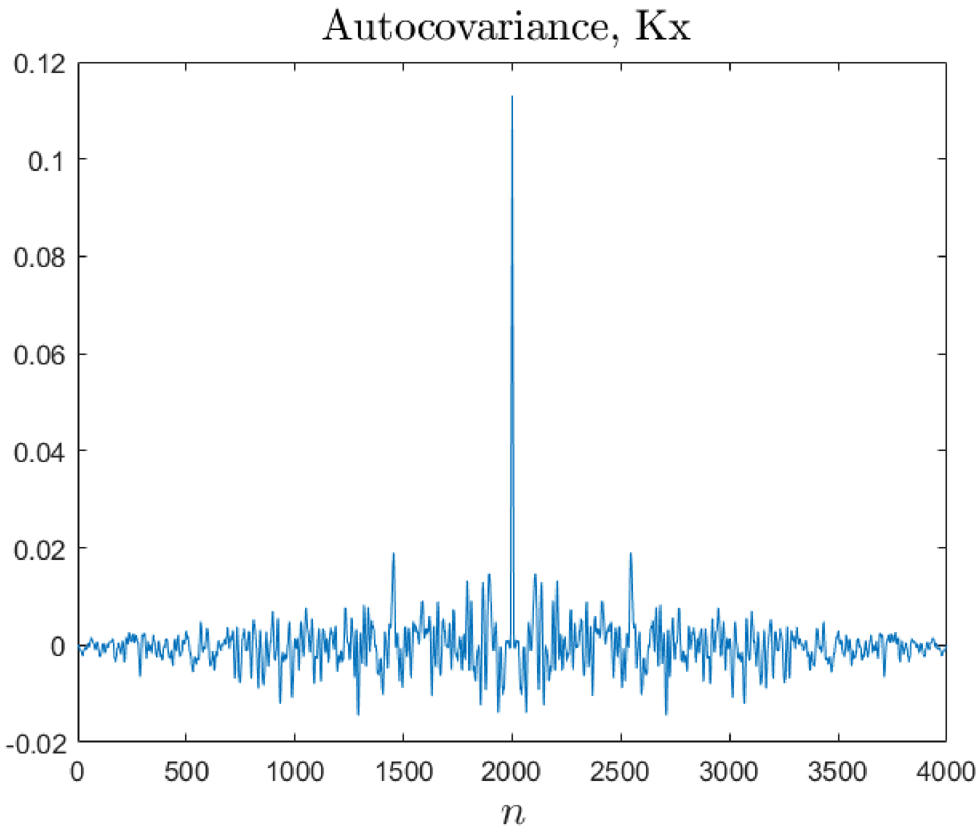

```
figure
plot( abs(fft(b)))
title('Filter b in frequency','interpreter', 'latex','fontsize',fontsize)
xlabel('$\nu$', 'interpreter', 'latex')
```




```
figure
plot(0:1/512:1-1/512, abs(fft(X, 512)))
title('Process X in frequency','interpreter', 'latex','fontsize',fontsize)
xlabel('$\nu$', 'interpreter', 'latex','fontsize',fontsize)
```

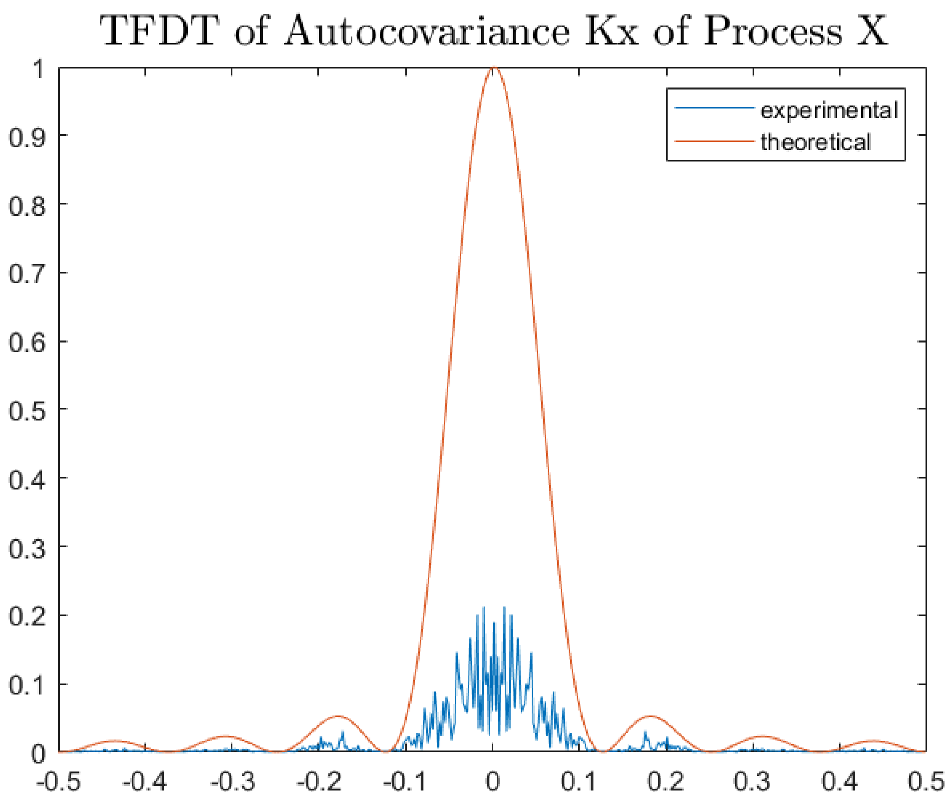



```
figure
covar = xcov(X);
plot(covar/N)
Kx0 = 20*log10(max(covar));
title('Autocovariance, Kx ','interpreter', 'latex','fontsize',fontsize)
xlabel('$n$', 'interpreter', 'latex','fontsize',fontsize)
```



Here, we see that the value of $K_X(0)$, the avearge power of the process is 0.125. This corresponds to the integration performed above, where the power would be $\sigma^2\,[h(n) * h(-n)](0) = \sigma^2 \cdot 0.125 = 0.125$.

```
F=-1/2:1/512:1/2-1/512;
TF_Ky = abs(fft(covar/N, 512));
TF_conv = abs(fft(my_conv', 512));
figure
plot(F,[TF_Ky(512/2:512); TF_Ky(1:512/2-1)])
%plot(TF_Ky)
hold on
%plot(TF_conv)
plot(F,[TF_conv(512/2:512); TF_conv(1:512/2-1)])
title('TFDT of Autocovariance Kx of Process X','interpreter', 'latex','fontsize',fontsize)
legend('experimental', 'theoretical')
```



We see that the amplitude of the TFTD of K_X diminished and increases approximately with the same speed as expected from the theoretical result. The modes are also present at the same frequencies. However, at $\nu = 0$, we see that the expected amplitude differs largeley from the observed amplitude.

4. Autoregressive Process

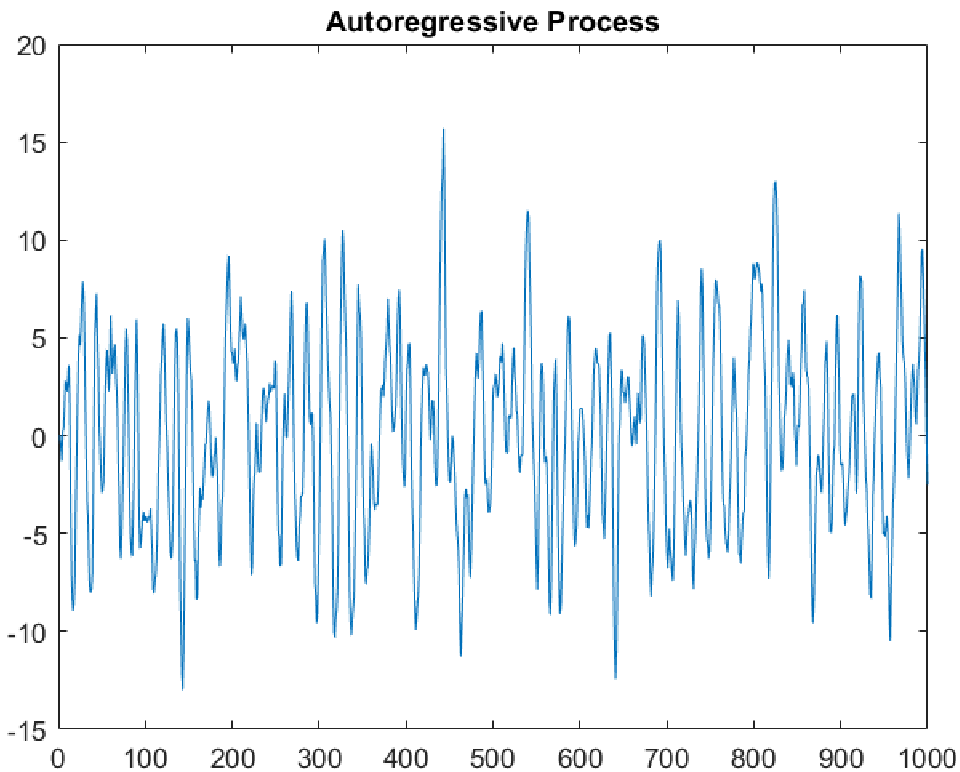
An Autoregressive Process, AR(2) is associated with the polynomial $A(z) = 1 + a_1 z^{-1} + a_2 z^{-2}$ whose poles are complex conjugates. Therefore, we can write the transfer function $H(z)$ and the poles of the system p_1 and p_2:

$$H(z) = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

$$\begin{cases} a_1 = -(p_1 + p_2) = -(\rho e^{-\phi} + \rho e^{+\phi}) = -2\rho\cos(\phi) \\ a_2 = p_1 p_2 = \rho^2 \end{cases}$$

```
rho = 0.9;
phi = 20;
N = 1000;
a1 = -2*rho*cosd(phi);
a2 = rho^2;
X = zeros(N,1);

for i=3:N
   X(i) = -a1*X(i-1)-a2*X(i-2)+normrnd(0,1);
end
```

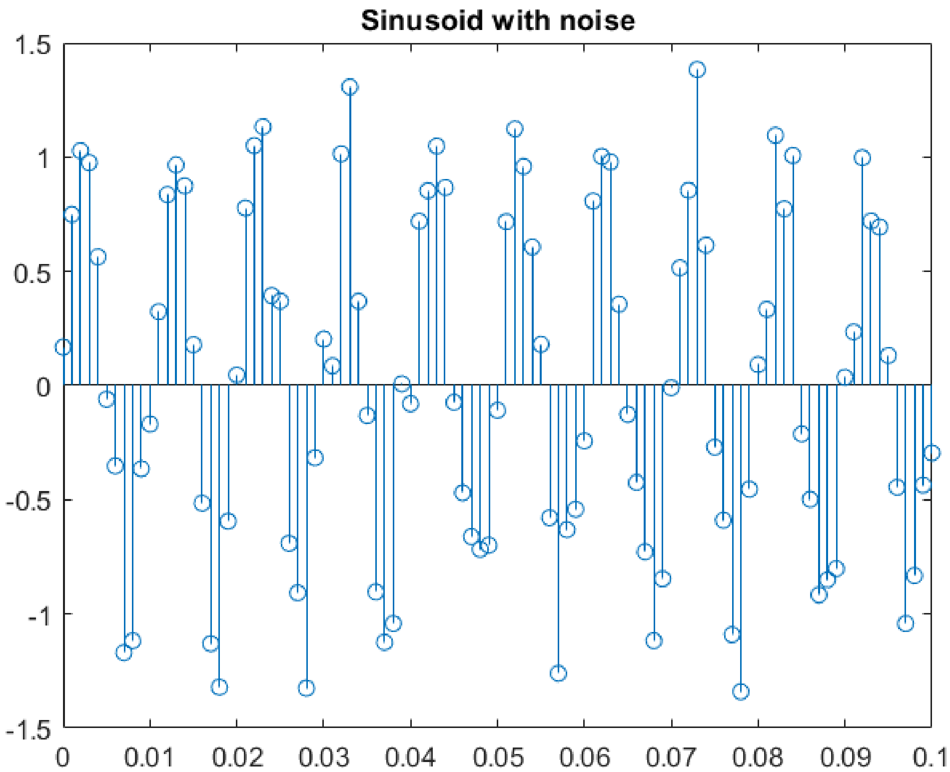
```
figure
plot(X)
title('Autoregressive Process')
```



5. Sinusoid with noise vs. AR(2) process

We sample the signal $y(t) = \sin(2\pi f_0 t) + b(t)$, where $f_0 = 100\,\mathrm{Hz}$ and $b(t)$ is a white gaussian noise of variance $\sigma^2 = 0.04$, with *N* points at the sampling frequency $f_s = 1000\,\mathrm{Hz}$. The result is shown below.

```
N = 100;
fe = 1000;
fo = 100;
var = 0.04;
numCycl = N*fo/fe;
t = 0:1/fe:numCycl*1/fo;
y = zeros(N, 1);
for i=1:N+1
    y(i) = sin(2*pi*fo*t(i)) + normrnd(0, sqrt(var));
end

figure
stem(t, y)
title('Sinusoid with noise')
```



To comapre the above with a AR(2) process, we have

$$x(n) + a_1 x(n-1) + a_2 x(n-2) = \epsilon(n)$$

where $\epsilon(n)$ is a white gaussian noise of variance σ^2. The resonant frequency associated with $x(n)$ is 100 Hz. We want to find the value of σ^2 for which the AR(2) process *x* and the sinusoid with noise *y* have the same average power.

We can show that $|a_1| < 1$ and $|a_2| < 1$, assuming that the poles of the transfer function are complex conjugates and the the filter is causal and stable. The filter has a maximum gain the the frequency ν_r:

$$\nu_r = \frac{1}{2\pi} \arccos\left(-\frac{a_1(1+a_2)}{4a_2}\right)$$

We know that the average power of *x* is equal to the autocorrelation function $r_x(0)$ since *x* is centered at 0. The Yule-Walker equations tell us:

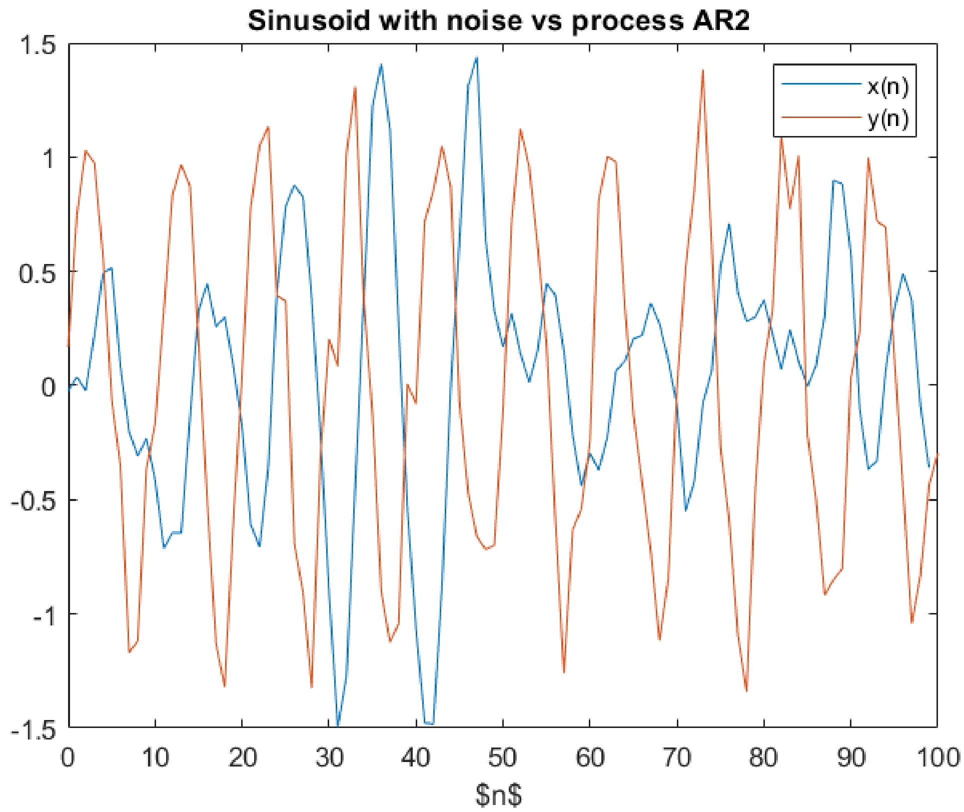
$$r_x(k) = -\sum_{i=1}^{2} a_i r_x(k-i) + \sigma^2 \sum_{j=0}^{0} b_j h(\bar{j-k})$$

Using $r_x(2)$ to replace in $r_x(0)$, we get the value for $r_x(1)$. Equating this with the value of $r_x(1)$ from the Yule Walker equation we finally get:

$$\sigma^2 = r_x(0)\left[1 - \frac{a_1^2}{(1+a_2)^2}\right](1 - a_2^2)$$

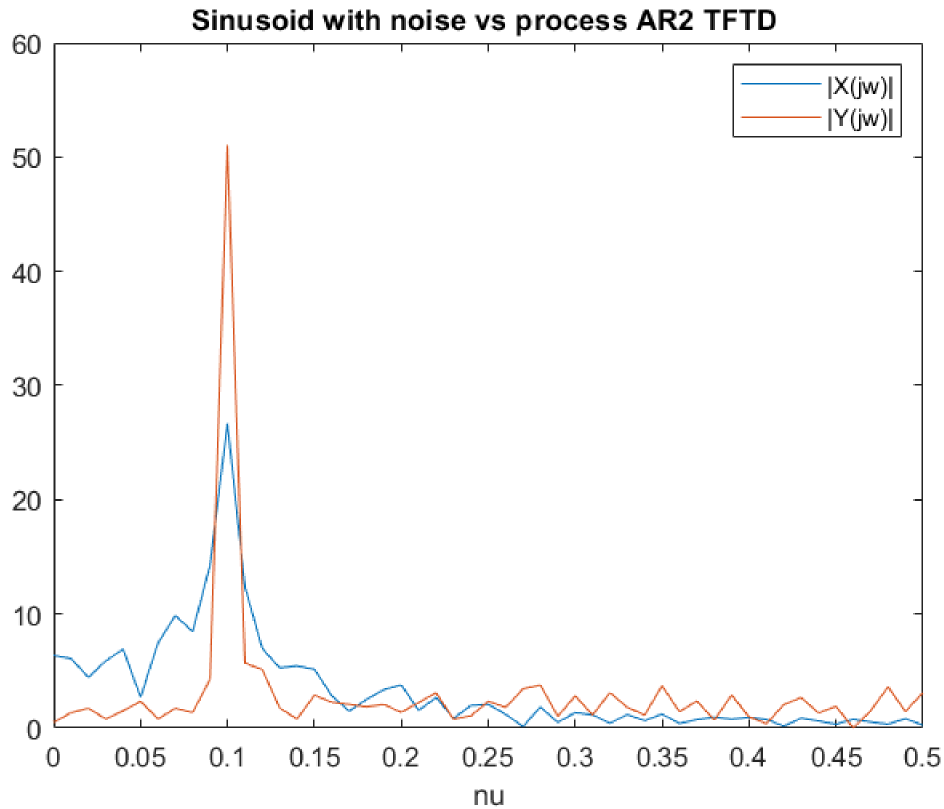
where $r_x(0) = P_{\mathrm{av}_y} = 1 + \sigma_b^2 = \frac{1}{2} + 0.04$. With the above equation and the one for ν_r, we have a system of 2 equations and 3 unknowns. We choose $a_2 = 0.83$, giving $\sigma^2 = 0.0599$. We show the results below:

```
a2 = 0.83; % tout valeur 0<a2<1 marche
a1 = -4*a2*cos(2*pi*fo/fe)/(1+a2);
nu_r = acos(-a1*(1+a2)/(4*a2))/(2*pi);
Pav = 1/2+ var;
sigma_carr = 1/2+var;
N = 100;
x = zeros(N, 1);
sigma_carr_x = Pav *(1 -( a1 ^2) /(1+ a2) ^2) *(1 - a2 ^2);
x (1) = normrnd (0, sqrt ( sigma_carr_x ));
x (2) = -a1*x (1) + normrnd (0, sqrt ( sigma_carr_x ));
for i=3:N
    x(i) = -a1*x(i-1)-a2*x(i-2)+normrnd(0,sqrt(sigma_carr_x));
end

figure
plot (0:N -1,x)
title ('Sinusoid with noise vs process AR2 ')
hold on
plot (0:N, y)
legend ('x(n)','y(n)')
xlabel ('$n$ ')
```



We see that the two processes are similar in terms of amplitude and frequency. The differences come becuase of the noise component. This becomes more visible in the frequency domain:

```
FT_x = abs( fft (x));
FT_y = abs( fft (y));
v = 0:1/( floor (N)) :0.5;
figure
plot (v, FT_x (1:N /2+1) );
title ('Sinusoid with noise vs process AR2 TFTD ')
hold on
plot (v, FT_y (1:N /2+1) )
legend ('|X(jw)|','|Y(jw)|')
xlabel ('nu')
```



Sinusoid with noise vs process AR2 TFTD
|X(jw)|
|Y(jw)|
60
50
40
30
20
10
0
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
nu