Ergodicity and Estimation in the Time Domain

1. Estimation of auto- and inter-correlation functions

The interference formula for K_{xy}: Given a filter with impulse response of $h(n)$, an input *X* and the output *Y*, we have the following relationship to calculate K_{XY}:

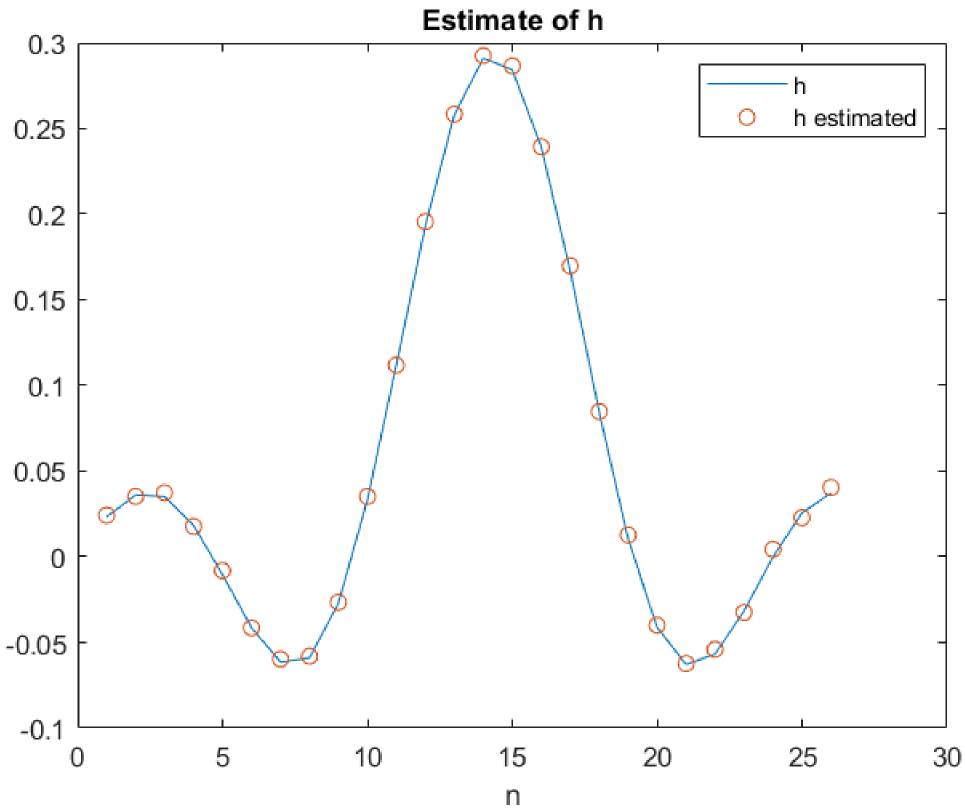
$$K_{XY}(\tau) = (h * K_X)(n) = \sum_{m=0}^{L} h(m) K_X(n - m - \tau)$$

For different values of $\tau \in Z$, we have the following matrix system:

$$\begin{bmatrix} K_x(n-0) & K_x(n-1) & \dots & K_x(n-(L-1)) \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ K_x(n-(L-1)) & \dots & \dots & K_x(n-0) \end{bmatrix} \begin{bmatrix} h(0) \\ \vdots \\ \vdots \\ h(L-1) \end{bmatrix} = \begin{bmatrix} K_{XY}(0) \\ \vdots \\ \vdots \\ K_{XY}(L-1) \end{bmatrix}$$

Method of moments for estimating the impulse response of an RIF Filter

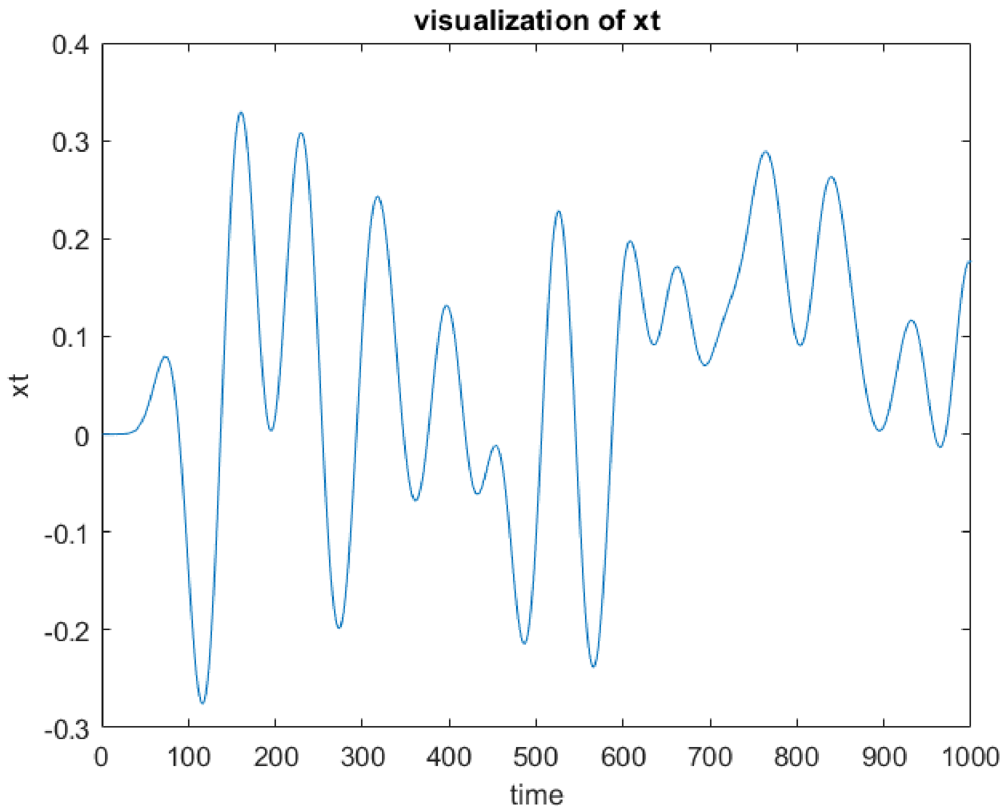
```
h = sin((-16:1.2:15)*(pi/5.8))./(-16:1.2:15)*(pi/5.8);
L = length(h);
num = [0.3 0.4 -0.2 0.1];
den = [1 -0.8 0.5];
X_predict = filter(num, den, randn(1,300));
Y = filter(h,1,X_predict);
nb = 300;
N = nb-L; %number of useful data

% les covariances
kX_hist = xcov(X_predict);
kYX = xcov(Y,X_predict);

r = kYX(L+1:N+L)';

R=zeros(N,L);
for i=1:N
    for j=1:L
        R(i,j)=kX_hist(L+i-(j-1));
    end
end
h_dash=(R'*R)\(R'*r);

% Comparaison par rapport au veritable h

figure
plot(h)
hold on
plot(h_dash, 'o')
legend('h','h estimated')
title('Estimate of h')
xlabel('n')
```



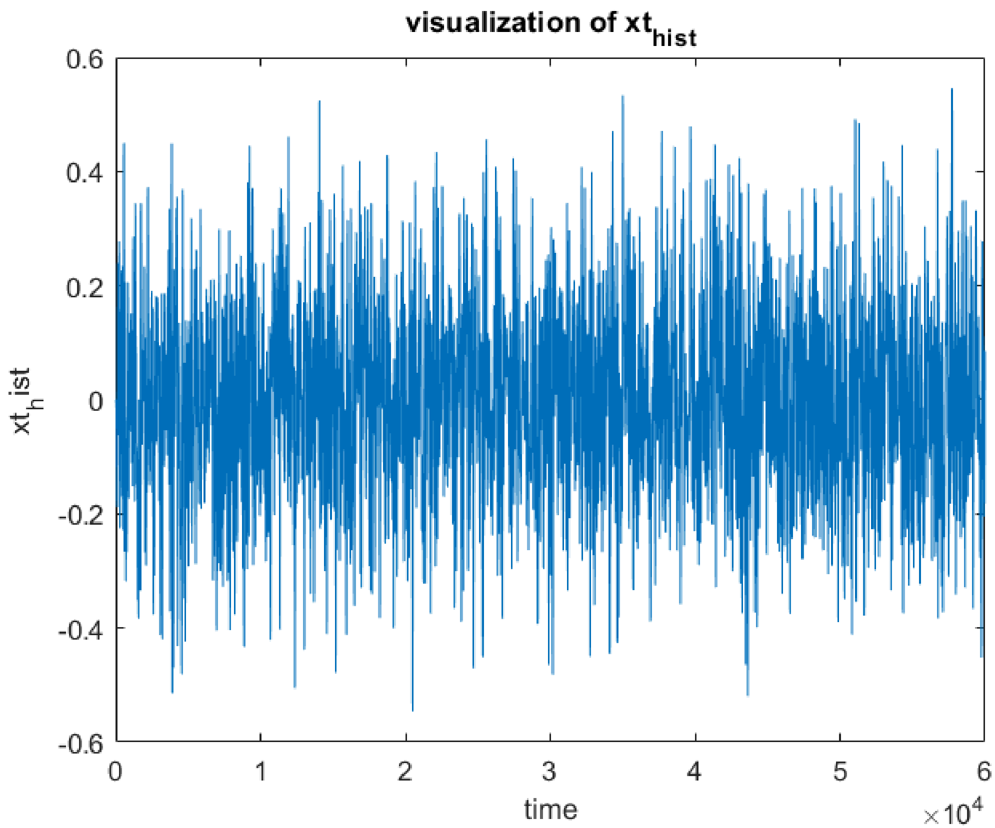
2. Application of the autocorrelation function

We have the trajectory of a body saved in `mouvement.mat`. Below, we see that the covaraince of the trajectory is similar to a dirac function. This justifies the use of a guassian preocess centered at 0 to model the signal.

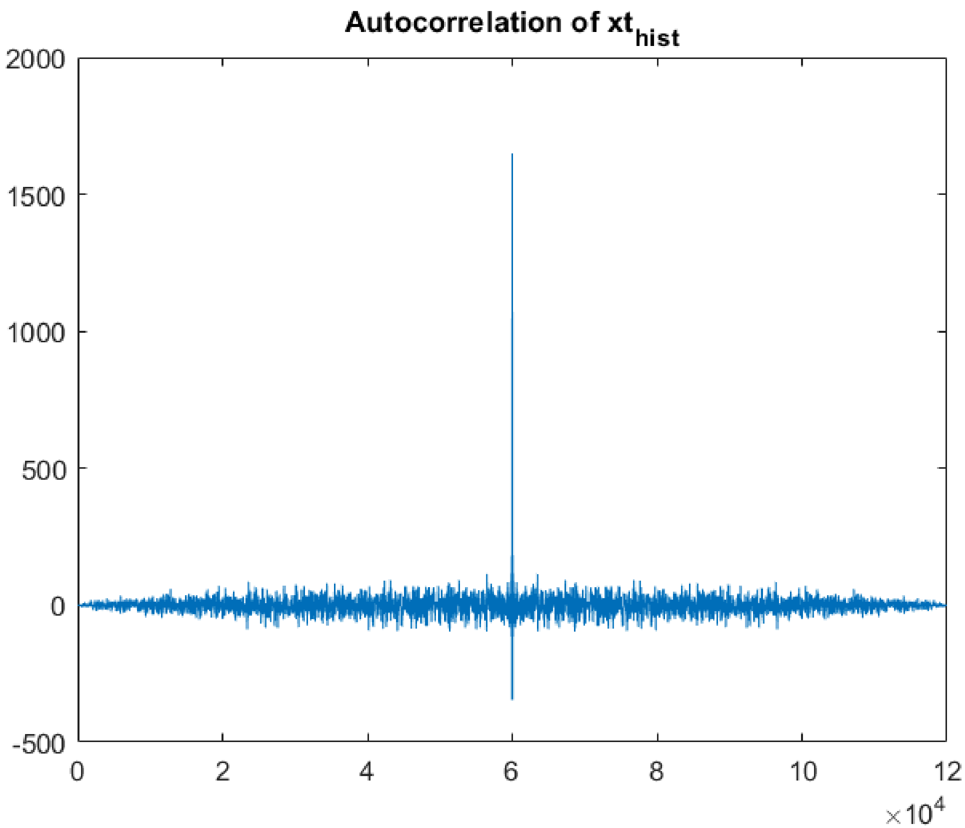
```
load('mouvement.mat')
load('mouvement_historique.mat')
figure
plot(xt)
title('visualization of xt')
xlabel('time')
ylabel('xt')
```




```
figure
plot(xt_hist)
title('visualization of xt_h_i_s_t')
xlabel('time')
ylabel('xt_hist')
```




```
rx_hist = xcorr(xt_hist);
L= length(xt_hist);
figure
plot(rx_hist)
title('Autocorrelation of xt_h_i_s_t')
```



We can use the historic trajectory to get a linear multivariate model to estimate the future trajectory. Therefore, we have:

$$\hat{X}_t = \sum_{i=1}^{n} a_{t,i} X_i \qquad , t > n$$

The $\hat{X}_t$ that minimizes the mean squared error is the projection of X_t on the plane generated by the vectorial space formed by $\{X_1, X_2, \ldots, X_n\}$. Hence,

$$\left(X_t - \hat{X}_t\right) X_l^T = 0, \qquad l \in [1, n]$$

$$\Rightarrow \left[\left(X_t - \hat{X}_t\right) X_l^T\right] = 0$$

$$\Rightarrow \left[X_t^T X_l\right] - E\left[\hat{X}_t^T X_l\right] = 0$$

$$\Rightarrow K_X(t, l) - \sum_{i=1}^{n} a_{t,i} E\left[X_i^T X_l\right] = 0$$

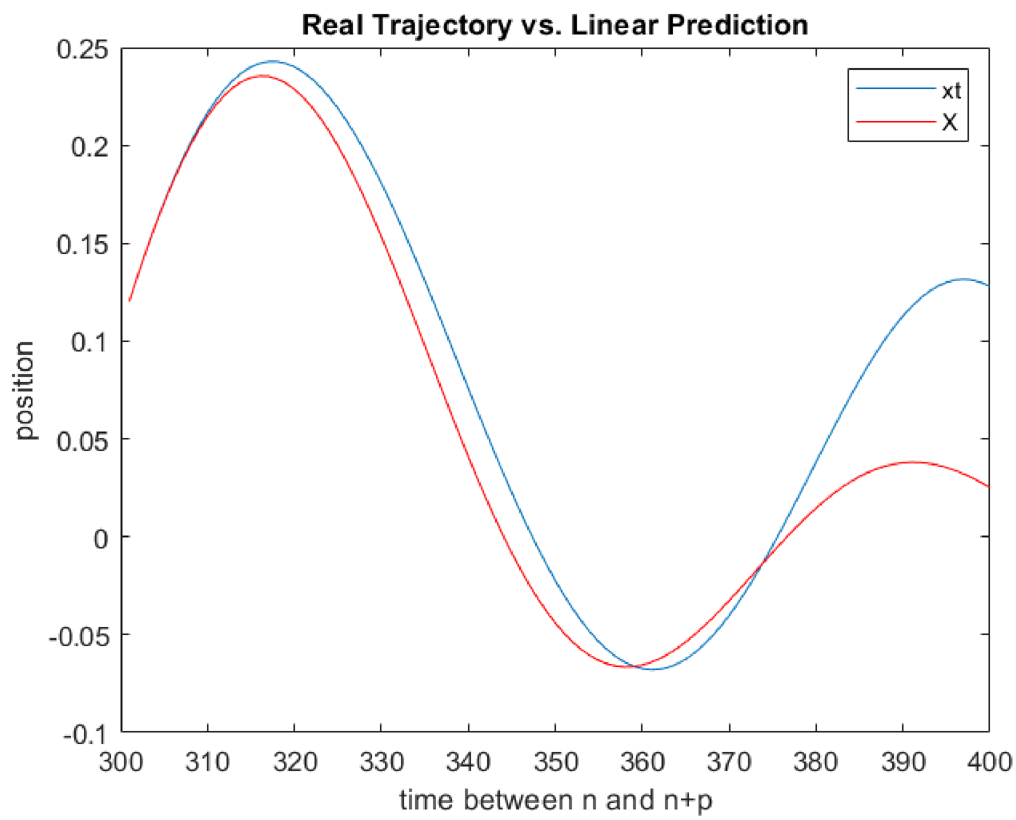
$$\Rightarrow K_X(t, l) - \sum_{i=1}^{n} a_{t,i} K_X(i, l) = 0$$

Again, writing this in the form of a matrix,

$$
\begin{bmatrix} K_X(1,1) & K_X(2,1) & \dots & K_X(n,1) \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ K_X(1,n) & \dots & \dots & K_X(n,n) \end{bmatrix} \begin{bmatrix} a_{t1} \\ \vdots \\ \vdots \\ a_{tn} \end{bmatrix} = \begin{bmatrix} K_X(t,1) \\ \vdots \\ \vdots \\ K_X(t,n) \end{bmatrix}
$$

Below, we use the values in `mouvement.mat`. Out of the 400 values, the first 300 values are used to predict the next 100 values.

```
n=300;
p=100;
t=n+p;

X_predict=zeros(1,t);
for i=1:n
    X_predict(i)=xt(i);
end

K=zeros(n,n);
for i=1:n
    for j=1:n
        K(i,j)= rx_hist(L+j-i);
    end
end

kt=zeros(n,p);
for t=n+1:t
    for j=1:n
        kt(j,t-n)=rx_hist(L+t-j);
    end
end

at = K\kt;
X_predict(n+1:n+p)=X_predict(1:n)*at;

figure;
plot(n+1:t,xt(n+1:t))
hold on;
plot(n+1:t,X_predict(n+1:t), 'r')
legend('xt','X')
title('Real Trajectory vs. Linear Prediction')
xlabel('time between n and n+p')
ylabel('position')
```



We see that the predictio is pretty accurate for around the first 20 values. After that, the error begins to become significant.

3. Applying the Levinson Algorithm

For the mathematical details, look at the Levinson Recursion Wikipedia article.

```
% Processus MA
sigma2 = 1;
N = 100;
e = wgn(N+2,1,sigma2);
X = zeros(N,1);
b1 = 1;
b2 = 1;
X(1) = e(1);
X(2) = e(1) + e(2);
for i = 3:N
    X(i) = e(i-2) + e(i-1) + e(i);
end

% Prediction de trajectoire
rX = xcorr(X);
p_max = 4; %nb de valeurs utilises par la prediction + 1
k = 60; %instant a partir de lequel on predit
h = zeros(p_max);
EQM = zeros(p_max, 1);
Xdash = ones(p_max, 1);
h(1,1) = rX(N+1)/rX(N) ;
```

```
Xdash(1) = h(1,1)*X(k-1);
EQM(1) = rX(N) - rX(N+1)*h(1,1);
alpha = zeros(p_max,1);

for p=1:p_max-1
   Coeff_p = h(p,1:p);
   Coeff_p_tilde = flip(Coeff_p);

   Vect_p = rX(N:N+p-1);
   Vect_p_tilde = flip(Vect_p);

   alpha(p) = rX(N+p+1) - Coeff_p*Vect_p_tilde;

   EQM(p+1) = EQM(p) -(alpha(p)^2)/EQM(p);
   h(p+1,1:p+1) = [squeeze(-Coeff_p +((alpha(p)^2)/EQM(p))*Coeff_p_tilde) -(alpha(p)^2)/EQM(p)]
   Xdash(p+1) = h(p+1,1:p+1)*X(k-1:-1:k-p-1);
end

X(k)
```

```
ans = 0.5283
```

```
Xdash
```

Xdash = 4×1

10^{22} ×

```
    0.0000
   -0.0000
   -0.0000
    7.1944
```

```
EQM
```

EQM = 4×1

10^{14} ×

```
    0.0000
    0.0000
   -0.0000
    1.2881
```