Machine Learning

A. G. Schwing & M. Telgarsky

University of Illinois at Urbana-Champaign, 2018

L7: Multiclass Classification and Kernel Methods

Announcements:

- Office hours on Friday 5-7pm (2TAs) ECEB 2015
- Office hours Instructors: right after this class; questions about any of the material

Goals of this lecture

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- Understanding multi-class logistic regression

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- Getting to know multi-class SVM

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Reading material

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Reading material

- K. Murphy; Machine Learning: A Probabilistic Perspective; Chapter 14

Recap:

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- Linear regression:

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$$\min_{\boldsymbol{w}} \frac{C}{2}\|\boldsymbol{w}\|_2^2 + \sum_{i\in\mathcal{D}} \frac{1}{2}(1 - y^{(i)} \underbrace{\boldsymbol{w}^T \phi(x^{(i)})}_{F(x^{(i)},\boldsymbol{w})})^2$$

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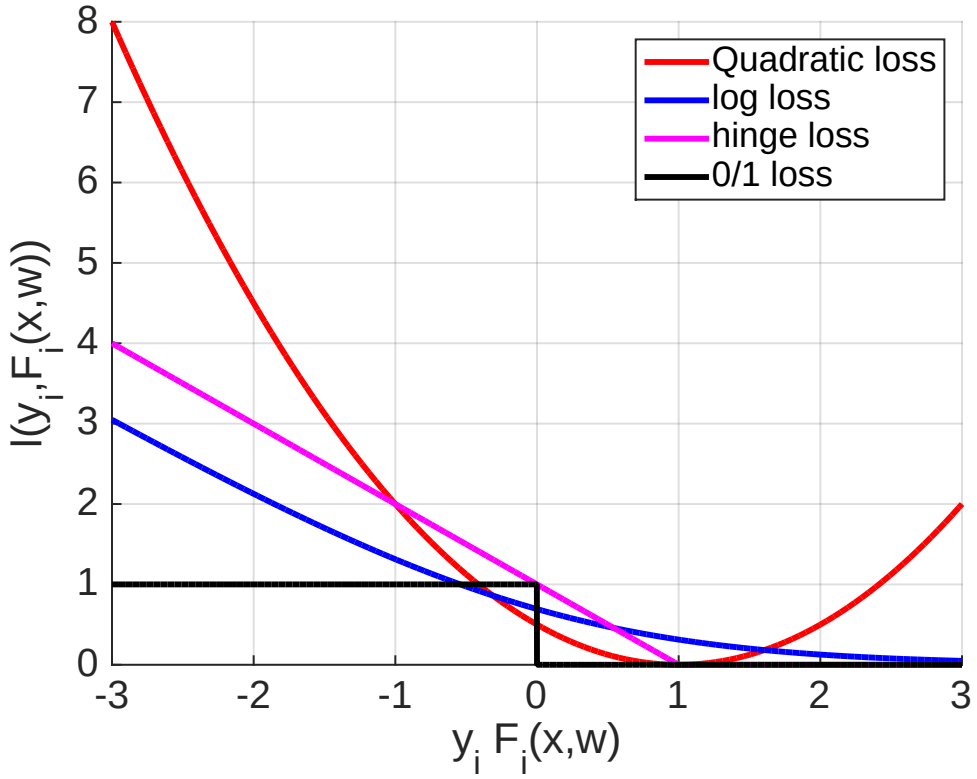
Quadratic loss
log loss
hinge loss
0/1 loss
8
7
6
5
4
3
2
1
0
-3
-2
-1
0
1
2
3
l(y_i,F_i(x,w))
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SVM as 0-temperature limit of logistic regression

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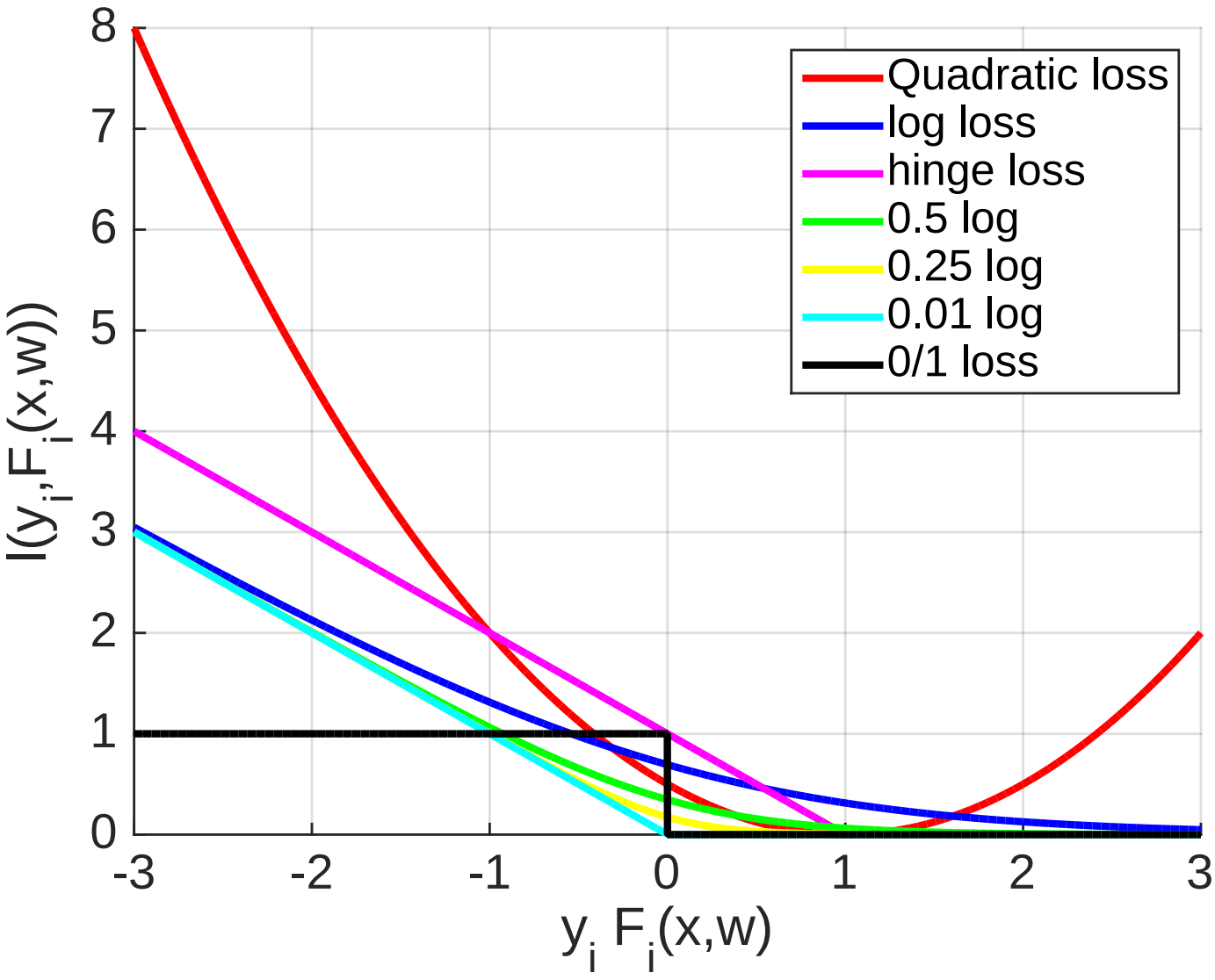
Different loss functions



Tasks we considered so far:

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- Object detection

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More than two answers (multiple classes) need to be considered.

Multiclass classification: How to classify between K classes?

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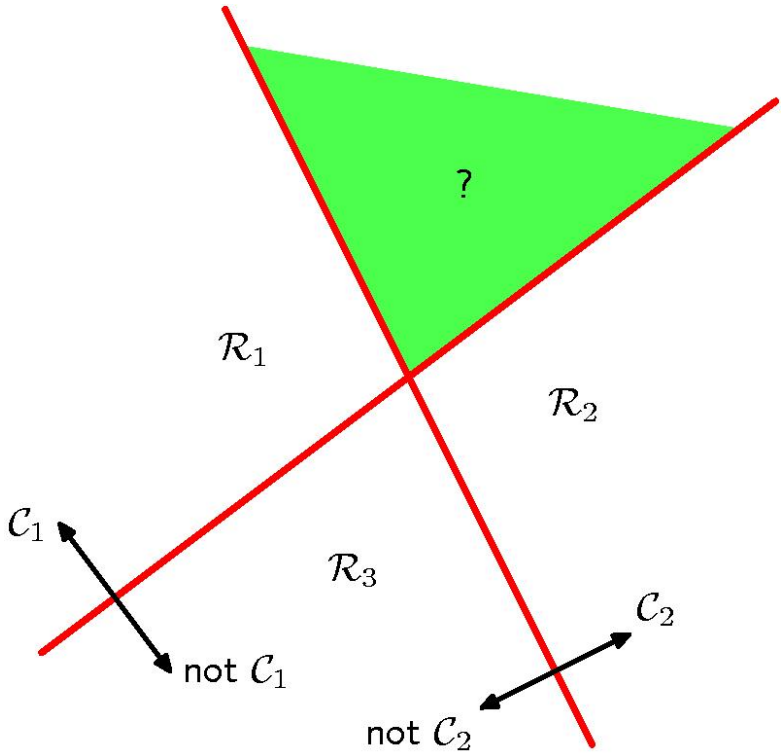

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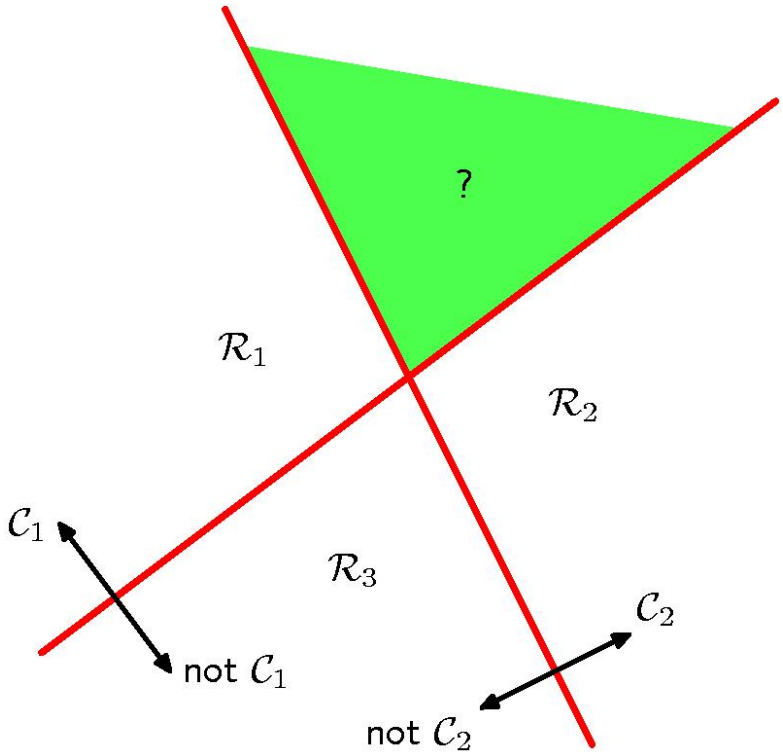

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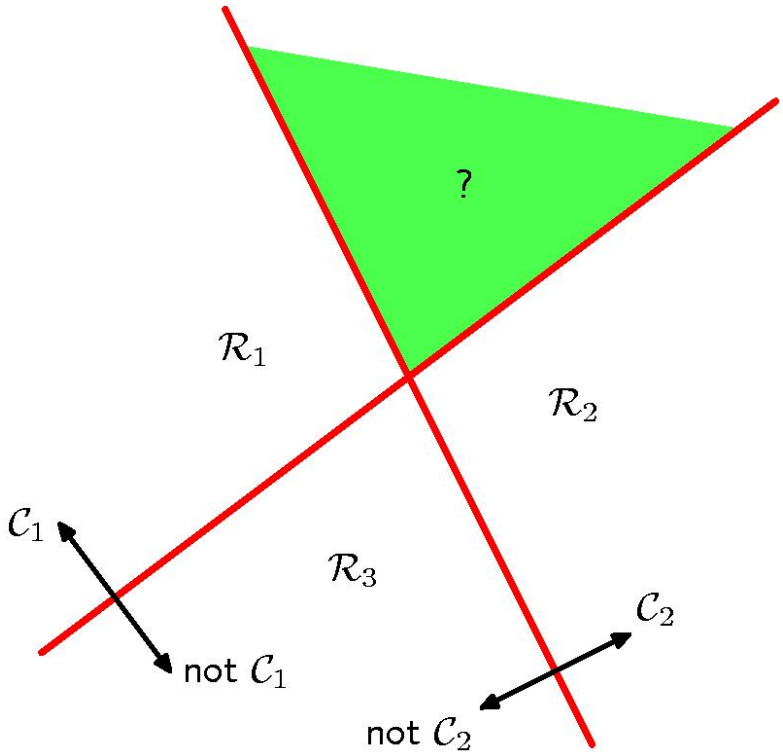

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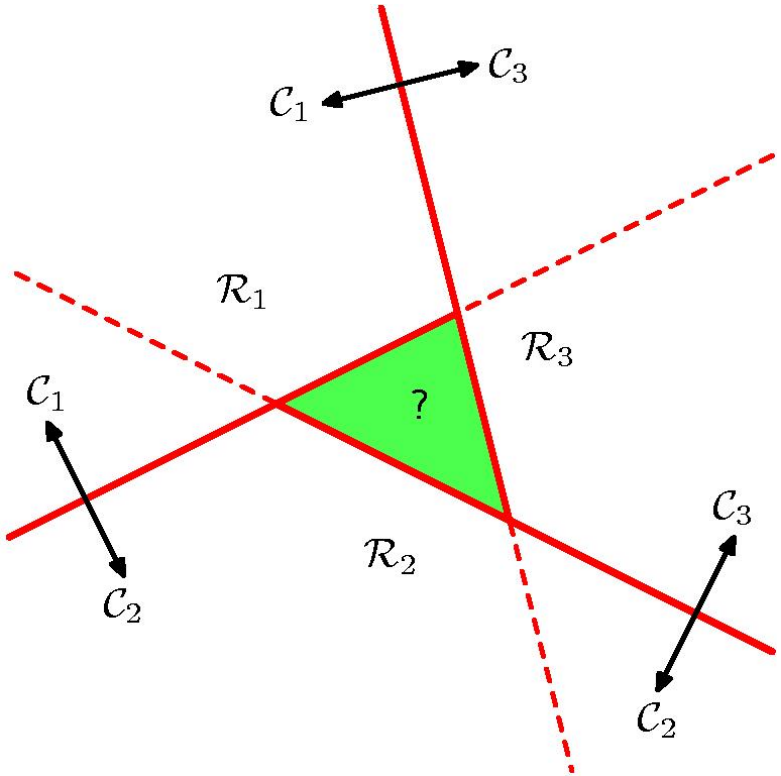

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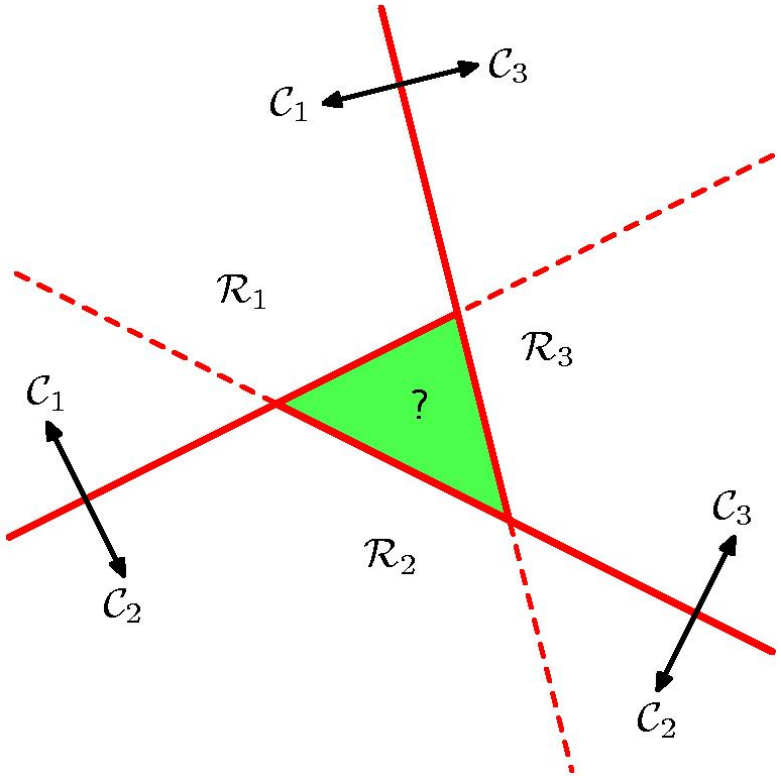

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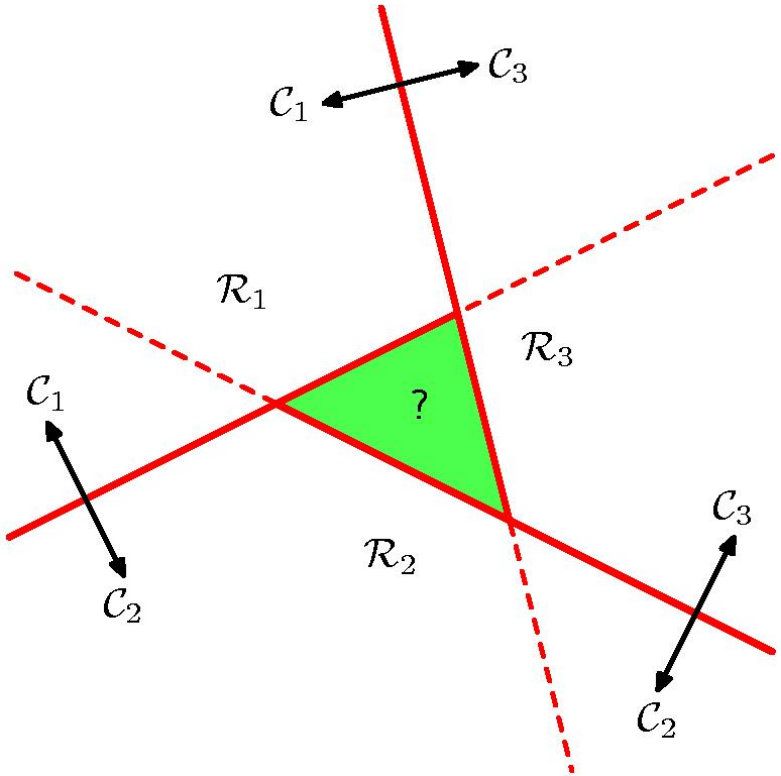

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$$\arg\max_{\boldsymbol{w}} \prod_{(x^{(i)},y^{(i)}) \in \mathcal{D}} p(y = y^{(i)}|x^{(i)}) = \arg\min_{\boldsymbol{w}} \sum_{(x^{(i)},y^{(i)}) \in \mathcal{D}} -\log p(y = y^{(i)}|x^{(i)})$$

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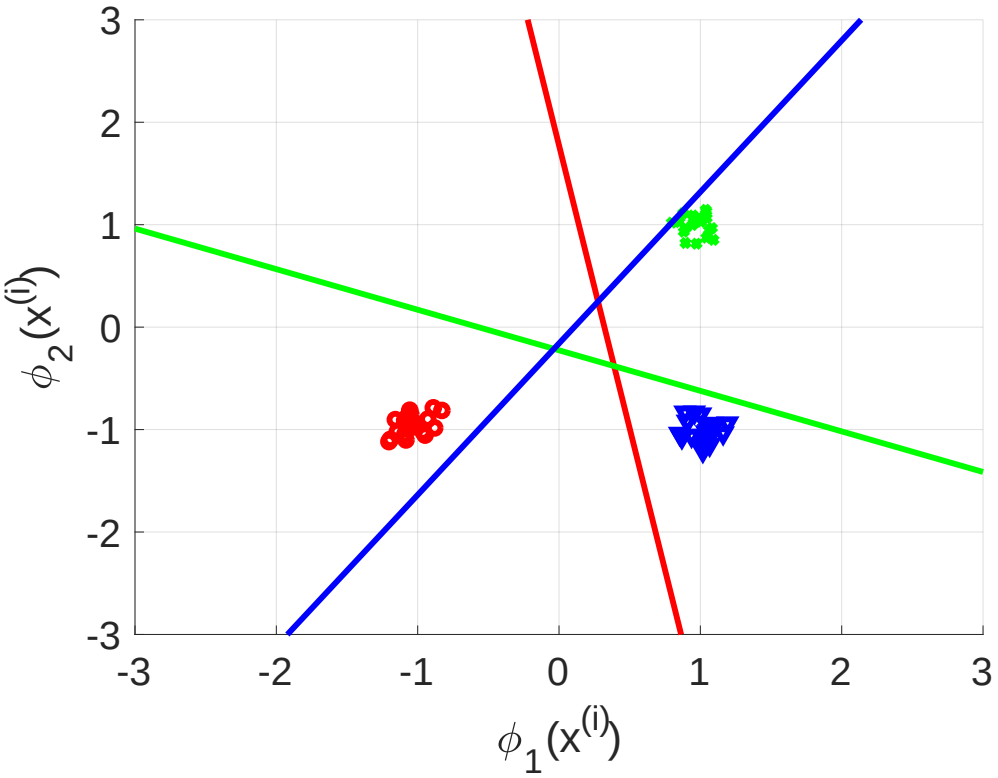
- How does this relate to our earlier binary setting?

Example: $(\phi_3(x^{(i)}) = 1)$



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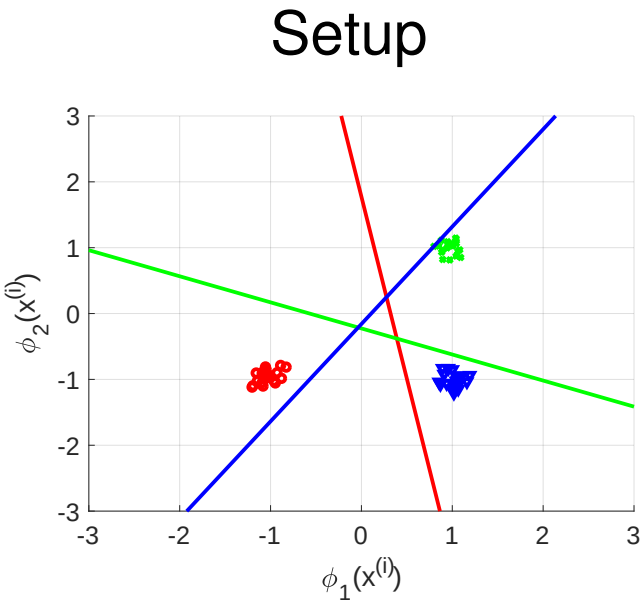

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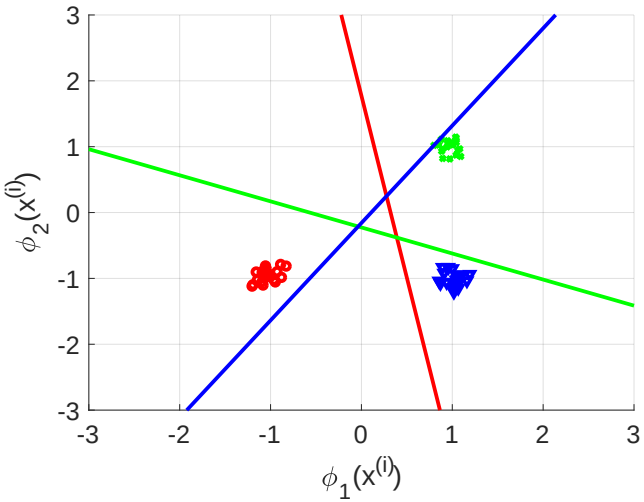

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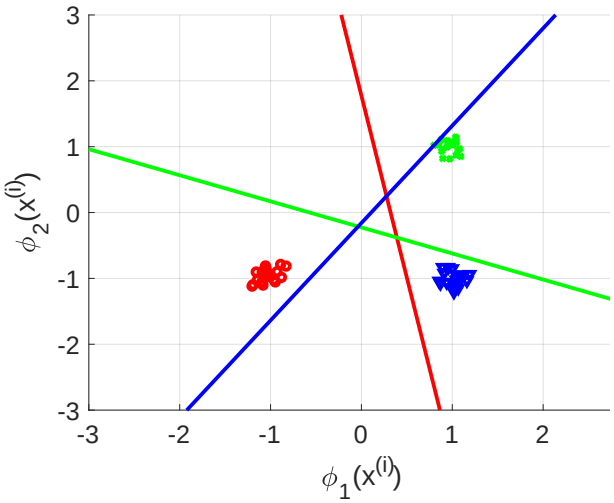

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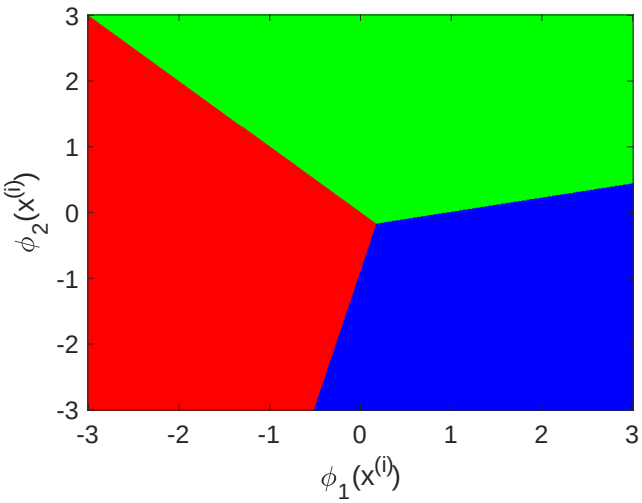

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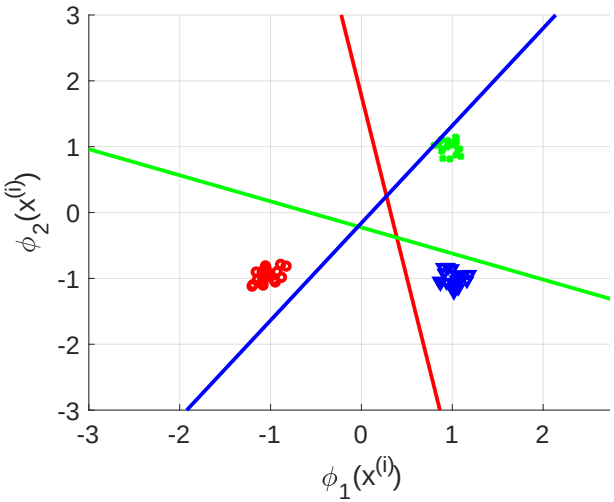
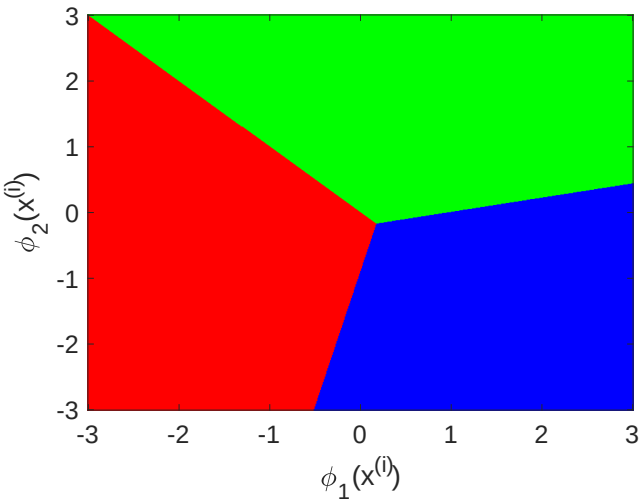


Decision Boundary Probability for $k = 1$



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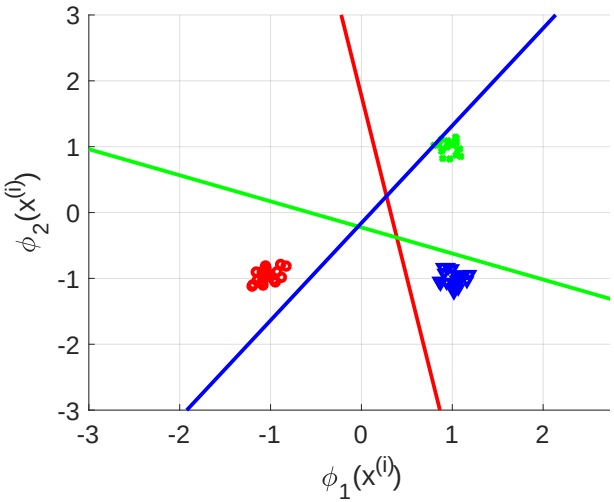

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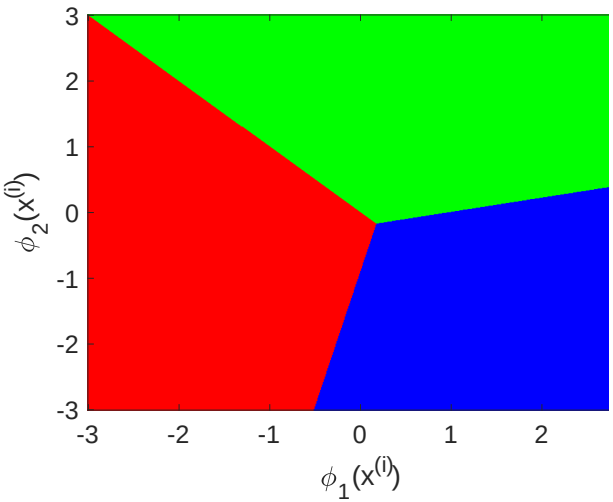

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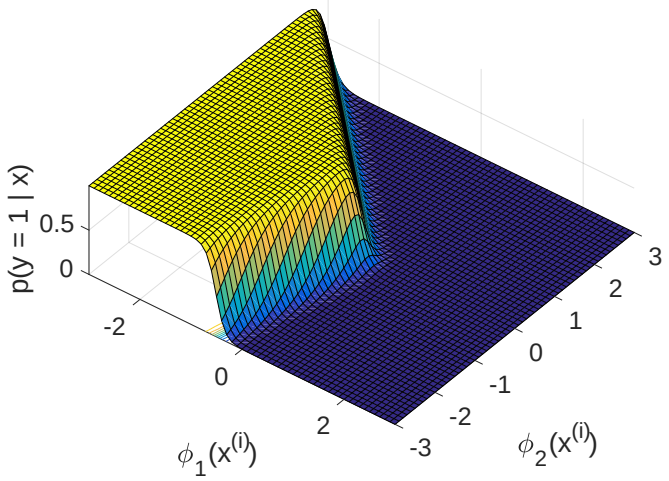

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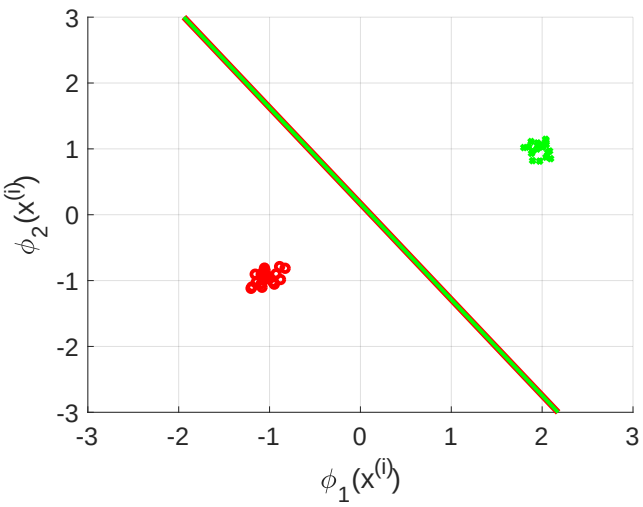

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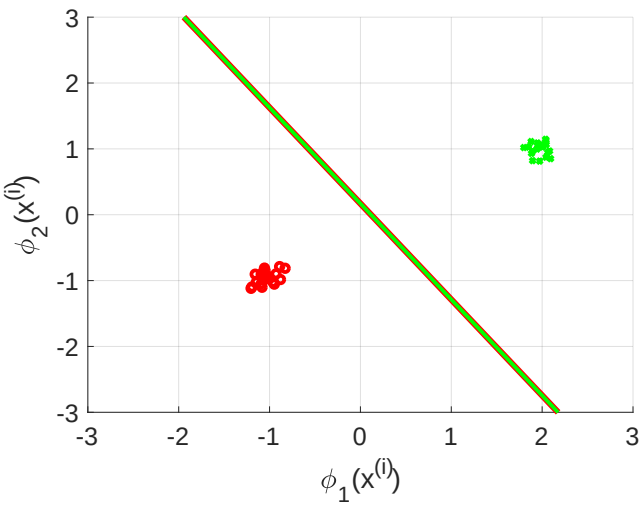

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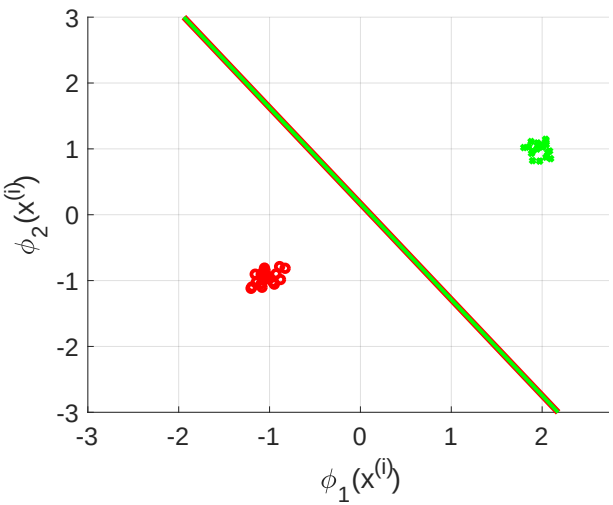

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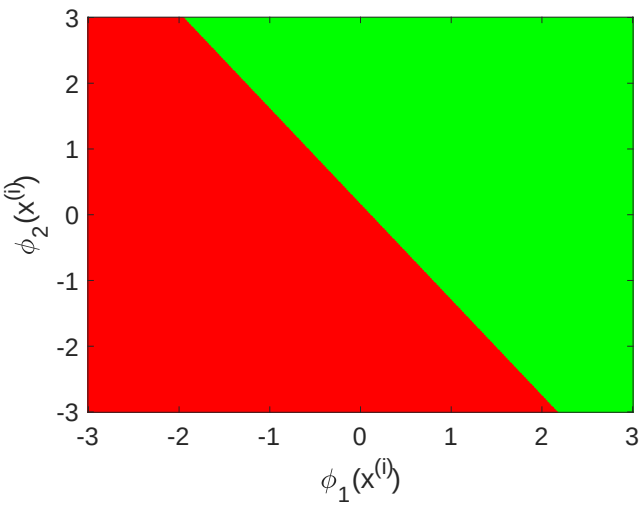

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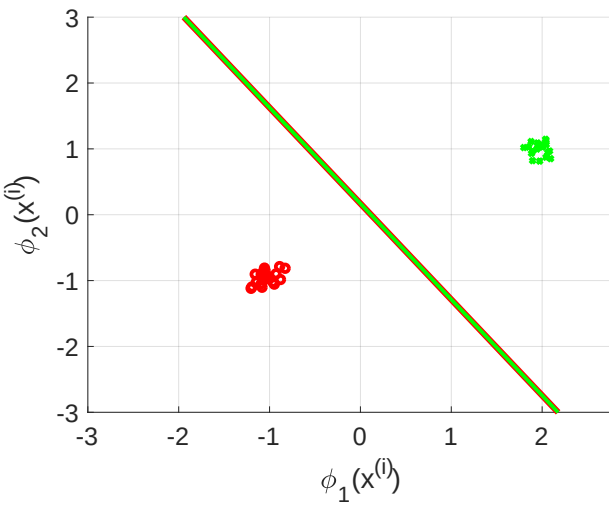

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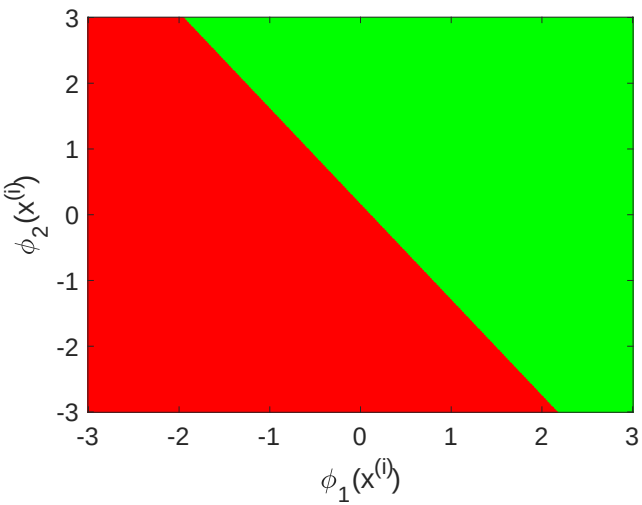

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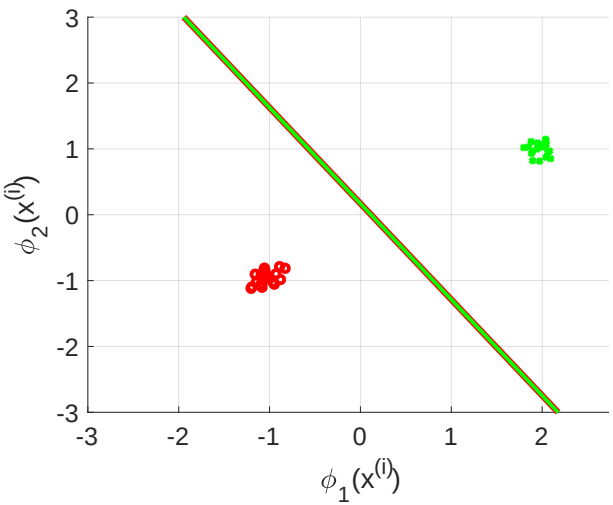

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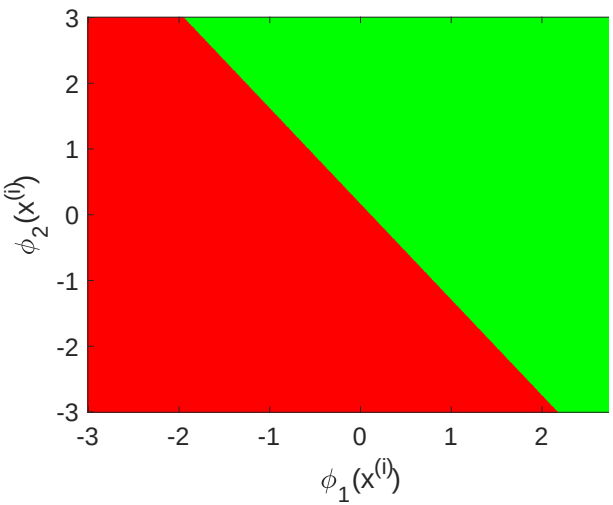

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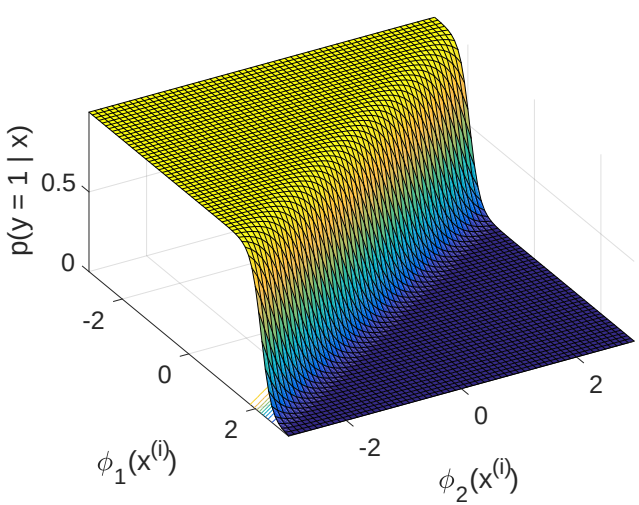

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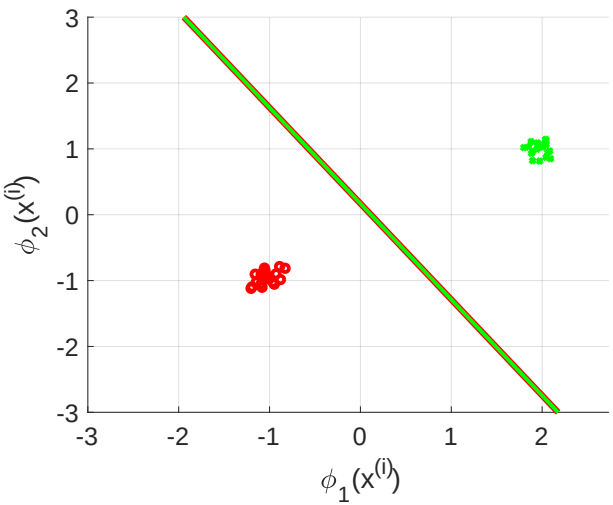

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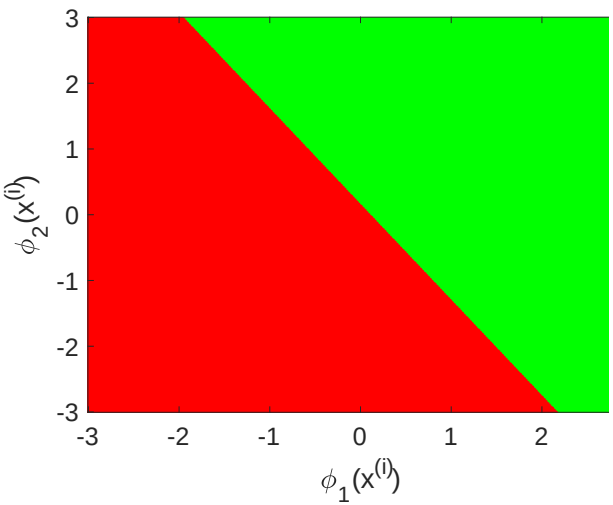

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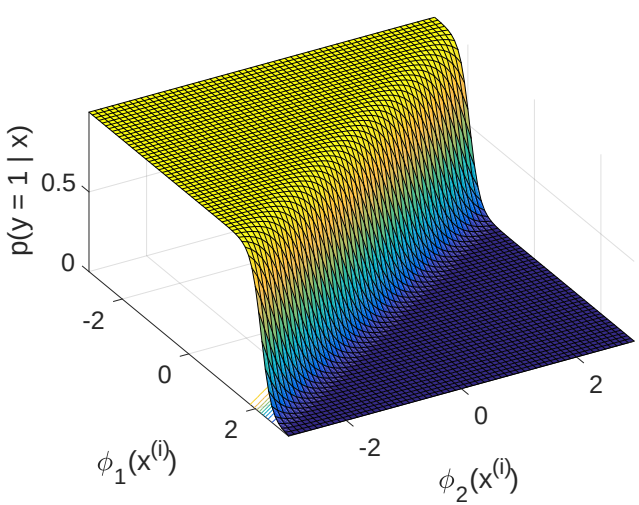
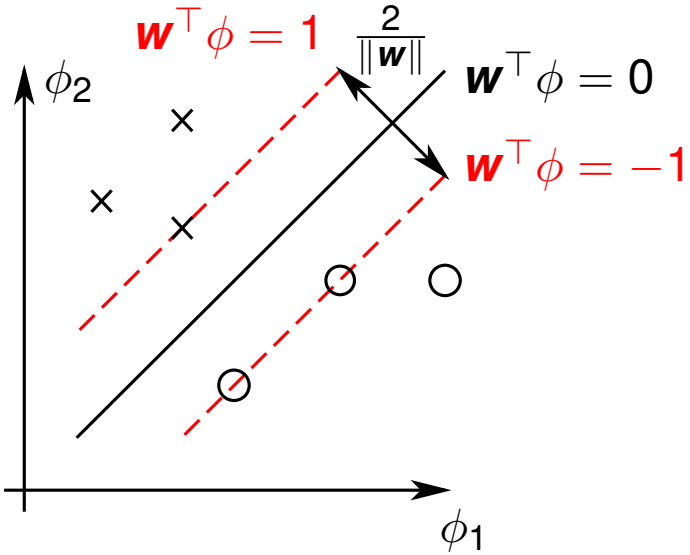


Lines coincide ($\boldsymbol{w}_{(1)} = -\boldsymbol{w}_{(2)}$)

How about multi-class SVM?

Recap: Binary SVM

Intuitively:



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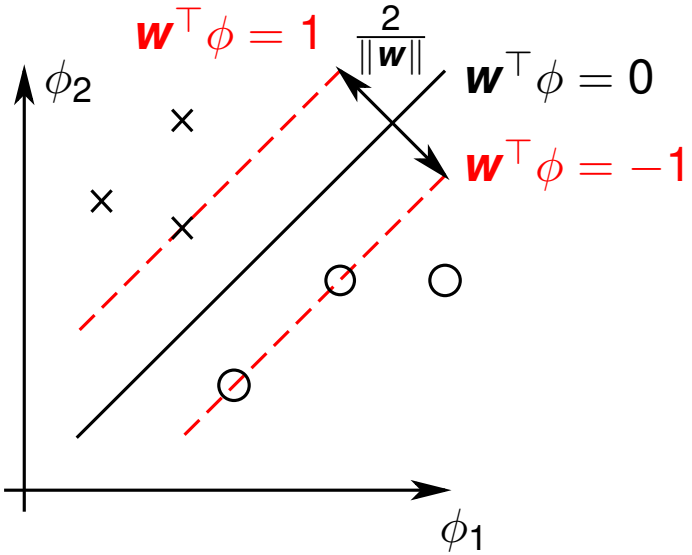

$$u_1 + c\boldsymbol{w} = u_2 \qquad \boldsymbol{w}^\top u_2 = -1$$

$$\boldsymbol{w}^\top (u_1 + c\boldsymbol{w}) = -1$$

$$c = \frac{-2}{\|\boldsymbol{w}\|_2^2}$$

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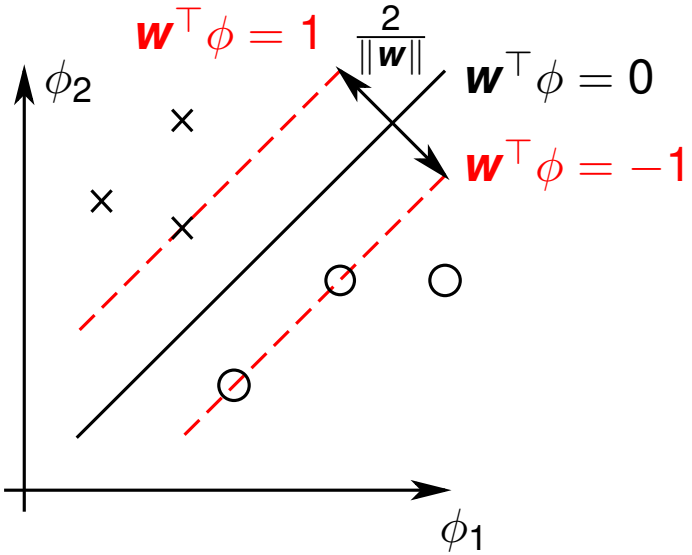

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Maximize margin $\frac{2}{\|w\|}$:

$$\min_w \frac{1}{2}\|w\|_2^2 \qquad \text{s.t.} \qquad y^{(i)} w^\top \phi(x^{(i)}) \geq \underbrace{1}_{\text{Taskloss: } L} \qquad \forall (x^{(i)}, y^{(i)}) \in \mathcal{D}$$

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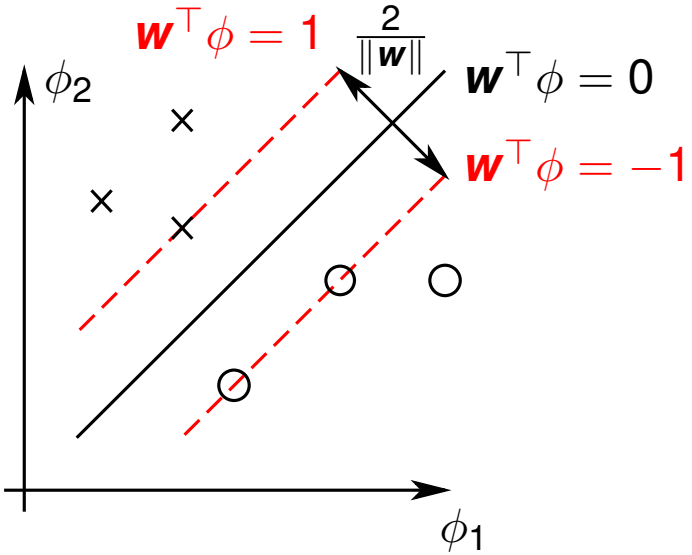

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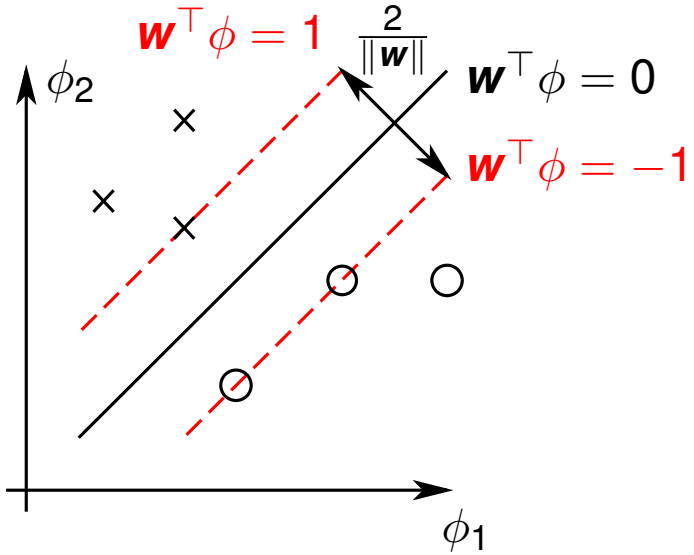

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Issue: what if data not linearly separable?

Recap: Binary SVM

Introduce slack variables $\xi^{(i)}$:

$$\min_{\boldsymbol{w},\xi^{(i)}\geq 0} \frac{C}{2}\|\boldsymbol{w}\|_2^2 + \sum_{i\in\mathcal{D}} \xi^{(i)} \qquad \text{s.t.} \qquad y^{(i)}\boldsymbol{w}^\top \phi(x^{(i)}) \geq 1 - \xi^{(i)} \quad \forall i \in \mathcal{D}$$

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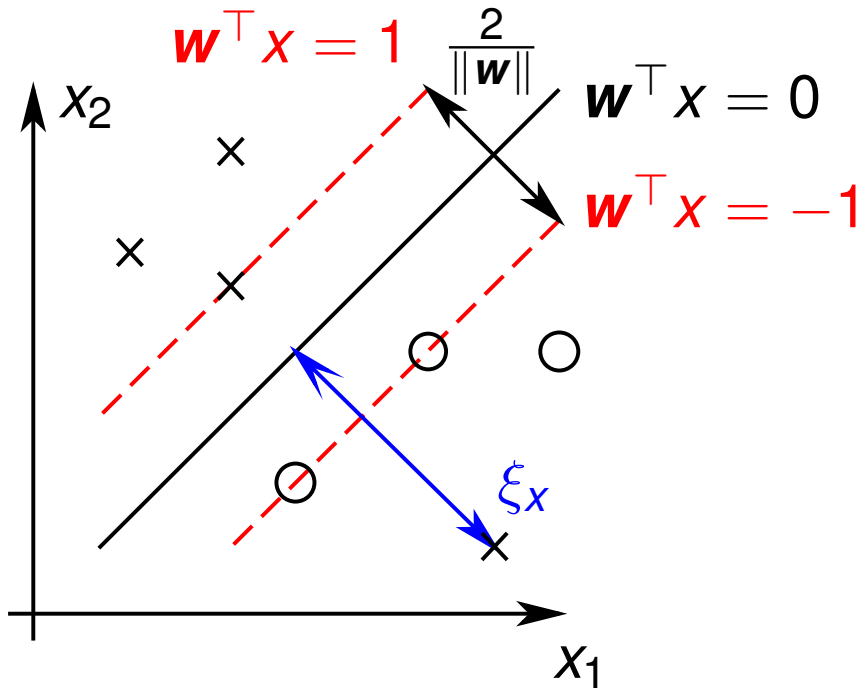

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Define mapping:

$$\boldsymbol{w} = \begin{bmatrix} \boldsymbol{w}_0 \\ \vdots \\ \boldsymbol{w}_{K-1} \end{bmatrix} \qquad \psi(x^{(i)}, y^{(i)}) = \begin{bmatrix} \phi(x^{(i)})\delta(y^{(i)} = 0) \\ \vdots \\ \phi(x^{(i)})\delta(y^{(i)} = K-1) \end{bmatrix}$$

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Equivalent formulation:

$$\min_{\boldsymbol{w},\xi^{(i)}\geq 0} \frac{C}{2}\|\boldsymbol{w}\|_2^2 + \sum_{i\in\mathcal{D}} \xi^{(i)} \quad \text{s.t.} \quad \boldsymbol{w}^T \left(\psi(x^{(i)}, y^{(i)}) - \psi(x^{(i)}, \hat{y})\right) \geq 1 - \xi^{(i)} \quad \forall i, \hat{y}$$

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How does multiclass SVM relate to multiclass logistic regression?

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Same temperature argument as in the last lecture.

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Homework

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Homework

Solution:

$$\min_{\boldsymbol{w}} \frac{C}{2}\|\boldsymbol{w}\|_2^2 + \sum_{i\in\mathcal{D}} \epsilon \ln \sum_{\hat{y}} \exp \frac{L(y^{(i)},\hat{y}) + \boldsymbol{w}^T\psi(x^{(i)},\hat{y}))}{\epsilon} - \boldsymbol{w}^T\psi(x^{(i)},y^{(i)})$$

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Use Kernels

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We need to ensure that κ is a valid kernel, i.e., it corresponds to an inner product in some feature space

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How to test without explicitly constructing feature vectors:

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- Decompose/Compose kernel into/from known kernels

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- Sigmoid kernel:

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Quiz:

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- What are kernels good for?

Important topics of this lecture

- Multi-class classification
- Kernels as non-linear feature mappings

Up next:

- Deep nets