Machine Learning

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University of Illinois at Urbana-Champaign, 2018

L3: Logistic Regression

Goals of this lecture

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- Understand logistic regression

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Reading Material

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- Contrast linear and logistic regression
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Reading Material

- K. Murphy; Machine Learning: A Probabilistic Perspective; Chapter 8

The Problem: Linear regression for classification

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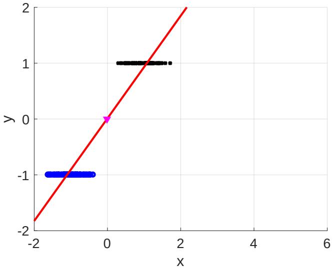

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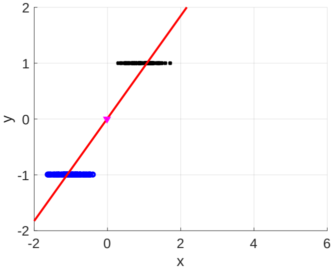

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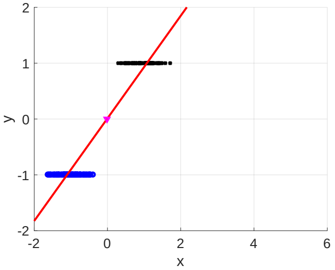

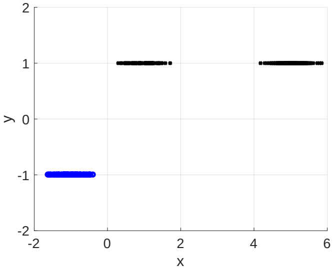

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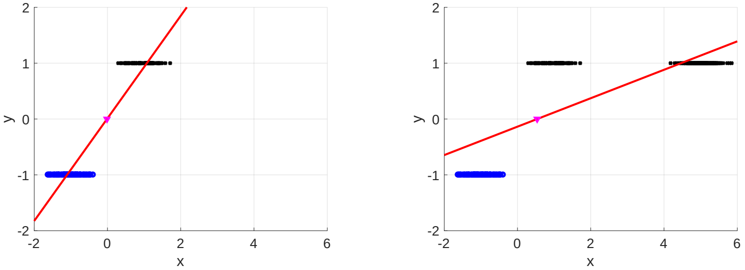

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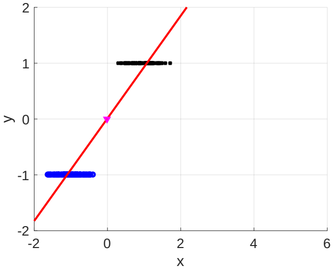

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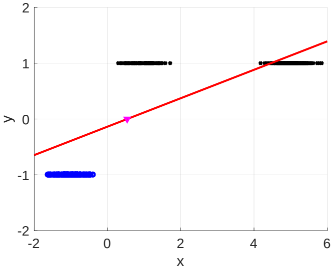

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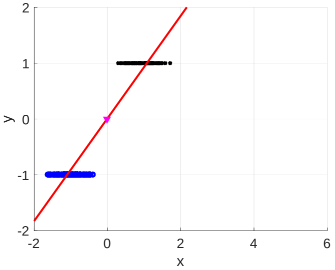

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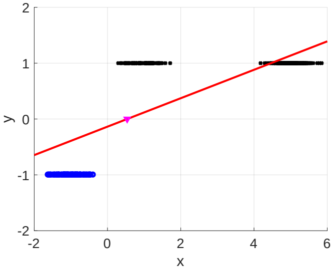

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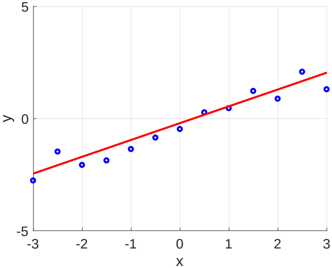

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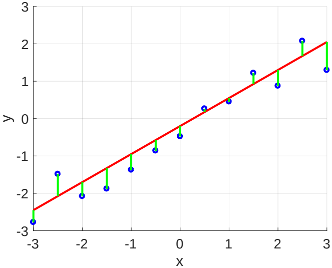
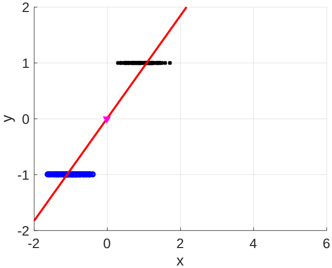


In our case:



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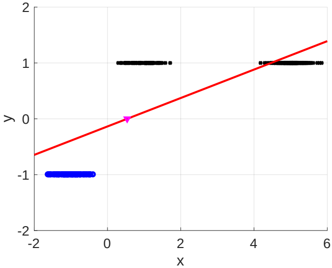

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Linear regression: Quadratic loss (recall $y^{(i)} \in \{-1, 1\}$)

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$$\begin{aligned}
\ell(y_i, \phi(x^{(i)})^\top \boldsymbol{w}) &= \frac{1}{2}(y^{(i)} - \phi(x^{(i)})^\top \boldsymbol{w})^2 \\
&\overset{\left(y^{(i)}\right)^2=1}{=} \frac{1}{2}(1 - \underbrace{y^{(i)} \underbrace{\phi(x^{(i)})^\top \boldsymbol{w}}_{F(x^{(i)},\boldsymbol{w})}}_{F(x^{(i)},\boldsymbol{w},y^{(i)})})^2
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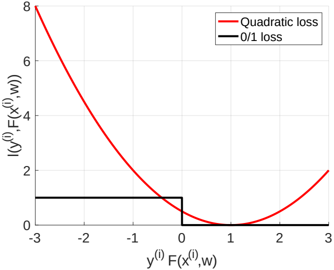

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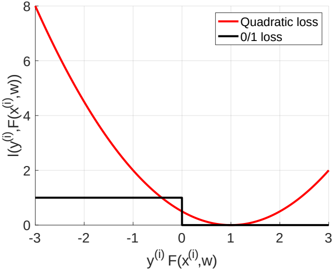

We penalize samples that are 'very easy to classify.'

How to fix this?

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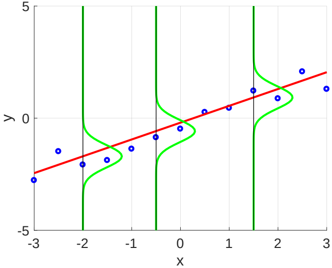

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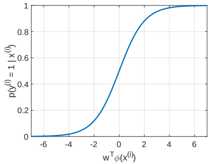

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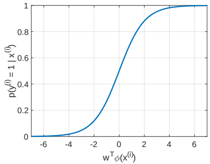

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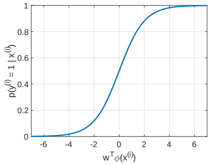

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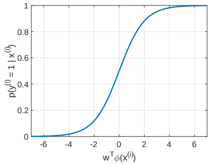

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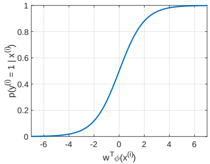

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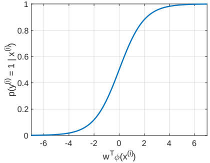

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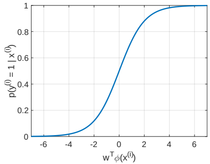

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What to do with this model?

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Task:

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$$\min_{\boldsymbol{w}} \sum_{(x^{(i)},y^{(i)})\in\mathcal{D}} \log\left(1 + \exp(-y^{(i)}\boldsymbol{w}^T\phi(x^{(i)}))\right)$$

Comparison

Linear regression

Program:

Logistic regression

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Program:

$$\min_{\boldsymbol{w}} \sum_{(x^{(i)},y^{(i)})\in\mathcal{D}} \frac{1}{2}\underbrace{(1-y^{(i)}\underbrace{\boldsymbol{w}^T\phi(x^{(i)})}_{F(x^{(i)},w)})}_{F(x^{(i)},w,y^{(i)})}{}^2$$

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Empirical risk minimization:

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$$\min_{\boldsymbol{w}} \sum_{(x^{(i)},y^{(i)})\in\mathcal{D}} \log\left(1 + \exp(-y^{(i)} \underbrace{\boldsymbol{w}^T\phi(x^{(i)})}_{F(x^{(i)},w)})\right)$$

Linear regression:

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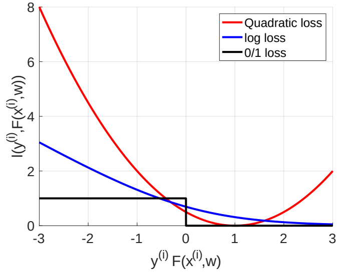

How to optimize

$$\min_{\boldsymbol{w}} \sum_{(x^{(i)},y^{(i)})\in\mathcal{D}} \log\left(1+\exp(-y^{(i)}\underbrace{\boldsymbol{w}^T\phi(x^{(i)})}_{F(x^{(i)},w)})\right)$$

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Can we set the gradient to zero and solve for $\boldsymbol{w}$?

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No analytic solution for $\boldsymbol{w}$ in general

How to optimize

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Gradient descent: (walking down a mountain)

How to optimize

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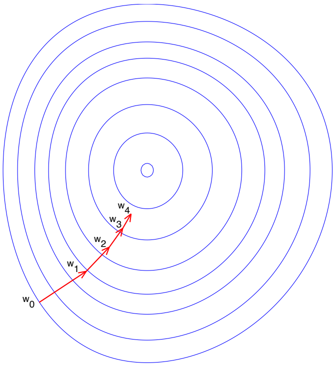

To solve

$$\min_{\boldsymbol{w}} f(\boldsymbol{w}) := \sum_{(x^{(i)},y^{(i)})\in\mathcal{D}} \log\left(1+\exp(-y^{(i)}\underbrace{\boldsymbol{w}^T\phi(x^{(i)})}_{F(x^{(i)},w)})\right)$$

we can use its gradient:

$$\nabla_{\boldsymbol{w}} f(\boldsymbol{w})$$

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Simple algorithm:

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Simple algorithm: Initialize $t=0$, $\boldsymbol{w}_t$, and stepsize α

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To solve

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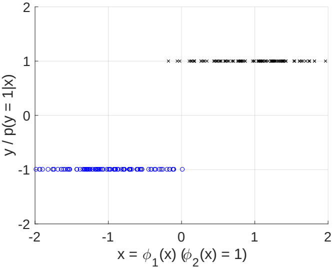
- Update $t \leftarrow t + 1$

More complex algorithms may be ‘better.’

Example:

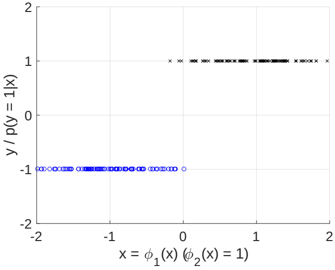
Data



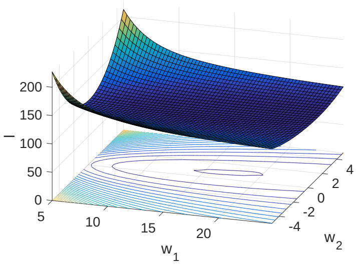
Loss

Example:

Data



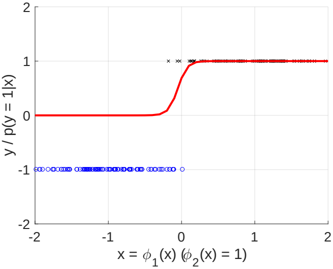
Loss



Movie

Example:

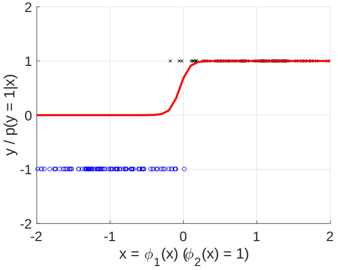
Data



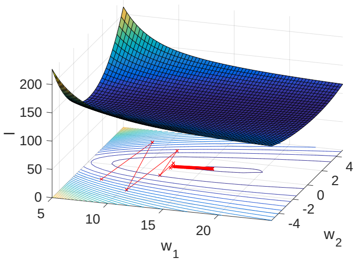
Loss

Example:

Data



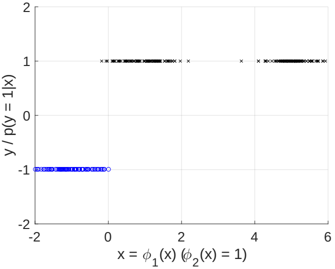
Loss



Example:

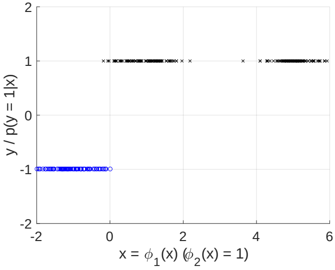
Data

Loss

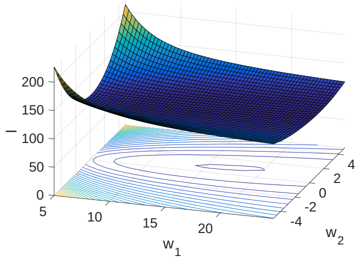


Example:

Data



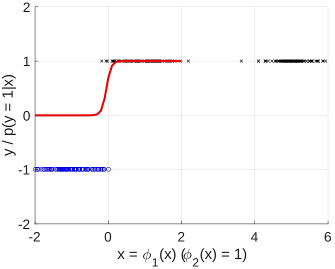
Loss



Movie

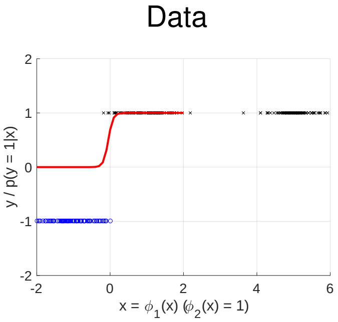
Example:

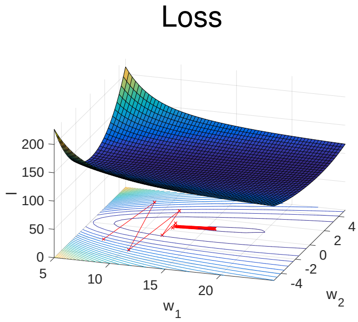
Data



Loss

Example:


Data
2
1
0
-1
-2
-2
0
2
4
6
y / p(y = 1|x)
x = ϕ_1(x) (ϕ_2(x) = 1)


Loss
200
150
100
50
0
l
5
10
15
20
w_1
4
2
0
-2
-4
w_2

Comparison:

Linear regression:

Logistic regression:

Comparison:

Linear regression:

- Closed form solution

Logistic regression:

Comparison:

Linear regression:

- Closed form solution
- Gaussian probability model

Logistic regression:

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- Not too well suited for classification

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Logistic regression:

- Well suited for binary classification

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Comparison:

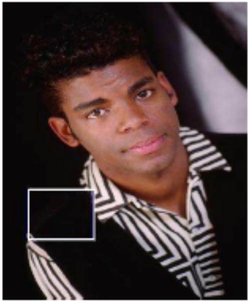
Linear regression:

- Closed form solution
- Gaussian probability model
- Not too well suited for classification

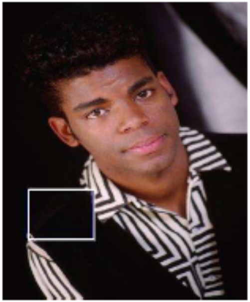
Logistic regression:

- Well suited for binary classification
- Logistic probability model
- No closed form solution

Application: Edge/Boundary detection: Issues?

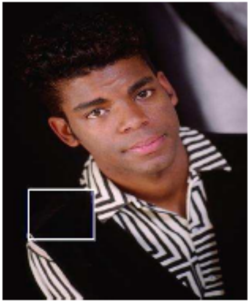


Application: Edge/Boundary detection: Issues?



Poor contrast

Application: Edge/Boundary detection: Issues?



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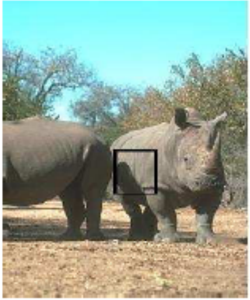

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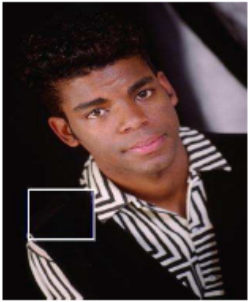

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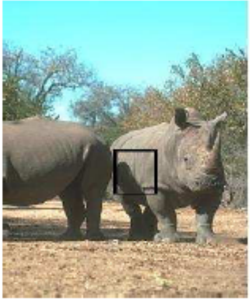
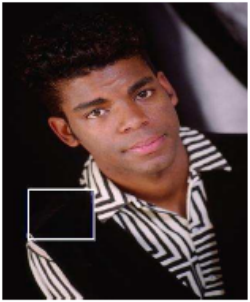


Shadow

Application: **Edge/Boundary detection:** Issues?



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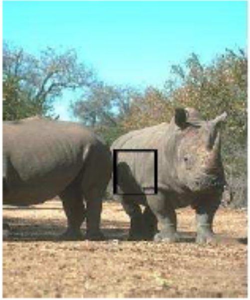

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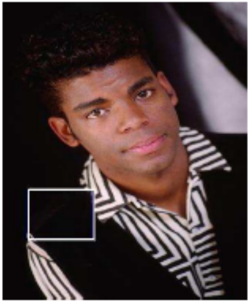

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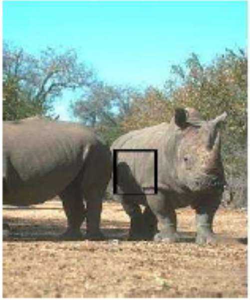
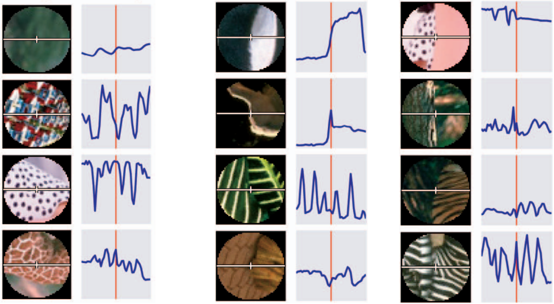


Shadow

Texture

Edge/Boundary detection: Why is it so difficult?

Let's look at a local image region and the corresponding intensities:



Edge/Boundary detection: Why is it so difficult?

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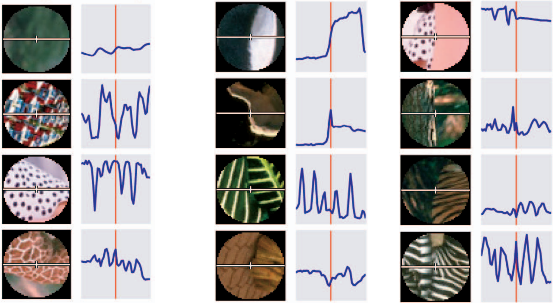

Non-Boundaries

Edge/Boundary detection: Why is it so difficult?
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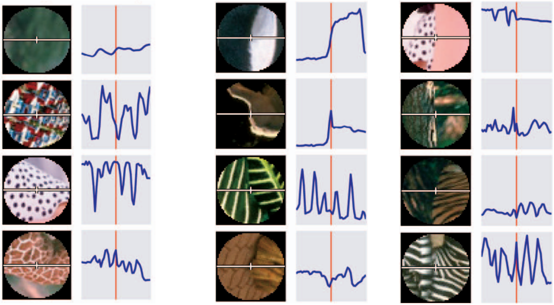

Non-Boundaries Boundaries

Edge/Boundary detection: Why is it so difficult?

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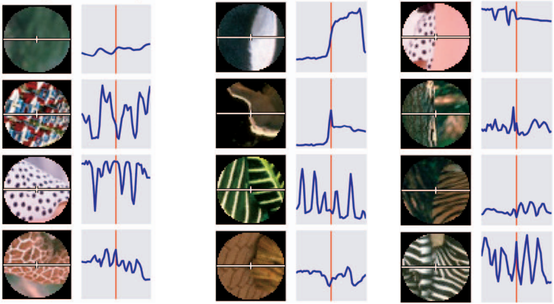

Non-Boundaries Boundaries

Intensity cue is not necessarily a good indicator for boundaries.

Edge/Boundary detection: What other image cues could be useful?

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- Boundary gradient (BG)

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Edge/Boundary detection: What other image cues could be useful?

- Boundary gradient (BG)
- Color gradient (CG)
- Texture gradient (TG)
- ...


Image
BG
CG
TG

How to combine all those cues?

How to combine all those cues? Learn a linear combination of cues

- What is $y^{(i)}$?

- What is $y^{(i)}$? Annotated pixel label

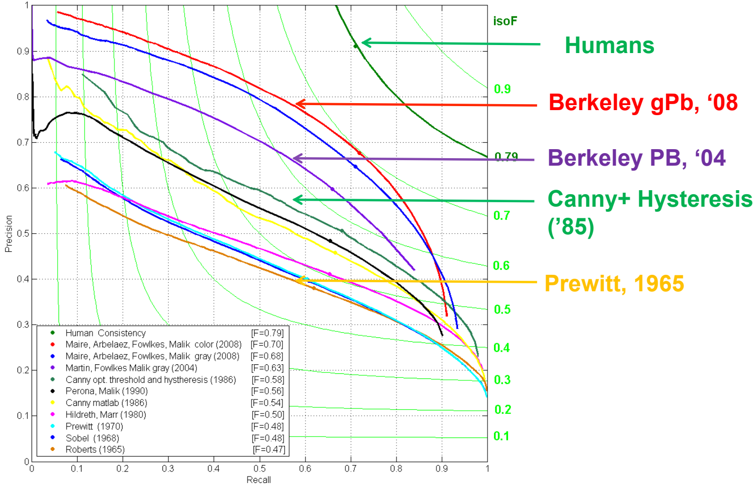
- What is $y^{(i)}$? Annotated pixel label
- What is $x^{(i)}$?

- What is $y^{(i)}$? Annotated pixel label
- What is $x^{(i)}$? Image

- What is $y^{(i)}$? Annotated pixel label
- What is $x^{(i)}$? Image
- What is $\phi(x^{(i)})$?

- What is $y^{(i)}$? Annotated pixel label
- What is $x^{(i)}$? Image
- What is $\phi(x^{(i)})$? Vector of features computed in the **neighborhood** of pixel i, e.g., intensity, texture gradient, oriented gradient etc.

Boundary detection performance:



Quiz:

Quiz:

- Which loss is used for logistic regression?

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- Which loss is used for logistic regression?
- What is the difference between logistic and linear regression?

Quiz:

- Which loss is used for logistic regression?
- What is the difference between logistic and linear regression?
- How to optimize linear and logistic regression?

Important topics of this lecture:

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- Linear regression

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- Linear regression
- Logistic regression

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- Linear regression
- Logistic regression

Up next:

- Basics about optimization techniques