A SCALARIZED VCG MECHANISM FOR SUPPLY-SIDE MARKETS

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CERTIFICATE

This seminar report entitled **A SCALARIZED VCG MECHANISM FOR SUPPLY-SIDE MARKETS** by Naman Aggarwal is approved by me for submission for fulfilment of requirements of BTP Phase-II. The report was further certified that to the best of my knowledge, represents the work carried out by the student.

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Abstract

Traditional approaches to supply-side markets have required market participants to submit supply functions as bids. These being infinite dimensional objects induce a large communication overhead in the mechanism. Other approaches have involved parameterization or piece-wise linear approximation of supply functions, which may not accurately capture the structure of true cost functions. We propose an efficient mechanism for allocation of production in supply-side markets that requires suppliers to only submit scalar bids and involves no parameterization of supply functions. Our mechanism combines one-dimensional signalling with the VCG mechanism via the construction of certain surrogate cost functions. The latter are parameterized by the scalar bids of the suppliers. Supply allocations as well as payments are calculated as in a VCG mechanism using these surrogate cost functions. We establish that Nash equilibria of the resulting game induce optimal allocations, implying that our mechanism, which incurs minimal communication overhead, suffers no efficiency loss.

Contents

Chapter 1

Introduction

The central goal of market design is to effect socially efficient outcomes. Its basic challenge lies in information asymmetry – outcomes are decided by the system planner, but the system planner does not have access to the utility functions of the market participants.

Market mechanisms serve as a means of soliciting information from market participants in order to determine outcomes. This opens the possibility that participants may misreport their information in order to induce outcomes that are more favorable to them. The fundamental challenge of market design lies in designing a system of *information exchange* and *payments* made by or to the market participants so that truthful revelation of information is incentivized and socially optimal outcomes are realized.

The famed VCG [16] mechanism achieves precisely this. Market participants are asked to report their utility functions, and the system planner computes the socially optimal outcome *based on the reported utility functions*. Payments are determined based on the *externality* caused by the market participants and computed using the reported utility functions. With these payments, truthful revelation of utility functions is the optimal response. Even more remarkable is that there is no assumption of structure – the VCG mechanism allows for *any* kind of utility functions and *any* space of outcomes.

However, practically speaking, there is a nontrivial difficulty in implementing the VCG mechanism. The "bids" of the VCG mechanism are *utility functions* and as such are infinite dimensional objects and incur massive communication overheads. In some markets this is not a major issue – the utility function is largely the same across multiple instances of the auction and only the incremental change needs to be communicated. For example, in markets for conventional energy, generators may only need to communicate if any units are unavailable due to maintenance or other operational reasons, and the resulting incremental offset in their supply curves. However, in markets for selling renewable energy, the utility function of the generators is technically the (negative of the) expected cost function, conditioned on the information available to the generator at the time of bidding. Since the information about the weather forecast varies rapidly, the cost function also varies, and has to be resubmitted each time.

In this paper we design a socially efficient mechanism that requires market participants to only submit *scalar* bids. We focus on the supply-side problem where a collection of suppliers compete to meet a known inelastic demand. The suppliers submit a nonnegative scalar as a bid, using which an allocation is determined by the system planner. The system planner accomplishes this by constructing *surrogate cost functions* parametrized by the bids and allocates the demand by minimizing the total (surrogate) cost of meeting the demand. The payment made to a supplier by the system planner equals the externality caused by the supplier *as calculated using the surrogate cost function.* This is the payment the supplier *would have received* in a VCG mechanism had all suppliers submitted the surrogate cost functions as their bids. Thus the entire mechanism works as if the surrogate cost functions were the bid cost functions themselves. Remarkably, we find that if the true cost functions of suppliers are convex, differentiable and increasing, then under some technical conditions, every Nash equilibrium of the resulting game (in the space of scalar bids) leads to an efficient allocation.

The scalarization and efficiency achieved by our mechanism resolves a significant stumbling block in the market design for renewable energy supply. Supply-side markets in energy [19] have traditionally employed supply functions as bids. Mathematically, these are infinite dimensional objects and immensely hard to specify (see, e.g., [1]). Real-world auctioneers have addressed the complexity of specifying supply functions by asking suppliers to submit piece-wise linear approximations of these supply functions (see [4, 2]). To the best of our knowledge, day-ahead electricity markets world over now follow this format. Our mechanism shows that one can dramatically simplify such auctions in certain cases. Moreover, our mechanism differs fundamentally from the approximation-based approach because it *does not* assume any parameterized form for the supply functions.

Of course, this is not a free lunch. The scalarization achieved in our mechanism comes at the expense of the generality of the setting. Unlike the VCG mechanism which allows arbitrary utility functions and abstract outcome spaces, our setting is much more specific. In particular, the assumptions of convexity and differentiability of the true cost functions play a key role in our results.

The last two decades have witnessed an explosion of studies on mechanisms where participants submit scalar bids. Most of these have focused on the consumer-side problem, i.e., allocation of a resource amongst multiple consumers. Unfortunately, these mechanisms are known to lead to efficiency loss. The famous Kelly mechanism [11] for resource allocation allocates resources proportional to the bid submitted; it was analyzed by Johari and Tsitsiklis [8] in a price-anticipative setting and the worst case efficiency loss was shown to be 25%. The same authors also analyzed a supply side problem with inelastic demand with scalar-parametrized supply function bidding and showed the loss of efficiency in this case to be at most $\frac{1}{N-2}$ times the minimum aggregate production cost where N is the number of suppliers [10].

Yang and Hajek [21] introduced a mechanism that they called VCG-Kelly, a combination of the Kelly mechanism and the VCG mechanism, that achieves social efficiency with scalar bids. The allocation of the mechanism is identical to that of the Kelly mechanism, but the payment charged is a VCG-style externality payment. Our paper takes inspiration from Yang and Hajek to design its payment rule. However, unlike the consumer-side setting where surrogate value functions were already defined by Kelly [11], in our case there is no precedent for the form of our surrogate cost functions. An important innovation in our work thus lies in identifying the right surrogate cost functions.

We conclude this introduction with a brief survey of related work. The model and background is presented in Chapter 2. Our mechanism and its analysis is presented in Chapter 3. We conclude in Chapter 4.

Related work

Supply function equilibrium (SFE) models have been studied extensively in literature. [5] and [6] consider SFE models with continuous and differentiable supply functions, although in practice, producers participating in the electricity market submit discretized supply functions as discussed earlier. The work of Klemperer and Meyer [12] sparked activity in the electricity market modelling literature. Their model required the competing firms to have identical cost functions. The study of supply function equilibria when the firms have dissimilar cost functions is a significantly harder problem. Linearity assumption on the bid supply functions is required to tackle such market settings as discussed in [3]. Johari [10] [8] [7] develops a single-price market clearing mechanism which leads to efficient allocations in the price-taking setting without any assumption on the firms' cost functions. There is significant evidence in the literature that price-anticipating behaviour can lead to loss of efficiency (see [17]). Johari proposes a mechanism for the supplier side problem akin to the Kelly mechanism [11] on the consumer side and shows in [7] that the devised mechanism minimizes efficiency loss among a certain class of market-clearing mechanisms.

Efficiency loss in scalar parameterized mechanisms has also been widely studied for the resource allocation setting on the consumer side [14] [13] [15] [18] [20]. As mentioned in the introduction, Yang and Hajek [21] developed a scalarized mechanism and adopted a VCG-like payment structure to show that their mechanism results in efficient allocations at all Nash equilibrium points (NEPs). Johari and Tsitsiklis [9] also developed a family of VCG-like mechanisms for the consumer side using simple scalar bids per player in a much more general setting by considering convex constraints on the allocation. In both of these works, simplicity of payments (single-price like payments as in Kelly) was sacrificed for efficiency at the NEPs.

Inspired by this progress in mechanism design on the consumer side, we develop a mechanism for the supplier side that achieves full efficiency at NEPs at the expense of the simplicity of the payment structure.

Chapter 2

Model and Background

We consider a market with a set of $N > 2$ producers, $\mathcal{N} = \{1, 2, \ldots, N\}$ competing to satisfy an infinitely divisible, inelastic demand, D. The cost function of the producer j is represented by $C_j(x_j)$, which is a strictly increasing, convex and continuously differentiable function of its production allocation, x_j over $\mathbb{R}$ with the assumption that $C_j(x_j) = 0, \ \forall \ x_j \leq 0$. A market planner is interested in solving the following problem termed the System Problem,

$$\min_{\mathbf{x} \in X_F} \sum_{j \in [N]} C_j(x_j) \tag{S}$$

where $X_F = \{\mathbf{x} \mid x_j \geq 0, \ \sum_{j \in [N]} x_j = D\}$. The system problem is a convex optimization problem with a compact feasible region, hence an optimal solution exists. Associated with every optimal allocation $\mathbf{x}^*$, there exists a Lagrange multiplier λ^* satisfying the following Karush-Kuhn-Tucker (KKT) conditions for our problem:

$$C_j^{'}(x_j^*) \geq \lambda^*, \qquad x_j^* = 0 \tag{2.1}$$

$$C_j^{'}(x_j^*) = \lambda^*, \qquad x_j^* > 0 \tag{2.2}$$

$$\text{and,} \quad \sum_j x_j^* = D \tag{2.3}$$

These conditions are necessary as well as sufficient conditions of optimality, that is, any $\mathbf{x}^*$ along with an associated λ^* which satisfies the above conditions is an optimal solution to (S).

As mentioned in the introduction, a producer's cost function is its private knowledge and is not known to the system planner who attempts to solve (S). This necessitates a mechanism by which the system planner elicits information from the producers such that the producers truthfully reveal their information, and thus the solution of (S) is achieved.

One can employ the celebrated Vickrey-Clarke-Groves (VCG) mechanism as a start. In the VCG mechanism, producers bid their cost functions and the market planner issues payments to the producers in such a way, that each producer is better off reporting its true cost function irrespective of what other producers report. In other words, truth telling is a dominant strategy equilibrium under the VCG mechanism. Since producers report their true cost functions, the allocation made by the market planner in equilibrium is efficient with respect to the System Problem (S). The VCG mechanism gives us desirable behaviour of the producers but requires submission of infinite-dimensional bids.

One simple method to reduce the communication overhead is to let the producers submit scalar bids which can be interpreted as a parametrization of their private cost functions. This has been done for the consumer side problem in the Kelly mechanism, which we discuss next.

2.0.1 Kelly Mechanism

Though originally proposed for a different problem setting – resource allocation over a network to maximise the total utility of all players with private utility functions, the idea of Kelly mechanism is still of relevance here. The System Problem for the Kelly mechanism is,

$$\max_{\mathbf{x}\in X_C} \sum_{j\in[N]} U_j(x_j) \tag{C}$$

where $X_C = \{\mathbf{x}|x_j \geq 0, \sum_{j\in[N]} x_j \leq D\}$ and the utility function for a player r, $U_r(x_r)$ is an increasing, concave and continuously differentiable function of the allocation x_r. In the Kelly Mechanism, communication between players and the market planner involves the players submitting scalar bids $\mathbf{w} = \{w_r\}_r$ to the planner and the planner returning the allocation, $x_r(\mathbf{w})$ to each player and the price per unit resource $\lambda(\mathbf{w})$, where $\mathbf{w}$ is the bid vector. The planner makes allocations based on the following rule,

$$\mathbf{x}(\mathbf{w}) = \begin{cases} \text{argmax}_{\mathbf{x}\in X_C} \sum\limits_{r\in[N]} w_r \log(x_r), & \text{if} \quad \mathbf{w} \neq 0 \\ 0, & \text{if} \quad \mathbf{w} = 0 \end{cases}$$

where $w_r \log(x_r)$ serves as a proxy for the true utility function $U_r(.)$ of player r. We can reduce the above allocation rule for a player r to $x_r(w_r, \mathbf{w}_{-r}) = \left(w_r/(\sum_{k\in[N]} w_k)\right) D, w_r > 0$ and $x_r(0, \mathbf{w}_{-r}) = 0$. Price per unit resource, $\lambda(\mathbf{w})$ as a function of the bid vector is determined as, $\lambda(\mathbf{w}) = \left(\sum_{k\in[N]} w_k\right)/D$. Total price charged to player r is, $m_r(w_r, \mathbf{w}_{-r}) = \lambda(w_r, \mathbf{w}_{-r})x_r(w_r, \mathbf{w}_{-r}) = w_r$. The payoff for player r therefore is,

$$\Pi_r(w_r, \mathbf{w}_{-r}) = U_r(x_r(w_r, \mathbf{w}_{-r})) - w_r$$

This induces a game among the players and we can characterise the efficiency loss, if at all, at the Nash equilibrium bid vectors for this game (existence and conditions on uniqueness/non-uniqueness of Nash equilibrium bid vectors for this problem has been widely studied and hence not repeated here). Johari and Tsitiklis [8] showed that for strategic buyers using the Kelly mechanism, the efficiency loss is atmost 25%. Efficiency loss is the price we pay for scalarization and captures the trade-off between efficiency and communication complexity.

We now give an overview of the work of Johari and Tsitiklis [10] on scalarized bidding for the supplier side problem. This is well aligned with the theme and setting of this paper as opposed to the above detour on Kelly mechanism which was used to introduce the idea of parametrization of private information using scalar bids and capture the trade-off between efficiency and communication complexity that such a scalarization implies.

2.0.2 Supply Function Bidding using Scalar Bids

Johari and Tsitiklis [10] assume a supply function of the form, $S(p, w) = D - \frac{w}{p}$ parametrized by w where p is the price charged by the market maker. Each firm submits a scalar bid w which parametrizes its supply function followed by the market planner determining the market clearing price $p(\mathbf{w})$ at which the aggregate supply exactly equals the inelastic demand, D. Payoff $\Pi_n(w_n; \mu)$ for a price-taking firm n at market price μ is given as, $\Pi_n(w_n; \mu) = \mu S(\mu, w_n) - C_n(S(\mu, w_n)) = (\mu D - w_n) - C_n(D - \frac{w_n}{\mu})$. A price-taking firm takes the market price set by the planner to be binding and does not anticipate its power to influence it. A pair $(\mathbf{w}, \mu)$ where $\mathbf{w} \geq 0$ and $\mu \geq 0$ is said to be in a competitive equilibrium in this price-taking setting, if the firms maximize their payoff and the market price is determined as follows,

$$\Pi_n(w_n; \mu) \geq \Pi_n(\overline{w}_n; \mu), \overline{w}_n \geq 0, \forall n,$$
$$\mu = \frac{\sum_n w_n}{D(N-1)}.$$

It is shown in [10] that for price-taking firms, their exists a competitive equilibrium at which the resulting allocation is efficient, meaning, it minimizes the aggregate production cost. The price-taking assumption models reality well when the number of participating firms is very large and the effect of one firm on the market clearing price is negligible, for all practical purposes. This is not true when the firms are aware of their power to influence the market clearing price.

Writing the market price μ explicitly as a function of the bid vector $\mathbf{w}$ for the price-anticipating case, we have: $\mu(\mathbf{w})$. For $w_n > 0$, the supply for a firm n, hence becomes $S(\mu(\mathbf{w}), w_n) = D\left(1 - \frac{w_n(N-1)}{w_n + \sum_{k \neq n} w_k}\right)$ and for $w_n = 0$, the authors adopt the convention that $S(\mu(\mathbf{w}), 0) = D \;\forall\; \mathbf{w}_{-n}$. Writing payoff for firm n,

$$\Pi_n(w_n, \mathbf{w}_{-n}) = \begin{cases} (\frac{\sum_k w_k}{N-1} - w_n) - & w_n > 0 \\ C_n(D(1 - \frac{w_n(N-1)}{\sum_k w_k})), & \\ \frac{\sum_{k \neq n} w_k}{N-1} - C_n(D), & w_n = 0 \end{cases}$$

Johari and Tsitiklis [10] establish the existence and uniqueness of Nash equilibria in the above setting when the firms are price-anticipating. They also derive an upper bound on the efficiency loss and show that the ratio of aggregate production cost at Nash equilibria to the minimum possible aggregate production cost is atmost $1+\frac{1}{N-2}$. To bridge this gap, we aim to develop a scalarized mechanism for the producer side that results in efficient allocations at the NEPs.

We now discuss the work of Yang and Hajek [21] who introduced a VCG inspired payment structure in a scalar mechanism for the consumer side which they described as the VCG-Kelly mechanism.

2.0.3 VCG-Kelly Mechanism

Yang and Hajek [21] developed this mechanism for the consumer side problem setting described in Section 2.0.1. Consumers submit one-dimensional bids to the planner from which it constructs surrogate valuation functions for each player on which the allocations are based, as is done in the Kelly mechanism. The valuation function of a consumer r with bid w_r is defined as, $V_r(w_r, x_r) = w_r f(x_r)\ \forall r$ where f_r's are strictly increasing, strictly concave and twice differentiable with $f_r(0+) = +\infty$. $f_r(0) = -\infty$ is allowed with the convention that, $V_r(0,0) = 0$. f_r's mimic the behaviour of the log function, however the authors introduce a generalization beyond the log-valuation function and consider the following family of surrogate valuations functions,

$$\phi^{(\alpha)}(x) = \begin{cases} (1-\alpha)^{-1}x^{1-\alpha}, & \text{if} \quad \alpha \in (0,1) \\ \log x, & \text{if} \quad \alpha = 1 \end{cases}.$$

We have the following allocation rule,

$$\mathbf{x}^{VCGK}(\mathbf{w}) = \underset{\mathbf{x}\in X_C}{\operatorname{argmax}} \sum_{r\in[N]} V_r(w_r, x_r)$$

where X_C is as defined in Section 2.0.1. The payment rule is inspired from the VCG mechanism and the payment made by a consumer r is the loss in utility of all other consumers because of its presence.

$$\begin{aligned} m_r^{VCGK}(\mathbf{w}) = \max_{\mathbf{x}\in X_C, x_r=0} &\sum_{j\neq r} V_j(w_j, x_j) - \\ &\sum_{j\neq r} V_j(w_j, x_j^{VCGK}(\mathbf{w})) \end{aligned}$$

Yang and Hajek [21] showed under a certain regulatory assumption on the buyers that NEPs exist and correspond to efficient allocations with respect to the System Problem. The assumption on the participating consumers was that there exist atleast two consumers such that their valuation function satisfies $U_r'(0) = +\infty$.

The mechanism designed by Yang and Hajek [21], does not automatically apply to the supplier side formulation. Superficially the problem (C) is similar to the problem (S) (one can take $U_j = -C_j$), however there are subtleties in the details. VCG-Kelly requires that

U_j is a concave increasing function, and since C_j is a convex increasing function, $-C_j$ would be a concave decreasing function, which is incompatible with VCG-Kelly. This necessitates a fresh design of a communication efficient and also economically efficient mechanism for the problem (S).

Chapter 3

A Scalarized VCG mechanism for efficient allocations

In this chapter, we present a mechanism with low communication overhead which achieves efficiency on the supplier side at the Nash equilibrium points (NEPs). Each producer r submits a one-dimensional bid $w_r \geq 0$ to the market maker, which then constructs a surrogate cost function as a proxy for the true cost function of the producer as follows:

$$V(w_r, x_r) = \begin{cases} w_r f(x_r) & \text{if} \quad w_r > 0 \\ 0 & \text{if} \quad w_r = 0 \end{cases}, \tag{3.1}$$

where $f(x_r) = \log(D/(D - x_r))$ and D is the a priori known inelastic demand. The domain of $V(w_r, \cdot)$ is $(-\infty, D]$.

As it turns out, this surrogate cost function is closely related to the supply function $S(p, w) = D - w/p$, which has been used previously in supply-side market modeling [7]. Formally, the supply function, $S(p, w)$ is defined as the maximum amount a producer is willing to supply at price p. Since the producer supplies an amount that maximizes its payoff at the given market price, $S(p, w)$ is given by

$$S(p, w) = \underset{x}{\operatorname{argmax}} \left(px - C(w, x)\right). \tag{3.2}$$

Equating the w-parameterized cost function $C(w, x)$ to our surrogate cost function $wf(x)$ for some positive scalar w and on solving (3.2), we get $S(p, w) = D - w/p$.

3.0.1 Allocation and Pricing Rule

Our mechanism makes allocations according to the following rule, where $\mathbf{w} = (w_r, r \in [N])$ denotes the vector of bids from the suppliers.

$$\mathbf{x}(\mathbf{w}) = \begin{cases} \operatorname{argmin}_{\mathbf{x} \in X_F} \sum\limits_{j \in [N]} V(w_j, x_j), & \text{if} \quad \mathbf{w} \neq 0 \\ \frac{D}{N}\mathbf{1}, & \text{if} \quad \mathbf{w} = 0 \end{cases} \tag{3.3}$$

where $X_F = \{\mathbf{x} \mid x_j \leq D \ \forall\ j,\ \sum_{j \in [N]} x_j = D\}$ and $\mathbf{1}$ is the vector of all ones. As is apparent from the above rule, if everybody submits a zero bid, the inelastic demand is met equally by all the producers. On the other hand, when $\mathbf{w} \neq 0$, the allocations are as follows.

Lemma 3.0.1. *If $\mathbf{w} \neq 0$, the allocation corresponding to producer j is given by*

$$x_j(\mathbf{w}) = D\left(1 - \frac{w_j(N-1)}{w_j + \sum_{k \neq j} w_k}\right). \tag{3.4}$$

Proof. Writing the Lagrangian function for the optimization problem posed in the allocation rule,

$$L(\mathbf{x}, \mu, \lambda) = \sum_{j:w_j>0} w_j f(x_j) + \sum_j \mu_j (x_j - D) - \lambda(\sum_j x_j - D)$$

where $\mu_j \geq 0$. Let $\mathbf{Z}$ be the set of all producers with strictly positive bids. For $j \in \mathbf{Z}$, we have $w_j f'(x_j) + \mu_j = \lambda$. If the allocation $x_j = D$, from complementary slackness, we have $\mu_j \geq 0$ which gives us $w_j f'(D) \leq \lambda$ which is not possible since λ exists and is a finite value. Therefore, for $j \in \mathbf{Z}$, $x_j < D$ which gives us $w_j f'(x_j) = w_j/(D - x_j) = \lambda$. Since, $w_j > 0$ for $j \in \mathbf{Z}$ and $x_j < D$, λ is strictly positive and we can write the allocation as, $x_j = D - \frac{w_j}{\lambda}$. Now for $k \in [N] \setminus \mathbf{Z}$, differentiating the Lagrangian with respect to x_k, we get $\mu_k = \lambda$. Since, $\lambda > 0$, we get $\mu_k > 0$ and from complementary slackness, $x_k = D$. Producers which submit a zero bid get allocated the entire inelastic demand D by the market planner. We obtain λ as a function of the bid vector $\mathbf{w}$ using the capacity constraint

$$\sum_{j \in \mathbf{Z}} \left(D - \frac{w_j}{\lambda}\right) + \sum_{j:w_j=0} D = D,$$

which implies

$$\lambda(\mathbf{w}) = \frac{\sum_{j \in [N]} w_j}{D(N-1)}.$$

From $x_j = D - w_j/\lambda$ for $w_j > 0$, and $\lambda(\mathbf{w})$ as derived above, for $j \in \mathbf{Z}$, we get (3.4). Note that the formula (3.4) also holds for producers making a zero bid (so long as not all producers bid zero); their allocation is D which is compatible with the formula. ■

We note here that our allocation rule does not guarantee non-negative allocations for all players. However, as we show in Lemma 3.0.5, under a certain technical assumption on the producers, any Nash equilibrium allocation to each player resulting from our mechanism is always non-negative.

We adopt a VCG-like externality payment structure based on our surrogate cost function. The payment to any player r as a function of the bid vector profile $\mathbf{w}$, for $\mathbf{w}_{-r} \neq 0$ is defined as,

$$\begin{aligned} m_r(w_r, \mathbf{w}_{-r}) = & \min_{x_r=0, \mathbf{x} \in X_F} \sum_{j \neq r} V_j(w_j, x_j) \\ & - \sum_{j \neq r} V_j(w_j, x_j(w_r, \mathbf{w}_{-r})). \end{aligned} \tag{3.5}$$

Note that the above payment captures the externality associated with producer r, *measured using the surrogate cost functions.* For $\mathbf{w}_{-r} = 0$, $m_r(w_r, \mathbf{w}_{-r}) = 0$. Simplifying the above expression for $\mathbf{w}_{-r} \neq 0$,

$$\begin{aligned} m_r(w_r, \mathbf{w}_{-r}) = &\sum_{j \neq r, w_j > 0} w_j \log\left(\frac{\sum_{k \neq r} w_k}{w_j(N-2)}\right) \\ &- \sum_{j \neq r, w_j > 0} w_j \log\left(\frac{w_r + \sum_{k \neq r} w_k}{w_j(N-1)}\right) \\ = &\sum_{j \neq r, w_j > 0} w_j \log\left(\frac{(N-1)\sum_{k \neq r} w_k}{(N-2)(w_r + \sum_{k \neq r} w_k)}\right) \end{aligned}$$

The term inside the log is a constant, and hence taking it outside the summation,

$$m_r(w_r, \mathbf{w}_{-r}) = (\sum_{j \neq r} w_j) \log\left(\frac{(N-1)\sum_{k \neq r} w_k}{(N-2)(w_r + \sum_{k \neq r} w_k)}\right)$$

The allocation (3.3) and pricing rule (3.5), together with the true cost functions, $(C_j : j \in [N])$ define a game among the producers. The payoff of a producer r is given by,

$$\Pi_r(w_r, \mathbf{w}_{-r}) = m_r(w_r, \mathbf{w}_{-r}) - C_r(\mathbf{x}_r(w_r, \mathbf{w}_{-r}))$$

A bid vector $\mathbf{w}^{NE}$ is an NEP of the game if and only if,

$$\Pi_r(w_r^{NE}, \mathbf{w}_{-r}^{NE}) \geq \Pi_r(w_r, \mathbf{w}_{-r}^{NE}), \ \forall \ w_r \geq 0, \ r \in [n].$$

3.0.2 Characterising Nash Equilibrium Bids

In this section, we derive several properties of Nash equilibrium bid vectors under the proposed scalarized VCG mechanism. The first is that an all-zero bid vector is not an equilibrium.

Lemma 3.0.2. *A zero bid-vector is not a Nash equilibrium.*

Proof. Say for a producer r, we have $\mathbf{w}_{-r} = 0$. $m_r(w_r, \mathbf{0}) = 0$ from the definition of our pricing rule. Payoff of producer r therefore is,

$$\Pi_r(w_r, \mathbf{w}_{-r}) = \begin{cases} -C_r((2-N)D) & \text{if} \quad w_r > 0 \\ -C_r(D/N) & \text{if} \quad w_r = 0 \end{cases}.$$

Since $N > 2$, therefore the allocation for $w_r > 0$ is negative and by the definition of cost function $C_r(.)$, $\Pi_r(w_r, \mathbf{w}_{-r}) = 0$ when $w_r > 0$. By comparing the two payoffs, producer r has a unilateral incentive to submit a positive bid. Therefore, a zero bid-vector can never be a Nash equilibrium for our mechanism. ■

To obtain futher properties of NEPs under our mechanism, we make the following assumption.

Assumption 3.0.1. *There are at least two producers with cost functions satisfying $C'(D) = +\infty$. These producers are called* special producers*.*

Special producers are incapable of meeting the entire demand D by themselves. As we show below, special producers are guaranteed to make strictly positive bids at any NEP. Assumption 3.0.1 is needed to argue the efficiency of NEPs under our mechanism; indeed, we can demonstrate instances of the game that admit inefficient Nash equilibria when this assumption does not hold.

Lemma 3.0.3. *Under Assumption 3.0.1, at least two coordinates of any Nash equilibrium bid vector are strictly positive.*

Proof. We show that special producers always submit strictly positive bids at Nash equilibrium. To do this, we show that the best response of any special producer j satisfies $w_j > 0$ irrespective of $\mathbf{w}_{-j}$.

Case 1: $\mathbf{w}_{-j} = 0$. In this case, Lemma 3.0.2 implies that the best response of any player is to submit a positive bid.

Case 2: $\mathbf{w}_{-j} \neq 0$. In this case, the payoff of the special producer is given by

$$\Pi_j(w_j, \mathbf{w}_{-j}) = (\sum_{k\neq j} w_k) \log\left(\frac{(N-1)\sum_{k\neq j} w_k}{(N-2)(w_j + \sum_{k\neq j} w_k)}\right) \\ - C_j\left(D\left(1 - \frac{w_j(N-1)}{w_j + \sum_{k\neq j} w_k}\right)\right).$$

$\Pi_j(w_j, \mathbf{w}_{-j})$ is a continuously differentiable function of w_j. Writing the partial derivative of the payoff with respect to w_j,

$$\frac{\partial \Pi_j}{\partial w_j}(w_j, \mathbf{w}_{-j}) = \frac{D(N-1)\sum_{k\neq j} w_k}{(w_j + \sum_{k\neq j} w_k)^2}\Bigg[C_j'(\mathbf{x}_j(w_j, \mathbf{w}_{-j})) \\ - \frac{w_j + \sum_{k\neq j} w_k}{D(N-1)}\Bigg].$$

$\mathbf{x}_j(w_j, \mathbf{w}_{-j})$ is a continuously differentiable and a monotone decreasing function of the bid w_j ($\mathbf{w}_{-j} \neq 0$). This and the fact that $C_j(.)$ is convex gives us that $C_j'(\mathbf{x}_j(w_j, \mathbf{w}_{-j}))$ is a decreasing function of w_j which makes the term inside square brackets a strictly decreasing, continuous function of w_j.

Since j is a special producer, $C_j'(\mathbf{x}_j(0, \mathbf{w}_{-j})) = C_j'(D) = +\infty$. This means that $\frac{\partial \Pi_j}{\partial w_j}(w_j, \mathbf{w}_{-j})$ tends to ∞ as $w_j \downarrow 0$. Therefore, the payoff is maximized for some $w_j > 0$ where the partial derivative is equal to zero.

Hence proved that any special producer j always submits a strictly positive bid for both $\mathbf{w}_{-j} \neq 0$ and $\mathbf{w}_{-j} = 0$. Since, we have atleast two special producers from our assumption, we have atleast two strictly positive bids for any bid-vector at Nash equilibrium. ■

An immediate corollary of Lemma 3.0.3 is the following.

Corollary 3.0.4. *Under Assumption 3.0.1, at any NEP, we would have* $\mathbf{w}_{-r} \neq 0$ *for all producers* r.

The preceding results allow us to argue that NEPs correspond to non-negative allocations.

Lemma 3.0.5. *Under Assumption 3.0.1, at any NEP, we have* $w_j \leq (\sum_{k \neq j} w_k)/(N-2)$ *for all producers* j*. Moreover, all the resulting allocations are non-negative.*

Proof. From Corollary 3.0.4, for any producer j, $\mathbf{w}_{-j} \neq 0$ at Nash equilibrium. Thus, the allocation for any producer j is given by

$$\begin{aligned} \mathbf{x}_j(w_j, \mathbf{w}_{-j}) &= D\left(1 - \frac{w_j(N-1)}{w_j + \sum_{k \neq j} w_k}\right) \\ &= D\left(\frac{\sum_{k \neq j} w_k - (N-2)w_j}{w_j + \sum_{k \neq j} w_k}\right). \end{aligned} \tag{3.6}$$

We first prove for all j that $w_j \leq \sum_{k \neq j} w_k/(N-2)$ at Nash equilibrium. We proceed with a proof by contradiction. Say, this condition is violated for some producer j, that is $w_j > \sum_{k \neq j} w_k/(N-2)$. Since $\mathbf{x}_j(w_j, \mathbf{w}_{-j}) < 0$, we have $C_j(\mathbf{x}_j(w_j, \mathbf{w}_{-j})) = 0$. Payoff for that producer j is thus

$$\Pi_j(w_j, \mathbf{w}_{-j}) = (\sum_{k \neq j} w_k) \log\left(\frac{(N-1)\sum_{k \neq j} w_k}{(N-2)(w_j + \sum_{k \neq j} w_k)}\right).$$

Since this is a monotone decreasing function of w_j the producer can increase its payoff by unilaterally reducing its bid w_j. Therefore, a bid-vector $\mathbf{w}$ with $w_j > \sum_{k \neq j} w_k/(N-2)$ for any producer j is not a Nash equilibrium.

Having shown that $w_j \leq \sum_{k \neq j} w_k/(N-2)$ $\forall$ j at Nash equilibrium, it follows from (3.6) that the resulting allocations are non-negative. ■

Finally, the relationship between equilibrium bids given by Lemma 3.0.5 further implies that equilibrium bid vectors are necessarily of the following two types.

Corollary 3.0.6. *Under Assumption 3.0.1, at any NEP, either all bids are strictly positive, or a single bid is zero and rest are all equal and strictly positive.*

Proof. Consider an NEP. Say producer r submits a zero bid. Let us derive a condition on the rest of the bids. From Lemma 3.0.5, for $j \neq r$,

$$w_j \leq \frac{\sum_{k \neq j} w_k}{N-2}$$

Adding $\frac{w_j}{N-2}$ to both sides,

$$w_j \leq \frac{\sum_{k \in [N]} w_k}{N-1} = \frac{\sum_{k \neq r} w_k}{N-1}$$

Consider the vector $\mathbf{w}_{-r}$. The above inequality states that each entry in this vector is less than or equal to the average of all entries. This is only possible if all the bids of $\mathbf{w}_{-r}$ are equal. Further, from Corollary 3.0.4, it follows that all the entries in $\mathbf{w}_{-r}$ must be equal and strictly positive. ∎

3.0.3 Main Results

Using the preceding properties of NEPs under the proposed scalarized VCG mechanism, we are now ready to prove our main results. The first guarantees the existence of NEPs (note that an NEP corresponds to a *pure* Nash equilibrium), and also provides an exact characterization of NEPs.

Theorem 3.0.7. *Under Assumption 3.0.1, at least one Nash equilibrium vector exists. A vector* $\mathbf{w}$ *is a NEP if and only if the following conditions hold:*

$$C'_j(\mathbf{x}_j(w_j, \mathbf{w}_{-j})) = \frac{\sum_{k \in [N]} w_k}{D(N-1)}, \quad \text{if} \quad w_j > 0, \tag{3.7}$$

$$C'_j(D) \leq \frac{\sum_{k \in [N]} w_k}{D(N-1)}, \quad \text{if} \quad w_j = 0. \tag{3.8}$$

Proof. To show existence of an NEP, it suffices to show that for any producer j and for any $\mathbf{w}_{-j}$, $\Pi_j(\cdot, \mathbf{w}_{-j})$ is quasi-concave [21]. This is easy to verify when $\mathbf{w}_{-j} = 0$. For $\mathbf{w}_{-j} \neq 0$, the payoff for producer j is,

$$\begin{aligned}\Pi_j(w_j, \mathbf{w}_{-j}) = (\sum_{k \neq j} w_k) \log \left(\frac{(N-1) \sum_{k \neq j} w_k}{(N-2)(w_j + \sum_{k \neq j} w_k)} \right) \\ - C_j \left(D \left(1 - \frac{w_j (N-1)}{w_j + \sum_{k \neq j} w_k} \right) \right)\end{aligned}$$

$\Pi_j(w_j, \mathbf{w}_{-j})$ is a continuously differentiable function of producer j's bid, w_j. Writing the partial derivative of the payoff with respect to w_j,

$$\frac{\partial \Pi_j}{\partial w_j}(w_j, \mathbf{w}_{-j}) = \frac{D(N-1)\sum_{k\neq j} w_k}{(w_j + \sum_{k\neq j} w_k)^2}\Bigg[C_j'(\mathbf{x}_j(w_j, \mathbf{w}_{-j})) \\ - \frac{w_j + \sum_{k\neq j} w_k}{D(N-1)}\Bigg].$$

The term inside square brackets is a strictly decreasing, continuous function of w_j. The term outside square brackets is positive for all w_j, therefore the payoff is a strictly quasiconcave function of w_j.

Finally, for $\mathbf{w}_{-j} \neq 0$ fixed, $w_j = 0$ is the best response of producer j if and only if $\frac{\partial \Pi_j}{\partial w_j}(0, \mathbf{w}_{-j}) \leq 0$. To have a positive bid as the best response, there must exist $w_j > 0$ such that $\frac{\partial \Pi_j}{\partial w_j}(w_j, \mathbf{w}_{-j}) = 0$. These conditions, given by (3.7) and (3.8), are necessary and sufficient. ■

Having established the existence of NEPs, the following result shows that all NEPs are efficient, validating the proposed scalarized VCG mechanism.

Theorem 3.0.8. *Under Assumption 3.0.1, all Nash equilibrium bid vectors result in efficient allocations.*

Proof. We need to show that the allocation resulting from a bid-vector $\mathbf{w}$ at Nash equilibrium satisfies equations (2.1)–(2.3). For any Nash equilibrium bid-vector, conditions (3.7)–(3.8) are satisfied as shown in Theorem 3.0.7. Fixing $\mathbf{w}$, we write $\mathbf{x}_j(w_j, \mathbf{w}_{-j})$ as x_j in the arguments below. Based on Corollary 3.0.6, it suffices to consider the following two cases.

Case 1: $\mathbf{w} > 0$. In this case, $x_j(\mathbf{w}) \in [0, D)\ \forall\ j$. Condition (3.7) is applicable, rewriting which we have

$$C_j'(x_j) = \frac{\sum_{k\in[N]} w_k}{D(N-1)}, \quad x_j \in [0, D).$$
$$\sum_{k\in[N]} x_k = D.$$

Conditions (2.1)–(2.3) are thus satisfied taking $\lambda^* = \frac{\sum_{k\in[N]} w_k}{D(N-1)}$. Hence, a Nash equilibrium bid vector with strictly positive bids always results in an efficient allocation.

Case 2: $w_r = 0$ ***and*** $\mathbf{w}_{-r} = w\mathbf{1} > 0$**.** From our allocation rule, $x_r(\mathbf{w}) = D$ and

$x_j(\mathbf{w}) = 0 \ \forall \ j \neq r$. From (3.7)–(3.8), we obtain the following conditions,

$$\begin{aligned} C_j'(0) &= \frac{\sum_{k\in[N]} w_k}{D(N-1)} \quad \forall \quad j \neq r, \\ C_r'(D) &\leq \frac{\sum_{k\in[N]} w_k}{D(N-1)}, \\ \sum_{k\in[N]} x_k &= D. \end{aligned} \tag{3.9}$$

Since $C_r'(x_r) \leq \frac{\sum_{k\in[N]} w_k}{D(N-1)}$, $\exists z_r \geq 0$ such that $C_r'(x_r) = \frac{\sum_{k\in[N]} w_k}{D(N-1)} - z_r$ where z_r is a slack variable. It follows from the above inequalities that $C_j'(x_j) = \frac{\sum_{k\in[N]} w_k}{D(N-1)} \geq \frac{\sum_{k\in[N]} w_k}{D(N-1)} - z_r$ for all $j \neq r$. Thus, equations (2.1)–(2.3) are satisfied for $\lambda^* = \frac{\sum_{k\in[N]} w_k}{D(N-1)} - z_r$. Therefore, all such Nash equilibrium bid vectors as described in this case result in efficient allocations. ■

Chapter 4

Concluding remarks

In this paper, we presented a scalarized mechanism for efficient production allocation among a group of suppliers, each associated with their own (private) cost function. Allocations are made based on surrogate cost functions constructed using the bids made by the suppliers, and the payments are based on a VCG-like externality calculation using the same surrogate cost functions. Our main result is that Nash equilibria between the suppliers under this mechanism are *optimal*, meaning there is no loss of efficiency due to the information asymmetry with respect to the suppliers' cost functions.

This work motivates extensions along various directions. A first step would be to modify the mechanism to ensure non-negative allocations for any vector of bids. It is also natural to explore other surrogate cost functions that also preserve efficiency of NEPs. But the most interesting extension would be to marry the supply side problem studied here with the demand side problem analysed before (see [21, 8]) to capture a *two-sided market*. Whether or not an efficient, low-overhead market mechanism can be designed in this setting is an important open problem.

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