A Scalarized VCG Mechanism for Supply-side Markets

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***Abstract*—**

I. Introduction

The central goal of market design is to effect socially efficient outcomes. Its basic challenge lies in information asymmetry – outcomes are decided by the system planner, but the system planner does not have access to the utility functions of the market participants. *Market mechanisms* serve as a means of soliciting information from market participants in order to determine outcomes. This opens the possibility that participants may misreport their information in order to induce outcomes that are more favorable to them. The fundamental challenge of market design lies in designing a system of *information exchange* and *payments* made by or to the market participants so that truthful revelation of information is incentivized and socially optimal outcomes are realized.

The famed VCG [1] mechanism achieves precisely this. Market participants are asked to report their utility functions, and the system planner computes the socially optimal outcome based on the *reported* utility functions. Payments are determined in such a way that truthful revelation of utility functions is the optimal response. This socially efficient allocations are obtained. A remarkable feature of the VCG mechanism is the sweeping generality with which it can be applied – the VCG mechanism allows for *any* kind of utility functions and *any* space of outcomes.

However, practically speaking, there is a nontrivial difficulty in implementing the VCG mechanism. The "bids" of the VCG mechanism are *utility functions* and as such are infinite dimensional objects and incur massive communication overheads. In some markets this is not a major issue – the utility function is largely the same across multiple instances of the auction and only the incremental change needs to be communicated. For example, in markets for conventional energy, generators may only need to communicate if any units are unavailable due to maintenance or other operational reasons, and the resulting incremental offset in their supply curves. However, in markets for selling renewable energy, the utility function of the generators is technically the (negative of the) expected cost function, conditioned on the information available to the generator at the time of bidding. Since the information about the weather forecast varies rapidly, the cost function also varies, and has to be resubmitted each time.

In this paper we design a socially efficient mechanism that requires market participants to submit only *scalar* bids. We focus on the supply-side problem where a collection of suppliers compete to meet a known inelastic demand. The suppliers submit a nonnegative scalar as a bid, using which an allocation is determined by the system planner. This induces a game between the suppliers wherein these scalar bids are strategies. We show that if the true cost functions of suppliers are convex, differentiable and increasing, then under some technical conditions, every Nash equilibrium of this game leads to a socially efficient allocation.

The scalarization and efficiency achieved by our mechanism resolves a significant stumbling block in the market design for renewable energy supply. Supply-side markets in energy [2] have traditionally employed supply functions as bids. Mathematically, these are infinite dimensional objects and immensely hard to specify (see, e.g., [3]). Real-world auctioneers have addressed the complexity of specifying supply functions by asking suppliers to submit piece-wise linear approximations of these supply functions (see [4], [5]). To the best of our knowledge, day-ahead electricity markets world over now follow this format. Our mechanism shows that one can dramatically simplify such auctions in certain cases. Moreover, our mechanism differs fundamentally from the approximation-based approach because it *does not* assume any parameterized form for the supply functions.

At the heart of our mechanism is the construction of *surrogate cost functions*. The system planner constructs surrogate cost functions parametrized by the bids submitted by suppliers and allocates the demand by minimizing the total (surrogate) cost of meeting the demand. The payment made to a supplier by the system planner equals the externality caused by the supplier *as calculated using the surrogate cost function*. In other words, this is the payment the supplier *would have received* in a VCG mechanism had all suppliers submitted the surrogate cost functions as their bids. Thus the entire mechanism works as if the surrogate cost functions were the bid cost functions themselves. Under certain assumptions on the true cost functions, this mechanism induces efficient allocations in equilibrium.

Needless to say, this is not a free lunch. The scalarization achieved in our mechanism comes at the expense of the generality of the setting. Unlike the VCG mechanism

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which allows arbitrary utility functions and abstract outcome spaces, our setting is much more specific. In particular, the assumptions of convexity and differentiability of the true cost functions play a key role in our results.

The last two decades have witnessed an explosion of studies on mechanisms where participants submit scalar bids. Most of these have focused on the demand-side problem, i.e., allocation of a resource amongst multiple *consumers*. Unfortunately, these mechanisms are known to lead to efficiency loss. The famous Kelly mechanism [6] for resource allocation allocates resources proportional to the bid submitted; it was analyzed by Johari and Tsitsiklis [7] in a price-anticipative setting and the worst case efficiency loss was shown to be 25%. The same authors also analyzed a supply side problem with inelastic demand with scalar-parametrized supply function bidding and showed the loss of efficiency in this case to be at most $\frac{1}{N-2}$ times the minimum aggregate production cost where N is the number of suppliers [8].

Yang and Hajek [9] introduced a mechanism that they called *VCG-Kelly*, a combination of the Kelly mechanism and the VCG mechanism, that achieves social efficiency with scalar bids. The allocation of the mechanism is identical to that of the Kelly mechanism, but the payment charged is a VCG-style externality-based payment. Our paper takes inspiration from Yang and Hajek to design its payment rule. However, unlike the demand-side setting where surrogate value functions were already defined by Kelly [6], in our case there is no precedent for the form of our surrogate cost functions. An important innovation in our work thus lies in identifying the right surrogate cost functions.

We conclude this introduction with a brief survey of some additional related work. The model and background is presented in Section II. Our mechanism and its analysis is presented in Section III. We conclude in Section **??**.

Related work

Supply function equilibrium (SFE) models have been studied extensively in literature. [10] and [11] consider SFE models with continuous and differentiable supply functions, although in practice, producers participating in the electricity market submit discretized supply functions as discussed in Section I. The work of Klemperer and Meyer [12] sparked activity in the electricity market modelling literature. Their model required the competing firms to have identical cost functions. The study of supply function equilibria when the firms have dissimilar cost functions is a significantly harder problem. Linearity assumption on the bid supply functions is required to tackle such market settings as discussed in [13]. Johari [8] [7] [14] develops a single-price market clearing mechanism which leads to efficient allocations in the price-taking setting without any assumption on the firms' cost functions. There is significant evidence in the literature that price-anticipating behaviour can lead to loss of efficiency (see [15]). Johari proposes a mechanism for the supplier side problem akin to the Kelly mechanism [6] on the consumer side and shows in [14] that the devised mechanism minimizes efficiency loss among a certain class of market-clearing mechanisms.

Efficiency loss in scalar parameterized mechanisms has also been widely studied for the resource allocation setting on the consumer side [16] [17] [18] [19] [20]. As mentioned in the introduction, Yang and Hajek [9] developed a scalarized mechanism and adopted a VCG-like payment structure to show that their mechanism results in efficient allocations at all Nash equilibrium points (NEPs). Johari and Tsitsiklis [21] also developed a family of VCG-like mechanisms for the consumer side using simple scalar bids per player in a much more general setting by considering convex constraints on the allocation. In both of these works, simplicity of payments (single-price like payments as in Kelly) was sacrificed for efficiency at the NEPs.

Inspired by this progress in mechanism design on the consumer side, we develop a mechanism for the supplier side that achieves full efficiency at NEPs at the expense of the simplicity of the payment structure.

II. Model and Background

We consider a market with a set of $N > 2$ producers, $\mathcal{N} = \{1, 2, \ldots, N\}$ competing to satisfy an infinitely divisible, inelastic demand, D. The cost function of producer j is represented by $C_j(x_j)$, which is a strictly increasing, convex and continuously differentiable function of its production allocation, x_j over $\mathbb{R}^+$ and $C_j(x_j) = 0,\ \forall\ x_j \leq 0$. A market planner is interested in solving the following problem termed the System Problem,

$$\min_{\mathbf{x} \in X_F} \sum_{j \in \mathcal{N}} C_j(x_j) \tag{S}$$

where $X_F = \{\mathbf{x} \mid x_j \geq 0,\ \sum_{j \in \mathcal{N}} x_j = D\}$. The system problem is a convex optimization problem with a compact feasible region, hence an optimal solution exists. Associated with every optimal allocation $\mathbf{x}^*$, there exists a Lagrange multiplier λ^* satisfying the following Karush-Kuhn-Tucker (KKT) conditions for our problem:

$$C_j^{'}(x_j^*) \geq \lambda^*, \qquad x_j^* = 0 \tag{1}$$

$$C_j^{'}(x_j^*) = \lambda^*, \qquad x_j^* > 0 \tag{2}$$

$$\text{and,} \quad \sum_j x_j^* = D \tag{3}$$

These conditions are necessary as well as sufficient conditions of optimality for our model, meaning that any $\mathbf{x}^*$ along with an associated λ^* which satisfies the above conditions is an optimal solution to (S).

As mentioned in the introduction, a producer's cost function is its private knowledge and is not known to the system planner who attempts to solve (S). This necessitates a mechanism by which the system planner elicits information from the producers such that the producers truthfully reveal their information, and thus the solution of (S) is achieved.

One can employ the celebrated Vickrey-Clarke-Groves (VCG) mechanism as a start. In the VCG mechanism, producers bid their cost functions and the market planner issues payments to the producers in such a way, that each producer is better off reporting its true cost function irrespective of what other producers report. In other words, truth telling is a dominant strategy equilibrium under the VCG mechanism. Since producers report their true cost functions, the allocation made by the market planner in equilibrium is efficient with respect to the System Problem (S). The VCG mechanism gives us desirable behaviour of the producers but requires submission of infinite-dimensional bids.

One simple method to reduce the communication overhead is to let the producers submit scalar bids which can be interpreted as a parametrization of their private cost functions. This has been done for the consumer side problem in the Kelly mechanism, which we discuss next.

A. Kelly Mechanism

Though originally proposed for a different problem setting – resource allocation over a network to maximise the total utility of all players with private utility functions, the idea of Kelly mechanism is still of relevance here. The System Problem for the Kelly mechanism is,

$$\max_{\mathbf{x}\in X_C} \sum_{j\in[N]} U_j(x_j) \tag{C}$$

where $X_C = \{\mathbf{x} | x_j \geq 0, \sum_{j\in[N]} x_j \leq D\}$ and the utility function for a player r, $U_r(x_r)$ is an increasing, concave and continuously differentiable function of the allocation x_r. In the Kelly Mechanism, communication between players and the market planner involves the players submitting scalar bids $\mathbf{w} = \{w_r\}_r$ to the planner and the planner returning the allocation, $x_r(\mathbf{w})$ to each player and the price per unit resource $\lambda(\mathbf{w})$, where $\mathbf{w}$ is the bid vector. The planner makes allocations based on the following rule,

$$\mathbf{x}(\mathbf{w}) = \begin{cases} \text{argmax}_{\mathbf{x}\in X_C} \sum\limits_{r\in[N]} w_r \log(x_r), & \text{if} \quad \mathbf{w} \neq 0 \\ 0, & \text{if} \quad \mathbf{w} = 0 \end{cases}$$

where $w_r \log(x_r)$ serves as a proxy for the true utility function $U_r(.)$ of player r. We can reduce the above allocation rule for a player r to $x_r(w_r, \mathbf{w}_{-r}) = \left(w_r/(\sum_{k\in[N]} w_k)\right) D, w_r > 0$ and $x_r(0, \mathbf{w}_{-r}) = 0$. Price per unit resource, $\lambda(\mathbf{w})$ as a function of the bid vector is determined as, $\lambda(\mathbf{w}) = \left(\sum_{k\in[N]} w_k\right)/D$. Total price charged to player r is, $m_r(w_r, \mathbf{w}_{-r}) = \lambda(w_r, \mathbf{w}_{-r}) x_r(w_r, \mathbf{w}_{-r}) = w_r$. The payoff for player r therefore is,

$$\Pi_r(w_r, \mathbf{w}_{-r}) = U_r(x_r(w_r, \mathbf{w}_{-r})) - w_r$$

This induces a game among the players and we can characterise the efficiency loss, if at all, at the Nash equilibrium bid vectors for this game (existence and conditions on uniqueness/non-uniqueness of Nash equilibrium bid vectors for this problem has been widely studied and hence not repeated here). Johari and Tsitiklis [7] showed that for strategic buyers using the Kelly mechanism, the efficiency loss is atmost 25%. Efficiency loss is the price we pay for scalarization and captures the trade-off between efficiency and communication complexity.

We now give an overview of the work of Johari and Tsitiklis [8] on scalarized bidding for the supplier side problem. This is well aligned with the theme and setting of this paper as opposed to the above detour on Kelly mechanism which was used to introduce the idea of parametrization of private information using scalar bids and capture the trade-off between efficiency and communication complexity that such a scalarization implies.

B. Supply Function Bidding using Scalar Bids

Johari and Tsitiklis [8] assume a supply function of the form, $S(p, w) = D - \frac{w}{p}$ parametrized by w where p is the price charged by the market maker. Each firm submits a scalar bid w which parametrizes its supply function followed by the market planner determining the market clearing price $p(\mathbf{w})$ at which the aggregate supply exactly equals the inelastic demand, D. Payoff $\Pi_n(w_n; \mu)$ for a price-taking firm n at market price μ is given as, $\Pi_n(w_n; \mu) = \mu S(\mu, w_n) - C_n(S(\mu, w_n)) = (\mu D - w_n) - C_n(D - \frac{w_n}{\mu})$. A price-taking firm takes the market price set by the planner to be binding and does not anticipate its power to influence it. A pair $(\mathbf{w}, \mu)$ where $\mathbf{w} \geq 0$ and $\mu \geq 0$ is said to be in a competitive equilibrium in this price-taking setting, if the firms maximize their payoff and the market price is determined as follows,

$$\Pi_n(w_n; \mu) \geq \Pi_n(\overline{w}_n; \mu), \overline{w}_n \geq 0, \forall n,$$
$$\mu = \frac{\sum_n w_n}{D(N-1)}.$$

It is shown in [8] that for price-taking firms, their exists a competitive equilibrium at which the resulting allocation is efficient, meaning, it minimizes the aggregate production cost. The price-taking assumption models reality well when the number of participating firms is very large and the effect of one firm on the market clearing price is negligible, for all practical purposes. This is not true when the firms are aware of their power to influence the market clearing price.

Writing the market price μ explicitly as a function of the bid vector $\mathbf{w}$ for the price-anticipating case, we have: $\mu(\mathbf{w})$. For $w_n > 0$, the supply for a firm n, hence becomes $S(\mu(\mathbf{w}), w_n) = D\left(1 - \frac{w_n(N-1)}{w_n + \sum_{k\neq n} w_k}\right)$ and for $w_n = 0$, the authors adopt the convention that $S(\mu(\mathbf{w}), 0) = D \;\forall\; \mathbf{w}_{-n}$. Writing payoff for firm n,

$$\Pi_n(w_n, \mathbf{w}_{-n}) = \begin{cases} (\frac{\sum_k w_k}{N-1} - w_n) - & w_n > 0 \\ C_n(D(1 - \frac{w_n(N-1)}{\sum_k w_k})), & \\ \frac{\sum_{k\neq n} w_k}{N-1} - C_n(D), & w_n = 0 \end{cases}$$

Johari and Tsitiklis [8] establish the existence and uniqueness of Nash equilibria in the above setting when the firms are price-anticipating. They also derive an upper bound on the efficiency loss and show that the ratio of aggregate production cost at Nash equilibria to the minimum possible aggregate production cost is atmost $1 + \frac{1}{N-2}$. To bridge this gap, we aim to develop a scalarized mechanism for the producer side that results in efficient allocations at the NEPs.

We now discuss the work of Yang and Hajek [9] who introduced a VCG inspired payment structure in a scalar mechanism for the consumer side which they described as the VCG-Kelly mechanism.

C. VCG-Kelly Mechanism

Yang and Hajek [9] developed this mechanism for the consumer side problem setting described in Section II-A. Consumers submit one-dimensional bids to the planner from which it constructs surrogate valuation functions for each player on which the allocations are based, as is done in the Kelly mechanism. The valuation function of a consumer r with bid w_r is defined as, $V_r(w_r, x_r) = w_r f(x_r)\ \forall r$ where f_r's are strictly increasing, strictly concave and twice differentiable with $f_r(0+) = +\infty$. $f_r(0) = -\infty$ is allowed with the convention that, $V_r(0,0) = 0$. f_r's mimic the behaviour of the $\log$ function, however the authors introduce a generalization beyond the log-valuation function and consider the following family of surrogate valuations functions,

$$\phi^{(\alpha)}(x) = \begin{cases} (1-\alpha)^{-1}x^{1-\alpha}, & \text{if} \quad \alpha \in (0,1) \\ \log x, & \text{if} \quad \alpha = 1 \end{cases}.$$

We have the following allocation rule,

$$\mathbf{x}^{VCGK}(\mathbf{w}) = \underset{\mathbf{x}\in X_C}{\operatorname{argmax}} \sum_{r\in[N]} V_r(w_r, x_r)$$

where X_C is as defined in Section II-A. The payment rule is inspired from the VCG mechanism and the payment made by a consumer r is the loss in utility of all other consumers because of its presence.

$$m_r^{VCGK}(\mathbf{w}) = \max_{\mathbf{x}\in X_C, x_r=0} \sum_{j\neq r} V_j(w_j, x_j) - \sum_{j\neq r} V_j(w_j, x_j^{VCGK}(\mathbf{w}))$$

Yang and Hajek [9] showed under a certain regulatory assumption on the buyers that NEPs exist and correspond to efficient allocations with respect to the System Problem. The assumption on the participating consumers was that there exist atleast two consumers such that their valuation function satisfies $U_r'(0) = +\infty$.

The mechanism designed by Yang and Hajek [9], does not automatically apply to the supplier side formulation. Superficially the problem (C) is similar to the problem (S) (one can take $U_j = -C_j$), however there are subtleties in the details. VCG-Kelly requires that U_j is a concave increasing function, and since C_j is a convex increasing function, $-C_j$ would be a concave decreasing function, which is incompatible with VCG-Kelly. This necessitates a fresh design of a communication efficient and also economically efficient mechanism for the problem (S).

III. A SCALARIZED VCG MECHANISM FOR EFFICIENT ALLOCATIONS

In this section, we present a mechanism with low communication overhead which achieves efficiency on the supplier side at the Nash equilibrium points (NEPs). Each producer r submits a one-dimensional bid $w_r \geq 0$ to the market maker, which then constructs a surrogate cost function as a proxy for the true cost function of the producer as follows:

$$V_r(w_r, x_r) = \begin{cases} w_r f_r(x_r) & \text{if} \quad w_r > 0 \\ 0 & \text{if} \quad w_r = 0 \end{cases}, \tag{4}$$

where f_r's defined over $(-\infty, D]$ are twice differentiable, strictly increasing and strictly convex functions with $f_r'(D-) = +\infty$. The domain of $V_r(w_r, \cdot)$ is also $(-\infty, D]$.

Examples: $f_r(x_r) = \log\left(\frac{D}{D-x_r}\right)$, $x_r \leq D$.

A. Allocation and Pricing Rule

Our mechanism makes allocations according to the following rule, where $\mathbf{w} = (w_r, r \in \mathcal{N})$ denotes the vector of bids from the suppliers.

$$\mathbf{x}(\mathbf{w}) = \begin{cases} \operatorname{argmin}_{\mathbf{x}\in X_F} \sum\limits_{j\in\mathcal{N}} V_j(w_j, x_j), & \text{if} \quad \mathbf{w} \neq 0 \\ \frac{D}{N}\mathbf{1}, & \text{if} \quad \mathbf{w} = 0 \end{cases} \tag{5}$$

where $X_F = \{\mathbf{x} \mid x_j \leq D\ \forall\ j,\ \sum_{j\in\mathcal{N}} x_j = D\}$ and $\mathbf{1}$ is the vector of all ones. As is apparent from the above rule, if everybody submits a zero bid, the inelastic demand is met equally by all the producers. On the other hand, when $\mathbf{w} \neq 0$, the allocation is unique which can be easily verified by the strict convexity of the optimization objective and is characterized as the solution to the following set of KKT conditions as described in Lemma 3.1 below.

Lemma 3.1: If $\mathbf{w} \neq 0$, KKT conditions corresponding to the allocation rule are as follows:

$$x_r = D, \quad \text{if} \quad w_r = 0 \tag{6}$$

$$w_r f_r'(x_r) = \lambda, \quad \text{if} \quad w_r > 0 \tag{7}$$

$$\sum_{j\in\mathcal{N}} x_j = D \tag{8}$$

where $\lambda > 0$.

Proof: Writing the Lagrangian function for the optimization problem posed in the allocation rule,

$$L(\mathbf{x}, \mu, \lambda) = \sum_{j:w_j>0} w_j f_j(x_j) + \sum_j \mu_j(x_j - D) - \lambda(\sum_j x_j - D)$$

where $\mu_j \geq 0$. Let $\mathbf{Z}$ be the set of all producers with strictly positive bids. For $j \in \mathbf{Z}$, we have $w_j f_j'(x_j) + \mu_j = \lambda$. If the allocation $x_j = D$, from complementary slackness, we have $\mu_j \geq 0$ which gives us $w_j f_j'(D) \leq \lambda$ which is not possible since λ exists and is a finite value. Therefore, for $j \in \mathbf{Z}$, $x_j < D$ which gives us $w_j f_j'(x_j) = \lambda$. Since, $w_j > 0$ for $j \in \mathbf{Z}$ and $x_j < D$, λ is strictly positive. Now for $k \in \mathcal{N} \setminus \mathbf{Z}$, differentiating the Lagrangian with respect to x_k, we get $\mu_k = \lambda$. Since, $\lambda > 0$, we get $\mu_k > 0$ and from complementary slackness, $x_k = D$. Producers which submit a zero bid get allocated the entire inelastic demand D by the market planner. ■

We adopt a VCG-like externality payment structure based on our surrogate cost function. The payment to any player r as a function of the bid vector profile $\mathbf{w}$, for $\mathbf{w}_{-r} \neq 0$ is defined as,

$$m_r(w_r, \mathbf{w}_{-r}) = \min_{x_r=0, \mathbf{x} \in X_F} \sum_{j \neq r} V_j(w_j, x_j) - \sum_{j \neq r} V_j(w_j, x_j(w_r, \mathbf{w}_{-r})). \quad (9)$$

Note that the above payment captures the externality associated with producer r, *measured using the surrogate cost functions*. Extending the above payment rule to the case $\mathbf{w}_{-r} = 0$ results in payment $m_r(w_r, \mathbf{0}) = 0$.

Example: Let's derive expressions for allocation and payment rule for our example surrogate cost function for clarity. Consider the example surrogate cost function $f_r(x_r) = \log\left(\frac{D}{D-x_r}\right)$ defined over $(-\infty, D]$ $\forall$ r. From (7), for $j \in \{k : w_k > 0\}$,$x_j < D$ and $w_j f_j'(x_j) = w_j/(D - x_j) = \lambda$. Re-arranging to get, $x_j = D - w_j/\lambda$. Similarly from (6), $x_j = D$ for all producers j with bid $w_j = 0$. We obtain λ as a function of the bid vector $\mathbf{w}$ using the capacity constraint (8),

$$\sum_{j \in \mathbf{Z}} \left(D - \frac{w_j}{\lambda}\right) + \sum_{j : w_j = 0} D = D,$$

which implies

$$\lambda(\mathbf{w}) = \frac{\sum_{j \in [N]} w_j}{D(N-1)}.$$

The above is a closed form expression for $\lambda(\mathbf{w})$. Substituting this into $x_j = D - w_j/\lambda$ for producers with a positive bid, we obtain the allocation as

$$x_j(w_j, \mathbf{w}_{-j}) = D\left(1 - \frac{w_j(N-1)}{w_j + \sum_{k \neq j} w_k}\right) \quad (10)$$

Lemma 3.2: For $\mathbf{w} \neq 0$, both the allocation vector $\mathbf{x}(\mathbf{w})$ and $\lambda(\mathbf{w})$ are continuously differentiable functions of the bid vector $\mathbf{w}$. Fixing $\mathbf{w}_{-r}$, the partial derivatives of x_r and λ with respect to w_r for any producer r are strictly negative and strictly positive respectively.

Proof: The KKT conditions as described in the previous lemma can be written as:

$$w_r f_r'(x_r) = \lambda, \forall\ r \in \mathcal{N} \quad (11)$$

$$\sum_{j \in [N]} x_j = D \quad (12)$$

Note that $w_r f_r'(x_r) = \lambda$ holds even for $w_r = 0$, since $f_r'(D) = +\infty$. The allocations $\{x_r\}_r$ and the bids $\{w_r\}_r$ are implicitly related to each other, and their relation is expressed through a series of KKT conditions as above. The conditions can be written as, $F(\mathbf{x}, \mathbf{w}, \lambda) = 0$, where

$$F(\mathbf{x}, \mathbf{w}, \lambda) = \begin{pmatrix} w_r f_r'(x_r) - \lambda,\ r \in \mathcal{N} \\ \sum_{j \in \mathcal{N}} x_j - D \end{pmatrix}$$

The function F is continuously differentiable in it's arguments $\mathbf{x}$, $\mathbf{w}$ and λ. We have,

$$\left.\begin{pmatrix} \frac{\partial F}{\partial \mathbf{x}} & \frac{\partial F}{\partial \lambda} \end{pmatrix}\right|_{\mathbf{w}} = \begin{pmatrix} D & -\mathbf{1} \\ \mathbf{1}^T & 0 \end{pmatrix} \quad (13)$$

where $D = \text{diag}(w_r f_r''(x_r))$ and $\mathbf{1}$ is the vector of all ones. D is positive definite because of our assumptions on the surrogate cost function f_r's. *It is easy to prove the invertibility of the above matrix*. Therefore, the Implicit function theorem implies that $\mathbf{x}(\mathbf{w})$ and $\lambda(\mathbf{w})$ are continuously differentiable functions of the bid vector $\mathbf{w}$ for $\mathbf{w} \neq 0$ (since the KKT conditions are not applicable for the $\mathbf{w} = 0$ case).

Lemma 3.3: Matrix $\begin{pmatrix} D & -\mathbf{1} \\ \mathbf{1}^T & 0 \end{pmatrix}$ defined above is invertible.

Proof: We proceed by proving that the null space of this matrix is empty. Consider a *non-zero* vector $v = (v_1\ v_2 \cdots v_N\ v_{N+1})^T$ that lies in the null space of our given matrix. Therefore,

$$\begin{pmatrix} D & -\mathbf{1} \\ \mathbf{1}^T & 0 \end{pmatrix} v = 0$$

This gives us,

$$v_r(w_r f_r''(x_r)) = v_{N+1},\ r \in \mathcal{N} \quad (14)$$

$$\sum_{j \in \mathcal{N}} v_j = 0 \quad (15)$$

$w_r f_r''(x_r) > 0,\ r \in \mathcal{N}$. Substituting (14) in (15),

$$v_{N+1}\left(\frac{1}{w_1 f_1''(x_1)} + \cdots + \frac{1}{w_N f_N''(x_N)}\right) = 0 \quad (16)$$

(16) is only possible if $v_{N+1} = 0$, since the term inside the bracket is strictly positive. From (14), for $v_{N+1} = 0$, we have $v_r = 0,\ r \in \mathcal{N}$. But this contradicts our assumption that v is a non-zero vector. Therefore, no non-zero vector lies in the null space of the given matrix. Hence proved that the given matrix is invertible.

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Inverse of the matrix in (13) can be verified out to be:

$$\begin{pmatrix} H & D^{-1}\mathbf{1}Q \\ -Q\mathbf{1}^T D^{-1} & Q \end{pmatrix}$$

where Q is defined by $Q = (\mathbf{1}^T D^{-1}\mathbf{1})^{-1}$ (exists) and $H = D^{-1} - D^{-1}\mathbf{1}Q\mathbf{1}^T D^{-1}$. The partial derivatives are given as,

$$\begin{pmatrix} \frac{\partial \mathbf{x}}{\partial \mathbf{w}} \\ \frac{\partial \lambda}{\partial \mathbf{w}} \end{pmatrix} = -\left.\begin{pmatrix} \frac{\partial F}{\partial \mathbf{x}} & \frac{\partial F}{\partial \lambda} \end{pmatrix}^{-1}\right|_{\mathbf{w}} \frac{\partial F}{\partial \mathbf{w}}$$

It can be shown that,

$$\left(\frac{\partial \mathbf{x}}{\partial \mathbf{w}}\right) = -H \operatorname{diag}(f_r') \quad (17)$$

$$\left(\frac{\partial \lambda}{\partial \mathbf{w}}\right) = Q\mathbf{1}^T D^{-1} \operatorname{diag}(f_r') \quad (18)$$

$\text{diag}(f_r')$ is a positive definite matrix, since f_r's are strictly increasing. Multiplication by it does not change the sign of the diagonal entries of the resulting matrix. Therefore, we look at the diagonal entries of H to determine the sign of

$\left(\frac{\partial x_r}{\partial w_r}\right)$.

It can be seen that Q defined as $Q = (\mathbf{1}D^{-1}\mathbf{1}^T)^{-1}$ is a positive scalar quantity, and simplies to $Q = \left(\sum_{r\in\mathcal{N}} 1/(w_r f_r''(x_r))\right)^{-1}$. Similarly, H defined as $H = D^{-1} - D^{-1}\mathbf{1}Q\mathbf{1}^T D^{-1}$ simplifies to $H = D^{-1} - A$ where the ij'th element of A is given as $A_{ij} = \left(\sum_{r\in\mathcal{N}} 1/(w_r f_r''(x_r))\right)^{-1} \left(1/(w_i f_i''(x_i))\right)\left(1/(w_j f_j''(x_j))\right)$. Therefore, looking at the diagonal entries of H, we get $H_{rr} = \left(1/(w_r f_r''(x_r))\right)\left(1 - \frac{1/w_r f_r''(x_r)}{1/w_1 f_1''(x_1)+\cdots+1/w_N f_N''(x_N)}\right) > 0$. This implies that the diagonal entries of $\left(\frac{\partial \mathbf{x}}{\partial \mathbf{w}}\right)$ are strictly negative, that is $\frac{\partial x_r}{\partial w_r} < 0 \ \forall\ r$. It is also easy to verify that each entry of the row vector $\frac{\partial \lambda}{\partial \mathbf{w}}$ is strictly positive, that is $\frac{\partial \lambda}{\partial w_r} > 0 \ \forall\ r$.

In summary, our lemma establishes the allocation vector, $\mathbf{x}$ and λ as explicit continuously differentiable functions of the bid vector $\mathbf{w}$, for $\mathbf{w} \neq 0$ by invoking the Implict function theorem on the KKT conditions for our allocation rule. It also characterizes the nature of the partial derivatives, $\frac{\partial x_r}{\partial w_r}$ and $\frac{\partial \lambda}{\partial w_r}$ for $\mathbf{w} \neq 0$. ∎

Remark: We note here that our allocation rule does not guarantee non-negative allocations for all players. However, as we show in Lemma 3.7, under a certain technical assumption on the producers, any Nash equilibrium allocation resulting from our mechanism is always non-negative for each player.

B. Characterising Nash Equilibrium Bids

In this section, we derive several properties of Nash equilibrium bid vectors under the proposed scalarized VCG mechanism. The first is that an all-zero bid vector is not an equilibrium.

Lemma 3.4: A zero bid-vector is not a Nash equilibrium.

Proof: Say for a producer r, we have $\mathbf{w}_{-r} = 0$. $m_r(w_r, \mathbf{0}) = 0$ from the definition of our pricing rule. Payoff of producer r therefore is,

$$\Pi_r(w_r, \mathbf{w}_{-r}) = \begin{cases} -C_r((2-N)D) & \text{if} \quad w_r > 0 \\ -C_r(D/N) & \text{if} \quad w_r = 0 \end{cases}.$$

Since $N > 2$, therefore the allocation for $w_r > 0$ is negative and by the definition of cost function $C_r(.)$, $\Pi_r(w_r, \mathbf{w}_{-r}) = 0$ when $w_r > 0$. By comparing the two payoffs, producer r has a unilateral incentive to submit a positive bid. Therefore, a zero bid-vector can never be a Nash equilibrium for our mechanism. ∎

To obtain futher properties of NEPs under our mechanism, we make the following assumption.

Assumption 3.1: There are at least two producers with cost functions satisfying $C'(D) = +\infty$. These producers are called *special producers*.

Special producers are incapable of meeting the entire demand D by themselves. As we show below, special producers are guaranteed to make strictly positive bids at any NEP. Assumption 3.1 is needed to argue the efficiency of NEPs under our mechanism; indeed, we can demonstrate instances of the game that admit inefficient Nash equilibria when this assumption does not hold.

Lemma 3.5: Under Assumption 3.1, at least two coordinates of any Nash equilibrium bid vector are strictly positive.

Proof: We show that special producers always submit strictly positive bids at Nash equilibrium. To do this, we show that the best response of any special producer j satisfies $w_j > 0$ irrespective of $\mathbf{w}_{-j}$.

Case 1: $\mathbf{w}_{-j} = 0$. In this case, Lemma 3.4 implies that the best response of any player is to submit a positive bid.

Case 2: $\mathbf{w}_{-j} \neq 0$. In this case, the payoff of the special producer can be written as

$$\Pi_j(w_j, \mathbf{w}_{-j}) = h(\mathbf{w}_{-j}) - \sum_{i\neq j, w_i>0} w_i f_i(x_i(w_j, \mathbf{w}_{-j})) - C_j(x_j(w_j, \mathbf{w}_{-j}))$$

$\Pi_j(w_j, \mathbf{w}_{-j})$ is a continuously differentiable function of w_j. Writing the partial derivative of the payoff with respect to w_j,

$$\begin{aligned}\frac{\partial \Pi_j}{\partial w_j}(w_j, \mathbf{w}_{-j}) &= -\lambda(\mathbf{w}) \sum_{i\neq j, w_i>0} \frac{\partial x_i}{\partial w_j}(w_j, \mathbf{w}_{-j}) \\ &\quad - C_j'(x_j(w_j, \mathbf{w}_{-j}))\frac{\partial x_j}{\partial w_j}(\mathbf{w}) \\ &= \left(-\frac{\partial x_j}{\partial w_j}(\mathbf{w})\right)\left[C_j'(x_j(w_j, \mathbf{w}_{-j})) - \lambda(\mathbf{w})\right]\end{aligned}$$

$\mathbf{x}_j(w_j, \mathbf{w}_{-j})$ is a continuously differentiable and a monotone decreasing function of the bid w_j ($\mathbf{w}_{-j} \neq 0$). This and the fact that $C_j(.)$ is convex gives us that $C_j'(\mathbf{x}_j(w_j, \mathbf{w}_{-j}))$ is a decreasing function of w_j which makes the term inside square brackets a strictly decreasing, continuous function of w_j.

Since j is a special producer, $C_j'(\mathbf{x}_j(0, \mathbf{w}_{-j})) = C_j'(D) = +\infty$. This means that $\frac{\partial \Pi_j}{\partial w_j}(w_j, \mathbf{w}_{-j})$ tends to ∞ as $w_j \downarrow 0$. Therefore, the payoff is maximized for some $w_j > 0$ where the partial derivative is equal to zero.

Hence proved that any special producer j always submits a strictly positive bid for both $\mathbf{w}_{-j} \neq 0$ and $\mathbf{w}_{-j} = 0$. Since, we have atleast two special producers from our assumption, we have atleast two strictly positive bids for any bid-vector at Nash equilibrium. ∎

An immediate corollary of Lemma 3.5 is the following.

Corollary 3.6: Under Assumption 3.1, at any NEP, we would have $\mathbf{w}_{-r} \neq 0$ for all producers r.

The preceding results allow us to argue that NEPs correspond to non-negative allocations.

Lemma 3.7: Under Assumption 3.1, allocation at any NEP is non-negative for all producers j.

Proof: Consider any producer j. From Corollary 3.6, $\mathbf{w}_{-j} \neq 0$ at Nash equilibrium. Payoff for producer j is given as

$$\Pi_j(w_j, \mathbf{w}_{-j}) = h(\mathbf{w}_{-j}) - \sum_{i\neq j, w_i>0} w_i f_i(x_i(w_j, \mathbf{w}_{-j})) - C_j(x_j(w_j, \mathbf{w}_{-j}))$$

Say $x_j(\tilde{w}_j, \mathbf{w}_{-j}) = 0$. Since $\frac{\partial x_j}{\partial w_j}(w_j, \mathbf{w}_{-j})$ is strictly negative $\forall\ w_j \geq 0$, we have that $x_j(w_j, \mathbf{w}_{-j}) < 0 \ \forall\ w_j > \tilde{w}_j$.

We show that $w_j < \tilde{w_j}$ at Nash equilibrium and we do so with a proof by contradiction. Let us assume that the best response for producer j is such that $w_j > \tilde{w_j}$. Therefore, $x_j(w_j, \mathbf{w}_{-j}) < 0$ and hence $C_j(x_j(w_j, \mathbf{w}_{-j})) = 0$. Payoff for our producer j can then be written as

$$\Pi_j(w_j, \mathbf{w}_{-j}) = h(\mathbf{w}_{-j}) - \sum_{i \neq j, w_i > 0} w_i f_i(x_i(w_j, \mathbf{w}_{-j}))$$

Therefore,

$$\frac{\partial \Pi_j}{\partial w_j}(w_j, \mathbf{w}_{-j}) = \lambda(\mathbf{w}) \frac{\partial x_j}{\partial w_j}(w_j, \mathbf{w}_{-j})$$

$\frac{\partial \Pi_j}{\partial w_j}(w_j, \mathbf{w}_{-j})$ is therefore strictly negative. We cannot therefore have our best response as any $w_j > \tilde{w_j}$ and hence the allocation at NEP is non-negative. Since this holds for any producer j, hence proved that allocations to all the producers at any NEP are non-negative. ■

Finally, the relationship between equilibrium bids given by Lemma 3.7 further implies that equilibrium bid vectors are necessarily of the following two types.

Corollary 3.8: Under Assumption 3.1, at any NEP, either all bids are strictly positive, or a single bid is zero and rest are all strictly positive.

Proof: We show that we can have atmost one zero bid in any Nash equilibrium bid vector. Having more than one zero bid will imply that some of the allocations are negative to satisfy our constraint, which we have shown is not possible for any Nash equilibrium bid vector. Even for one zero bid, allocation to all the remaining producers is zero from the capacity constraint which implies that the rest of the producers must submit strictly positive bids at Nash equilibrium. ■

C. Main Results

Using the preceding properties of NEPs under the proposed scalarized VCG mechanism, we are now ready to prove our main results. The first guarantees the existence of NEPs (note that an NEP corresponds to a *pure* Nash equilibrium), and also provides an exact characterization of NEPs.

Theorem 3.9: Under Assumption 3.1, at least one Nash equilibrium vector exists. A vector $\mathbf{w}$ is a NEP if and only if the following conditions hold:

$$C'_j(\mathbf{x}_j(w_j, \mathbf{w}_{-j})) = \lambda(\mathbf{w}), \quad \text{if} \quad w_j > 0, \tag{19}$$

$$C'_j(D) \leq \lambda(\mathbf{w}), \quad \text{if} \quad w_j = 0. \tag{20}$$

Proof: To show existence of an NEP, it suffices to show that for any producer j and for any $\mathbf{w}_{-j}$, $\Pi_j(\cdot, \mathbf{w}_{-j})$ is quasiconcave [9]. This is easy to verify when $\mathbf{w}_{-j} = 0$. For $\mathbf{w}_{-j} \neq 0$, the payoff for producer j is,

$$\begin{aligned}\Pi_j(w_j, \mathbf{w}_{-j}) = h(\mathbf{w}_{-j}) - \sum_{i \neq j, w_i > 0} w_i f_i(x_i(w_j, \mathbf{w}_{-j})) \\ - C_j(x_j(w_j, \mathbf{w}_{-j}))\end{aligned}$$

$\Pi_j(w_j, \mathbf{w}_{-j})$ is a continuously differentiable function of producer j's bid, w_j. Writing the partial derivative of the payoff with respect to w_j,

$$\frac{\partial \Pi_j}{\partial w_j}(w_j, \mathbf{w}_{-j}) = \left(-\frac{\partial x_j}{\partial w_j}(\mathbf{w})\right)\left[C'_j(x_j(w_j, \mathbf{w}_{-j})) - \lambda(\mathbf{w})\right]$$

The term inside square brackets is a strictly decreasing, continuous function of w_j. The term outside square brackets is positive for all w_j, therefore the payoff is a strictly quasiconcave function of w_j.

Finally, for $\mathbf{w}_{-j} \neq 0$ fixed, $w_j = 0$ is the best response of producer j if and only if $\frac{\partial \Pi_j}{\partial w_j}(0, \mathbf{w}_{-j}) \leq 0$. To have a positive bid as the best response, there must exist $w_j > 0$ such that $\frac{\partial \Pi_j}{\partial w_j}(w_j, \mathbf{w}_{-j}) = 0$. These conditions, given by (19) and (20), are necessary and sufficient. ■

Having established the existence of NEPs, the following result shows that all NEPs are efficient, validating the proposed scalarized VCG mechanism.

Theorem 3.10: Under Assumption 3.1, all Nash equilibrium bid vectors result in efficient allocations.

Proof: We need to show that the allocation resulting from a bid-vector $\mathbf{w}$ at Nash equilibrium satisfies equations (1)–(3). For any Nash equilibrium bid-vector, conditions (19)–(20) are satisfied as shown in Theorem 3.9. Fixing $\mathbf{w}$, we write $\mathbf{x}_j(w_j, \mathbf{w}_{-j})$ as x_j in the arguments below. Based on Corollary 3.8, it suffices to consider the following two cases.

Case 1: $\mathbf{w} > 0$. In this case, $x_j(\mathbf{w}) \in [0, D) \ \forall \ j$. Condition (19) is applicable, rewriting which we have

$$\begin{aligned} &C'_j(x_j) = \lambda(\mathbf{w}), \quad x_j \in [0, D). \\ &\sum_{k \in [N]} x_k = D. \end{aligned}$$

Conditions (1)–(3) are thus satisfied taking $\lambda^* = \lambda(\mathbf{w})$. Hence, a Nash equilibrium bid vector with strictly positive bids always results in an efficient allocation.

Case 2: $w_r = 0$ ***and*** $\mathbf{w}_{-r} > 0$***.*** From our allocation rule, $x_r(\mathbf{w}) = D$ and $x_j(\mathbf{w}) = 0 \ \forall \ j \neq r$. From (19)–(20), we obtain the following conditions,

$$\begin{aligned} &C'_j(0) = \lambda(\mathbf{w}) \quad \forall \quad j \neq r, \\ &C'_r(D) \leq \lambda(\mathbf{w}), \\ &\sum_{k \in [N]} x_k = D. \end{aligned}$$

Since $C'_r(x_r) \leq \lambda(\mathbf{w})$, $\exists z_r \geq 0$ such that $C'_r(x_r) = \lambda(\mathbf{w}) - z_r$ where z_r is a slack variable. It follows from the above inequalities that $C'_j(x_j) = \lambda(\mathbf{w}) \geq \lambda(\mathbf{w}) - z_r$ for all $j \neq r$. Thus, equations (1)–(3) are satisfied for $\lambda^* = \lambda(\mathbf{w}) - z_r$. Therefore, all such Nash equilibrium bid vectors as described in this case result in efficient allocations. ■

IV. Two Sided Markets

Until now, we have explored the behaviour of multiple producers with their private cost models competing to meet a certain inelastic demand for some resource, a problem which we refer to as the Supplier-side problem. The analogous case on the other side of the market is the Consumer-side problem in which multiple consumers with their private utility functions compete with each other to consume a certain inelastic supply of some resource.

Motivated by situations arising in real life, one is often interested to study and analyse a market in which both consumers and producers participate simultaneously.

A. Model

In this subsection, we discuss the model of our two-sided marketplace.

1) Consumers: Consider a set of N_c consumers, $\mathcal{N}_c = \{1, 2, \ldots, N_c\}$. The utility function of consumer j is represented by $U_j(x_j)$, which is a *strictly increasing, strictly concave* and continuously differentiable function of its consumption allocation, x_j over $\mathbb{R}^+$. We also make an assumption on the consumer side of our market, something we'll refer from now on as the *Special Buyers Assumption* (see [9]): There are atleast two buyers participating in our market such that their utility functions satisfy $U'(0) = +\infty$.

2) Producers: Consider a set of $N_p > 2$ producers, $\mathcal{N}_p = \{1, 2, \ldots, N_p\}$. Cost function of any producer j is represented by the function $C_j(.)$ over it's domain $\mathcal{D}_j = (-\infty, 0] \cup \mathcal{D}_j^+$. $C_j(x_j)$ is a *strictly increasing, strictly convex* and continuously differentiable function of it's production allocation, x_j over $\mathcal{D}_j^+$ and $C_j(x_j) := 0,\ \forall\ x_j \in (-\infty, 0]$.

Consider a subset $\mathcal{S} \subset \mathcal{N}_p$ which we refer to as the set of *"special producers"*. Special producers have a markedly different characterization than the rest of the producers. For $j \in \mathcal{N}/\mathcal{S}$, $\mathcal{D}_j^+ = \mathbb{R}^+$ whereas for $j \in \mathcal{S}$, $\mathcal{D}_j^+ = [0, d_j]$ and $C_j'(d_j) = +\infty$ where $0 < d_j \leq D_0$. Here D_0 (*assumed known to the market planner*) is a lower bound on the optimal production level for the two-sided market D^* (see IV-B for details). Moreover, $C_j(x_j) := +\infty$ for $x_j \in (d_j, +\infty)\ \forall j \in \mathcal{S}$. We make the following assumption in this two-sided setting:

Assumption 4.1: There are atleast two special producers participating in the market, i.e. $|\mathcal{S}| \geq 2$.

The significance and the role of this assumption will become clear in subsequent sections. In this sub-section, we described the model of our two-sided market and that of the market participants. In the next sub-section, we discuss what is called the System problem for this two-sided setting.

B. System Problem

The market planner is trying to achieve equilibrium in the two-sided market in a way that maximizes the *"net utility"* for the entire system. This goal can be captured by the following optimization problem (S2):

$$\max \sum_{j \in \mathcal{N}_c} U_j(x_{c,j}) - \sum_{k \in \mathcal{N}_p} C_k(x_{p,k}) \tag{S2}$$

subject to $\sum_{j \in \mathcal{N}_c} x_{c,j} = \sum_{k \in \mathcal{N}_p} x_{p,k}$ and $\mathbf{x_c}, \mathbf{x_p} \geq 0$ and $x_{p,j} \leq d_j\ \forall\ j \in \mathcal{S}$. Here, $\mathbf{x_c}$ is the consumption allocation vector for the buyer's side and $\mathbf{x_p}$ is the supply allocation vector for the producer's side.

Alternatively, we can look at the following optimization problem (S3) where a consensus has already been reached on the total amount of good to be consumed or produced:

$$\max \sum_{j \in \mathcal{N}_c} U_j(x_{c,j}) - \sum_{k \in \mathcal{N}_p} C_k(x_{p,k}) \tag{S3}$$

subject to $\sum_{j \in \mathcal{N}_c} x_{c,j} = \sum_{k \in \mathcal{N}_p} x_{p,k} = D$ and $\mathbf{x_c}, \mathbf{x_p} \geq 0$ and $x_{p,j} \leq d_j\ \forall\ j \in \mathcal{S}$. This problem is different from (S2) in the sense that you're looking to allocate supply and consumption in an optimal way for a pre-determined amount, D of the good in (S3). This can very much be different from the optimal amount of good, D^* that ends up maximizing the net system utility in (S2).

(S3) provides us with a nice framework to solve the problem posed in (S2). One can start with an initial production level and hope to converge to the optimal production level by following an iterative process. The subsequent sections will focus on (S3) and not (S2).

In the subsequent sections, we'll discuss a market mechanism to solve (S3) and also how to actually go about the iteration process to solve (S2) through (S3). Before moving on, let's translate (S3) to a more workable form:

$$\max_{\sum_j x_{c,j} = D} \sum_{j \in \mathcal{N}_c} U_j(x_{c,j}) - \min_{\sum_k x_{p,k} = D} \sum_{k \in \mathcal{N}_p} C_k(x_{p,k}) \tag{S3-II}$$

subject to $\mathbf{x_c}$ and $\mathbf{x_p} \geq 0$, $x_{p,j} \leq d_j\ \forall\ j \in \mathcal{S}$ for the max and min optimization problems respectively. (S3-II) and (S3) are equivalent optimization problems and this nice decoupling is possible because the level of consumption (or production) is pre-determined or fixed.

C. Mechanism

We aim to design a market mechanism that solves (S3-II) considering the utility and cost models as private information of the players. Remember that the total amount of good to be consumed or produced, say D is fixed or given in this setting.

The nice decoupling of the consumer-side and producer-side problems as represented by (S3-II) enables the market planner to solve the consumer-side and producer-side problems separately and independently of each other for the given amount of good D. Therefore, our mechanism consists of a separate bidding process on the consumer and the producer side respectively. The allocation and pricing rules for the consumer-side bidding process are defined by the VCG-Kelly mechanism of Yang and Hajek [9] and for the producer-side bidding process, are defined by the mechanism proposed in this paper in Section III.

Recall that our *Special Producer's Assumption* (Assumption 4.1) in this two-sided market setting is different from the one made in Section III (Assumption 3.1). We now discuss a series of results exactly as we did in Section III for our modified model. *Note that the given production level, D is assumed to be greater than or equal to D_0 (defined in Section IV-A) for the rest of the analysis in this sub-section.*

Lemma 4.1: A zero bid-vector is not a Nash equilibrium.

Proof: Same as for Lemma 3.4. ∎

Lemma 4.2: Under Assumption 4.1, at least two coordinates of any Nash equilibrium bid vector are strictly positive.

Proof: Our approach is similar to the proof of Lemma 3.5. The only change is that instead of having atleast two producers s.t. $C_j'(D) = +\infty$, we have atleast two producers s.t. $C_j'(d_j) = +\infty$ where $0 < d_j \leq D_0 \leq D$ and d_j is not necessarily equal to D. For a special producer j, $C_j(x_j)$ is *strictly increasing, strictly convex* and a continuously differentiable function for $x_j \in [0, d_j]$ and is defined as $C_j(x_j) = +\infty$ for $x_j > d_j$ as defined in IV-A.

We prove that special producers always submit strictly positive bids at Nash equilibrium by showing that the best response of any special producer j satisfies $w_j > 0$ irrespective of $\mathbf{w}_{-j}$.

Case 1: $\mathbf{w}_{-j} = 0$. In this case, Lemma 4.1 implies that the best response of any player is to submit a positive bid.

Case 2: $\mathbf{w}_{-j} \neq 0$. $x_j(w_j, \mathbf{w}_{-j})$ is a monotone decreasing function of bid w_j for a given $\mathbf{w}_{-j}$ (from Lemma 3.2). Let's say $x_j(\tilde{w}_j, \mathbf{w}_{-j}) = d_j$. Therefore, payoff of the special producer can be written as:

$$\begin{aligned}\Pi_j(w_j, \mathbf{w}_{-j}) &= -\infty, \quad w_j \leq \tilde{w}_j \\ &= h(\mathbf{w}_{-j}) - \sum_{i \neq j, w_i > 0} w_i f_i(x_i(w_j, \mathbf{w}_{-j})) \\ &\quad - C_j(x_j(w_j, \mathbf{w}_{-j})), \quad w_j > \tilde{w}_j\end{aligned}$$

The special producer being a rational agent will submit a bid w_j such that $w_j > \tilde{w}_j > 0$ to avoid the $-\infty$ payoff. Hence proved that any special producer j always submits a strictly positive bid for both $\mathbf{w}_{-j} \neq 0$ and $\mathbf{w}_{-j} = 0$. Since, we have atleast two special producers from our assumption, we have atleast two strictly positive bids for any Nash equilibrium bid-vector. ∎

Corollary 4.3: Under Assumption 4.1, at any NEP, we would have $\mathbf{w}_{-r} \neq 0$ for all producers r.

The preceding results allow us to argue that NEPs correspond to non-negative allocations.

Lemma 4.4: Under Assumption 4.1, allocation at any NEP is non-negative for all producers j.

Proof: Similar to the proof of Lemma 3.7. ∎

Finally, the relationship between equilibrium bids given by Lemma 4.4 further implies that equilibrium bid vectors are necessarily of the following two types.

Corollary 4.5: Under Assumption 4.1, at any NEP, either all bids are strictly positive, or a single bid is zero and rest all are strictly positive.

Proof: Same as for Corrolary 3.8. ∎

We now discuss the existence and, necessary and sufficient conditions for a Nash equilibrium bidding strategy under Assumption 4.1.

Theorem 4.6: Under Assumption 4.1, at least one Nash equilibrium vector exists. A vector $\mathbf{w}$ is a NEP if and only if the following conditions hold:

$$C_j'(\mathbf{x}_j(w_j, \mathbf{w}_{-j})) = \lambda(\mathbf{w}), \quad \text{if} \quad w_j > 0, x_j \in [0, D), j \in \mathcal{N}_p/\mathcal{S} \tag{21}$$

$$C_j'(\mathbf{x}_j(w_j, \mathbf{w}_{-j})) = \lambda(\mathbf{w}), \quad \text{if} \quad w_j > 0, x_j \in [0, d_j), j \in \mathcal{S} \tag{22}$$

$$C_j'(D) \leq \lambda(\mathbf{w}), \quad \text{if} \quad w_j = 0. \tag{23}$$

Proof: To show existence of an NEP, it suffices to show that for any producer j and for any $\mathbf{w}_{-j}$, $\Pi_j(\cdot, \mathbf{w}_{-j})$ is quasiconcave [9].

First let's talk about non-special producers i.e. those producers whose cost models are strictly increasing, strictly convex and continuously differentiable over the entire domain $[0, D]$. From Corollary 4.3, $\mathbf{w}_{-j} \neq 0$, therefore the payoff for producer j is,

$$\begin{aligned}\Pi_j(w_j, \mathbf{w}_{-j}) = h(\mathbf{w}_{-j}) - \sum_{i \neq j, w_i > 0} w_i f_i(x_i(w_j, \mathbf{w}_{-j})) \\ - C_j(x_j(w_j, \mathbf{w}_{-j}))\end{aligned} \tag{24}$$

$\Pi_j(w_j, \mathbf{w}_{-j})$ is a continuously differentiable function of producer j's bid, w_j. Writing the partial derivative of the payoff with respect to w_j,

$$\frac{\partial \Pi_j}{\partial w_j}(w_j, \mathbf{w}_{-j}) = \left(-\frac{\partial x_j}{\partial w_j}(\mathbf{w})\right)\left[C_j'(x_j(w_j, \mathbf{w}_{-j})) - \lambda(\mathbf{w})\right] \tag{25}$$

The term inside square brackets is a strictly decreasing, continuous function of w_j. The term outside square brackets is positive for all w_j, therefore the payoff is a strictly quasiconcave function of w_j.

Finally, for $\mathbf{w}_{-j} \neq 0$ fixed, $w_j = 0$ is the best response of producer j if and only if $\frac{\partial \Pi_j}{\partial w_j}(0, \mathbf{w}_{-j}) \leq 0$. To have a positive bid as the best response, there must exist $w_j > 0$ such that $\frac{\partial \Pi_j}{\partial w_j}(w_j, \mathbf{w}_{-j}) = 0$. These conditions, given by (19) and (20), are necessary and sufficient.

Now let's discuss special producers. Consider a special producer j such that $C_j'(d_j) = +\infty$. For $\mathbf{w}_{-j} \neq 0$, $x_j(w_j, \mathbf{w}_{-j})$ is a monotone decreasing function of the bid w_j. Let's consider the bid $\tilde{w}_j$ uniquely determined by $\mathbf{w}_{-j}$ such that $x_j(\tilde{w}_j, \mathbf{w}_{-j}) = d_j$. At Nash equilibrium, $w_j > \tilde{w}_j$ and correspondingly $x_j(w_j, \mathbf{w}_{-j}) < d_j$. The payoff for special producer j for $w_j \geq \tilde{w}_j$ is given by (24). The partial derivative with respect to w_j is given by (25). The term inside square brackets is a strictly decreasing, continuous function of w_j and $\frac{\partial \Pi_j}{\partial w_j}(\tilde{w}_j, \mathbf{w}_{-j}) = +\infty > 0$, therefore there must exist $w_j > \tilde{w}_j$ such that $\frac{\partial \Pi_j}{\partial w_j}(w_j, \mathbf{w}_{-j}) = 0$. Therefore, the necessary and sufficient condition for a special producer is given by (22). Hence, proved. ∎

Having established the existence of NEPs, the following result shows that all NEPs are efficient, validating the proposed scalarized VCG mechanism under Assumption 4.1.

Theorem 4.7: Under Assumption 4.1, all Nash equilibrium bid vectors result in efficient allocations.

Proof: KKT conditions for the supplier-side System problem are as follows:

$$C_j'(x_j) \geq \lambda^*, \quad \text{if} \quad x_j = 0 \tag{26}$$
$$C_j'(x_j) = \lambda^*, \quad \text{if} \quad j \in \mathcal{S} \quad \text{and} \quad x_j \in (0, d_j) \tag{27}$$
$$C_j'(x_j) = \lambda^*, \quad \text{if} \quad j \in \mathcal{N}_p/\mathcal{S} \quad \text{and} \quad x_j \in (0, D] \tag{28}$$
$$\sum_{j \in \mathcal{N}_p} x_j = D \tag{29}$$

We need to show that the allocation resulting from a bid-vector $\mathbf{w}$ at Nash equilibrium satisfies equations (26)–(29). The necessary and sufficient conditions for a Nash equilibrium bid vector are described in Theorem 4.6. Fixing $\mathbf{w}$, we write $\mathbf{x}_j(w_j, \mathbf{w}_{-j})$ as x_j in the arguments below. Based on Corollary 4.5, it suffices to consider the following two cases.

Case 1: $\mathbf{w} > 0$. In this case, $x_j(\mathbf{w}) \in [0, D) \ \forall \ j$. Conditions (21)–(22) are applicable, rewriting which we have

$$C_j'(x_j) = \lambda(\mathbf{w}), \quad x_j \in [0, D), \ j \in \mathcal{N}_p/\mathcal{S}$$
$$C_j'(x_j) = \lambda(\mathbf{w}), \quad x_j \in [0, d_j), \ j \in \mathcal{S}$$
$$\sum_{k \in \mathcal{N}_p} x_k = D.$$

Conditions (26)–(29) are thus satisfied taking $\lambda^* = \lambda(\mathbf{w})$. Hence, a Nash equilibrium bid vector with strictly positive bids always results in an efficient allocation.

Case 2: $w_r = 0$ ***and*** $\mathbf{w}_{-r} > 0$. From our allocation rule, $x_r(\mathbf{w}) = D$ and $x_j(\mathbf{w}) = 0 \ \forall \ j \neq r$. From (19)–(20), we obtain the following conditions,

$$C_j'(0) = \lambda(\mathbf{w}) \quad \forall \quad j \neq r,$$
$$C_r'(D) \leq \lambda(\mathbf{w}),$$
$$\sum_{k \in \mathcal{N}_p} x_k = D.$$

Since $C_r'(x_r) \leq \lambda(\mathbf{w})$, $\exists z_r \geq 0$ such that $C_r'(x_r) = \lambda(\mathbf{w}) - z_r$ where z_r is a slack variable. It follows from the above inequalities that $C_j'(x_j) = \lambda(\mathbf{w}) \geq \lambda(\mathbf{w}) - z_r$ for all $j \neq r$. Thus, equations (26)–(29) are satisfied for $\lambda^* = \lambda(\mathbf{w}) - z_r$. Therefore, all such Nash equilibrium bid vectors as described in this case result in efficient allocations. ■

D. Gradient Ascent Scheme for Optimal Production Level

In this sub-section, we discuss how to solve (S2) through (S3-II). Consider the function $V(D)$ defined as follows:

$$V(D) = \max_{\mathbf{x}_c \in X_c(D)} \sum_{j \in \mathcal{N}_c} U_j(x_{c,j}) - \min_{\mathbf{x}_p \in X_p(D)} \sum_{k \in \mathcal{N}_p} C_k(x_{p,k}) \tag{30}$$

where $X_c(D)$ and $X_p(D)$ are constraint sets parametrized by D, defined as $X_c(D) = \{\mathbf{x} \mid x_j \geq 0 \ \forall \ j, \sum_j x_j = D\}$ and $X_p(D) = \{\mathbf{x} \mid x_j \geq 0 \ \forall \ j, x_j \leq d_j \ \forall \ j \in \mathcal{S}, \sum_j x_j = D\}$. Let $\mu_c^*(D)$ and $\mu_p^*(D)$ be Lagrange multipliers for the optimization problem posed above, corresponding to the capacity constraint on the supplier-side and the producer-side respectively.

Lemma 4.8: V(D) is a continuously differentiable, concave function of D, and $V'(D) = \mu_c^*(D) - \mu_p^*(D)$.

Proof: The lemma follows directly from the Envelope theorem [22]. ■

We follow a gradient ascent type scheme to update the production level, D as follows:

$$D^{k+1} = D^k + \alpha^k(\mu_c^*(D^k) - \mu_p^*(D^k))$$

where $\{\alpha^k\}$ is an appropriate step-size sequence. Note that the initial condition for our iteration on the production level, $D^0 = D_0$ where D_0 is a lower bound on the optimal production level D^* assumed known to the market planner as defined in IV-A. For appropriate step-size sequences, $D^k > D_0 \ \forall \ k$.

The market planner cannot directly observe $\mu_c^*(D^k)$ and $\mu_p^*(D^k)$ since it doesn't have access to the private utility and cost models of the market participants.

Fortunately for us, since our mechanism, both the supply-side and the consumer-side induce optimal allocations at Nash equilibrium, such non-trivial information about the market participants is revealed to the market planner at Nash equilibrium.

For consumer-side, $\mu_c^*(D^k) = \mu_c(\mathbf{w}_c)$ (proved in [9]) where $\mathbf{w}_c$ is the consumer-side Nash equilibrium bid vector for production level D^k and $\mu_c(\mathbf{w}_c)$ is the capacity constraint Lagrange multiplier corresponding to the supply-side allocation rule for the Nash equilibrium bid vector at D^k.

For producer-side, we discuss two different cases as highlighted in Theorem 4.7:

Case-I: If for production level D^k, $\mathbf{w}_p > 0$ (all bids strictly positive) is the Nash equilibrium bid vector, then $\mu_p^*(D^k) = \mu_p(\mathbf{w}_p)$ where $\mu_p(\mathbf{w}_p)$ is the capacity constraint Lagrange multiplier corresponding to the supply-side allocation rule for the Nash equilibrium bid vector at D^k.

Case-II: For production level D^k, let $\mathbf{w}_p$ be the Nash equilibrium bid vector such that $\mathbf{w}_{p,r} = 0$ for some supplier r and $\mathbf{w}_{p,-r} > 0$. This case is only possible if all producers $j \neq r$ have the same startup cost, $C_j'(0)$ as evident from the KKT conditions. $\mu_p^*(\mathbf{w}_p)$ cannot be determined from the knowledge of $\mu_p(\mathbf{w}_p)$ in this case. We therefore introduce a penalty payment structure defined in the following sub-section such that supplying the entire demand is not feasible for any one user.

E. Penalty Payments Scheme

We introduce a penalty payment scheme to prevent any one producer submitting a zero bid at Nash equilibrium and supplying the entire demand by itself. Think of this as a way to disincentivize a producer (if there exists any of the sort), that can otherwise supply the entire demand in lieu of it's very low private cost model (under the original payment scheme) and as a result undercut the rest of the market. Such a producer is considered as an outlier and the penalty payment scheme is a method for the market planner to operate the market in a fair manner for all the market participants.

The net payoff of a producer r with penalty payments is the payment it receives from the market planner minus the cost and the penalty payment corresponding to the allocated supply:

$$\Pi_r(w_r, \mathbf{w}_{-r}) = m_r(\mathbf{w}) - p(x_r(\mathbf{w})) - C_r(x_r(\mathbf{w}))$$

where $p(x)$ is the penalty payment function for x amount of supply allocation. $p(x)$ is a *strictly convex, strictly increasing* function for $x > 0$ and $p(x) \coloneqq 0,\ x \leq 0$.

Lemma 4.9: Designing a penalty payment function $p(.)$ such that $p'(D) - p'(0) > \max_{i,j} \left\{C'_i(0) - C'_j(0)\right\}$ will result in a Nash equilibrium bid vector with all strictly positive bids.

Proof: Say that for the given market participants on the supplier side, producer r is badly behaving and the Nash equilibrium bid vector $\mathbf{w}$ without penalty payments is such that $w_r = 0$ and $\mathbf{w}_{-r} > 0$. Recall that this is possible only when $C'_j(0) = C'(0)\ \forall\ j \neq r$ and $C'_r(0) < C'(0)$.

We proceed by showing that with penalty payments, submitting a zero bid is not possible for any producer at Nash equilibrium. We first state the modified Nash equilibrium conditions with penalty payments (*can be easily derived*):

$$C'_j(\mathbf{x}_j(w_j, \mathbf{w}_{-j})) = \lambda(\mathbf{w}) - p'(x_j(\mathbf{w})), \quad \text{if} \quad w_j > 0 \tag{31}$$

$$C'_j(D) \leq \lambda(\mathbf{w}) - p'(D), \quad \text{if} \quad w_j = 0. \tag{32}$$

For the badly behaving producer r to submit a zero bid at Nash equilibrium under this new payment scheme, (32) must be true.

From our assumption on the penalty payment scheme, $p'(D) - p'(0) > \max_{i,j} \left\{C'_i(0) - C'_j(0)\right\} \geq C'(0) - C'_r(0)$. Rearranging, this gives us $C'_r(0) > [C'(0) + p'(0)] - p'(D)$. From (31), $C'(0) + p'(0) = \lambda(\mathbf{w})$, therefore $C'_r(0) > \lambda(\mathbf{w}) - p'(D)$. Therefore (31) cannot be true and hence the badly behaving producer r no longer submits a zero bid at Nash equilibrium. Hence, proved. ■

F. Remarks

This section is reserved for some final remarks on the two-sided market setting to be filled before submission.

Appendix

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