The Jacobi Method. For each $k \geq 1$, generate the components $x_i^{(k)}$ of $\boldsymbol{x}^{(k)}$ from $\boldsymbol{x}^{(k-1)}$ by

$$x_i^{(k)} = \frac{1}{a_{ii}}\left[\sum_{\substack{j=1,\\ j\neq i}}^{n}(-a_{ij}x_j^{(k-1)}) + b_i\right], \qquad \text{for } i = 1,2,, \dots n$$

Example. Apply the Jacobi method to solve

$$\begin{aligned} 5x_1 - 2x_2 + 3x_n &= -1 \\ -3x_1 + 9x_2 + x_n &= 2 \\ 2x_1 - x_2 - 7x_n &= 3 \end{aligned}$$

Continue iterations until two successive approximations are identical when rounded to three significant digits.

Solution To begin, rewrite the system

$$\begin{aligned} x_1 &= \frac{-1}{5} + \frac{2}{5}x_2 - \frac{3}{5}x_3 \\ x_2 &= \frac{2}{9} + \frac{3}{9}x_1 - \frac{1}{9}x_3 \\ x_3 &= -\frac{3}{7} + \frac{2}{7}x_1 - \frac{1}{7}x_2 \end{aligned}$$

Choose the initial guess $x_1 = 0, x_2 = 0, x_3 = 0$

The first approximation is

$$\begin{aligned} x_1^{(1)} &= \frac{-1}{5} + \frac{2}{5}(0) - \frac{3}{5}(0) = -0.200 \\ x_2^{(1)} &= \frac{2}{9} + \frac{3}{9}(0) - \frac{1}{9}(0) = 0.222 \\ x_3^{(1)} &= -\frac{3}{7} + \frac{2}{7}(0) - \frac{1}{7}(0) = -0.429 \end{aligned}$$

The Jacobi Method

***Two assumptions made on Jacobi Method*:**

1. The system given by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nn}x_n &= b_n \end{aligned}$$

Has a unique solution.

2. The coefficient matrix A has no zeros on its main diagonal, namely, $a_{11}, a_{22}, \dots, a_{nn}$ are nonzeros.

Main idea of Jacobi

To begin, solve the 1st equation for x_1, the 2nd equation for x_2 and so on to obtain the rewritten equations:

$$\begin{aligned} x_1 &= \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3 - \cdots a_{1n}x_n) \\ x_2 &= \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3 - \cdots a_{2n}x_n) \\ &\vdots \\ x_n &= \frac{1}{a_{nn}}(b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots a_{n,n-1}x_{n-1}) \end{aligned}$$

Then make an initial guess of the solution $\boldsymbol{x}^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, \dots x_n^{(0)})$. Substitute these values into the right hand side the of the rewritten equations to obtain the *first approximation,* $\left(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \dots x_n^{(1)}\right)$.

This accomplishes one **iteration**.

In the same way, the *second approximation* $\left(x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, \dots x_n^{(2)}\right)$ is computed by substituting the first approximation's x-vales into the right hand side of the rewritten equations.

By repeated iterations, we form a sequence of approximations $\boldsymbol{x}^{(k)} = \left(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, \dots x_n^{(k)}\right)^t, \quad k = 1,2,3, \dots$

Continue iteration, we obtain

n	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$x_1^{(k)}$	0.000	-0.200	0.146	0.192			
$x_2^{(k)}$	0.000	0.222	0.203	0.328			
$x_2^{(k)}$	0.000	-0.429	-0.517	-0.416			

Numerical Algorithm of Jacobi Method

Input: $A = [a_{ij}]$, $\boldsymbol{b}, \boldsymbol{XO} = \boldsymbol{x}^{(0)}$, tolerance TOL, maximum number of iterations N.

Step 1 Set $k = 1$

Step 2 while $(k \leq N)$ do Steps 3-6

Step 3 For for $i = 1,2,, \dots n$

$$x_i = \frac{1}{a_{ii}}\left[\sum_{\substack{j=1,\\ j\neq i}}^{n}(-a_{ij}\boldsymbol{XO}_j) + b_i\right],$$

Step 4 If $\|\boldsymbol{x} - \boldsymbol{XO}\| < TOL$, then OUTPUT $(x_1, x_2, x_3, \dots x_n)$;

STOP.

Step 5 Set $k = k + 1$.

Step 6 For for $i = 1,2,, \dots n$

Set $\boldsymbol{XO}_i = x_i$.

Step 7 OUTPUT $(x_1, x_2, x_3, \dots x_n)$;

STOP.

Another stopping criterion in Step 4: $\frac{\|\boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k-1)}\|}{\|\boldsymbol{x}^{(k)}\|}$

Problem 1: Write a menu driven program in C / C++ code for the above algorithm. Menu should be of your own choice. Use this algorithm as a function and solve m simultaneous equations with n unknowns. Find the conditions for unique and no solution. If solution exists, find whether it is degenerate or not. Try to draw the graph for it.

Problem 2:

(a) 3x - y = 8, x + y = 8 (Sol: (1, 2))
(b) 6x - 10y = 5, 2x - 3y = 1 (Sol: (7/2, -2))
(c) 2x - 8y = 9, x - 4y = -6 (Sol: No solution)