Design and Simulation of LQR and LQG Controller for Crane System

ENPM 667 – CONTROL OF ROBOTIC SYSTEMS

FINAL PROJECT REPORT

SUBMITTED BY

MOUMITA PAUL- 116860970

NAMAN GUPTA - 116949330

SUBMITTED TO:

Dr. Waseem Malik

Introduction

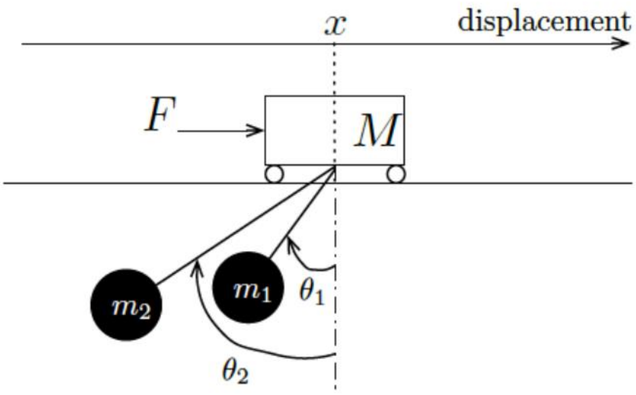


Fig: Cart with two suspended loads moving in 1 dimensional direction

Variables:

- M: Mass of the cart
- F: Force applied to the cart
- m1: mass of load 1
- m2: mass of load 2
- l1 = length of cable 1 through which mass 1 is suspended
- l2 = length of cable 2 through which mass 2 is suspended

Assumptions:

- The payload is regarded as a material particle.
- The rope is considered as an inflexible rod.
- Compared with the payload mass, the rope mass is ignored.
- The trolley moves in the x-direction.
- The payload moves on the x–y surface.
- No friction exists in the system.
- For simulations, you should have MATLAB, SIMULINK installed, and the Simulink should also have a Luenberger Observer Block

A) EQUATIONS OF MOTION

Now using the above assumptions, we obtain the equations of motion of the system and thus obtain the equations of the non-linear system

Analytical modeling of the given cart and pendulum system can be done by two techniques; Newton-Euler and Euler-Lagrange method. In this project, Euler-Lagrange technique is used which requires Kinetic and Potential Energies for formulating equations of motion of the system. The Euler-Lagrange equation is given as,

$$\frac{d}{dt}\left[\frac{\partial \boldsymbol{\mathcal{L}}}{\partial \dot{q}}\right] - \frac{\partial \boldsymbol{\mathcal{L}}}{\partial q} = f$$

where, $\boldsymbol{\mathcal{L}} = T - V$ is the Lagrange operator, q is the generalized coordinates, T denotes the Kinetic energy and V denotes potential energy of the system, f is the acting force of system in the its generalized coordinate.

Lagrangian equation

The generalized coordinates in our given system are x, θ_1 and θ_2. Using the frame of reference shown in Fig., position of mass 1 and mass 2 can be computed as follows,

$$X_{m1} = x - l_1 S_1$$

$$Y_{m1} = -l_1 C_1$$

$$X_{m2} = x - l_2 S_2$$

$$X_{m2} = -l_2 C_2$$

Here, x is the displacement of cart in positive-x direction.

To compute the velocities associated with each pendulum, derivatives of the above equations are computed.

$$\dot{X_{m1}} = \dot{x} - l_1 C_1 \dot{\theta}_1$$

$$\dot{X_{m2}} = \dot{x} - l_2 C_2 \dot{\theta}_2$$

$$\dot{Y_{m1}} = l_1 S_1 \dot{\theta}_1$$

$$\dot{Y_{m2}} = l_2 S_2 \dot{\theta}_2$$

These equations will provide the component of velocities of pendulum in x-direction and y-direction. The magnitude of velocity vector associated with the pendulums can be computed as follows,

$$v_1^2 = \dot{X_{m1}}^2 + \dot{Y_{m1}}^2$$

$$= \left(\dot{x} - l_1 C_1 \dot{\theta}_1\right)^2 + \left(l_1 S_1 \dot{\theta}_1\right)^2$$

$$= \dot{x}^2 + l_1 \dot{\theta}_1^{\ 2} - 2 l_1 C_1 \dot{x} \dot{\theta}_1$$

$$\mathrm{v}_2^2 = \dot{X_{m2}}^2 + \dot{Y_{m2}}^2$$

$$= \left(\dot{x} - l_2 C_2 \dot{\theta}_2\right)^2 + \left(l_2 S_2 \dot{\theta}_2\right)^2$$

$$= \dot{x}^2 + l_2\dot{\theta_2}^2 - 2l_2C_2\dot{x}\dot{\theta_2}$$

The displacement of cart is only in positive-x direction. So, the velocity of cart is given by $\dot{x}$. Using Eqn. () and Eqn. (), the Kinetic Energy of the system can be written as sum of the kinetic energy associated with the cart as well as both the pendulums.

$$K.E. = \frac{1}{2}\dot{x}^2(M) + \frac{1}{2}(m_1{v_1}^2) + \frac{1}{2}(m_2{v_2}^2)$$

Upon substituting the values of x, v_1, v_2 in the above equation, we get the following result.

$$= \frac{1}{2}\dot{x}^2(m_1 + m_2 + M) + \frac{1}{2}\left(m_1l_1\dot{\theta_1}^2\right) + \frac{1}{2}\left(m_2l_2\dot{\theta_2}^2\right) - m_1l_1cos_1\dot{x}\dot{\theta_1} - m_2l_2C_2\dot{x}\dot{\theta_2}$$

For computing the Potential Energy of the system, the cart height is taken as reference. Therefore, it consists of components from the pendulums only and is given by,

$$P.E. = -m_1gl_1C_1 - m_2gl_2C_2$$

Now, the Lagrange of the system can be calculated as,

$$\boldsymbol{\mathcal{L}} = K.E. - P.E.$$

$$\boldsymbol{\mathcal{L}} = \frac{1}{2}\dot{x}^2(m_1 + m_2 + M) + \frac{1}{2}\left(m_1l_1\dot{\theta_1}^2\right) + \frac{1}{2}\left(m_2l_2\dot{\theta_2}^2\right) - m_1l_1cos_1\dot{x}\dot{\theta_1} - m_2l_2C_2\dot{x}\dot{\theta_2} + m_1gl_1C_1 + m_2gl_2C_2$$

Next, we compute the derivative of the Lagrangian with respect to $\dot{x}, \dot{\theta}_1, \dot{\theta}_2$

$$\frac{d}{dt}\left[\frac{\partial\boldsymbol{\mathcal{L}}}{\partial\dot{x}}\right] - \frac{\partial\boldsymbol{\mathcal{L}}}{\partial x} = \mathrm{F}$$

$$\frac{d}{dt}\left[\frac{\partial\boldsymbol{\mathcal{L}}}{\partial\dot{\theta}_1}\right] - \frac{\partial\boldsymbol{\mathcal{L}}}{\partial\theta_1} = 0$$

$$\frac{d}{dt}\left[\frac{\partial\boldsymbol{\mathcal{L}}}{\partial\dot{\theta}_2}\right] - \frac{\partial\boldsymbol{\mathcal{L}}}{\partial\theta_2} = 0$$

$$\frac{\partial\boldsymbol{\mathcal{L}}}{\partial\dot{x}} = \dot{x}(m_1 + m_2 + M) - m_1l_1C_1(\dot{\theta_1}) - m_2l_2C_2(\dot{\theta_2})$$

$$\frac{\partial\boldsymbol{\mathcal{L}}}{\partial x} = 0$$

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{x}}\right] = \ddot{x}(m_1 + m_2 + M) - m_1 l_1\left(\ddot{\theta}_1 C_1 - S_1 \dot{\theta}_1{}^2\right) - m_2 l_2\left(\ddot{\theta}_2 C_2 - S_2 \dot{\theta}_2{}^2\right) = \mathrm{F}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = m_1 l_1{}^2 \dot{\theta}_1 - m_1 l_1 C_1 \dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) = m_1 l_1{}^2 \ddot{\theta}_1 - m_1 l_1\left(C_1 \ddot{x} - S_1 \dot{x}\dot{\theta}_1\right)$$

$$\frac{\partial \mathcal{L}}{\partial \theta_1} = S_1 m_1 l_1 m_1 \dot{x}\dot{\theta}_1 - m_1 g l_1 S_1$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} = m_2 l_2{}^2 \dot{\theta}_2 - m_2 l_2 C_2 \dot{x}$$

$$\frac{\partial \mathcal{L}}{\partial \theta_2} = S_2 m_2 l_2 m_2 \dot{x}\dot{\theta}_2 - m_2 g l_2 S_2$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right) = m_2 l_2{}^2 \ddot{\theta}_2 - m_2 l_2\left(C_2 \ddot{x} - S_2 \dot{x}\dot{\theta}_2\right)$$

$Thus\ we\ get,$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) - \frac{\partial \mathcal{L}}{\partial \theta_1} = m_1 l_1{}^2 \ddot{\theta}_1 - m_1 l_1\left(C_1 \ddot{x} - S_1 \dot{x}\dot{\theta}_1\right) - S_1 m_1 l_1 m_1 \dot{x}\dot{\theta}_1 + m_1 g l_1 S_1 = 0$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) - \frac{\partial \mathcal{L}}{\partial \theta_1} = m_1 l_1{}^2 \ddot{\theta}_1 - m_1 l_1 C_1 \ddot{x} + m_1 g l_1 S_1$$

Now we assume that for very small angle we take the following assumptions:
$C_1 = 1, S_1 = \theta_1, \dot{\theta}_1{}^2 = 0, \dot{\theta}_2{}^2 = 0$

Thus upon linearizing the previous equation, we get: $m_1 l_1{}^2 \ddot{\theta}_1 - m_1 l_1 \ddot{x} + m_1 g l_1 \theta_1 = 0$

Similarly

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right) - \frac{\partial \mathcal{L}}{\partial \theta_2} = m_2 l_2{}^2 \ddot{\theta}_2 - m_2 l_2\left(C_2 \ddot{x} - S_2 \dot{x}\dot{\theta}_2\right) - S_2 m_2 l_2 m_2 \dot{x}\dot{\theta}_2 + m_2 g l_2 S_2$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right) - \frac{\partial \mathcal{L}}{\partial \theta_2} = m_2 l_2{}^2 \ddot{\theta}_2 - m_2 l_2 C_2 \ddot{x} + m_2 g l_2 S_2$$

Linearizing the above equation, we get: $m_2 l_2{}^2 \ddot{\theta}_2 - m_2 l_2 \ddot{x} + m_2 g l_2 \theta_2 = 0$

$$F = \ddot{x}(m_1 + m_2 + M) - m_1 l_1\left(\ddot{\theta}_1 C_1 - S_1 \dot{\theta}_1{}^2\right) - m_2 l_2\left(\ddot{\theta}_2 C_2 - S_2 \dot{\theta}_2{}^2\right)$$

$$m_1 l_1{}^2 \ddot{\theta}_1 - m_1 l_1 C_1 \ddot{x} + m_1 g l_1 S_1 = 0$$

$$m_2 l_2{}^2 \ddot{\theta}_2 - m_2 l_2 C_2 \ddot{x} + m_2 g l_2 S_2 = 0$$

$$\ddot{\theta}_1 = \frac{C_1\ddot{x} - gS_1}{l_1}$$
$$\ddot{\theta}_2 = \frac{C_2\ddot{x} - gS_2}{l_2}$$

Next, we substitute the obtained values of $\ddot{\theta}_1, \ddot{\theta}_2$ into the equation of $\ddot{x}$ as follows:

$$\ddot{x} = \frac{F + m_1 l_1\left(\ddot{\theta_1} C_1 - S_1\dot{\theta}_1{}^2\right) + m_2 l_2\left(\ddot{\theta_2} C_2 - S_2\dot{\theta}_2{}^2\right)}{(m_1 + m_2 + M)}$$

$$\ddot{x} = \frac{F + m_1 l_1\left(C_1\left(\frac{C_1\ddot{x} - gS_1}{l_1}\right) - S_1\dot{\theta}_1{}^2\right) + m_2 l_2\left(C_2\left(\frac{C_2\ddot{x} - gS_2}{l_2}\right) - S_2\dot{\theta}_2{}^2\right)}{(m_1 + m_2 + M)}$$

$$\ddot{x} = \frac{F + m_1\left(C_1{}^2\ddot{x} - gS_1C_1 - l_1S_1\dot{\theta}_1{}^2\right) + m_2(C_2{}^2\ddot{x} - gS_2C_2 - l_2S_2\dot{\theta}_2{}^2)}{(m_1 + m_2 + M)}$$

$$(m_1 + m_2 + M)\ddot{x} - m_1C_1{}^2\ddot{x} - m_2C_2{}^2\ddot{x} = F - m_1\left(gS_1C_1 + l_1S_1\dot{\theta}_1{}^2\right) - m_2\left(gS_2C_2 + l_2S_2\dot{\theta}_2{}^2\right)$$

$$\left(M + m_1(1 - C_1{}^2) + m_2(1 - C_2{}^2)\right)\ddot{x} = F - m_1\left(gS_1C_1 + l_1S_1\dot{\theta}_1{}^2\right) - m_2\left(gS_2C_2 + l_2S_2\dot{\theta}_2{}^2\right)$$

Upon separating $\ddot{x}$ from the equation we get:

$$\boldsymbol{\ddot{x} = \frac{F - m_1\left(gS_1C_1 + l_1S_1\dot{\theta}_1{}^2\right) - m_2\left(gS_2C_2 + l_2S_2\dot{\theta}_2{}^2\right)}{\left(M + m_1(S_1{}^2) + m_2(S_2{}^2)\right)}}$$

Next, we substitute the value of $\ddot{x}$ in the previously derived equations of $\ddot{\theta}_1$, $\ddot{\theta}_2$ we get

$$\boldsymbol{\ddot{\theta}_1 = \frac{C_1}{l_1}\left[\frac{F - m_1\left(gS_1C_1 + l_1S_1\dot{\theta}_1{}^2\right) - m_2\left(gS_2C_2 + l_2S_2\dot{\theta}_2{}^2\right)}{\left(M + m_1(S_1{}^2) + m_2(S_2{}^2)\right)}\right] - g\frac{S_1}{l_1}}$$

$$\boldsymbol{\ddot{\theta}_2 = \frac{C_2}{l_2}\left[\frac{F - m_1\left(gS_1C_1 + l_1S_1\dot{\theta}_1{}^2\right) - m_2\left(gS_2C_2 + l_2S_2\dot{\theta}_2{}^2\right)}{\left(M + m_1(S_1{}^2) + m_2(S_2{}^2)\right)}\right] - g\frac{S_2}{l_2}}$$

Non-linear state space representation

Thus, the equations derived above are the non-linear equations of the system.

Thus, nonlinear state space representation of the above given system is:

$$\dot{X} = AX + BU$$

$$\boldsymbol{\dot{X}} = \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dfrac{F - m_1\left(gS_1C_1 + l_1S_1\dot{\theta_1}^2\right) - m_2\left(gS_2C_2 + l_2S_2\dot{\theta_2}^2\right)}{\left(M + m_1(S_1{}^2) + m_2(S_2{}^2)\right)} \\ \dot{\theta}_1 \\ \dfrac{C_1}{l_1}\left[\dfrac{F - m_1\left(gS_1C_1 + l_1S_1\dot{\theta_1}^2\right) - m_2\left(gS_2C_2 + l_2S_2\dot{\theta_2}^2\right)}{\left(M + m_1(S_1{}^2) + m_2(S_2{}^2)\right)}\right] - g\dfrac{S_1}{l_1} \\ \dot{\theta}_2 \\ \dfrac{C_2}{l_2}\left[\dfrac{F - m_1\left(gS_1C_1 + l_1S_1\dot{\theta_1}^2\right) - m_2\left(gS_2C_2 + l_2S_2\dot{\theta_2}^2\right)}{\left(M + m_1(S_1{}^2) + m_2(S_2{}^2)\right)}\right] - g\dfrac{S_2}{l_2} \end{bmatrix}$$

B) LINEARIZING EQUATIONS OF MOTION

Linearization can be done by two methods:

1. Jacobian Linearization
2. Linearization by small angle approximation

Now we take the linearized equations derived earlier:

$$F = \ddot{x}(m_1 + m_2 + M) - m_1 l_1 \ddot{\theta}_1 - m_2 l_2 \ddot{\theta}_2$$

$$\ddot{\theta}_1 = \frac{m_1 l_1 \ddot{x} - m_1 g l_1 \theta_1}{m_1 l_1^2}$$

$$\ddot{\theta}_1 = \frac{\ddot{x} - g\theta_1}{l_1}$$

$$\ddot{\theta}_2 = \frac{\ddot{x} - g\theta_2}{l_2}$$

$$\ddot{\theta}_2 = \frac{m_2 l_2 \ddot{x} - m_2 g l_2 \theta_2}{m_2 l_2^2}$$

Upon substituting the values of $\ddot{\theta}_1$, $\ddot{\theta}_2$ we get

$$F = \ddot{x}(m_1 + m_2 + M) - m_1 l_1 \ddot{\theta}_1 - m_2 l_2 \ddot{\theta}_2$$
$$= \ddot{x}(m_1 + m_2 + M) - m_1 l_1 \left[\frac{\ddot{x} - g\theta_1}{l_1}\right] - m_2 l_2 \left[\frac{\ddot{x} - g\theta_2}{l_2}\right]$$
$$= M\ddot{x} + m_1 g\theta_1 + m_2 g\theta_2$$

Thus,

$$\ddot{x} = \frac{F - m_1 g\theta_1 - m_2 g\theta_2}{M}$$

$$\ddot{\theta}_1 = \frac{\frac{F - m_1 g\theta_1 - m_2 g\theta_2}{M}}{l_1} - \frac{g\theta_1}{l_1}$$

$$\ddot{\theta}_1 = -\left(\frac{m_1 g}{M l_1} + \frac{g}{l_1}\right)\theta_1 - \left(\frac{m_2 g}{M l_1}\right)\theta_2 + \frac{F}{M l_1}$$

$$\ddot{\theta}_2 = \frac{\frac{F - m_1 g\theta_1 - m_2 g\theta_2}{M}}{l_2} - \frac{g\theta_2}{l_2}$$

$$\ddot{\theta}_2 = -\left(\frac{m_2 g}{M l_2} + \frac{g}{l_2}\right)\theta_2 - \left(\frac{m_1 g}{M l_2}\right)\theta_2 + \frac{F}{M l_2}$$

The second method with which we can do linearization is Jacobian linearization.

$$A = \begin{bmatrix} \frac{\delta f_1}{\delta X_1} & \frac{\delta f_1}{\delta X_2} & \frac{\delta f_1}{\delta X_3} & \frac{\delta f_1}{\delta X_4} & \frac{\delta f_1}{\delta X_5} & \frac{\delta f_1}{\delta X_6} \\ \frac{\delta f_2}{\delta X_1} & \frac{\delta f_2}{\delta X_2} & \frac{\delta f_2}{\delta X_3} & \frac{\delta f_2}{\delta X_4} & \frac{\delta f_2}{\delta X_5} & \frac{\delta f_2}{\delta X_6} \\ \frac{\delta f_3}{\delta X_1} & \frac{\delta f_3}{\delta X_2} & \frac{\delta f_3}{\delta X_3} & \frac{\delta f_3}{\delta X_4} & \frac{\delta f_3}{\delta X_5} & \frac{\delta f_3}{\delta X_6} \\ \frac{\delta f_4}{\delta X_1} & \frac{\delta f_4}{\delta X_2} & \frac{\delta f_4}{\delta X_3} & \frac{\delta f_4}{\delta X_4} & \frac{\delta f_4}{\delta X_5} & \frac{\delta f_4}{\delta X_6} \\ \frac{\delta f_5}{\delta X_1} & \frac{\delta f_5}{\delta X_2} & \frac{\delta f_5}{\delta X_3} & \frac{\delta f_5}{\delta X_4} & \frac{\delta f_5}{\delta X_5} & \frac{\delta f_5}{\delta X_6} \\ \frac{\delta f_6}{\delta X_1} & \frac{\delta f_6}{\delta X_2} & \frac{\delta f_6}{\delta X_3} & \frac{\delta f_6}{\delta X_4} & \frac{\delta f_6}{\delta X_5} & \frac{\delta f_6}{\delta X_6} \end{bmatrix}$$

The state space representation of the above equations is as follows:

$$\begin{bmatrix} \dot{\boldsymbol{x}} \\ \ddot{\boldsymbol{x}} \\ \dot{\boldsymbol{\theta}}_1 \\ \ddot{\boldsymbol{\theta}}_1 \\ \dot{\boldsymbol{\theta}}_2 \\ \ddot{\boldsymbol{\theta}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{-gm_1}{M} & 0 & \dfrac{-gm_2}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{-g(M + m_1)}{Ml_1} & 0 & \dfrac{-gm_2}{Ml_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{-gm_1}{Ml_2} & 0 & \dfrac{-g(M + m_2)}{Ml_2} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \dot{\boldsymbol{x}} \\ \boldsymbol{\theta}_1 \\ \dot{\boldsymbol{\theta}}_1 \\ \boldsymbol{\theta}_2 \\ \dot{\boldsymbol{\theta}}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{M} \\ 0 \\ \dfrac{1}{Ml_1} \\ 0 \\ \dfrac{1}{Ml_2} \end{bmatrix} F$$

C) CONTROLLABILITY

Next, we obtain the conditions for which the system is controllable. The A and B matrices obtained above are independent of time and thus the system is an LTI system. An LTI system is controllable if the controllability matrix obtained has full rank condition. The dimensions of the controllability matrix ‘C’ are n x nm, hence its rank should be equal to n. Hence,

$$rank(C) = rank[B\ AB\ A^2B\ A^3B\ A^4B\ A^5B] = n$$

Controllability matrix is obtained using MATLAB.

$$C = \begin{pmatrix} 0 & \frac{1}{M} & 0 & \sigma_3 & 0 & \sigma_6 \\ \frac{1}{M} & 0 & \sigma_3 & 0 & \sigma_6 & 0 \\ 0 & \frac{1}{M\,l_1} & 0 & \sigma_1 & 0 & \sigma_4 \\ \frac{1}{M\,l_1} & 0 & \sigma_1 & 0 & \sigma_4 & 0 \\ 0 & \frac{1}{M\,l_2} & 0 & \sigma_2 & 0 & \sigma_5 \\ \frac{1}{M\,l_2} & 0 & \sigma_2 & 0 & \sigma_5 & 0 \end{pmatrix}$$

Where ,

$$\sigma_1 = -\frac{g\ (M\, l_2 + l_1\, m_2 + l_2\, m_1)}{M^2\, {l_1}^2\, l_2}$$

$$\sigma_2 = -\frac{g\ (M\, l_1 + l_1\, m_2 + l_2\, m_1)}{M^2\, l_1\, {l_2}^2}$$

$$\sigma_3 = -\frac{g\ (l_1\, m_2 + l_2\, m_1)}{M^2\, l_1\, l_2}$$

$$\sigma_4 = \frac{g^2\ \left(M^2\, {l_2}^2 + M\, {l_1}^2\, m_2 + M\, l_1\, l_2\, m_2 + 2\, M\, {l_2}^2\, m_1 + \sigma_8 + 2\, l_1\, l_2\, m_1\, m_2 + \sigma_7\right)}{M^3\, {l_1}^3\, {l_2}^2}$$

$$\sigma_5 = \frac{g^2\ \left(M^2\, {l_1}^2 + 2\, M\, {l_1}^2\, m_2 + M\, l_1\, l_2\, m_1 + M\, {l_2}^2\, m_1 + \sigma_8 + 2\, l_1\, l_2\, m_1\, m_2 + \sigma_7\right)}{M^3\, {l_1}^2\, {l_2}^3}$$

$$\sigma_6 = \frac{g^2\ \left(\sigma_8 + M\, {l_1}^2\, m_2 + 2\, l_1\, l_2\, m_1\, m_2 + \sigma_7 + M\, {l_2}^2\, m_1\right)}{M^3\, {l_1}^2\, {l_2}^2}$$

$$\sigma_7 = {l_2}^2\, {m_1}^2$$

$$\sigma_8 = {l_1}^2\, {m_2}^2$$

For the above given controllability matrix to be of full rank, its determinant should not be equal to be zero i.e. $det(C) \neq 0$ i.e.

$$det(C) = \frac{-g^6(l_1^2 - l_2^2)}{M^6 l_1^6 l_2^6} \neq 0$$

The determinant of C matrix won't be equal to zero only when $l_1^2 - l_2^2 \neq 0$

Or, $l_1 \neq l_2$

Thus, the given system is controllable only when the lengths of the cables of the crane isn't equal.

D) LQR CONTROLLER

We are given that $M = 1000\ Kg$, $m_1 = m_2 = 100\ Kg$ $l_1 = 20m$ $l_2 = 10m$. Thus, we get the values of A matrix as follows:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.9800 & 0 & -0.9800 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -0.5390 & 0 & -0.049 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -0.0980 & 0 & -1.078 & 0 \end{bmatrix}$$

The eigen values of A matrix before applying the LQR controller are:

```
0.0000 + 0.0000i
0.0000 + 0.0000i
0.0000 + 0.7282i
0.0000 - 0.7282i
0.0000 + 1.0425i
0.0000 - 1.0425i
```

We can see that all the eigenvalues are on the imaginary axis.

LQR controller theory:

Using LQR controller we try to bring the desired state to zero. System eigenvalue that make a regulator work well are the same system eigenvalues that make a tracker work well. When you give a reference that is not zero the system acts as a tracker that follows an external reference input. However, we can think of the LQR controller as a regulator problem which has no additional input that tries to bring the system states to zero.

In the system if the A and B matrices are controllable then we can put the eigenvalues of the system anywhere we want. The problem in changing the position of the system eigenvalues is that we don't know which eigenvalue will result in what kind of dynamical response.

Controlling by altering the position of the eigen values:
From second order systems which aren't perfect, we won't get a perfect sense of the time response by just placing the eigen values. If we try to place our eigen values to far left, then in real life the problem is that the system will require huge amounts of input. And providing such huge amounts isn't feasible.

The above process is not at all intuitive. We don't know where to place the eigen values. Even though we know that the system is controllable (i.e. we can place the eigen values wherever we want) we don't know exactly where to put them. In order to deal with this problem, we use an LQR controller. It tries to make the choice of K a little more intuitive to the designer.

$$\dot{X} = AX + BU \text{ where } U = -KX$$

Thus,

$$\dot{X} = (A - BK)X$$

The LQR controller problem is sometimes also called an Infinite Horizon problem because we integrate the cost function from zero to infinity. The reason for choosing Infinity is because we want the system to work perfectly over all the time instants instead of just a specific interval of time.

Our aim is to minimize the cost function given below, by adjusting the weights of the states given in the Q matrix. Here the Q matrix belongs is multiplied with the states of the system, and the R matrix is multiplied with the input of the system

$$J = \int_0^{\infty} (\vec{X}(t)^T Q \vec{X}(t) \ + \ (\vec{U}(t)^T R \vec{U}(t))dt$$

Here the dimensions of the matrices are as follows:

$$\vec{X}(t) = n * 1$$
$$\vec{X}(t)^T = 1 * n$$
$$Q = n * n$$
$$\vec{U}(t) = p * 1$$
$$\vec{U}(t)^T = 1 * p$$
$$R = p * p$$
$$J = 1 * 1$$

The name of the Controller is LQR because:

- L → The controller is applied to an approximated linear system as U = -KX
- Q → In the controller, the quadratic cost function J is minimized. The cost J also has a minima if plotted on a graph
- R → The R stands for the regulatory behavior of the system as the controller tries to bring the states to zero.

We can say that:

$$X \epsilon R^2$$

$$Q \ = \begin{bmatrix} q_1 & q_3 \\ q_3 & q_2 \end{bmatrix} \ X \ = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

$$X^T Q X \ = \ q_1 {X_1}^2 \ + \ q_2 {X_2}^2 \ + \ 2 q_3 X_1 X_2$$

Here Q_1 penalizes the first state X_1, Q_2 penalizes the second state. And if Q is 3*3 then Q_3 would penalize the third state. The diagonal elements penalize the state and the off-diagonal elements penalize the combination of the states. Also, the values of the off-diagonal states need to be lesser that that of the diagonal states.

We always want $\vec{X}(t)^T Q \vec{X}(t)$ to be a positive number. If we get it as negative, it will minimize the cost function J, but we won't get the desired results, rather we will get some weird absurd results. Thus $\boldsymbol{\vec{X}(t)^T Q \vec{X}(t)}$ **must be positive definite.**

Thus $\vec{X}(t)^T Q \vec{X}(t) \ \geq \ 0$ for all $X \ \neq \ 0$

We know that Q and R, are positive definite. So, in Algebra Ricatti Equation (ARE) we give an A, B, P and R and it will give us a P matrix.

Hence P is given by

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

Where,

$$K = R^{-1}B^T P$$
$$also, U = -KX$$

Upon substituting the value of K obtained in (A-BK), we get a stabilizable controller that has proper system eigen values, which also minimizes the cost J calculated earlier.

The eigen values after applying the LQR controller are:

```
-35.4675 + 0.0000i
 -0.4311 + 0.0000i
 -0.1798 + 0.3577i
 -0.1798 - 0.3577i
 -0.1269 + 0.7769i
 -0.1269 - 0.7769i
```

According to the lecture notes, Lyapunov's indirect method states that we first linearize the original system around the equilibrium point of interest and then check the eigen values as shown above. Our eigen values of A matrix have negative real part, and hence the original system is at least locally stable, around equilibrium point. In this case, a Lyapunov function for the linearized system will be valid at least locally.

The initial states of the system were taken to be 10 degrees for mass 1 and 5 degrees for mass 2.

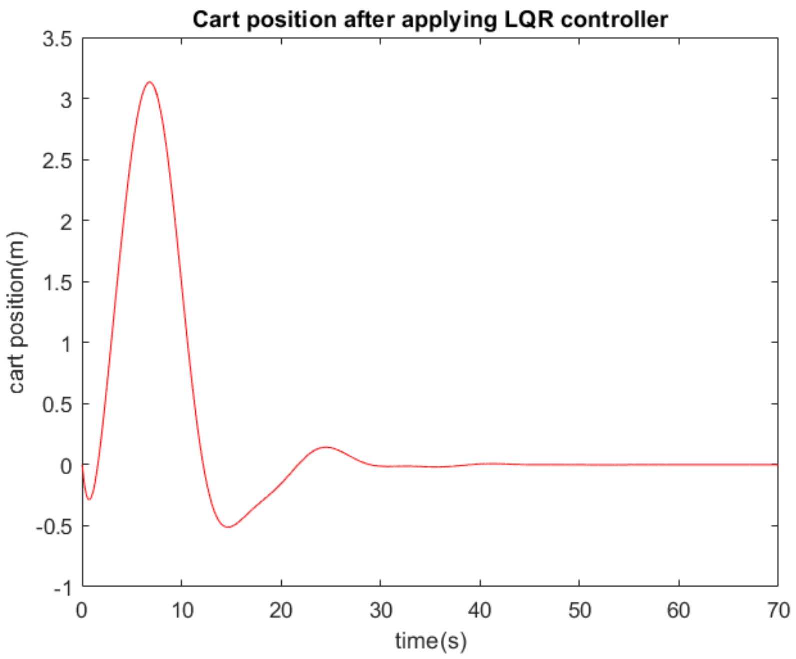


Fig: Position of cart after applying LQR controller

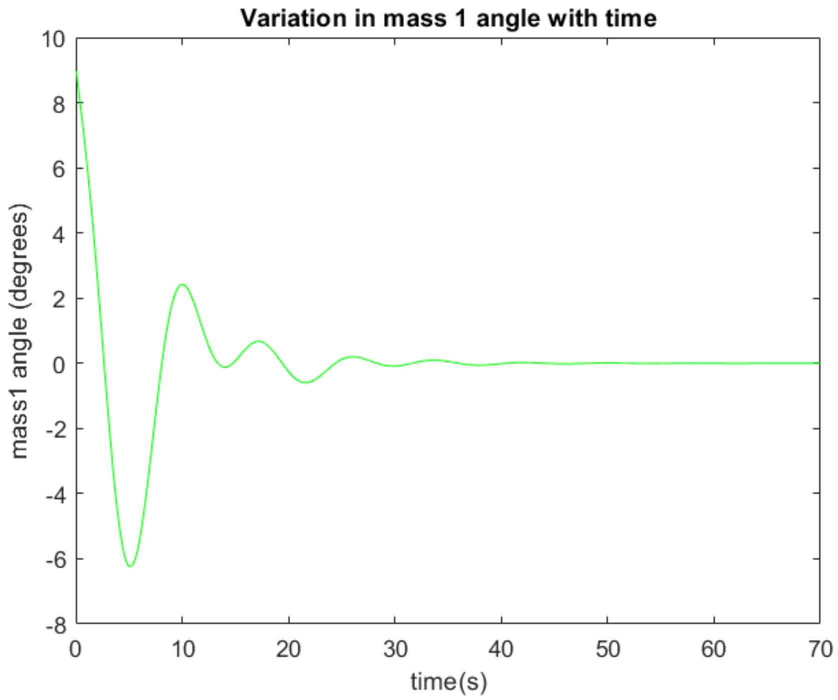


Fig: Angle of mass 2 after applying LQR controller

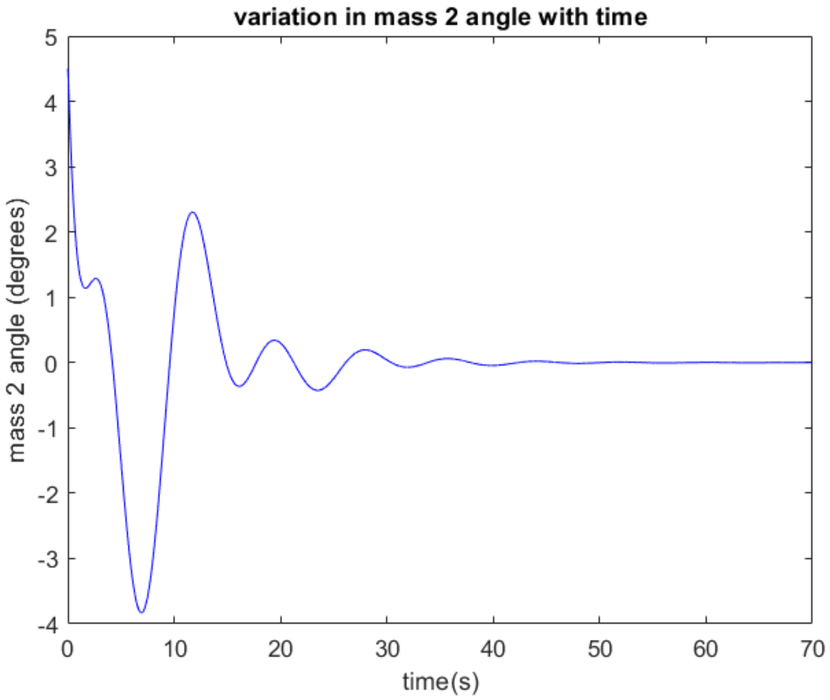


Fig: Angle of mass 2 after applying LQR controller

From the above graphs we can clearly see that the system is stabilized within 40s.

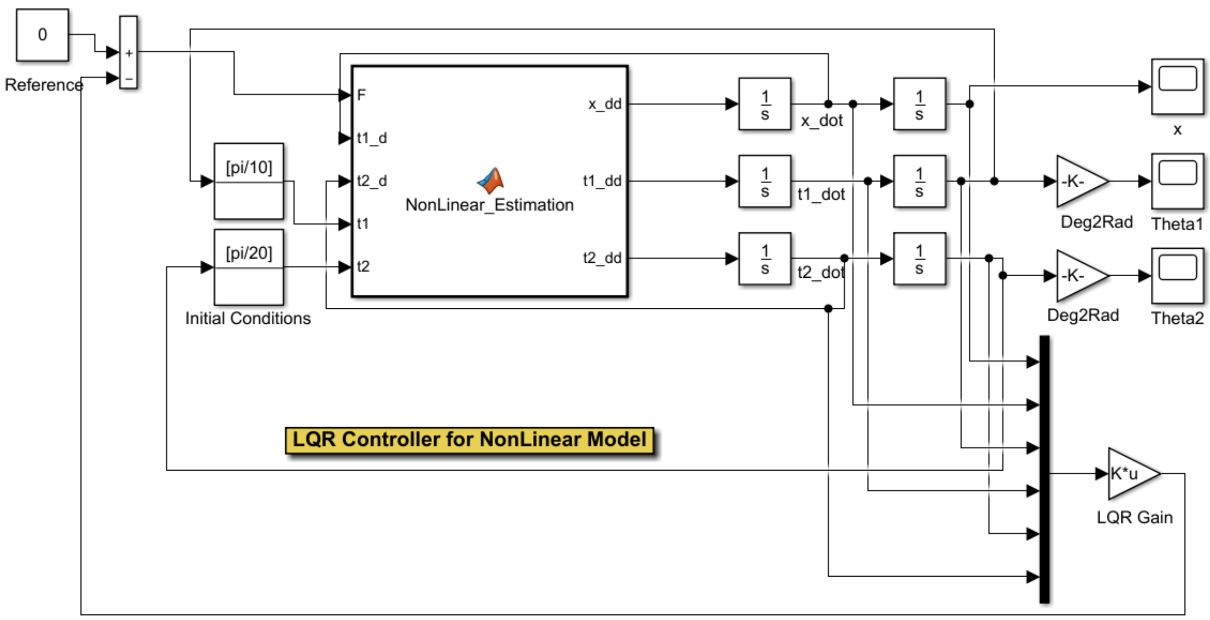


Fig: LQR Controller Simulink Diagram for Non-Linear Plant Model

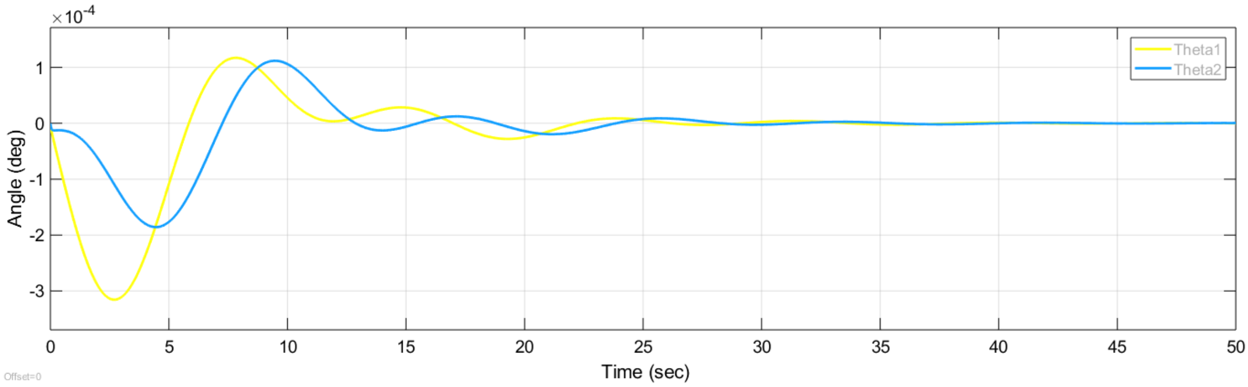


Fig: Angle of pendulums after applying LQR controller on NonLinear

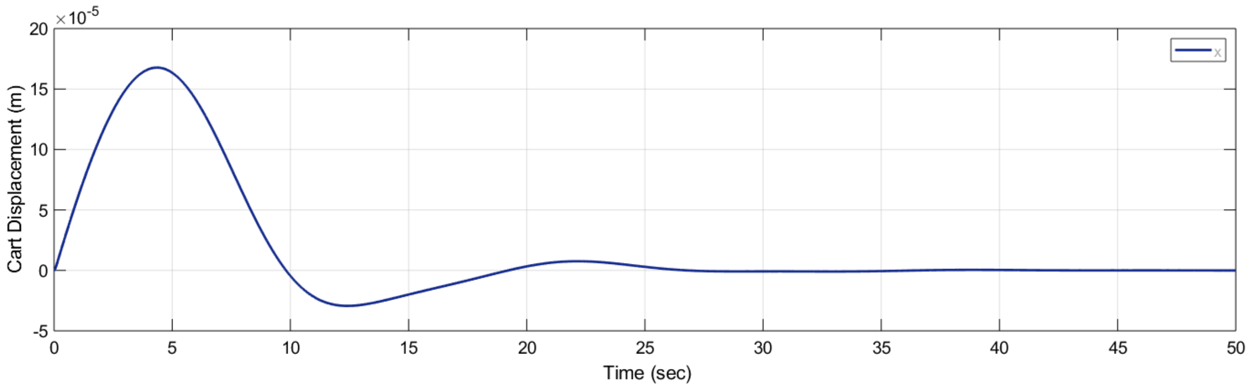


Fig: Position of cart after applying LQR controller on NonLinear

We can see from the above graphs that the nonlinear system is also getting stabilized after applying the LQR controller to it.

SECOND COMPONENT:

E) Observable Vectors

Here we have four output vectors: $x(t)$, $(\theta_1(t); \theta_2(t))$, $(x(t); \theta_2(t))$ and $(x(t); \theta_1(t); \theta_2(t))$.

• For vector: x(t),

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

• For vector: $(\theta_1(t); \theta_2(t))$,

$$C_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix};$$

• For vector: ($x(t)$; $\theta_2(t)$),

$$C_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix};$$

• For output vector: ($x(t)$, $\theta_1(t)$, $\theta_2(t)$),

$$C_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix};$$

Defining variables

```
syms m1 g m2 M L1 L2
```

Observability Check

```
A = [0 1 0 0 0 0; 0 0 -m1*g/M 0 -m2*g/M 0; 0 0 0 1 0 0; 0 0 -((M*g)+(m1*g))/(M*L1) 0 -g*m2/(M
*L1) 0; 0 0 0 0 0 1; 0 0 -m1*g/(M*L2) 0 -((M*g)+(m2*g))/(M*L2) 0];
B = [0; 1/M; 0; 1/(L1*M); 0; 1/(L2*M)];
c1 = [1 0 0 0 0 0; 0 0 0 0 0 0; 0 0 0 0 0 0];
c2 = [0 0 0 0 0 0; 0 0 1 0 0 0; 0 0 0 0 1 0];
c3 = [1 0 0 0 0 0; 0 0 0 0 0 0; 0 0 0 0 1 0];
c4 = [1 0 0 0 0 0; 0 0 1 0 0 0; 0 0 0 0 1 0];
Obs1 = rank([c1' A'*c1' ((A')^2)*c1' ((A')^3)*c1' ((A')^4)*c1' ((A')^5)*c1'])
Obs2 = rank([c2' A'*c2' ((A')^2)*c2' ((A')^3)*c2' ((A')^4)*c2' ((A')^5)*c2'])
Obs3 = rank([c3' A'*c3' ((A')^2)*c3' ((A')^3)*c3' ((A')^4)*c3' ((A')^5)*c3'])
Obs4 = rank([c4' A'*c4' ((A')^2)*c4' ((A')^3)*c4' ((A')^4)*c4' ((A')^5)*c4'])
```

From the code, we get the rank for respective output vector as 6,4,6,6. So the output vector ($\theta_1(t)$; $\theta_2(t)$) is not observable.

F) Observer Design

The Luenberger Observer is written in state-space representation as:

$$\dot{\hat{X}}(t) = A\hat{x}(t) + B_k U_k(t) + L(Y(t) - C\hat{x}(t));$$

Here, $\hat{x}(t)$ is state estimator, L is observer gain matrix, $Y(t) - C\hat{x}(t)$ is correction term and $\hat{x}(0) = 0$. The estimation error $X_e(t) = X(t) - \hat{X}(t)$ has the following state space representation:

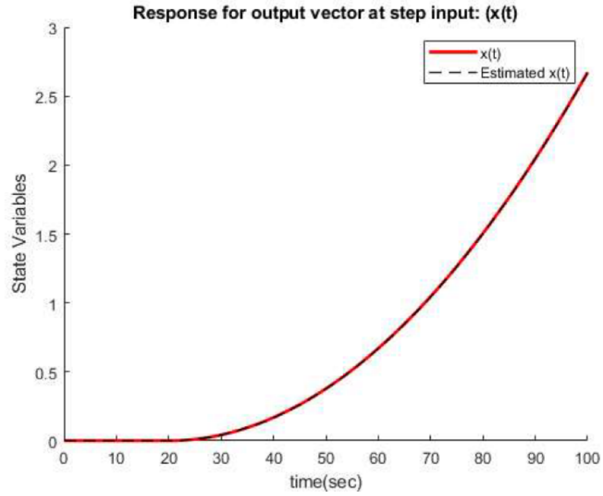
$$\dot{X}_e(t) = \dot{X}(t) - \dot{\hat{X}}(t);$$

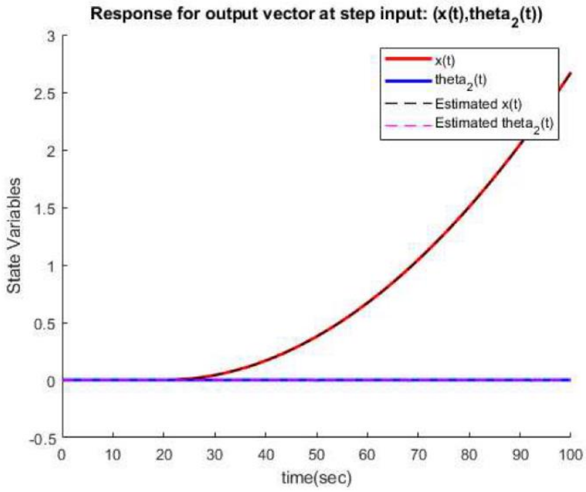
$$\dot{X}_e(t) = AX_e(t) - L\big(Y(t) - C\hat{x}(t)\big) + B_d U_d(t);$$

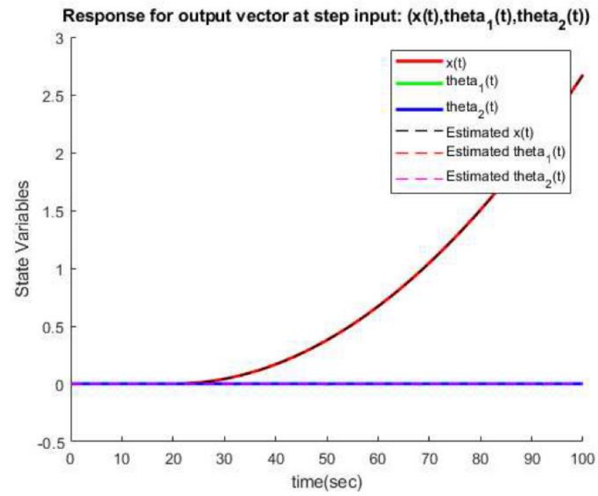
Here, we assume D = 0, $Y(t) = Cx(t)$. Therefore, the equation can be written as

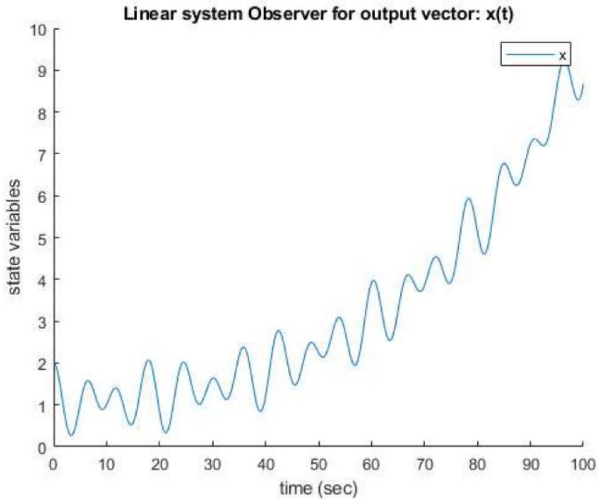
$$\dot{X}_e(t) = (A - LC)X_e(t) + B_d U_d(t);$$

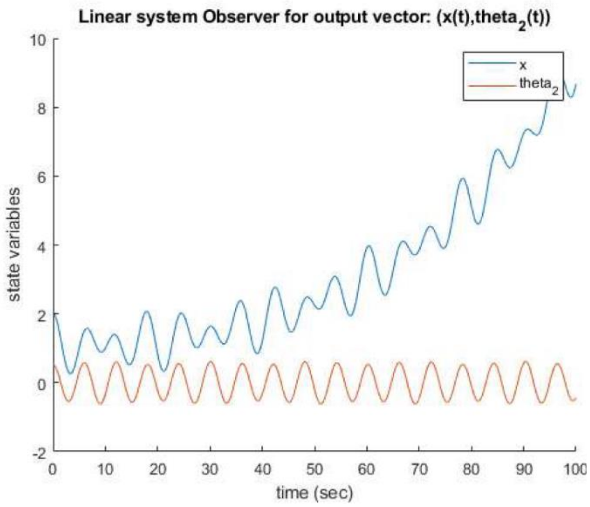
Now, let’s see the MATLAB code and its output respectively.

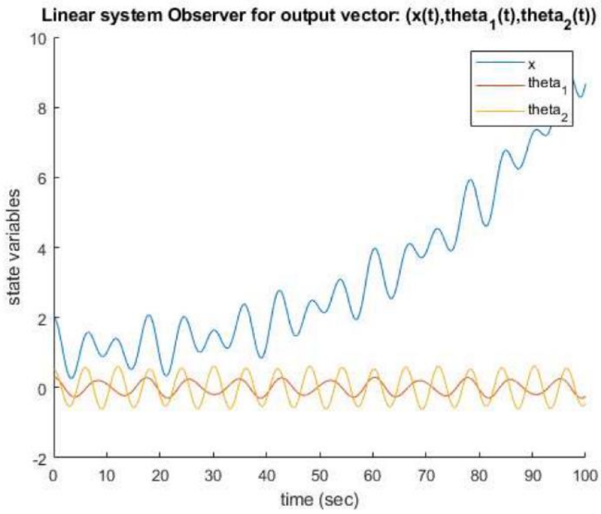












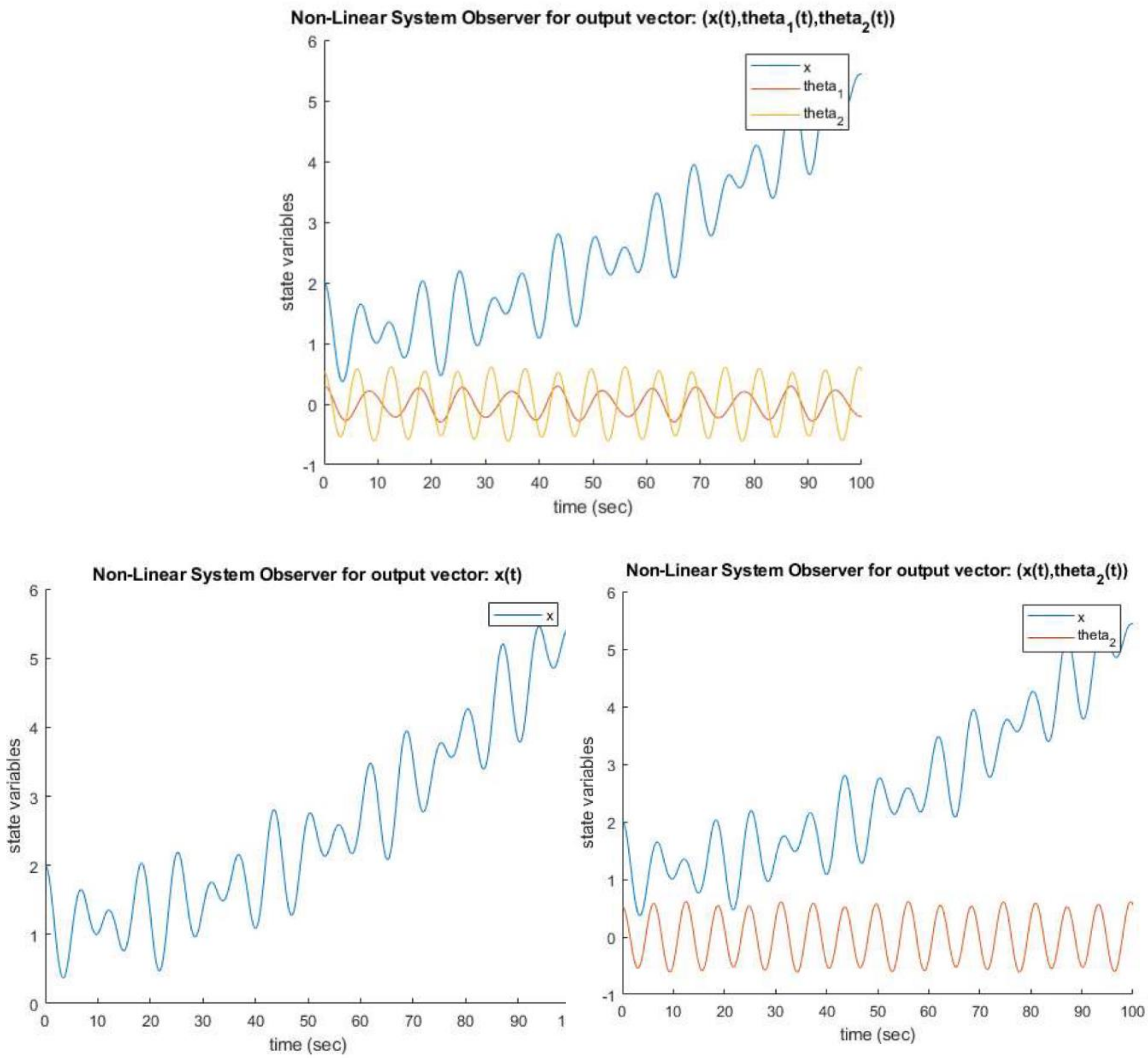


Fig: Output response of system with unit step input, linear system and non-linear system at initial condition x = 2m, 17º and 30º respectively.

The code for the above response can be found in the Appendix-F.

G) LQG Design

The LQG controller is blend of LQR controller and the Kalman filter. On substituting Q = 1, R = 0.1, noise Bd = 0.1 and Vd = 0.01, we get the following response curve of the output. The code for the following response is in appendix-G.

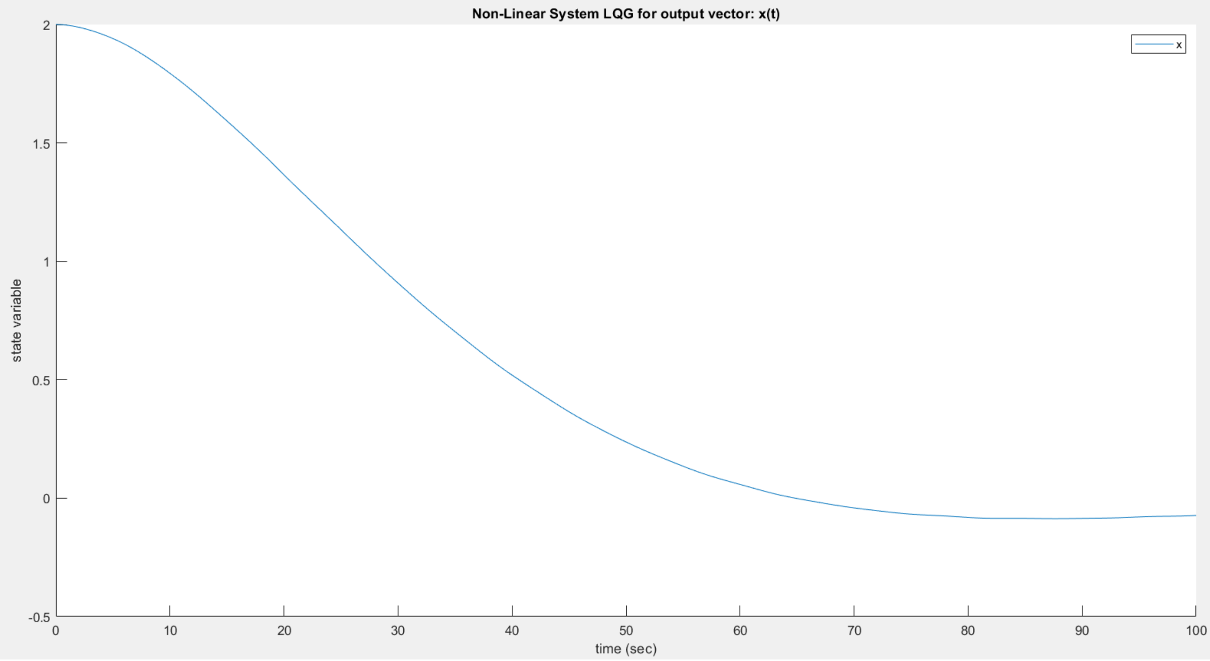


Fig: Non-Linear LQG response for output vector: x(t)

We will reconfigure our controller by providing a desired x vector and tune the LQR controller accordingly to get the feedback according to the desired output. Yes, our design can reject constant force disturbances applied on the cart. If you increase the disturbance noise in the Kalman filter, you will notice that the LQR controller is strong enough to stabilize the x(t) cart position within few seconds.

APPENDIX-F

```
%% Kalman Estimator Design
Bd = 0.1*eye(6);                %Process Noise
Vn = 0.01;                      %Measurement Noise
[Lue1,P,E] = lqe(A,Bd,c1,Bd,Vn*eye(3));
[Lue3,P,E] = lqe(A,Bd,c3,Bd,Vn*eye(3));
[Lue4,P,E] = lqe(A,Bd,c4,Bd,Vn*eye(3));

Ac1 = A-(Lue1*c1);
Ac3 = A-(Lue3*c3);
Ac4 = A-(Lue4*c4);
e_sys1 = ss(Ac1,[B Lue1],c1,0);
e_sys3 = ss(Ac3,[B Lue3],c3,0);
e_sys4 = ss(Ac4,[B Lue4],c4,0);
% step(e_sys1)
% step(e_sys3)
% step(e_sys4)

%% Generating plot for step input
unitStep = 0*tspan;
unitStep(200:length(tspan)) = 1;
```

```
[y1,t] = lsim(sys1,unitStep,tspan);
[x1,t] = lsim(e_sys1,[unitStep;y1'],tspan);

[y3,t] = lsim(sys3,unitStep,tspan);
[x3,t] = lsim(e_sys3,[unitStep;y3'],tspan);

[y4,t] = lsim(sys4,unitStep,tspan);
[x4,t] = lsim(e_sys4,[unitStep;y4'],tspan);

figure();
hold on
plot(t,y1(:,1),'r','Linewidth',2)
plot(t,x1(:,1),'k--','Linewidth',1)
ylabel('State Variables')
xlabel('time(sec)')
legend('x(t)','Estimated x(t)')
title('Response for output vector at step input: (x(t)')
hold off

figure();
hold on
plot(t,y3(:,1),'r','Linewidth',2)
plot(t,y3(:,3),'b','Linewidth',2)
plot(t,x3(:,1),'k--','Linewidth',1)
plot(t,x3(:,3),'m--','Linewidth',1)
ylabel('State Variables')
xlabel('time(sec)')
legend('x(t)','theta_2(t)','Estimated x(t)','Estimated theta_2(t)')
title('Response for output vector at step input: (x(t),theta_2(t))')
hold off

figure();
hold on
plot(t,y4(:,1),'r','Linewidth',2)
plot(t,y4(:,2),'g','Linewidth',2)
plot(t,y4(:,3),'b','Linewidth',2)
plot(t,x4(:,1),'k--','Linewidth',1)
plot(t,x4(:,2),'r--','Linewidth',1)
plot(t,x4(:,3),'m--','Linewidth',1)
ylabel('State Variables')
xlabel('time(sec)')
legend('x(t)','theta_1(t)','theta_2(t)','Estimated x(t)','Estimated
theta_1(t)','Estimated theta_2(t)')
title('Response for output vector at step input:
(x(t),theta_1(t),theta_2(t))')
hold off
%% Linear Model Observer Response
[t,q1] = ode45(@(t,q)linearObs1(t,q,Lue1),tspan,q0);
figure();
hold on
plot(t,q1(:,1))
ylabel('state variables')
xlabel('time (sec)')
title('Linear system Observer for output vector: x(t)')
legend('x')
hold off

[t,q3] = ode45(@(t,q)linearObs3(t,q,Lue3),tspan,q0);
figure();
hold on
plot(t,q3(:,1))
```

```
plot(t,q3(:,5))
ylabel('state variables')
xlabel('time (sec)')
title('Linear system Observer for output vector: (x(t),theta_2(t))')
legend('x','theta_2')
hold off

[t,q4] = ode45(@(t,q)linearObs4(t,q,Lue4),tspan,q0);
figure();
hold on
plot(t,q4(:,1))
plot(t,q4(:,3))
plot(t,q4(:,5))
ylabel('state variables')
xlabel('time (sec)')
title('Linear system Observer for output vector:
(x(t),theta_1(t),theta_2(t))')
legend('x','theta_1','theta_2')
hold off
%% Non-linear Model Observer Response
[t,q1] = ode45(@(t,q)nonLinearObs1(t,q,1,Lue1),tspan,q0);
figure();
hold on
plot(t,q1(:,1))
ylabel('state variables')
xlabel('time (sec)')
title('Non-Linear System Observer for output vector: x(t)')
legend('x')
hold off

[t,q3] = ode45(@(t,q)nonLinearObs3(t,q,1,Lue3),tspan,q0);
figure();
hold on
plot(t,q3(:,1))
plot(t,q3(:,5))
ylabel('state variables')
xlabel('time (sec)')
title('Non-Linear System Observer for output vector: (x(t),theta_2(t))')
legend('x','theta_2')
hold off

[t,q4] = ode45(@(t,q)nonLinearObs4(t,q,1,Lue4),tspan,q0);
figure();
hold on
plot(t,q4(:,1))
plot(t,q4(:,3))
plot(t,q4(:,5))
ylabel('state variables')
xlabel('time (sec)')
title('Non-Linear System Observer for output vector:
(x(t),theta_1(t),theta_2(t))')
legend('x','theta_1','theta_2')
hold off
```

APPENDIX-G

```
clear all

%% Defining variables
syms m1 g m2 M L1 L2 x dx
m1 = 100;
m2 = 100;
M = 1000;
L1 = 20;
L2 = 10;
g = 9.81;
tspan = 0:0.1:100;
% q = [x dx t1 dt1 t2 dt2];
%Enter initial conditions
q0 = [2 0 deg2rad(0) 0 deg2rad(0) 0];

%% Linearized Model
A = [0 1 0 0 0 0; 0 0 -m1*g/M 0 -m2*g/M 0; 0 0 0 1 0 0; 0 0 -
((M*g)+(m1*g))/(M*L1) 0 -g*m2/(M*L1) 0; 0 0 0 0 0 1; 0 0 -m1*g/(M*L2) 0 -
((M*g)+(m2*g))/(M*L2) 0];
B = [0; 1/M; 0; 1/(L1*M); 0; 1/(L2*M)];
c1 = [1 0 0 0 0 0; 0 0 0 0 0 0; 0 0 0 0 0 0];
d = [1;0;0];
sys1 = ss(A,B,c1,d);

%% LQR Controller
Q = [1 0 0 0 0 0; 0 0 0 0 0 0; 0 0 0 0 0 0; 0 0 0 0 0 0; 0 0 0 0 0 0; 0 0 0
0 0 0];
R = 0.1;
[K,S,P] = lqr(A,B,Q,R);
sys = ss(A-B*K,B,c1,d);
% step(sys,200);

%% Kalman Estimator Design
Bd = 0.1*eye(6);                %Process Noise
Vn = 0.01;                      %Measurement Noise
[Lue1,P,E] = lqe(A,Bd,c1,Bd,Vn*eye(3)); %Considering vector output: x(t)
Ac1 = A-(Lue1*c1);
e_sys1 = ss(Ac1,[B Lue1],c1,0);

%% Non-linear Model LQG Response
[t,q1] = ode45(@(t,q)nonLinearObs1(t,q,-K*q,Lue1),tspan,q0);
figure();
hold on
plot(t,q1(:,1))
ylabel('state variable')
xlabel('time (sec)')
title('Non-Linear System LQG for output vector: x(t)')
legend('x')
hold off
```