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Question 10.31

A new section of a highway is opened, and X = 4 accidents occurred there during one month. The number of accidents has Poisson(θ) distribution, where θ is the expected number of accidents during one month. Experience from the other sections of this highway suggests that the prior distribution of θ is Gamma(5,1). Find the Bayes estimator of θ under the squared-error loss and find its posterior risk.

Solution

Given

$\overline{X} = 4$

$\pi(\theta)$ follows Gamma(5,1) and $f(x/\theta)$ follows Poisson(θ)

Gamma is conjugate to Poisson, so posterior is given by

$$\pi(\theta/x) = Gamma(\alpha + n\overline{X}, \lambda + n)$$

Here, $\alpha = 5, n = 1,\ \overline{X} = 4, \lambda = 1$

$$\pi(\theta/x) = Gamma(9, 2)$$

Bayers Estimator of θ: $E(\theta|x) = \frac{\alpha+n\overline{X}}{\lambda+n} = \frac{9}{2} = 4.5$

Posterior Risk: $\rho(\theta) = Var(\theta|x) = \frac{9}{2^2} = 2.25$

Question 10.36

In Example 9.13 on p. 251, we constructed a confidence interval for the population mean μ based on the observed Normally distributed measurements. Suppose that prior to the experiment we thought this mean should be between 5.0 and 6.0 with probability 0.95.

a. *Find a conjugate prior distribution that fully reflects your prior beliefs.*
b. *Derive the posterior distribution and find the Bayes estimator of μ. Compute its posterior risk.*
c. *Compute a 95% HPD credible set for μ. Is it different from the 95% confidence interval? What causes the differences?*

Solution

a. We know that, $\theta\ follows\ Normal(\mu, \tau)$, then $\mu\ \pm Z_{0.025}\tau = [5, 6]$
   Solving the 2 equations($\mu + 1.96\tau = 5\ and\ \mu - 1.96\tau = 6$), we get

   $$2\mu = 11 => \ \mu = 5.5 \quad and$$
   $$3.92\tau = \ 1 => \ \tau = 0.255$$

   Therefore, it is a Normal(5.5, 0.2555)

b. From example,

   $$\overline{X} = 6.5, \qquad \sigma = 2.2, \qquad n = 6$$

   Normal is conjugate to itself, thus

   $$\mu_x = \frac{\left(\frac{n\overline{X}}{\sigma^2}+\frac{\mu}{\tau^2}\right)}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}} = \frac{\left(\frac{6*6.5}{2.2^2}+\frac{5.5}{0.255^2}\right)}{\frac{6}{2.2^2}+\frac{1}{0.255^2}} = 5.579$$

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$$\tau = \sqrt{\frac{1}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}} = \sqrt{\frac{1}{\frac{6}{2.2^2} + \frac{1}{0.255^2}}} = 0.060$$

$$\pi(\theta/x) = Normal(5.575, 0{,}060)$$

Bayers Estimator: $\widehat{\theta_B} = 5.575$

Posterior Risk: $\rho(\widehat{\theta_B}) = 0.060$

c. 95% of credible set

$$\begin{aligned} \mu_x \pm Z_{0.025}\tau_x &= 5.575 \pm (1.96 * 0.245) \\ &= 5.575 \pm 0.480 \\ &= [5.095, 6.055] \end{aligned}$$

Question 10.37

If ten coin tosses result in ten straight heads, can this coin still be fair and unbiased?
By looking at a coin, you believe that it is fair (a 50-50 chance of each side) with probability 0.99. This is your prior probability. With probability 0.01, you allow the coin to be biased, one way or another, so its probability of heads is Uniformly distributed between 0 and 1. Then you toss the coin ten times, and each time it turns up heads. Compute the posterior probability that it is a fair coin.

Solution

Prior Probability,

$$\pi(\theta) = \begin{pmatrix} fair & 0.99 \\ biased & 0.01 \end{pmatrix}$$

f(x / $\theta = fair$) is binomial distribution of n=10 and p=0.5

f(x / $\theta = biased$) is uniform distribution of a=0 and b=1

$$\begin{aligned} \pi(\theta = fair \mid x) &= \frac{\pi(x \mid \theta = fair\,) * \pi(fair)}{\pi(x)} \\ &= \frac{0.5^{10} * 0.99}{0.5^{10} * 0.99 + \int_0^1 \theta^{10} d\theta (0.01)} \\ &= \frac{0.5^{10} * 0.99}{0.5^{10} * 0.99 + \frac{0.01}{11}} = 0.5154 \end{aligned}$$

$$\begin{aligned} \pi(\theta = biased \mid x) &= 1 - 0.5154 \\ &= 0.4846 \end{aligned}$$