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The Duality Principle for Unitary Systems

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Yale-NUS College Capstone Project

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Abstract

B.Sc (Hons)

The Duality Principle for Unitary Frames

by Naman KEDIA

The duality principle allows us to identify the relationships at the level of operators from matrix representations of these operators. It expresses this relationship in Hilbert spaces in terms of frames, which are generalizations of a basis to spanning sets. This capstone project serves as an introductory perspective to frames and the duality principle.

With an elementary understanding of linear algebra as a pre-requisite, the report provides a lucid and accessible exposition of the duality principle, a concept that, sometimes, is not clearly understood from mathematical literature.

Specifically, this project applies the duality principle to Gabor type unitary systems with a wandering vector by presenting a novel argument that could be used to prove a result previously established in Han and Larson, 2000. The approaches outlined in this report are more intuitive since they use the duality principle instead of von Neumann algebras to show this result, and could help unify this example with the general theory relating to the duality principle.

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Chapter 1

Introduction

Gabor frames are an extremely important class of frames as discussed in Han and Larson, 2000, Fan, Heinecke, and Shen, 2016 and are extensively covered in books such as Christensen, 2008 and Feichtinger and Strohmer, 2003. These frames are extremely useful in fields such as image processing, as seen in Ji, Shen, and Zhao, 2017, and are therefore generalized to Gabor type unitary systems, which are widely studied in papers such as Dai and Larson, 1998. Han and Larsen proved that for Gabor type unitary systems with a wandering vector, a frame vector is equivalent to a Riesz basis vector, however, did so using von Neumann algebras. It would be more useful to show the same result using the duality principle which has been studied in Fan, Ji, and Shen, 2016, since finding the adjoint system is not trivial for a tight-frame vector in this case. Therefore, in this capstone, we wish to use the duality principle to prove this result, since it states that for a given system, the adjoint system appears as the rows of a matrix representation of its synthesis operator.

Chapter 2 will introduce the concept of adjoint systems using the analysis and synthesis operators for systems in finite dimensions through the example of orthonormal bases. It will discuss notions such as matrix representations and unitary operators while viewing these from the lens of the orthonormal basis specified by the discrete Fourier transform matrix. Chapter 3 will introduce the concepts of frames, Riesz sequences, and the duality principle in finite dimensions. It will also consider examples of the "Mercedes-Benz" and "discrete Fourier" frames. Chapter 4 revisits properties and results from previous chapters in the context of infinite-dimensional Hilbert spaces. It extends definitions of operators, frames, and Riesz sequences to Hilbert spaces while introducing bi-infinite matrices. Chapter 5 builds on this expository knowledge by considering the specific case of Gabor type unitary systems with a wandering vector formed using projective unitary representations. It also posits original approaches, which could be used to prove that a wandering vector generates a Riesz basis if and only if it generates a frame, using the duality principle.

Chapter 2

Orthonormal Bases

We begin this chapter with a discussion on orthonormal bases and their properties which motivate the definition of frames. These frames will be the subject of this capstone and it is of great importance to properly outline these properties and its relationship to the duality principle. More material relating to this chapter can be found in a foundational linear algebra textbook such as Axler, 1997.

Section 2.1 outlines elementary yet important results regarding orthonormal bases. Section 2.2.1 and Section 2.2.2 provide the definitions for the analysis and synthesis operators of a system and provides a matrix representation for each. Section 2.2.3 introduces the adjoint system and discusses key properties of adjoint operators and matrix representations. Section 2.3 defines a Gramian and dual Gramian for a system while illustrating some results in the case of orthonormal bases. Section 2.4.1 discusses some results regarding unitary operators and Section 2.4.2 introduces the concept of unitary equivalence, which is important for this capstone. Section 2.5 uses the example of the discrete Fourier transform to highlight the concepts introduced in this chapter.

2.1 Properties of Orthonormal Bases

Let V be a vector space equipped with a complex inner product, and $(v_i)_{i=1}^n \subset V$ be an orthonormal basis for the space. This complex inner product is a **sesquilinear** form, which means that it is linear in one argument and antilinear in the other.

Throughout this capstone, we follow the convention that the inner product is linear in the first argument and antilinear in the second argument. It is also important to note, that any inner product and norm will have a subscript indicating the vector space for which it holds, however, the subscript will be omitted in case the vector space referred to is apparent.

We see that for all indices $i, j \in \mathbb{N}$ where $i, j \leq n$

$$\langle v_i, v_j \rangle = \delta_{i,j} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \tag{2.1}$$

by definition of orthonormality where $\delta_{\cdot,\cdot}$ is called the Kronecker Delta function.

Consider an arbitrary $x \in V$, since $(v_i)_{i=1}^n$ is a basis, one has that for some unique coefficients $(a_i)_{i=1}^n$, $x = \sum_{i=1}^n a_i v_i$. The orthonormality of this specific basis implies that $\langle x, v_i \rangle = a_i$ and that

$$x = \sum_{i=1}^{n} \langle x, v_i \rangle v_i. \tag{2.2}$$

We would like to refer to the property outlined in Equation (2.2) as "**perfect reconstruction**". This property is significant because, for a finite-dimensional basis without pairwise orthonormal elements, one would have to perform Gaussian elimination in order to determine the values of the coefficient sequence $(a_i)_{i=1}^n$ which is a costly process. The advantage provided by perfect reconstruction is that these coefficients can be efficiently computed as the inner products of the arbitrary vector with the basis elements.

Furthermore, this is equivalent to saying that

$$\|x\|^2 = \sum_{i=1}^{n} |\langle x, v_i \rangle|^2 \tag{2.3}$$

using the property that $\langle x, x \rangle = \|x\|^2$, and an analogous derivation can be seen in Section 3.2.1. This is equation is called **Parseval's Equality**.

Another elementary result is that any orthonormal collection of vectors is linearly independent, and the interested reader may refer to Axler, 1997, Ch 6 for its proof.

2.2 Analysis and Synthesis Operators

We refer to a collection of vectors as a **system**. We can express the perfect reconstruction property seen in (2.2) using an operator, with respect to a system. To do so, we will first define the analysis and synthesis operators for a general system and discuss results specific to when the system is a given orthonormal basis $(v_i)_{i=1}^n$.

In order to aid our exposition we will consider the vector space W, consisting of sequences of length $\#X$, where $\#X$ refers to the cardinality of a system $X \subset V$. We would like to note that W is isomorphic to $\mathbb{R}^{\#X}$ or

$\mathbb{C}^{\#X}$ depending on whether the scalars of the vector space are from $\mathbb{R}$ or $\mathbb{C}$ respectively and is an alternative way of viewing the vector space W.

2.2.1 Analysis Operator

Definition 2.2.1. *Let V be a vector space and $X \subset V$ be a system. The **analysis operator**, $T^* : V \to W$, is defined with respect to X by*

$$T^* v = (\langle v, x \rangle)_{x \in X}, \qquad \forall v \in V. \tag{2.4}$$

This operator accepts inputs of any vector $v \in V$ and returns a sequence of inner products of v with all the vectors in the system X. The matrix representation of the analysis operator with respect to the standard basis is given with the rows being the complex conjugate of the vectors of the system X. This is due to the sesquilinearity of the complex inner product with the vectors from the system in the second argument.

We now consider the case when $X = (v_i)_{i=1}^n$. The matrix representation of T^* corresponding to $(v_i)_{i=1}^n$ with respect to the standard basis, $(e_i)_{i=1}^n$, is given by the matrix in the equation

$$\begin{bmatrix} \text{———} & \overline{v_1} & \text{———} \\ \text{———} & \overline{v_2} & \text{———} \\ & \vdots & \\ \text{———} & \overline{v_n} & \text{———} \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} \langle x, v_1 \rangle \\ \langle x, v_2 \rangle \\ \vdots \\ \langle x, v_n \rangle \end{bmatrix}. \tag{2.5}$$

2.2.2 Synthesis Operator

Definition 2.2.2. *Let V be a vector space and $X \subset V$ be a system. The **synthesis operator**, $T : W \to V$ is defined with respect to X by*

$$T\left(\alpha(x)\right)_{x\in X} = \sum_{x\in X} \alpha(x)x, \qquad \forall \left(\alpha(x)\right)_{x\in X} \in W. \tag{2.6}$$

This operator linearly combines the elements of the system using an input sequence of length $\#X$ and returns a vector in V. The matrix representation of the synthesis operator with respect to the standard basis is given with the columns being the vectors of the system X. To understand why this is the case, we recall that matrix-vector multiplication is the linear combination of the columns of a matrix with coefficients from a vector.

Considering the case when $X = (v_i)_{i=1}^n$, the columns of our matrix representation must be the orthonormal basis vectors. Thus the matrix representation of T corresponding to $(v_i)_{i=1}^n$ with respect to the standard basis $(e_i)_{i=1}^n$ is given by the matrix in the equation

$$\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} | \\ \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n \\ | \end{bmatrix}. \tag{2.7}$$

2.2.3 Adjoint Systems

Definition 2.2.3. *The **adjoint system** for a system X, given by X^*, is the system for which the analysis operator for X is the synthesis operator for the X^*.*

We denote an adjoint system using the superscript (*), indicating that the system X^* is the adjoint system of X. Furthermore, we modify our notation of analysis and synthesis operators to include a subscript to indicate the system for which the operator is defined. Thus for system X, the analysis operator is T_X^* and we can see that, at an operator level, the synthesis operator of the adjoint system is $T_{X^*} = T_X^*$.

We now show an important result relating to the analysis operator and synthesis operators of a system. Let $\alpha = (\alpha(x))_{x \in X} \in W$ and $v \in V$ be arbitrary vectors in their respective vector spaces, then we have that

$$\begin{aligned} \langle T\alpha, v\rangle_V = \langle \sum_{x\in X} \alpha(x)x, v\rangle_V &= \sum_{x\in X} \alpha(x)\overline{\langle v, x\rangle} \\ &= \langle(\alpha, (\langle v, x\rangle)_{x\in X}\rangle_W = \langle \alpha, T^* v\rangle_W. \end{aligned} \tag{2.8}$$

We say that the analysis operator is the **adjoint operator** for the synthesis operator and our previously defined notation for the analysis and synthesis operators account for this relationship. Therefore, adjoint systems can also be characterized by considering the synthesis operator for X and analysis operator for X^* in Definition 2.2.3. We restate (2.8) in a more general manner below.

Statement 2.2.4. *For an operator A, the adjoint operator A^*, is the unique operator such that for all $v \in V$ and for all $w \in W$,*

$$\langle Av, w\rangle_W = \langle v, A^* w\rangle_V. \tag{2.9}$$

We use Statement 2.2.4 to show that the matrix representation of a synthesis operator, $\mathcal{A}$, is the *conjugate transpose* of the matrix representation of the analysis operator, $\mathcal{A}^*$, for all $v \in V$ and for all $w \in W$ by

$$\langle \mathcal{A}v, w\rangle_W = (\mathcal{A}v)^T\overline{w} = v^T\mathcal{A}^T\overline{w} = v^T\overline{\mathcal{A}^*w} = \langle v, \mathcal{A}^*w\rangle_V. \tag{2.10}$$

It is important to note that a matrix representation of an operator depends on the choice of basis used to represent it and for a different choice of basis, there is a different matrix representation of the operator. (2.10) is independent of the choice of basis used for both of the matrix representations, however, holds only if both the matrix representations are in terms of the same bases.

We see that the matrix representations of the analysis and synthesis operators seen in (2.5) and (2.7) are conjugate transposes of each other, which implies that their operators are adjoint operators for each other. This result ensures that the relationship between the matrix representations for these operators remains consistent with their relationship at the level of operators.

Furthermore, these matrix representations will play a crucial part in understanding the duality principle. This duality principle follows directly from the above definition of the adjoint system and shall be explained further in the next chapter.

2.3 Gramian and Dual Gramian

The **frame operator** S is defined such that $S = TT^*$.

For any orthonormal basis $(v_i)_{i=1}^n$, it encodes the property of perfect reconstruction namely that for all $x \in V$,

$$Sx = TT^*x = \sum_{i=1}^{n} \langle x, v_i \rangle v_i = Ix = x \tag{2.11}$$

where I denotes the identity operator, implying that $S = I$. As a note, throughout the capstone, we will denote I as the identity operator and $\mathcal{I}$ as the identity matrix.

We now use terminology introduced in Fan, Ji, and Shen, 2016 and refer to any matrix representation of the synthesis operator as a **pre-Gramian**, denoted by $\mathcal{J}$. Any matrix representation of the frame operator TT^* is called a **dual Gramian**, given by $\mathcal{J}\mathcal{J}^*$.

From (2.11) we know that, in the case of an orthonormal basis, $\mathcal{J}\mathcal{J}^* = \mathcal{I}$ and we use (2.5) and (2.7) to show that

$$\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \text{———} & \overline{v_1} & \text{———} \\ \text{———} & \overline{v_2} & \text{———} \\ & \vdots & \\ \text{———} & \overline{v_n} & \text{———} \end{bmatrix} = \begin{bmatrix} | & | & & | \\ e_1 & e_2 & \cdots & e_n \\ | & | & & | \end{bmatrix}.$$

Similarly any matrix representation of the operator T^*T is referred to as the **Gramian** and is given by $\mathcal{J}^*\mathcal{J}$. Furthermore, we can see that $\mathcal{J}^*\mathcal{J} = \mathcal{I}$ since

$$\begin{bmatrix} \text{———} & \overline{v_1} & \text{———} \\ \text{———} & \overline{v_2} & \text{———} \\ & \vdots & \\ \text{———} & \overline{v_n} & \text{———} \end{bmatrix} \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} \delta_{1,1} & \delta_{1,2} & \cdots & \delta_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{n,1} & \delta_{n,2} & \cdots & \delta_{n,n} \end{bmatrix}.$$

Although, both $\mathcal{J}^*\mathcal{J} = \mathcal{I}$ and $\mathcal{J}\mathcal{J}^* = \mathcal{I}$, it is important to note that the reasons for them being the identity matrix are very different. The Gramian is the identity matrix due to the orthonormality of the columns of the pre-Gramian and the dual Gramian is the identity matrix due to perfect reconstruction. This property that both the Gramian and the dual Gramian of a system are the identity matrix is very special and motivates the discussion seen in the following section.

2.4 Unitary Operators and Equivalence

2.4.1 Unitary Operators

An invertible operator $U : V \to V$ is known as **unitary** if and only if $U^* = U^{-1}$. An equivalent characterization of this is seen by $UU^* = U^*U = I$.

We note that a unitary operator is an isometry, this is because for all $x \in V$, $\|Ux\|^2 = \langle Ux, Ux \rangle = \langle U^*Ux, x \rangle = \|x\|^2$.

We see that any pre-Gramian of an orthonormal basis is unitary since $\mathcal{J}\mathcal{J}^* = \mathcal{J}^*\mathcal{J} = \mathcal{I}$. This result motivates the following elementary Lemma, which is proved here using the Gramian and dual Gramian and helps develop an intuition of how the rows and columns of a matrix are interrelated.

Lemma 2.4.1. *If $\mathcal{A}$ is an invertible matrix in finite dimensions, the rows of $\mathcal{A}$ are an orthonormal basis if and only if its columns are an orthonormal basis.*

Proof. If the columns are pairwise orthonormal, we have that $\mathcal{A}^*\mathcal{A} = \mathcal{I}$ because $\mathcal{A}^*\mathcal{A}$ has the inner products of the columns of $\mathcal{A}$ as its entries. Since $\mathcal{A}$ is invertible, $\mathcal{A}^* = \mathcal{A}^{-1}$ due to uniqueness of inverse. This implies that $\mathcal{A}\mathcal{A}^* = \mathcal{I}$, and therefore we have that the rows are orthonormal by letting $\mathcal{A}^* = \mathcal{B}$. This is because $\mathcal{B}^*\mathcal{B} = \mathcal{I}$ and the columns of $\mathcal{B}$ are the conjugate transpose of the rows of $\mathcal{A}$. Since the number of rows is equal to the dimension of the co-domain and their orthonormality implies that they are linearly independent, the rows form an orthonormal basis for the co-domain. The converse can be shown by first letting $\mathcal{A}^* = \mathcal{B}$ and applying the same argument to $\mathcal{B}$. □

Lemma 2.4.1 provides us with a flavor of the duality principle, where we see a relationship between the rows and the columns of the matrix by considering adjoint systems.

2.4.2 Unitary Equivalence

The following definition allows us to identify properties of operators from matrices without concerning ourselves with the specific choice of basis and is from Horn and Johnson, 2012, Ch 2.2.

Definition 2.4.2. *Operators A and B are **unitarily equivalent**, if there exists a bijective unitary operator U where U such that $A = UBU^*$.*

Since orthonormal bases are unitary, we can view unitary equivalence as different matrix representations of the same operator with a change of basis. Thus one can use matrix representations to determine properties relating to operators independent of the choice of basis, by considering them up to unitary equivalence.

2.5 Discrete Fourier Transform Matrix

The Discrete Fourier Transform (DFT) matrix is given by $(e^{-\frac{2\pi i\, j\cdot k}{N}})_{j,k\in 0,\ldots,N-1}$ and is a linear mapping from $\mathbb{C}^N \to \mathbb{C}^N$. It is the matrix representation in

$\mathbb{C}^N$ of one of the most important linear operators, the Fourier transform operator. The Fourier transform operator is one of the most widely studied operators and has a large range of applications ranging from data compression to signal processing. Letting $e^{-\frac{2\pi i}{N}} = \omega$, with ω being the principal N-th root of unity, the DFT matrix is given by

$$\frac{1}{\sqrt{N}} \begin{bmatrix} \omega^{0\cdot 0} & \omega^{0\cdot 1} & \cdots & \omega^{0\cdot(N-1)} \\ \omega^{1\cdot 0} & \omega^{1\cdot 1} & \cdots & \omega^{1\cdot(N-1)} \\ \omega^{2\cdot 0} & \omega^{2\cdot 1} & \cdots & \omega^{2\cdot(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \omega^{(N-1)\cdot 0} & \omega^{(N-1)\cdot 1} & \cdots & \omega^{(N-1)\cdot(N-1)} \end{bmatrix}. \tag{2.12}$$

Let v_j be the j-th column of the DFT matrix, then

$$\langle v_j, v_j \rangle = \frac{1}{N} \sum_{i=1}^{N} \omega^{ji} \overline{\omega^{ji}} = \frac{1}{N} \sum_{i=1}^{N} \omega^0 = 1. \tag{2.13}$$

We also get to see that for $k \neq j$,

$$\langle v_j, v_k \rangle = \frac{1}{N} \sum_{i=1}^{N} \omega^{ji} \overline{\omega^{ki}} = \sum_{i=1}^{N} \omega^{(j-k)i} = 0. \tag{2.14}$$

The last equality holds since $j \neq k$ and the sum of the N-th roots of unity evaluates to zero.

Thus, the columns of the matrix are pairwise orthonormal and this is consistent with the fact that the Fourier Basis in infinite dimensions is an orthonormal basis. We consider this orthonormal basis, $(v_i)_{i=1}^N$ to be the system X.

Furthermore, using (2.13) and (2.14), we can also compute that the rows of the matrix are also pairwise orthonormal. We consider the complex conjugate of these rows vectors to be the orthonormal system, X^*, and see that they form a basis using Lemma 2.4.1.

This helps us conclude that the DFT matrix is unitary and explains why the Gramian and dual Gramian for X are identity matrices ($\mathcal{J}_X \mathcal{J}_X^* = \mathcal{J}_X^* \mathcal{J}_X = \mathcal{I}$).

This relationship between the rows and columns of a matrix allows us to identify an adjoint system using the rows of the matrix. When we see that the DFT matrix is a complex-symmetric matrix, we are already noting that the basis X^* is very closely related to the basis X. In fact, the basis, X^*, is the complex conjugate of the basis X and this tells us that $T_{X^*} = CT_X$ where C is the antilinear operator for complex conjugation. The relationship between the rows and columns provides us with insights at the operator level with results from the matrix level and is part of the reason why the duality principle is so important.

Chapter 3

Finite Dimensional Frames

Section 3.1 provides a motivating example of the "Mercedes-Benz" frame in order to build intuition regarding frames. Section 3.2.1 and Section 3.2.2 introduce the concept of frames and Riesz sequences respectively while specifying terminology and outlining some key results regarding them. Section 3.3 formally introduces the duality principle in finite dimensions and provides alternative characterizations of the same. Section 3.4 modifies the example of the DFT matrix in Section 2.5 to a case involving tight frames and examines the tight frame through the lens of the duality principle.

To explore the information covered in this chapter in greater detail, one should refer to Christensen et al., 2003 and Christensen, 2008.

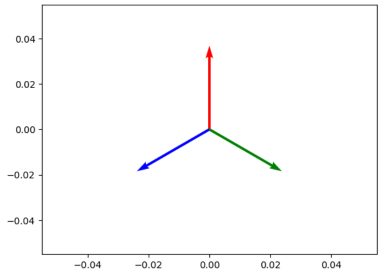

FIGURE 3.1: The "Mercedes-Benz" frame

3.1 The Mercedes-Benz Frame

We begin our discussion of frames with a motivating example of the frame colloquially referred to as the "Mercedes-Benz" frame seen in Figure 3.1. This frame is a list of three vectors, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -\frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{bmatrix}$ and $\begin{bmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{bmatrix}$, denoted by v_1, v_2 and v_3 respectively, with their plot bearing likeness to the logo of the popular car manufacturer, Mercedez-Benz.

It can be shown that these 3 vectors span the space $\mathbb{R}^2$, however, they are not linearly independent since $v_1 = -(v_2 + v_3)$. Since they are not linearly independent they cannot be an orthonormal set. Interestingly, it can be verified that for all $x \in \mathbb{R}^2$

$$x = \frac{2}{3} \sum_{i=1}^{3} \langle x, v_i \rangle v_i. \tag{3.1}$$

This implies that the frame consisting of each of these vectors, scaled by a factor of $\sqrt{\frac{2}{3}}$, has the property of perfect reconstruction despite being neither an orthonormal set nor a basis.

3.2 Frames and Riesz Sequences

A frame is a generalization of a basis in a vector space, endowed with an inner product, to a set of vectors that do not necessarily have to be linearly independent. By considering frames, we now consider spanning lists instead of bases (least spanning lists) and are able to generalize some useful properties of these least spanning lists to spanning lists. An example of such a property is perfect reconstruction, which is discussed below.

3.2.1 Frames

We begin by considering the case when we observe perfect reconstruction in the case of frames. An example of this is seen in Section 3.1 and we generalize it as follows.

Let V be a vector space and let $X \subset V$ be a system, then for all $v \in V$,

$$v = \sum_{x \in X} \langle v, x \rangle x. \tag{3.2}$$

This is called a **1-tight frame**, and we note that the frame operator for such a frame is the identity operator, $T_X T_X^* = I$. This is done by constructing the analysis and synthesis operators as outlined in Section 2.2, with the sole distinction that the system of vectors now forms a 1-tight frame.

The sequence of coefficients in the sum shown in (3.2) is not unique, since the list is not linearly independent, implying that $\ker(T) \neq \emptyset$. This fact generalizes to all frames which are not a basis for the space.

Parseval's equality holds for 1-tight frames since for all $v \in V$,

$$\|v\|^2 = \langle v, v\rangle = \langle v, \sum_{x\in X} \langle v, x\rangle x\rangle = \sum_{x\in X} \overline{\langle v, x\rangle}\langle v, x\rangle = \sum_{x\in X} |\langle v, x\rangle|^2 \quad (3.3)$$

where the second equality holds due to the perfect reconstruction of a 1-tight frame and the third equality holds due to the sesquilinearity of the complex inner product.

To show that a system that satisfies Parseval's equality implies that it has the property of perfect reconstruction, and is thus a 1-tight frame, we prove the contrapositive of this statement. If a system does not have the perfect reconstruction property, then for all $(a(x))_{x\in X}$ where it holds that $v = \sum_{x\in X} a(x)x$, it follows that $(a(x))_{x\in X} \neq (\langle v, x\rangle)_{x\in X}$. Therefore,

$$\|v\|^2 = \langle v, v\rangle = \langle v, \sum_{x\in X} a(x)x\rangle \neq \langle v, \sum_{x\in X} \langle v, x\rangle x\rangle = \sum_{x\in X} |\langle v, x\rangle|^2, \quad (3.4)$$

which means that the system does not satisfy Parseval's equality. Thus, a 1-tight frame can be characterized by any system that satisfies Parseval's equality.

Definition 3.2.1. *Let V be a vector space and $X \subset V$ be a system. X is a **frame** if there exist $A, B > 0$, such that for all $v \in V$,*

$$A\|v\|^2 \leq \|T_X^* v\|^2 \leq B\|v\|^2 \quad (3.5)$$

*where T_X^* is the analysis operator of the system and A, B are known as **frame bounds**.*

We say that a frame is **overcomplete** if it is not a basis. If $A = B$, then we have a ***A*-tight frame**. In this capstone, all subsequent mentions of tight frame refer to a 1-tight frame since we can easily construct a 1-tight frame from an A-tight frame by scaling each vector in the frame by $\sqrt{\frac{1}{A}}$. We also note that $\|T^*v\|^2 = \sum_{x\in X} |\langle v, x\rangle|^2$, which means that (3.3) is a specific case of (3.5) and the previous mention of a 1-tight frame is coherent with the definition of a frame.

Note that frame bounds given by $\sup\{\sqrt{A}|A$ is a lower frame bound$\}$ and $\inf\{\sqrt{B}|B$ is an upper frame bound$\}$, are the optimum bounds of the analysis operator, T_X^*. These operator bounds correspond to the smallest and largest singular values of T_X^* respectively.

We would like to note that a system X is a tight frame if and only if $T_X T_X^* = I$. This arises from the perfect reconstruction property. Furthermore, it is of note to notice that the analysis operator for a tight frame is a linear isometry, since $\|T_X^*v\| = \|v\|$. Thus its only eigenvalue is 1.

Similarly, a system X is a frame and if and only if $T_X T_X^*$ is invertible. This is because the smallest singular value of $T_X T_X^*$ has a modulus greater than zero implying the matrix is non-singular.

(3.5) is motivated by the idea that even if a system does not satisfy Parseval's equality, the norm of the inner product sequence from the analysis operator is within some frame bounds. This has many benefits such as using the canonical dual frame to achieve perfect reconstruction for any such frame, which can be understood from Christensen et al., 2003, Ch 1.

3.2.2 Riesz Sequences

In a manner analogous to frames, we define Riesz sequences, which are another key feature of frame theory. We let W be the vector space of sequences of length $\#X$ as defined in Chapter 2.

Definition 3.2.2. *Let V be a vector space and $X \subset V$ be a system. X is a **Riesz sequence**, if there exists $C, D > 0$, such that for all $\alpha \in W$,*

$$C\|\alpha\|^2 \leq \|T_X\alpha\|^2 \leq D\|\alpha\|^2 \tag{3.6}$$

*where T_X is the synthesis operator of the system and C, D are known as **Riesz bounds**.*

If the Riesz sequence spans the space V, then it is known as a **Riesz basis**. If $C = D = 1$ it is an **orthonormal Riesz sequence** and additionally if $\text{span}(X) = V$, then it is an **orthonormal basis**. This understanding of an orthonormal basis is consistent with the discussion in Chapter 2.

Similar to Section 3.2.1, we note that $\sup\{\sqrt{C}|C \text{ is a lower frame bound}\}$ and $\inf\{\sqrt{D}|D \text{ is an upper frame bound}\}$ are the optimal bounds for T_X which correspond to the smallest and largest singular values of T_X respectively.

We would like to note that we have an orthonormal Riesz sequence if and only if $T_X^* T_X = I$. This is done by considering a Gramian for the operator. We also have that the System is a Riesz sequence if and only if $T_X^* T_X$ is invertible since the smallest singular value of T^*T has a modulus greater than zero implying that the matrix is non-singular.

3.3 The Duality Principle

Statement 3.3.1. (Duality Principle in Finite Dimensions) *Let V be a finite-dimensional vector space with system X, then:*

(i) X is a frame if and only if X^* is a Riesz sequence,

(ii) *X is a 1-tight frame if and only if X^* is an orthonormal Riesz sequence.*

This duality principle arises from the definition of the adjoint system as seen in Section 2.2.3. A system X satisfies (3.5) if and only if the system X^* satisfies (3.6) with $C = A$ and $B = D$, using the fact $T_X^* = T_{X^*}$ from Definition 2.2.3. This means that X is a frame if and only if X^* is a Riesz sequence. (i) $\implies$ (ii) with $A = B$.

It follows that, in finite dimensions, X^* is a Riesz basis if and only if X is a basis. This holds because if X is overcomplete, the number of rows of $\mathcal{J}_X$ is strictly less than the number of its columns. It follows that $\mathcal{J}_{X^*}$ has fewer columns than the dimension of the vector space W ($\#X^* < \#X$). Thus X^* is not a spanning set for W and therefore it cannot be a Riesz basis.

The duality principle also characterized by

$$\begin{aligned}\langle x, T_{X^*}^* T_{X^*} x\rangle = \langle T_{X^*} x, T_{X^*} x\rangle = \|T_{X^*} x\|^2 \\ = \|T_X^* x\|^2 = \langle T_X^* x, T_X^* x\rangle = \langle x, T_X T_X^* x\rangle\end{aligned}$$

which implies that a dual Gramian of a system is a Gramian of its adjoint system and vice versa, by considering matrix representations of the operators.

Matrix representations also allow us to view the duality principle through the lens of the following statement adapted from Fan, Heinecke, and Shen, 2016. The columns of $\mathcal{J}_X$ are associated with the system X and the rows of it are associated with the system X^*. Thus, no matter how difficult it is to find an adjoint system at the operator level, using this principle, we can easily find it at the matrix level. Chapter 5 will consider an example of such a system.

3.4 Discrete Fourier Frame

To continue the example from the previous section, we consider the DFT matrix when $N = 3$, and choose to omit the second row (The same idea applies to a different choice of N or when other rows are omitted). The resultant matrix $\mathcal{J}$ is

$$\begin{bmatrix} e^{-\frac{2\pi}{3}0\cdot 0} & e^{-\frac{2\pi}{3}0\cdot 1} & e^{-\frac{2\pi}{3}0\cdot 2} \\ e^{-\frac{2\pi}{3}2\cdot 0} & e^{-\frac{2\pi}{3}2\cdot 1} & e^{-\frac{2\pi}{3}2\cdot 2} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} - i\frac{\sqrt{3}}{2} & -\frac{1}{2} + i\frac{\sqrt{3}}{2} \end{bmatrix} \tag{3.7}$$

multiplied by a scalar factor of $\frac{1}{\sqrt{3}}$, which we will refer to as a "discrete Fourier frame".

Since the rows of (2.12) were pairwise orthonormal, it directly follows that $\mathcal{J}$ has orthonormal rows. However, the new linear map is from $\mathbb{C}^3 \to \mathbb{C}^2$.

Since the rows are orthonormal, it follows that $\mathcal{J}\mathcal{J}^* = \mathcal{I}$, however by calculating the matrix product, we see that $\mathcal{J}^*\mathcal{J} \neq \mathcal{I}$.

Since $\mathcal{J}\mathcal{J}^* = \mathcal{I}$, we see that the columns of $\mathcal{J}$ form a tight frame due to perfect reconstruction and, in order to build intuition regarding frames, it has been manually verified in Appendix A.

Letting $\mathcal{A} = \mathcal{J}^*$, we have that $\mathcal{A}^*\mathcal{A} = I$, which implies that the columns of $\mathcal{A}$ form an orthonormal Riesz sequence. The orthonormal Riesz sequence does not span the adjoint space $\mathbb{C}^3$ and is thus not an orthonormal basis for the space. By definition of $\mathcal{A}$, these correspond to the rows of $\mathcal{J}$ up to conjugation.

This highlights the duality principle, where the columns of this matrix form a tight frame and the rows form an orthonormal Riesz sequence.

Chapter 4

Infinite Dimensional Frames

Section 4.1.1 introduces the concepts of Hilbert spaces, Hamel bases, and Schauder bases while discussing some important properties and mentioning relevant results. Section 4.1.2 constructs the analysis and synthesis operators for a system in a Hilbert space. Section 4.2.1 discusses different ways of representing bi-infinite matrices. Section 4.2.2 provides a matrix representation of the synthesis operator and discusses the case for analysis operators. Section 4.3.1 introduces the concept of Bessel sequences and discusses important properties and results regarding frames and Riesz sequences in Hilbert spaces. Section 4.3.2 restates the duality principle for Hilbert spaces. Section 4.3.3 shows a Gramian and dual Gramian in Hilbert space and discusses issues relating to its convergence.

Many results from this section are important results from functional analysis and can be found in a textbook such as Kreyszig, 1991 or Ovchinnikov, 2018. More results regarding frames and Riesz sequences can be seen in Christensen et al., 2003 and the definitions of the analysis and synthesis operators are adapted from Fan, Heinecke, and Shen, 2016.

4.1 Operators on Hilbert Spaces

4.1.1 Hilbert Spaces

Banach spaces are an important class of vector spaces in functional analysis and are defined to be *complete normed vector spaces*. A vector space is **complete** if and only if every Cauchy sequence of vectors in the space converges to a vector in the space.

Hilbert spaces are Banach spaces where the norm arises from an inner product defined on the space. Let H be a Hilbert space, then for all $x \in H$, the norm is given by the relationship $\|x\| = \sqrt{\langle x, x \rangle}$.

It is a well-known result that all finite-dimensional inner product spaces with dimension n are isomorphic to either $\mathbb{R}^n$ or $\mathbb{C}^n$, conditional on the field of scalars being $\mathbb{R}$ or $\mathbb{C}$ respectively, and the proof of this can be seen in Axler, 1997, Ch 3. Since $\mathbb{R}^n$ and $\mathbb{C}^n$ are complete, we see that all finite-dimensional inner product spaces are Hilbert spaces. Therefore, the definitions in this chapter remain consistent with definitions from previous chapters and are generalizations to infinite-dimensional Hilbert spaces. Since we have already discussed finite-dimensional Hilbert spaces in great detail in Chapter 2 and Chapter 3, we will now consider infinite-dimensional Hilbert spaces.

The fundamental result that every vector space has a basis called a **Hamel basis** can be found in a functional analysis book such as Ovchinnikov, 2018, Ch 1. This includes infinite-dimensional Hilbert spaces and we note that every vector in this infinite-dimensional space can be represented as finite linear combinations of vectors from the Hamel basis.

Theorem 4.15 of Ovchinnikov, 2018 provides a proof of how Hamel bases in infinite-dimensional Banach spaces are always an uncountable collection of vectors. This makes it unclear as to how one can find a matrix representation for operators in terms of this Hamel basis. To address this, we consider a **Schauder basis** which is a countable system X in a Hilbert space H, such that for all $v \in H$, there exists a unique sequence $(\alpha(x))_{x \in X}$ such that

$$v = \sum_{x \in X} \alpha(x) x. \tag{4.1}$$

Series using Schauder bases converge, however, may not converge unconditionally. For the purpose of this capstone, all bases will be assumed to be Schauder bases.

Folland, 1999, Ch 5 shows that every Hilbert space has an orthonormal basis, and also shows that every separable Hilbert space has a Schauder basis. A **separable Hilbert space** is one which has a *countable dense subset* and for the purpose of this capstone, all Hilbert spaces will be assumed to be separable.

4.1.2 Analysis and Sythesis Operators

Let H be a Hilbert space and $X \subset H$ be a countable system. $\ell_0(X)$ is the Hilbert space of finitely supported sequences, i.e. infinite dimensional sequences with finite non-zero terms, indexed by elements of X. $\ell_2(X)$ is the Hilbert space of square-summable infinite-dimensional sequences indexed by elements of X, and we can show that $\ell_0(X)$ is dense in $\ell_2(X)$.

We first define the synthesis operator $T'_X : \ell_0(X) \to H$, which accepts a finitely supported sequence and linearly combines that with the vectors in the system using the sequence as coefficients. This operator is well-defined and reduces the series to a finite sum, and in case this operator is bounded above, we would like to continuously extend it to $\ell_2(X)$.

This extension $T_X : \ell_2(X) \to H$ is given for all $\alpha \in \ell_2(X)$ by

$$T_X\alpha = \begin{cases} T'_X\alpha & \text{if } \alpha \in \ell_0(X) \\ \lim\limits_{i\to\infty} T'_X\alpha_i & \text{if } \alpha \notin \ell_0(X) \end{cases} \tag{4.2}$$

where $(\alpha)_{i\in\mathbb{N}} \subset \ell_0(X)$ such that $\lim\limits_{i\to\infty} \alpha_i = \alpha$. We now show that if T'_X is bounded above with bound B, then T_X is continuous. This is true since for all $x, y \in \ell_2(X)$, $\|T_Xx - T_Xy\| = \|T_X(x-y)\| \le B\|x-y\|$ and as $\|x-y\| \to 0$, $\|T_Xx - T_Xy\| \to 0$. The uniqueness of the operator follows from the uniqueness of the limit. Thus T_X is the unique continuous extension of T'_X and is given for the system X by

$$T_X : \ell_2(X) \to H : \alpha \mapsto \sum_{x\in X} \alpha(x)x. \tag{4.3}$$

For system X, the analysis operator is given by

$$T^*_X : H \to \ell_2(X) : v \mapsto (\langle v, x\rangle)_{x\in X} \tag{4.4}$$

and it is bounded if and only if the synthesis operator is bounded, since $\|T^*\| = \|T\|$.

4.2 Bi-Infinite Matrices

4.2.1 Bi-Infinite Matrix Representations

We first consider a **bi-infinite** matrix which is a matrix in infinite-dimensional space. We note that it can be represented by

$$\begin{bmatrix} a_{1,1} & \cdots \\ \vdots & \ddots \end{bmatrix} \tag{4.5}$$

where $a_{1,1}$ is an element of the matrix. This matrix extends infinitely along two semi-axes, and therefore it is not meaningful to represent matrices in this manner since we cannot possibly list out all of its entries.

Therefore, we represent a bi-infinite matrix using the notation $A = (a_{i,j})_{i \in K, j \in K'}$ where K and K' are countable index sets that are used to index the rows and columns of the matrix respectively. $a_{i,j}$ corresponds the entry of the matrix in the i-th row and j-th columns. We can alternatively index the rows and columns of the matrix using systems in case each entry of the matrix is uniquely determined by a pair of vectors or using groups in case each entry is uniquely determined by a pair of group elements.

4.2.2 Operators as Bi-Infinite Matrices

To understand how to represent an operator as a bi-infinite matrix, we begin by considering the synthesis operator of a system. We let K be a countable index set such that $(x_k)_{k \in K} = X$. Furthermore, let $(e_k)_{k \in K}$ be the standard basis for $\ell_2(X)$, where the only non-zero entry in e_k is 1, at the k-th position. Therefore we have that for all $k \in K, Te_k = x_k$. Let K' be a countable index set such that $(o_{k'})_{k' \in K'}$ is an orthonormal basis for H. Since a matrix representation for T_X is a pre-Gramian we denote it using $\mathcal{J}_X$ and have that

$$\mathcal{J}_X = (\langle Te_k, o_{k'} \rangle)_{k' \in K', k \in K} = (\langle x_k, o_{k'} \rangle)_{k' \in K', k \in K} . \tag{4.6}$$

The idea that $(\mathcal{J}_X)_{k',k} = \langle Te_k, o_{k'} \rangle$ follows from the fact that

$$Te_k = \sum_{k' \in K'} (\mathcal{J}_X)_{k',k} o_{k'} \tag{4.7}$$

since the coefficients for an orthonormal basis are inner products.

Using the definition of the adjoint system, we can find a matrix representation of the analysis operator by constructing a matrix representation of the synthesis operator for the adjoint system. One can do it easily by considering the conjugate transpose of this matrix $\mathcal{J}_X$ since that is exactly equal to $\mathcal{J}_X^*$.

4.3 Frames and the Duality Principle

4.3.1 Frames and Riesz Sequences

If a linear operator is bounded above, then the system which generates the operator is known as a **Bessel sequence**. Christensen et al., 2003, Ch 3 states that all series of Bessel sequences converge unconditionally, given that the coefficient sequence is from the space $\ell_2(X)$. We note that frames and Riesz sequences are examples of Bessel sequences.

In finite dimensions, the existence of an upper bound on linear operators generated by systems is guaranteed due to the Cauchy-Schwarz inequality as that upper bound is a finite sum. However, it is not so trivial to find such an upper bound in the infinite-dimensional case as the series might diverge. This implies that Bessel sequences are a special class of sequences in infinite dimensions which further implies it is less trivial to have frames and Riesz sequences in infinite-dimensional space, in comparison to finite-dimensional space. We see that for a Hilbert space, {Bessel sequences} $\supseteq$ {frames} $\supseteq$ {tight frames} $\supseteq$ {orthonormal bases}.

Frames and Riesz sequences in Hilbert spaces are defined analogously to how they are defined in Section 3.2. A system X is a frame if it satisfies (3.5) and it is a Riesz sequence if it satisfies (3.6), with the analysis and synthesis operators corresponding to their respective definitions for Hilbert spaces.

Proposition 4.3.1 is an important result from Christensen et al., 2003, Ch 3 which states that every Riesz basis is also a frame and Corollary 4.3.2 follows directly from it.

Proposition 4.3.1. *If system X is a Riesz basis for Hilbert space H, then there exists constants $A, B > 0$ such that for all $v \in H$,*

$$A\|v\|^2 \leq \sum_{x \in X} |\langle v, x \rangle|^2 \leq B\|v\|^2. \tag{4.8}$$

Corollary 4.3.2. *Every orthonormal basis in a Hilbert space is a tight frame.*

Note that the converse does not hold and that every tight frame is not an orthonormal basis. An example of this fact is seen in Section 3.4, where the columns are an overcomplete tight frame.

4.3.2 The Duality Principle

We restate the duality principle for Hilbert spaces by saying that a system X, is a (tight) frame if and only if X^* is a (orthonormal) Riesz sequence up to unitary equivalence.

4.3.3 Gramian and Dual Gramian

We begin by examining the dual Gramian constructed from the pre-Gramian given in (4.6) and see that

$$(\mathcal{J}\mathcal{J}^*)_{k',j\in K'} = \sum_{k\in K} \langle x_k, o_{k'} \rangle \overline{\langle x_k, o_j \rangle}. \tag{4.9}$$

Since K might be an index set with a countably infinite number of elements, problems relating to the convergence of the series may arise. Since a divergent series cannot be an entry of the dual-Gramian, we avoid situations like this by specifying that the columns of the pre-Gramian must be square-summable, i.e. $\sum_{k\in K} |a_k|^2, \sum_{k\in K} |b_k|^2 < \infty$ where $a_k = \langle x_k, o_{k'} \rangle$ and $b_k = \langle x_k, o_j \rangle$.

This constraint solves the issue, since, due to the Cauchy-Schwarz Inequality, we have that

$$|\sum_{k\in K} a_k \overline{b_k}| \leq \left(\sum_{k\in K} |a_k|^2 \right)^{\frac{1}{2}} \left(\sum_{k\in K} |b_k|^2 \right)^{\frac{1}{2}} < \infty. \tag{4.10}$$

A similar argument works for the Gramian if the rows of the pre-Gramian are square-summable. Therefore, in this capstone, we assume that all rows and columns of the pre-Gramian are square-summable, raising no issues of convergence.

We also mention the Gramian constructed from the pre-Gramian in (4.6) and see that

$$(\mathcal{J}^*\mathcal{J})_{k,j\in K} = \sum_{k'\in K'} \overline{\langle x_k, o_{k'}\rangle}\langle x_j, o_{k'}\rangle. \tag{4.11}$$

We reiterate that the duality principle characterization of a Gramian and a dual Gramian for adjoint systems holds true even in the case of infinite-dimensional Hilbert spaces and remember that unitary equivalence accounts for differences in orthonormal bases.

Chapter 5

Unitary Systems

Section 5.1.1 introduces the concept of projective unitary representations from which one obtains the Gabor type unitary systems discussed in Section 5.1.2. Section 5.1.3 defines unitary systems generated by a wandering vector and Section 5.1.4 contains an original derivation that defines another orthonormal basis used in a later section. Section 5.2 states the main result of the capstone and we provide original approaches to prove this result using the duality principle. Section 5.3 discusses how one may use the above results to better understand Gabor type systems with a wandering vector.

The content from this section is largely adapted from Han and Larson, 2000, Fan, Heinecke, and Shen, 2016, and Dai and Larson, 1998.

5.1 Preliminaries

5.1.1 Projective Unitary Representation

Let G be a countable group and H be a separable Hilbert space. The unitary operators on H form a group under composition.

Definition 5.1.1. *A **unitary representation** of G on H is a group homomorphism into the group $\mathcal{U}(H)$ of unitary operators on H.*

Definition 5.1.2. *A mapping $\pi\colon G \to \mathcal{U}(H)$ is a **projective unitary representation** if it is a unitary representation up to unimodular multipliers, i.e.,*

$$\pi(g)\pi(h) = \mu(g,h)\pi(gh) \quad \text{for all } g,h \in G, \tag{5.1}$$

*where $\mathbb{T} := \mu\colon G \times G \to \{z \in \mathbb{C}\colon |z| = 1\}$ is called the **multiplier** of π.*

The unitary representations can be intuitively understood as a group of unitary operators indexed by some other group that is homomorphic to it. Thus, for some $g \in G$, $\pi(g)$ is a unitary operator.

The unimodular multiplier has the property that $\overline{\mu} = \mu^{-1}$. Since group homomorphisms map the identity element to the identity element, it implies that $\pi(g)\pi(g^{-1}) = \mu(g,g^{-1})I$ and thus, we see that $\pi(g^{-1}) = \mu(g,g^{-1})\pi(g)^*$.

5.1.2 Gabor Type Unitary Systems

This definition of projective unitary representations is motivated by Gabor representations which have numerous applications in signal processing. We note that $L_2(\mathbb{R}^d)$ refers to the Hilbert space of measurable functions with domain $\mathbb{R}^d$ that are square-integrable.

Example 5.1.3. *Let $K, L \subset \mathbb{R}^d$ be the ranges of bijective linear maps $A_K, A_L\colon \mathbb{Z}^d \to \mathbb{R}^d$. Let E^t be the translation and M^t the modulation operators on $L_2(\mathbb{R}^d)$, i.e., $E^t f(x) = f(x-t)$ and $M^t f(x) = e^{it\cdot x} f(x)$ for $t \in \mathbb{R}^d$. Then*

$$\pi_{(K,L)}\colon \mathbb{Z}^d \times \mathbb{Z}^d \to \mathcal{U}(L_2(\mathbb{R}^d))\colon (z_1,z_2) \mapsto E^{A_K z_1} M^{A_L z_2} \tag{5.2}$$

is a projective unitary representation.

The fact that Example 5.1.3 is a projective unitary representation can be verified, and in the case that $d = 1$ and $A_K, A_L = I$, for some $g = (g_1, g_2)$ and $h = (h1, h2)$, we can derive that $\mu(g, h) = e^{-ig_2h_1}$.

Example 5.1.3 can be further generalized to the example below, which yields the Gabor type unitary systems $\{U^m V^n\}_{m,n\in\mathbb{Z}}$ described in Han and Larson, 2000, Ch 4.

Example 5.1.4. *Let $U, V \in \mathcal{U}(H)$ such that $UV = \lambda VU$ for some complex $|\lambda| = 1$, then*

$$\pi\colon \mathbb{Z} \times \mathbb{Z} \to \mathcal{U}(H)\colon (m, n) \mapsto U^m V^n \tag{5.3}$$

is a projective unitary representation.

If $g = (g_1, g_2), h = (h_1, h_2) \in G = \mathbb{Z}^2$, then from Example 5.1.4 we have that $\mu(g, h) = \lambda^{-g_2h_1}$ and $\pi(g)\pi(h) = \lambda^{-g_2h_1+g_1h_2}\pi(h)\pi(g)$.

5.1.3 Wandering Vectors

A vector $\eta \in H$ is called the **generator** of system $\pi(G)\eta := (\pi(g)\eta)_{g\in G}$ and its synthesis operator and analysis operators are defined in the manner outlined in Section 4.1.2.

The generator is called a **wandering vector**, or complete wandering vector if $\pi(G)\eta$ is an orthonormal basis for H and a **Riesz (resp. frame) vector** if $\pi(G)\eta$ is a Riesz basis (resp. frame).

Note that the domain of $T^*_{\pi(G)\eta}$ is $\{y \in H\colon \sum_{g\in G} |\langle y, \pi(g)\eta\rangle|^2 < \infty\}$, since the sequences must be square-summable.

5.1.4 Pre-Gramian of a Wandering Vector System

Let $(\pi(g)\psi)_{g\in G}$ be an orthonormal basis for H and $\eta \in H$. Then for all $h \in G$,

$$\begin{aligned}\sum_{g\in G} |\langle \pi(g)\eta, \pi(h)\psi\rangle|^2 &= \sum_{g\in G} |\langle \eta, \pi(g)^{-1}\pi(h)\psi\rangle|^2 \\ &= \sum_{g\in G} |\langle \eta, \overline{\mu(g,g^{-1})}\pi(g^{-1})\pi(h)\psi\rangle|^2 \\ &= \sum_{g\in G} |\langle \eta, \overline{\mu(g,g^{-1})}\mu(g^{-1},h)\pi(g^{-1}h)\psi\rangle|^2 \\ &= \sum_{g\in G} |\langle \eta, \pi(g^{-1}h)\psi\rangle|^2 = \sum_{g\in G} |\langle \eta, \pi(g)\psi\rangle|^2 = \|\eta\|_H^2.\end{aligned}$$

The convergence of the above series implies that

$$(\psi_{g,h})_{g\in G} := (\overline{\mu(g,g^{-1})}\mu(g^{-1},h)\pi(g^{-1}h)\psi)_{g\in G} \tag{5.4}$$

is an orthonormal basis for H. We have that $(\psi_{g,h})_{g\in G}$ is normalized since $|\mu(g,h)| = 1$ and is a complete wandering vector because $Gh = G$ since $h \in G$ and G is a group.

Since we see that the inner products seen above are square-summable, we can consider the pre-Gramian of $(\pi(g)\eta)_{g\in G}$ with respect to the orthonormal basis $(\pi(h)\psi)_{h\in G}$ given by

$$\mathcal{J}_\eta := (\langle \pi(g)\eta, \pi(h)\psi\rangle)_{h,g\in G} = (\langle \eta, \psi_{g,h}\rangle)_{h,g\in G}. \tag{5.5}$$

This acts as an operator on $\ell_2(G)$ equipped with the standard unit vector basis $(e_g)_{g\in G}$.

5.2 Duality Principle in Spaces with Wandering Vectors

The following result is a restatement of Corollary 3.4, Corollary 4.4, and Corollary 4.5 proved in Han and Larson, 2000. It is important to note that in this capstone we refer to complete wandering vectors as wandering vectors since we would only like to consider the case when the wandering vector forms an orthonormal basis.

Proposition 5.2.1. *Let π be a projective unitary representation of a group G on H, with a wandering vector. Then $\eta \in H$ generates a*

(i) frame if and only if it generates a Riesz basis.

(ii) tight frame if and only if it generates an orthonormal basis.

Han and Larsen made use of von Neumann algebras to show this very result, however, it is useful to prove this result using the duality principle as well by considering adjoint systems and a pre-Gramian.

5.2.1 Approaches

Approach 1: Diagonal Argument

The proof of Proposition 5.2.1, using a diagonal argument requires that the pre-Gramian of the system generated by η is injective. The injectivity of $\mathcal{J}_\eta$ implies that the $\mathcal{J}_\eta^*$ is surjective. This means that the adjoint system of $(\pi(g)\eta)_{g\in G}$ is a spanning set and thus forms a basis for the co-domain.

Conjecture 5.2.2. *Let π be a projective unitary representation of G on H which allows a wandering vector. If $\eta \in H$ is a frame vector, then $\mathcal{J}_\eta$ is injective.*

Remark 5.2.3. *We have been unable to show the injectivity of this matrix yet, however, if we are able to show this result, then Proposition 5.2.1 can be proved as seen below.*

Strategy Towards a Potential Proof of Proposition 5.2.1

We denote the system $(\pi(g)\eta)_{g\in G}$ using X. If X is a Riesz basis then, using Proposition 4.3.1, we have that X is a frame. If X is a frame then, from the duality principle, we have that X^* is a Riesz sequence. Furthermore, from Conjecture 5.2.2 which means that $\mathcal{J}_\eta$ is injective, we have that $\mathcal{J}_\eta^*$ is surjective. By the duality principle, X^* is a spanning set, and thus X^* is a Riesz Basis. Then using Proposition 4.3.1, we have that X^* is a frame, which implies that X is a Riesz basis, due to the duality principle.

(i) $\implies$ (ii) since tight frames and orthonormal bases are frames that have equal upper and lower frame bounds.

The diagonal argument of the above proof is outlined in Figure 5.1 for the case of tight frames.

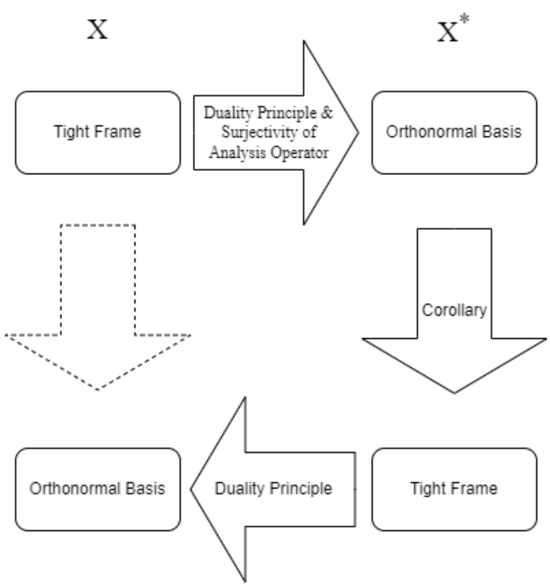

FIGURE 5.1: Diagonal Argument

Approach 2: Dual Gramian Analysis

We note that the inner products in every row and every column of $\mathcal{J}_\eta$ contain (up to a unimodular factor) every element of the basis $(\psi_{g,h})_{g\in G}$ exactly once. Therefore, by rearranging inner products in a matrix representation if we can show that the system is its own adjoint system up to unitary equivalence, then the result follows from the duality principle.

Remark 5.2.4. *The proof we consider in this approach relies on an extra assumption that the group G is abelian.*

Consider the system $(\pi(g^{-1})\eta)_{g\in G}$, just so that notation is later more convenient and since Bessel sequences have unconditional series with square-summable coefficients. Using the basis $(\pi(h)\psi)_{h\in G}$ the wandering vector generates, the pre-Gramian of $(\pi(g^{-1})\eta)_{g\in G}$ is

$$\begin{aligned}\mathcal{J}_\eta &= \Big(\langle \pi(g^{-1})\eta, \pi(h)\psi\rangle\Big)_{h,g} = \Big(\langle \eta, \overline{\mu}(g^{-1},g)\pi(g)\pi(h)\psi\rangle\Big)_{h,g}\\ &= (\langle \eta, \pi(g)\pi(h)\psi\rangle)_{h,g}\, U = (\langle \eta, \mu(g,h)\overline{\mu}(h,g)\pi(h)\pi(g)\psi\rangle)_{h,g}\, U\end{aligned}$$

where U is the unitary diagonal matrix with diagonal $(\overline{\mu}(g^{-1},g))_{g\in G}$. Therefore with $s_{g,h} := \mu(g,h)\overline{\mu}(h,g)$,

$$(\langle \eta, \pi(g)\pi(h)\psi\rangle)_{h,g} = \big(\langle \eta, s_{g,h}\pi(h)\pi(g)\psi\rangle\big)_{h,g}\,.$$

In general, it follows that the numbers $s_{g,h}$ are not all equal to 1, however, in the rare case that is seen here where ψ is a wandering vector providing an additional structure which, after further analysis, might imply that $s_{g,h} = 1$. If that result were true, the remaining argument is as follows

$$\mathcal{J}_\eta = (\langle \eta, \pi(g)\pi(h)\psi\rangle)_{h,g} U = (\langle \eta, \pi(h)\pi(g)\psi\rangle)_{h,g} U = (\mathcal{J}_\eta)^\top U.$$

So $(\pi(g^{-1})\eta)_{g\in G}$ is its own adjoint system, and Proposition 5.2.1 follows from the duality principle.

Approach 3: Co-Isometries

We restate Proposition 3.1 from Han and Larson, 2000 below.

Proposition 5.2.5. *Suppose that ψ is a wandering vector for a unitary system $\mathcal{U}$, then a vector η is a normalized tight frame vector if and only if there exists a unique co-isometry $A \in C_\psi(\mathcal{U})$ such that $A\psi = \eta$.*

Here $C_\psi(\mathcal{U})$ refers to the local commutant of the unitary system and is defined as $\{T \in B(H) \mid (TU - UT)\psi = 0, \forall U \in \mathcal{U}\}$.

We also consider Proposition 1.3 from Dai and Larson, 1998 which states that all wandering vectors of a unitary system are unitarily equivalent to one another and these unitary operators are in the local commutant of the system and there is a bijection between the number of unitary operators in the local commutant and the number of wandering vectors in the system.

Since every orthonormal basis is a tight frame, all that is left to show is that if η is a tight frame vector then η is a wandering vector since a wandering vector, by definition, generates an orthonormal basis.

Thus, if we can show that the unique co-isometry A is in fact unitary, we will have shown that η is unitarily equivalent to ψ and thus is a wandering vector.

Proposition 3.5, Corollary 3.6, and the comment following Corollary 3.6 from Han and Larson, 2000 may be used in order to similarly prove results for (i) in Proposition 5.2.1.

This would show that the system generated by η would be its own adjoint system and up to unitary equivalence and thus the result would follow due to the duality principle.

5.3 Future Work

If we can prove that any of the suggested approaches outlined in Section 5.2.1 are valid, then one may be able to construct an argument using the duality principle to prove Proposition 5.2.1.

In order to work on Approach 1, one would have to prove injectivity by perhaps considering the matrix representation of the operator and evaluating whether its null space is trivial. An alternative characterization of the injectivity of a frame operator might be useful to do so, however, a clear characterization as such is not obvious.

In order to work on Approach 2, one would have to examine the structure provided by a wandering vector to the system. This is examined in Han and Larson, 2000, Ch 4 using the specific case of Example 5.1.3. However, this approach uses the assumption that the group G is abelian, which may not be the case.

In order to work on Approach 3, one would have to refer to the construction of the co-isometry A from the proof of Proposition 5.2.5 and consider a matrix representation to show that it is unitary or invertible for the tight frame or frame case respectively.

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Appendix A

Discrete Fourier Frame Calculation

Recall that the columns of the frame are given by $v_1 = \frac{1}{\sqrt{3}}(1,1)^T, v_2 = \frac{1}{\sqrt{3}}(1, e^{-\frac{2\pi}{3}})^T$ and $v_3 = \frac{1}{\sqrt{3}}(1, e^{-\frac{4\pi}{3}})^T$.

Let $x = y + iz$ where $y, z \in \mathbb{R}^2$. Then

$$y + iz = x = \sum_{i=1}^{3} \langle x, v_i \rangle v_i = \sum_{i=1}^{3} \langle y, v_i \rangle v_i + i \sum_{i=1}^{3} \langle z, v_i \rangle v_i$$

due to linearity of inner product in the first argument. Thus if perfect representation holds for $y \in \mathbb{R}^2$, it will hold for $x \in \mathbb{C}^2$ due to linearity. Letting $y = (a,b)^T$, we evaluate the individual inner products to get that $\langle y, v_1 \rangle v_1 = \frac{1}{3}(a+b, a+b)^T, \langle y, v_2 \rangle v_2 = \frac{1}{3}(a + e^{\frac{2\pi}{3}} b, e^{-\frac{2\pi}{3}} a + b)^T, \langle y, v_3 \rangle v_3 = \frac{1}{3}(a + e^{\frac{4\pi}{3}} b, e^{-\frac{4\pi}{3}} a + b)^T$.

Thus,

$$\begin{aligned}
\begin{bmatrix} a \\ b \end{bmatrix} &= \frac{1}{3}\begin{bmatrix} a+b \\ a+b \end{bmatrix} + \frac{1}{3}\begin{bmatrix} a+e^{\frac{2\pi}{3}}b \\ e^{-\frac{2\pi}{3}}a+b \end{bmatrix} + \frac{1}{3}\begin{bmatrix} a+e^{\frac{4\pi}{3}}b \\ e^{-\frac{4\pi}{3}}a+b \end{bmatrix} \\
&= \begin{bmatrix} a+\frac{b}{3}(1+e^{\frac{2\pi}{3}}+e^{\frac{4\pi}{3}}) \\ (1+e^{-\frac{2\pi}{3}}+e^{-\frac{4\pi}{3}})a+b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}
\end{aligned}$$

with the terms in curved brackets evaluating to zero since it is the sum of the cube roots of unity.