Math mini Quiz solution

January 21, 2023

Q1.1: $f(x) = \frac{1}{2x+a}, x = e^{3u} + u$ what is $\frac{\partial f}{\partial u}$

$$\begin{aligned} \frac{\partial f}{\partial u} &= \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} \\ &= [-1(2x+a)^{-2} \cdot 2] \cdot [(3e^{3u} + 1)] \\ &= \frac{-6e^{3u} - 2}{(2e^{3u} + 2u + a)^2} \end{aligned}$$

Q1.2: this function takes min $y = -3$ at $x = -2$ and $x = 0$, takes max at $x = 2, y = 13$

Q1.3 What is the gradient $\nabla f(x, y)$ of the function $f(x, y) = e^x - 2xy$?

$$\nabla f(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} e^x - 2y \\ -2x \end{pmatrix}$$

Q1.4

$$\frac{1}{2} ln|x^2 + 1| + C$$

Q2.1 While it is true that if X and Y are independent then $E[XY] = E[X]E[Y]$, the converse is not true in general. $E[XY] = E[X]E[Y]$ only suggests X and Y are uncorrelated. You can see a specific example here

Q2.2 $Var[Y] = Var[X_1] + Var[X_2] + 2Cov(X_1, X_2)$. Only expectation has linearity. Variance of the sum of 2 random variables generally involves an additional covariance term. However, if X_1 and X_2 are independent that covariance term goes away.

Q2.3

$$\begin{aligned} E[\bar{X}] &= E[\frac{1}{n}\sum_{i=1}^{i=n} X_i] \\ &= \frac{1}{n} E[\sum_{i=1}^{i=n} X_i] \\ &= \frac{1}{n}\sum_{i=1}^{i=n} E[X_i] \quad \text{linearity of expectation} \\ &= \frac{1}{n} \cdot n \cdot \frac{2}{3} \\ &= \frac{2}{3} \end{aligned}$$

Q2.4

$$\begin{aligned}
Var[\bar{X}] &= Var[\frac{1}{n}\sum_{i=1}^{i=n} X_i] \\
&= \frac{1}{n^2} Var[\sum_{i=1}^{i=n} X_i] \\
&= \frac{1}{n^2}\sum_{i=1}^{i=n} Var[X_i] \quad \text{due to independence of each flip} \\
&= \frac{1}{n^2}\cdot n \cdot \frac{1}{3}\cdot\frac{2}{3} \\
&= \frac{2}{9n}
\end{aligned}$$

Q3.1: pdf of normal distribution with variance $\tilde{\sigma}^2$ is $\frac{1}{\tilde{\sigma}\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\tilde{\sigma}})^2}$
Substitute $\tilde{\sigma}$ with $\sqrt{2}\sigma$ gives the desired expression. Thus the variance is $\tilde{\sigma}^2 = (\sqrt{2}\sigma)^2 = 2\sigma^2$

Q3.2 Expected number of event $\lambda = \frac{60}{12} = 5$

$$\begin{aligned}
\mathbf{P}(\text{num meteors} >= 2) &= 1 - \mathbf{P}(\text{num meteors} < 2) \\
&= 1 - \mathbf{P}(\text{num meteors} = 0) - \mathbf{P}(\text{num meteors} = 1) \\
&= 1 - \frac{5^0 e^{-5}}{0!} - \frac{5^1 e^{-5}}{1!}
\end{aligned}$$

Q3.3: number of ways you can get 2 out of 3 multiplied by each event's probability

$$\mathbf{P}(2 \text{ out of } 3) = \binom{3}{2} * p^2 * q^1 = 3 * 0.75^2 * 0.25$$

Q4.1: Chain rule

Q4.2: Apply Bayes Theorem

$$\begin{aligned}
\mathbf{P}(gold|A) &= \frac{\mathbf{P}(\text{gold} ; A)}{\mathbf{P}(\text{gold})} \\
&= \frac{\mathbf{P}(\text{gold} ; A)}{\mathbf{P}(\text{gold} ; A) + \mathbf{P}(\text{gold} ; B)} \\
&= \frac{\mathbf{P}(\text{gold}|A)\cdot\mathbf{P}(A)}{\mathbf{P}(\text{gold}|A)\cdot\mathbf{P}(A) + \mathbf{P}(\text{gold}|B)\cdot\mathbf{P}(B)} \\
&= \frac{(0.2 * 1)}{(0.2 * 1) + (0.8 * 0.25)} \\
&= 0.5
\end{aligned}$$

Q4.3: The general intuition is that the conditional expectation of $\mathbf{E}[Y|Z=2]$ is not the same as the original expectation of Y. A larger than expected Z will change our understanding on both X and Y at the same time. Since this is a True/False question, we dont need to calculate the actual value. It is not hard to imagine Y should also take larger values: $\mathbf{E}(Y|Z=2) > 0$, when $Z = 2$

$$\begin{aligned}
E(X|Z=2) &= E(3Z - 3Y|Z=2) \\
&= 6 - 3 * E(Y|Z=2) \\
&<= 6
\end{aligned}$$

This quantity can also be estimated explicitly: a good read

$$\begin{aligned}
E(X|Z=2) &= E(X) + \frac{Cov(X,Z)}{Var(Z)} * (z - E(Z)) \\
&= 0 + \frac{\frac{1}{3}}{\frac{10}{9}} * 2 \\
&= \frac{3}{5}
\end{aligned}$$

Q5.1

$$det(A) = [0 * (0 * (-4) - (-3) * 1)] - [1 * (0 * (-4) - 0 * 1))] + [0 * (0 * (-3) - 0 * 0)] = 0$$

Q5.2

$$tr(A) = 0 + 0 + (-4) = -4$$

Q5.3 characteristic function: $det(A - \lambda I) = 0$

$$(-\lambda) * [\lambda * (4 + \lambda) + 3] = 0$$

$$(\lambda) * (\lambda + 3) * (\lambda + 1) = 0$$

Thus eigenvalues are (0, -3, -1)

Q5.4 plug in $\lambda = -1$ and solve for $(A - \lambda I)x = 0$

Q5.5 $f(x) = b^T x + 5$. Then $\frac{\partial f}{\partial x} = b$ regarless of what value x takes

Q6.1 Q6.2 Q6.3 vector inner product, l2 norm and outer product

Q7.1 rank can only be smaller or equal to min(row number, column number) = 12

Q7.2

$$v^T Mv = v^T (AA^T)v = (v^T A)(A^T v) = (A^T v)^T (A^T v) = (A^T v)^2 \geq 0$$

Q8

$$\frac{f(n)}{g(n)} = \frac{ln(n)}{\log_2(n)} = \frac{\frac{\log_2 n}{\log_2 e}}{\log_2(n)} = \frac{1}{\log_2(e)}$$

$$f(n) * \log_2(n) = g(n)$$

Since the two functions are just off by a constant, both expressions are correct.