__Example__ minimize $\frac{1}{2}(x_1^2+x_2^2+x_3^2)$ — (P)

s.t. $x_1+x_2+x_3=3$

$f(x) = \frac{1}{2}(x_1^2+x_2^2+x_3^2)$ — strictly convex

$\mathcal{H} = \{x: x_1+x_2+x_3=3\}$ — convex, closed

We showed in Lec 13 using LM's that $x^* = (1,1,1)$ is the unique solution to (P)

We can also apply necessary and sufficient condition for minimizing convex function over closed convex set (lec 10) to verify:

$\nabla f(x) = (x_1, x_2, x_3)$.

For all $x \in \mathcal{H}$,

$$\nabla f(x^*)^T(x-x^*) = \sum_{i=1}^{3} x_i^*(x_i - x^*)$$

$$\xrightarrow{x^*=(1,1,1)} = \sum_{i=1}^{3}(x_i - 1)$$

$$= \sum_{i=1}^{3} x_i - 3 = 3-3 = 0$$

Therefore $x^* = (1,1,1)$ is a global min

Uniqueness follows from strict convexity

__Example__ Nonlinear equality constraint

$$\text{minimize} \quad x_1 + x_2$$

$$\text{s.t.} \quad x_1^2 + x_2^2 - 8 = 0$$

$$\nabla f(x) = (1, 1), \qquad \nabla^2 f(x) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\nabla h(x) = (2x_1, 2x_2), \qquad \nabla^2 h(x) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$\nabla h(x) \neq 0$, $\forall x \in \mathcal{H}$ $\Rightarrow$ all $x \in \mathcal{H}$ are regular

x^*, λ satisfy:

$$(x_1^*)^2 + (x_2^*)^2 = 8 \quad — (1)$$

$$1 + 2\lambda x_1^* = 0 \quad — (2)$$

$$1 + 2\lambda x_2^* = 0 \quad — (3)$$

$$(2), (3) \Rightarrow x_1^* = x_2^* = -\frac{1}{2\lambda}$$

$$(1) \Rightarrow (x_1^*)^2 = (x_2^*)^2 = 4$$

Two possible solutions: $x_1^* = x_2^* = 2$

and $x_1^* = x_2^* = -2$

For $x^* = (2, 2)$, $\lambda = -\frac{1}{4}$

For $x^* = (-2, -2)$, $\lambda = \frac{1}{4}$

Two possible candidates for local min:

$x^* = (2, 2)$, $\lambda = -\frac{1}{4}$ and $x^* = (-2\ -2)$, $\lambda = \frac{1}{4}$

Check second order condition:

$$\nabla^2_{xx} L(x^*, \lambda) = \nabla^2 f(x^*) + \lambda \nabla^2 h(x^*)$$

$$= 0 + 2\lambda I$$

$$= \begin{cases} -\frac{1}{2} I \prec 0 & \text{if } x^* = (2,2) \\ +\frac{1}{2} I \succ 0 & \text{if } x^* = (-2,-2) \end{cases}$$

I.e., $x^* = (-2, -2)$ is a (strict) local min by second order suff. condition

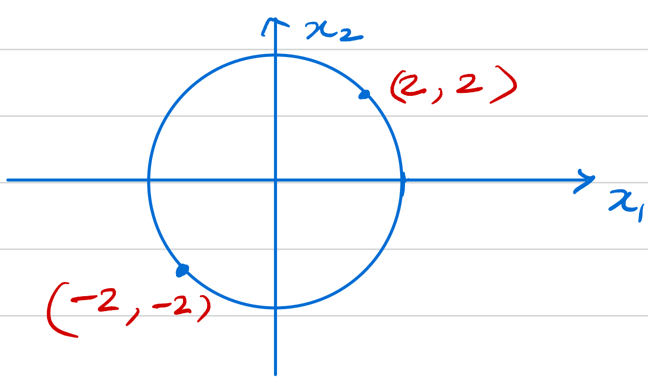
Note that $x^* = (2, 2)$ cannot be a local min by second order necc. condition

So $x^* = (-2, -2)$ is the unique local min

Is $x^* = (-2, -2)$ also the global min?

Yes, by WT since $\mathcal{H}$ is compact.

By considering minimizing $-f(x)$ over $\mathcal{H}$, can show that $(2, 2)$ is global max



INEQUALITY CONSTRAINTS

$$\text{minimize } f(x)$$

$$\text{s.t. } h_i(x) = 0, \quad i = 1, \dots, m \qquad \text{— (ICP)}$$
$$g_j(x) \leq 0, \quad j = 1, \dots, r$$

$\boxed{f, h_i, g_j \text{ cont. diff.}}$

constraints can be written compactly as:

$$h(x) = 0 \qquad h : \mathbb{R}^n \to \mathbb{R}^m$$
$$g(x) \leq 0 \qquad g : \mathbb{R}^n \to \mathbb{R}^r$$

Active v. Inactive Inequality constraints

The constraint $g_j(x) \leq 0$ is said to be active at x if $g_j(x) = 0$, and inactive if $g_j(x) < 0$

$$A(x) = \{ j \in \{1, \dots r\} : g_j(x) = 0 \}$$

↳ set of active inequality constraints

Claim: If x^* is a local min for (ICP), then x^* is also a local min for

$$\text{minimize } f(x)$$

$$\text{s.t. } h_i(x) = 0, \quad i = 1, \dots, m \qquad \text{— (ECP)}$$
$$g_j(x) = 0, \quad \forall j \in A(x^*)$$

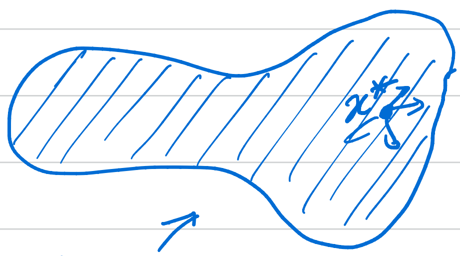
i.e., we can discard inactive inequality constraints in (ICP) in search for local minima

Intuition for (ICP) $\rightarrow$ (ECP)

Consider

$$\text{minimize } f(x) \quad \text{s.t. } g(x) \le 0, \quad g : \mathbb{R} \to \mathbb{R}. \qquad — (1)$$

If x^* is a local min of (1) and $g(x^*) < 0$



$\{x : g(x) \le 0\}$

By continuity of g,

$\{x : g(x) < 0\}$ is open

$\Rightarrow$ if $g(x^*) < 0$, then all directions are locally feasible at x^*

$\Rightarrow \quad \nabla f(x^*) = 0$ (lec 3)

- Constraint does not affect necessary condition.

This argument carries over to multiple $g_j(x^*) < 0$.

$\Rightarrow$ only active inequality constraints at x^* matter

If x^* is regular for (ECP), i.e., if

$\nabla h_i(x^*)$, $i = 1, \ldots, m$, and $\nabla g_j(x^*)$, $j \in A(x^*)$

are linearly independent, then $\exists$ Lagrange multipliers $\lambda_1, \ldots, \lambda_m$, μ_j, $j \in A(x^*)$ s.t.

$$\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla h_i(x^*) + \sum_{j \in A(x^*)} \mu_j \nabla g_j(x^*) = 0$$

- Follows from solution in lec 12 – 13.

It also turns out that $\boxed{\mu_j \ge 0, \; j \in A(x^*).}$

Intuition for why $\mu_j \geq 0$ $j \in A(x^*)$

$$\text{minimize } f(x)$$
$$\text{s.t. } h_i(x) = 0, \ i = 1, \dots, m$$
$$g_j(x) \leq 0, \ j = 1, \dots, r$$

If j^{th} constraint $g_j(x) \leq 0$ is relaxed to $g_j(x) \leq u_j$ for $u_j > 0$ (small), then

$$f(x^*(u_j)) \leq f(x^*(0)) = f(x^*)$$

Why? Optimization is over larger set

Now if $j \in A(x^*)$, then $g_j(x^*) = 0$.

By sensitivity interpretation of Lagrange multipliers for equality constraints (lec 13):

$$f(x^*(u_j)) = f(x^*) - \mu_j u_j + o(u_j)$$

$$\Rightarrow \quad -\mu_j u_j + o(u_j) = f(x^*(u_j)) - f(x^*) \leq 0$$

Dividing by u_j and letting $u_j \to 0 \Rightarrow \mu_j \geq 0$.

Complementary Slackness

$$\mu_j = 0 \quad \forall j \notin A(x^*)$$

$$\Leftrightarrow \quad \mu_j = 0 \text{ whenever } g_j(x^*) \neq 0, \ j = 1, \dots, r$$

$$\Leftrightarrow \quad \mu_j \, g_j(x^*) = 0, \ j = 1, \dots, r$$

Karush-Kuhn-Tucker (KKT) Necessary Conditions

Lagrangian function for (ICP):

$$L(x,\lambda,\mu) = f(x) + \sum_{i=1}^{m} \lambda_i h_i(x) + \sum_{j=1}^{r} \mu_j g_j(x)$$

Proposition (KKT) Let x^* be a local min. of (ICP) and assume that x^* is regular for (ECP). Then $\exists$ unique Lagrange multipliers $\lambda = (\lambda_1, \ldots, \lambda_m)$ and $\mu = (\mu_1, \ldots, \mu_r)$ s.t.

$$\nabla_x L(x^*, \lambda, \mu) = 0$$

$$\mu_j \geq 0, \quad j = 1, \ldots, r$$

$$\mu_j = 0, \quad \forall j \notin A(x^*)$$

If f, h_i, g_j are twice cont. diff., then

$$y^T \nabla^2_{xx} L(x^*, \lambda, \mu) y \geq 0$$

for all $y \in \mathbb{R}^n$ s.t.

$$\nabla h_i(x^*)^T y = 0, \ i = 1, \ldots, m, \quad \nabla g_j(x^*)^T y = 0, \ \forall j \in A(x^*)$$

Proof Convert (ICP) to:

minimize $f(x)$

s.t. $h_i(x) = 0, \ i = 1, \ldots, m$ — (P3)

$g_j(x) + z_j^2 = 0, \ j = 1, \ldots, r$

Auxilliary variables $z = (z_1, \ldots, z_r)$, $z_j \geq 0$

$$\text{minimize } f(x)$$
$$\text{s.t. } h_i(x) = 0, \quad i = 1, \dots, m \qquad \text{— (P3)}$$
$$g_j(x) + z_j^2 = 0, \quad j = 1, \dots, r$$

(P3) can be considered as optimization problem over x and $z = (z_1, \dots, z_r)$, with z only appearing in constraint.

Let x^* be a local min. for (ICP), then (x^*, z^*) is a local min. for (P3) with
$$z_j^* = (-g_j(x^*))^{\frac{1}{2}}, \quad j = 1, \dots, r.$$

Define Lagrangian for (P3):
$$L(x, z, \lambda, \mu) = f(x) + \sum_{i=1}^{m} \lambda_i h_i(x) + \sum_{j=1}^{r} \mu_j g_j(x) + \sum_{j=1}^{r} \mu_i z_i^2$$

From first-order necessary condition (lec 12), assuming that (x^*, z^*) is regular for (P3),

$$\nabla L(x^*, z^*, \lambda, \mu) = 0 \quad (\text{Note: } \nabla \text{ w.r.t. both } x \text{ and } z)$$

$$\Rightarrow \nabla_x f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla_x h_i(x^*) + \sum_{j=1}^{r} \mu_j \nabla_x g_j(x^*) = 0$$

$$\text{and} \quad \sum_{j=1}^{r} \mu_j \nabla_z (z_j^2) \Big|_{z = z^*} = 0$$

$$\nabla_x f(x) + \sum_{i=1}^{m} \lambda_i \nabla_x h_i(x) + \sum_{j=1}^{r} \mu_j \nabla_x g_j(x) = 0 \quad -(1)$$

and $$\sum_{j=1}^{r} \mu_j \nabla_z (z_j^2)\Big|_{z=z^*} = 0 \quad -(2)$$

Note: $\nabla_z (z_j^2) = [0, \cdots, 0, 2z_j, 0 \cdots 0]^T$

$\uparrow$ j^{th} component

(1) $\Rightarrow$ $\nabla_x L(x^*, \lambda, \mu) = 0$ with

$$L(x, \lambda, \mu) = f(x) + \sum_{i=1}^{m} \lambda_i h_i(x) + \sum_{j=1} \mu_j g_j(x)$$

(2) $\Rightarrow$ $$\begin{bmatrix} 2\mu_1 z_1^* \\ \vdots \\ 2\mu_r z_r^* \end{bmatrix} = 0 \Rightarrow \mu_j z_j^* = 0 \text{ for } j = 1, \ldots, r.$$

But $z_j^* = (-g_j(x^*))^{\frac{1}{2}} > 0 \quad \forall j \notin A(x^*)$

$\Rightarrow \mu_j = 0$ for $j \notin A(x^*)$

<u>Second-order necessary conditions for (P3):</u>

$$\begin{bmatrix} y \\ w \end{bmatrix}^T \nabla^2 L(x^*, z^*, \lambda, \mu) \begin{bmatrix} y \\ w \end{bmatrix} \geq 0$$

$\forall y \in \mathbb{R}^n$, $w \in \mathbb{R}^r$ satisfying

$$\nabla_x h_i(x^*)^T y = 0 \qquad i = 1, \ldots, m$$

and $$\nabla_x g_j(x^*)^T y + 2 z_j^* w_j = 0, \quad j = 1, \ldots, r.$$

$$L(x, z, \lambda, \mu) = f(x) + \sum_{i=1}^{m} \lambda_i h_i(x) + \sum_{j=1}^{r} \mu_i (g_i(x) + z_j^2)$$

$$\nabla^2 L(x, z, \lambda, \mu) = \begin{bmatrix} \nabla^2_{xx} L(x, \lambda, \mu) & 0 & \cdots & & 0 \\ 0 & \ddots & 2\mu_1 & 0 & 0 \\ \vdots & & & \ddots & \vdots \\ \vdots & & \ddots & & 0 \\ 0 & - & \cdots & 0 & 2\mu_r \end{bmatrix}_{(n+r) \times (n+r)}$$

$n \times n$

For every $j \in A(x^*)$ select (y, w) with

$y = 0, \; w_j \neq 0$ and $w_k = 0 \;\; \forall k \neq j$

Then (y, w) satisfies $\nabla h_i(x^*)^T y = 0, \; i = 1, \ldots, m$

and $\nabla g_k(x^*)^T y + 2 z_k^* w_k = 0 \quad \forall k \neq j$

and $\nabla g_j(x^*)^T y + \underbrace{2 z_j^*}_{=0} w_j = 0$ since $j \in A(x^*)$

Thus for this choice of (y, w)

$$\begin{bmatrix} y & w \end{bmatrix}^T \begin{bmatrix} \nabla^2_{xx} L(x, \lambda, \mu) & 0 & \cdots & & 0 \\ 0 & \ddots & 2\mu_1 & 0 & 0 \\ \vdots & & & \ddots & \vdots \\ \vdots & & \ddots & & 0 \\ 0 & - & \cdots & 0 & 2\mu_r \end{bmatrix} \begin{bmatrix} y \\ w \end{bmatrix} \geq 0$$

$$\Rightarrow \quad 2\mu_j w_j^2 \geq 0 \quad \Rightarrow \quad \mu_j \geq 0$$

Similarly we can show $\mu_j \geq 0 \;\; \forall j \in A(x^*)$.

Now choose $y \in \mathbb{R}^n$ s.t.

$$\nabla_x h_i(x^*)^T y = 0, \; i=1,\dots,m, \quad \nabla_x g_j(x^*)^T y = 0, \; \forall j \in A(x^*)$$

and w s.t.

$$w_j = \begin{cases} 0 & \text{if } j \in A(x^*) \\ -\dfrac{\nabla_x g_j(x^*)^T y}{2 z_j^*} & \text{if } j \notin A(x^*) \end{cases}$$

Then $\nabla_x g_j(x^*)^T y + 2 z_j^* w_j = 0 \quad \forall j \notin A(x^*)$

and $\nabla_x g_j(x^*)^T y + 2 z_j^* w_j = 0 \quad \forall j \in A(x^*)$

Thus $\nabla_x g_j(x^*)^T y + 2 z_j^* w_j = 0 \quad \forall j = 1, \dots, r$

Also, $w_j = 0 \;\; \forall j \in A(x^*)$ and $\mu_j = 0 \;\; \forall j \notin A(x^*)$

$$\Rightarrow \quad \mu_j w_j = 0 \quad \forall j = 1, \dots, r.$$

Thus, for the above choice of (y, w):

$$[y \;\; w]^T \begin{bmatrix} \nabla^2_{xx} L(x,\lambda,\mu) & 0 & \cdots & & 0 \\ 0 & 2\mu_1 & 0 & & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & & 0 & 2\mu_r \end{bmatrix} \begin{bmatrix} y \\ w \end{bmatrix} \geq 0$$

$$\Rightarrow \quad y^T \nabla^2_{xx} L(x, \lambda, \mu) y \geq 0$$

QED