Optimization with Equality Constraints

$$\text{minimize } f(x)$$

$$\text{subject to } \left.\begin{aligned} h_1(x) &= 0 \\ h_2(x) &= 0 \\ &\vdots \\ h_m(x) &= 0 \end{aligned}\right\} \text{ combined into } h(x) = 0$$

h ← vector-valued

$$f : \mathbb{R}^n \to \mathbb{R}, \quad h_i : \mathbb{R}^n \to \mathbb{R}, \quad h : \mathbb{R}^n \to \mathbb{R}^m$$

- If h is continuous, then $\mathcal{H} = \{x : h(x) = 0\}$ is closed but not necessarily convex
- Assume that h_i's are consistent so that $\mathcal{H}$ is non-empty

Examples

1) $h(x) = a^T x - b$

   $\mathcal{H} = \{x : a^T x = b\}$ — hyper-plane — convex

2) $h(x) = \|x\| - 1$

   $\mathcal{H} = \{x : \|x\| = 1\}$ — surface of sphere — not convex

Assume that f, h_i's are continuously differentiable. Under appropriate conditions, we will show that if x^* is a local min, then

$\exists\ \lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{R}$ s.t.

$$\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla h_i(x^*) = 0 \quad \text{— } (*)$$

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Two ways to interpret $(*)$:

(1) $\nabla f(x^*) = -\sum_{i=1}^{m} \lambda_i \nabla h_i(x^*)$, meaning that

$$\nabla f(x^*) \in \mathcal{G}(x^*) = \text{span}\{\nabla h_i(x^*),\ i=1,2,\dots,m\}$$

Note that $\mathcal{G}(x^*)$ is m-dimensional iff $\nabla h_i(x^*)$ are linearly independent (l.i.)

(2) $\nabla f(x^*)$ is orthogonal to the subspace:

$$\mathcal{V}(x^*) = \{z : \nabla h_i(x^*)^T z = 0,\ i=1,2,\dots,m\}$$

$\mathcal{V}(x^*)$ is the space of first-order feasible variations at x^*

Why? Feasible x must satisfy $h(x)=0$.

If z is a first-order feasible variation at x^*, $h(x^* + \alpha z) = 0$ for α sufficiently small

$$\Rightarrow \quad 0 = h_i(x^* + \alpha z) = \underbrace{h_i(x^*)}_{=0} + \alpha \nabla h_i(x^*)^T z + o(\alpha)$$

Dividing by α and taking $\alpha \to 0$

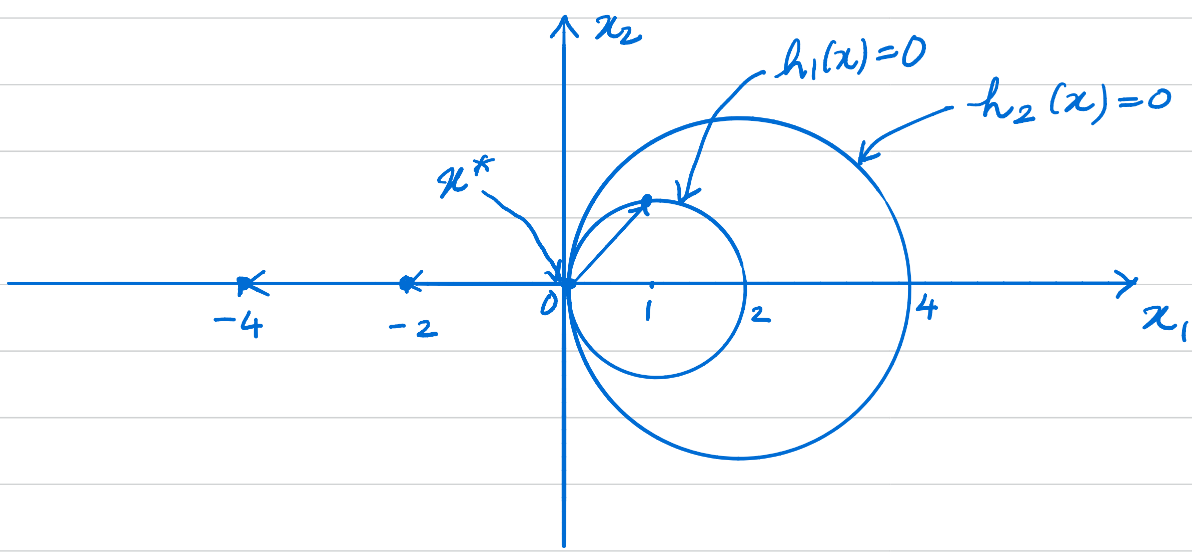
$$\nabla h_i(x^*)^T z = 0$$

Example (A problem with no Lagrange multipliers)

$$\text{minimize } f(x) = x_1 + x_2$$
$$\text{subject to } h_1(x) = (x_1-1)^2 + x_2^2 - 1 = 0$$
$$h_2(x) = (x_1-2)^2 + x_2^2 - 4 = 0$$

$\nabla f(x) = (1, 1)$

$\nabla h_1(x) = (2(x_1-1), 2x_2)$ $\quad \nabla h_2(x) = (2(x_1-2), 2x_2)$



Only feasible point $x = (0,0) \Rightarrow x^* = (0,0)$

But $\nabla f(x^*) = (1,1)$

$\nabla h_1(x^*) = (-2, 0)$ and $\nabla h_2(x^*) = (-4, 0)$

$\Rightarrow$ There is no choice of λ_1, λ_2 that makes

$$\nabla f(x^*) + \lambda_1 \nabla h_1(x^*) + \lambda_2 \nabla h_2(x^*) = 0$$

Reason: $\nabla h_1(x^*)$ and $\nabla h_2(x^*)$ are linearly dependent

Definition A (feasible) $x \in \mathcal{X}$, i.e., $h(x) = 0$, is called **regular** if $\nabla h_1(x), \nabla h_2(x), \ldots, \nabla h_m(x)$ are **linearly independent**.

Note For $\nabla h_1(x), \nabla h_2(x), \ldots, \nabla h_m(x)$ to be l.i., m has to be $\leq n$.

Lagrange Multiplier Theorem - First-order necessary condition: Let x^* be a local min. of $f(x)$ subject to $h(x) = 0$. Assume that $\nabla h_1(x^*), \nabla h_2(x^*), \ldots, \nabla h_m(x^*)$ are linearly independent. Then $\exists$ a unique $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ s.t.

$$\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla h_i(x^*) = 0 \quad — (*)$$

Remark If $m = n$, and $\nabla h_1(x^*), \ldots, \nabla h_m(x^*)$ are l.i., then $(*)$ holds trivially since $\nabla h_1(x^*), \ldots, \nabla h_m(x^*)$ form a basis for $\mathbb{R}^n$. So we need to only consider the case $m < n$ in the proof.

Proof of LMT (Penalty Approach)

Consider sequence of functions:

$$g^{(k)}(x) = f(x) + \underbrace{\frac{k}{2} \|h(x)\|^2}_{\text{penalty for violating } h(x)=0} + \overbrace{\frac{\alpha}{2}}^{>0} \underbrace{\|x - x^*\|^2}_{\text{strictly convex around } x^*}, \quad k = 1, 2, \ldots$$

$$g^{(k)}(x) = f(x) + \frac{k}{2}\|h(x)\|^2 + \frac{\alpha}{2}\|x - x^*\|^2, \quad k = 1, 2, \ldots$$

x^* local min. of $f(x)$ over $\mathcal{H} = \{x : h(x) = 0\}$ $\Rightarrow$ $\exists\, \varepsilon > 0$ s.t. $f(x^*) \le f(x)$ for all $x \in \mathcal{H} \cap \mathcal{S}$,

$\mathcal{S} = \{x : \|x - x^*\| \le \varepsilon\}$

Let $x^{(k)}$ be an optimal solution to:

$$\begin{aligned} &\text{minimize} && g^{(k)}(x) \\ &\text{subject to} && x \in \mathcal{S} \end{aligned} \qquad \text{—} \quad (P^{(k)})$$

$\mathcal{S}$ is compact $\Rightarrow$ Optimal solution to $(P^{(k)})$ exists by Weierstrass' Theorem

<u>Lemma</u> $x^{(k)} \longrightarrow x^*$ as $k \to \infty$

<u>Proof</u> $g^{(k)}(x^{(k)}) = f(x^{(k)}) + \frac{k}{2}\|h(x^{(k)})\|^2 + \frac{\alpha}{2}\|x^{(k)} - x^*\|^2$

since $x^{(k)}$ minimizes $g^{(k)}$ over $\mathcal{S}$ and $x^* \in \mathcal{S}$ $\longrightarrow$ $\le g^{(k)}(x^*) = f(x^*)$ — (1)

Now, since $f(x^{(k)})$ is bounded over $\mathcal{S}$, $\forall k$, we must have $\lim_{k\to\infty} \|h(x^{(k)})\| = 0$. Otherwise LHS of (1) will blow up to ∞ as $k \to \infty$.

Thus every limit point of $x^{(k)}$, $\bar{x}$ must satisfy $h(\bar{x}) = 0$, i.e. $\bar{x} \in \mathcal{H}$ (feasible set)

$$(1) \Rightarrow \quad f(x^{(k)}) + \frac{\alpha}{2}\|x^{(k)} - x^*\|^2 \le f(x^*)$$

$$f(x^{(k)}) + \frac{\alpha}{2}\|x^{(k)} - x^*\|^2 \leq f(x^*)$$

$\Rightarrow$ $f(\bar{x}) + \frac{\alpha}{2}\|\bar{x} - x^*\|^2 \leq f(x^*)$ for every limit point $\bar{x}$.

Now $\bar{x} \in \mathcal{H}$, and $\bar{x} \in \mathcal{S}$ (since $x^{(k)} \in \mathcal{S}, \forall k$) and $\mathcal{S}$ is closed

Thus $\bar{x} \in \mathcal{H} \cap \mathcal{S}$, and therefore

$$f(x^*) \leq f(\bar{x})$$

$$\Rightarrow f(\bar{x}) + \frac{\alpha}{2}\|\bar{x} - x^*\|^2 \leq f(\bar{x})$$

$$\Rightarrow \|\bar{x} - x^*\| = 0 \Longrightarrow \bar{x} = x^*.$$

Thus $\lim_{k\to\infty} x^{(k)}$ exists and $= x^*$. ∎

Recall $\mathcal{S} = \{x : \|x - x^*\| \leq \varepsilon\}$

$\Rightarrow$ x^* is an interior point of $\mathcal{S}$

$\Rightarrow$ $x^{(k)}$ is an interior point of $\mathcal{S}$ for k sufficiently large

$\Rightarrow$ $\nabla g^{(k)}(x^{(k)}) = 0$ for k suff. large

Recall $g^{(k)}(x) = f(x) + \frac{k}{2}\sum_{i=1}^{m}(h_i(x))^2 + \frac{\alpha}{2}\|x - x^*\|^2$

$$\nabla g^{(k)}(x) = \nabla f(x) + k\sum_{i=1}^{m} h_i(x)\nabla h_i(x) + \alpha(x - x^*)$$

$$\nabla g^{(k)}(x) = \nabla f(x) + k \sum_{i=1}^{m} h_i(x) \nabla h_i(x) + \alpha (x - x^*)$$

Define $\underbrace{\nabla h(x)}_{n \times m \text{ matrix}} = [\nabla h_1(x) \ \nabla h_2(x) \ \cdots \ \nabla h_m(x)]$

$\nabla h(x^*)$ has rank m (columns are l.i.)

$\Rightarrow$ $\nabla h(x^{(k)})$ has rank m for suff. large k

by continuity of $\nabla h_i(x)$, $i = 1, \ldots, m$.

If $A_{n \times m}$ $(n \geq m)$ has rank m, then $A^T A$ is invertible. $(A^T A)^{-1} A^T$ is called the pseudo-inverse of A, denoted as A^+, and

$$A z = b$$

has the unique solution $z = (A^T A)^{-1} A^T b = A^+ b$

For k suff. large

$$\nabla g^{(k)}(x) = \nabla f(x) + k \nabla h(x) h(x) + \alpha (x - x^*)$$

$$0 = \nabla g^{(k)}(x^{(k)}) = \nabla f(x^{(k)}) + k \nabla h(x^{(k)}) h(x^{(k)}) + \alpha (x^{(k)} - x^*)$$

$$\Rightarrow k \nabla h(x^{(k)}) h(x^{(k)}) = -\left(\nabla f(x^{(k)}) + \alpha (x^{(k)} - x^*)\right)$$

$$\Rightarrow k h(x^{(k)}) = -\left(\nabla h(x^{(k)})\right)^+ \left(\nabla f(x^{(k)}) + \alpha (x^{(k)} - x^*)\right)$$

$$\Rightarrow \lim_{k \to \infty} k h(x^{(k)}) = -\left(\nabla h(x^*)\right)^+ \nabla f(x^*) \triangleq \lambda$$

(uniqueness from uniqueness of limit)

$$0 = \nabla g^{(k)}(x^{(k)}) = \nabla f(x^{(k)}) + k\,\nabla h(x^{(k)})\,h(x^{(k)}) + \alpha\,(x^{(k)} - x^*)$$

$$\Rightarrow \nabla f(x^{(k)}) + \nabla h(x^{(k)})\,(k\,h(x^{(k)})) + \alpha\,(x^{(k)} - x^*) = 0$$

Taking limits as $k \to \infty$

$$\Rightarrow \nabla f(x^*) + \nabla h(x^*)\,\lambda = 0$$

$$\boxed{\text{i.e.} \quad \nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla h_i(x^*) = 0}$$