

Power Allocation Problem

P = total power available at $Tx > 0$

x_i = power allocated to channel $i \geq 0, \forall i$

σ_i^2 = noise power on channel $i > 0, \forall i$

$$\# \text{ bits carried by channel } i = \log\left(1 + \frac{x_i}{\sigma_i^2}\right)$$

Goal Maximize $\sum_{i=1}^n \log\left(1 + \frac{x_i}{\sigma_i^2}\right)$

s.t. $x_i \geq 0, \forall i, \quad \sum_{i=1}^n x_i \leq P.$

Note Often functions that we are optimizing are only well-defined on subsets of \mathbb{R}^n . The subset of \mathbb{R}^n for which $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is well-defined is called the **domain** of f , denoted $\text{dom}(f)$

Examples

- $f: \mathbb{R} \rightarrow \mathbb{R} \quad f(x) = \log(x) \quad \text{dom}(f) = \{x: x > 0\}$

- $f: \mathbb{R} \rightarrow \mathbb{R} \quad f(x) = \frac{1}{x} \quad \text{dom}(f) = \{x: x \neq 0\}$

- $f: \mathbb{R}^n \rightarrow \mathbb{R}, \quad f(x) = \sum_{i=1}^n \log\left(1 + \frac{x_i}{\sigma_i^2}\right)$

$$\text{dom}(f) = \left\{x: 1 + \frac{x_i}{\sigma_i^2} \geq 0, i=1, \dots, n\right\}$$

$$\text{PAP} \quad \text{Maximize} \quad \sum_{i=1}^n \log \left(1 + \frac{x_i}{\sigma_i^2} \right)$$

s.t. $x_i \geq 0, \forall i, \quad \sum_{i=1}^n x_i \leq P.$

$$\text{Define } f(x) = - \sum_{i=1}^n \log \left(1 + \frac{x_i}{\sigma_i^2} \right)$$

$$g_i(x) = -x_i, \quad i=1, \dots, n$$

$$g_o(x) = \sum_{i=1}^n x_i - P$$

Then PAP can be written as :

$$\begin{aligned} & \text{minimize} && f(x) \\ & x \in \text{dom}(f) \end{aligned}$$

$$\text{s.t. } g_i(x) \leq 0, \quad i=0, 1, \dots, n$$

$$L(x, \mu) = - \sum_{i=1}^n \log \left(1 + \frac{x_i}{\sigma_i^2} \right) - \sum_{i=1}^n \mu_i x_i + \mu_0 \left(\sum_{i=1}^n x_i - P \right)$$

$$\frac{\partial L(x, \mu)}{\partial x_i} = -\frac{1}{1 + \frac{x_i}{\sigma_i^2}} \cdot \frac{1}{\sigma_i^2} - \mu_i + \mu_0$$

$$\text{For } i=1, \dots, n, \quad \nabla g_i(x) = (0, \dots, 0, \overset{i\text{th position}}{-1}, 0, \dots, 0)$$

$$\nabla g_o(x) = (1, 1, \dots, 1)$$

Therefore, $\nabla g_o(x), \nabla g_1(x), \dots, \nabla g_n(x)$ are l.d.

But constraints cannot all be active simultaneously!

Why? If $x_i = 0, \forall i=1, \dots, n$, then $\sum x_i \neq P > 0$

Also, any strict subset of $\nabla g_o(x), \dots, \nabla g_n(x)$ are l.i.

\Rightarrow all feasible x are regular

$$\text{KKT conditions: } \nabla_x L(x^*, \mu) = 0$$

$$\mu_i \geq 0, \quad i=0, 1, \dots, n$$

$$\mu_i x_i^* = 0, \quad i=1, \dots, n$$

$$\mu_0 \left(\sum_{i=1}^n x_i^* - P \right) = 0$$

$$\nabla L(x^*, \mu) = 0 \Rightarrow -\frac{1}{1 + \frac{x_i^*}{\sigma_i^2}} \cdot \frac{1}{\sigma_i^2} - \mu_i + \mu_0 = 0, \quad i=1, \dots, n$$

$$\Rightarrow \mu_i = \mu_0 - \frac{1}{\sigma_i^2 + x_i^*}, \quad i=1, \dots, n$$

$$\mu_i \geq 0 \Rightarrow \mu_0 \geq \frac{1}{\sigma_i^2 + x_i^*}, \quad i=1, \dots, n$$

$$> 0 \quad \text{since } x_i^* \geq 0.$$

$$\text{Thus } \sum_{i=1}^n x_i^* = P \quad (\text{constraint is active})$$

For $i=1, \dots, n$, if $\mu_i > 0$, then $x_i^* = 0$

$$\Rightarrow \mu_i = \mu_0 - \frac{1}{\sigma_i^2} > 0$$

$$\Rightarrow \mu_0 > \frac{1}{\sigma_i^2} \Rightarrow \frac{1}{\mu_0} < \sigma_i^2$$

$$\text{Therefore, } \frac{1}{\mu_0} \geq \sigma_i^2 \Rightarrow \mu_i = 0$$

$$\Rightarrow \mu_0 = \frac{1}{\sigma_i^2 + x_i^*}$$

$$\Rightarrow x_i^* = \frac{1}{\mu_0} - \sigma_i^2$$

$$\text{Set } x_i^* = \left(\frac{1}{M_0} - \sigma_i^{-2} \right)^+ = \max\{0, \frac{1}{M_0} - \sigma_i^{-2}\}$$

with M_0 satisfying :

$$P = \sum_{i=1}^n x_i^* = \sum_{i=1}^n \left(\frac{1}{M_0} - \sigma_i^{-2} \right)^+.$$

Check KKT complementary slackness For $i=1, \dots, n$,

$$x_i^* = 0 \Rightarrow \frac{1}{M_0} - \sigma_i^{-2} \leq 0$$

$$\Rightarrow M_i = M_0 - \frac{1}{\sigma_i^{-2}} \geq 0$$

$$x_i^* > 0 \Rightarrow x_i^* = \frac{1}{M_0} - \sigma_i^{-2}$$

$$\Rightarrow M_i = M_0 - \frac{1}{\sigma_i^{-2} + x_i^*} = M_0 - M_0 = 0$$

$$\text{Thus } x_i^* M_i = 0$$

$$L(x, \mu) = - \sum_{i=1}^n \log \left(1 + \frac{x_i}{\sigma_i^{-2}} \right) - \sum_{i=1}^n M_i x_i + M_0 \left(\sum_{i=1}^n x_i - P \right)$$

↑ convex in x

(strictly) convex in μ

$$\text{dom}(L) = \text{dom}(f) = \{x: 1 + \frac{x_i}{\sigma_i^{-2}} > 0\} \quad \begin{matrix} \text{convex} \\ \text{open} \end{matrix}$$

Since $\nabla L(x^*, \mu) = 0$ and $x^* \in \text{dom}(L)$,

$$L(x^*, \mu) = \min_{x \in \text{dom}(f)} L(x, \mu)$$

By general sufficiency x^* is the global min for PAP

Alternative Way to solve PAP

$$f(x) = - \sum_{i=1}^n \log \left(1 + \frac{x_i}{\sigma_i^2} \right)$$

$$g_i(x) = -x_i, \quad i=1, \dots, n$$

$$g_0(x) = \sum_{i=1}^n x_i - P$$

minimize $f(x)$

$x \in \text{dom}(f)$

s.t. $g_i(x) \leq 0, \quad i=0, 1, \dots, n$

Define $\mathcal{S} = \{x : g_i(x) \leq 0, i=1, \dots, n\}$.

$= \{x : x_i \geq 0, i=1, \dots, n\}$

Note that $\mathcal{S} \subset \text{dom}(f) = \{x_i : 1 + \frac{x_i}{\sigma_i^2} > 0\}$.

Therefore we can rewrite PAP as :

minimize $f(x)$

$x \in \mathcal{S}$

s.t. $g_0(x) \leq 0$

Only one inequality constraint.

$$L(n, \mu) = - \sum_{i=1}^n \log \left(1 + \frac{x_i}{\sigma_i^2} \right) + \mu \left(\sum_{i=1}^n x_i - P \right)$$

\mathcal{S} is a closed and convex set.

Using General Sufficiency to solve PAP directly

x^*, μ need to satisfy:

$$\sum_{i=1}^n x_i^* \leq P$$

$$\mu \geq 0$$

$$\mu \left(\sum_{i=1}^n x_i - P \right) = 0$$

$$L(x^*, \mu) = \min_{x \in \mathcal{S}} L(x, \mu)$$

To find x^*, μ that satisfy conditions, assume first that $\mu > 0$ and see if this works.

$$L(x, \mu) = - \sum_{i=1}^n \log \left(1 + \frac{x_i}{\sigma_i^2} \right) + \mu \left(\sum_{i=1}^n x_i - P \right)$$

L is (strictly) convex in x , and L is closed, convex
 \Rightarrow nec. and sufficient condition for x^* to be (unique)
minimizer of $L(x, \mu)$ over \mathcal{S} is (see lec 10):

$$\nabla L(x^*, \mu)^T (x - x^*) \geq 0 \quad \forall x \in \mathcal{S} \quad -(1)$$

To find x^* that satisfies (1), set

$$\nabla L(x^*, \mu) = 0$$

i.e. $\frac{\partial}{\partial x_i} L(x^*, \mu) = \mu - \frac{1}{x_i^* + \sigma_i^2} = 0, i=1, \dots, n$

$$\Rightarrow x_i^* = \frac{1}{\mu} - \sigma_i^2$$

But $x_i^* = \frac{1}{\mu} - \sigma_i^{-2} \geq 0$ only if $\frac{1}{\mu} \geq \sigma_i^{-2}$

So set $x_i^* = \left(\frac{1}{\mu} - \sigma_i^{-2} \right)^+$

To check that $x_i^* = \left(\frac{1}{\mu} - \sigma_i^{-2} \right)^+, \forall i$ satisfies

$$\nabla L(x^*, \mu)^T (x - x^*) \geq 0 \quad \forall x \in \mathcal{S},$$

Note that for i s.t. $\frac{1}{\mu} > \sigma_i^{-2}$, $\frac{\partial}{\partial x_i} L(x^*, \mu) = 0$

For i s.t. $\frac{1}{\mu} \leq \sigma_i^{-2}$, $x_i^* = 0$

$$\Rightarrow \frac{\partial}{\partial x_i} L(x^*, \mu) = \mu - \frac{1}{\sigma_i^{-2} + 0} \geq 0$$

Therefore,

$$\begin{aligned} \nabla L(x^*, \mu)^T (x - x^*) &= \sum_{i=1}^n \frac{\partial}{\partial x_i} L(x^*, \mu) \cdot (x_i - x_i^*) \\ &= \underbrace{\sum_{i: x_i^* = 0} \frac{\partial}{\partial x_i} L(x^*, \mu) \cdot x_i}_{\geq 0} \underbrace{\geq 0}_{\text{on } \mathcal{S}} \end{aligned}$$

$$\Rightarrow \nabla L(x^*, \mu)^T (x - x^*) \geq 0 \quad \forall x \in \mathcal{S}$$

$$\Rightarrow L(x^*, \mu) = \min_{x \in \mathcal{S}} L(x, \mu).$$

(and x^* is unique)

Only one remaining condition to check:

$$\mu \left(\sum_{i=1}^n x_i^* - p \right) = 0$$

Since we assumed $\mu > 0$, need to see if we can find μ s.t. $\sum_{i=1}^n x_i^* = p$.

i.e., we need to see if $\exists \mu > 0$ s.t.

$$\sum_{i=1}^n \left(\frac{1}{\mu} - x_i^2 \right)^+ = p \quad - (2)$$

But as μ goes from 0 to ∞ , LHS monotonically decreases from ∞ to 0.

$\Rightarrow \exists$ unique μ that satisfies (2).

How about the case where constraint is inactive,

i.e. $\mu = 0$ and $\sum_{i=1}^n x_i^* < p$?

(Don't need to check since minimizer x^* is unique.)

Let $\varepsilon \in (0, p - \sum_{i=1}^n x_i^*)$, and $\bar{x}_i = \begin{cases} x_i^* + \varepsilon & \text{if } i=1 \\ x_i^* & \text{if } i=2, \dots, n \end{cases}$

Then $\bar{x} \in S$ and $f_0(\bar{x}) \leq 0$ (feasible), but

$$-\log(1 + \bar{x}_1) < -\log(1 + x_1^*), \quad \forall \varepsilon \in (0, p - \sum_{i=1}^n x_i^*)$$

$$\Rightarrow f(\bar{x}) < f(x^*), \quad \forall \varepsilon \in (0, p - \sum_{i=1}^n x_i^*)$$

$\Rightarrow x^*$ cannot be a local min $\Rightarrow \Leftarrow$

$$x_i^* = \left(\frac{1}{\mu} - \sigma_i^2 \right)^+ \text{ and } \mu: \sum_{i=1}^n \left(\frac{1}{\mu} - \sigma_i^2 \right)^+ = P$$

satisfy the general sufficiency conditions:

$$\sum_{i=1}^n x_i^* \leq P$$

$$\mu \geq 0$$

$$\mu \left(\sum_{i=1}^n x_i^* - P \right) = 0$$

$$L(x^*, \mu) = \min_{x \in \mathbb{R}} L(x, \mu)$$

$\Rightarrow x^*$ is global min for PAP.

Water-Filling Interpretation

