Barrier Method

Computational method to solve inequality constrained problems. Applies to problems of form:

$$\text{minimize } f(x) \qquad \text{— (P)}$$
$$\text{s.t. } x \in \mathcal{X} \text{ and } g_j(x) \leq 0, \; j = 1, \dots, r$$

where $\mathcal{X}$ is closed set.

<u>Barrier Function</u> $B(x)$ is a function that is continuous and $\to \infty$ as any $g_j(x) \to 0$.

<u>Examples</u>

$$B(x) = -\sum_{j=1}^{r} \ln(-g_j(x))$$

$$B(x) = -\sum_{j=1}^{r} \frac{1}{g_j(x)}$$

Note that if $g_j(x)$ is convex for all j, then both of these barrier functions are convex

In Barrier Method, choose sequence $\{\varepsilon_k\}$ s.t.

$$0 < \varepsilon_{k+1} < \varepsilon_k, \quad k = 0, 1, \dots$$

and $\quad \varepsilon_k \to 0 \quad$ as $\quad k \to \infty$.

Define feasible set $\mathcal{F} = S \cap \{g_j(x) \leq 0, \forall j\}$
Note that $\mathcal{F}$ is closed.

Let $x^{(k)}$ be a solution to:

$$\underset{x \in \mathcal{F} \cap \mathrm{dom}(B)}{\text{minimize}} \quad f(x) + \varepsilon_k B(x) \quad \text{—— } (P_k)$$

Since $B(x) \to \infty$ on the boundary of $\mathcal{F}$, $x^{(k)}$ must be an interior point of $\mathcal{F}$

$$\Rightarrow \quad \nabla f(x^{(k)}) + \varepsilon_k \nabla B(x^{(k)}) = 0$$

Therefore, if we have a initial point in the interior of $\mathcal{F}$, we can choose step size of any unconstrained GD method to stay in interior of $\mathcal{F}$ for all iterations and ~~solve~~ (P_k)

As $k \to \infty$, $\varepsilon_k \to 0$, and barrier becomes inconsequential, and we expect $x^{(k)}$ to approach minimum of original problem (P)

Proposition Every limit point $\bar{x}$ of $\{x^{(k)}\}$ is a global min of (P).

Proof Let $\bar{x} = \lim\limits_{\substack{k \to \infty \\ k \in \mathcal{K}}} x^{(k)}$.

Since $x^{(k)} \in \mathcal{F}$ for all k, and $\mathcal{F}$ is closed, $\bar{x} \in \mathcal{F}$.

Suppose x^* is a global min of (P) and x^* is in interior of $\mathcal{F}$, and

$f(x^*) < f(\bar{x})$, i.e. $\bar{x}$ is not global min for (P).

Then, by definition of $x^{(k)}$,

$$f(x^{(k)}) + \varepsilon_k B(x^{(k)}) \leq f(x^*) + \varepsilon_k B(x^*)$$

Taking limit as $k \to \infty$, $k \in \mathcal{K}$, since $|B(x^*)| < \infty$

$$f(\bar{x}) + \lim_{\substack{k\to\infty \\ k \in \mathcal{K}}} \varepsilon_k B(x^{(k)}) \leq f(x^*) + 0$$

If $\bar{x}$ is in interior of $\mathcal{F}$, then $|B(\bar{x})| < \infty$

$$\Rightarrow \lim_{\substack{k\to\infty \\ k \in \mathcal{K}}} \varepsilon_k B(x^{(k)}) = 0$$

If $\bar{x}$ is on boundary of $\mathcal{F}$, then $B(x^{(k)}) \to \infty$

$$\Rightarrow \lim_{\substack{k\to\infty \\ k \in \mathcal{K}}} \varepsilon_k B(x^{(k)}) \geq 0 .$$

Therefore $f(\bar{x}) \leq f(x^*)$ $\Rightarrow\Leftarrow$

i.e., $\bar{x}$ is a global min for (P)

If x^* is not in interior of $\mathcal{F}$, then we can assume that $\exists$ an interior point $\tilde{x}$ which can be made arbitrarily close to x^*. Proof holds using $\tilde{x}$ in place of x^*.

Barrier Method Applied to Linear Programs

$$\min \; c^T x$$
$$\text{s.t.} \quad Ax \leq b \text{ and } x \geq 0 .$$

Linear Program (LP) in Standard form

- LP's arise in many applications:
  - Agriculture — crops / quantity to maximize revenue
  - Transportation — airline routing, mail / package routing (travelling salesman problem)
  - Efficient Manufacturing
  - Optimizing power grid.
  - ...

- In this context, Barrier Method is also called Interior Point Method.

  - Common technique for solving LP's is simplex method (Dantzig, 1947)

  - Karmarkar (1984) showed that interior point method (with projective scaling) can solve large LP's much faster than simplex method

Barrier Method Example

$$\text{minimize} \quad f(x) = \frac{1}{2}(x_1^2 + x_2^2) \qquad \text{— (P)}$$
$$\text{s.t.} \quad x_1 \geq 2 .$$

Solution using KKT Conditions

$g(x) = 2 - x_1$.

$\nabla g(x) = (-1, 0) \Rightarrow$ All feasible x are regular

$\nabla f(x) = (x_1, x_2)$

$L(x, \mu) = f(x) + \mu g(x)$

$\nabla L(x, \mu) = \nabla f(x) + \mu \nabla g(x) = (x_1 - \mu, x_2)$.

Case 1 : Constraint inactive, i.e. $\mu = 0$

$\nabla L(x, \mu) = 0 \Rightarrow x = (0, 0)$

But $x = (0,0)$ does not satisfy $x_1 \geq 2 \Rightarrow$ infeasible

Case 2 : Constraint active

$\nabla L(x, \mu) = 0 \Rightarrow x_1 - \mu = 0, \; x_2 = 0$

$g(x) = 0 \Rightarrow x_1 = 2$

$\Rightarrow x^* = (2, 0), \; \mu = 2$ satisfies first-order KKT

$L(x, \mu)$ is strictly convex on $\mathbb{R}^2$.

$$\Rightarrow L(x^*, 2) = \min_{x \in \mathbb{R}^2} L(x, 2)$$

By general Sufficiency Condition $x^* = (2, 0)$ is the global min for (P).

$$\text{minimize } f(x) = \frac{1}{2}(x_1^2 + x_2^2)$$
$$\text{s.t. } x_1 \geq 2. \qquad \text{— (P)}$$

Logarithmic Barrier $B(x) = -\ln(-g(x)) = -\ln(x_1 - 2)$

$$x^{(k)} \in \arg\min_{x \in dom(g^{(k)})} g^{(k)}(x)$$

$$g^{(k)}(x) = \frac{1}{2}(x_1^2 + x_2^2) - \varepsilon_k \ln(x_1 - 2)$$

$g^{(k)}(x)$ is convex in x over $dom(g^{(k)}) = \{x : x_1 > 2\}$.

$$\nabla g^{(k)}(x) = 0 \Rightarrow x_1 - \frac{\varepsilon_k}{x_1 - 2} = 0$$
$$\text{and } x_2 = 0$$

i.e., $x_1^2 - 2x_1 - \varepsilon_k = 0$

$\Rightarrow x_1 = 1 \pm \sqrt{1 + \varepsilon_k}$. But $x_1 < 0$ not possible

$\Rightarrow x_1 = 1 + \sqrt{1 + \varepsilon_k}$

Therefore, $x^{(k)} = (1 + \sqrt{1 + \varepsilon_k}, 0)$

and as $k \to \infty$, $\varepsilon_k \to 0$ and $x^{(k)} \to (2, 0) = x^*$.

Penalty Method Computational method for solving:

$$\begin{aligned} &\text{minimize } f(x) \\ &\text{s.t. } x \in \mathcal{X} \\ &\quad h_i(x) = 0, \quad i = 1, \dots, m \end{aligned} \quad \text{— } (P)$$

Algorithm

① choose an increasing positive sequence $\{c_k\}$ s.t. $c_k \to \infty$ as $k \to \infty$.

② Solve for $x^{(k)}$ to:

$$\underset{x \in \mathcal{X}}{\text{minimize}} \; f(x) + c_k \|h(x)\|^2 \quad \text{— } (P_k)$$

Note: $\|h(x)\|^2 = \sum_{i=1}^{m} (h_i(x))^2$

Result Every limit point $\bar{x}$ of $\{x^{(k)}\}$ is a global min of (P) if $\mathcal{X}$ is closed.

Proof Let $\bar{x} = \lim_{\substack{k \to \infty \\ k \in \mathcal{K}}} x^{(k)}$.

$$f^* = \min_{\substack{x \in \mathcal{X} \\ h(x)=0}} f(x) = \min_{\substack{x \in \mathcal{X} \\ h(x)=0}} f(x) + c_k \|h(x)\|^2$$

$$\geq \min_{x \in \mathcal{X}} f(x) + c_k \|h(x)\|^2$$

$$= f(x^{(k)}) + c_k \|h(x^{(k)})\|^2$$

$$\Rightarrow c_k \|h(x^{(k)})\|^2 \leq f^* - f(x^{(k)}) \quad \text{—— } (1)$$

By continuity of f, $\lim_{\substack{k\to\infty \\ k\in\mathcal{K}}} f(x^{(k)}) = f(\bar{x})$

Thus, as $k\to\infty$, $k\in\mathcal{K}$, the RHS of (1) goes to $f^* - f(\bar{x})$ which is finite.

Since $c_k\to\infty$ as $k\to\infty$, $k\in\mathcal{K}$,

$$\lim_{\substack{k\to\infty \\ k\in\mathcal{K}}} \|h(x^{(k)})\|^2 = 0$$

By continuity of $\|h(x)\|^2$,

$$\lim_{\substack{k\to\infty \\ k\in\mathcal{K}}} \|h(x^{(k)})\|^2 = \|h(\bar{x})\|^2 = 0$$

Now, since $\mathcal{S}$ is closed, and $x^{(k)}\in\mathcal{S}$ for all k, $\bar{x}\in\mathcal{S}$ as well.

Now from (1),

$$f^* - f(x^{(k)}) \geq c_k \|h(x^{(k)})\|^2 \geq 0$$

$$\Rightarrow f(x^{(k)}) \leq f^*$$

$$\Rightarrow \lim_{\substack{k\to\infty \\ k\in\mathcal{K}}} f(x^{(k)}) \leq f^*$$

$$\Rightarrow f(\bar{x}) \leq f^*$$

But $\bar{x}\in\mathcal{S}$ and $h(\bar{x})=0$, which means $\bar{x}$ is feasible and $f(\bar{x}) \leq f^*$

$\Rightarrow \bar{x}$ is a global min for (P).

Example minimize $f(x) = x_1 + x_2 - 2x_1x_2$

s.t. $x_1 + x_2 - 2 = 0$

Easy to show using LMT that $x^* = (1,1)$ is global min.

$$g^{(k)}(x) = f(x) + c_k \|h(x)\|^2$$
$$= x_1 + x_2 - 2x_1x_2 + c_k(x_1 + x_2 - 2)^2$$

$$\nabla g^{(k)}(x) = \begin{bmatrix} 1 - 2x_2 + 2c_k(x_1 + x_2 - 2) \\ 1 - 2x_1 + 2c_k(x_1 + x_2 - 2) \end{bmatrix}$$

$$\nabla^2 g^{(k)}(x) = \begin{bmatrix} 2c_k & 2(c_k - 1) \\ 2(c_k - 1) & 2c_k \end{bmatrix} = 2\begin{bmatrix} c_k & c_k - 1 \\ c_k - 1 & c_k \end{bmatrix}$$

$$\det\left(\nabla^2 g^{(k)}(x)\right) = 2\left(c_k^2 - (c_k - 1)^2\right)$$
$$= 4c_k - 2$$

If $c_k > \frac{1}{2}$, then $\nabla^2 g^{(k)}(x) > 0$

$\Rightarrow x^{(k)}$ is solution to $\nabla g^{(k)}(x) = 0$

$$2c_k x_1 + 2(c_k - 1)x_2 + 1 - 4c_k = 0$$
$$2c_k x_2 + 2(c_k - 1)x_1 + 1 - 4c_k = 0$$

$$\Rightarrow x_1^{(k)} = x_2^{(k)} = \frac{4c_k - 1}{4c_k - 2}$$

As $k \to \infty$, $c_k \to \infty$, and $x^{(k)} \to (1,1) = x^*$