Duality

$$\text{minimize } f(x)$$
$$\text{s.t. } x \in \mathcal{X}$$
$$h_i(x) = 0 \text{ for } i = 1, \dots, m$$
$$g_j(x) \leq 0 \text{ for } j = 1, \dots, r$$

$$L(x, \lambda, \mu) = f(x) + \underbrace{\sum_{i=1}^{m} \lambda_i h_i(x)}_{\lambda^T h(x)} + \underbrace{\sum_{j=1}^{r} \mu_j g_j(x)}_{\mu^T g(x)}$$

Dual Function

$$D(\lambda, \mu) = \min_{x \in \mathcal{X}} L(x, \lambda, \mu) \quad \text{(assuming min. exists)}$$

on convex set

$$\mathcal{C} = \{(\lambda, \mu) : \lambda \in \mathbb{R}^m, \ \mu_j \geq 0, j = 1, \dots, r\}$$

<u>Result</u> $D(\lambda, \mu)$ is concave on $\mathcal{C}$

<u>Proof</u> Let (λ, μ) and $(\tilde{\lambda}, \tilde{\mu}) \in \mathcal{C}$.

For $\alpha \in [0, 1]$,

$$D(\alpha\lambda + (1-\alpha)\tilde{\lambda}, \alpha\mu + (1-\alpha)\tilde{\mu})$$

$$= \min_{x \in \mathcal{X}} f(x) + (\alpha\lambda + (1-\alpha)\tilde{\lambda})^T h(x) + (\alpha\mu + (1-\alpha)\tilde{\mu})^T g(x)$$

$$= \min_{x \in \mathcal{X}} \alpha[f(x) + \lambda^T h(x) + \mu^T g(x)] + (1-\alpha)[f(x) + \tilde{\lambda}^T h(x) + \tilde{\mu}^T g(x)]$$

$$\geq \min_{x \in \mathcal{X}} \alpha[f(x) + \lambda^T h(x) + \mu^T g(x)] + \min_{x \in \mathcal{X}} (1-\alpha)[f(x) + \tilde{\lambda}^T h(x) + \tilde{\mu}^T g(x)]$$

Thus,

$$D(\alpha\lambda + (1-\alpha)\tilde{\lambda}, \alpha\mu + (1-\alpha)\tilde{\mu})$$
$$\geq \alpha D(\lambda, \mu) + (1-\alpha) D(\tilde{\lambda}, \tilde{\mu}) \qquad \boxtimes$$

Weak Duality

Define the feasibility set

$$\mathcal{F} = \{x : x \in \mathcal{S}, h(x) = 0, g(x) \leq 0\}$$

$$\max_{(\lambda,\mu) \in \mathcal{C}} D(\lambda, \mu) \leq \min_{x \in \mathcal{F}} f(x)$$

Proof For $(\lambda, \mu) \in \mathcal{C}$, $x \in \mathcal{F}$,

$$L(x,\lambda,\mu) = f(x) + \lambda^T h(x) + \mu^T g(x) \leq f(x)$$

$$\Rightarrow \min_{x \in \mathcal{F}} L(x, \lambda, \mu) \leq f(x) \text{ for all } x \in \mathcal{F}$$
$$\leq \min_{x \in \mathcal{F}} f(x) = f^*$$

Since $\mathcal{F} \subseteq \mathcal{S}$,

$$\min_{x \in \mathcal{S}} L(x, \lambda, \mu) \leq f^*$$

i.e. $D(\lambda, \mu) \leq f^* \quad \forall (\lambda, \mu) \in \mathcal{C}$

$$\Rightarrow \max_{(\lambda,\mu) \in \mathcal{C}} D(\lambda, \mu) \leq f^*.$$

Strong Duality Under some conditions, equality holds, i.e.,

$$\underbrace{\max_{(\lambda,\mu)\in \mathcal{C}} D(\lambda,\mu)}_{\text{dual problem}} = \underbrace{\min_{x\in\mathcal{F}} f(x)}_{\text{primal problem}}$$

Result Suppose f is convex, h_i's are affine, g_j's are convex, and $\mathcal{S} = \mathbb{R}^n$. If x^* is an optimal solution for primal problem, x^* is regular, and (λ^*, μ^*) are corresponding Lagrange multipliers, then strong duality holds and (λ^*, μ^*) maximize $D(\lambda, \mu)$.

Proof Under regularity assumption, using first-order KKT necessary conditions,

$$\nabla_x L(x^*, \lambda^*, \mu^*) = 0 \quad \text{--- (1)}$$

and

$$\mu^{*T} g(x^*) = 0 \quad \text{--- (2)}$$

Since f is convex, h_i's are affine, g_j's are convex, $L(x, \lambda^*, \mu^*)$ is convex in x. Thus, by (1),

$$L(x^*, \lambda^*, \mu^*) = \min_{x\in\mathbb{R}^n} L(x, \lambda^*, \mu^*)$$

$$\leq \max_{(\lambda,\mu)\in\mathcal{C}} \min_{x\in\mathbb{R}^n} L(x,\lambda,\mu) \quad \text{--- (3)}$$

Furthermore,

$$L(x^*, \lambda^*, \mu^*) = f(x^*) + \lambda^{*T}\underbrace{h(x^*)}_{=0} + \underbrace{\mu^{*T}g(x^*)}_{=0}$$

$$= f(x^*)$$

$$\geq f(x^*) + \lambda^T h(x^*) + \mu^T g(x^*)$$

for all $(\lambda, \mu) \in \mathcal{C}$

$$= L(x^*, \lambda, \mu), \quad \forall (\lambda, \mu) \in \mathcal{C}$$

$$\Rightarrow \quad L(x^*, \lambda^*, \mu^*) \geq \max_{(\lambda,\mu)\in\mathcal{C}} L(x^*, \lambda, \mu)$$

$$\geq \min_{x\in\mathbb{R}^n} \max_{(\lambda,\mu)\in\mathcal{C}} L(x, \lambda, \mu) \quad -(4)$$

From (3) and (4),

$$\min_{x\in\mathbb{R}^n} \max_{(\lambda,\mu)\in\mathcal{C}} L(x,\lambda,\mu) \leq L(x^*, \lambda^*, \mu^*) \leq \max_{(\lambda,\mu)\in\mathcal{C}} \min_{x\in\mathbb{R}^n} L(x,\lambda,\mu) \quad -(5)$$

<u>Lemma</u> Consider function $g(y,z)$, $y \in \mathcal{Y}$, $z \in \mathcal{Z}$.

$$\max_{z\in\mathcal{Z}} \min_{y\in\mathcal{Y}} g(y,z) \leq \min_{y\in\mathcal{Y}} \max_{z\in\mathcal{Z}} g(y,z)$$

<u>Proof</u> HW 5

By Lemma,

$$\max_{(\lambda,\mu)\in \mathcal{C}} \min_{x\in\mathbb{R}^n} L(x,\lambda,\mu) \leq \min_{x\in\mathbb{R}^n} \max_{(\lambda,\mu)\in\mathcal{C}} L(x,\lambda,\mu) \quad -(6)$$

From (5), (6) we get

$$\max_{(\lambda,\mu)\in \mathcal{C}} \min_{x\in\mathbb{R}^n} L(x,\lambda,\mu) = \min_{x\in\mathbb{R}^n} \max_{(\lambda,\mu)\in\mathcal{C}} L(x,\lambda,\mu) = L(x^*,\lambda^*,\mu^*)$$

$$\max_{(\lambda,\mu)\in\mathcal{C}} L(x,\lambda,\mu) = \max_{(\lambda,\mu)\in\mathcal{C}} f(x) + \lambda^T h(x) + \mu^T g(x)$$

$$= \begin{cases} \infty \text{ if } h(x)\neq 0 \text{ or } g(x) \nleq 0, \text{ i.e. } x\notin \mathcal{F} \\ f(x) \text{ if } x\in\mathcal{F} \end{cases}$$

$$\Rightarrow \min_{x\in\mathbb{R}^n} \max_{(\lambda,\mu)\in\mathcal{C}} L(x,\lambda,\mu) = \min_{x\in\mathcal{F}} f(x)$$

Also $\displaystyle\max_{(\lambda,\mu)\in\mathcal{C}} \min_{x\in\mathbb{R}^n} L(x,\lambda,\mu) = \max_{(\lambda,\mu)\in\mathcal{C}} D(\lambda,\mu)$

Therefore $\displaystyle\max_{(\lambda,\mu)\in\mathcal{C}} D(\lambda,\mu) = \min_{x\in\mathcal{F}} f(x)$

Furthermore,

$$\max_{(\lambda,\mu)\in\mathcal{C}} D(\lambda,\mu) = L(x^*,\lambda^*,\mu^*) = D(\lambda^*,\mu^*).$$

i.e., (λ^*,μ^*) solve dual problem.

If the optimization problem is a linear program and it is feasible, Then strong duality holds (always).

Why? Lagrange Multipliers always exist

$$L(x,\lambda,\mu) = \underbrace{f(x) + \sum_{i=1}^{m} \lambda_i h_i(x) + \sum_{j=1}^{r} \mu_j g_j(x)}_{\text{convex}}$$

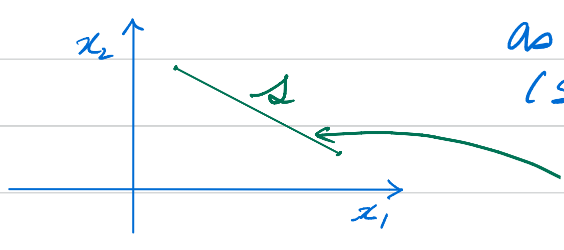
All Conditions of strong Duality Theorem met

Strong Duality result can be generalized to $\mathcal{S}$ being a convex subset of $\mathbb{R}^n$, but in this Case, we need the existence of a point x in the <u>relative interior</u> of $\mathcal{S}$ s.t. $g_j(x) < 0 \ \forall j$ and $h_i(x) = 0 \ \forall i$.

This is called <u>Slater's Condition</u>

<u>Relative Interior</u> of convex set $\mathcal{S}$.

Interior of $\mathcal{S}$ when it is viewed as a subset of its affine hull (See Appendix B.1 of text)



This $\mathcal{S}$ has no interior in $\mathbb{R}^2$ but has rel. int. (on line)

Example (From HW 4)

$$\text{minimize } x_1^2 + x_2^2 - 4x_1 - 2x_2 + 2$$

$$\text{s.t. } \quad x_1 + x_2 \leq 2, \quad x_1 + 2x_3 \leq 3$$

We showed that $x^* = \left(\frac{3}{2}, \frac{1}{2}\right)$ is the global min with $\mu^* = (1, 0)$.

$$f^* = f(x^*) = -\frac{5}{2}.$$

What is the dual of this convex program?

$$L(x, \mu) = x_1^2 + x_2^2 - 4x_1 - 2x_2 + 2 + \mu_1(x_1 + x_2 - 2) + \mu_2(x_1 + 2x_2 - 3).$$

$$= x_1^2 + x_2^2 + (\mu_1 + \mu_2 - 4)x_1 + (\mu_1 + 2\mu_2 - 2)x_2 + 2 - 2\mu_1 - 3\mu_2$$

$$D(\mu) = \min_{x \in \mathbb{R}^2} L(x, \mu) \longleftarrow \text{convex in } x$$

min satisfies $\nabla_x L(x, \mu) = 0$

i.e.
$$2x_1 + (\mu_1 + \mu_2 - 4) = 0$$
$$2x_2 + (\mu_1 + 2\mu_2 - 2) = 0$$

i.e. $x_1^* = \dfrac{-(\mu_1 + \mu_2 - 4)}{2}$, $x_2^* = \dfrac{-(\mu_1 + 2\mu_2 - 2)}{2}$

$$D(\mu) = -\left(\frac{\mu_1+\mu_2-4}{2}\right)^2 - \left(\frac{\mu_1+2\mu_2-2}{2}\right)^2$$
$$+ 2 - 2\mu_1 - 3\mu_2$$

Dual Problem

$$\underset{\mu_1 \geq 0,\ \mu_2 \geq 0}{\text{maximize}} \quad D(\mu) = \underset{\mu_1 \geq 0,\ \mu_2 \geq 0}{\text{minimize}} \quad -D(\mu)$$

Note: $$-\nabla D(\mu) = \begin{bmatrix} \mu_1 + \frac{3\mu_2}{2} - 1 \\ \frac{3\mu_1}{2} + \frac{5\mu_2}{2} - 1 \end{bmatrix}$$

$$-\nabla^2 D(\mu) = \begin{bmatrix} 1 & \frac{3}{2} \\ \frac{3}{2} & \frac{5}{2} \end{bmatrix} \succ 0$$

$\Rightarrow$ $-D(\mu)$ is (strictly) convex (as it should be)

For the optimization problem,

$$\min_{\mu_1 \geq 0,\ \mu_2 \geq 0} \quad -D(\mu)$$

$f(\mu) = -D(\mu)$, $g_1(\mu) = -\mu_1$, $g_2(\mu) = -\mu_2$

$\nabla g_1(\mu) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $\nabla g_2(\mu) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ — l.i

$\Rightarrow$ all μ are regular

Let the Lagrangian parameters be $\nu = (\nu_1, \nu_2)$

$$L(\mu, \nu) = -D(\mu) - \nu_1 \mu_1 - \nu_2 \mu_2$$

$$\nabla_\mu L(\mu, \nu) = -\nabla D(\mu) - \nu$$

KKT cond.: $\nabla_\mu L(\mu, \nu) = 0$, $\nu_i \geq 0$, $\nu_i \mu_i = 0$, $i = 1, 2$

$$\nabla_\mu L(\mu, \nu) = 0 \Rightarrow \left.\begin{aligned} \mu_1 + \tfrac{3}{2}\mu_2 - 1 - \nu_1 &= 0 \\ \tfrac{3}{2}\mu_1 + \tfrac{5}{2}\mu_2 - 1 - \nu_2 &= 0 \end{aligned}\right\} - (1)$$

does not reveal which constraints active/inactive

Case 1 Both inactive: $(1) \Rightarrow \mu_1 + \frac{3}{2}\mu_2 = 1 \Rightarrow \mu_2 = -\frac{1}{3}$

$(\nu_1 = 0, \nu_2 = 0)$ $\quad \mu_1 + \frac{5}{2}\mu_2 = \frac{2}{3}$ invalid

Case 2 1 active, 2 inactive: $(1) \Rightarrow \frac{3}{2}\mu_2 - \nu_1 = 1 \Rightarrow \nu_1 = \frac{-2}{5}$

$(\mu_1 = 0, \nu_2 = 0)$ $\quad \frac{5}{2}\mu_2 = 1$ invalid

Case 3 1 inactive, 2 active: $(1) \Rightarrow \mu_1 = 1$

$(\nu_1 = 0, \mu_2 = 0)$ $\quad \frac{3}{2}\mu_1 - \nu_2 = 1$

$\Rightarrow \mu_1 = 1, \nu_2 = \frac{1}{2}$ candidate for local min

Case 4 Both active: $(1) \Rightarrow \left.\begin{aligned} \nu_1 &= -1 \\ \nu_2 &= -1 \end{aligned}\right)$ Invalid

$\mu_1 = \mu_2 = 0$

Since $L(\mu, \nu)$ is (strictly) convex in μ, by general sufficiency, $\mu_1^* = 1$, $\mu_2^* = 0$ maximizes $D(\mu)$

$$D(\mu^*) = -\frac{9}{4} - \frac{1}{4} + 2 - 2 = -\frac{5}{2} = f(x^*)$$