

## Constrained Optimization & Gradient Projection

Lec 10

$$\min_{x \in S} f(x)$$

where  $S$  is a (non-empty) closed and convex subset of  $\mathbb{R}^n$

- Assume that  $f$  is continuously differentiable on  $S$ .



Definition  $x^*$  is a local min. of  $f$  over  $S$  if  $\exists \varepsilon > 0$

s.t.  $f(x^*) \leq f(x) \quad \forall x \in S$  with  $\|x - x^*\| < \varepsilon$ .

$x^*$  is global min if  $f(x^*) \leq f(x) \quad \forall x \in S$ .

### Proposition (Optimality conditions)

(a) If  $x^*$  is a local min of  $f$  over  $S$ , then

$$\nabla f(x^*)^\top (x - x^*) \geq 0 \quad \forall x \in S \quad \text{--- (A)}$$

(b) If  $f$  is convex over  $S$ , then (A) is also sufficient for  $x^*$  to be a (global) min.

Proof (a) Suppose  $x^*$  is a local min. and  
 $\nabla f(x^*)^\top (x - x^*) < 0$  for some  $x \in \mathcal{S}$ .

Let  $g(\varepsilon) = f(x^* + \varepsilon(x - x^*))$ .

$$\text{Then } g'(\varepsilon) = \nabla f(x^* + \varepsilon(x - x^*))^\top (x - x^*).$$

By MVT,  $g(\varepsilon) = g(0) + \varepsilon g'(0)$  for some  $\beta \in [0, 1]$

$$\Rightarrow f(x^* + \varepsilon(x - x^*)) = f(x^*) + \varepsilon \nabla f(x^* + \beta\varepsilon(x - x^*))^\top (x - x^*)$$

for some  $\beta \in [0, 1]$ .

Since  $\nabla f$  is continuous, we have that for all suff.

$$\text{small } \varepsilon > 0, \quad \nabla f(x^* + \beta\varepsilon(x - x^*))^\top (x - x^*) < 0$$

$$\Rightarrow f(x^* + \varepsilon(x - x^*)) < f(x^*)$$

Since  $x^* + \varepsilon(x - x^*) = \varepsilon x + (1-\varepsilon)x^* \in \mathcal{S}$  (convexity)

$x^*$  cannot be a local min over  $\mathcal{S}$   $\iff$ .

(b) convexity of  $f$  over  $\mathcal{S}$

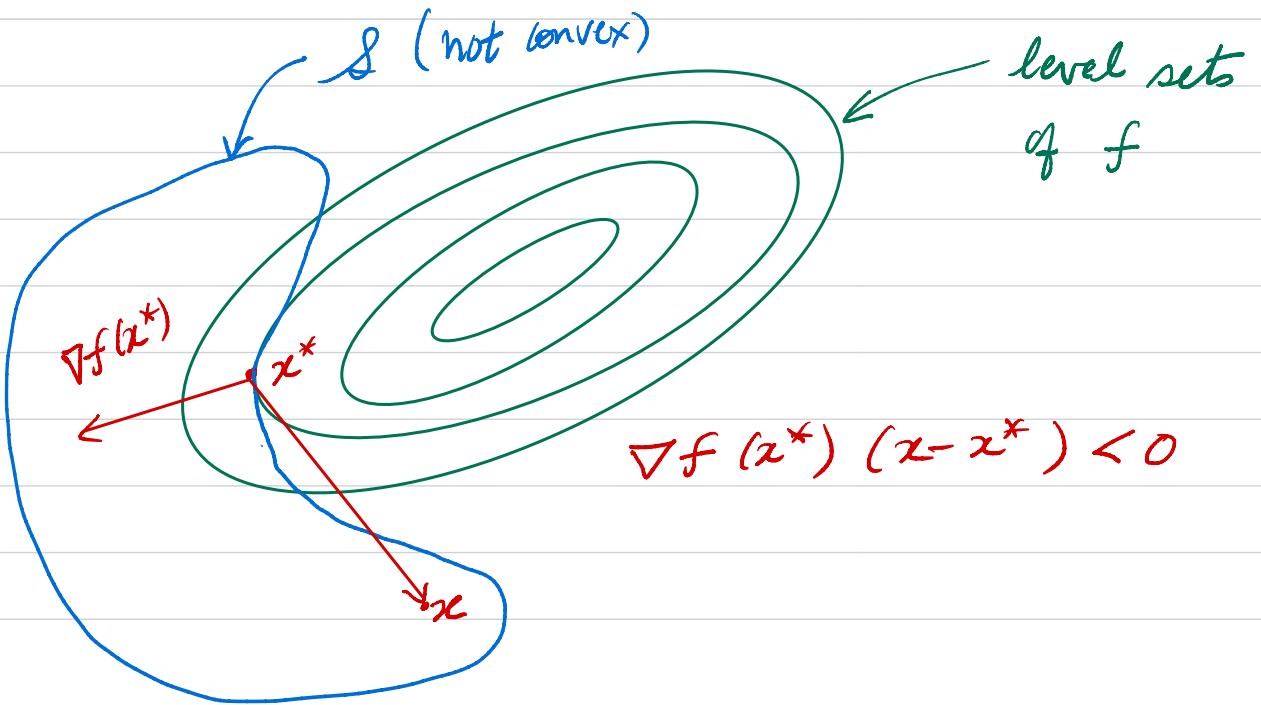
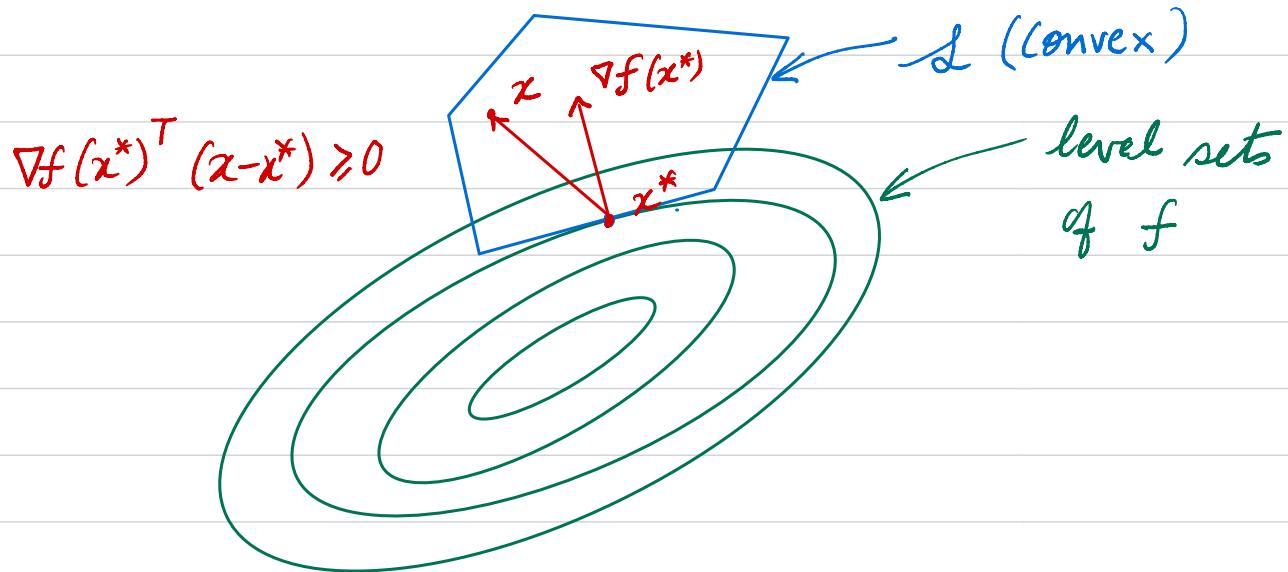
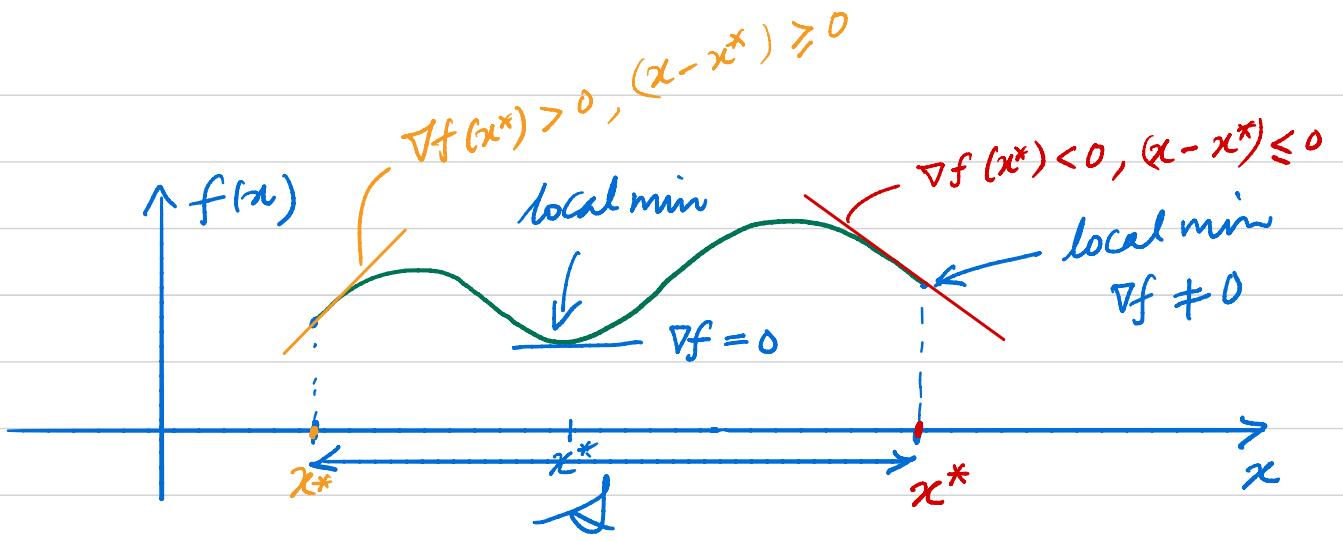
$$\Rightarrow f(x) \geq f(x^*) + \nabla f(x^*)^\top (x - x^*), \quad \forall x \in \mathcal{S}.$$

Thus, with convexity

$$\nabla f(x^*)^\top (x - x^*) \geq 0 \quad \forall x \in \mathcal{S}$$

$$\Rightarrow f(x) \geq f(x^*) \quad \forall x \in \mathcal{S}$$

$\Rightarrow x^*$  is a global min of  $f$  over  $\mathcal{S}$



Definition  $y$  is an interior point of  $S$  if  $\exists \varepsilon > 0$   
 s.t.  $B_\varepsilon = \{x : \|y-x\| < \varepsilon\} \subset S$ .

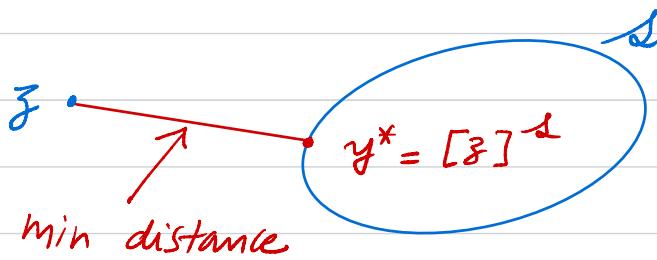
Remark If  $x^*$  is an interior point of  $S$ , then from lec 3,  
 $x^*$  is local min  $\Rightarrow \nabla f(x^*) = 0$ . Also, if  $f$  is convex,  
 $x^*$  is a global min  $\Leftrightarrow \nabla f(x^*) = 0$  (lec 4)

## Projection onto Closed Convex Sets

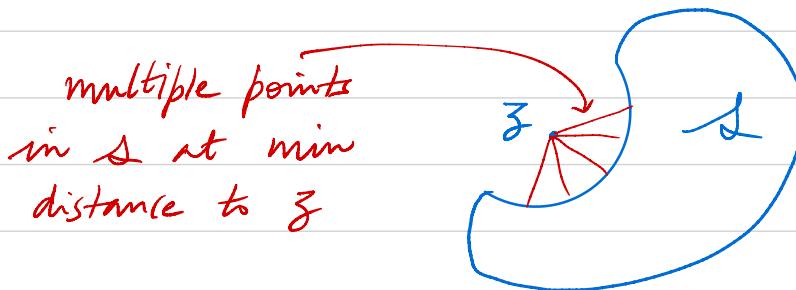
Definition Let  $S$  be a closed convex subset of  $\mathbb{R}^n$ . Then, for  $z \in \mathbb{R}^n$ , the projection of  $z$  on  $S$  is denoted by  $[z]^S$  and is given by

$$[z]^S = \arg \min_{y \in S} \|z-y\|^2$$

(" = " justified since we will show that  $[z]^S$  exists and is unique.)



If  $S$  is not convex minimizer may not be unique



(not necessarily bounded)

Result (Existence and Uniqueness of Projection)

Let  $S$  be a closed convex subset of  $\mathbb{R}^n$ .

Then, for every  $z \in \mathbb{R}^n$ , there exists a unique  $[z]^\perp$ .

Proof Need to show that  $\min_{y \in S} \|y - z\|^2$  exists and is unique.

Let  $x$  be some element of  $S$ . Then

minimizing  $\|z - y\|^2$  over all  $y \in S$

$\equiv$  minimizing  $\|z - y\|^2$  over the set

$$A = \{y \in S : \|z - y\|^2 \leq \|z - x\|^2\}$$

$g(y) = \|z - y\|^2$  is strictly convex on set  $S$

$\Rightarrow A$  is convex set and  $g$  is convex on  $A$

Also  $g$  is continuous  $\Rightarrow A$  is closed

Finally,  $y \in A \Rightarrow \|y\| = \|y - z + z\|$

$$\begin{aligned} &\xrightarrow{\text{Triangle ineq}} \|y - z\| + \|z\| \\ &\leq \|z - x\| + \|z\| \end{aligned}$$

$\Rightarrow A$  is bounded

Thus,  $g(y) = \|z - y\|^2$  is strictly convex over set  $A$ , which is compact. Therefore,

$$\min_{y \in S} \|z - y\|^2 = \min_{y \in A} \|z - y\|^2 \text{ exists and is unique (WT)}$$

(strict convexity)

Result (Necessary and Sufficient Condition for Projection)

Let  $S$  be a closed convex subset of  $\mathbb{R}^n$ . Then

$$\begin{aligned} [\bar{z}]^S = y^* &\iff (\bar{z} - y^*)^T (y - y^*) \geq 0, \quad \forall y \in S \\ &\iff (\bar{z} - y^*)^T (y - y^*) \leq 0, \quad \forall y \in S. \end{aligned}$$

Proof

$$[\bar{z}]^S = \arg \min_{y \in S} g(y), \text{ with } g(y) = \|\bar{z} - y\|^2$$

$\uparrow$  strictly convex

$$\nabla g(y) = 2(y - \bar{z})$$

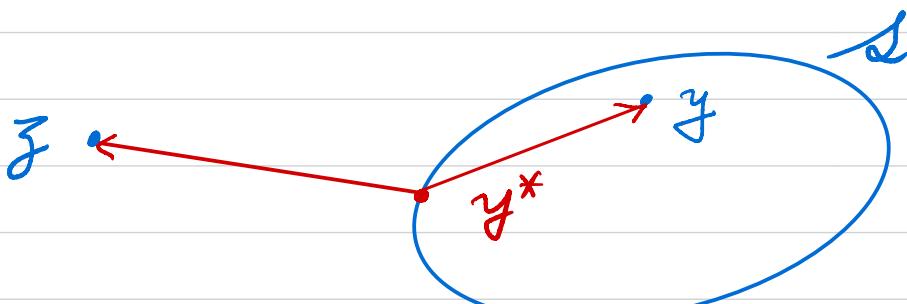
From optimality condition for minimizing a convex function over closed convex set  $S$ ,

$y^*$  is the unique minimizer of  $g(y)$  over  $S$

$$\iff \nabla g(y^*)^T (y - y^*) \geq 0 \quad \forall y \in S$$

$$\iff (\bar{z} - y^*)^T (y - y^*) \geq 0 \quad \forall y \in S,$$

$$\iff (\bar{z} - y^*)^T (y - y^*) \leq 0 \quad \forall y \in S$$



$$(\bar{z} - y^*)^T (y - y^*) \leq 0.$$

Result (Projection is non-expansive)

Let  $S$  be a closed convex subset of  $\mathbb{R}^n$ . Then for  $x, z \in \mathbb{R}^n$ ,

$$\|x^\perp - z^\perp\| \leq \|x - z\|$$

Proof From previous result,

$$(1) \cdots (x^\perp - x)^T (y - x^\perp) \geq 0 \quad \forall y \in S$$

$$(2) \cdots (z^\perp - z)^T (y - z^\perp) \geq 0 \quad \forall y \in S$$

Setting  $y = z^\perp$  in (1) and  $y = x^\perp$  in (2), and adding, we get

$$(z^\perp - x^\perp)^T (x^\perp - x + z - z^\perp) \geq 0$$

$$\Rightarrow (z^\perp - x^\perp)^T (z - x) \geq \|z^\perp - x^\perp\|^2$$

Applying Cauchy-Schwarz inequality to LHS,

$$\|z^\perp - x^\perp\|^2 \leq \|z^\perp - x^\perp\| \|z - x\| \quad \blacksquare$$

