University of Illinois **Spring 2023**

ECE 490: Problem Set 5

Due: Tuesday, April 11 through Gradescope by 11 AM
Reading: Lecture Notes 16-19; Secs 4.3, 5.1, 5.2, and 6.1 of text.

1. **[General Sufficiency Condition]**
   In this problem you will establish a sufficiency condition for the optimization problem:

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & x \in \mathcal{S} \\ & h_i(x) = 0,\ i = 1, \ldots, m \\ & g_j(x) \leq 0,\ j = 1, \ldots, r \end{aligned} \tag{1}$$

Let

$$L(x, \lambda, \mu) = f(x) + \sum_{i=1}^{m} \lambda_i h_i(x) + \sum_{j=1}^{r} \mu_j g_j(x) = f(x) + \lambda^\top h(x) + \mu^\top g(x).$$

Suppose (x^*, λ, μ) satisfy:

$$\begin{aligned} x^* &\in \mathcal{S} \\ h_i(x^*) &= 0,\ i = 1, \ldots, m \\ g_j(x^*) &\leq 0,\ j = 1, \ldots, r \\ \mu_j &\geq 0,\ j = 1, \ldots, r \\ \mu_j g_j(x^*) &= 0,\ j = 1, \ldots, r \end{aligned}$$

and

$$L(x^*, \lambda, \mu) = \min_{x \in \mathcal{S}} L(x, \lambda, \mu). \tag{2}$$

Then, x^* is a global minimum for (1).

   (a) Show that

$$f(x^*) = L(x^*, \lambda, \mu)$$

   (b) Now use (2) along with part (a) to show that:

$$f(x^*) \leq \min_{x \in \mathcal{S}, h(x)=0, g(x) \leq 0} L(x, \lambda, \mu)$$

   (c) Conclude that x^* is a global minimum for (1).

2. **[Penalty Method]**
   Consider the following optimization problem:

$$\begin{aligned} \text{minimize} \quad & x_1^2 + x_2^2 \\ \text{subject to} \quad & x_1 + x_2 = 2. \end{aligned}$$

   (a) Find the global minimum using the Lagrange Multiplier Theorem.
   (b) Now apply the basic quadratic penalty method from Lec 17 to the optimization problem and verify that it converges to the same solution as in part (a).

3. **[min-max v. max-min]**
Consider the function $g(y, z)$, with $y \in \mathcal{Y}$ and $z \in \mathcal{Z}$. Assuming both sides of the inequality exist, show that:

$$\max_{y \in \mathcal{Y}} \min_{z \in \mathcal{Z}} g(y, z) \leq \min_{z \in \mathcal{Z}} \max_{y \in \mathcal{Y}} g(y, z).$$

4. **[KKT]**
Using the KKT conditions, find the global minimum for the following optimization problem:

$$\begin{aligned} \text{minimize} \quad & x_1^2 + x_2^2 - 4x_1 - 4x_2 \\ \text{subject to} \quad & 2x_1 \leq x_2 \\ & x_1 + x_2 = 3 \end{aligned}$$

5. **[Duality]**
Consider the optimization problem given above in Problem 4.

   (a) Find the dual function $D(\lambda, \mu)$.

   (b) Using the KKT conditions, solve the dual problem :

$$\begin{aligned} \text{maximize} \quad & D(\lambda, \mu) \\ \text{subject to} \quad & \mu \geq 0 \end{aligned}$$

   Note: To verify that your solution is correct, you can check the strong duality condition.

6. **[Dual of Linear Program with Inequality Constraints]**
Find the dual of the following linear program:

$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{subject to} \quad & Ax - b \leq 0 \end{aligned}$$