University of Illinois **Spring 2022**

ECE 490: Problem Set 1

Due: Tuesday January 31 through Gradescope by 11 AM
Reading: Lecture Notes 1-4; Appendix A, Appendix B, and Sec 1.1 of text.

1. **[Open and Closed Sets]**
   Let $g : \mathbb{R}^n \to \mathbb{R}$ be continuous function.
   (a) Show that the set $\mathcal{H} = \{x : g(x) = 0\}$ is a closed set.
   Hint: Use the "limit-point" definition of closure. Suppose there is a sequence $\{y_k\} \subset \mathcal{H}$ that converges to y, i.e., $\lim_{k\to\infty} y_k = y$, then show that $y \in \mathcal{H}$.
   (b) Show that the set $\mathcal{A} = \{x : g(x) \leq 0\}$ is a closed set.
   Hint: It may be easier to show that $\mathcal{A}^c = \{x : g(x) > 0\}$ is open. For this you need to show that for every $y \in \mathcal{A}^c$, there exists $\delta > 0$ such that $B_\delta(y) = \{x : \|x - y\| < \delta\} \subset \mathcal{A}^c$. Use the "$\epsilon - \delta$" definition of continuity to show this.
   (c) Show that the set $\mathcal{S} = \{x : g(x) \geq 0\}$ is a closed set.

2. **[Minima and Maxima]**
   Suppose $f(x) = f(x_1, x_2, x_3) = x_1^2 + x_2^2 + 2e^{-x_3} + e^{x_3}$.
   (a) Does f achieve its minimum and maximum over the set:
   $$\mathcal{S}_1 = \{x \in \mathbb{R}^3 : x_1^2 + 2x_2^2 + 4x_3^2 \leq 10\}?$$
   (b) Does f achieve its minimum and maximum over $\mathbb{R}^3$?
   (c) Does f achieve its minimum and maximum over the set:
   $$\mathcal{S}_2 = \{x \in \mathbb{R}^3 : x_1^2 + 2x_2^2 \leq 3\}?$$

3. **[Trace]**
   Prove the following things about the traces of square matrices.
   (a) Let matrices A, B, and C be such that ABC is a square matrix. Then show that:
   $$\mathrm{Tr}(ABC) = \mathrm{Tr}(CAB) = \mathrm{Tr}(BCA).$$
   (b) Using eigen-decomposition, show that for a symmetric $n \times n$ matrix A with eigen-values $\lambda_1, \lambda_2, \ldots, \lambda_n$,
   $$\mathrm{Tr}(A) = \sum_{i=1}^{n} \lambda_i$$

4. **[PSD, PD, NSD, ND]**
   Identify which of these matrices is PSD, PD, NSD, or ND, or none of the above (indefinite):
   $$A = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} -3 & -3 & 0 \\ -3 & -5 & 1 \\ 0 & 1 & -8 \end{bmatrix}, \quad D = \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix}.$$

5. **[Convex Sets]**
We showed in class that if $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function, then:

$$\mathcal{S} = \{x \in \mathbb{R}^n : f(x) \leq a\}$$

is a convex set for all $a \in \mathbb{R}$. Give an example to illustrate that if f is not convex, the above set $\mathcal{S}$ may not be convex.

6. **[Using Optimality Conditions to Find Optima]**
Consider $f(x) = x^4 - 18x^2 + 81,\ x \in \mathbb{R}$.

   (a) Find all the extreme (stationary) points of f.

   (b) Identify which of these stationary points is a local minimum, local maximum, or neither.

   (c) Find the global minimum and global maximum of f if they exist.

7. **[Convexity]**
Show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by:

$$f(x) = \ln\left(e^{x_1} + e^{x_2}\right)$$

is convex.

8. **[Convexity of Quadratic Function]**
Determine if the following function is convex, concave, or neither:

$$f(x) = f(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 + 0.5x_3^2 + 2x_1x_3 + 3x_2x_3 + x_1 + x_2 + x_3 + 2.$$

9. **[Minimum and Maximum of Quadratic Function]**
Suppose:

$$f(x) = f(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 + 0.5x_3^2 + 2x_1x_2 - 2x_1 - 4x_2 - x_3 + 3.$$

Find the minimum and maximum of f over $\mathbb{R}^3$ if they exist.