University of Illinois **Spring 2023**

ECE 490: Problem Set 4

Due: Tuesday March 28 through Gradescope by 11 AM
Reading: Lecture Notes 12-16; Secs 4.2 and 4.3 of text.

1. **[Optimization with Linear Equality Constraint]**
   In this problem you will use the Lagrange Multiplier Theorem to find the global minimum of

   $$f(x) = 2x_1 + x_2 - x_1x_2$$

   subject to the constraint $2x_1 + x_2 = 2$.

   (a) Let $\mathcal{H} = \{x \in \mathbb{R}^2 : 2x_1 + x_2 = 2\}$ denote the constraint set. Show that f achieves its (global) minimum on $\mathcal{H}$.

   (b) Use the necessary conditions from the Lagrange Multiplier Theorem to find candidates for local minima of f on $\mathcal{H}$. Don't forget to check regularity conditions.

   (c) Use the second order sufficiency condition to find the global minimum if it exists.

2. **[Optimization with Quadratic Equality Constraint]**
   Use the Lagrange Multiplier Theorem to find the global minimum of $x_1 + 2x_2$ subject to $x_1^2 + x_2^2 = 5$.

3. **[Optimization with Linear Inequality Constraints - I]**
   Consider the function:

   $$f(x) = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}$$

   and convex set

   $$\mathcal{S} = \{x \in \mathbb{R}^3 : x_i > 0,\ i = 1, 2, 3\}.$$

   Our goal here is to find the global minimum for the following optimization problem:

   $$\begin{aligned} \underset{x \in \mathcal{S}}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & x_1 + x_3 \le 1 + \frac{1}{\sqrt{2}} \\ & x_2 + x_3 \le 1 + \frac{1}{\sqrt{2}} \end{aligned}$$

   (a) Use the KKT necessary conditions to find candidates for the local minima for this optimization problem. Don't forget to check regularity conditions.

   (b) Use the general sufficiency condition to find the global minimum for the optimization problem if it exists.

4. **[Optimization with Linear Inequality Constraints - II]**
   Using the KKT conditions, find the global minimum for the following optimization problem:

   $$\begin{aligned} \text{minimize} \quad & x_1^2 + x_2^2 - 4x_1 - 2x_2 + 2 \\ \text{subject to} \quad & x_1 + x_2 \le 2 \\ & x_1 + 2x_2 \le 3 \end{aligned}$$

5. **[KKT Conditions]**
   Assuming that $n \ge 2$, use the KKT conditions to find the global minimum for the following optimization

problem:

$$\begin{array}{ll} \text{minimize} & -\log(1+x_n) - \sum_{i=1}^{n-1} \log x_i \\ \text{subject to} & x_i + x_n \leq 1,\ i = 1, \ldots, n-1 \\ & x_i \geq 0,\ i = 1, \ldots, n \end{array}$$

Hint: First note that x_i cannot be 0 for $i = 1, \ldots, n-1$, i.e., the constraints $x_i \geq 0$ are *inactive* for $i = 1, \ldots, n-1$. Then show that the constraints $x_i + x_n \leq 1$, $i = 1, \ldots, n-1$ all have to be *active*. Then you will be left with only two cases to consider, i.e., whether the constraint $x_n \geq 0$ is *active* or *inactive.*