University of Illinois **Spring 2023**

ECE 490: Problem Set 6

Due: Thursday, April 27 through Gradescope by 11 AM
Reading: Lecture Notes 19-23; Secs 3.6, 5.2, 7.5 of text.

1. **[Convergence of Augmented Lagrangian Method]**
Consider the optimization problem:

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & h(x) = 0. \end{aligned} \tag{1}$$

In Lec 20 we discussed solving this optimization problem using the augmented Lagrangian:

$$L_c(x, \lambda) = f(x) + \lambda^\top h(x) + c\|h(x)\|^2, \quad \lambda \in \mathbb{R}^m, c > 0.$$

Now suppose $\{c_k\}$ is a sequence of positive numbers that increases to ∞ as $k \to \infty$, and let

$$x^{(k)} \in \arg\min_x L_{c_k}(x, \lambda).$$

Then show that every limit point $\bar{x}$ of the sequence $\{x^{(k)}\}$ is a global minimum for (1) (assuming that the global min exists).

Hint: Follow the steps for the proof of a similar result given in Lec 18.

2. **[Properties of Subgradients]**
Prove the following properties of subgradients (here f, f_1 and f_2 are convex functions):
   (i) Scaling: For scalar $a > 0$, $\partial(af) = a\partial f$, i.e., g is a subgradient of f at x iff ag is a subgradient of af at x.
   (ii) Addition: If g_1 is a subgradient of f_1 at x, and g_2 is a subgradient of f_2 at x, then $g_1 + g_2$ is subgradient of $f_1 + f_2$ at x.
   (iii) Affine Combination: Let $h(x) = f(Ax + b)$, with A being a square, invertible matrix. Then $\partial h(x) = A^\top \partial f(Ax + b)$, i.e., g is a subgradient of f at $Ax + b$ iff $A^\top g$ is a subgradient of h at x.

3. **[Subgradient Computation]**
Consider the continuous function

$$f(x) = \begin{cases} -(x+1) & \text{if } x < -1 \\ 0 & \text{if } -1 \leq x \leq 1 \\ x^2 - 1 & \text{if } x > 1 \end{cases}$$

   (a) Show that f is a convex function.
   (b) Draw a plot of $f(x)$ versus x.
   (c) Based on the plot, write down your conjecture for the subdifferential $\partial f(x)$ for all $x \in \mathbb{R}$.
   (d) Prove that all the values in the set that you identified in part (c) as $\partial f(-1)$ are sub-gradients at $x = -1$.
   (e) Prove the converse of the result in part (d), i.e., show that any scalar outside of the set you identified in part (b) as $\partial f(-1)$ cannot be a sub-gradient at $x = -1$.

4. **[Subgradients in Two Dimensions]**
Consider

$$f(x_1, x_2) = |x_1| + |x_2|.$$

Show that any vector of the form (a, b), with $a \in [-1, 1]$ and $b \in [-1, 1]$ is a subgradient for f at the point $(0, 0)$. (You are not required to give the converse argument here.)

5. **[Subgradient of the Negative Dual]**
Consider the dual $D(\lambda)$ of the optimization problem:

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & h(x) = 0. \end{aligned}$$

We know that $D(\lambda)$ is a concave function. Let $b(\lambda) = -D(\lambda)$, which is convex. Now suppose

$$x(\lambda) \in \arg\min_x f(x) + \lambda^\top h(x).$$

Show that for every solution $x(\lambda)$, $-h(x(\lambda))$ is a subgradient of $b(\lambda)$.

6. **[Convergence with Diminishing Step-Size]**
Consider the steepest descent algorithm applied to a function f:

$$x_{k+1} = x_k - \alpha_k \nabla f(x_k), \quad k = 0, 1, 2, \ldots$$

As in Lec 6, assume that f has L Lipschitz gradients, and that $f(x) \geq f_{\min} > -\infty$, for all $x \in \mathbb{R}^n$. The step-sizes satisfy the following conditions:

$$\alpha_k > 0, \forall k, \quad \lim_{k\to\infty} \alpha_k = 0, \quad \sum_{k=0}^{\infty} \alpha_k = \infty.$$

(a) Show using the Descent Lemma (Lec 6) that there exists a positive integer m such that

$$f(x_{k+1}) \leq f(x_k) - \frac{\alpha_k}{2} \|\nabla f(x_k)\|^2, \quad \forall k \geq m.$$

(b) Use the result of part (a) to show that:

$$\sum_{k=m}^{\infty} \alpha_k \|\nabla f(x_k)\|^2 < \infty$$

which implies that

$$\sum_{k=0}^{\infty} \alpha_k \|\nabla f(x_k)\|^2 < \infty.$$

(c) Now use the result of part (b), along with the fact that $\sum_{k=0}^{\infty} \alpha_k = \infty$ to show that if the sequence $\{\|\nabla f(x_k)\|\}$ converges, then it must converge to 0, i.e.,

$$\lim_{k\to\infty} \|\nabla f(x_k)\| = 0.$$

Hint: Consider any $a > 0$. Show by contradiction that the sequence $\{\|\nabla f(x_k)\|\}$ cannot converge to a.