Mathematical Foundations MFE 2023 Assignment 1

By: Naman Singhal (naman.singhal@berkeley.edu)

Solved Alone

Q1) (a) The value of this bond for T periods would be:

$$PV = \frac{c}{1+r} + \frac{c(1+g)}{(1+r)^2} + \frac{c(1+g)^2}{(1+r)^3} \cdots + \frac{c(1+g)^{T-1}}{(1+r)^T} + \frac{P}{(1+r)^T}$$

$$PV = \frac{c}{1+r}\left[1 + \left(\frac{1+g}{1+r}\right) + \left(\frac{1+g}{1+r}\right)^2 \cdots \left(\frac{1+g}{1+r}\right)^T\right) + \frac{P}{(1+r)^T}$$

$$= \frac{c}{1+r} \times \frac{1-\left(\frac{1+g}{1+r}\right)^{T+1}}{1-\left(\frac{1+g}{1+r}\right)} + \frac{P}{(1+r)^T}$$

$$PV = \frac{c}{r-g}\left[1-\left(\frac{1+g}{1+r}\right)^{T+1}\right] + \frac{P}{(1+r)^T}$$

As $T \to \infty$ $\quad \left(\frac{1+g}{1+r}\right)^{T+1} \to 0$ (because $g < r$)

and $\quad \left(\frac{1}{1+r}\right)^T \to 0$

$\therefore$ Formula for growing perpetuity is

$$\boxed{PV = \frac{c}{r-g}}$$

(b) The present value after T terms, where each term is 76 years long is

$$PV = \sum_{t=0}^{\infty} \frac{2061 + t*76}{(1+r)^{47+t*76}}$$

$$= \sum_{t=0}^{\infty} \frac{2061}{(1+r)^{47}}\left(\frac{1}{(1+r)^{76t}}\right) + \frac{76}{(1+r)^{47}}\left[\sum_{0}^{\infty} \frac{t}{(1+r)^{76t}}\right]$$

$$= \frac{2061}{(1+r)^{\cancel{47}}} \times \frac{(1+r)^{\cancel{76}\,29}}{(1+r)^{76}-1} + \frac{76}{(1+r)^{47}} \cdot \frac{(1+r)^{\cancel{76}\,29}}{((1+r)^{76}-1)^2}$$

$$\boxed{= 2512.91775}$$

$A = \frac{1}{n} + \frac{2}{n^2} + \frac{3}{n^3} \cdots$

$nA = 1 + \frac{2}{n} + \frac{3}{n^2} \cdots$

$(n-1)A = 1 + \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} \cdots$

$A = \frac{n}{(n-1)^2}$

Q2) (a) $\Delta_{P_K} = \dfrac{\partial P_K}{\partial S} = \dfrac{\partial}{\partial S}\Big[K e^{-rT}\left(1 - N(x - \sigma\sqrt{T})\right) - S\left[1 - N(x)\right] \Big]$

$$= \frac{\partial}{\partial S}\left[K e^{-rT}\left(1-N(x-\sigma\sqrt{T})\right) - S\left[1-N(x)\right]\right]$$

$$= K e^{-rT}\left(-\frac{e^{-\frac{(x-\sigma\sqrt{T})^2}{2}}}{\sqrt{2\pi}}\right)\frac{\partial x}{\partial S} - \left[1-N(x)\right] + S\left(\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\right)\frac{\partial x}{\partial S}$$

$$= (N(x)-1) + \frac{\partial x}{\partial S}\left[-K\,\frac{e^{-rT} \times e^{-\frac{x^2}{2}} \times e^{-\frac{\sigma^2 T}{2}} \times e^{x\sigma\sqrt{T}}}{\sqrt{2\pi}} + S\,\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\right]$$

$$= (N(x)-1) + \frac{\partial x}{\partial S} \times \frac{e^{-x^2/2}}{\sqrt{2\pi}}\underbrace{\left[S - K e^{x\sigma\sqrt{T} - (r+\sigma^2/2)T}\right]}_{=0}$$

$$\boxed{\Delta_{P_K} = N(x) - 1}$$

(b)

$$\Gamma_{P_K} = \frac{\partial^2 P_K}{\partial^2 S} = \frac{\partial \Delta_{P_K}}{\partial S}$$

$$= \frac{\partial (N(x)-1)}{\partial S}$$

$$= N'(x) \times \frac{\partial x}{\partial S}$$

$$= N'(x) \times \frac{\partial}{\partial S}\left[\frac{\ln(S/K)}{\sigma\sqrt{T}} + \frac{(r+\sigma^2/2)T}{\sigma\sqrt{T}}\right]$$

$$= N'(x) \times \frac{1}{S\sigma\sqrt{T}}$$

$$\boxed{\Gamma_{P_K} = \frac{N'(x)}{S\sigma\sqrt{T}}}$$

(c) $V(s+\Delta s) \approx h_s(s+\Delta s) + h_1\left(P_{K_1} + \Delta_{P_{K_1}} \cdot \Delta s + \Gamma_{P_{K_1}} \cdot \frac{\Delta s^2}{2}\right)$

$$+ h_2\left(P_{K_2} + \Delta_{P_{K_2}} \cdot \Delta s + \Gamma_{P_{K_2}} \cdot \frac{\Delta s^2}{2}\right)$$

Setting $V(s+\Delta s) - V(s) = 0$

$$\Delta s\left(h_s + h_1 \Delta_{P_{K_1}} + h_2 \Delta_{P_{K_2}}\right)$$

$$+ \frac{\Delta s^2}{2}\left(h_1 \Gamma_{P_{K_1}} + h_2 \Gamma_{P_{K_2}}\right) = 0$$

This can be written as

$$\begin{bmatrix} \Delta_{P_{K_1}} & \Delta_{P_{K_2}} \\ \Gamma_{P_{K_1}} & \Gamma_{P_{K_2}} \end{bmatrix} \times \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} -\Delta_s h_s \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \frac{1}{\Delta_{P_{K_1}} \Gamma_{P_{K_2}} - \Delta_{P_{K_2}} \Gamma_{P_{K_1}}} \begin{bmatrix} \Gamma_{P_{K_2}} & -\Delta_{P_{K_2}} \\ -\Gamma_{P_{K_1}} & \Delta_{P_{K_1}} \end{bmatrix} \begin{bmatrix} -\Delta_s h_s \\ 0 \end{bmatrix}$$

$$\boxed{\begin{aligned} h_1 &= \frac{-\Gamma_{P_{K_2}} \Delta_s h_s}{\Delta_{P_{K_1}} \Gamma_{P_{K_2}} - \Delta_{P_{K_2}} \Gamma_{P_{K_1}}} \\ h_2 &= \frac{\Gamma_{P_{K_1}} \Delta_s h_s}{\Delta_{P_{K_1}} \Gamma_{P_{K_2}} - \Delta_{P_{K_2}} \Gamma_{P_{K_1}}} \end{aligned}}$$

Q3)

(a) Price of portfolio at $t=0$ = $V_0 = h^T S^0$

$$= (1,1,0)\begin{pmatrix}2\\3\\16\end{pmatrix}$$

$$= 5$$

(b) The market is complete if the row space of D is R^2

Row space $\Rightarrow \left\{ h_1\begin{pmatrix}1\\1\end{pmatrix} + h_2\begin{pmatrix}1\\2\end{pmatrix} + h_3\begin{pmatrix}8\\4\end{pmatrix},\ h \in R^3 \right\}$

since $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\2\end{pmatrix}$ are linearly independent,

it is easy to see that the row space is R^2

$\therefore$ The market is complete.

(c) LOOP holds if $g^T D = 0 \Rightarrow g^T S_0 = 0$, $g \in R^3$

$$(g_1\ g_2\ g_3)\begin{pmatrix}1 & 1\\1 & 2\\8 & 4\end{pmatrix} = 0$$

$$g_1 + g_2 + 8g_3 = 0$$

$$g_1 + 2g_2 + 4g_3 = 0$$

$$\Downarrow$$

$$g_2 = -g_3$$

$$g_1 = -7g_3$$

$g_2 + g_3 = 0$

$g_2 = -g_3$

$g_1 + 7g_3 = 0$

$g_1 = -7g_3$

$\therefore g^T D = 0$ for g in $\left\{ \alpha\begin{pmatrix}-7\\-1\\1\end{pmatrix} \text{ for } \alpha \in R \right\}$

$$g^T S_0 = (-7\ -1\ \ 1)\begin{pmatrix}2\\3\\16\end{pmatrix} = -14 - 3 + 16$$

$$= -1 \neq 0$$

$\therefore$ LOOP does not hold for this market!

Q4) (a) (I) State price vector exists $\longrightarrow$ LooP Holds

There exists a ψ s.t. $h^T s^0 = h^T D \psi \quad \forall\, h \in \mathbb{R}^N$

If $h^T D = 0$

then $h^T s^0 = h^T D \psi$
$= \bar{0}\,\bar{\psi}$
$= 0$

$\therefore$ If $h^T D = 0 \Rightarrow h^T s^0 = 0$
and LooP holds

(II) LooP Holds $\longrightarrow$ State price vector exists

is same as

State price vector doesn't exists $\longrightarrow$ LooP doesn't hold.

$\downarrow$

$\exists h_1 \in \mathbb{R}^N$ s.t. $h_1^T \bar{D}\,[1; \psi] \neq 0 \qquad \forall\, \psi \in \mathbb{R}^M$

$h_1^T s^0 - h_1^T D \psi \neq 0 \quad \forall\, \psi \in \mathbb{R}^M \longrightarrow (1)$

Let $h_1^T D = z,\ z^T \in \mathbb{R}^M$

$\therefore\ z\psi \in \mathbb{R}$

Since $\psi \in \mathbb{R}^M$ if $z \neq 0$, Range$(z\psi) = \mathbb{R}$

We can easily find a ψ where
$h_1^T s^0 - h_1^T D \psi = 0$

$\therefore\ z = 0$ for the condition (1) to hold.

$h_1^T D = 0$

This implies

$h_1^T s^0 - 0\,\psi \neq 0$
$h_1^T s^0 \neq 0$

$\therefore$ LooP does not hold.

Hence LooP holds if and only if there exists a state price vector.

(b)

$$h^T s^0 = h^T D \varphi$$

If we have two possible values of $\varphi \Rightarrow \varphi_1, \varphi_2$
we have

$$h^T D \varphi_1 = h^T s^0$$
$$h^T D \varphi_2 = h^T s^0$$
$$h^T D (\varphi_1 - \varphi_2) = 0$$

$\therefore$ φ_1 and φ_2 can be distinct if and only if there exist a $z \neq 0$ with

$$h^T D z = 0$$

This implies, if $N(D) = \{0\}$ we have a unique φ

$\therefore$ Unique $\varphi \Leftrightarrow N(D) = \{0\}$

We also know that completeness can be represented as

$$\text{Range}(D^T) = R^M$$

Using the fundamental theorem of linear algebra:

$$N(D^T) + \text{Range}(D) = R^M$$

$\therefore$ If φ is unique $\rightarrow N(D^T) = \{0\} \rightarrow \text{Range}(D) = R^M$
$\downarrow$
Market is complete.

and

If Market is complete $\rightarrow \text{Range}(D) = R^M \rightarrow N(D^T) = \{0\}$
$\downarrow$
φ is unique.

Q5) (a)

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

(a)

$$\|A - \lambda I\| = 0$$

$$\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)^2 = 1$$

$$3 - \lambda = \pm 1$$

$$\underline{\lambda = 2, 4} \Rightarrow \underline{\text{Eigen Values} = 4, 2}$$

For $\lambda = 4$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0$$

$$c_2 - c_1 = 0$$
$$c_1 - c_2 = 0 \Rightarrow c_1 = c_2 = \alpha$$

Eigen Vector $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For $\lambda = 2$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0$$

$$c_1 = -c_2 = \alpha$$

Eigen Vector $= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\therefore$ $\underline{\text{Eigen Vectors} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

(b) $$\phi = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \Lambda = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$$

(c) Yes, A is positive definite because both eigen values are > 0.

Q6) (a) $XX^T = \begin{bmatrix} 0.7155 & 1.42 \\ 1.42 & 3.1269 \end{bmatrix}$

For this matrix, we have two eigen values

$d = \{0.05834, \; 3.7841\}$

Using the second eigen value, the eigen vector is

optimal $u = \begin{bmatrix} -0.066 \\ -0.204 \end{bmatrix}$

(b) We can calculate the c vector as

$$c_i = \frac{\langle v, u \rangle}{\|u\|^2} = \langle v, u \rangle \qquad (\|u\|^2 = 1)$$

$$\therefore \text{Error} = \frac{1}{15} \sqrt{\sum_m \| v^m - x \langle v, x \rangle \|_2^2}$$

$$= 0.0161$$

(c) Yes, it is possible to represent v's in a 2-D substace.