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Assignment2 (Maths)

Q1 (a) This probability will be $\Phi^2_{1,8}$ which is same as

$R_1 \cdot C_8$, where R_i is the row vectors and C_j is the column vector of Φ

$= 0.0021869\%$

(b) Assuming the matrix has a long term distribution, Φ^* will have an eigen value 1 and the related eigen vector $\bar{p}$ such that

$$\Phi^* p = p$$

$$(\Phi^* - 1)\, p = 0$$

Representing $\Phi^* - 1$ based on just the sign of values and if it is zero, we have

$$\begin{bmatrix} - & + & + & 0 & 0 & 0 & 0 & 0 \\ + & - & + & + & + & + & 0 & 0 \\ + & + & - & + & + & + & + & 0 \\ + & + & + & - & + & + & + & 0 \\ + & + & + & + & - & + & + & 0 \\ 0 & + & + & + & + & - & + & 0 \\ 0 & 0 & + & + & + & + & - & 0 \\ 0 & 0 & + & + & + & + & + & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \end{bmatrix} = 0$$

We know all $p_i \geq 0$

so from the last equation we have $p_3 = p_4 = p_5 = p_6 = p_7 = 0$

Similarly, it is easy to see that $p_1 = p_2 = 0$ from the rest of equations

so eigen vector is $P = (0\ 0\ 0\ 0\ 0\ 0\ 0\ p_8)^T$

since $\Sigma p_i = 1$ so $p_8 = 1$.

The long state stationary distribution is

$(0\ 0\ 0\ 0\ 0\ 0\ 0\ 1)$

\# This seems obvious looking carefully at Φ. All bonds have a >0 probability to change to Default over k period for $k \geq 2$, but the probability of going from Default to any other state is always zero. So all bonds tend to Default as $k \to \infty$

Q2)

$A = 1^T \Sigma^{-1} 1$
$B = 1^T \Sigma^{-1} \mu$
$C = \mu^T \Sigma^{-1} \mu$

$\Delta = AC - B^2 > 0$

$w_A = \frac{1}{A} \Sigma^{-1} 1$

$w_B = \frac{1}{B} \Sigma^{-1} \mu$

Constraints

$\mu_0 = h^T \mu$

$h^T 1 = 1$

To maximize

$U(h) = h^T \mu - \frac{\gamma}{2} h^T \Sigma h$

Lagrange form is:

$$\max_{h, d_1, d_2} \; U(h) + d_1 (h^T 1 - 1) + d_2 (h^T \mu - \mu_0)$$

$$\frac{\partial}{\partial h} : \quad \mu - \gamma \Sigma h + d_1 1 + d_2 \mu = 0$$

$$\frac{\partial}{\partial d_1} : \quad h^T 1 = 1$$

$$\frac{\partial}{\partial d_2} : \quad h^T \mu = \mu_0$$

$$\gamma \Sigma h = (1 + d_2) \mu + d_1 1$$

$$h = \Sigma^{-1} \left[\left(\frac{1+d_2}{\gamma} \right) \mu + \frac{d_1}{\gamma} 1 \right]$$

$$h = \left(\frac{1+d_2}{\gamma} \right) \Sigma^{-1} \mu + \frac{d_1}{\gamma} \Sigma^{-1} 1 \quad \text{— } (1)$$

$$1^T h = 1$$

$$\left(\frac{1+d_2}{\gamma} \right) B + \frac{d_1}{\gamma} A = 1 \quad \text{— } (2)$$

$$\mu^T h = \mu_0$$

$$\left(\frac{1+d_2}{\gamma} \right) C + \frac{d_1}{\gamma} \underbrace{\mu^T \Sigma^{-1} 1}_{B = 1^T \Sigma^{-1} \mu} = \mu_0$$

$$\left(\frac{1+d_2}{\gamma} \right) C + \frac{d_1}{\gamma} B = \mu_0 \quad \text{— } (3)$$

Solving (2) and (3), we have

$$\left(\frac{1+d_2}{\gamma} \right) \left(\frac{C}{B} - \frac{B}{A} \right) = \frac{\mu_0}{B} - \frac{1}{A}$$

$$\left(\frac{1+d_2}{\gamma} \right) = \frac{(A\mu_0 - B)}{\Delta} \longrightarrow (4)$$

Using (4) in (3), we have

$$\frac{d_1}{\gamma} = \left(\mu_0 - \frac{(A\mu_0 - B)C}{AC - B^2} \right) \frac{1}{B}$$

$$= \frac{C - \mu_0 B}{\Delta} \quad \text{—— (5)}$$

Using 4,5 and 1, we can find h

$$h^* = \left(\frac{A\mu_0 - B}{\Delta} \right) \Sigma^{-1} \mu + \left(\frac{C - \mu_0 B}{\Delta} \right) \Sigma^{-1} \mathbf{1}$$

$$h^* = \frac{1}{\Delta} \begin{pmatrix} AB w_B \mu_0 & -B^2 w_B \\ AC w_A & -AB w_A \mu_0 \end{pmatrix}$$

$$w_A^T \Sigma w_A = \frac{1}{A^2} (\Sigma^{-1}\mathbf{1})^T \Sigma (\Sigma^{-1} \mathbf{1}) = \frac{1}{A}$$

$$w_B^T \Sigma w_B = \frac{1}{B^2} (\Sigma^{-1}\mu)^T \Sigma (\Sigma^{-1}\mu) = \frac{C}{B^2}$$

$$w_A^T \Sigma w_B = \frac{1}{AB} (\Sigma^{-1}\mathbf{1})^T \Sigma (\Sigma^{-1}\mu) = \frac{1}{A} = w_B^T \Sigma w_A$$

$$\boxed{h^* = w_A \, A\frac{(C - \mu_0 B)}{\Delta} + w_B \, B\left(\frac{A\mu_0 - B}{\Delta} \right)}$$

$$w_A - w_B = \frac{1}{A} \Sigma^{-1} \mathbf{1} - \frac{1}{B} \Sigma^{-1} \mu$$

(b)

$$\sigma_h^2 = h^T \Sigma h$$

$$= \left[w_A^T \left(\frac{A(C - \mu_0 B)}{\Delta} \right) + w_B^T \left(\frac{B(A\mu_0 - B)}{\Delta} \right) \right] \Sigma \left[w_A \left(\frac{A(C - \mu_0 B)}{\Delta} \right) + w_B \left(\frac{B(A\mu_0 - B)}{\Delta} \right) \right.$$

$$= \frac{1}{\cancel{A}} \left(\frac{A^{\cancel{2}}(C - \mu_0 B)^2}{\Delta^2} + \frac{2\cancel{A}B(C - \mu_0 B)(A\mu_0 - B)}{\Delta^2} \right) + \frac{C}{\cancel{B^2}} \left(\frac{\cancel{B^2}(A\mu_0 - B)^2}{\Delta^2} \right)$$

$$= \frac{1}{\Delta^2} \begin{pmatrix} AC^2 + \cancel{AB^2\mu_0^2} - 2\cancel{ABC}\mu_0 + 2\cancel{ABC}\mu_0 \\ - 2B^2C - \cancel{2}AB^2\mu_0^2 + 2B^3\mu_0 \end{pmatrix} + \frac{C(A\mu_0 - B)^2}{\Delta^2}$$

$$= \frac{1}{\Delta^2} \left(-AB^2\mu_0^2 + 2B^3\mu_0 + AC^2 - 2B^2C + CA^2\mu_0^2 + CB^2 - 2ABC\mu_0 \right)$$

$$= \frac{1}{\Delta^2} \left(\mu_0^2 A \Delta + \mu_0 2B(B^2 - AC) + C(AC - B^2) \right)$$

$$\boxed{\sigma_h^2 = \frac{A\mu_0^2 - 2B\mu_0 + C}{\Delta}}$$

(C)

Expressing Utility in μ_0 terms, we have

$$U(\mu_0) = \mu_0 - \frac{\gamma}{2}\left(\frac{A\mu_0^2 - 2B\mu_0 + C}{\Delta}\right)$$

$$\frac{d\,U(\mu_0)}{d\mu_0} = 1 - \frac{\gamma}{\cancel{2}}\left(\frac{\cancel{2}A\mu_0 - \cancel{2}B}{\Delta}\right)$$

$$0 = 1 - \gamma\left(\frac{A\mu_0 - B}{\Delta}\right)$$

$$\mu_0 = \frac{\Delta}{\gamma A} + \frac{B}{A}$$

$$\boxed{\therefore\ \mu_0 = \frac{1}{A}\left(\frac{\Delta}{\gamma} + B\right)}$$

Q3 (a)

$$\frac{d\,U(h)}{dh} = 0$$

$$(\mu - R\mathbf{1}) - \gamma \Sigma h = 0$$

$$h = \frac{\Sigma^{-1}}{\gamma}(\mu - R\mathbf{1})$$

$$w = \frac{h}{\mathbf{1}h} = \frac{\Sigma^{-1}(\mu - R\mathbf{1})}{\mathbf{1}\Sigma^{-1}\mu - R\,\mathbf{1}\Sigma^{-1}\mathbf{1}}$$

$$\boxed{w = \frac{\Sigma^{-1}(\mu - R\mathbf{1})}{B - RA}}$$

(b) Expected Return $= R + h^T(\mu - R\mathbf{1})$

$$= R + (\mu - R\mathbf{1})^T h$$

$$= R + \frac{(\mu - R\mathbf{1})^T \Sigma^{-1}(\mu - R\mathbf{1})}{\gamma}$$

$$\boxed{= R + \frac{(C - 2RB + R^2 A)}{\gamma}}$$

∴ If risk aversion is high, expected return is close to R.

Q4)

$$W^T \Sigma W = W^T \Sigma \left(\frac{1}{B - AR} \Sigma^{-1} (\mu - R1) \right)$$

$$W^T \Sigma W = \frac{W^T \cancel{\Sigma} \cancel{\Sigma^{-1}} (\mu - R1)}{1^T \Sigma^{-1} (\mu - R1)}$$

$$\frac{w^T (\mu - R1)}{1^T \Sigma^{-1} (\mu - R1) \, w^T \Sigma w} = 1 \longrightarrow (1)$$

$$\beta w^T (\mu - R1) = \frac{\Sigma w \, w^T (\mu - R1)}{w^T \Sigma w}$$

$$= \frac{(\mu - R1) \, w^T (\mu - R1)}{1^T \Sigma^{-1} (\mu - R1) \, w^T \Sigma w} = 1$$

$$\boxed{\beta w^T (\mu - R1) = (\mu - R1)}$$