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Q1

$$f(x,y) = \begin{cases} \frac{2}{5}(2x+3y) & 0 \le x \le 1, \; 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$X \rightarrow$ Mathematics

$Y \rightarrow$ Music

(a) $A = \{ X > 0.8 \}$

$$P(A) = \int_{0.8}^{1}\int_{0}^{1} \frac{2}{5}(2x+3y)\, dy\, dx$$

$$= \int_{0.8}^{1} \frac{2}{5}\left(2xy + \frac{3y^2}{2}\right)\Big|_{y=0}^{1} dx$$

$$= \int_{0.8}^{1} \frac{2}{5}\left(2x + \frac{3}{2}\right)$$

$$= \frac{2}{5}\left(x^2 + \frac{3}{2}x\right)\Big|_{x=0.8}^{1}$$

$$= \frac{2}{5}\left(1^2 - (0.8)^2 + \frac{3}{10}\right)$$

$$\boxed{= 0.264}$$

(b) $P(x > 0.8 \mid y = 0.3) = \dfrac{f_{x,y}(x > 0.8,\ y = 0.3)}{f_Y(y = 0.3)}$

Marginal probability of y is:

$$f(y) = \int_0^1 \frac{2}{5}(2x + 3y)\,dx$$

$$= \begin{cases} \frac{2}{5}(1 + 3y) & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f(y = 0.3) = 0.76$$

$$f_{xy}(x > 0.8,\ y = 0.3) = \int_{0.8}^{1} \frac{2}{5}(2x + 0.9)\,dx$$

$$= \frac{2}{5}\left((1 - 0.8^2) + 0.18\right)$$

$$= 0.216$$

$P(x > 0.8 \mid y = 0.3) =$

$$= \frac{0.216}{0.76} = 0.16416$$

(c) $P(y > 0.8 \mid x = 0.3) = \dfrac{f_{xy}(x=0.3,\ y>0.8)}{f_x(x=0.3)}$

Marginal probability of x is:

$$f_x(x) = \int_0^1 \frac{2}{5}(2x + 3y)\,dy$$

$$f_x(x) = \frac{2}{5}\left(2x + \frac{3}{2}\right)$$

$$\underline{f_x(x = 0.3) = 0.84}$$

$$f_{xy}(x=0.3,\ y>0.8) = \int_{0.8}^{1} \frac{2}{5}(0.6 + 3y)\,dy$$

$$= \frac{2}{5}\left(0.12 + 3(0.36)\right)$$

$$= 0.48$$

$$P(y > 0.8 \mid x = 0.3) = 0.57142.$$

Q2] (a) $f(y_1, y_2, y_3 \cdots y_n)$

$$= \int_{-\infty}^{\infty} f(y_1, y_2, y_3 \cdots y_n \mid x)\, f(x)\, dx$$

$$= \int_{-\infty}^{\infty} g(y_1 \mid x)\, g(y_2 \mid x) \cdots g(y_n \mid x)\, f(x)\, dx$$

$$= \int_{y_{max}}^{\infty} \frac{1}{x^n} \times \frac{x^n e^{-x}}{n!}\, dx$$

$0 < y_1 < x$,
$0 < y_2 < x$,
$0 < y_3 < x$
$\vdots$
can be combined into
$y_{max} = \max(y_1, y_2 \cdots y_n)$
$x > y_{max}$
$y_i > 0 \quad \forall i$

$$= \left. \frac{-e^{-x}}{n!} \right|_{y_{max}}^{\infty}$$

$$\boxed{f(y_1, y_2 \cdots y_n) = \frac{e^{-y_{max}}}{n!}}$$

$y_{max} = \max(y_1, y_2 \cdots y_n)$

and $y_i > 0$ for all $0 \le i \le n$

(b) $$f(x \mid y_1, y_2, y_3 \cdots y_n) = \frac{f(y_1, y_2 \cdots y_n \mid x)\, f(x)}{f(y_1, y_2 \cdots y_n)}$$

$$= \frac{\left(\prod_i g(y_i \mid x)\right) f(x)}{f(y_1, y_2 \cdots y_n)}$$

$$= \frac{(1/x^n)\, \frac{1}{n!}\, x^n e^{-x}}{\frac{e^{-y_{max}}}{n!}}$$

$$\boxed{= e^{y_{max} - x} \qquad x \geq y_{max}}$$

Q3

$$f(x) = \begin{cases} \frac{1}{2}x & \text{for } 0 < x < 2 \\ 0 \end{cases}$$

$$Y = 4 - x^3$$

CDF of y will be

$$F_Y(y) = P(Y \leq y)$$

$$= P(4 - x^3 \leq y)$$

$$= P((4-y)^{1/3} \leq X)$$

$$= 1 - F_X((4-y)^{1/3})$$

$$= 1 - \int_0^{(4-y)^{1/3}} \frac{1}{2} x \, dx$$

$$F_Y(y) = 1 - \frac{(4-y)^{2/3}}{4} \qquad -4 \leq y \leq 4$$

$$f_Y(y) = \frac{d F_Y(y)}{dy} = \frac{(4-y)^{-1/3}}{6} \qquad -4 \leq y \leq 4$$

Q4] Consider $f(x) = \begin{cases} \frac{3}{2} x^{-5/2} & x \geq 1 \\ 0 & x < 1 \end{cases}$

$$E(x) = \int_1^\infty \frac{3}{2} x^{-3/2} \, dx$$

$$= \left(\frac{3}{2}\right) \frac{x^{-1/2}}{(-1/2)} \Big|_1^\infty$$

$$E(x) = 3 \quad \text{(finite)}$$

$$E(x^2) = \int_1^\infty \frac{3}{2} x^{-1/2} \, dx$$

$$= \frac{3}{2} \frac{x^{1/2}}{(1/2)} \Big|_1^\infty$$

$$E(x^2) = \infty$$

$$Var(x) = E(x^2) - E(x)^2 = \infty \quad \text{(infinite)}$$

Q5] $\Psi_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$

$= \int_{1}^{\infty} e^{tx}/x^2 \, dx$

If $t > 0$ we have.

$\frac{e^{tx}}{x^2} \geq e^{tx}$ $\left(\text{because } 0 \leq (1/x^2) \leq 1 \text{ for } 1 \leq x < \infty\right.$

$\int_1^\infty \frac{e^{tx}}{x^2} dx \geq \int_1^\infty e^{tx} dx$

$\Psi_x(t) \geq \frac{e^{tx}}{t}\Big|_1^\infty$

Since the RHS tends to ∞

$\Psi_x(t)$ tents to ∞ for $t > 0$.

If $t \leq 0$ we have

$\frac{e^{tx}}{x^2} \leq \frac{1}{x^2}$ $\left(\text{because } 0 < e^{tx} < 1 \text{ for } 1 \leq x \leq \infty\right)$

$\int_1^\infty \frac{e^{tx}}{x^2} dx \leq \int_1^\infty \frac{1}{x^2} dx$

$$\psi_x(t) \leq \frac{-1}{x} \Big|_1^{\infty}$$

$$\psi_x(t) \leq 1$$

Since RHS is finite, $\psi_x(t)$ is also finite for $t \leq 0$.

$\therefore$ mgf is finite for all $t \leq 0$ but not for $t > 0$.

Q6

$$f(x,y) = \begin{cases} \frac{1}{3}(x+y) & 0 \le x \le 1, \quad 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Calculating Marginals

$$f_x(x) = \frac{1}{3}\int_0^2 (x+y)\,dy$$

$$= \frac{1}{3}\left(xy + \frac{y^2}{2}\right)\Big|_0^2$$

$$= \frac{1}{3}(2x+2) = \frac{2}{3}(x+1)$$

$$f_y(y) = \frac{1}{3}\int_0^1 (x+y)\,dx = \frac{1}{3}\left(\frac{1}{2}+y\right)$$

$$= \frac{y}{3} + \frac{1}{6}$$

$$E(X) = \int_0^1 x\, f_x(x)\,dx = \frac{2}{3}\int_0^1 (x^2+x)\,dx = \frac{2}{3}\left(\frac{x^3}{3}+\frac{x^2}{2}\right)\Big|_0^1$$

$$\boxed{E(X) = \frac{5}{9}}$$

$$E(Y) = \int_0^2 y\left(\frac{y}{3}+\frac{1}{6}\right)dx = \frac{y^3}{9} + \frac{y^2}{12}\Big|_0^2 = \boxed{\frac{11}{9}}$$

$$E(x^2) = \int_0^1 x^2 f(x)\,dx = \frac{2}{3}\int_0^1 (x^3 + x^2)\,dx$$

$$= \frac{2}{3}\left(\frac{x^4}{4} + \frac{x^3}{3}\right)\Big|_0^1$$

$$= \frac{7}{18}$$

$$E[Y^2] = \int_0^2 y^2 f(y)\,dy = \int_0^2 \frac{y^3}{3} + \frac{y^2}{6}\,dy$$

$$= \frac{y^4}{12} + \frac{y^3}{18}\Big|_0^2$$

$$= \frac{16}{9}$$

$$V(X) = E(X^2) - E(X)^2 = \frac{7}{18} - \left(\frac{5}{9}\right)^2 = \frac{13}{162}$$

$$V(Y) = E(Y^2) - E(Y)^2 = \frac{16}{9} - \left(\frac{11}{9}\right)^2 = \frac{23}{81}$$

$$E(XY) = \frac{1}{3}\int_0^2\int_0^1 (x^2y + y^2x)\,dx\,dy$$

$$= \frac{1}{3}\int_0^2 \left(\frac{x^3y}{3} + \frac{y^2x^2}{2}\right)\Big|_0^1 dy$$

$$= \frac{1}{3}\int_0^2 \left(\frac{y}{3} + \frac{y^2}{2}\right) dy$$

$$= \frac{1}{3} \left(\frac{y^2}{6} + \frac{y^3}{6} \right) \Big|_0^2$$

$$= 2/3$$

$$Cov(X,Y) = E(XY) - E(X)\,E(Y)$$

$$= \frac{2}{3} - \left(\frac{5}{9}\right)\left(\frac{11}{9}\right)$$

$$= \frac{-1}{81}$$

Finally. $V(2X - 3Y + 8) = V(2X - 3Y)$

$$= 2^2 V(X) + 3^2 V(Y) - 2 \times 2 \times 3\, Cov(XY)$$

$$= 4\left(\frac{13}{162}\right) + 9\left(\frac{23}{81}\right) - 12\left(\frac{-1}{13}\right)$$

$$\boxed{V(2X - 3Y + 8) = \frac{245}{81}}$$

Q7) (d) $x = \mu_x + \sigma_x Z_1$

$$\sim \mu_x + \sigma_x N(0,1)$$

$$\sim \mu_x + N(0, \sigma_x^2)$$

$$\boxed{x \sim N(\mu_x, \sigma_x^2)}$$

$$y = \mu_y + \sigma_Y(\rho Z_1 + \sqrt{1-\rho^2}\, Z_2)$$

$$y = (\mu_y + \rho\sigma_Y Z_1) + \sqrt{1-\rho^2}\,\sigma_y Z_2$$

$$y \sim N(\mu_y, \rho^2\sigma_Y^2) + N(0, (1-\rho^2)\sigma_y^2)$$

$$\boxed{y \sim N(\mu_y, \sigma_Y^2)}$$

Since x and y are Normally dist. (x,y) is also bivariate normal.

$E(X) = \mu_x$, $E(Y) = \mu_Y$,

$$E(XY) = E\Big(\mu_x\mu_Y + \mu_x\sigma_Y \cancel{(\rho Z_1 + \sqrt{1-\rho^2}\, Z_2)}^{0}$$

$$+ \mu_Y\sigma_x \cancel{Z_1}^{0} + \sigma_x\sigma_Y(\rho Z_1^2 + \cancel{\sqrt{1-\rho^2}\, Z_1 Z_2}^{0})\Big)$$

Using $E(Z_1) = 0$, $E(Z_2) = 0$, $E(Z_1 Z_2) = 0$, $E(Z_1^2) = 1$

$$E(XY) = \mu_x\mu_Y + \sigma_x\sigma_Y\rho$$

$$\boxed{\begin{aligned} Cov(X,Y) &= E(XY) - E(X)\,E(Y) \\ &= \sigma_X \sigma_Y \rho \end{aligned}}$$

(b) $E(y|x) = E(y|(\mu_x + \sigma_x z_1))$

$= E(y|z_1)$ (because μ_x & σ_x are constants)

$= E(\mu_y + \sigma_Y(\rho z_1 + \sqrt{1-\rho^2}\, z_2 \,|\, z_1)$

$= \mu_Y + \sigma_Y(\rho\, E(z_1|z_1) + \sqrt{1-\rho^2}\, E(z_2|z_1)$

$E(z_1|z_1) = z_1$, $E(z_2|z_1) = E(z_2) = 0$

$E(y|x) = \mu_Y + \rho\, \sigma_Y\, z_1$

$$\boxed{E(y|x) = \mu_Y + \rho \frac{\sigma_Y}{\sigma_X}(x - \mu_X)}$$

(c)

$$\text{Var}(y|x) = \text{Var}\left(\mu_y + \sigma_Y\left(\rho z_1 + \sqrt{1-\rho^2}\, z_2 \;\middle|\; z_1\right)\right)$$

$\text{Var}(c) = 0$ where c is a const

$$= \sigma_Y^2 \,\text{Var}\left(\rho z_1 + \sqrt{1-\rho^2}\, z_2 \;\middle|\; z_1\right)$$

Since z_1 and z_2 are iid

$$= \sigma_Y^2 \left[\rho^2 \text{Var}(z_1|z_1) + (1-\rho^2)\,\text{Var}(z_2|z_1)\right]$$

$\text{Var}[z_1|z_1] = 0$ and $\text{Var}(z_2|z_1) = \text{Var}(z_2) = 1$

$$\boxed{\text{Var}(y|x) = \sigma_Y^2 (1-\rho^2)}$$