積分問題集

積分問題 1

問題

$$\int \frac{\sin x}{\sin x + \cos x} dx$$

計算

$$\begin{align}
\int \frac{\sin x}{\sin x + \cos x} dx &= \int \frac{\sin x}{\sqrt{2}\sin(x + \frac{\pi}{4})} dx \tag{1} \\
&= \frac{1}{\sqrt{2}} \int \frac{\sin(t - \frac{\pi}{4})}{\sin t} dt \tag{2} \\
&= \frac{1}{\sqrt{2}} \int \frac{\sin t \cos \dfrac{\pi}{4} - \cos t \sin \dfrac{\pi}{4}}{\sin t} dt \tag{3} \\
&= \frac{1}{2} \int \left(1 - \frac{\cos t}{\sin t}\right) dt \tag{4} \\
&= \frac{1}{2} t - \frac{1}{2} \log(\sin t) + C \tag{5} \\
&= \frac{1}{2}\left(x + \frac{\pi}{4}\right) - \frac{1}{2} \log \sin\left(x + \frac{\pi}{4}\right) + A \tag{6}
\end{align}$$

答え

$$\int \frac{\sin x}{\sin x + \cos x} dx = \frac{1}{2} x - \frac{1}{2} \log \sin\left(x + \frac{\pi}{4}\right) + C \tag{7}$$

積分問題 2

問題

$$\int \frac{1}{\sin^2 x \cos^2 x} dx$$

計算

$$\int \frac{1}{\sin^2 x \cos^2 x} dx = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx \tag{8}$$

$$= \int \left(\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} \right) dx \tag{9}$$

$$= \tan x - \frac{1}{\tan x} + C \tag{10}$$

答え

$$\int \frac{1}{\sin^2 x \cos^2 x} dx = \tan x - \frac{1}{\tan x} + C \tag{11}$$

積分問題 3

問題

$$\int_0^{\frac{\pi}{2}} \log(\sin x) dx$$

計算

$$I = \int_0^{\frac{\pi}{2}} \log(\sin x) dx \tag{12}$$

$$2I = \int_0^{\frac{\pi}{2}} \log(\sin x) dx + \int_0^{\frac{\pi}{2}} \log(\cos x) dx \tag{13}$$

$$= \int_0^{\frac{\pi}{2}} \log(\frac{\sin x}{2}) dx \tag{14}$$

$$= \int_0^{\frac{\pi}{2}} \log(\sin 2x) dx - \int_0^{\frac{\pi}{2}} \log 2 dx \tag{15}$$

$$= \frac{1}{2} \int_0^{\pi} \log(\sin u) du - \int_0^{\frac{\pi}{2}} \log 2 dx \tag{16}$$

$$= \int_0^{\frac{\pi}{2}} \log(\sin x) dx - \int_0^{\frac{\pi}{2}} \log 2 dx \tag{17}$$

$$= I - \frac{\pi}{2} \tag{18}$$

答え

$$I = \int_0^{\frac{\pi}{2}} \log(\sin x) dx = -\frac{\pi}{2} \log 2 \tag{19}$$

積分問題 4

問題

$$\int_0^1 \log x dx$$

計算

$$\int_0^1 \log x dx = \int_{-\infty}^0 te^t dt \tag{20}$$

$$= \left[te^t\right]_{-\infty}^0 - \int_{-\infty}^0 e^t dt \tag{21}$$

$$= -\left[e^t\right]_{-\infty}^0 \tag{22}$$

$$= -1 \tag{23}$$

答え

$$\int_0^1 \log x dx = -1 \tag{24}$$

積分問題 5

問題

$$\int \sqrt{x^2+1} dx$$

計算

$$I = \int \sqrt{x^2+1}dx \tag{25}$$

$$= x\sqrt{x^2+1} - \int \frac{x^2}{\sqrt{x^2+1}}dx \tag{26}$$

$$= x\sqrt{x^2+1} - \int \frac{x^2+1-1}{\sqrt{x^2+1}}dx \tag{27}$$

$$= x\sqrt{x^2+1} - I + \int \frac{dx}{\sqrt{x^2+1}} \tag{28}$$

$$= x\sqrt{x^2+1} - I + \log(x+\sqrt{x^2+1}) \tag{29}$$

答え

$$I = \int \sqrt{x^2+1}dx = \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\log(x+\sqrt{x^2+1}) \tag{30}$$

積分問題 6

問題

$$\int_{-\infty}^{\infty} e^{-x^2}dx$$

計算

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx \tag{31}$$

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy \tag{32}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dxdy \tag{33}$$

$$= \int_{0}^{\infty} \int_{0}^{2\pi} e^{-r^2} rdrd\theta \tag{34}$$

$$= 2\pi \left[-\frac{1}{2} e^{-r^2} \right]_0^{\infty} \tag{35}$$

$$= \pi \tag{36}$$

答え

$$\int_{-\infty}^{\infty} e^{-x^2} = \sqrt{\pi} \tag{37}$$

積分問題 7

問題

$$\int \left(\frac{1}{\log x} + \log(\log x) \right) dx$$

計算

$$\int\left(\frac{1}{\log x}+\log(\log x)\right)dx=\int\frac{dx}{\log x}+\int\log(\log x)dx \tag{38}$$

$$=\int\frac{dx}{\log x}+x\log(\log x)-\int\frac{x}{x\log x}dx \tag{39}$$

$$=\int\frac{dx}{\log x}+x\log(\log x)-\int\frac{dx}{\log x} \tag{40}$$

$$=x\log(\log x)+C \tag{41}$$

答え

$$\int\left(\frac{1}{\log x}+\log(\log x)\right)dx=x\log(\log x)+C \tag{42}$$

積分問題 8

問題

$$\int\sqrt{x^2-1}dx$$

計算

$$I = \int \sqrt{x^2-1}dx \tag{43}$$

$$= x\sqrt{x^2-1} - \int \frac{x^2}{\sqrt{x^2-1}}dx \tag{44}$$

$$= x\sqrt{x^2-1} - \int \frac{\sqrt{x^2-1}^2+1}{\sqrt{x^2-1}}dx \tag{45}$$

$$= x\sqrt{x^2-1} - \int \sqrt{x^2-1}dx + \int \frac{dx}{\sqrt{x^2-1}} \tag{46}$$

$$= x\sqrt{x^2-1} - I - \log|x+\sqrt{x^2+1}| \tag{47}$$

答え

$$I = \int \sqrt{x^2-1}dx = \frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\log|x+\sqrt{x^2+1}| \tag{48}$$

積分問題 9

問題

$$\int_0^\infty \frac{x}{e^x-1}dx$$

計算

$$\int_0^\infty \frac{x}{e^x-1}dx = \int_0^\infty \frac{x}{e^x(1-\frac{1}{e^x})}dx \tag{49}$$

$$= \int_0^\infty xe^{-x}\sum_{n=0}^{\infty} e^{-nx}dx \tag{50}$$

$$= \sum_{n=0}^{\infty}\int_0^\infty xe^{-(n+1)x}dx \tag{51}$$

$$= \sum_{n=0}^{\infty}\int_0^\infty \frac{t}{n+1}\cdot e^{-t}\cdot\frac{dt}{n+1} \tag{52}$$

$$= \sum_{n=0}^{\infty}\frac{1}{(n+1)^2}\int_0^\infty te^{-t}dt \tag{53}$$

$$= \sum_{n=0}^{\infty}\frac{1}{(n+1)^2} \tag{54}$$

$$= \frac{\pi^2}{6} \tag{55}$$

答え

$$\int_0^\infty \frac{x}{e^x-1}dx = \frac{\pi^2}{6} \tag{56}$$

積分問題 10

問題

$$\int_a^b (x-a)^m(x-b)^n dx$$

計算

$$x = (b-a)t + a$$

と置換。

$$\int_a^b (x-a)^m (x-b)^n dx = \int_0^1 (b-a)^m t^m (b-a)^n (t-1)^n (b-a) dt \tag{57}$$

$$= (-1)^n (b-a)^{m+n+1} \int_0^1 t^m (1-t)^n dt \tag{58}$$

$$= (-1)^n (b-a)^{m+n+1} B(m+1, n+1) \tag{59}$$

$$= (-1)^n (b-a)^{m+n+1} \frac{\Gamma\ (m+1)\ \Gamma\ (n+1)}{\Gamma\ (m+n+2)} \tag{60}$$

$$= \frac{(-1)^n m! n!}{(m+n+1)!} (b-a)^{m+n+1} \tag{61}$$

答え

$$\int_a^b (x-a)^m (x-b)^n dx = \frac{(-1)^n m! n!}{(m+n+1)!} (b-a)^{m+n+1} \tag{62}$$