ベクトル解析

1　円柱座標

1.1　準備

基本事項の整理。直交座標との関係性。

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned} \tag{1}$$

上から分かる以下二つの式。

$$\begin{aligned} r &= \sqrt{x^2+y^2} \\ \theta &= \tan^{-1}\frac{y}{x} \end{aligned} \tag{2}$$

1.2　勾配

勾配の成分を一つ一つ計算していく。まずは x 成分から

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial z}{\partial x}\frac{\partial}{\partial z} \tag{3}$$

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\sqrt{x^2+y^2} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r} = \cos\theta \tag{4}$$

$$\frac{\partial \theta}{\partial x} = \frac{\partial}{\partial x}\tan^{-1}\frac{y}{x} = -\frac{y}{x}\cdot\frac{1}{1+(\frac{y}{x})^2} = -\frac{y}{x^2+y^2} = -\frac{\sin\theta}{r} \tag{5}$$

$$\frac{\partial z}{\partial x} = 0 \tag{6}$$

(4)(5)(6) を (3) の式に代入すると

$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \tag{7}$$

y 成分も同様に計算します。

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial}{\partial z} \tag{8}$$

$$\frac{\partial r}{\partial y} = \frac{\partial}{\partial y}\sqrt{x^2+y^2} = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{r} = \sin\theta \tag{9}$$

$$\frac{\partial \theta}{\partial y} = \frac{\partial}{\partial y}\tan^{-1}\frac{y}{x} = \frac{1}{x}\cdot\frac{1}{1+(\frac{y}{x})^2} = \frac{x}{x^2+y^2} = \frac{\cos\theta}{r} \tag{10}$$

$$\frac{\partial z}{\partial y} = 0 \tag{11}$$

(9)(10)(11) を (8) の式に代入すると

$$\frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta} \tag{12}$$

z 成分は

$$\frac{\partial}{\partial z} \tag{13}$$

以上より

$$\begin{aligned}
\nabla &= \frac{\partial}{\partial x}\boldsymbol{e_x} + \frac{\partial}{\partial y}\boldsymbol{e_y} + \frac{\partial}{\partial z}\boldsymbol{e_z} \\
&= \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\boldsymbol{e_x} + \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)\boldsymbol{e_y} + \frac{\partial}{\partial z}\boldsymbol{e_z} \\
&= \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)(\boldsymbol{e_r}\cos\theta - \boldsymbol{e_\theta}\sin\theta) + \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)(\boldsymbol{e_r}\sin\theta + \boldsymbol{e_\theta}\cos\theta) + \frac{\partial}{\partial z}\boldsymbol{e_z} \\
&= \frac{\partial}{\partial r}\boldsymbol{e_r} + \frac{1}{r}\frac{\partial}{\partial \theta}\boldsymbol{e_\theta} + \frac{\partial}{\partial z}\boldsymbol{e_z}
\end{aligned} \tag{14}$$

1.3 発散

$$\boldsymbol{A} = A_r \boldsymbol{e_r} + A_\theta \boldsymbol{e_\theta} + A_z \boldsymbol{e_z} \tag{15}$$

ここで

$$\nabla \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{16}$$

右辺第一項

$$\begin{aligned}\frac{\partial A_x}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial A_x}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial A_x}{\partial \theta} + \frac{\partial z}{\partial x}\frac{\partial A_x}{\partial z} \\ &= \cos\theta \frac{\partial A_x}{\partial r} - \frac{\sin\theta}{r}\frac{\partial A_x}{\partial \theta} \\ &= \cos\theta\left(\frac{\partial A_r}{\partial r}\cos\theta - \frac{A_\theta}{\partial r}\sin\theta\right) - \frac{\sin\theta}{r}\left(-A_r\sin\theta + \frac{\partial A_r}{\partial \theta}\cos\theta - A_\theta\cos\theta - \frac{A_\theta}{\partial \theta}\sin\theta\right)\end{aligned} \tag{17}$$

右辺第二項

$$\begin{aligned}\frac{\partial A_y}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial A_y}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial A_y}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial A_y}{\partial z} \\ &= \sin\theta \frac{\partial A_y}{\partial r} + \frac{\cos\theta}{r}\frac{\partial A_y}{\partial \theta} \\ &= \sin\theta\left(\frac{\partial A_r}{\partial r}\sin\theta + \frac{A_\theta}{\partial r}\cos\theta\right) + \frac{\cos\theta}{r}\left(A_r\cos\theta + \frac{\partial A_r}{\partial \theta}\sin\theta - A_\theta\sin\theta + \frac{A_\theta}{\partial \theta}\cos\theta\right)\end{aligned} \tag{18}$$

右辺第三項

$$\frac{\partial A_z}{\partial z} \tag{19}$$

これらの変形において、(4)(5)(6) を使用した。これらの結果を、(16) に代入して整理すると

$$\begin{aligned}\nabla \cdot \boldsymbol{A} &= \frac{\partial A_r}{\partial r} + \frac{A_r}{r} + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z} \\ &= \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}\end{aligned} \tag{20}$$

1.4　ラプラシアン

$$\Delta f = \nabla \cdot (\nabla f) = \nabla \cdot \left(\frac{\partial f}{\partial r} \boldsymbol{e_r} + \frac{1}{r} \frac{\partial f}{\partial \theta} \boldsymbol{e_\theta} + \frac{\partial f}{\partial z} \boldsymbol{e_z} \right) \tag{21}$$

ただし、勾配の式を利用した。ここで

$$\begin{aligned} A_r &= \frac{\partial f}{\partial r} \\ A_\theta &= \frac{1}{r} \frac{\partial f}{\partial \theta} \\ A_z &= \frac{\partial f}{\partial z} \end{aligned} \tag{22}$$

と考えて、(20) を適用すると

$$\Delta f = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} \tag{23}$$

2　極座標

基本事項の整理。直交座標との関係性。

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \end{aligned} \tag{24}$$

上から分かる以下 3 つの式。

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2} \\ \phi &= \tan^{-1} \frac{y}{x} \\ \theta &= \cos^{-1} \frac{z}{\sqrt{x^2 + y^2 + z^2}} \end{aligned} \tag{25}$$

2.1 勾配

勾配の成分を一つ一つ計算していく。まずは x 成分から

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi} \tag{26}$$

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2} = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r} = \sin\theta\cos\phi \tag{27}$$

$$\frac{\partial \theta}{\partial x} = \frac{\partial}{\partial x}\cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{1}{r\sin\theta}\cdot\frac{r\cos\theta}{r^2}\cdot(r\sin\theta\cos\phi) = \frac{\cos\theta\cos\phi}{r} \tag{28}$$

$$\frac{\partial \phi}{\partial x} = -\frac{y}{x^2+y^2} = -\frac{r\sin\theta\sin\phi}{r^2\sin^2\theta} = -\frac{\sin\phi}{r\sin\theta} \tag{29}$$

(27)(28)(29) を (26) の式に代入すると

$$\frac{\partial}{\partial x} = \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\phi\frac{\partial}{\partial \theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial \phi} \tag{30}$$

y 成分も同様に計算します。

$$\frac{\partial}{\partial y} = \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\phi\frac{\partial}{\partial \theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial \phi} \tag{31}$$

z 成分は

$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \tag{32}$$

以上より

$$\nabla = \frac{\partial}{\partial x}\boldsymbol{e_x} + \frac{\partial}{\partial y}\boldsymbol{e_y} + \frac{\partial}{\partial z}\boldsymbol{e_z} \tag{33}$$

を計算していく。ここで基底変換に対して以下の式が成立。

$$\begin{aligned}
\boldsymbol{e_x} &= \sin\theta\cos\phi\boldsymbol{e_r} + \cos\theta\cos\phi\boldsymbol{e_\theta} - \sin\phi\boldsymbol{e_\phi} \\
\boldsymbol{e_y} &= \sin\theta\sin\phi\boldsymbol{e_r} + \cos\theta\sin\phi\boldsymbol{e_\theta} + \cos\phi\boldsymbol{e_\phi} \\
\boldsymbol{e_z} &= \cos\theta\boldsymbol{e_r} - \sin\theta\boldsymbol{e_\theta}
\end{aligned} \tag{34}$$

よって (30)、(31)、(32)、(34) を (33) に代入して整理すると

$$\nabla f = \frac{\partial f}{\partial r}\boldsymbol{e_r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\boldsymbol{e_\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\boldsymbol{e_\phi} \tag{35}$$

2.2 発散

$$\boldsymbol{A} = A_r\boldsymbol{e_r} + A_\theta\boldsymbol{e_\theta} + A_\phi\boldsymbol{e_\phi} \tag{36}$$

ここで

$$\nabla \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{37}$$

右辺第一項

$$\frac{\partial A_x}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial A_x}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial A_x}{\partial \theta} + \frac{\partial \phi}{\partial x}\frac{\partial A_x}{\partial \phi} \tag{38}$$

$$= \sin\theta\cos\phi\frac{\partial A_x}{\partial r} + \frac{1}{r}\cos\theta\cos\phi\frac{\partial A_x}{\partial \theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial A_x}{\partial \theta}$$

$$= \sin\theta\cos\phi\left(\sin\theta\cos\phi\frac{\partial A_r}{\partial r} + \cos\theta\cos\phi\frac{A_\theta}{\partial r} - \sin\phi\frac{\partial A_\phi}{\partial r}\right)$$

$$+ \frac{1}{r}\cos\theta\cos\phi\left(\cos\theta\cos\phi A_r + \sin\theta\cos\phi\frac{\partial A_r}{\partial \theta} - \sin\theta\cos\phi A_\theta + \cos\theta\cos\phi\frac{A_\theta}{\partial \theta} - \sin\phi\frac{\partial A_\phi}{\partial \theta}\right)$$

$$- \frac{\sin\phi}{r\sin\theta}\left(-\sin\theta\sin\phi A_r + \sin\theta\cos\phi\frac{\partial A_r}{\partial \phi} - \cos\theta\sin\phi A_\theta + \cos\theta\cos\phi\frac{A_\theta}{\partial \phi} - \cos\phi A_\phi - \sin\phi\frac{\partial A_\phi}{\partial \phi}\right)$$

右辺第二項も同様に計算する。

$$
\begin{aligned}
\frac{\partial A_y}{\partial y} &= \sin\theta\sin\phi\left(\frac{\partial A_r}{\partial r}\sin\theta\sin\phi + \frac{\partial A_\theta}{\partial r}\cos\theta\sin\phi + \frac{\partial A_\phi}{\partial r}\cos\phi\right) \\
&+ \frac{1}{r}\cos\theta\sin\phi\left(A_r\cos\theta\sin\phi + \sin\theta\sin\phi\frac{\partial A_r}{\partial\theta} - A_\theta\sin\theta\sin\phi + \frac{\partial A_\theta}{\partial\theta}\cos\theta\sin\phi + \frac{\partial A_\phi}{\partial\theta}\cos\phi\right) \\
&+ \frac{\cos\phi}{r\sin\theta}\left(A_r\sin\theta\cos\phi + \frac{\partial A_r}{\partial\theta}\sin\theta\sin\phi + A_\theta\cos\theta\cos\phi + \frac{\partial A_\theta}{\partial\phi}\cos\theta\sin\phi - A_\phi\sin\phi + \frac{\partial A_\phi}{\partial\phi}\cos\phi\right)
\end{aligned} \tag{39}
$$

右辺第三項

$$
\frac{\partial A_z}{\partial z} = \cos\theta\left(\frac{\partial A_r}{\partial r}\cos\theta - \frac{\partial A_\theta}{\partial r}\sin\theta\right) - \frac{\sin\theta}{r}\left(-A_r\sin\theta + \frac{\partial A_r}{\partial\theta}\cos\theta - A_\theta\cos\theta - \frac{\partial A_\theta}{\partial\theta}\sin\theta\right) \tag{40}
$$

これらの結果を、(37) に代入して整理すると

$$
\begin{aligned}
\nabla\cdot\boldsymbol{A} &= \frac{\partial A_r}{\partial r} + \frac{2}{r}A_r + \frac{\cos\theta}{r\sin\theta}A_\theta + \frac{1}{r}\frac{\partial A_\theta}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}
\end{aligned} \tag{41}
$$

2.3 ラプラシアン

$$
\Delta f = \nabla\cdot(\nabla f) = \nabla\cdot\left(\frac{\partial f}{\partial r}\boldsymbol{e_r} + \frac{1}{r}\frac{\partial f}{\partial\theta}\boldsymbol{e_\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\boldsymbol{e_\phi}\right) \tag{42}
$$

ただし、勾配の式を利用した。

$$
\begin{aligned}
A_r &= \frac{\partial f}{\partial r} \\
A_\theta &= \frac{1}{r}\frac{\partial f}{\partial\theta} \\
A_z &= \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}
\end{aligned} \tag{43}
$$

と考えて、(41) を適用すると

$$
\begin{aligned}
\Delta f &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial f}{\partial\theta}\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\right) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}
\end{aligned} \tag{44}
$$