2. Describe an algorithm of O(m + n) to determine whether a given flow network contains a unique minimum (s, t)-cut. Explain.

Solution:
I will first proof this statement: let E = $\{e_1, e_2, ..., e_i\}$ be the set of cut-edges in the minimum st-cut of our flow network.

The original min st-cut is not unique

$$\Longleftrightarrow$$

$\exists$ an edge in E that when increased, the max flow doesn't increase

Proof:
$\rightarrow$
Lets say you increase the capacity of one edge in set E (edge e) by 1 and recalculate the max max-flow and the max-flow is the same as it was in the original graph. This means that edge e is not in the new min st-cut since the capacity of the of e has increased by 1 since it was in the old flow network and if it were in the new min st-cut the value of the st-cut would increase by at least 1. Since e is not in the new min-cut and the new min st-cut has the same value of the old st-cut, this means that the original graph has at least two different min st-cut and the original st-cut is not unique.
$\leftarrow$
Lets say that we know that the original cut is not unique and we have two min st-cuts that give us two sets of cut-edges E = $\{e_1, e_2, ..., e_k\}$, E$'$ = $\{e'_1, e'_2, ..., e'_k\}$. You can find an edge e that is in E but not in E' and if you just increase the capacity of e by 1. When you recalculate the min-cut of the new graph, we still have E$'$ on the flow network. Since E$'$ doesn't contain edge e, max flow remains the same.

My Algorithm:
Now we will first compute the st-cut for the given max flow, let's say we have a minimum st-cut (C) with a set of cut-edges E = $\{e_1, e_2, ..., e_k\}$. If this minimum (C) cut is not unique, then there exists some other minimum cut (C$'$) with a set of cut-edges E$'$ = $\{e'_1, e'_2, ..., e'_k\}$, such that E$'$!= E. If so, we can go through each edge in E, add to its capacity, recalculate the max flow (we don't need to use FF again, I'll say how later), and check if it increased. By the statement I proved above, If there exists an edge in E that when it's capacity is increased the max flow doesn't increase then the st-cut is not unique otherwise the st-cut is unique.
We don't need to recalculate the max flow for each edge in E, since we are given the max flow (the Prof. told us that in the lecture) we can just preform our computations on the residual graph of the max flow. So for each edge in E, increase the capacity in the edge in the residual graph and see if we can find a path from s to t, if so this means that we can increase the max flow, and if not

it means that we have no way to increase the max flow. This is what I propose:

Pseudo code:
C = min st-cut in network
FOR EACH (cut edge e in C) DO
    add to the capacity of e in the residual graph given
    check if you have an path from s to t in the new residual graph (using DFS)
    if yes (this means you can increase the max flow) → min st-cut not unique
st-cut is unique

Complexity:
We are running DFS at each point on the residual graph to see if we can find a path from s to t, DFS is O(m+n). The rest of the computations in my algorithm are constant time. So out running time will be O(m)*O(m+n) = O(m+n).

Resources used:
https://stanford.edu/ rezab/classes/cme305/W14/Midterm/pmidtermIsoln.pdf
https://stackoverflow.com/questions/7673711/determining-the-uniqueness-of-a-min-cut