
4. The edge connectivity of an undirected graph is the minimum number k of edges that must be removed to disconnect the graph. Show how the edge connectivity of an undirected graph $G = (V, E)$ can be determined by running Ford-Falkerson algorithm on at most $|V|$ flow networks, each having $O(|V|)$ vertices and $O(|E|)$ edges.

Solution:

I refer to the Ford-Falkerson algorithm as FF.

To find the minimum edge connectivity of a graph we must first make a set of flow network of the graph and then run FF on each of them to find the global min cut.

To build a flow network G' , first pick an arbitrary vertex to be the Source (s) and then consider every possible selection of vertices other than s to be the sink (t). We will have $n - 1$ choices for t , that means that we can build $n-1$ possible flow networks which is less than $|V|$, and each flow network has n vertices or $O(V)$. For each flow network built, we will replace each un-directed edge of the original graph with two directed edges with capacity 1 going in opposite directions, so for each edge (u,v) in the original graph we will add two directed edges (u,v) and (v,u) each with a capacity of 1. This means that each flow network has $2m$ edges which is $O(E)$.

Now let's say we have a flow network G' made out of the graph G , G' has an arbitrary sink and source. If we run the FF algorithm on G' then we will get a max flow which is equal to the value of the min-cut between s and t . Since all the edges in our flow network have 1 as their capacity then the value of the cut is equal to the number of edge cuts in the flow network. This means that in each of the flow networks

$$\text{max flow} = \text{min cut} = \text{minimum number of edges to disconnect } s \text{ and } t$$

And the max flow will give us the min number of edges that need to be removed for s and t to be in different components.

I claim that if we run the FF algorithm on all the $n-1$ flow networks that we've made, the smallest number is the edge-connectivity of the graph. To prove this, I shall prove that running FF on all the $n-1$ flow networks will give us the global min cut. Let k be the edge-connectivity of the graph, E be a set of k edges whose removal disconnects the graph, and let A be the connected component containing s in the disconnected graph resulting from the removal of the edges in E . A is the smallest min cut between all the flow networks and has capacity k and contains s , we must find A to find the edge connectivity of our graph.

In at least one iteration of the $n-1$ algorithms we run we have a flow network such that t is not in A , which means that A is a cut containing s with capacity k , and so when the FF algorithm finds a minimum st -cut in the network it must find a cut of capacity k . This means that, for at least one t , the cut A is a smallest min-cut and thus indicates the edge-connectivity.

So the minimum value of all the max flows gotten by running FF on all the flow networks created from the graph will give us the edge connectivity of the original graph.