Solution 1 (using strong duality)

(3). max

$$Z = 2x_1 + 3x_2 + x_3$$

S/t

$$x_1 + x_2 + x_3 \leq 40$$

$$2x_1 + x_2 - x_3 \geq 10 \quad = \quad -2x_1 - x_2 + x_3 \leq -10$$

$$-x_2 + x_3 \geq 10 \quad = \quad x_2 - x_3 \leq -10$$

$$x_1, x_2, x_3 \geq 0$$

re write :

$$\max \; Z = 2x_1 + 3x_2 + x_3$$

S/t

$$x_1 + x_2 + x_3 \leq 40$$

$$-2x_1 - x_2 + x_3 \leq -10$$

$$x_2 - x_3 \leq -10$$

$\left.\begin{array}{l} x_1 = x_2 = 10 \\ x_3 = 20 \end{array}\right\}$ is a feasible solution

it creates a lower bound for the optimal sol.

$\Rightarrow$

$$Z = 2x_1 + 3x_2 + x_3 =$$

$$20 + 30 + 10 \Rightarrow$$

$$Z^* \geq 70$$

Z^* being the optimal solution

if we 2 times the first constrait to the third constaint

we get

$$2(x_1 + x_2 + x_3) \leq 80$$

$$+ \quad (x_2 - x_3) \leq -10$$

$$\Rightarrow \quad 2x_1 + 3x_2 + x_3 \leq 70$$

we also know

$$Z^* = 2x_1 + 3x_2 + x_3 \leq 2x_1 + 3x_2 + x_3 \leq 70$$

Q3. Solution 2 (using complementary slackness)

primal LP :

$$\max Z = 2x_1 + 3x_2 + x_3$$

s/t

$$x_1 + x_2 + x_3 \leq 40$$

$$2x_1 + x_2 - x_3 \geq 10 \rightarrow -2x_1 - x_2 + x_3 \leq -10$$

$$-x_2 + x_3 \geq 10 \longrightarrow +x_2 - x_3 \leq -10$$

$$x_1, x_2, x_3 \geq 0$$

$$\left\{ \begin{array}{l} C = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ A = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \quad b = \begin{bmatrix} 40 \\ -10 \\ -10 \end{bmatrix} \end{array} \right.$$

from the lectures:

P)

max $C^T x$

s/t $Ax \leq b$

$x \geq 0$

D) min $y^T b$

s/t $A^T y \geq c$

$y \geq 0$

dual LP:

let $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$\min \quad y^T b \Rightarrow$$

$$\min \quad [y_1 \; y_2 \; y_3] \begin{bmatrix} 40 \\ -10 \\ -10 \end{bmatrix} \Rightarrow$$

$$\min \quad 40y_1 - 10y_2 - 10y_3$$

s/t

$$A^T y \geq c \Rightarrow$$

$$\begin{bmatrix} 1 & -2 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \geq \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \Rightarrow$$

$$\begin{cases} y_1 - 2y_2 \geq 2 \\ y_1 - y_2 + y_3 \geq 3 \\ y_1 + y_2 - y_3 \geq 1 \end{cases}$$

$$y_1, y_2, y_3 \geq 0$$

dual LP:

$$\min \quad 40y_1 - 10y_2 - 10y_3$$

s/t

$$y_1 - 2y_2 \geq 2$$

$$y_1 - y_2 + y_3 \geq 3$$

$$y_1 + y_2 - y_3 \geq 1$$

$$y_1, y_2, y_3 \geq 0$$

Since x_1, x_2, x_3 are all non-zero this means that

the first, second, and third constraints of the dual

have no slack;

$$y_1 - 2y_2 = 2$$

$$y_1 - y_2 + y_3 = 3$$

$$y_1 + y_2 - y_3 = 1$$

I'll find the sontion to these equations

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 2 \\ 1 & -1 & 1 & 3 \\ 1 & 1 & -1 & 1 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\Rightarrow \begin{cases} y_1 = 2 \\ y_2 = 0 \\ y_3 = 1 \end{cases}$$ these are the optimal solution for the dual

and these values satisfy the dual constraints

let's check if $x_1 = x_2 = 10$ is the optimal solution
$x_3 = 20$

first I'll check if the values satisfy the constraints in the primal problem

$10 + 10 + 20 \leq 40$ ✓

$2(10) + 10 - 20 \geq 10$ ✓

$-10 + 20 \geq 10$ ✓

$10, 10, 20 \geq 0$ ✓

the values satisfy the constraints

Now I'll check what the complementary slackness tells us about the optimal solution y_1, y_2, y_3 of the dual the values for the primal and dual are both feasible and give the optimal solution of 70 for both. So yes

$x_1 = x_2 = 10$ is optimal and
$x_3 = 20$

gives the value of 70