3.

Solution:

It's absolutely possible to have a problem that is in NP-hard but not in NP-complete. We know from the lectures, that to show that a problem is in NP-complete it's enough to show that it's in NP and NP-hard.

I believe the professor is wrong, given that it's simple to see that the problem is in NP (proof coming up). So if the problem is in NP-hard it'll definitely be in NP-complete, given that it's in NP.

Proof that the problem is in NP:

Given a certificate, the shortest path from s to t, we can check in polynomial time whether the path is of length k or not. So we can certify this in O(k) where k is the length of the shortest path given to us and the problem is definitely in NP.

Now that we know the problem is in NP, we can say for sure the professor is wrong, since if the problem is in NP-hard it'll definitely be in NP-complete. But is the problem in NP-hard?

I believe the problem is in NP-hard since the Hamiltonian Path problem which is in NP-complete and be reduced to it.

Proof that the problem is in NP-hard:

Assume we have a polynomial time algorithm for the problem above. To solve the Hamiltonian Path problem, we can call the algorithm for every pair of node in the graph ($\mathrm{O}(\mathrm{V}^2)$) and with K = ($\|V\|$-1).

Our algorithm will check if for every pair of node in our graph there is a shortest path of size ($\|V\|$-1), so it'll check if we have a path from s to t that passes thought all other nodes. Since the algorithm is checking for the shortest path we can be sure we will not use each edge and vertex more than once since using them more than once would create a cycle which would increase the length of the path which contradicts the shortest path.

If at any point the output of the algorithm is true then it means that we can find a path that visits every vertex exactly once and return true.

To solve the Hamiltonian Path problem we will have to make at most $\|V\|^2$ calls to the polynomial time algorithm of the problem we have in this question

So we have proven that a NP-complete problem reduces to the problem here which is also in NP.

So by the proofs above the problem in in NP and NP-hard thus is also in NP-complete.