Intro

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1 Introduction

Imagine yourself as a pre-historic human, perhaps collecting berries. You collect a red berry from a bush, then another, then another. You move on to another bush, collect a black berry, then another. And you notice, even though these berries are different, they are similar enough to combine them in regards to satiating your appetite.

Humans are really good at grouping items, we do it all the time while talking. So it is only natural that we notice we can combine similar items in our mind. Yet this is a most fundamental leap of abstraction. Thinking about our thinking, we might try to formalize this mental layer of abstraction through some terminology. We might say a group of similar items is a *set* composed of items of some *type* T.

Depending on the type of a set, these sets can be very different: for instance, a herd of cows is a different thing than a litter of cats, though at the same time they are also similar in that they are mammals. Yet there is a kind of property common to all sets: their size or *cardinality*. You might abstract from this and define a representation of possible cardinalities and call them *numbers*, and you might invent a method of determining the cardinality of a set by *counting*, in other words, using the equivalence between ordinals and cardinals to determine determine cardinality by ordinality. But then you might start to describe places by the number of steps needed to get to them, so to notice that the same means that allow you to measure cardinality also allow you to measure length. From this point on it is only a question of time until you invent real numbers.

A lot of time passes, and your study of numbers becomes a study of equations, then the study of structures. But what are these structures built on? You notice that, if you allow yourself to group mathematical entities into sets like you do with real ones, it gives the degree of recursivity that makes it very powerful as a base for any further constructions you might need. However, it also complicates the meaning of “set”: sets of physical structures are grouped together by their properties, can we do the same for mathematical structures? Can we make a set of all mathematical structures that are similar in some way?

And here you run into Russell's paradox: can we make a set of sets that are similar in that they do not contain themselves? This seems logically impossible. So we have to restrict our set theory somehow.

Along comes ZFC. Now you're safe: there is only one elementary set, the empty set, and there are some operations to derive other sets from each other, each of which seems fairly innocuous, which together give you a large enough universe to define most mathematical structures you might need. But you've also severed the contact between mathematics and nature.

If sets of real objects were mathematical, we could apply mathematical operations to them and it would make sense. As it stands, the sets of ZFC and a set of oranges seem to have very little to do with each other, and there would not be any reason that a law that holds for mathematical sets would hold for real sets. The unreasonable effectiveness of mathematics does indeed seem unreasonable if mathematical and real objects seem to have nothing in common. A reader might object that these two sets do have something in common, namely their cardinality, but determining the cardinality of a set should be a set-theoretic operation, so it cannot be applied to objects outside of set theory like sets of real objects.

ZFC is interesting in that it is foremost a theory of ordinals that are used for cardinal purposes, the order relation being what retains the link to previous theories of sets. $ETCS$ throws out the order-theoretic aspect and demonstrates that it is not needed to provide a home for mathematics. But in doing that, it further underlines the abstraction from reality that is already inherent in ZFC. It is a theory of *abstract* sets, wherein an "element" is just a morphism from the terminal object, which is just a point.

However, while set theory has always been with us, it has never been sufficient to encapsulate our thinking process. It allows too much: any two sets can be combined into a set, regardless of whether they are similar at all. This is already far from human behaviour. We generally do not group arbitrary objects together, but similar ones: we group black berries together because they are black, and we group black and red berries together because they are berries, so when we unify sets of objects what we are actually doing is not an abstract operation on sets, it is a change of the type of objects considered.

This is where type theory comes in, which, since it has models in categories of geometric objects, contains an internal aspect that is absent from set theory. A geometric space is more than just the sum of its points, thus to understand a space, it is important to probe it with more objects than just with the one-point space. A categorical calculus of spaces generally proceeds along the same lines everywhere: there always exists some form of projection and embedding, and a morphism factors as a projection followed by an embedding, so that a morphism $X \to Y$ can be understood as a (possibly degenerated) appearance of X in Y.

This is rather more like another ancient line of thought: the idea of *ideas*. According to this conception, objects in our world are degenerated instances (shadows) of ideas that are outside of this universe. So we might view our universe as a space $\mathcal{U}$ and any object in it as a morphism $O \to \mathcal{U}$ from the idea of that object into $\mathcal{U}$. Please note that a type here has the character of an object, not a family of objects like in set theory.

However, one might reasonably object that what we are really doing is not grouping berries together because they are shadows of the idea of a berry but because they have similar properties, such as being full of nutrients. Thus what we are actually doing when adding our two red berries to our three black berries is extending the range of objects under consideration by restricting the property used to classify objects to one shared by both black and red berries.

Properties are also often considered in geometry. In convenient situations, a property of some type on a space X is given by a morphism $X \to P$ of X into a *classifying space* of properties of that type. A simple example is temperature: up to a choice of unit (so a positive scalar), a temperature on a space X can be identified by a morphism from X into the positive real numbers, mapping each point p of X to the temperature at p.

But how would one then quantify berries? If we had an idea B of a berry, it would be simple, just take the set of functions $B \to \mathcal{U}$ from the idea of a berry into our universe. This straightforward interpretation doesn't work for properties since properties of a space are exactly not objects in X but morphisms from X into somewhere else. Furthermore, a property is not generally shared over an entire space, or it wouldn't be useful for quantifying objects in that space.

So what we would like is for there to be a kind of representability relationship between objects and properties: each property p should have an object $\overline{p}$ such that any object having the property p receives a morphism from $\overline{p}$, so is an instance of the idea $\overline{p}$.

So let's try this out with a toy model! Let's assume P is a classifying space for colors, so a color space, and we are looking at the slice topos of a topos of spaces over p so a topos of colored spaces. Let's say p is a point in that color space representing the color red (realistically "red" would be a whole subset of that color space, but let's worry about that later). Then an object would be red if and only if the terminal morphism into P factored through the point p. Conversely, the red point p could be mapped into any space containing a part that is red. It wouldn't really classify spaces that are solely red, but every instance of red anywhere would admit a morphism from the red point.

However, color is a purely local property. What about the property of having a hole? Well, the ∞-topos $\mathcal{H}$ of homotopy types has an object classifier $\hat{\mathcal{H}}_0$. The property of "having a hole" could be seen as meaning "receiving a morphism from the ideal hole" S^1, which corresponds to a morphism into $\mathcal{H}_0$. We might say more generally that this classifier classifies homotopic properties, which is quite a bit. However, the example also highlights a problem: according to this definition, every object with a point in it contains a hole, although perhaps a trivial one.

This goes hand in hand with another problem: for every pair of ideas, an idea should exist that is the largest commonality of these ideas, mirroring the generalization from *red berries* and *black berries* to *berries*. A natural candidate would be the cartesian product, which is a generalization of an intersection. However, the cartesian product of the ideal circle S^1 with itself is a torus, which doesn't mirror the largest commonality of the cartesian circle with itself at all! Both of these problems would disappear if we could focus our attention on monomorphisms, but S^n as homotopy types occupy all levels of the ∞-topos, so no notion of n-monomorphism encompasses all of them for finite n, and every morphism is an ∞-monomorphism. This problem would go away in a cohesive ∞-topos, where we have an actual geometry with a zero-truncated circle whose shape is S^1.

But it seems we are no closer to finding an idea of a berry. It seems unlikely that an inherent characteristics could suffice to define a berry, if anything, it would seem it could best be defined by its function: a berry is a means for a plant to procreate. Of course to really parse this sentence, we would have to know what all these words mean. Furthermore, there might be edge cases, where one would say a thing is a berry and another would not. The idea that we could find an unambiguous mathematical object for every linguistic notion seems silly. As it stands we can only express some basic intrinsic characteristics of objects.

At this point we might call it quits and say that, while we might not be able to restate complex cognitive concepts using the elementary language of mathematics, no harm should follow from assuming that there are types representing them, even just as purely formal objects, to close the gap between mathematics and reality. However, it could be hoped that category theory, which is all about functions and relations, could say a bit more about how things can be defined.