Hegel in Mathematics

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2 Introduction

One might be surprised to find Hegel's thought in the middle of a deep intersection between current research of logic, mathematics and physics, yet it would not have surprised Hegel, who saw in his method something with universal applicability. That he even *had* a method and was not just using obtuse language without or with trivial meaning was often called into question, particularly by the analytical school, starting with Russell in his *History of Western Philosophy* [20]:

> The great mathematicians of the seventeenth century were optimistic and anxious for quick results; consequently they left the foundations of analytical geometry and the infinitesimal calculus insecure. Leibnitz believed in actual infinitesimals, but although this belief suited his metaphysics it has no sound basis in mathematics. Weierstrass, soon after the middle of the nineteenth century, showed how to establish the calculus without infinitesimals, and thus at last made it logically secure. Next came Georg Cantor, who developed the theory of continuity and infinite numbers. "Continuity" had been, until he defined it, a vague word, convenient for philosophers like Hegel, who wished to introduce metaphysical muddles into mathematics. Cantor gave a precise significance to the word, and showed that continuity, as he defined it, was the concept needed by mathematicians and physicists.

It is ironic that the concepts Russell used in his attack, infinitesimals and "continuity", have been particularly useful, once re-evaluated through the recent formalization of Hegel's thought in the context of modal homotopy type theory. This was work started by William Lawvere in the 1985s [16]:

> In early 1985, while I was studying the foundations of homotopy theory, it occurred to me that the explicit use of a certain simple categorical structure might serve as a link between mathematics and philosophy.

Lawvere went on to provide a formally strict logical calculus that tries to capture Hegelian dialectics and started the formalization of Hegel's objective logic [15]. However, the mathematical power of Lawvere's formalization, though already considerable, was restricted by the mathematics it was founded on.[1] More recently, a new foundation of mathematics was developed under the initiative of Vladimir Voevodsky in homotopy type theory, which provides an alternative to set theory and a setting for logic based on a radical interpretation of equality that amplifies its power and has been found to show remarkable similarity to Hegel's thought. However, the importance of these devopments exceeds mathematics and reaches into philosopy. Lawvere stated in 1992 [13]:

> It is my belief that in the next decade and in the next century the technical advances forged by category theorists will be of value to dialectical philosophy, lending precise form with disputable mathematical models to ancient philosophical distinctions such as general vs. particular, objective vs. subjective, being vs. becoming, space vs. quantity, equality vs. difference, quantitative vs. qualitative etc. In turn the explicit attention by mathematicians to such philosophical questions is necessary to achieve the goal of making mathematics (and hence other sciences) more widely learnable and useable. Of course this will require that philosophers learn mathematics and that mathematicians learn philosophy.

This article aims to provide an exposition of the current state of the formalization of Hegel's thought, as continued in particular by Urs Schreiber, as it appeared first in [25] and [22]. A preliminary dictionary has

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[1]There is also some indication that Lawvere might not actually appreciate the current extension of his work due to his dislike of ∞-categories.

been set up in [27], [2] but it assumes an understanding of several new mathematical disciplines. Usually it is impossible to explain current mathematical research to a layperson, but the particularly intuitive character of the disciplines in question might allow doing so in this special case. This is attempted here.

2.1 How to read this text

A work like this is inevitably torn between contradictory priorities. Should it be easily understandable or exact? Exactness demands mathematics, and since the intended audience is one of non-mathematicians, it has to be sacrificed for digestibility at certain points. However, mathematical notation, once it is understood, greatly aids clarity, hence it has been adapted. So some necessary mathematical notation is introduced throughout the text, sometimes with a, more or less exact, definition, sometimes dialectical, so that their meaning might not be immediately understood but becomes clarified throughout the text. Terms, when they are introduced, will be written in cursive, in which case the context in which they are used should be taken as a first implicit definition or description of the notion they are denoting. Readers are advised to use the meaning of a word they intuit, and allow it to be modified throughout the text. Objects will often be denoted by letters, with similar letters denoting similar objects. For instance, propositions will be denoted by p and q, while types will be denoted T and S.

Then, should the focus be on the current theory, or on Hegel's relationship to it? The foremost purpose of this work is its applicability, thus exposition on the theory is generally preferred. However, quotes which are relevant for the identification of Hegel's terms with mathematical constructions are usually given, both in the original German and English, though this had the side effect of increasing the length of the work. We will be using the translation of A. V. Miller and importing its paragraph numbering.

It should be noted that this text is an exposition of the current (and ongoing, though somewhat stalled) formalization of Hegel's work as extended by Urs Schreiber and others, *not* of Lawvere's original formalization, so when terminology between Lawvere and Schreiber is in conflict, current terminology is preferred over Lawvere's.

Since it cannot describe the mathematics it is rooted in, this work is inevitably incomplete. As such, curious readers are invited to look up the details for themselves. Almost all the material presented here is contained in [27], which also includes some philosophical reflections in the introduction which will not be recapitulated since they can be read with ease. If any take up the invitation, this work should form a good companion piece to aid the endeavour by providing intuitive guidance.

2.2 Why read this text?

The importance of a mathematical formalization of Hegel for anyone interested in Hegel's thought lies in, first of all, simplicity and exactness, making it completely understandable without any ambiguity, and thus undeniable without denying mathematics. This formalization, furthermore, opens connections between Hegelian thought and current physics and mathematics, and can be used as a means to understand, in particular, geometry on a highly conceptual level. As noted by Schreiber, a formalization of Hegel has to inevitably contain some degree of arbitrariness. However, if Hegel claims to define a kind of logic, then his work should be interpretable from a logical context. The fact that this can be done is significant.

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[2]see also [24], [26] for an exposition with particular focus on the role of homotopy type theory, [23], particularly for more detail on the mathematical structures and the physics, and section 5 in [2].

3 Category-theoretic preliminaries

Over the past century, mathematics went through a rapid increase in expressiveness, as it subsumed not just *quantities* like numbers, but also *qualitative* meaning, as the focus shifted first from concrete entities, like numbers, to the structures they inhabit, like groups, and on to the relationships between those structures and the realms they inhabit, which are categories. However, abstraction did not stop there, but went on to define a whole slew of higher categories. These higher categories are where the constructions behind our definitions reside. We will however only need the barest fundamentals of category theory, for a more comprehensive introduction we refer to [17], see also [12] for a philosophical treatment. An category,[3] consists of a *type* of *objects* X and a type of *morphisms* $X \xrightarrow{f} Y$, such that each morphism has a *source* and a *target.* Furthermore, a morphism, usually called f or g and denoted as an arrow $f : X \to Y$ from its source X to a target Y can be composed with a morphism $g : Y \to Z$ with source Y to a morphism $f \,\mathbin{;}\, g : X \to Z$, and every object X has a morphism $id_X : X \to X$ called its *identity*, such that, whenever a morphism is composed with id_X, the result is equal to the morphism itself: $id_X \,\mathbin{;}\, f = f$ and $g \,\mathbin{;}\, id_X = g$. Such objects can be identified with Hegel's *Dinge*, expressed as in §1065a (of *Science of Logic* [7]):
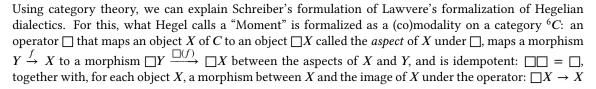
> Diese mehreren verschiedenen Dinge stehen in wesentlicher Wechselwirkung durch ihre Eigenschaften; die Eigenschaft ist diese Wechselbeziehung selbst, und das Ding ist nichts außer derselben;

> These many different things stand in essential reciprocal action via their properties; the property is this reciprocal relation itself and apart from it the thing is nothing;

as the "nature" of an object in category theory is completely determined by its relations to other objects.[4] In this sense, category theory can also be seen as a formalization of mathematical *structuralism* (see [1]).

We will mostly be focused on one particular (kind of) category, the *universe of types*, where types are the structures as first developed by Russell and nowadays commonly used in computer science.[5] There are several possible candidates for such a category, and we will not make a choice yet. Rather, we will outline the *theory* that has to be satisfied by such a choice, and choose a *model* later.

4 Calculus of Hegelian dialectics

Using category theory, we can explain Schreiber's formulation of Lawvere's formalization of Hegelian dialectics. For this, what Hegel calls a "Moment" is formalized as a (co)modality on a category [6]C: an operator $\Box$ that maps an object X of C to an object $\Box X$ called the *aspect* of X under $\Box$, maps a morphism $Y \xrightarrow{f} X$ to a morphism $\Box Y \xrightarrow{\Box(f)} \Box X$ between the aspects of X and Y, and is idempotent: $\Box\Box = \Box$, together with, for each object X, a morphism between X and the image of X under the operator: $\Box X \to X$

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[3]The following definition yields categories when applied to normal types, as they are usually understood. However, when we apply the same definition to homotopy type theory, as explained in 6 we get $(\infty, 1)$-categories, thus we don't have to define them separately.

[4]However, a mathematical object can live in different categories at once, if it allows, or necessitates, several different kinds of structures. For instance, the "real numbers" are a field, and as such live in the category of fields, but they are also a total order, so they live in the category total orders. If an object lives in several categories at once, the internal properties of the object often give rise to relations between the categories it is living in.

[5]Of course, this is a *mathematical* universe that houses *mathematical* structures, we will only get to our physical universe at the very end.

[6]In later sections we will mainly look at universes of types, but the potential applicability of Lawvere's calculus is much more general.

or $X \rightarrow \bigcirc(X)$. The direction of this morphism depends on whether the moment is a modality or a comodality. We will call moments of the form $\square(X) \rightarrow X$ *previous* moments and denote them by $\square$, while we call moments of the form $X \rightarrow \bigcirc(X)$ *successive* moments.[7][8]

Due to idempotency, any type that lies in the image of a moment is *purely of that moment* (and vice versa): $Y = \square X \Leftrightarrow \square Y = Y$. Thus, the types that are invariant under a moment form their own sub-category: $C_\square \hookrightarrow C$. The image $\square X$ of a type X under that moment should be thought of as "the best possible approximation of X in $C_\square$" or the $\square$-aspect of X, and the transformation $\square X \rightarrow X$ (resp. $X \rightarrow \bigcirc X$), depending on the exact nature of the moment, either a deduction of X from $\square X$ ($\bigcirc X$ from X), an embedding of $\square X$ into X (X into $\bigcirc X$), or a construction/deformation that makes $\square X$ into X (X into $\bigcirc X$).

From a category-theoretic perspective, this state of affairs can be understood as saying that a moment $\square$ can be decomposed into a *projection* of C onto another category $C_\square$, identifying objects if they have are the same in regards to the aspect $\square$:

$$C \twoheadrightarrow C_\square$$

and an *embedding* of $C_\square$ into C, assigning to each aspect a representative in C:

$$C_\square \hookrightarrow C.$$

Now, the concept of a *unity of opposites* is translated into a pair of moments $\triangle_1 \dashv \triangle_2$ that fulfill the *adjointness condition*: for any two objects X, Y and each morphism $\triangle_1 Y \rightarrow X$, there exists a morphism $Y \rightarrow \triangle_2 X$, and these morphisms are subject to a further naturality condition. Adjunctions are deep structures that can be expressed in a variety of ways, this being the most concrete, and the reader cannot be expected to immediately grasp the meaning of the adjointness condition. However, some of its consequences can be used for better understanding. For instance, it follows that if one of the $\triangle_i$ is a preceding moment $\square$, then the other is a successive moment $\bigcirc$ and vice versa, so that a unity of opposites is either a *unity of a preceding to an opposite successive moment* (or *ps-unity* for short) of the form $\square \dashv \bigcirc$ or a *unity of a successive to an opposite preceding moment* (or *sp-*unity) $\bigcirc \dashv \square$. So a unity of opposites is made up of moments that are actually of opposite kinds (preceding to successive), and two opposite kinds of unities of opposites exist. Furthermore, one part of a unity of opposites uniquely determines the other in that if, for a moment $\square$ both $\square \dashv \bigcirc_1$ and $\square \dashv \bigcirc_2$ hold, then $\bigcirc_1$ and $\bigcirc_2$ are equal, and the other way around (however, a moment can have both a left and a right opposite, so the opposition is actually directed). Each type X sits in between its preceeding and successive aspects (let us call this the *aspect sequence of the unity*):

$$\square X \rightarrow X \rightarrow \bigcirc X.$$

Finally, the two sub-universes $C_\square$, $C_\bigcirc$ determined by the modalities are both equal *without the context of the surrounding universe C*, so their opposition actually lies in their relationship to the larger category, and

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[7]The *preceding* and *successive* terminology, as well as definitions building on it, are original work.

[8]Please note that these dialectics are *not* of a temporal, but *causal* manner. These two overlap in our universe, but in general, causality does not require time, which is why logic can be used when reasoning about mathematical objects. This is in line with Hegel's use of the word "moment", which he always used in the neutral gender i.e. "das Moment", meaning an aspect of something, while he also used "der Moment" with usual temporal meaning (see also p. 43- 44 in [11]). However, the analogy to time inherent in the word is welcome for consequent nomenclature and gives the term a kind of double meaning.

to be opposites they have to be the same. More precisely, we can, recalling the decomposition of moments into a projection and an embedding we saw before, say that the opposites of a unity share each one of their morphisms and describe the situation in the following way: a *ps*-unity $\Box \dashv \bigcirc$ consists of *two* embeddings of the same sub-universe $C_\Box = C_\bigcirc$ into the universe C and *one* projection from C onto $\mathscr{V}$:

$$C_\Box \overset{\longrightarrow}{\underset{\twoheadleftarrow}{\hookrightarrow}} C$$

while an *sp*-unity consists of *one* embedding of $\mathscr{V}$ into C and *two* projections of C onto $\mathscr{V}$:

$$C_\Box \overset{\twoheadleftarrow}{\underset{\twoheadleftarrow}{\hookrightarrow}} C$$

From this follows in particular that in an *sp*-unity $\Box \bigcirc X = \Box\Box Y = \Box Y = \bigcirc X$ and vice versa, so $\Box\bigcirc = \bigcirc$ and $\bigcirc\Box = \Box$. Thus, a type is purely of one moment if and only if it is purely of the other.

Adjunctions are category-theoretic concepts and thus can be applied in any suitable 2-category, so that an adjunction between a unity of opposites might itself be adjoint to a another unity of opposites [3]. This *opposite to a unity* of opposites $\triangle_1 \dashv \triangle_2$ is another unity of opposites, and their relation will be denoted

$$
\begin{array}{ccc}
\triangle_3 & \dashv & \triangle_4 \\
\bot & & \bot \\
\triangle_1 & \dashv & \triangle_2
\end{array} \ .
$$

From the uniqueness of adjoints follows that in such a configuration the left moment of the first unity has to be equal to the right moment of the second unity: $\triangle_1 = \triangle_4$. So each unity of opposites of unities takes the form of a string of modalities $\diamond \dashv \Box \dashv \bigcirc$ or $\bigstar \dashv \bigcirc \dashv \Box$. We could ask for even higher opposites, but examples are rare and largely in categories with exotic or no logics.[9]

The other significant relation between unities of opposites is that of Aufhebung. Following (Schreiber's formalization of) Hegel, we say a unity of opposites $\triangle_3 \dashv \triangle_4$ is a *higher sphere*[10] of a unity of opposites $\triangle_2 \dashv \triangle_1$, denoted

$$
\begin{array}{ccc}
\triangle_3 & \dashv & \triangle_4 \\
\vee & & \vee \\
\triangle_1 & \dashv & \triangle_2
\end{array}
$$

if $\triangle_1$ is contained in $\triangle_3$: $\triangle_3\triangle_1 = \triangle_1$ and $\triangle_2$ is contained in $\triangle_4$: $\triangle_4\triangle_2 = \triangle_2$. A higher sphere $\Box_2 \to \bigcirc_2$ of a unity $\Box_1 \to \bigcirc_1$ is a *right Aufhebung* of $\Box_1 \to \bigcirc_1$ if furthermore $\Box_1$ is contained in $\bigcirc_2$: $\bigcirc_2\Box_1 = \Box_1$ and a *left Aufhebung* if $\bigcirc_1$ is contained in $\Box_2$: $\Box_2\bigcirc_1 = \bigcirc_1$. Both kinds can be referred to as *Aufhebungen*. We can similarly define Aufhebungen for *sp*-unities $\bigcirc \dashv \Box$, however, since in an *sp*-unity

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[9]For instance, the forgetful functor of the category of modules of a projective algebra is part of a five-fold adjunction. It could be interesting to see how it interacts with their internal type theory.

[10]This translation of the German word "Sphäre" is problematic because the meaning of "Sphäre" as a ball is less pronounced relative to the meaning "höhere Sphäre" as in "higher sphere of existence", compared to the English word "sphere". The topos-theoretic concept that Lawvere's dialectics is based on is called "higher level", which might be a better translation, since "sphere" too much invokes the image of something round.

both moments project into the same sub-universe, each higher sphere of $\bigcirc \dashv \square$ is already both a left and a right Aufhebung. We will denote a right Aufhebung as

$$
\begin{array}{ccc}
\square_2 & \dashv & \bigcirc_2 \\
\vee & & \vee \\
\square_1 & \dashv & \bigcirc_1
\end{array}
$$

and a left Aufhebung as

$$
\begin{array}{ccc}
\square_2 & \dashv & \bigcirc_2 \\
\vee & & \vee \\
\square_1 & \dashv & \bigcirc_1
\end{array}
$$

So an Aufhebung of $\square_1 \dashv \bigcirc_1$ is a unity of opposites $\square_2 \dashv \bigcirc_2$ such that $\square_1$ is a special aspect of $\square_2$ and $\bigcirc_1$ is a special aspect of $\bigcirc_2$, but where *also* both $\bigcirc_1$ and $\bigcirc_2$ are special aspects of one of the aspects of $\square_2 \dashv \bigcirc_2$, let's say $\bigcirc_2$, so that they are unified in $\bigcirc_2$ and both on the same side of a greater opposition. From this follows that the opposition $\square_2 \dashv \bigcirc_2$ is trivial on, in this case, $\square_1$: $\square_2\square_1 = \square_1 = \bigcirc_2\square_1$. Please note that this construction captures all three meanings of the German word *Aufheben*: to lift, to preserve and to negate.

Generally, Aufhebungen are not unique, the same unity can have several of them. The *minimal (left/right) Aufhebung* of a unity $\square \dashv \bigcirc$, the smallest sphere that fulfills the Aufhebungs-condition would be unique, but does not always exist.[11]

5 *Subjektive Logik* as Type Theory

Following the work of Martin-Löf in [18], Schreiber identifies the part of Hegelian logic which he calls "subjektive Logik" and which has been understood as classical logic, in his formalization with the deductive part of type theory, and extends it to type theory in general. A *Begriff* in Hegel's terminology is here understood as a type. For this identification it should be noted that to each logical proposition p a type T can be associated as the "type of reasons p is true", so that a *constituent t* of T can be understood as a "reason" p is true. We will call this the *internal logical perspective on types*. From this perspective, type theory is an extension of first order logic (often assumed without the *law of the excluded middle*, that either a proposition is true or its negation $p \vee \neg p$ or equivalently that from the untruth of the negation of a proposition follows the proposition itself $\neg\neg p \Rightarrow p$, a point we will return to): if T contains the reasons a proposition p is true and S contains the reasons q is true, then a morphism $m : T \rightarrow S$, which associates to each constituent t of T a constituent $m(t)$ of S, can be understood as a *deduction* of q from p, interpreting each reason p is true as a reason q is true, and be identified with Hegel's *Urteil* (judgement). Similarly, logical conjunction, disjunction and negation have type-theoretic equivalents. The passage from "reasons" for a proposition p to simply asking whether p is true consists then of a simple operation that can be accomplished within type theory.

However, a type can *also* be understood as a *context* for logic. We will call this the *external perspective on types*, and it is closely interlinked with the internal perspective. In this understanding, a proposition

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[11]As usual, we are following [27] in our terminology. Lawvere *defines* an Aufhebung as being minimal, see [13]. However, in light of the following derivation of unities this seems too restrictive a notion.

p corresponds to the largest sub-context $U \hookrightarrow X$ in which it is true. For this identification, it should be noted that the mathematical term "type" is abstracted from "type of mathematical structure", and one way to understand a type is as the container of a type of mathematical structure. Then type theory is a means of applying logic in the context of a type of structure. This way, type theory can be used as an alternative to *set theory*.

6 *Wesen* as Homotopy Type Theory

We will now discuss homotopy type theory, which we identify with Hegel's notion of *Wesen*,[12] though we will only fully justify the identification at the end of 7.5. Of course, we can only give a very brief introduction to homotopy type theory, a more complete one is given in [28].

The revolutionary advancement of homotopy type theory over traditional logic is to treat equality not as a property but as a structure. Concretely, this means the following: for any two constituents s, t of a type T,[13] there exists a type $s =_T t$ of *equalities* of s and t. Applying this requirement to itself we see that for any two constituents of $s =_T t$ a similar type of equalities has to exist, and so on. One way equality has traditionally been understood is as the strongest possible equivalence relation. In homotopy type theory, this is translated to saying that, given an equality $e \ : \ s =_T t$ between two constituents s and t, each property that holds for a constituent s has to also hold for constituent t *once it has been transported along e*. In particular, if there are two equalities $e \ : \ s =_T t$ and $f \ : \ t =_T s$, then f *yields an equality* $e^{-1} \ ; f \ : \ t =_T t$ *of t to itself once it has been transported along e*. Furthermore, there is always at least one equality id_t for each constituent t of a type T and a morphism always maps id_t to $id_{m(t)}$.[14]

We can now define a *morphism of homotopy types $m \ : \ S \to T$* recursively as a mapping that associates to each constituent s of S a constituent $m(s)$ of T along with, for each pair of constituents s, s', a morphism of homotopy types from $s =_S s'$ to $m(s) =_T m(s')$, satisfying a condition that expresses that it does not matter whether an equality is transported along another equality before or after applying a morphism.

The homotopy-theoretic interpretation of equality can be visualized by imagining a type T as a space, constituents $s, t \ : \ T$ as points within this space and equalities $e, f \ : \ s =_T t$ as paths between the points s and t. The canonical identity id_t of a constituent t then corresponds to the constant path on a point. An equality $f \ : \ s =_T t$ that is transported along another equality $e \ : \ t =_T s$ to give a self-equality $e ; f \ : \ t =_T t$ is visualized as, in a space with two points s, t and two paths e, f between them, first going along e from t to s, then along f from s to t, so going around a loop that starts and stops at t. A higher equality $\alpha \ : \ s =_T t$ could be visualized as a surface between the paths e and f, and so on. In particular, T contains only one constituent if and only if its visualization is *connected*. A morphism between homotopy types then maps points to points, paths to paths, surfaces to surfaces, and so on. Two homotopy types are equivalent if and only if they both have the same form once their fabric has been contracted as much as possible without gluing in new fabric anywhere, or, equivalenty, if they have the same n-dimensional holes. The condition that a property should hold for any equal constituents once it has been transported along an equality can be translated to stating that each property can be transported along any path in the space.

Understanding spaces through their points and paths between them is called homotopy theory,[15] and our

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[12]Translated as "essence" by Miller.

[13]But *not* for constituents of different types, in other words, constituents of different types are not just not equal, the question of whether they are equal cannot even be asked, at least without the context of a morphism between the two types.

[14]Please note the similarity between the definition of equalities of constituents and that of morphisms of a category.

[15]See [19] for an introduction.

mental image can be formalized into the statement that homotopy types can be modeled by topological spaces up to homotopy. It should be stressed that *this is a model*. If s and t are constituents which are equal, they cannot be different points in a space, since they then would not share every property. However, this model is important as it, conversely, allows us to model geometric shapes through equalities. We will return to this.

Homotopy theory has been of great use in modern mathematics. Each space can be assigned an underlying homotopy type, which captures a lot of information about that space. In a short while we will come across homotopy types that have *additional* geometric structure and study how Hegelian thought can be used to describe the relationship between a space and its underlying homotopy type.

It should be in this way that each equality that has appeared in this exposition up to now is understood, not as *saying* that its two expressions are equal, but as explicitly providing an equality that generally will not be the same as any other given equality between them. Another way to understand these equalities is as *equivalences*, in that they provide a means by which two structures can be exhibited as fulfilling the same function. This is a philosophical problem that has plagued modern mathematics since the formalization of its structures: often two structures are equivalent (or "isomorphic")[16] in the sense that they fulfill the same function, but not equal. Furthermore, making them equal would lose valuable information. Intuitively, the problem is that, classically, there can be at most one equality between two structures. In homotopy type theory, equivalence and equality can be treated equivalently (and are thus equal). This is in line with Hegel's dictum on *Verschiedenheit*:

> Alle Dinge sind verschieden, oder: Es giebt nicht zwei Dinge, die einander gleich sind.

> All things are different; or: there are no two things like each other.

Where difference between things is equated with the non-existence of an equivalence between them.

A way to express this *Law of Equivalence*[17] is to posit the existence of a "type of types". Such a type would be a reflection u of the universe $\mathcal{U}$ inside itself in that a type T would correspond to a constituent $'T'$ of u, and an equality between types S and T to a constituent e of the type $'S' =_u' T'$. This state of affairs reflects Hegel's dictum that (§115)

> Das Wesen scheint in sich, oder ist reine Reflexion...

> At first, essence shines or shows within itself, or is pure reflection...

Then the *Law of Equivalence* could be expressed as the *Univalence Axiom* that, for any two types S, T, such that S and T contain the reasons for propositions p and q, an equivalence exists between the type of *outer* equivalences $S \simeq T$, which contains the reasons for $p \Leftrightarrow q$, and the type $'S' =_u T'$ of *inner* equalities between $'S'$ and $'T'$, which correspond to equalities between S and T. This again can be understood in Hegel's terms as in §1149:

> Das innere ist als die Form der reflektierten Unmittelbarkeit oder des Wesens, gegen das Aussere als die Form des Seins bestimmt, aber beide sind nur eine Identitaet.

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[16]Actually, isomorphism is only the first in a hierarchy of increasingly weaker notions of equivalence that all collapse to equality if homotopy type theory is assumed. This is useful for the purposes of higher category theory, where these weaker equivalences are what actually matters, since isomorphism would be too strong a notion, similarly to how equality is too strong for lower-categorical structures.

[17]This is actually the strongest possible form of a "law" among category-theoreticians that no definition should treat equivalent things differently: if equivalent things *are* equal, then they cannot be treated differently.

> The inner is determined as the form of reflected immediacy or of essence against the outer as the form of being, but the two are only one identity.

However, doing this naively we would run into problems similar to Russell's paradox. To deal with this issue, usually a hierarchy of *Grothendieck universes* $\mathcal{U}_i$ is assumed, each containing the previous one, so that the universe $\mathcal{U}_{i+1}$ contains a type u_i, which contains all types within $\mathcal{U}_i$. We will still denote our universe $\mathcal{U}$ and assume it is one of the $\mathcal{U}_{i+1}$ on the ladder of universes, which contains a type of $\mathcal{U}_i$-types u.

One immediate consequence of the *Law of Equivalence* is a dramatic simplification of the type-theoretic universe $\mathcal{U}$, since any equivalent types have to be equal. For comparison, in a model $\mathcal{M}$ for the standard set theory mathematicians use as a foundation, ZFC, each class of equivalent, but non-equal, sets (except the one containing only the empty set), contains more members than any set in $\mathcal{M}$. Another important consequence of the existence of a type of types is that it allows a type theory that quantifies over sub-types of types and can thus be used to implement higher-order logic.

7 Objective Logic

We will now apply our formalization of Hegel's dialectics to the universe of types $\mathcal{U}$, which results in a process of structures derived from each other starting with very elementary assumptions. This process of derivation runs roughly parallel to the one Hegel sketched from his intuition in *Science of Logic*. However, the current process is strictly formalized and understandable through a fairly normal logical and geometric framework. The notions at the beginning of this process are also what led Lawvere to notice the similarity between mathematical structures and Hegel's. We will go through the mathematical process and see how well it lines up with that of Hegel when we try to identify the resulting mathematical notions with Hegel's (following Schreiber).

7.1 *Sein* and *Nichts*

For the process to begin, we first have to assume the existence of a (unique) type with one constituent $*$. This type is identified with Hegel's *Sein* (being). Hegel says about pure being, "it has no diversity within itself nor any with a reference outwards". Accordingly, everything within the one-constituent type $*$ is equal. Furthermore, $*$ has no diversity with a reference outwards in that every morphism $m : * \to T$ from $*$ to any other type is of the same form: a constituent $m(*)$ of T, which can (and, in a sense, should, as we will see) be thought of as a point in a completely homogenous, completely symmetric space of points. Notice also that for each type T, exactly one morphism $T \to *$ to $*$ exists, mapping each constituent of T to $*$. This can be thought of as everything in T becoming equal.[18]

Next we introduce the *empty type* $\varnothing$. This is a type that contains no constituents and is identified with Hegel's *Nichts* (nothing).[19] Inspecting the possibilities for morphisms on $\varnothing$, we notice that a unique mor-

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[18] Please note here that this becoming equal can be understood as projecting all of T onto a single element, or as inserting equalities between any pair of constituents of T, inserting equalities between those equalities and so on, until everything is equal. According to homotopy type theory, these two processes are the same.

[19] It should be noted here that there is a canonical filtration of any ∞-topos $\mathcal{U}$ (such as the universe of homotopy types) by truncation modalities $\tau_{\leq n}$, where $\tau_{\leq -1}(\mathcal{U})$ is the total order of truth values. In classical logic only two truth values exist, *true*, which we identify with $*$, and *false*, which we identify with $\varnothing$, so the image of $\tau_{\leq -1}$ has the concrete form of $\varnothing \to *$. Furthermore, the image $\tau_{-2}(\mathcal{U})$ is the one object subcategory on $\star$. In this sense, Hegel's introduction of being before nothing is correct. Reasoning within homotopy type theory, the fact that being is in this sense more fundamental than nothing can be understood as saying that if nothing (identified

phism exists from the empty type to any other type, with empty source and therefore empty image. This can be taken as a generalization of the logical proposition that "ex falsio quot libet".

Though we have introduced $*$ and $\varnothing$ as axioms, we could also *deduce* their existence from that of finite products and coproducts in the universe $\mathcal{U}$ in question, which is equivalent to its internal logic allowing the formation of arbitrary finite conjunctions resp. disjunctions. $*$ is then the product over nothing, the conjunction of no propositions, which corresponds well to Hegel's description of being as[20]

> Sein, reines Sein, ohne alle weitere Bestimmung.

> Being, pure being, without any further determination.

Dually, $\varnothing$ is the coproduct over nothing. Here an interpretation is more subtle, since binary disjunction in a universe of types is not actually the binary coproduct, but the (-1)-truncation of the binary coproduct. For this reason it is better in this context to think of the coproduct over nothing not in terms of propositions but view it as expressing a disjoint union, which means then that $\varnothing$ is the disjoint union of no sets/types.

We can now form our first unity of opposites. The left moment will be denoted $\varnothing$ and is defined by $\varnothing T := \varnothing$. Similarly the right moment will be denoted $*$ and is defined by $*T := *$. The adjunction condition is trivially fulfilled: $Hom(\varnothing, T) = * = Hom(T, *)$. This unity $\varnothing \dashv *$ we take as *becoming* as in Hegel's description

> Das reine Seyn und das reine Nichts ist also dasselbe. Was die Wahrheit ist, ist weder das Seyn, noch das Nichts, sondern daß das Seyn in Nichts, und das Nichts in Seyn,—nicht übergeht,—sondern übergegangen ist. Aber eben so sehr ist die Wahrheit nicht ihre Ununterschiedenheit, sondern daß sie nicht dasselbe, daß sie absolut unterschieden, aber ebenso ungetrennt und untrennbar sind, und unmittelbar jedes in seinem Gegentheil verschwindet. Ihre Wahrheit ist also diese Bewegung des unmittelbaren Verschwindens des einen in dem andern; das Werden; eine Bewegung, worin beide unterschieden sind, aber durch einen Unterschied, der sich eben so unmittelbar aufgelöst hat.

> Pure Being and pure nothing are, therefore, the same. What is the truth is neither being nor nothing, but that being — does not pass over but has passed over — into nothing, and nothing into being. But it is equally true that they are not undistinguished from each other, that, on the contrary, they are not the same, that they are absolutely distinct, and yet that they are unseparated and inseparable and that each immediately vanishes in its opposite. Their truth is therefore, this movement of the immediate vanishing of the one into the other: becoming, a movement in which both are distinguished, but by a difference which has equally immediately resolved itself.

and the aspect sequence of the unity of becoming $\varnothing = \varnothing T \to T \to *T = *$ formalizes Hegel's statement that

> ...es gar nichts giebt, das nicht ein Mittelzustand zwischen Seyn und Nichts ist.

> ...there is nothing which is not an intermediate state between being and nothing.

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with the empty type $\varnothing$) exists, this nothing is equal to itself by $id_\varnothing$, thus the existence of nothing implies the existence of being. Notice the similarity to Hegel's description of nothing as "simply equality with itself".

[20]This correspondence is original work.

7.1.1 Negatives

The existence of $*$ allows us to define another structure we will need later on: From a preceding moment $\square$, we can form a successive moment $\overline{\square}$ given by, for each type T, the cofiber $\overline{\square}$ of the morphism $\square T \to T$, which trivializes the the image of $\square T$ in T by collapsing it to a point. This is, perhaps somewhat more loosely, identified with the Hegelian *negative* based on the paragraphs 911:

> Diese in sich reflektirte Gleichheit mit sich, die in ihr selbst die Beziehung auf die Ungleichheit enthält, ist das Positive; so die Ungleichheit die in ihr selbst die Beziehung auf ihr Nichtseyn, die Gleichheit enthält, ist das Negative.—Oder beide sind das Gesetztseyn; insofern nun die unterschiedene Bestimmtheit als unterschiedene bestimmte Beziehung des Gesetztseyns auf sich genommen wird, so ist der Gegensatz eines Theils das Gesetztseyn in seine Gleichheit mit sich reflektirt; andern Theils dasselbe in seine Ungleichheit mit sich reflektirt; das Positive und Negative.—Das Positive ist das Gesetztseyn als in die Gleichheit mit sich reflektirt; aber das reflektirte ist das Gesetztseyn, das ist, die Negation als Negation, so hat diese Reflexion in sich die Beziehung auf das Andere zu ihrer Bestimmung. Das Negative ist das Gesetztseyn als in die Ungleichheit reflektirt; aber das Gesetztseyn ist die Ungleichheit selbst, so ist diese Reflexion somit die Identität der Ungleichheit mit sich selbst und absolute Beziehung auf sich.—Beide also, das in die Gleichheit mit sich reflektirte Gesetztseyn hat die Ungleichheit, und das in die Ungleichheit mit sich reflektirte Gesetztseyn hat auch die Gleichheit an ihm.

> This self-likeness reflected into itself that contains within itself the reference to unlikeness, is the positive; and the unlikeness that contains within itself the reference to its non-being, to likeness, is the negative. Or, both are a positedness; now in so far as the differentiated determinateness is taken as a differentiated determinate self-reference of positedness, the opposition is, on the one hand, positedness reflected into its likeness to itself and on the other hand, positedness reflected into its unlikeness to itself — the positive and the negative. The positive is positedness as reflected into self-likeness; but what is reflected is positedness, that is, the negation as negation, and so this reflection-into-self has reference-to-other for its determination. The negative is positedness as reflected into unlikeness; but the positedness is unlikeness itself, and this reflection is therefore the identity of unlikeness with itself and absolute self-reference. Each is the whole; the positedness reflected into likeness-to-self contains unlikeness, and the positedness reflected into unlikeness-to-self also contains likeness.

and 938:

> Das Negative ist also die ganze, als Entgegensetzung auf sich beruhende Entgegensetzung, der absolute sich nicht auf Anderes beziehende Unterschied; er schließt als Entgegensetzung die Identität von sich aus; aber somit sich selbst, denn als Beziehung auf sich bestimmt er sich als die Identität selbst, die er ausschließt.

> The negative is, therefore, the whole opposition based, as opposition, on itself, absolute difference that is not related to an other; as opposition, it excludes identity from itself — but in doing so excludes itself; for as self-relation it is determined as the very identity that it excludes.

Dually, we can also define a notion $\overline{\bigcirc}$ of the negative of a successive moment: the fiber of the morphism $T \to \bigcirc T$. However, this fiber depends on a chosen basepoint, so it only makes sense on types which have only one constituent (but possibly this constituent has (higher) equalities), or, thought of homotopically, have only one connected component. In this case, $\overline{\bigcirc} T$ contains that part of the structure of T that is trivialized by $T \to \bigcirc T$. Note that $\overline{\square}$ and $\square$ (or the dual notions) do *not* form a unity of opposites. However,

for each object T, a sequence $\overline{\Box}T \to T \to \Box T$ exists and this sequence *decomposes* each T, in the sense that T could be reconstructed from its aspects under a moment and its negative, as well as their relation. This is not generally true for unities of opposites.

7.2 Quantity and Quality

Given the unity of becoming $\varnothing \dashv *$, we can now derive a sequence of new unities starting with its minimal Aufhebung. We begin this sequence of derivations with the smallest moment that contains both $\varnothing$ and $*$, which is the *double negation* modality, mapping a type T_p of reasons a proposition p is true to the type $T_{\neg\neg p}$ of reasons its double negation is true. [10].[21] The sub-universe $\mathcal{U}_\sharp$ that is invariant under $\sharp$ is the largest sub-universe in which the law of the excluded middle holds. Thus, if our new unity is supposed to be non-trivial, the internal logic of our universe $\mathcal{U}$ cannot be classical. *If the law of the excluded middle is assumed to hold universally, then the entire sequence of structures we are about to derive collapses into triviality.* We will therefore assume that the internal logic we are operating in is *intuitionistic*, meaning it is like classical logic but without the law of the excluded middle. The action of the double negation modality can again be understood both from an internal and an external perspective.

Internally, say a type T is the context we are operating in, and a proposition p defines the largest sub-context $U \hookrightarrow T$ in which it holds. Then the negation $\neg p$ of p similarly defines the largest sub-context $\neg_T U$ on which it holds, and each constituent in $* \to T$ is either contained in U or in $\neg_T U$. However, in spite of this, T is *not* just the union of U and $\neg_T U$, which corresponds to the proposition $p \vee \neg_T p$, the law of the excluded middle in T. What is missing from this union is a structure of T that lives between U and $\neg U$ and is not made up of points. This is a kind of *spatial* structure, which is not just determined by points, but goes beyond them, kinds of local neighborhoods of points that build up the spatial structure of T. U and $\neg_T U$ slice up T between them but in doing so cut out the spatial structure that connects them. The double negation $\neg\neg T$ of T is then a space with the same constituents as T, but where no information is lost by slicing it up, which implies that the spatial structure between any constituents of T is trivial in some way.

Externally, since in intuitionistic logic it is assumed that a proposition implies its double negation even if the law of the excluded middle is not, each constituent t of a type T, which contains the reasons a proposition p is true, gives rise to a reason $\sharp_T(t) : \sharp T$ for why $\neg\neg p$ is true. This mapping constitutes the morphism $T \to \sharp T$ that is the expression of $p \Rightarrow \neg\neg p$. If t_1 and t_2 are non-equal constituents of T corresponding to reasons for p, then $\sharp_T(t_1)$ and $\sharp_T(t_2)$ correspond to non-equal reasons for $\neg\neg p$, thus $\sharp_T$ does not identify points. The failure of the law of the excluded middle to hold is then the failure for this map to be an equivalence. Since both T and $\sharp T$ have the same points, this failure must be due to spatial structure that is not determined by their points. Furthermore, since it is provable in intuitionistic logic that $\neg\neg\neg p$ is equivalent to $\neg p$, their corresponding types are equal. So the failure of the law of the excluded middle can be measured by the amount of non-equal types T that are mapped to the same double negation $\sharp T$ (at least two, T and $\sharp T$ for some T if the law of the excluded middle fails), which corresponds to the amount of different spatial structures that can be put onto the same points. Furthermore, since each T maps to its double negation, $\sharp T$ must have the coarsest possible geometry among all types that it is the double negation of, in the sense that any morphism m into any T yields a map into its double negation, given by its composition with the double negation morphism $m \,\text{\textcorner}\, \sharp_T$. This means that no spatial structure can exist to distinguish the points of $\sharp T$, they all exist in the same place, but they are not equal, so that any morphism into T is already *continuous*. *So the failure of the law of the excluded middle in a context T lies precisely in the geometry of T.* This state of affairs fits remarkably well Hegel's description of continuity:

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[21]This terminology stems from mathematical logic and is not to be confused with the Hegelian negative.

Die Kontinuität ist also einfache, sich selbst gleiche Beziehung auf sich, die durch keine Grenze und Ausschließung unterbrochen ist, aber nicht unmittelbare Einheit, sondern Einheit der fürsichseienden Eins. Es ist darin das Außereinander der Vielheit noch enthalten, aber zugleich als ein nicht Unterschiedenes, Ununterbrochenes. Die Vielheit ist in der Kontinuität so gesetzt, wie sie an sich ist; die Vielen sind eins was andere, jedes dem anderen gleich, und die Vielheit daher einfache, unterschiedslose Gleichheit. Die Kontinuität ist dieses Moment der Sichselbstgleichheit des Außereinanderseins, das Sichfortsetzen der unterschiedenen Eins in ihre von ihnen Unterschiedenen.

Continuity is, therefore, simple, self-same self-relation, which is not interrupted by any limit or exclusion; it is not, however, an immediate unity, but a unity of ones which possess being-for-self. The asunderness of the plurality is still contained in this unity, but at the same time as not differentiating or interrupting it. In continuity, the plurality is posited as it is in itself; the many are all alike, each is the same as the other and the plurality is, consequently, a simple, undifferentiated sameness. Continuity is this moment of self-sameness of the asunderness, the self-continuation of the different ones into those from which they are distinguished.

We will therefore call $\sharp$ the *moment of continuity*. We'll furthermore call our types *spaces* from now on and denote them by X to remind us of their spatial structure. More abstractly, the moment of continuity can be described as extracting from each space its points and collapsing them to the same place. Similarly, its opposite, $\flat$, if it exists (which we demand it does), trivializes the spatial X-structure by extracting the points of X and putting each into its own piece of space, completely disconnected from any other piece, forming the *discrete* space $\flat X$. For each X, the aspect sequence of the unity $\flat X \to X \to \sharp X$ can then be visualized as first bringing the points of $\flat X$ together until they form X, then bringing them even closer until they all occupy the same place. This unity $\flat \dashv \sharp$ can be understood as the *unity of discreteness and continuity*, as described by Hegel §397:

Unmittelbar hat daher die Größe in der Kontinuität das Moment der Diskretion, - die Repulsion, wie sie nun Moment in der Quantität ist. Die Stetigkeit ist Sichselbstgleichheit, aber des Vielen, das jedoch nicht zum Ausschließenden wird; die Repulsion dehnt erst die Sichselbstgleichheit zur Kontinuität aus. Die Diskretion ist daher ihrerseits zusammenfließende Diskretion, deren Eins nicht das Leere, das Negative, zu ihrer Beziehung haben, sondern ihre eigene Stetigkeit, und diese Gleichheit mit sich selbst im Vielen nicht unterbrechen.

In continuity, therefore, magnitude immediately possesses the moment of discreteness — repulsion, as now a moment in quantity. Continuity is self-sameness, but of the Many which, however, do not become exclusive; it is repulsion which expands the selfsameness to continuity. Hence discreteness, on its side, is a coalescent discreteness, where the ones are not connected by the void, by the negative, but by their own continuity and do not interrupt this self-sameness in the many.

Hegel writes of this unity in §398:

Die Quantität ist die Einheit dieser Momente, der Kontinuität und Diskretion.

Quantity is the unity of these moments of continuity and discreteness.

Since the spatial structure of $\sharp X$ is trivial, morphisms into it are completely determined by the values of their points. These objects[22] are identified with the *intensive quantities* of Lawvere [4], Grassmann [5],

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[22] As well as their subobjects.

and Hegel. Furthermore, the negative $\overline{\sharp}X$ of $\sharp X$ consists purely of the structure that makes X spatial, thus the local aspects of X. So these are identified with *extensive quantities*. A morphism on a space X into an intensive quantity $\sharp X$ can be described as something that associates to each point of X a quantity, while a function into $\overline{\sharp}X$ associates a quantity to volumina of X.

One way to understand how intensive and extensive quantities work is in relation to the decomposition of a system into subsystems [9]: given a system S_0 that can be decomposed into two[23] sub-systems S_1 and S_2, the value of an extensive quantity Q on S_0 is equal to the sum of the values of the quantity on S_1 and S_2. An example is the mass of a body or the length of a text. Meanwhile, an intensive quantity q has a value v_0 on the system S_0 if and only if it has the same value on both sub-systems S_1 and S_2. An example is the temperature of a body. However, if an intensive quantity q of a system is constant on one value v_0, the map $S_0 \to \sharp T$ lifts through the aspect sequence $\flat T \to T \to \sharp T$. In light of our understanding of the next unity in the process, we can see this as saying that *constant intensive quantities are qualities*.[24] This corresponds well to our intuition. For instance, color, understood as the wavelength of light, is an intensive quantity. However, we intuitively refer to an object as having a specific color if and only if all of its parts have the same color. Finally, both $\sharp$ and $\flat T$ express the *cardinality* of X, in its discrete and continuous form, so the description of this unity as quantity seems justified.

The $\flat$-moment is also identified with the *moment of repulsion* as in Hegel §342 and §343:

> ...so stößt das Eins sich selbst von sich ab. Die negative Beziehung des Eins auf sich ist Repulsion.

> Diese Repulsion, so als das Setzen der vielen Eins aber durch Eins selbst ist das eigne Außersichkommen des Eins, aber zu solchen außer ihm, die selbst nur Eins sind. Es ist dieß die Repulsion dem Begriffe nach, die an sich seyende. Die zweite Repulsion ist davon unterschieden, und ist die der Vorstellung der äußern Reflexion zunächst vorschwebende, als nicht das Erzeugen der Eins, sondern nur als gegenseitiges Abhalten vorausgesetzter, schon vorhandener Eins. Es ist dann zu sehen, wie jene an sich seyende Repulsion zur zweiten, der äußerlichen, sich bestimmt.

> ...the one repels itself from itself. The negative relation of the one to itself is repulsion.

> This repulsion as thus the positing of many ones but through the one itself, is the one's own coming-forth-from-itself but to such outside it as are themselves only ones. This is repulsion according to its Notion, repulsion in itself. The second repulsion is different from it, it is what is immediately suggested to external reflection: repulsion not as the generation of ones, but only as the mutual repelling of ones presupposed as already present.

and has its own left opposite $\smallint \dashv \flat$, so that $\smallint$ is, as by Hegel §395

> Die Attraktion ist auf diese Weise als das Moment der Kontinuität in der Quantität.

> Attraction is in this way the moment of continuity in quantity.

identified with the *moment of attraction*. Since $\smallint \dashv \flat$ is an *sp*-unity, $\smallint$ lands in the same spaces as $\flat$, so a space $\smallint X$ is discrete. In models we will demand that the modality $\smallint$ is the localization of the path object of the topos, in the sense that $\smallint X$ contracts all points that are connected to each other through a path to the same point, so $\smallint X$ becomes discrete, it remembers each path between points as an equality in its

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[23]Of course, a decomposition of any cardinality can be assumed.
[24]This identification is original work.

homotopical *shape*, as in Section 6.[25] Thus, the moment of attraction can be understood as the internal contraction within the fabric of X taking over and X contracting as much as possible while still not losing its shape. The unity of opposites $\int \dashv \flat$, and, more concretely, the *points to pieces transform* $\flat X \to X \to \int X$ can be understood as describing "the difference of the one and the many" as in Hegel §361:

> Der Unterschied von Einem und Vielen hat sich zum Unterschiede ihrer Beziehung auf einander bestimmt, welche in zwei Beziehungen, die Repulsion und die Attraktion, zerlegt ist, deren jede zunächst selbstständig außer der anderen steht, so daß sie jedoch wesentlich zusammenhängen. Die noch unbestimmte Einheit derselben hat sich näher zu ergeben.

> The difference of the one and the many is now determined as the difference of their relation to one another, with each other, a relation which splits into two, repulsion and attraction, each of which is at first independent of the other and stands apart from it, the two nevertheless being essentially connected with each other. Their as yet indeterminate unity is to be more precisely ascertained.

and is identified with *quality*, as, on the one hand, the $\int$-aspect of a type captures the quality of its shape, on the other hand its negative captures what makes it different from simply being the homotopy type of its shape. Overall, the system of modalities $\int \dashv \flat \dashv \sharp$ has been termed *cohesion* and the objects it acts on as *cohesive substance* by Lawvere,[14][26] and identified by Schreiber with *measure* as in Hegel 699:

> Im Maaße sind, abstrakt ausgedrückt, Qualität und Quantität vereinigt.

> Abstractly expressed, in measure quality and quantity are united.

and indeed $\int$ is a generalization of the Euler characteristic and the tools within this unity are already enough to define a very general notion of an integral[8].

7.3 The Aufhebung of Finiteness

Next we introduce an Aufhebung of the last derived unity $\int \dashv \flat$ given by an *sp*-unity $\mathfrak{I} \dashv \&$:

$$
\begin{array}{ccc}
\mathfrak{I} & \dashv & \& \\
\vee & & \vee \\
\int & \dashv & \flat
\end{array}
$$

Since $\mathfrak{I} \dashv \&$ is a higher sphere than $\int \dashv \flat$, the morphism $X \to \int X$ factors through $\mathfrak{I}$: $X \to \mathfrak{I}X \to \int X$. Thus, $\mathfrak{I}X$ can be understood as a contraction of X, where X is not contracted down to its basic shape, but only "small" paths in X are contracted. More precisely, what is contracted are *infinitesimal paths*, and, correspondingly, the negative $\overline{\mathfrak{I}}X$ of $\mathfrak{I}$ on a connected space is the *purely infinitesimal* part of X. We conclude this stage of the process by positing an opposite to $\mathfrak{I} \dashv \&$, given by a preceding moment $\mathfrak{R}$. These modalities $\mathfrak{R} \dashv \mathfrak{I} \dashv \&$ can together be taken to describe the infinitesimal structure of X. Hegel described the infinitesimal thus in §171:

> Das Angeführte ist auch dieselbe Dialektik, die der Verstand gegen den Begriff braucht, den die höhere Analysis von den unendlich-kleinen Größen giebt. Von diesem Begriffe wird weiter

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[25]This interaction between the two different spatial structures of X, that of its local structure (which is induced by its local elements) and its structure of homotopic equality, is of crucial importance in topology.

[26]More precisely, Lawvere defined the functors underlying the modalities, Schreiber identified them with modalities.

unten ausführlicher gehandelt.—Diese Größen sind als solche, bestimmt worden, die in ihrem Verschwinden sind, nicht vor ihrem Verschwinden, denn als dann sind sie endliche Größen;—nicht nach ihrem Verschwinden, denn alsdann sind sie nichts. Gegen diesen reinen Begriff ist eingewendet und immer wiederholt worden, daß solche Größen entweder Etwas seyen, oder Nichts; daß es keinen Mittelzustand (Zustand ist hier ein unpassender, barbarischer Ausdruck) zwischen Seyn und Nichtseyn gebe.—Es ist hierbei gleichfalls die absolute Trennung des Seyns und Nichts angenommen. Dagegen ist aber gezeigt worden, daß Seyn und Nichts in der That dasselbe sind, oder um in jener Sprache zu sprechen, daß es gar nichts giebt, das nicht ein Mittelzustand zwischen Seyn und Nichts ist. Die Mathematik hat ihre glänzendsten Erfolge der Annahme jener Bestimmung, welcher der Verstand widerspricht, zu danken.

The foregoing dialectic is the same, too, as that which understanding employs the notion of infinitesimal magnitudes, given by higher analysis. A more detailed treatment of this notion will be given later. These magnitudes have been defined as such that they are in their vanishing, not before their vanishing, for then they are finite magnitudes, or after their vanishing, for then they are nothing. Against this pre notion it is objected and reiterated that such magnitudes are either something or nothing; that there is no intermediate state between being and non-being ('state' is here an unsuitable, barbarous expression). Here too, the absolute separation of being and nothing is assumed. But against this it has been shown that being and nothing are, in fact, the same, or to use the same language as that just quoted, that there is nothing which is not an intermediate state between being and nothing. It is to the adoption of the said determination, which understanding opposes, that mathematics owes its most brilliant successes.

which is fairly on point: the *reduced* part $\Re X$ of a space X is the part of the space which *really* exists in that it contains points of X but no infinitesimals, and is embedded into X: $\Re X \to X$. Accordingly, the types purely of the negative $\overline{\Im}$ of $\Im$ are precisely infinitesimally thickened points. So both $\Re$ and $\Im$ are ways of trivializing infinitesimal structure, but, just like with $\flat$ and $\sharp$, are trivializing it in opposite ways: $\Re X$ contains the core of X that is spanned by its real part, but in this core infinitesimal structure is still contained, as long as it is induced by real structure. $\Im$ meanwhile identifies the infinitesimal neighborhoods of any points in X with each other, thereby contracting X to a shape that is *infinitesimally flat* or *crystalline*: every morphism from an infinitesimally thickened point to an $\Im$-modal space has to be constant. This is not true for $\Re$-modal spaces.

The unity $\Im \dashv \&$ is identified with *ideality* as the Aufhebung of quality, as in Hegel 305, where *infinite* is taken to mean *infinitesimal* i.e. infinitely small:

In Beziehung auf Realität und Idealität wird aber der Gegensatz des Endlichen und Unendlichen so gefaßt, daß das Endliche für das Reale gilt, das Unendliche aber für das Ideelle gilt; wie auch weiterhin der Begriff als ein Ideelles und zwar als ein nur Ideelles, das Daseyn überhaupt dagegen als das Reale betrachtet wird...

Die Idealität kann die Qualität der Unendlichkeit genannt werden; aber sie ist wesentlich der Proceß des Werdens und damit ein Übergang, wie des Werdens in Daseyn, der nun anzugeben ist. Als Aufheben der Endlichkeit, d. i. der Endlichkeit als solcher und ebenso sehr der ihr nur gegenüberstehenden, nur negativen Unendlichkeit ist diese Rückkehr in sich, Beziehung auf sich selbst, Seyn. Da in diesem Seyn Negation ist, ist es Daseyn, aber da sie ferner wesentlich Negation der Negation, die sich auf sich beziehende Negation ist, ist sie das Daseyn, welches Fürsichseyn genannt wird.

With reference to reality and ideality, however, the opposition of finite and infinite is grasped in such a manner that the finite ranks as the real but the infinite as the 'ideal' [das Ideelle]; in the same way that further on the Notion, too, is regarded as an 'ideal', that is, as a mere 'ideal', in contrast to determinate being as such which is regarded as the real...

Ideality can be called the quality of infinity; but it is essentially the process of becoming, and hence a transition — like that of becoming in determinate being — which is now to be indicated. As a sublating of finitude, that is, of finitude as such, and equally of the infinity which is merely its opposite, merely negative, this return into self is self-relation, being. As this being contains negation it is determinate, but as this negation further is essentially negation of the negation, the self-related negation, it is that determinate being which is called being-for-self.

Consequently, & is identified with *Seyn-für-eines* as in Hegel 322:

Dieß Moment drückt aus, wie das Endliche in seiner Einheit mit dem Unendlichen oder als Ideelles ist. Das Fürsichseyn hat die Negation nicht an ihm als eine Bestimmtheit oder Grenze, und damit auch nicht als Beziehung auf ein von ihm anderes Daseyn. Indem nun dieß Moment als Seyn-für-Eines bezeichnet worden, ist noch nichts vorhanden, für welches es wäre,—das Eine nicht, dessen Moment es wäre. Es ist in der That dergleichen noch nicht im Fürsichseyn fixirt; das für welches Etwas (—und ist hier kein Etwas—) wäre, was die andere Seite überhaupt seyn sollte, ist gleicherweise Moment, selbst nur Seyn-für-Eines, noch nicht Eines.—Somit ist noch eine Ununterschiedenheit zweier Seiten, die im Seyn-für-eines vorschweben können, vorhanden; nur Ein Seyn-für-Anderes, und weil es nur Ein Seyn-für-Anderes ist, ist dieses auch nur Seyn-für-Eines; es ist nur die Eine ldealität dessen, für welches oder in welchem eine Bestimmung als Moment und dessen, was Moment in ihm seyn sollte. So machen Für-eines-seyn und das Fürsichseyn keine wahrhaften Bestimmtheiten gegeneinander aus. Insofern der Unterschied auf einen Augenblick angenommen und hier von einem Fürsichseyenden gesprochen wird, so ist es das Fürsichseyende, als Aufgehobenseyn des Andersseyns, selbst, welches sich auf sich als auf das aufgehobene Andere bezieht, also für-Eines ist; es bezieht sich in seinem Andern nur auf sich. Das Ideelle ist nothwendig für-Eines, aber es ist nicht für ein Anderes; das Eine, für welches es ist, ist nur es selbst.—Ich also, der Geist überhaupt, oder Gott, sind Ideelle, weil sie unendlich sind; aber sie sind ideell nicht, als für-sich-seyende, verschieden von dem, das für-Eines ist. Denn so wären sie nur unmittelbare, oder näher Daseyn und ein Seyn-für-Anderes, weil das, welches für sie wäre, nicht sie selbst, sondern ein Anderes wäre, wenn das Moment, für-eines zu seyn, nicht ihnen zukommen sollte. Gott ist daher für sich, insofern er selbst das ist, das für ihn ist.

Für-sich-seyn und Für-Eines-seyn sind also nicht verschiedene Bedeutungen der Idealität, sondern sind wesentliche, untrennbare Momente derselben.

This moment expresses the manner in which the finite is present in its unity with the infinite, or is an ideal being [Ideelles]. In being-for-self, negation is not present as a determinateness or limit, or consequently as a relation to a determinate being which is for it an other. Now though this moment has been designated as being-for-one, there is as yet nothing present for which it would be-no one, of which it would be the moment. There is, in fact, nothing of the kind as yet fixed in being-for-self; that for which something (and here there is no something) would be, whatever the other side as such might be, is likewise a moment, is itself only a being-for-one, not yet a one. Consequently, what we have before us is still an undistinguishedness of the two sides which may be suggested by being-for-one; there is only one being- orother,

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and because there is only one, this too is only a being-for-one; there is only the one ideality of that, for which or in which there is supposed to be a determination as moment, and of that which is supposed to be a moment in it. Being-for-one and being-for-self are, therefore, not genuinely opposed determinatenesses. If the difference is assumed for a moment and we speak of a being-for-self, then it is this itself which, as the sublatedness of otherness, relates itself to itself as the sublated other, and is therefore 'for one'; it is related in its other only to its own self. Ideal being [Ideelles] is necessarily 'for one', but it is not for an other; the one for which it is, is only itself. The ego, therefore, spirit as such, or God, are 'ideal' because they are infinite; but as being for themselves they are not 'ideally' different from that which is 'for one'. For if they were, they would be only immediate existences, or, more precisely, determinate being and a being-for-other, because that which would be for them would be, not themselves but an other, if they were supposed to lack the moment of being 'for one'. God is, therefore, for himself in so far as he himself is that which is for him.

To be 'for self' and to be 'for one' are therefore not different meanings of ideality, but are essential, inseparable moments of it.

This identification is connected to Hegel's critique of atomism:

> Die Atomistik hat den Begriff der Idealität nicht; sie faßt das Eins nicht als ein solches, das in ihm selbst die beiden Momente des Fürsichseyns und des Für-es-seyns enthält, also als ideelles, sondern nur als einfach, trocken Für-sich-seyendes.

> The atomistic philosophy does not possess the Notion of ideality; it does not grasp the one as an ideal being, that is, as containing within itself the two moments of being-for-self and being-for-it, but only as a simple, dry, real being-for-self.

Here, the *one* is identified with the smallest unit of a space, while *atoms* are identified with its points. The complaint is then that a space is not just made up of its points but these points are connected by *ideal* structure.[27] This is precisely the approach of synthetic differential geometry.

Since the unity $\mathfrak{I} \dashv \&$ expresses *ideality*, its opposite $\mathfrak{R} \dashv \mathfrak{I}$ expresses *reality*. A system of such modalities $\mathfrak{R} \dashv \mathfrak{I} \dashv \&$ is called *differential cohesion* and its objects *elastic substance*. Notice the similarities and differences between $\mathfrak{R} \dashv \mathfrak{I} \dashv \&$ and $\int \dashv \flat \dashv \sharp$: in particular, both contain a kind of shape modality, but in one it is the leftmost, in the other it is in the middle. Meanwhile, the reduction modality seems to have no equivalent in the cohesive context. This fits with the nature of Aufhebung: we have already analyzed how the containment $\int < \mathfrak{I}$ of the shape modality in the infinitesimal shape modality necessitates the conclusion that the infinitesimal shape modality equalizes certain paths, it stands to reason that similarly the containment of both $\int$ and $\flat$ in $\&$ implies that $\&$ is shares aspects of both. However, $\&$ can also be understood through its position as the rightmost member in a unity of unities, which makes it similar to $\sharp$ in that both are related to the *classification* of quantities. It has been found that certain spaces F classify physical fields in the sense that a field $\mathscr{F}$ on a space X corresponds to a morphism from X into a certain other type F, the *classifying space* for this kind of field.[28] If F is the classifying space of some type of field, then a morphism $f : X \to \&F$ corresponds to a morphism $\overline{f} : \mathfrak{I}X \to F$, so the field classified by f is given by the field on its infinitesimal shape $\mathfrak{I}X$, extended it a trivial way to the entirety of X.

In this elastic structure, notions such as distance and angle can be derived, so this is the first sphere in

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[27]See also §254b and §256b in *Philosophy of Nature*.
[28]This opens a very deep connection to physics that we can only mention in passing but see [23].

which Euclidean geometry exists. Furthermore, large swaths of theory that would take several semesters to introduce in a classical manner, can elegantly be defined the using these modalities.

7.4 Light and Matter

Here the process extends with a *left* Aufhebung of the unity $\mathfrak{R} \dashv \mathfrak{I}$:

$$
\begin{array}{ccc}
\rightsquigarrow & \dashv & Rh \\
\vee & & \vee \\
\mathfrak{R} & \dashv & \mathfrak{I}
\end{array}
$$

and an opposition to the unity $\rightsquigarrow\dashv Rh$ given by an operator $\Rrightarrow$. Since $\rightsquigarrow\dashv Rh$ is a higher sphere than $\mathfrak{R} \dashv \mathfrak{I}$, the reduction modality factors as $\mathfrak{R}X \to \overset{\rightsquigarrow}{X} \to X$. Thus, $\rightsquigarrow X$ removes some of the infinitesimal structure of X, but not all of it. It also follows from the definitions that the removed infinitesimal structure is non-commutative in some way, so that our spaces are normal spaces, but have an infinitesimal thickening that is non-commutative. In models we will demand that these non-commutative spaces are actually super-spaces, which model spaces with two different types of particles in them, *fermions* and *bosons*. The *bosonic modality* $\rightsquigarrow$ strips away the fermionic part of a space, while its negative, the *fermionic modality* $\Rrightarrow$ consists of the fermionic content. The primary differences between fermions and bosons are that bosons are massless, move at lightspeed and several of them can be at the same place even if they have the same properties, whereas fermions have mass, can never obtain lightspeed and no two of them can be at the same place if they otherwise have the same properties. This leads to fermions making up matter and bosons making up forces. Thus, in the dictionary, the bosonic modality is identified with *light*, as in §220 of *Philosophy of Nature* (PoN) [6]:

> Als das abstrakte Selbst der Materie ist das Licht das absolute-leichte, und als Materie ist sie unendliches Außersichsein aber als reines Manifestieren, materielle Idealität untrennbares und einfaches Außersichsein.

> As the abstract self of matter, light is absolutely lightweight, and as matter, infinite, but as material ideality it is inseparable and simple being outside of itself.

which is a fairly good description of bosons considering only light was the only known boson at the time. Meanwhile, the fermionic modality is identified with *rigidity* as in PN§279 of PoN:

> Das Dunkle, zunächst das Negative des Lichts, ist der Gegensatz gegen dessen abstrakte-identische Idealität, – der Gegensatz an ihm selbst; er hat materielle Realität und er zerfällt in sich in die Zweihet, α) der körperlichen Verschiedenheit, d.i. des materiellen Fürsichseins, der Starrheit, β) der Entgegensetzung.

> Darkness, as immediately the negative of light, is the opposition to light's abstractly identical ideality; it is this opposition in its own self. It has material reality and falls apart into a duality, namely (1) corporeal difference, i.e. material being-for-self, rigidity; (2) opposition...

or, more generally, *matter*.

Compare again the newly derived unity of unities to its predecessors. Like $\int \dashv \flat \dashv \sharp$, and unlike $\mathfrak{R} \dashv \mathfrak{I} \dashv$ &, it consists of an opposition of an *sp*-unity to a *ps*-unity. Again, the middle modality $\rightsquigarrow$ functions most obviously similar to its lower counterpart, the reduction modality $\mathfrak{R}$. The rightmost modality in all three

cases is linked to the classification of quantities. This time, the *rheonomic* modality Rh classifies fields on spaces X that are trivially extended from the bosonic sub-space $\to X$ to the entirety of X.

7.5 *Wesen* opposing itself

The calculus of unities might perhaps be extended further. In any case, the Aufhebung of *all* unities of opposites is given by the unity of oppositions of the identity with itself: $id \dashv id$, with $id\ X = X$. This is identified with the *opposition of the* Wesen *to itself*, as in §812:

> Das Wesen aber, wie es hier geworden ist, ist das, was es ist, nicht durch eine ihm fremde Negativität, sondern durch seine eigne, die unendliche Bewegung des Seyns. Es ist An-und-Fürsichseyn; absolutes Ansichseyn, indem es gleichgültig gegen alle Bestimmtheit des Seyns ist, das Andersseyn und die Beziehung auf anderes schlechthin aufgehoben worden ist.

> But essence as it has here come to be, is what it is, through a negativity which is not alien to it but is its very own, the infinite movement of being. It is being that is in itself and for itself; it is absolute being-in-itself in that it is indifferent to every determinateness of being, and otherness and relation-to-other have been completely sublated.

This is the last stage of the process before it becomes externalized. We will treat this in the next section.

8 Nature

This system of modal type theory is now identified with Hegel's *Idee* (idea) as in §1631:

> Die Idee ist der adäquate Begriff, das objektive Wahre oder das Wahre als solches. Wenn irgend etwas Wahrheit hat, hat es sie durch seine Idee, oder etwas hat nur Wahrheit, insofern es Idee ist.

> The Idea is the adequate Notion, that which is objectively true, or the true as such. When anything whatever possesses truth, it possesses it through its Idea, or, something possesses truth only in so far as it is Idea.

This idea is then externalized into a model, meaning an ∞-topos that satisfies its true propositions. Such a model is identified with *Nature* as in

> Die Natur hat sich als die Idee in der Form des Andersseins ergeben. Da die Idee so als das Negative ihrer selbst oder sich äußerlich ist, so ist die Natur nicht äußerlich nur relativ gegen diese Idee (und gegen die subjektive Existenz derselben, den Geist), sondern die Äußerlichkeit macht die Bestimmung aus, in welcher sie als Natur ist.

> Nature has presented itself as the Idea in the form of otherness. Since therefore the Idea is the negative of itself, or is external to itself Nature is not merely external in relation to this Idea (and to its subjective existence Spirit); the truth is rather that externality constitutes the specific character in which Nature, as Nature, exists.

The simplest non-trivial model is the ∞-topos of *super formal smooth ∞-groupoids* $\infty - SFS$, however, a refinement might be needed, for instance to incorporate singular cohesion into the theory. Regardless, we can assume for our purposes that the *rheonomic* modality Rh is represented by an object $\mathbb{R}^{0|1}$ in the sense that the action of Rh consists of trivializing any instance of $\mathbb{R}^{0|1}$ in any space into a simple point. This is

true in $\infty - SFS$. It follows from the definitions that $\mathbb{R}^{0|1}$ is purely fermionic and has no bosonic content. Thus we call it the *fermionic superpoint*. From this fermionic superpoint, through a series of canonical extensions, emanates a piece of 11-dimensional spacetime $\mathbb{R}^{10,1|32}$, which has precisely the properties that are demanded by M-theory, the joint generalization of string theory and supergravity. Our physical universe would then be determined by some manifold in $\infty - SFS$ or some extension thereof, with some force field on it that is the unified fundamental force which manifests as the four fundamental forces of our universe after M through a process called KK-compactification, which yields the four-dimensional spacetime we all know and love.

At this point mathematics becomes difficult to avoid. However, it should be noted that a conjecture by Sati and Schreiber called *Hypothesis H*, based on the observations by Sati in [21], states that such a field would be described by a cycle in the fourth differential equivariant J-twisted cohomotopy on X, so a differential equivariant refinement of a section of the J-morphism into the 4-dimensional spere. Given the importance of the n-dimensional spheres and the J-morphism in homotopy theory (and in general), this result would be highly satisfying.

9 Some reflection

In the beginning, it was noted how unlikely it might seem that following a famously obtuse philosopher's line of thought would yield a strict logical calculus powerful enough to express large swaths of mathematics and reach towards a description of physics. Yet the formalism cannot be denied. Neither can the fact that Lawvere was inspired by and saw his work as a formalization of Hegel's thought. Furthermore, insights that had partially been had independently (such as homotopy typee theory) and partially as an extension of the work of Lawvere have been found to fit Hegelian descriptions remarkably well. How precise this correspondence is and how far it goes is still not determined. Let us reflect on these insights once again. The first, and the one most identified with Hegel, is a logical calculus that can be, and has been, like any other kind of logic, deformalized and applied to any suitable situation. In the aspect that he made the first attempt, at least in western philosopy, to formalize a kind of logic of fundamental usefulness that had been used throughout history, Hegel can be compared to Aristotele. The second is the use of homotopy type theory, which allows equality to be identified with function by treating it as a structure on its own instead of a property, and forms a bridge between logic and geometry. The third is the derivation of a sequence of dialectical notions from the logical notions of homotopy type theory. Apart from the clarification of several mathematical concepts, this step yields some concepts that are of importance in everyday philosophy, such as the distinction of intensive and extensive quantities. The fourth insight is an interpretation of this logical calculus in a concrete topos, which gives them geometric, and even physical, meaning, that can be used to come very close to a theory of everything. This is achieved in a manner that is so elegant and natural that it can be explained to a general audience, even if only at a superficial manner. This has to be compared to descriptions of equivalent structures in conventional mathematics, which seem arbitrary and cumbersome. Indeed, the ever more impenetrable nature of conventional physics has given rise to a pessimism that the structure of the universe might not conform to our inherent notions of beauty and elegance. This new and highly intuitive path to a model of the universe challenges this resignation.

9.1 What is to be done

Cohesion and its refinements have already garnered some attention in the mathematical community (though not as much as it deserves), and its potential, which has only just become visible, is currently explored by

Urs Schreiber and his associates. However, the formalization of Hegel's thought poses a philosophical challenge to both his followers and his detractors that has largely gone unnoticed. Both analytical and Marxist philosophers were quick to dismiss Hegel's metaphysical speculations. Now that these cryptic metaphysics have shown themselves capable to be interpreted through formally exact logic, these dismissals should be reconsidered.

The challenge extends however to the Hegelians themselves. As it stands, Hegelian dialectics can mean almost anything, depending on who is asked. While this formalization cannot capture all dialectic reasoning, as formal logic cannot ever capture all logical reasoning, it might provide at least a standard that can universally be understood in the same way, so that different dialectic interpretations can be compared through their differences to the common logical method. On a more immediate level, the current dictionary should be checked, compared and extended *also* by philosophers.

Finally, the simplicity of cohesion, combined with its power in expressing even otherwise obscure mathematical and physical concepts, might provide a path to bridge the divide between mathematics and physics on the one side and philosophy and the humanities on the other.

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