# ASSIGNMENT ONE

Abstract. Induction and Pigeonhole Principle. Deadline - June 1, 2024

## 1. Induction

**Question 1.1.** (Easy) Given $n$ points, $v_1, , \dots, v_n$ and $n$ numbers $d_1, \dots, d_n$ such that $\sum_{i=1}^{n} d_i = 2n-2, d_i \geq 1$. Prove that the number of trees on the set $\{v_1, \dots, v_n\}$ in which $v_i$ has degree $d_i$ is $\frac{(n-2)!}{(d_1-1)! \dots (d_n-1)!}$.

**Question 1.2.** (Hard) A connected graph is 2-*edge-connected* if it remains connected whenever fewer than 2 edges are removed. A *circuit* is path that begins and ends at the same vertex. A circuit that doesn't repeat vertices is called a *cycle*. We say that an edge is *well fitted* with the circuit $C$ if it lies on $C$ or has no point in common with $G$. An independent set of edges is a set of edges of which no two have a vertex in common. Let $e_1, \dots, e_k$ be independent edges of a 2-edge-connected graph $G$. Show that there is a circuit $C$ into which all edges $e_1, \dots, e_k$ are well fitted.

## 2. Pigeonhole Principle

**Question 2.1.** (Easy) Let $G = (V, E)$ be a graph, and $(C_v)_{v \in V}$ be a sequence of sets. We can look at each set $C_v$ as a color set for the vertex $v$. Given such a list of color sets, we consider only colorings $c$ such that $c(v) \in C_v$ for all $v \in V$ , and call them list colorings of $G$. A coloring is legal if no two adjacent vertices receive the same color. The chromatic number $\chi(G)$ of graphs is the smallest number of colors we need in order to color the vertices of $G$ in such a way that no two adjacent vertices receive the same color. The list chromatic number $\chi_l(G)$ is the smallest number $k$ such that for any list of color sets $C_v$ with $|C_v| = k$ for all $v \in V$ , there always exists a legal list coloring of $G$. Of course, $\chi_l(G) \leq |V|$. Show that $\chi(G) \leq \chi_l(G) \leq \Delta(G) + 1$.

**Question 2.2.** (Hard) A graph $G = (V, E)$ in which each vertex is connected to every other vertex is called a *complete* graph. Consider a complete graph $G$ on $\lfloor \frac{3k+1}{2} \rfloor$ vertices. If we 2-color the edges of $G$, then prove that there is a monochromatic path of length $k$.