THE INDUCTION PRINCIPLE

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Abstract. Induction principle is a powerful tool which utilizes inherent discreteness to prove statements. It is one of the most commonly used techniques in Mathematics.

Pre-requisites

Elementary graph theory.

1. Statements

Induction: Let $P(n)$ be a property dependent on $n \in \mathbb{N}$, and we have shown that following

(1) $P(b)$ is true for some b
(2) $\forall k > b,\ P(k)$ can be shown true assuming $P(k-1)$ is true

Then $P(n)$ is true $\forall n \geq b$

Strong form of Induction: Instead of just using $P(k-1)$ to prove $P(k)$, all of $P(b), P(b+1), \cdots P(k-1)$ can also be used.

2. Warm Up

Example 2.1. Let F_n be a sequence such that $F_0 = 1, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$. Show that
$F_{n-1}F_{n+1} = F_n^2 + (-1)^{n-1}$

Proof Idea. Write the expressions for $P(k-1)$ and $P(k)$. Use the recursive formula and add them.

Example 2.2. For any $a \in \mathbb{N}$ and prime number p, $a^p - a$ is multiple of p

Proof Idea. Use binomial expansion. For the induction step, use the fact $\binom{p}{j}$ is divisible by p for all $1 \leq j \leq p-1$.

Example 2.3. Every Tree of n vertices has $n-1$ edges

Proof Idea. We use induction on n. For $n = 1$, an acyclic 1-vertex graph has no edge. For $n > 1$, let the theorem be true for graphs with fewer than n vertices. Given an acyclic connected graph G, argue that one can always find a vertex with degree $= 1$. We call such a vertex "leaf". For leaf v show that $G' = G \setminus v$ also is acyclic and connected. Applying the induction hypothesis to G' yields $|E'| = n-2$. Since only one edge is incident to v, we have $|E| = n-1$.

OR

Induct on n. Base case is trivial. Use the fact that any two vertices in a tree are connected by a unique path(Otherwise argue it would be cyclic which contradicts the acyclicity of the tree). Assume the hypothesis is true for all trees with less than n vertices. Let $(u, v) \in E$, argue that removing this edge makes the graph disconnected and it decomposes into two trees G_1 and G_2. Using the induction hypothesis prove the required claim.

Example 2.4. Given a maximum degree of a graph $G = (V, E)$ is k. Then it is possible to color the vertices using $k + 1$ colors, such that no edge has same color on both side.

Proof Idea. DONT INDUCT ON k (You can try out this approach, it won't lead you to the answer). Induct on the number of vertices n. For induction step, we have $|V| = n+1$ with the maximum degree of G being k. Argue that on removal of a particular vertex and it's incident edges, we get a graph that satisfies the given properties. Add the vertex again, justify that the new graph can still be coloured from the set of $k + 1$ distinct colours such that no edge has end points sharing the same colour.

Example 2.5. Show that for $n \geq 0$

$$\sum_{k=0}^{n} \binom{n+k}{k} \frac{1}{2^k} = 2^n$$

Proof Idea. Induct on n. You may use the identity $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$.

3. More Questions

Question 3.1. *Prove that for any positive integer n, it is possible to partition any triangle T into $3n + 1$ similar triangles.*

Question 3.2. *Let μ be a function on the set of non-negative integers such that $\mu(1) = 1$. We have $\mu(n) = 0$ if $n > 1$ and n is divisible by the square of an integer $a > 1$. Otherwise, if $n = p_1 p_2 \ldots p_k$, where the p_i are all distinct primes, then let $\mu(n) = (-1)^k$. Prove that for all positive integers $m > 1$, $Z_n = \sum_{d>0, d|n} \mu(d) = 0$*

min max

Question 3.3. *A threshold graph is a graph you can build from K_1 (a graph with only one vertex) by repeatedly doing one of two operations: (1) add a vertex adjacent to all other vertices and (2) add a vertex not adjacent to any vertices. Prove that if G is a threshold graph, you can give every vertex v a number $0 < x_v < 1$ such that two vertices v, w are adjacent exactly when $x_v + x_w \geq 1$.*

Question 3.4. *Find the error in the following proof that all horses are the same color.*

CLAIM : *In any set of h horses, all horses are the same color.*

PROOF : *By induction on h.*

Basis : For $h = 1$. In any set containing just one horse, all horses clearly are the same color.

Induction step : For k greater than or equal to 1, assume that the claim is true for $h = k$, and prove that it is true for $h = k + 1$. Take any set H of $k + 1$ horses. We show that all horses in this set are the same color. Remove one horse from this set to obtain the set H_1 with just k horses. By the induction hypothesis, all of the horses in H_1 are the same color. Now replace the removed horse and remove *a different one to obtain the set* H_2*. By the same argument, all the horses in* H_2 *are the same color. Therefore, all the horses in* H *must be the same color, and the proof is complete.*