Aditya Mathur   Assignment - 1   220068

Q1.1 Base case : $n = 2$ $v_1 \rightarrow v_2$

$d_1 = 1$, $d_2 = 1$

no. of trees $= 1 = \dfrac{(2-2)!}{(1-1)!\,(1-1)!} = 1$ ✓

Induction hypothesis

Let the given statement be true for n number of vertices and $\sum_{i=1}^{n} d_i = 2n-2$ such that the number of trees be

$$\frac{(n-2)!}{(d_1-1)!\,(d_2-1)!\cdots(d_n-1)!}$$

$(n+1)$ case $\{v_1, \ldots, v_{n+1}\}$ s.t.:

$$\sum_{i=1}^{n+1} d_i = 2(n+1) - 2$$

As $n \geq 2$, therefore there exists a vertex with degree $= 1$

let $d_k = 1$ $\xrightarrow{\text{connected to } v_{k+1}}$ then removing v_k from the set, we get

$$\sum_{\substack{i=1 \\ i \neq k}}^{n+1} d_i = 2(n+1) - 2 - \underbrace{(2)}_{\text{one from } v_k \text{ and one from the corresponding connecting vertex}}$$

$\Rightarrow$ n vertices $\Rightarrow \{v_1, \ldots, v_{n+1}\} \setminus \{v_k\}$ and $\{d_1, \ldots, d_{n+1}\}$ ↳ $d_{k+1}-1$ and no d_k

with $\sum_{\substack{i=1 \\ i \neq k}}^{n+1} d_i = 2(n) - 2$

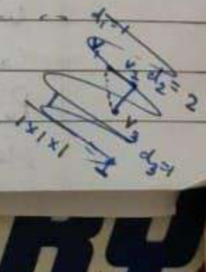
By induction hypothesis no. of such trees

$$= \frac{(n-2)!}{(d_1-1)!\,(d_2-1)!\cdots(d_{k+1}-2)!\cdots(d_{n+1}-1)!}$$

↳ remove d_k term.

Now, we can add leaf node v_k to any one of the $(n-1)$ vertices, thus and $(d_k-1)! = 1$, thus for total number of trees we have:

$$\text{Total} = \frac{(n-2)!}{(d_1-2)!\,(d_2-1)!\cdots(d_{n+1}-1)!} + \frac{(n-2)!}{(d_1-1)!\,(d_2-2)!\cdots(d_{n+1}-1)!} + \cdots + \frac{(n-2)!}{(d_1-1)!\,(d_2-1)!\cdots(d_{n+1}-1)!}$$



$$= \frac{(n-2)!\,[(d_1-1)+(d_2-1)+\cdots+(d_{n+1}-1)]}{(d_1-1)!\cdots(d_{n+1}-1)!}$$

because $\sum d_i - d_k = 2(n+1)-2-1 = 2n-1$

$$= \frac{(n-2)!\,(n-1)}{(d_1-1)!\cdots(d_{n+1}-1)!} = \frac{(n-1)!}{(d_1-1)!\cdots(d_{n+1}-1)! \times (d_k-1)!}$$

[because $d_k-1=0$]

$$= \frac{(n-1)!}{(d_1-1)!\cdots(d_{n+1}-1)!}$$

Hence, proved.

Q1.2 Induction on k : In 2-edge connected graph there exists atleast 2 distinct paths b/w any 2 points. → can be proven. — (★)

Base case : $k=1$

let $e_1 = (a_1, a_2)$

then $\exists$ a_3 as the graph is 2 edge connected. And so there is a path that connects a_2 to a_3 (name it P_1) and a_3 to a_1 - (name it P_2).

So e_1, P_1, P_2 forms the required circuit.

Induction hypothesis : let the statement : "independent edges $e_1, \ldots, e_k$ are well fitted in a circuit C" be true.

k+1 case.

let $\{e_1, \ldots, e_{k+1}\}$ be the set of independent edges in 2-edge connected graph G.

Start with $e_1 = (a_1, b_1)$. As G is 2-edge connected $\exists$ a path P_1 s.t. P_1 connects b_1 & a_2 where $e_2 = (a_2, b_2)$. s.t. P_1 doesn't have b_2 as a vertex (using ★)

So e_1, P_1, e_2 are connected. Similarly, e_1, P_1, e_2, P_2, e_3 can be taken into consideration where P_2 connects b_2 & a_3 and doesn't have b_3 as vertex.

Carrying on, we get $e_1, P_1, e_2, P_2, \ldots, e_{k+1}, P_{k+1}$ where P_{k+1} connects b_{k+1} to a_1. This forms the required circuit ; □.

Proof of ($*$):

To show: There exist 2 distinct paths (only common vertices are a & b) connecting a & b in a 2 edge connected

Assume there exists a common vertex in any 2 paths P_1 & P_2 namely v_k. Let P' be the corresponding path connecting a & v_k. Then removing one of the edge in P' leaves the graph unconnected. Contradiction, hence proved.

Q2.1 Let v be the vertex with maximum degree $= \Delta(G)$ then $\{v_1, \ldots, v_{\Delta(G)}\}$ are the neighbours of v. Then for lists of $(\Delta G+1)$ colours assigned to each vertex, there remains a colour left out that can be used to colour v as there are more number of colours than neighbouring vertices and using pigeonhole principle we get a legal colouring for v. For other vertices we can use induction on the number of vertices (or neighbours of a vertex) in G.

For base case: $|G| = 1$ is true.

Let it be true for n vertices in G.

Consider $(n+1)$ then remove a vertex v from it $\rightarrow G'$ is graph.

Now n vertices remain and so using induction hypothesis we get a legal colouring for $\Delta(G')+1$ colours in lists.

Now reinsert v. As $\Delta(G) \geq \Delta(G')$ thus with $\Delta(G)+1$ and v (max neighbours can be atmost $\Delta(G)$), there remains one colour for v. Hence, legal colouring.

$$\therefore \quad \chi_L(G) \leq \Delta(G) + 1 \quad \text{——} \quad (1)$$

Let $\chi(G) = n \Rightarrow \exists$ a legal coloring of G with k colours.

$\Rightarrow$ if we assign each vertex a list of k colours. then $\exists$ a legal colouring using colours from the lists.

therefore $\chi_L(G) \geq n$ and so

$$\chi(G) \leq \chi_L(G) \quad \text{——} \quad (2)$$

(1) & (2) proves the required result.

82.2. → Not able to do .