Group Theory (Handout - 3)

References : § Algebra Artin 2.7

1. Equivalence relations and partitions :-

An equivalence relation on a set S is a way to declare certain elements equivalent to each other ("$a \sim b$") yielding a smaller set of equivalence classes ("$S/\sim$")
(the quotient of S by $\sim$)

1.1 (Definition) An equivalence relation on a set S is a binary relation (i.e., a subset of $S \times S$) which is :-

$a \sim b$ iff $(a,b) \in R$

1) Reflexive : $\forall a \in S$, $a \sim a$

2) Symmetric : $\forall a, b \in S$, $a \sim b \Rightarrow b \sim a$

3) Transitive : $\forall a, b, c \in S$, if $a \sim b$ and $b \sim c$ then $a \sim c$.

- The equivalence class of $a \in S$ is $\{a' \in S \mid a \sim a'\}$ (denoted by $[a]$). (by transitivity, the elements of $[a]$ are all equivalent to each other.)

- The equivalence classes form a partition of S, i.e., these are mutually disjoint subsets of S whose union is S.

- The quotient of S by $\sim$ is the set of equivalence classes, $S/\sim \; = \{[a] \mid a \in S\} \subset P(S)$

This comes with a surjective map $\begin{aligned} S &\to S/\sim \\ a &\mapsto [a] \end{aligned}$

1.2 Examples :-

- $S = \mathbb{Z}$, given $n \in \mathbb{Z}_{>0}$, set $a \sim b$ iff $n \mid b-a$

There are n equivalence classes

$[0] = \{ \dots, -n, 0, n, 2n, \dots \}$
$= \mathbb{Z}n$

$[1] = \{ \dots, 1-n, 1, 1+n, 1+2n, \dots \}$

$\vdots$

$[n-1]$ "

The quotient is naturally in bijection with $\mathbb{Z}/n$

$$\begin{aligned} \mathbb{Z} &\twoheadrightarrow \mathbb{Z}/\sim \; \cong \mathbb{Z}/n \\ a &\mapsto [a] \end{aligned}$$

(We defined $\mathbb{Z}/n$ as $\{0, 1, \dots n-1\}$ only to avoid the language of equivalence classes)

But it makes more sense to redefine it as the quotient set

- Given a map $f: S \to T$, set $a \sim b$ iff $f(a) = f(b)$

  This is an equivalence relation, the partition into equivalence classes is

$$S = \bigsqcup_{t \in T} f^{-1}(t)$$

$\hookrightarrow = \{a \in S \mid f(a) = t\}$

$\hookrightarrow$ if f not surjective, only consider $t \in f(S) \subset T$

and f factors through quotient: $S \twoheadrightarrow S/\sim \hookrightarrow T$

$a \longmapsto [a] \longmapsto f(a)$

(if f surjective then $S/\sim \cong T$)

Using this construction: Equivalence relation on S

$\iff$ partition of S into disjoint subsets

$\iff$ surjective map from S to another set T (up to composition with a bijection $T \xrightarrow{\sim} T'$)

2. Back to Group Theory

Assume we have a surjective group homomorphism $\varphi: G \to H$

Recall, the kernel $K = \ker \varphi = \{a \in G \mid \varphi(a) = e_H\}$ is a subgroup of G.

Let's look at the partition of G induced by φ :-

$$\varphi(a) = \varphi(b) \iff \varphi(a)^{-1}\varphi(b) = e_H$$

$$\iff a^{-1}b \in K$$

let $k = a^{-1}b$, then $b = ak$

$$\iff b \in aK = \{ak \mid k \in K\}$$

2.1 (Definition) + (proposition) :-

Given any subgroup K of a group G,

- $aK = \{ak \mid k \in K\} \subset G$ is called the (left) coset of $K < G$ containing a.

- The relation $a \sim b \iff a^{-1}b \in K$ is an equivalence relation on G, whose equivalence classes are the left cosets.

- The quotient (the set of left cosets) is denoted by G/K. We have a partition $G = \bigsqcup_{aK \in G/K} aK$.

Proof : • $a^{-1}a = e \in K$, so $a \sim a \;\forall a \in G$

• if $a \sim b$ then $a^{-1}b \in K$, hence $(a^{-1}b)^{-1} = b^{-1}a \in K$

Hence $b \sim a$.

- If $a \sim b$ and $b \sim c$ then $a^{-1}b \in K$, $b^{-1}c \in K$, so $(a^{-1}b)(b^{-1}c) \in K$, $a \sim c$.

Also, $b \in aK \iff \exists k \in K$ s.t. $b = ak$

$\iff \exists k \in K$ s.t. $a^{-1}b = k$

$\iff a^{-1}b \in K \iff a \sim b$.

$\square$

2.2 Examples :-

$$\begin{aligned} \varphi : \mathbb{Z} &\twoheadrightarrow \mathbb{Z}/n \\ a &\mapsto a \bmod n \end{aligned}$$ has kernel $\mathbb{Z}n \subset \mathbb{Z}$

The cosets are $[k] = k + \mathbb{Z}n$ $(0 \leq k \leq n-1)$

and we have a bijection $\mathbb{Z}/\mathbb{Z}n \cong \mathbb{Z}/n$

$$[k] \mapsto k$$

This gives a group law on the quotient!

(Addition of cosets $\iff$ addition mod n)

When a subgroup K of the kernel of a homomorphism $\varphi : G \twoheadrightarrow H$, we get a bijection

$$G/K \cong H$$

$$aK \mapsto \varphi(a)$$

and we can use this bijection to get a group structure on G/K, essentially

$$(aK)\cdot(bK) = abK.$$

Then $G \twoheadrightarrow G/K$ is a group homomorphism
$a \mapsto aK$

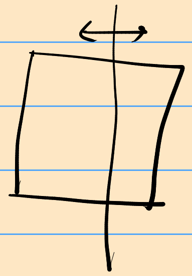
For a general subgroup $K \subset G$, however, trying to make G/K a group by setting $(aK)\cdot(bK) = abK$ might not work. The obstacle to this is:

Assume $a \sim a'$ ($\Leftrightarrow aK = a'K \Leftrightarrow a^{-1}a' \in K$) and $b \sim b'$ ($\Leftrightarrow bK = b'K \Leftrightarrow b^{-1}b' \in K$).

Does it follow that $ab \sim a'b'$? ($\Leftrightarrow abK = a'b'K$?)
(if not our operation is not well defined).

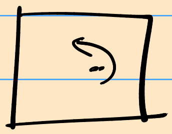
Example: $G = D_4$: symmetries of a square

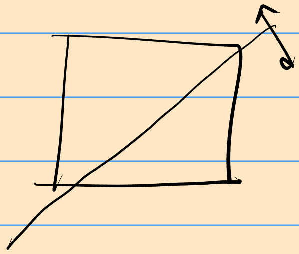
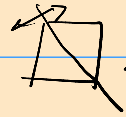
$H = \{e, h\}$ where h = horizontal flip



Then $e \sim h$ (coset $eH = hH$
$= \{e,h\}$)

but setting r = rotation by $90°$



$h \circ r =$  v/s the coset of $e \circ r = r$ is $\{ r, r \circ h =$  $\}$

$\Rightarrow$ $h \circ r \not\sim e \circ r$ even though $h \sim e$ (and $r \sim r$)

2.3 Right cosets v/s Left cosets :

Similarly to the left cosets
$$aK = \{ ak \,/\, k \in K \}$$
$$(a \sim b \iff a^{-1}b \in K)$$

We define right cosets $Ka = \{ ka \mid k \in K \}$, which compared to $a \sim b \iff ba^{-1} \in K$.

<u>Remark</u> : None of these are subgroups of G ! (except for K itself)
(They don't contain e !)

Denote : $aKa^{-1} = \{ aka \mid k \in K \}$

→ This is a subgroup

2.4 (Definition) $K \subset G$ is a **normal subgroup** if $\forall a \in G$, $aK = Ka$
or equivalently $\forall a \in G$, $aKa^{-1} = K$.

<u>Examples</u>:

- Any subgroup of an Abelian group is normal.

  ($a+K = K+a$ ✓).

- In D_4, the subgroup $H = \{e, h\}$ is not normal.

  horizontal reflection

$$\left(\begin{array}{l} rH = \{r, rh\} \\ \neq \quad Hr = \{r, hr\} \end{array} \right)$$

2.5 (Theorem) Given a group G and a subgroup $K \subset G$, the following are equivalent:

1) $\exists$ group homomorphism $\varphi : G \to H$ (some other group) with $\ker \varphi = K$

2) K is a normal subgroup

3) G/K has a group structure given by $(aK)\cdot(bK) = abK$