THE PIGEONHOLE PRINCIPLE

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ABSTRACT. We will apply the pigeonhole principle to study Ramsey's theorem for graphs. We'll also briefly cover the complexity of computing Ramsey Numbers and expose the readers to some of the outstanding open problems.

Theory Brief

The pigeonhole principle states the "obvious" fact that $n + 1$ pigeons cannot sit in n holes so that every pigeon is alone in its hole. As trivial as the pigeonhole principle itself may sound, it has numerous nontrivial applications. The hard part in applying this principle is to decide what to take as pigeons and what as pigeonholes.

Pre-requisites

Elementary graph theory and elementary discrete probability.

1. Warm Up

Pigeonhole Principle : If $kn + 1$ objects ($k \geq 1$ not necessarily finite) are distributed among n boxes, one of the boxes will contain at least $k + 1$ objects. More generally, given n pigeons in q pigeonholes, there has to be a pigeonhole with at least $\lceil n/q \rceil$ pigeons, and a pigeonhole with at most $\lfloor n/q \rfloor$ pigeons.

Example 1.1. Prove that every set of 10 two-digit integer numbers has two disjoint subsets with the same sum of elements. *(IMO 1972)*

Proof Idea. Let A be a set of 10 two-digit integer numbers. Argue that there are 1022 interesting subsets of S. Show that the sum of elements of any proper subset of A doesn't cross a suitable threshold, denote it by T. Show that T is less than 1022. The interesting subsets are pigeons and $[T]$ are pigeonholes. Since, the number of pigeons are more than the number of pigeonholes, we must have a pigeonhole that contains atleast two pigeons. Let A_1 and A_2 be two subsets of A that are kept in the same pigeonhole. Argue, why deleting the common elements gives us two disjoint sets with same sum.

Example 1.2. Given a set M of 1985 distinct positive integers, none of which has a prime factor greater than 23, prove that M contains at least one subset of four distinct elements whose product is the fourth power of an integer. *(IMO 1985)*

Proof Idea. (Taken from AOPS) We have that $x \in M \Rightarrow x = 2^{e_1}3^{e_2} \cdots 19^{e_8}23^{e_9}$. We need only consider the exponents. First, we consider the number of subsets of two elements, such that their product is a perfect square. There are $2^9 = 512$ different parity cases for the exponents $e_1, e_2, ..., e_9$. Thus, we have at least one pair of elements out of 1985 elements. Removing these two elements yields 1983 elements. By applying the Pigeonhole Principle again, we find that there exists another such subset. Continuing on like this yields at least 734 pairs of elements of M whose product is a perfect square. Let S be the set of the square roots of the products of each pair. Then, by the Pigeon Hole Principle again, there exist at least two elements whose product is a perfect square. Let the elements be x, y and let $x = \sqrt{ab}, y = \sqrt{cd}$ where $a, b, c, d \in M$. Then, we have $xy = z^2$ for some z which implies $abcd = z^4$ and the claim is proved.

Example 1.3. The points of the plane are colored by finitely many colors. Prove that one can find a rectangle with vertices of the same color.

Proof Idea. Consider a $(p+1) \times (n\binom{n+1}{2}+1)$ rectangular grid. By the pigeonhole principle, each of the $n\binom{p+1}{2}+1$ horizontal segments contains two contains two points of the same color. Find the maximum number of possible configurations of such monochromatic pairs. Show that this number is less than the number of horizontal segments. Conclude that the two pairs are the vertices of a monochromatic rectangle.

Example 1.4. Let G be an n-vertex graph. If every vertex has a degree of at least $(n-1)/2$ then G is connected.

Proof Idea. We'll show that every pair of vertices is either adjacent or shares a common neighbor. Pick two vertices u and v arbitrarily. If there is an edge between u and v, then we're done. If not, then u has at least $(n-1)/2$ neighbours and v isn't one of them. Same for v. Hence, let's look at the combined set of all edges of u and v, at least $n-1$ edges join them to the remaining vertices. These are pigeons. But, there are only $n-2$ other vertices represented by pigeonholes. The pigeonhole principle implies that one of them must be adjacent to both u and v.

Definition 1.5. Given $s \in \mathbb{N}$, let $R(s)$ be the minimum $n \in \mathbb{N}$ such that every red-blue colouring of the edges of K_n contains a subgraph isomorphic to K_s the edges of which all have the same color (refered to as being monochromatic)

Theorem 1.6. *(Ramsey, 1930) For every $s \in \mathbb{N}$, $R(s)$ is finite.*

Lemma 1.7. *For every $s \in \mathbb{N}$, $R(s) \leq 4^s$.*

Proof. Let $n = 2^{2s}$. We'll give a greedy algorithm to find a monochromatic K_s for an arbitrary (but fixed) red/blue edge-colouring $c : E(K_n) \mapsto \{red, blue\}$ of K_n. *Algorithm* : Pick an arbitrary vertex from the set of vertices still under consideration. Delete all neighbours whose edges are coloured with the less frequently appearing colour (we have at least half of the vertices remaining). Formally, let us set $S_0 := V$ and for every $i = 0, 1, ..., 2s-3$ do the following. Given a set S_i of size 2^{2s-i}, we select an arbitrary vertex in S_i, name it v_{i+1}, and let B_{i+1} and R_{i+1} denote the sets of those neighbors of v_{i+1} in S_i which are connected to it via a blue and a red edge, respectively. We have $\mid B_{i+1} \mid + \mid R_{i+1} \mid = |S_i| - 1$. We choose S_{i+1} to be the larger of B_{i+1} and R_{i+1}, so for its size we have $|S_{i+1}| \geq \lceil |B_i| + |R_i|/2 \rceil = \lceil (2^{2s-i}-1)/2 \rceil = 2^{2s-(i+1)}$, as desired. Now we will argue that this process terminates after finite number of steps i.e. v_{2s-2} can actually be selected. For that we need S_{2s-3} to be non-empty, which is the case since $|S_{2s-3}| = 2^{2s-(2s-3)} = 8$.

Definition 1.8. (Ramsey numbers) Given $s, t \in \mathbb{N}$, let $R(s,t)$ be the minimum $n \in \mathbb{N}$ such that every red-blue colouring of the edges of K_n contains either a red K_s or a blue K_t.

Theorem 1.9. *(Erdős–Szekeres, 1935) For every $s, t \in \mathbb{N}$, $R(s,t) \le \binom{s+t-2}{s-1}$. In particular, $R(s) = O(\frac{4^s}{\sqrt{s}})$*

Theorem 1.10. *(Ramsey, 1930). For any two-colouring of $\binom{\mathbb{N}}{2}$, there exists an infinite set $S \subset \mathbb{N}$ for which $\binom{\mathbb{S}}{2}$ is monochromatic.*

Proof. We apply the same greedy procedure as done in Lemma 1.7 infinitely often and hence create an infinite right-monochromatic sequence. The only adaptation we need here is that $S_i = B_{i+1} \cup R_{i+1}$ being infinite implies S_{i+1} being infinite. And the infinite right-monochromatic sequence gives rise to an infinite monochromatic clique.

Definition 1.11. (Multicolour Ramsey numbers). Given integers $r \ge 2$ and $t_1, t_2, \ldots, t_r \in \mathbb{N}$, let $R_r(t_1, t_2, \ldots, t_r)$ be the minimum $n \in \mathbb{N}$ such that for any colouring of the edges of K_n with colours from $[r]$, there is some index i for which there is a monochromatic K_{t_i} of colour i.

Theorem 1.12. *For any $r \ge 2$ and $t_1, t_2, \ldots, t_r \in \mathbb{N}$, $R_r(t_1, t_2, \ldots, t_r)$ is finite.*

Proof. We will induct on the number of colours, r. *Base case* : $r = 2$ (Theorem 1.6). For the induction step, suppose $r \ge 3$, and we have numbers $t_1, t_2, \ldots, t_r$. We will choose some suitable value of n and fix an arbitrary r-colouring c of the edges of K_n. Define coloring $c^* : E(K_n) \mapsto [r-1]$ from c. Let $c^*(xy) = r-1$ if $c(xy) = r$ and $c^*(xy) = c(xy)$ otherwise. By the induction hypothesis, $R_{r-1}(t_1, t_2, \ldots, t_{r-2}, R(t_{r-1}, t_r))$ is finite. Choose $n = R_{r-1}(t_1, \ldots, t_{r-2}, R(t_{r-1}, t_r))$. We have a monochromatic clique in one of the $r-1$ colors. If this monochromatic clique is in one of the first $r-2$ colours, then we are done, as we then have a monochromatic clique of size t_i in colour $i, 1 \le i \le r-2$. Otherwise we have a clique of size $R(t_{r-1}, t_r)$ that uses the combined colour. We now restore the original colouring, so that all of these edges are coloured either $r-1$ or r. By definition of $R(t_{r-1}, t_r)$, we also find the desired monochromatic clique in this case.

1.0.1. *Lower Bounds on Ramsey's theorem.*

Theorem 1.13. *(Erdős, 1947)* $R(t,t) \ge (1 - o(1))t/e\sqrt{2} \cdot 2^{t/2}$

Proof. We'll use the probabilistic method to prove the existence of large ramsey colouring. Colour each edge of K_n by red or blue with probability 1/2, such that these random choices are mutually independent of each other. We want to avoid a monochromatic K_t. So for each $R \in \binom{[n]}{t}$, i.e. each set R of t vertices, we define E_R be the event that the induced subgraph of K_n on R is monochromatic. The probability that there exists a monochromatic K_t can then be estimated by the union bound $P(\bigcup_{R \in \binom{[n]}{t}}) K_t) \le \sum_{R \in \binom{[n]}{t}} P(R_K) = \binom{n}{t} \cdot 2 \cdot (1/2)^{t/2} \le 2(en/t)^t (1/2)^{t/2}$. Taking the t-th root and rearranging we obtain that if $n < \frac{t}{2^{1/t} \cdot \sqrt{2}e} 2^{t/2}$, then $P(\text{there is a m.c. } K_t) < 1$. Therefore, there exists a red/blue-colouring without a monochromatic K_t on $n = \lfloor \frac{t}{2^{1/t} \cdot \sqrt{2}e} 2^{t/2} \rfloor$ vertices.

Theorem 1.14. $R(t,t) = (1 - o(1))t/e2^{t/2}$

Proof. We'll remove the constant factor of $\sqrt{2}$ now. Again colour the edges of K_n uniformly at random by either red or blue with probability 1/2 . Let X be the random variable that equals the number of monochromatic K_t's in this two-colouring. For each t-element set K, let X_K denote the indicator random variable of the event that K induces a monochromatic K_t. Then $X = \sum_{K\in\binom{[n]}{t}} X_K$ and by the linearity of expectation $E[X] = \sum_{K\in\binom{[n]}{t}} E[X_k] = \sum_{K\in\binom{[n]}{t}} P(X_K)\binom{n}{t}\cdot 2\cdot(1/2)^{t/2}$. Therefore, there exists a colouring c such that the number of monochromatic K_t's is at most $\binom{n}{t}\cdot 2\cdot(1/2)^{t/2}$. Fix such a colouring and delete one vertex from each monochromatic K_t. This gives us a red/blue-coloring on at least $n - \binom{n}{t}\cdot 2\cdot(1/2)^{t/2}$ vertices without any monochromatic K_t. Hence $R(k,k) > n - \binom{n}{t}\cdot 2\cdot(1/2)^{t/2} \geq n - (ne/t\cdot 2^{-(t-1)2+1/t})^t$ where we estimated $\binom{n}{t} \leq (ne/t)^t$. Substituting $n = \sqrt{2}^t\cdot t/e$, we obtain a red/blue-coloring on $\sqrt{2}^t\cdot t/e - (2^{1/2+1/t})^t = \sqrt{2}^t\cdot t/e(1-o(1))$ vertices.

1.0.2. *Computational Complexity of Ramsey numbers.* The computational complexity of determining Ramsey numbers is unknown. There is a famous statement made by Erdos that "Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack".

Definition 1.15. $F \mapsto (G,H)$, if and only if every two-color edge-coloring of the graph F contains the graph G in the first color or the graph H in the second color as a subgraph.

Definition 1.16. (Arrowing) Given F, G, and H, does $F \mapsto (G,H)$?

The Arrowing problem can be decided in $coNP^{NP}$. This is true because guessing a two colour edge-coloring of F , and then ask the oracle to verify whether the chosen coloring contains a monochromatic G or H. If it does not, reject the input. Schaefer proved that arrowing is $coNP^{NP}$-complete. The proof relied heavily on constructing a suitable graph F . Thus it is unlikely that a similar construction could be used for the case when F is a complete graph.

Definition 1.17. (Clique-arrowing) Given the graphs K_n, G, and H, does $K_n \mapsto (G,H)$?

The best known lower bound for the complexity of Clique-arrowing is due to Burr, who proved that the problem is NP-hard. In that proof Burr showed that, if H is a path, then the minimum n for which $K_n \mapsto (G,H)$ often depends on the chromatic number of G, determining which is NP-hard.

Definition 1.18. (Ramsey) *(Open)* Given k, l, and n, does $K_n \mapsto (K_k, K_l)$?

The complexity of the Ramsey problem varies based on how the inputs k, l, and n are encoded. When encoded in unary, the problem is in $coNP^{NP}$. However, with binary encoding, the input size becomes logarithmic, making it infeasible to guess a coloring in polynomial time due to the exponential size of the coloring.

2. Exercises

Question 2.1. Show there is a positive Fibonacci number that is divisible by 1000.

Question 2.2. Let B be a set of more than $2^{n+1}/n$ distinct points with coordinates of the form $(\pm 1, \pm 1, ..., \pm 1)$ in n-dimensional space with $n \geq 3$. Show that there are three distinct points in B which are the vertices of an equilateral triangle.

Question 2.3. (Chinese Remainder Theorem) If m and n are relatively prime, and $0 \leq a < m$ and $0 \leq b < n$, then there is an integer x such that $x \ mod \ m = a$ and $x \ mod \ n = b$.

Question 2.4. Let A and B be 2×2 matrices with integer entries such that $A, A + B, A + 2B, A + 3B$ and $A + 4B$ are all invertible matricse whose inverses have integer entries. Show that $A + 5B$ is invertible and that its inverse has integer entries.

Question 2.5. Show that $R(3, 4) = 9$

Question 2.6. (Extra Reading) *On the sets of integers which contain no three in arithmetic progression.* Proc. Nat. Acad. Sci., 1946 by Felix A. Behrend.

References

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