Problem Set 2

1. Let X be an inner product space equipped with an inner product $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ and P be a finite dimensional subspace of X. Let $\|f\| = \langle f, f \rangle^{1/2}$. For an $f \in X$, show that

   (a) there exists a $p \in P$ such that $\|f - p\| = \inf_{q \in P} \|f - q\|$.

   (b) the function $p \in P$ satisfies $\|f - p\| = \inf_{q \in P} \|f - q\|$ if and only if $\langle p, q \rangle = \langle f, q \rangle$ for all $q \in P$.

   (c) There is a unique $p \in P$ such that $\|f - p\| = \inf_{q \in P} \|f - q\|$.

2. For a positive integer k, let $C_0^k([a,b]) = \{\varphi \in C^k(a,b) \cap C([a,b]) : \varphi(a) = \varphi(b) = 0\}$ where $C^k(a,b)$ denotes the space of k-times continuously differentiable real valued functions on (a,b) and $C([a,b])$ consists of all continuous real valued function on $[a,b]$.

   (a) Find a constant $c \in [1/\pi^2, 1/2)$ such that

   $$\int_a^b \varphi(t)\varphi(t)\, dt \le c(b-a)^2 \int_a^b \varphi'(t)\varphi'(t)\, dt \tag{1}$$

   for all $\varphi \in C_0^1([a,b])$. Explain why a constant c satisfying (1) for all $\varphi \in C_0^1([a,b])$ can not lie in the interval $(-\infty, 1/\pi^2)$.

   (b) Show that the map $\langle \cdot, \cdot \rangle : C_0^1([a,b]) \times C_0^1([a,b]) \to \mathbb{R}$ given by

   $$\langle \varphi, \psi \rangle = \int_a^b \varphi'(t)\psi'(t)\, dt$$

   defines an inner-product on $C_0^1([a,b])$.

   (c) Consider the boundary value problem $u'' = f(t)$, $a < t < b$, with boundary conditions $u(a) = 0, u(b) = 0$, and let P be a finite dimensional subspace of $C_0^2([a,b])$. If $v \in P$ is the Galerkin solution to the boundary value problem, that is, $v \in P$ satisfies

   $$\int_a^b (f(t) - v''(t))w(t)\, dt = 0$$

   for all $w \in P$, then show that

   $$\|u - v\|_{new} = \inf_{w \in P} \|u - w\|_{new}$$

   where $\|\varphi\|_{new} = \langle \varphi, \varphi \rangle^{1/2}$.

(d) If $v \in P$ is the Galerkin solution to the boundary value problem in the previous part, show that $E(v) = \inf_{w\in P} E(w)$ where the functional $E : C_0^1([a,b]) \to \mathbb{R}$ is given by

$$E(w) = \frac{1}{2}\int_a^b w'(t)w'(t)\,dt + \int_a^b f(t)w(t)\,dt.$$

(e) Show that

$$\langle u, v\rangle_{H^1} = \int_a^b \left(u(t)v(t) + u'(t)v'(t)\right)\,dt$$

is an inner-product and

$$\|u\|_{H^1} = \left(\langle u, v\rangle_{H^1}\right)^{1/2}$$

is the norm induced by this inner-product.

3. Consider the boundary value problem

$$\begin{aligned} \Delta u &= f \quad \text{in } \Omega = (0,1)\times(0,1) \\ u &= g \quad \text{on } \Gamma, \end{aligned}$$

and the corresponding discrete problem

$$\begin{aligned} \Delta_h u_h &= f \quad \text{in } \Omega_h = (0,1)\times(0,1)\cap\{(mh,nh)) : m,n\in\mathbb{Z}\} \\ u_h &= g \quad \text{on } \Gamma_h, \end{aligned}$$

where $h = 1/N$ for some positive integer $N > 1$ and $\overline{\Omega}_h = \Omega_h \cup \Gamma_h$.

(a) Let v be a function on satisfying $\Delta_h v \geq 0$ on Ω_h. Then, show that $\max_{\Omega_h} v \leq \max_{\Gamma_h} v$, and equality holds if and only if v is constant.

(b) Let v be a function on satisfying $\Delta_h v \leq 0$ on Ω_h. Then, show that $\min_{\Omega_h} v \geq \min_{\Gamma_h} v$, and equality holds if and only if v is constant.

(c) There is a unique solution u_h to the discrete problem.

(d) The solution to the discrete problem, u_h, satisfies

$$\|u_h\|_{\infty,\overline{\Omega}_h} \leq \frac{1}{8}\|f\|_{\infty,\Omega_h} + \|g\|_{\infty,\Gamma_h}.$$

(e) Show that

$$\|u_h - u\|_{\infty,\overline{\Omega}_h} \leq \frac{1}{8}\|\Delta u - \Delta_h u\|_{\infty,\overline{\Omega}_h}.$$

(f) Let $L(\Omega_h) = \{u : \overline{\Omega}_h \to \mathbb{R} : u(x) = 0, x \in \Gamma_h\}$. Show that $\varphi_m(x_1,x_2) = \varphi_m(x_1)\varphi_n(x_2), m,n \in \{1,\ldots,N-1\}$ are orthogonal to each other with respect to the inner-product

$$\langle u, v\rangle_h = \sum_{m=1}^{N-1}\sum_{n=1}^{N-1} u(mh,nh)v(mh,nh)$$

where $\varphi_m(x) = \sin \pi m x$.

4. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$\varphi(x_1, x_2) = \begin{cases} 1 - x_1, & x_1 \le 1, x_2 \ge 0, x_2 - x_1 \le 0, \\ 1 - x_2, & x_1 \ge 0, x_2 \le 1, x_1 - x_2 \le 0, \\ 1 - (x_2 - x_1), & x_1 \le 0, x_2 \ge 0, x_2 - x_1 \le 1, \\ 1 + x_1, & x_1 \ge -1, x_2 \le 0, x_1 - x_2 \le 0, \\ 1 + x_2, & x_1 \le 0, x_2 \ge -1, x_2 - x_1 \le 0, \\ 1 - (x_1 - x_2), & x_1 \ge 0, x_2 \le 0, x_1 - x_2 \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Given $h > 0$, let $\varphi_{mn}(x_1, x_2) = \varphi((x_1 - mh)/h, (x_2 - nh)/h)$. Show that

(a)

$$\int_{\mathbb{R}^2} \varphi_{mn} = h^2.$$

(b)

$$\int_{\mathbb{R}^2} \nabla\varphi_{mn} \cdot \nabla\varphi_{k\ell} = \begin{cases} 4, & m = k, n = \ell, \\ -1, & m = k \pm 1, n = \ell \text{ or } m = k, n = \ell \pm 1, \\ 0, & \text{otherwise.} \end{cases}$$

(c) If $f \in C^2(\mathbb{R}^2)$, then

$$\frac{1}{h^2}\int_{\mathbb{R}^2} f\varphi_{mn} = f(mh, nh) + O(h^2).$$

5. Consider a nine-point difference approximation to the Laplacian of the form

$$\begin{aligned} \Delta_h^* v_{m,n} = \frac{1}{h^2} [&\alpha(v_{m-2,n} + v_{m+2,n} + v_{m,n-2} + v_{m,n+2}) \\ &+ \beta(v_{m-1,n} + v_{m+1,n} + v_{m,n-1} + v_{m,n+1}) + \gamma v_{m,n}] \end{aligned}$$

Show how to choose the constants α, β and γ so that the scheme $\Delta_h^* v = f$ is consistent to fourth order with the equation $\Delta u = f$.

6. Consider a nine-point difference approximation to the Laplacian of the form

$$\begin{aligned} \Delta_h^* v(x, y) = \frac{1}{h^2}[&\alpha\{v(x - h, y - h) + v(x - h, y + h) + v(x + h, y - h) + v(x + h, y + h)\} \\ &\beta\{v(x + h, y) + v(x - h, y) + v(x, y + h) + v(x, y - h)\} + \gamma v(x, y)]. \end{aligned}$$

Show that

(a) there is no choice of constants α, β, and γ so that the scheme is fourth order accurate.

(b) the coefficients can be chosen to give a fourth order scheme of the form $\Delta_h^* v = Rf$ where

$$Rf(x,y) = \frac{f(x+h,y) + f(x-h,y) + f(x,y+h) + f(x,y-h) + 8f(x,y)}{12}.$$

(c) with the same choice of coefficients as in part (b), the scheme $\Delta_h^* v = 0$ is a sixth order accurate approximation of the homogeneous equation $\Delta u = 0$.

7. (a) Let $\hat{T}$ be the triangle with vertices $\hat{a}_1 = (0,0)$, $\hat{a}_2 = (1,0)$ and $\hat{a}_3 = (0,1)$. Find the formulas for the three linear functions $\hat{\lambda}_i$ which satisfy $\hat{\lambda}_i(\hat{a}_j) = \delta_{ij}$.

(b) Let T be an arbitrary triangle with vertices a_1, a_2 and a_3. Find the formulas for the linear functions λ_i satisfying $\lambda_i(a_j) = \delta_{ij}$.

8. Consider the following method for $u_t = cu_{xx}$,

$$\frac{u_n^{j+1} - u_n^j}{k} = \frac{1}{2}c\frac{u_{n+1}^{j+1} - 2u_n^{j+1} + u_{n-1}^{j+1}}{h^2} + \frac{1}{2}c\frac{u_{n+1}^j - 2u_n^j + u_{n-1}^j}{h^2}.$$

for all solutions u if $\lambda = ck/h^2 \leq 1$.

(a) Show that this scheme satisfies the *maximum norm stability*

$$\|u^{j+1}\|_\infty \leq \|u^j\|_\infty$$

Hint: Show that if u_{n0}^{j+1} is the largest value of u_n^{j+1}, then

$$u_{n0}^{j+1} \leq -\frac{ck}{2h^2}u_{n0-1}^{j+1} + \left(1 + \frac{ck}{h^2}\right)u_{n0}^{j+1} - \frac{ck}{2h^2}u_{n0+1}^{j+1} \leq \|u^j\|_\infty.$$

(b) Find the order of the scheme.

9. Show that schemes of the form

$$u_n^{j+1} = \alpha u_{n+1}^j + \beta u_{n-1}^j$$

are stable if $|\alpha| + |\beta| \leq 1$. Conclude, in particular, that Lax-Friedrichs scheme is stable for $|\lambda| \leq 1$.

10. Show that schemes of the form

$$\alpha u_{n+1}^{j+1} + \beta u_{n-1}^{j+1} = u_n^j$$

are stable if and only if $||\alpha| - |\beta|| \leq 1$.

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