Lesson 4

Numerical Solution of PDE

4.1 BVP for 2nd Order Elliptic PDE

- Finite Difference Method

- **Finite Element Method**

Akash Anand
MATH, IIT KANPUR

Now, lets try to solve the Poisson's equation with homogeneous boundary condition

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that is

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We, therefore, see that the Galerkin approximation error is bounded by a constant multiple of the best approximation error for u by functions in V_h!

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- **Finite Element Method**

    - **Construction of FEM Approximation Spaces**

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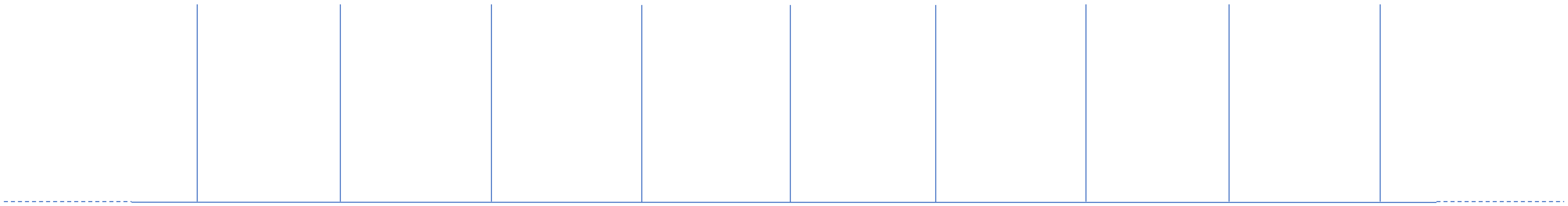

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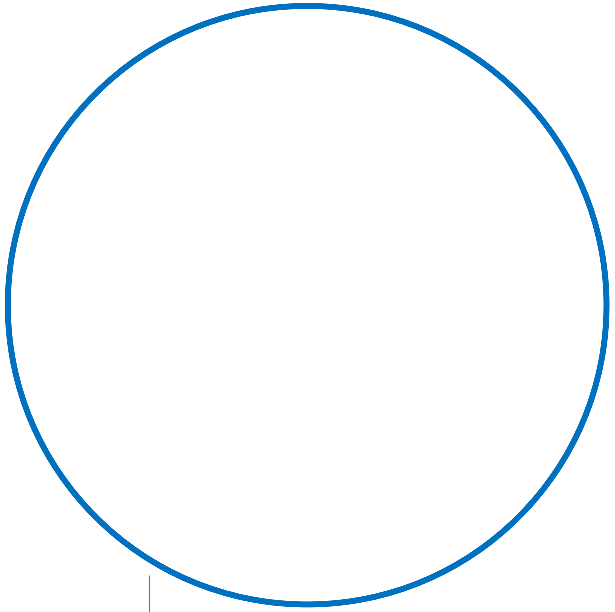

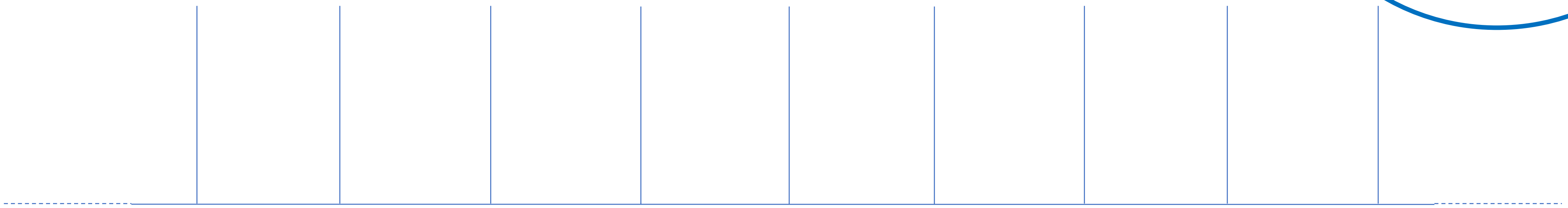

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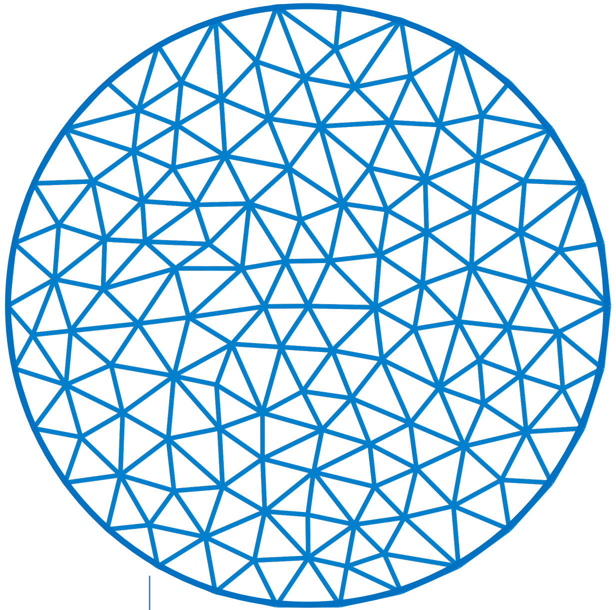

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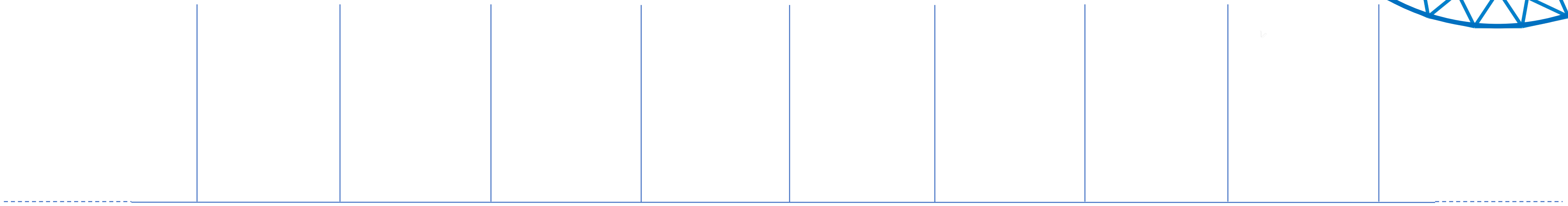

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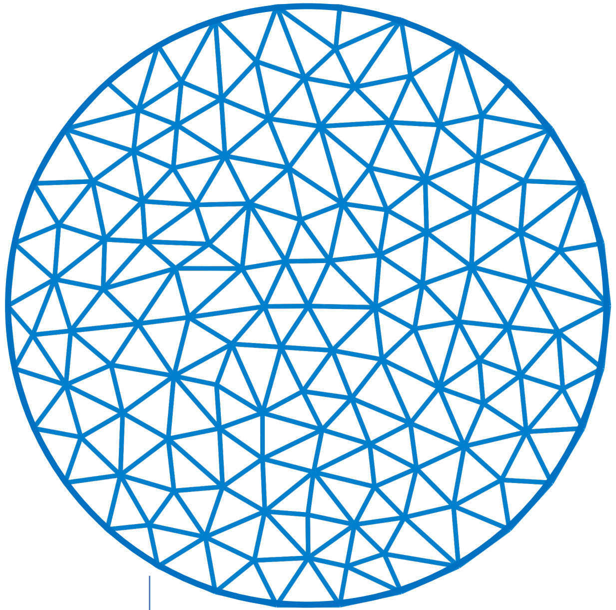

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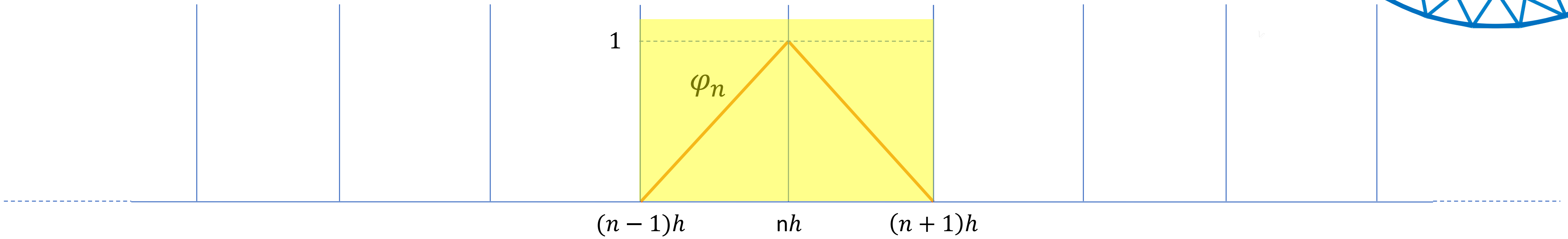

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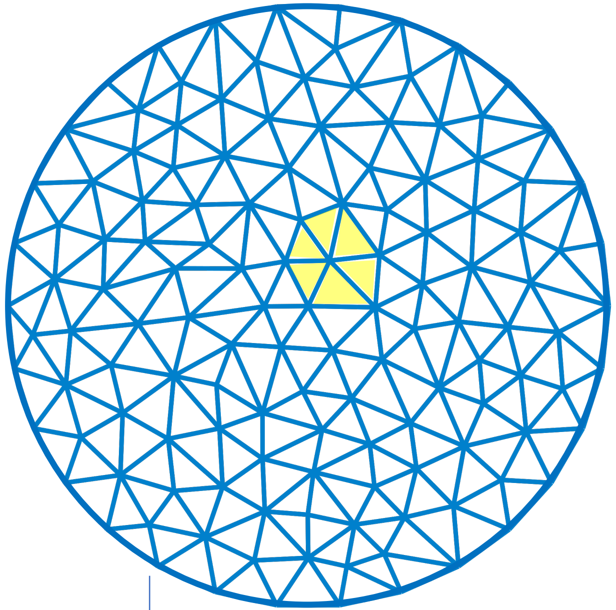

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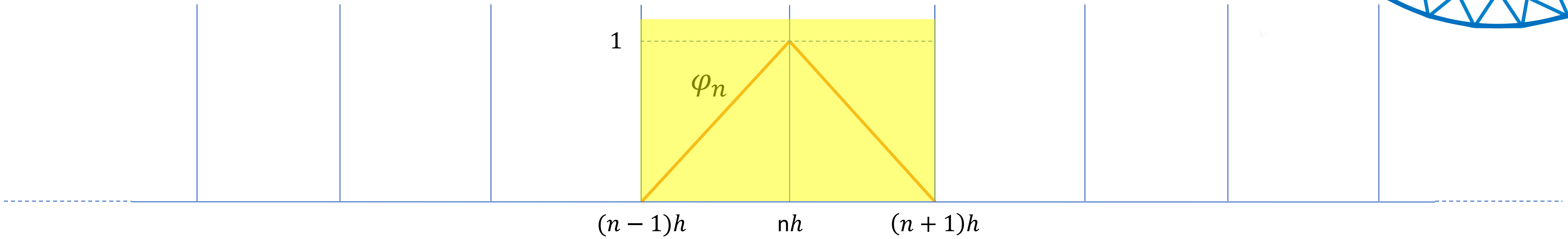

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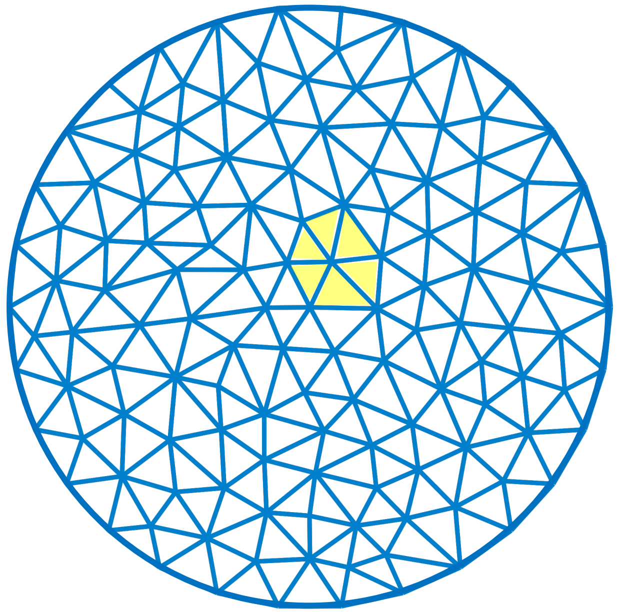

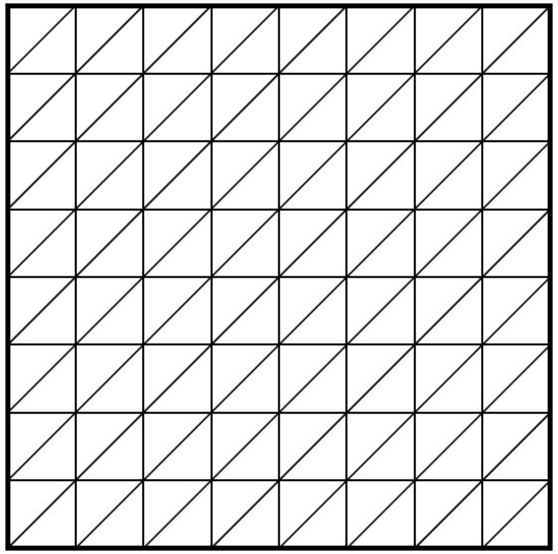

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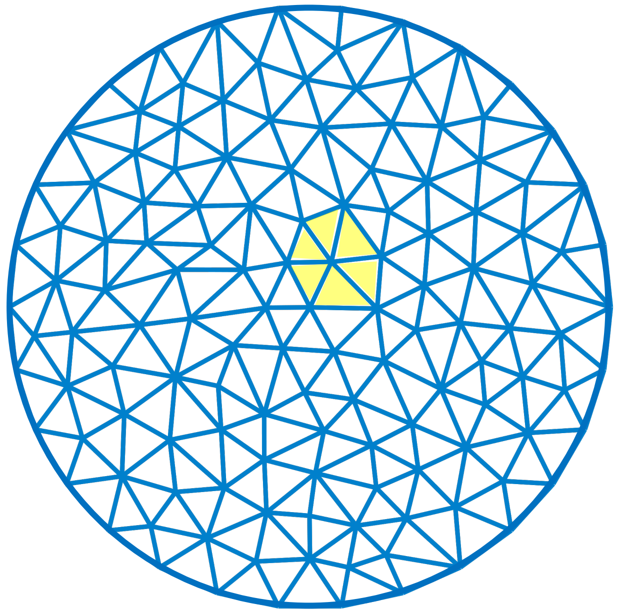

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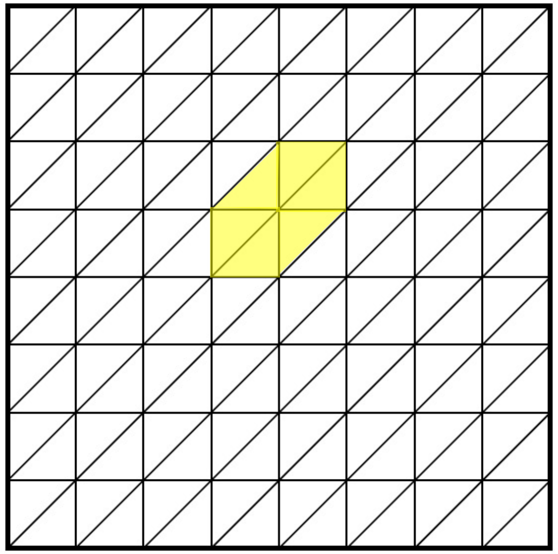

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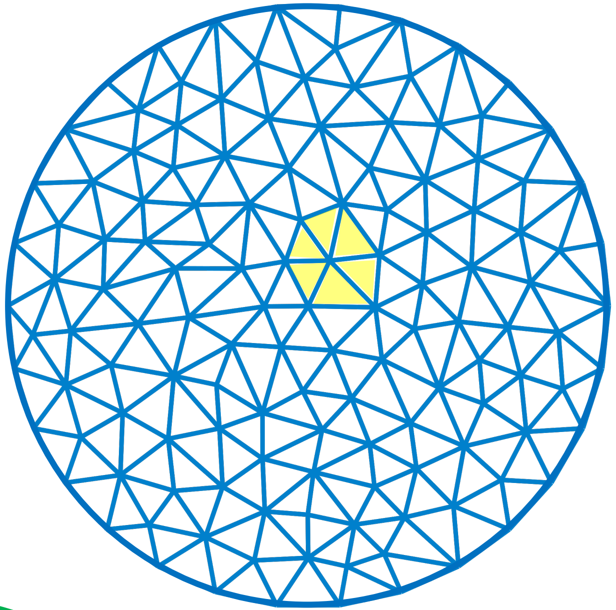

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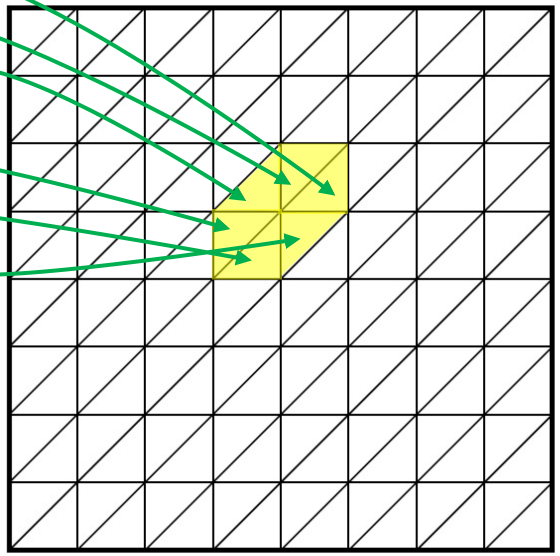
$$\varphi_{mn}(x_1, x_2) = \begin{cases} 1 - (x_1 - mh)/h \\ 1 - (x_2 - nh)/h \\ 1 + (x_1 - mh)/h - (x_2 - nh)/h \\ 1 + (x_1 - mh)/h \\ 1 + (x_2 - nh)/h \\ 1 - (x_1 - mh)/h + (x_2 - nh)/h \end{cases}$$



Numerical Methods for PDE: 2nd Order Elliptic PDE

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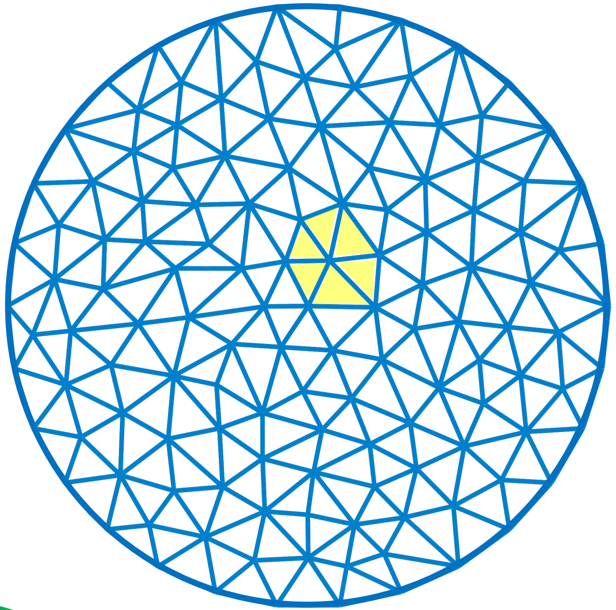
Back to the Poisson's equation with homogeneous boundary condition

$$\Delta u = f, \qquad \text{in } \Omega,$$
$$u = 0, \qquad \text{on } \Gamma.$$

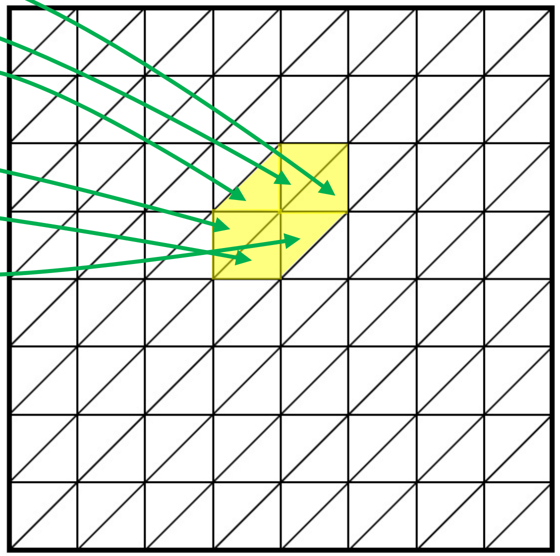


How do we construct a finite element (that is, locally supported) basis for V_h?

Recall, that in one dimensions, we partitioned Ω (an interval) into subintervals and constructed a basis for piecewise linear functions with respect to the partition.

For simplicity, we take $\Omega = (0,1)\times(0,1)$ and the triangulation shown in the figure. Then,

$$\varphi_{mn}(x_1, x_2) = \begin{cases} 1 - (x_1 - mh)/h \\ 1 - (x_2 - nh)/h \\ 1 + (x_1 - mh)/h - (x_2 - nh)/h \\ 1 + (x_1 - mh)/h \\ 1 + (x_2 - nh)/h \\ 1 - (x_1 - mh)/h + (x_2 - nh)/h \end{cases}$$



The functions φ_{mn} form a basis for subspace of piecewise linear functions with respect to the given partition/triangulation.

Note that (exercise)

$$\int_\Omega \nabla\varphi_{mn}\cdot\nabla\varphi_{kl} = \begin{cases} 4, & m=k, n=l,\\ -1, & m=k\pm 1, n=l \text{ or } m=k, n=l\pm 1,\\ 0, & otherwise.\end{cases}$$

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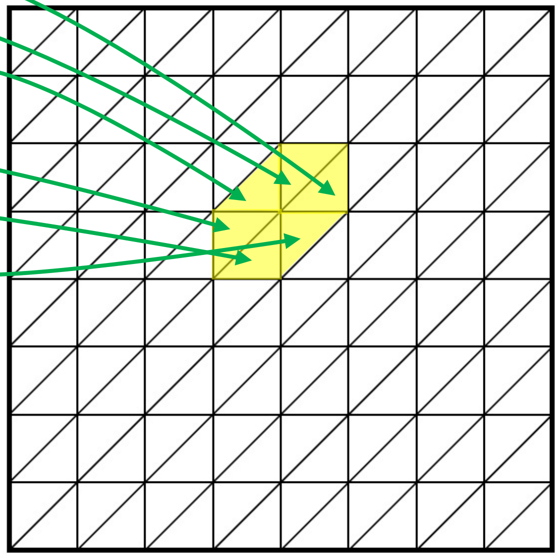
$$\varphi_{mn}(x_1,x_2) = \begin{cases} 1-(x_1-mh)/h \\ 1-(x_2-nh)/h \\ 1+(x_1-mh)/h-(x_2-nh)/h \\ 1+(x_1-mh)/h \\ 1+(x_2-nh)/h \\ 1-(x_1-mh)/h+(x_2-nh)/h \end{cases}$$



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Thus, for $u_h = \sum u_{mn}\varphi_{mn}$, *the linear system reads*

$$\frac{u_{m-1,n}+u_{m+1,n}+u_{m,n-1}+u_{m,n+1}-4u_{mn}}{h^2} = \frac{1}{h^2}\int_\Omega f\varphi_{mn} = \tilde{f}_{mn}, \qquad 1\le m,n\le N.$$

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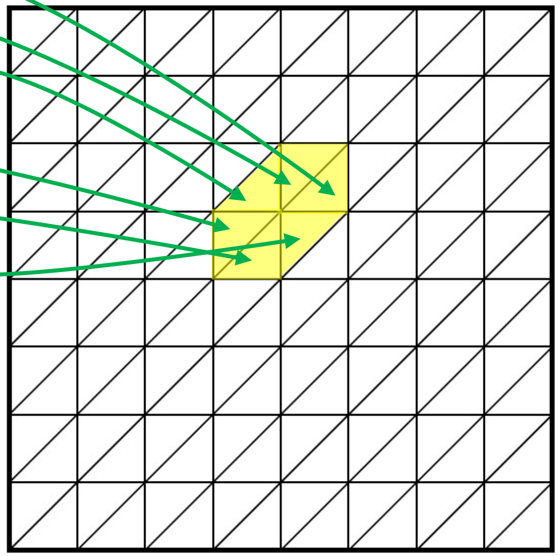

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Numerical Methods for PDE: 2nd Order Elliptic PDE

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We see that matrix on the left hand side of the linear matrix (called stiffness matrix) for the piecewise linear finite elements for the Laplace operator on the unit square using a uniform mesh is exactly the matrix of the 5-point Laplacian.

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$$\tilde{f}_{mn} = \frac{1}{h^2}\int_\Omega \Big(f(mh,nh) + f_{x_1}(mh,nh)(x_1-mh) + f_{x_2}(mh,nh)(x_2-nh) + O(h^2)\Big)\varphi_{mn} = f_{mn} + O(h^2)$$

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Therefore, at vertices, we see that the finite element method converges with order 2 (why?).

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Therefore, at vertices, we see that the finite element method converges with order 2 (why?).

Recall that, from the general error analysis, we have

$$\|u_h-u\|_{H^1} \le (1+C/\gamma)\inf_{w\in V_h}\|u-w\|_{H^1}.$$

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Exercise: Find C *and* γ *for the piecewise linear finite elements.*

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Exercise: Find C and γ for the piecewise linear finite elements.

From the approximation theory, we have the following result on the best approximation error.

Theorem

Let there be given a family of triangulations $\{\mathcal{T}_h\}$ of a polygonal domain Ω and let $h = \max_{T \in \mathcal{T}_h} \operatorname{diam}(T)$. Let r be a positive integer. For each h let $P_h : C(\Omega) \to M_0^r(\mathcal{T}_h)$ denote the nodal interpolant, where $M_0^r(\mathcal{T}_h)$ is the space of continuous functions which restrict to polynomials of degree at most r when restricted to any triangle $T \in \mathcal{T}_h$. Then, there is a constant c such that

$$\|u - P_h u\|_{L^\infty(\Omega)} \leq c h^{r+1} \left\|u^{(r+1)}\right\|_{L^\infty(\Omega)}, \qquad u \in C^{r+1}(\overline{\Omega}),$$

$$\|u - P_h u\|_{L^2(\Omega)} \leq c h^{r+1} \left\|u^{(r+1)}\right\|_{L^2(\Omega)}, \qquad u \in H^{r+1}(\Omega).$$

Moreover, if the family of triangulations are shape regular (the minimal angle of each triangulation is bounded below uniformly), then there is a constant C such that

$$\|\nabla(u - P_h u)\|_{L^\infty(\Omega)} \leq C h^{r} \left\|u^{(r+1)}\right\|_{L^\infty(\Omega)}, \qquad u \in C^{r+1}(\overline{\Omega}),$$

$$\|\nabla(u - P_h u)\|_{L^2(\Omega)} \leq C h^{r} \left\|u^{(r+1)}\right\|_{L^2(\Omega)}, \qquad u \in H^{r+1}(\Omega).$$

Lesson 4

Numerical Solution of PDE

4.2 Parabolic PDE

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Numerical Methods for PDE: Parabolic PDE

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As a model problem, consider the heat equation on a spatial domain Ω *for a time interval* $[0, T]$*. The solution* u *satisfies*

$$\frac{\partial u}{\partial t} = \mathrm{c}\Delta u + f, \qquad \mathrm{x} \in \Omega, \mathrm{t} \in [0, T], \qquad c > 0.$$

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To obtain a well-posed problem, we need to give boundary conditions

$$u = 0, \qquad \mathrm{x} \in \Gamma, \mathrm{t} \in [0, T]$$

and an initial condition

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Let's assume, for simplicity, that $\Omega = (0,1)\times(0,1)$.

Lesson 4

Numerical Solution of PDE

4.2 Parabolic PDE

- Semi-discretization

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Numerical Methods for PDE: Parabolic PDE

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Let's assume, for simplicity, that $\Omega = (0,1)\times(0,1)$. Then, we have the 5-point Laplacian Δ_h as the discretization of the Laplacian Δ. Thus, we the solution u_h satisfying

$$\begin{aligned} \frac{\partial u_h}{\partial t} &= c\Delta_h u_h + f, && \mathrm{x} \in \Omega_\mathrm{h}, \mathrm{t} \in [0, T] \\ u_h &= 0, && \mathrm{x} \in \Gamma_\mathrm{h}, \mathrm{t} \in [0, T] \\ u_h &= u_0, && \mathrm{x} \in \Omega_\mathrm{h}, \mathrm{t} = 0. \end{aligned}$$

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We may view this as an initial value problem for a system of $(N-1)^2$ ODEs.

Numerical Methods for PDE: Parabolic PDE

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The process of reducing the evolutionary PDE to a system of ODEs by using a finite difference approximation of the spatial operator is called semi-discretization or the method of lines.

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The process of reducing the evolutionary PDE to a system of ODEs by using a finite difference approximation of the spatial operator is called semi-discretization or the method of lines.

This is not a full discretization as we still have to choose a numerical method to solve the ODEs.

Lesson 4

Numerical Solution of PDE

4.2 Parabolic PDE

- Semi-discretization
- **Finite difference discretization**

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To further simplify the presentation, let's drop down to one space dimension, that is, take $\Omega = (0,1)$.

Numerical Methods for PDE: Parabolic PDE

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Numerical Methods for PDE: Parabolic PDE

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To solve the ODEs, we first employ the Euler method.

Writing $u_n^j = u_h(nh, jk)$, *the fully discrete system reads*

$$\frac{u_n^{j+1} - u_n^j}{k} = c\,\frac{u_{n+1}^j - 2u_n^j + u_{n-1}^j}{h^2} + f_n^j, \qquad 0 < n < N, j = 0,1,\dots,M-1,$$

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We call this the forward-centered difference method for the heat equation. Since the Euler's method is explicit, we don't need to solve a linear system:

$$u_n^{j+1} = (1-2\lambda)u_n^j + \lambda u_{n+1}^j + \lambda u_{n-1}^j + k f_n^j, \qquad j = 0,1,\dots,M-1,$$

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To analyze the method, consider the local (truncation) error

$$\ell_n^j = \frac{u(nh,(j+1)k) - u(nh,jk)}{k} - c\,\frac{u\big((n+1)h,jk\big) - 2u(nh,jk) + u\big((n-1)h,jk\big)}{h^2} - f_n^j.$$

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By Taylor's theorem,

$$\ell_n^j = \frac{k}{2}\frac{\partial^2 u}{\partial t^2} - c\,\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}$$

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Let $e_n^j = u_n^j - u(nh,jk)$. Then,

$$e_n^{j+1} = (1-2\lambda)e_n^j + \lambda e_{n+1}^j + \lambda e_{n-1}^j - k\ell_n^j.$$

To analyze the method, consider the local (truncation) error

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$$\ell_n^j = \frac{k}{2}\frac{\partial^2 u}{\partial t^2} - c\,\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}$$

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If $\lambda \le 1/2$, *and* $\ell = \max_{n,j}\left|\ell_n^j\right|$, *we get*

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Numerical Methods for PDE: Parabolic PDE

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for all j*. As* $jk \le T$ *and* $k \le h^2/(2c)$*, we have*

$$\max_{n,j}\left|e_n^j\right| \le T\ell \le C(k+h^2) \le C(1+1/(2c))h^2.$$

Lesson 4

Numerical Solution of PDE

4.2 Parabolic PDE

- *Semi-discretization*
- *Full finite difference discretization*
- ***Fourier Analysis***

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Recall that, on $L(\mathrm{I_h})$, *we define the inner product*

$$\langle u, v\rangle_h = h\sum_{k=1}^{N-1} u(kh)v(kh)$$

with the corresponding norm $\|v\|_h$ *and* $\varphi_m(x) = \sin \pi m x, m = 1, \dots, N-1,$ *form an orthogonal basis.*

Also,

$$D_h^2\varphi_m = -\lambda_m\varphi_m, \qquad \lambda_m = \frac{2}{h^2}(\cos \pi m h - 1) = \frac{4}{h^2}\sin^2\frac{\pi m h}{2},$$

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We write the semi-discrete solution $u_h(t,\cdot) = \sum_{m=1}^{N-1} a_m^h\varphi_m$*. Then,*

$$\frac{\partial u_h}{\partial t} = \sum_{m=1}^{N-1}\frac{da_m^h}{dt}\varphi_m, \qquad D_h^2 u_h = -\sum_{m=1}^{N-1} a_m^h\lambda_m\varphi_m\,.$$

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The numbers $a_m^h(0)$ are the coefficients in the discrete Fourier sine transform of the initial data:

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The difference equation then gives

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In the matrix form, we write

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where $\left(I + ckD_h^2\right)$ *is a symmetric operator on* $L(I_h)$. *Thus,*

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From stability, we obtain the convergence result in the same way as earlier. Let $e_n^j = u_n^j - u(nh, jk)$*, we have*

$$\frac{e_n^{j+1} - e_n^j}{k} = c\,\frac{e_{n+1}^j - 2e_n^j + e_{n-1}^j}{h^2} - \ell_n^j, \qquad 0 < n < N, j = 0,1,\dots,M-1,$$
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The stability result then yields

$$\left\|e^j\right\|_h \le \left(\max_m |1 - ck\lambda_m|\right)^j \|e^0\|_h + Mk \max_j \left\|\ell^j\right\|_h \le T \max_j \left\|\ell^j\right\|_h.$$