MTH 430: Numerical Analysis & Scientific Computing II
Department of Mathematics and Statistics
Indian Institute of Technology - Kanpur October 4, 2019

LAB 9

SUBMISSION INSTRUCTIONS

Submit your lab files `lab9_exercise1.m` and `lab9_exercise2.m` by attaching them to an **email with the subject Lab9-ID** where ID stands for your roll number (for example, if your roll number is 123456, then the subject will be `Lab9_123456`) and send it to

mth430.iitk@gmail.com

before 11:59 pm on October 17, 2019.

LAB EXERCISES

Consider the problem

$$
\begin{aligned}
&u_t(t,x) + u_x(t,x) = 0,\ x \in (-1,1),\ t > 0,\\
&u(0,x) = \exp(-100x^2),\ x \in [-1,1],\\
&u(t,-1) = \exp(-100(1+t)^2),\ t > 0,\\
&u(t,1) = \exp(-100(1-t)^2),\ t > 0.
\end{aligned}
$$

Clearly, $u_e(t,x) = e^{-100(x-t)^2}$ is the exact solution. Write a function `lab9_exercise1` in the following format

```
function [ ] = lab9_exercise1(nt,nx,cfl)
```

to implement its numerical solution using the forward-centered difference scheme

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right).$$

The input parameters `nt` and `nx`, respectively, specify the number of time steps N_t to reach the final time $T = N_t\Delta t$, and the number N_x used to define $\Delta x = 2/N_x$ so that the x-grid points $x_j = -1 + j\Delta x, j = 0, \ldots, N_x$. The input `cfl` specifies the ratio $\Delta t/\Delta x$.
Next, solve the same problem using the upstream/upwind (forward-backward) method that reads

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left(u_j^n - u_{j-1}^n\right).$$

Your implementation of this scheme should read

```
function [ ] = lab9_exercise2(nt,nx,cfl)
```

where the input parameters remain the same as those in `lab9_exercise1.m`. Both these implementations should display the evolution of the solution as a movie by plotting the solution at successive time steps on the same figure.

OTHER EXERCISES

1. To show that the finite difference scheme

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right)$$

is divergent, complete the following steps:

(a) Show that $e^{\beta t}e^{i\alpha x}$ is a solution to the difference equation if

$$e^{\beta\Delta t} = 1 + i\frac{\Delta t}{\Delta x}\sin(\alpha\Delta x).$$

(b) Consider the initial data

$$u_0(x) = \sum_{k=0}^{\infty} 2^{-2k}\cos(2^{k-1}\pi x)$$

Show that

$$u(t,x) = \mathrm{Re}\sum_{k=0}^{\infty} 2^{-2k}e^{\beta_k t}e^{i\alpha_k x}$$

solves the difference equation along with the initial data where, for $\alpha_k = 2^{k-1}\pi$, β_k is chosen such that it satisfies the identity in part (a).

(c) Take $\Delta x = 2^{-n}$. Show that the term $k = n$ dominates the sum of all the other terms as $n \to \infty$, in

$$u(t,0) \geq -\sum_{k=n+1}^{\infty} 2^{-2k} + \mathrm{Re}\sum_{k=0}^{n} 2^{-2k}\left(1 + i\frac{\Delta t}{\Delta x}\sin(2^{k-1-n}\pi)\right)^{t/\Delta t}.$$

Conclude that $u(t,0) \to \infty$ as $n \to \infty$.

2. Investigate the consistency, order, and stability of the following schemes

(a)

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right)$$

(b)

$$u_j^{n+1} = \frac{1}{2}u_{j+1}^n + \frac{1}{2}u_{j-1}^n - \frac{\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right)$$