Numerical Analysis & Scientific Computing II

Module 2

Initial Value Problems

2.1 Well-posedness

2.2 Stability

2.3 Euler's method

- Errors and error propagation

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Initial Value Problems: Euler's method

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Example

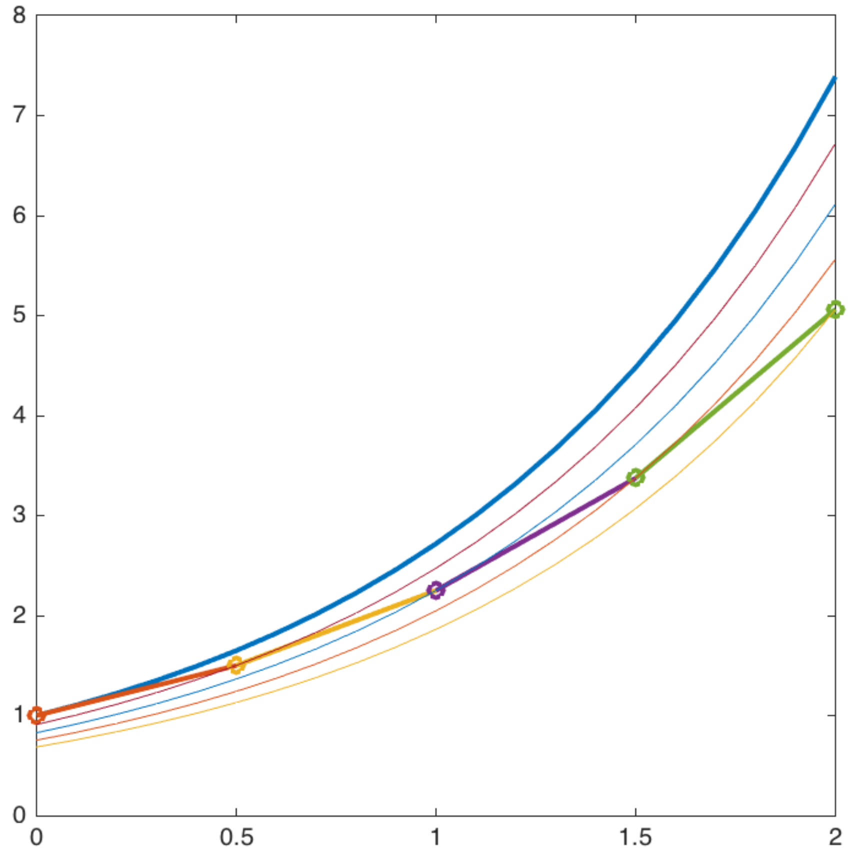
Let us solve $y' = y,\ y(0) = 1$ *using the Euler's method taking the uniform step size* $h = h_k = t_{k+1} - t_k = 0.5.$

$$y_1 = 1 + h,\quad y_2 = y_1 + hy_1 = (1+h)^2,\quad y_3 = y_2 + hy_2 = (1+h)^3, \dots,\quad y_k = (1+h)^k, \dots$$

An error is introduced at each step of the method – one step error or single step error.

These local errors accumulate over time to produce a global error.

Note that how local errors accumulate is not always clear and it depends on the underlying solution.



With each step, we hop from one solution of the ODE to another.

Local errors get amplified if the solutions to the ODE are unstable.

Initial Value Problems: Euler's method

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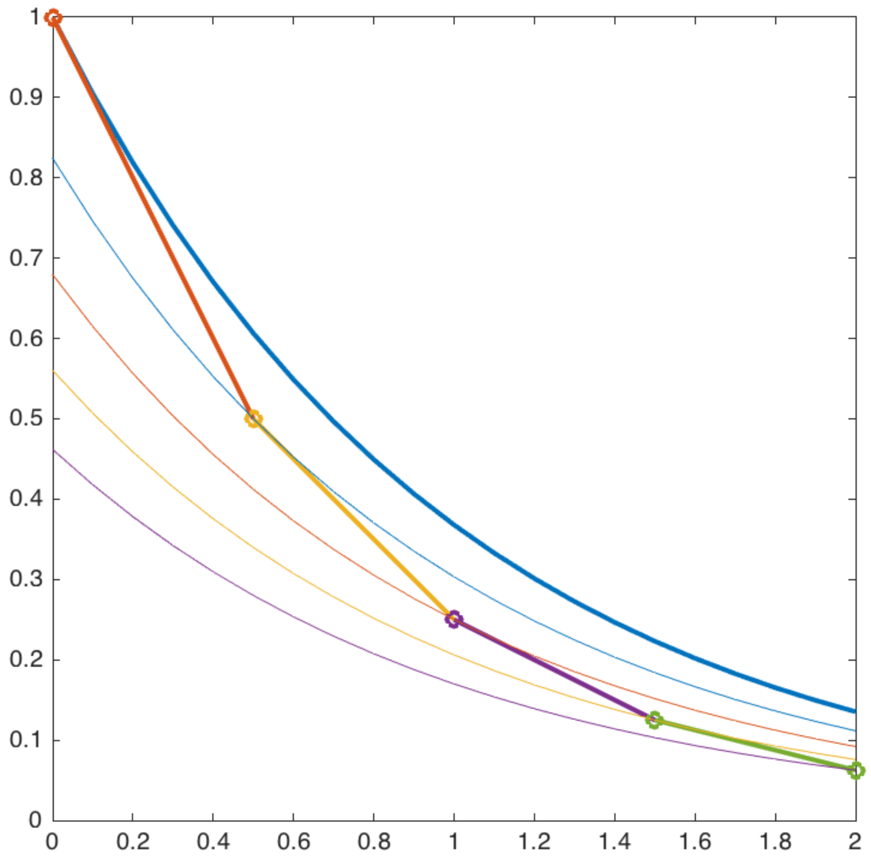
Example

Let us solve $y' = -y,\ y(0) = 1$ *using the Euler's method taking the uniform step size* $h = 0.5$.

$$y_1 = 1 - h,\ \ y_2 = y_1 - hy_1 = (1-h)^2,\ \ y_3 = y_2 - hy_2 = (1-h)^3, \ldots,\ \ y_k = (1-h)^k, \ldots$$



For an equation with stable solutions, the errors in the numerical solution do not grow, and for equations with asymptotically stable solutions, the errors diminish with time.

Initial Value Problems: Accuracy and Stability

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Sources of error

Rounding error –

… due to truncation of data (e.g., real numbers requiring infinite space is represented using a finite amount of space); finite precision of the floating point arithmetic.

Truncation error –

… due to truncation of infinite mathematical processes to finite processes or algorithms (e.g., derivative replaced by a finite difference approximation, or integration replaced by a quadrature).

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In most practical situations, truncation error dominates other sources and, therefore, in analysis of numerical methods, we will focus exclusicely on truncation error.

Global error

… is the difference

$$e_k = y_k - y(t_k),$$

where y_k is the computed solution at $t = t_k$ and $y(t)$ is the exact/true solution of the ODE passing through the initial point (t_0, y_0).

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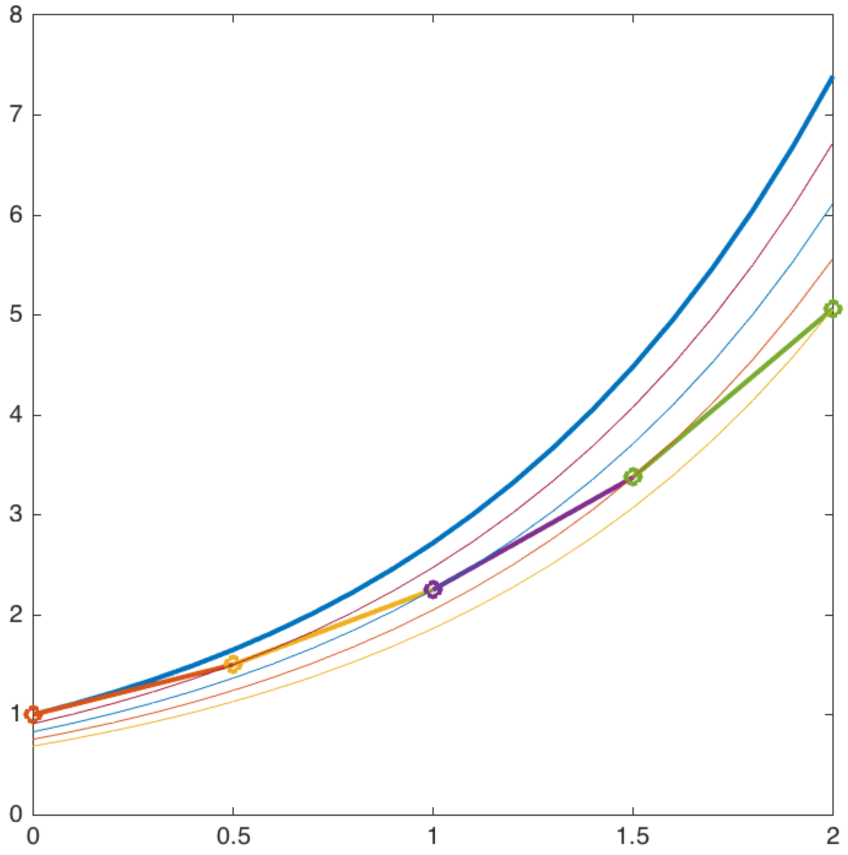

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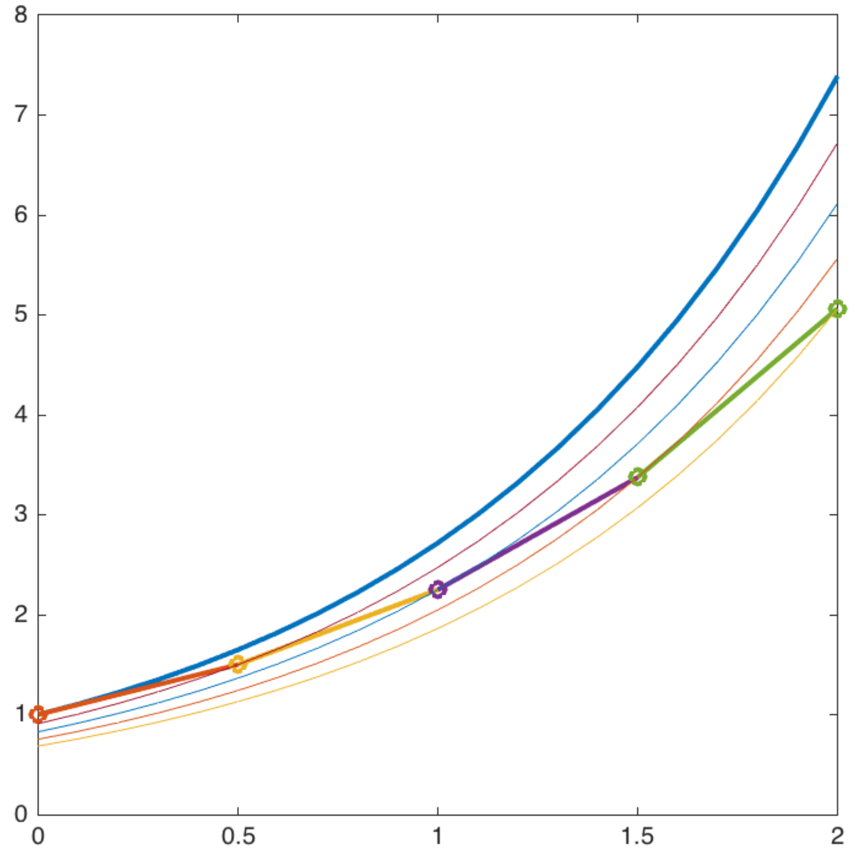

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We see that the local error at a given time step is simply the amount by which the solution of the ODE fails to satisfy the method.

Initial Value Problems: Accuracy and Stability

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More generally, for a one step method

$$y_{k+1} = y_k + h_k \phi(t_k, y_k, h_k)$$

the local error is

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If solutions of the ODE are diverging, then the local errors at each time step is magnified over time.

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In order to assess the effectiveness of a numerical method, we need to characterize both

a) its local error (accuracy), and
b) the compounding effects over multiple steps (stability).

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The accuracy of a numerical method is said to be of order p *if*

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The local error per unit step is $\ell_k / h_{k-1} = O(h_{k-1}^p)$ and under reasonable conditions, the global error is $O(h^p)$ where h is the average step size.

Why is there a difference of 1 in the definition?

To develop an intuition, let's consider the IVP $y' = \lambda y$, $y(0) = y_0$ and apply Euler's method with time step h:

$$y_k = (1 + \lambda h)^k y_0$$

The local error is given by

$$\ell_k = u_{k-1}(t_{k-1}) + h_{k-1} f(t_{k-1}, u_{k-1}(t_{k-1})) - u_{k-1}(t_k)$$
$$y_{k-1} + h_{k-1}(\lambda y_{k-1}) - y_{k-1} e^{\lambda(t_k - t_{k-1})}$$
$$y_{k-1}\left(1 + \lambda h - e^{\lambda h}\right) = y_{k-1}\left(-\tfrac{1}{2}\lambda^2 h^2 - \cdots\right) = O(h^2)$$

On the other hand, the error at time $t = T = nh$, we have

$$e_n = y_n - y(t_n) = (1 + \lambda h)^n y_0 - y_0 e^{\lambda n h}$$
$$= y_0 \left(1 + n\lambda h + \frac{n(n-1)}{2}\lambda^2 h^2 + \cdots - 1 - \lambda n h - \frac{\lambda^2 n^2 h^2}{2} + \cdots\right) = y_0\left(-\frac{n}{2}\lambda^2 h^2 + \cdots\right)$$
$$= -\frac{T}{2} y_0 \lambda^2 h + \cdots = O(h)$$

The local error per unit step is $\ell_k / h_{k-1} = O(h_{k-1}^p)$ and under reasonable conditions, the global error is $O(h^p)$ where h is the average step size. Euler's method is first-order accurate!

Initial Value Problems: Accuracy and Stability

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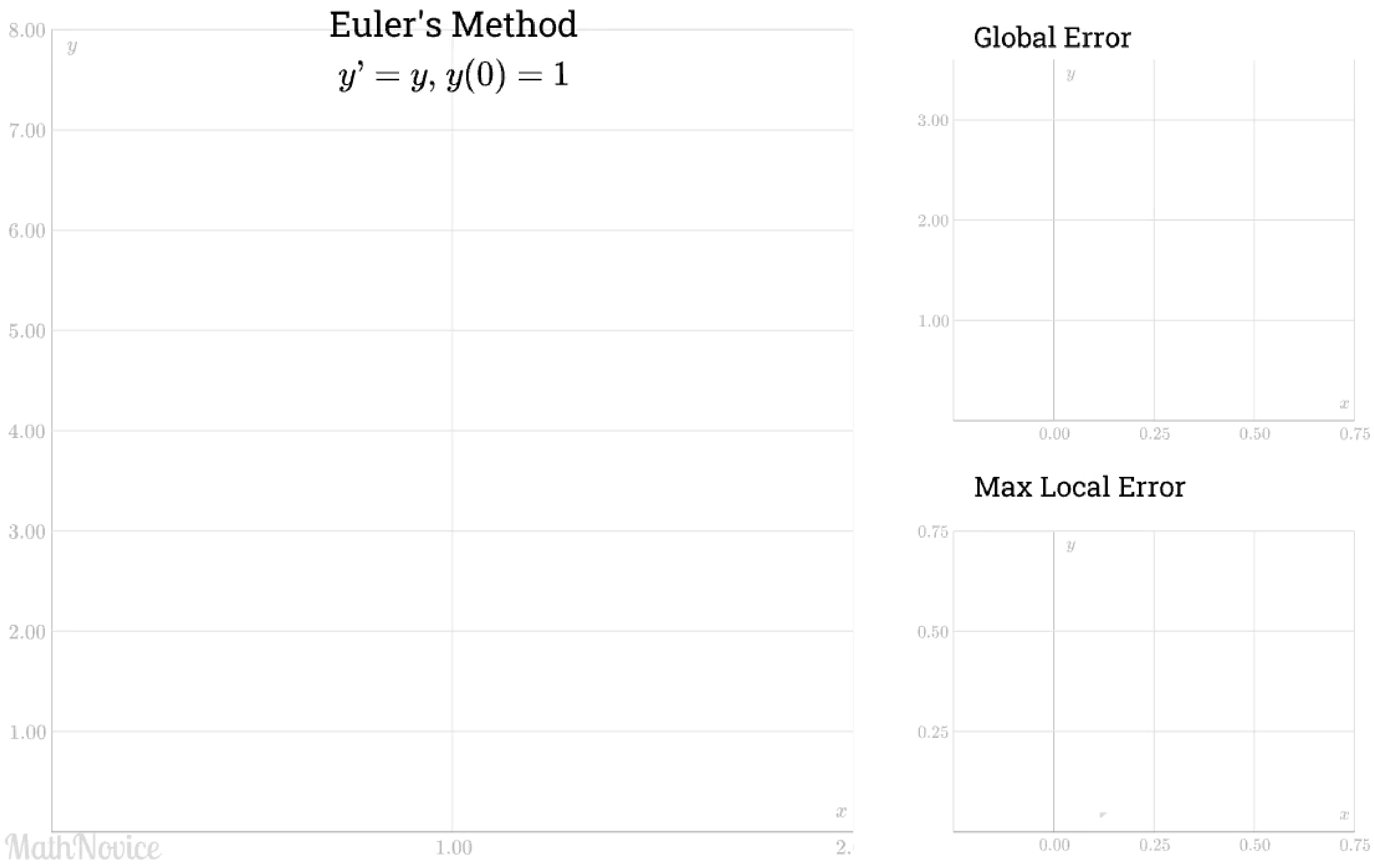
MATH, IIT KANPUR



Source: https://www.youtube.com/watch?v=ivyNR1w9-zk

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However, if $|1 + \lambda h| > 1$, *then* $|y_k| \to \infty$ *as* $k \to \infty$ *regardless of the sign of* $Re(\lambda)$. *This means that Euler's method is unstable even when the exact solution is stable.*

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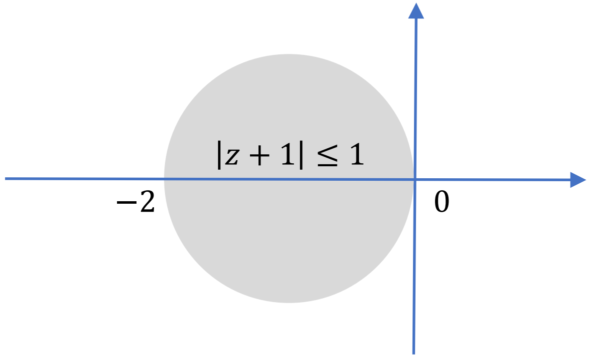

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More generally, for $y' = f(t, y)$, the global error for Euler's method is

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Thus the global error is multiplied at each step by the factor $(I + h_k f')$ *which is called the growth factor or amplification factor.*

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which is satisfied if all eigenvalues of $h_k f'$ lie inside the circle in the complex plane of radius 1 and centered at -1.