Lesson 4

Numerical Solution of PDE

4.3 Hyperbolic PDE

- Finite Difference Methods

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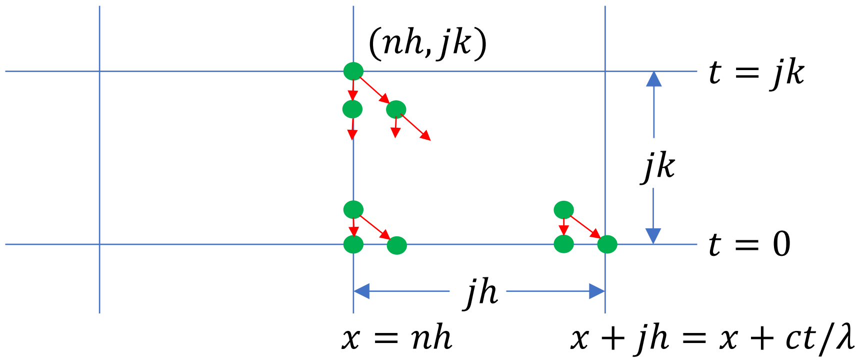

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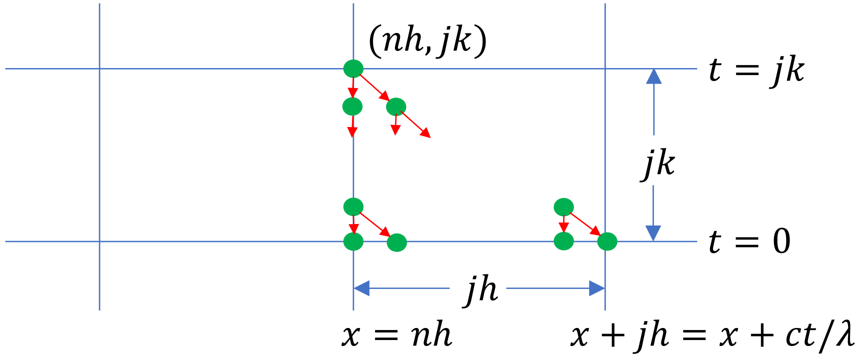

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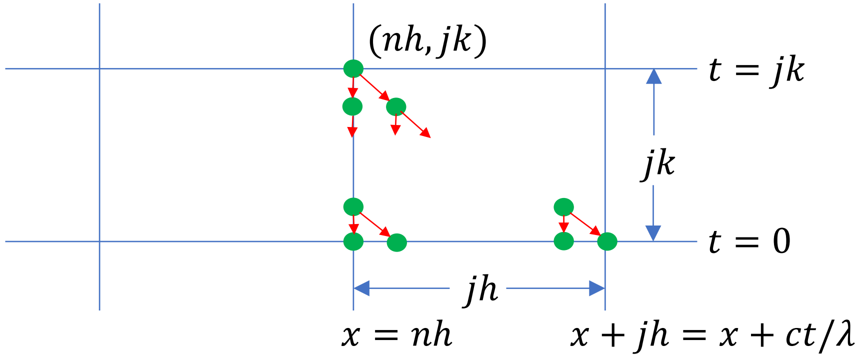

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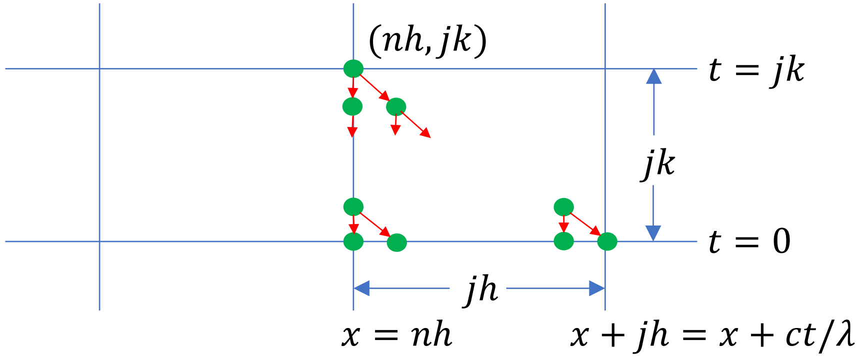

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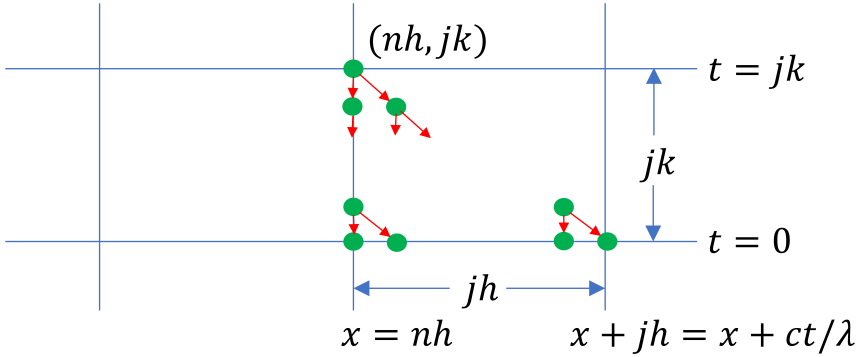

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This necessary condition, which fails for forward-forward difference method, is called the Courant-Friedrichs-Levy condition, or CFL condition.

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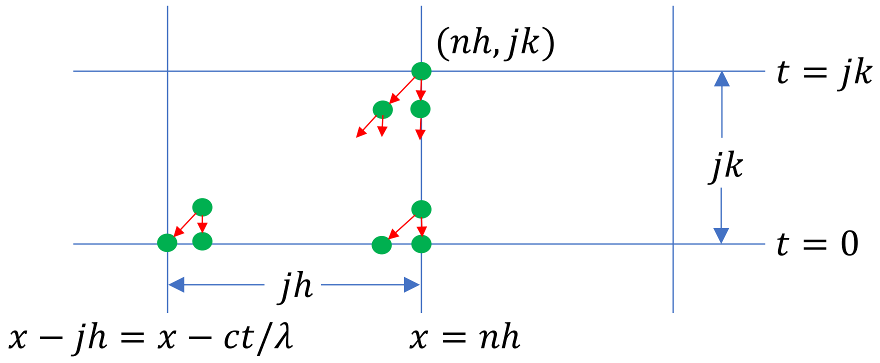

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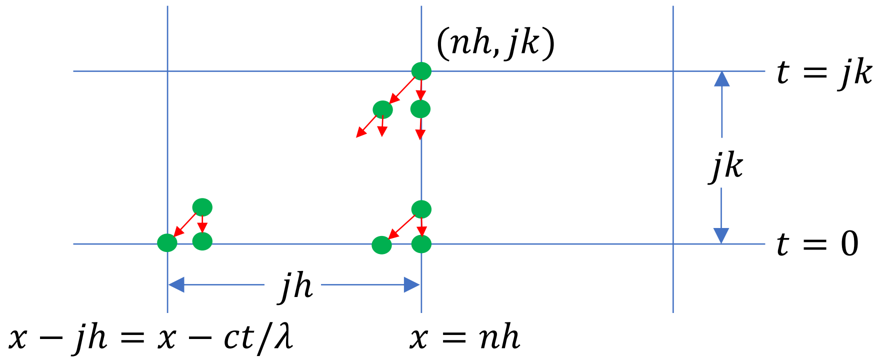

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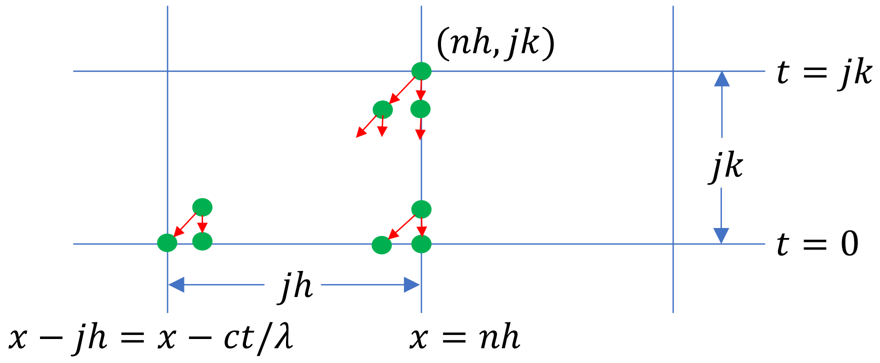

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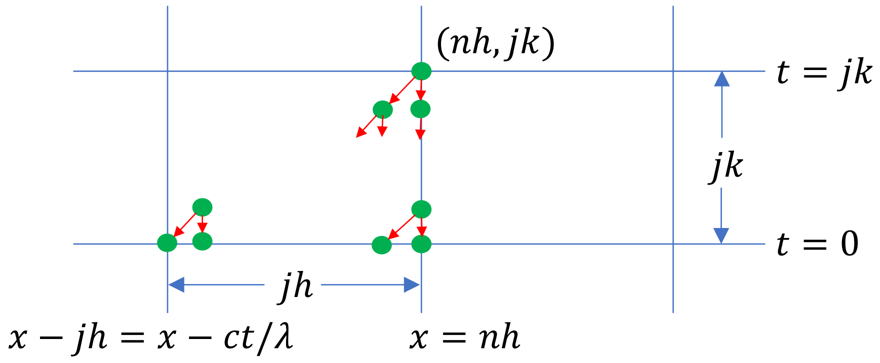

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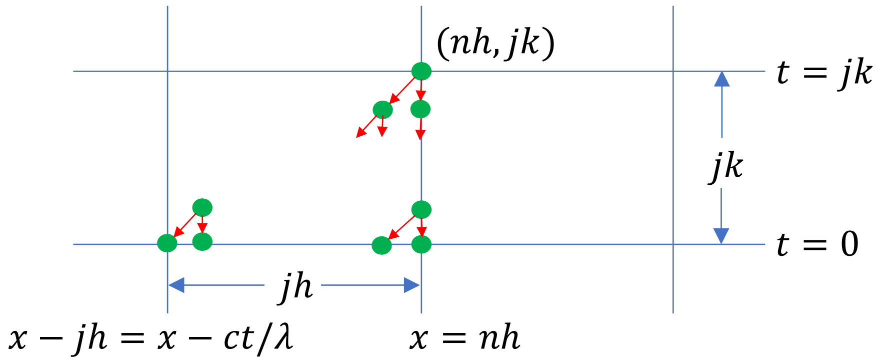

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In general, however, CFL is not sufficient for convergence. It turns out that, for forward-central scheme, while the CFL condition is $|\lambda| \leq 1$, the method is unconditionally unstable.

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- ***Stability Analysis***

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For simplicity, let's consider a 1-periodic problem rather that a boundary value problem:

$$\begin{aligned} \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} &= 0, \qquad x \in \mathbb{R}, t > 0, \\ u(x,0) &= u_0(x), \qquad x \in \mathbb{R}, \\ u(x+1,t) &= u(x,t), \qquad x \in \mathbb{R}, t > 0. \end{aligned}$$

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The ψ_m are orthogonal with respect to the inner product (exercise)

$$\langle \phi, \psi \rangle_h = h \sum_{n=0}^{N-1} \phi(nh)\,\overline{\psi(nh)}.$$

Note that ψ_m is an eigenvector for the forward difference operator D_h^+, the backward difference operator D_h^- and the centered difference operator D_h. For example,

$$D_h^- \psi_m(x) = \frac{\psi_m(x) - \psi_m(x-h)}{h} = \frac{1 - e^{-2\pi i m h}}{h} \psi_m(x).$$

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For the forward-centered method for advection equation, the eigen-values are

$$-\frac{\lambda}{2} e^{2\pi i m h} + 1 + \frac{\lambda}{2} e^{-2\pi i m h} = 1 - i\lambda \sin(2\pi m h).$$

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As every eigenvalue has magnitude greater that one, the method is indeed unconditionally unstable.

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Remark

Note that the Lax-Friedrichs scheme can be rewritten as

$$\frac{u_n^{j+1} - u_n^j}{k} + c\,\frac{u_{n+1}^j - u_{n-1}^j}{2h} - \frac{ch}{2\lambda}\,\frac{u_{n+1}^j - 2u_n^j + u_{n-1}^j}{h^2} = 0.$$

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Thus the method suggests discretization of the equation $\partial u/\partial t + c\partial u/\partial x - (ch/2\lambda)\partial^2 u/\partial x^2 = 0$. This is an advection-diffusion equation. The method can be viewed as a forward-centered scheme with small diffusion added.

Lesson 5

Integral Equations

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MATH, IIT KANPUR

Lesson 5

Integral Equations

5.1 Some solutions of boundary value problems for PDEs via integral equations

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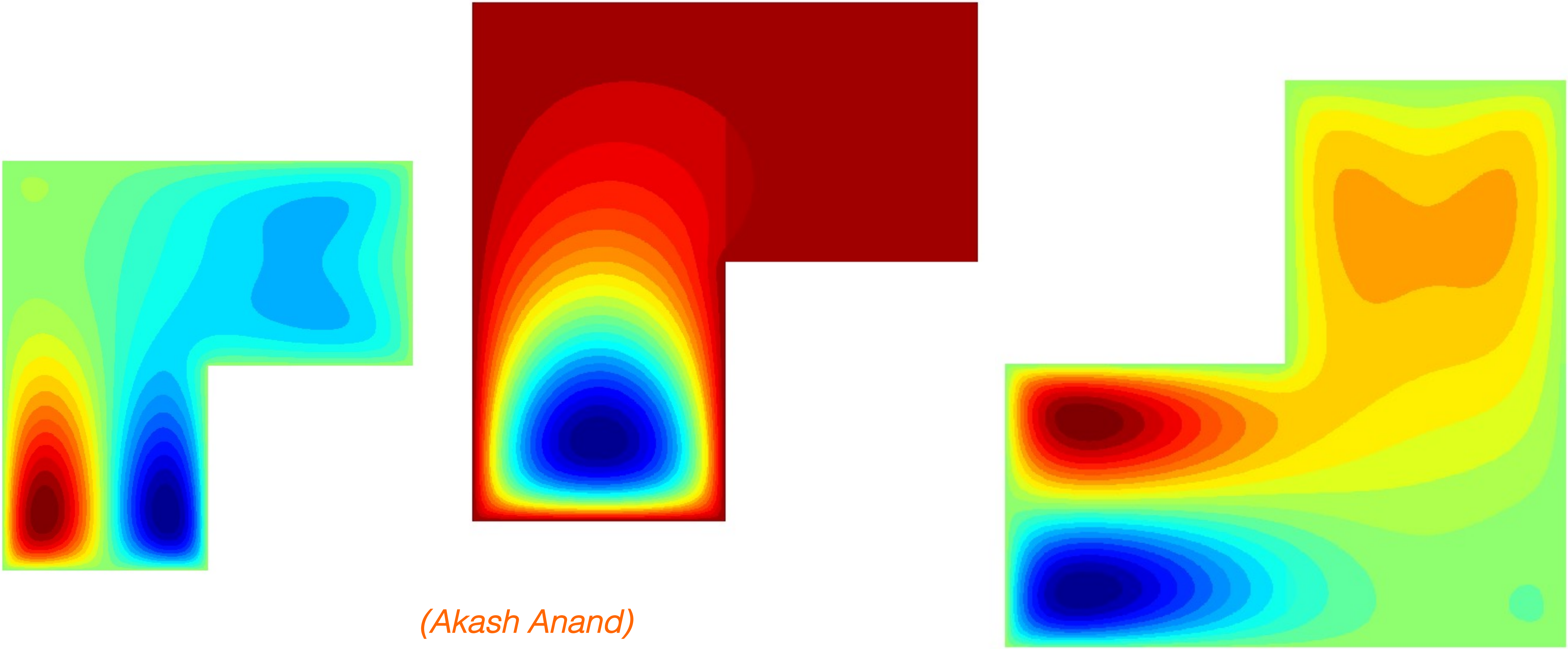
Integral Equations: Some examples

Some solutions of integral Fredholm integral equations



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Integral Equations: Some examples

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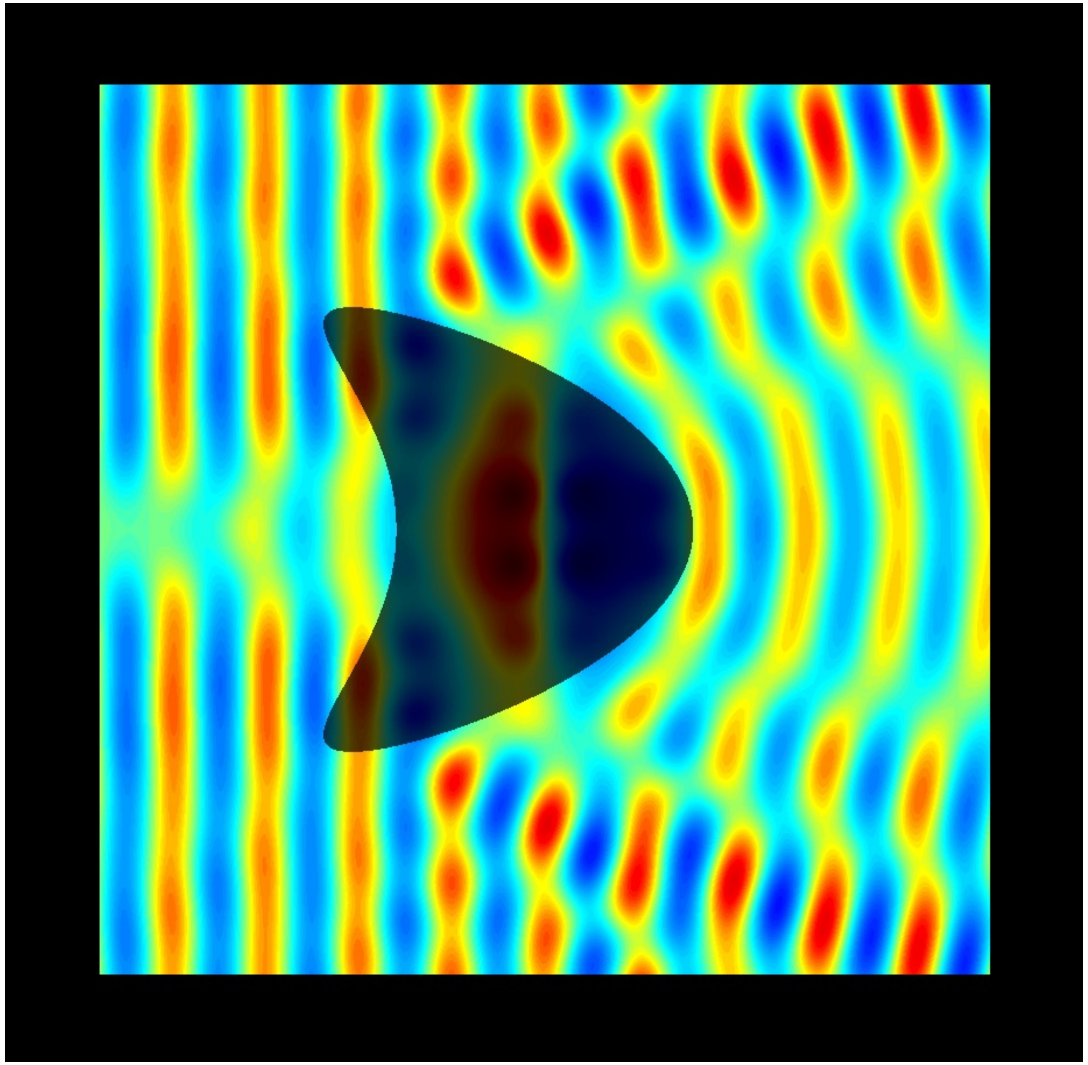
Some solutions of integral Fredholm integral equations in wave scattering



(Ambuj Pandey, Akash Anand)

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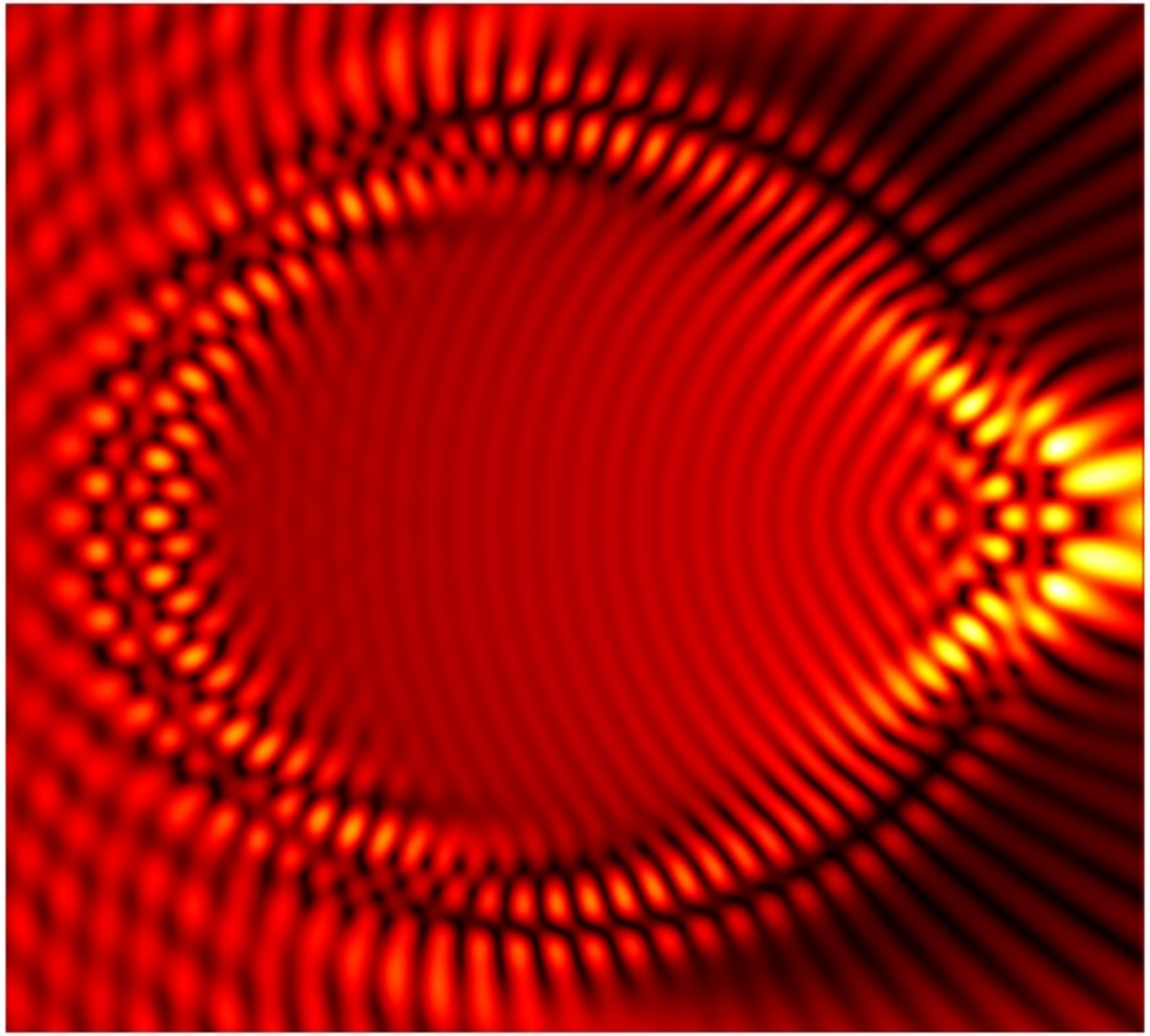

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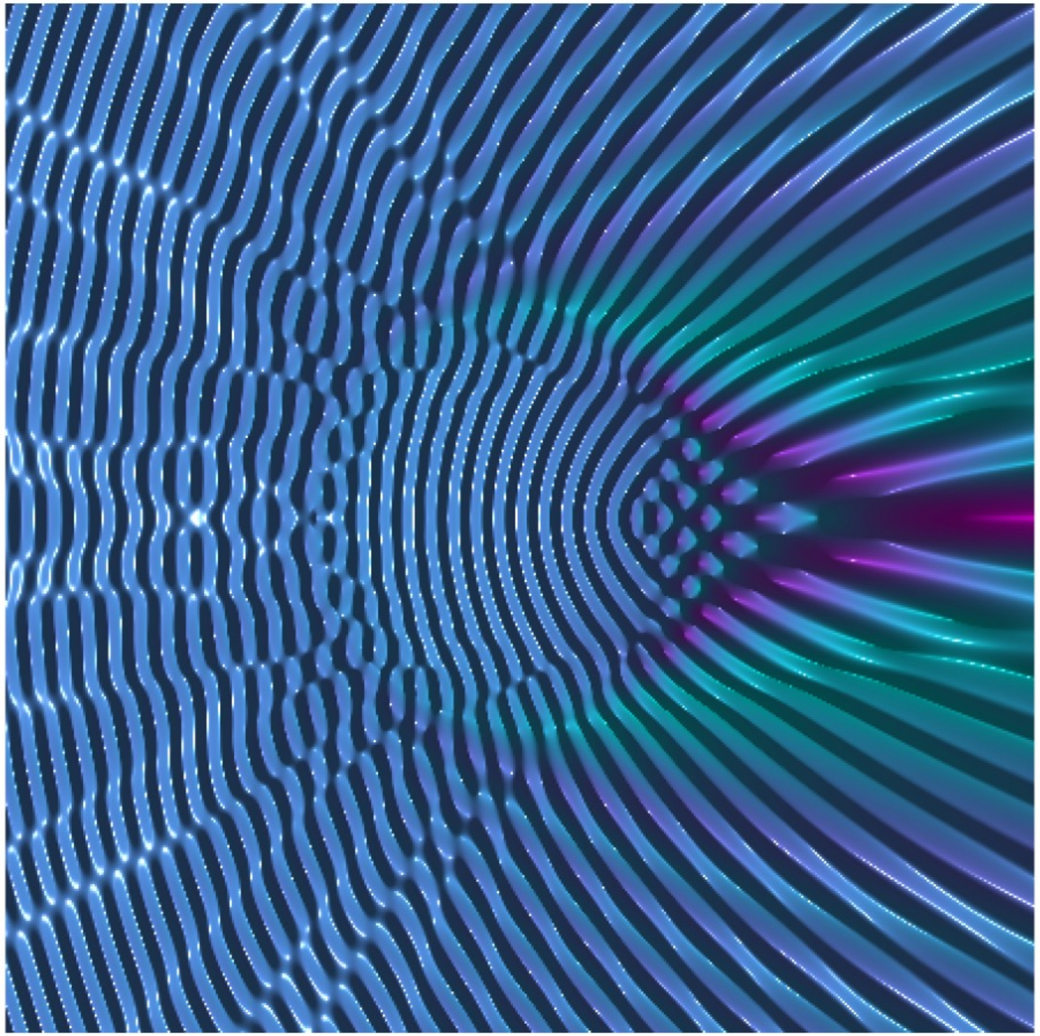

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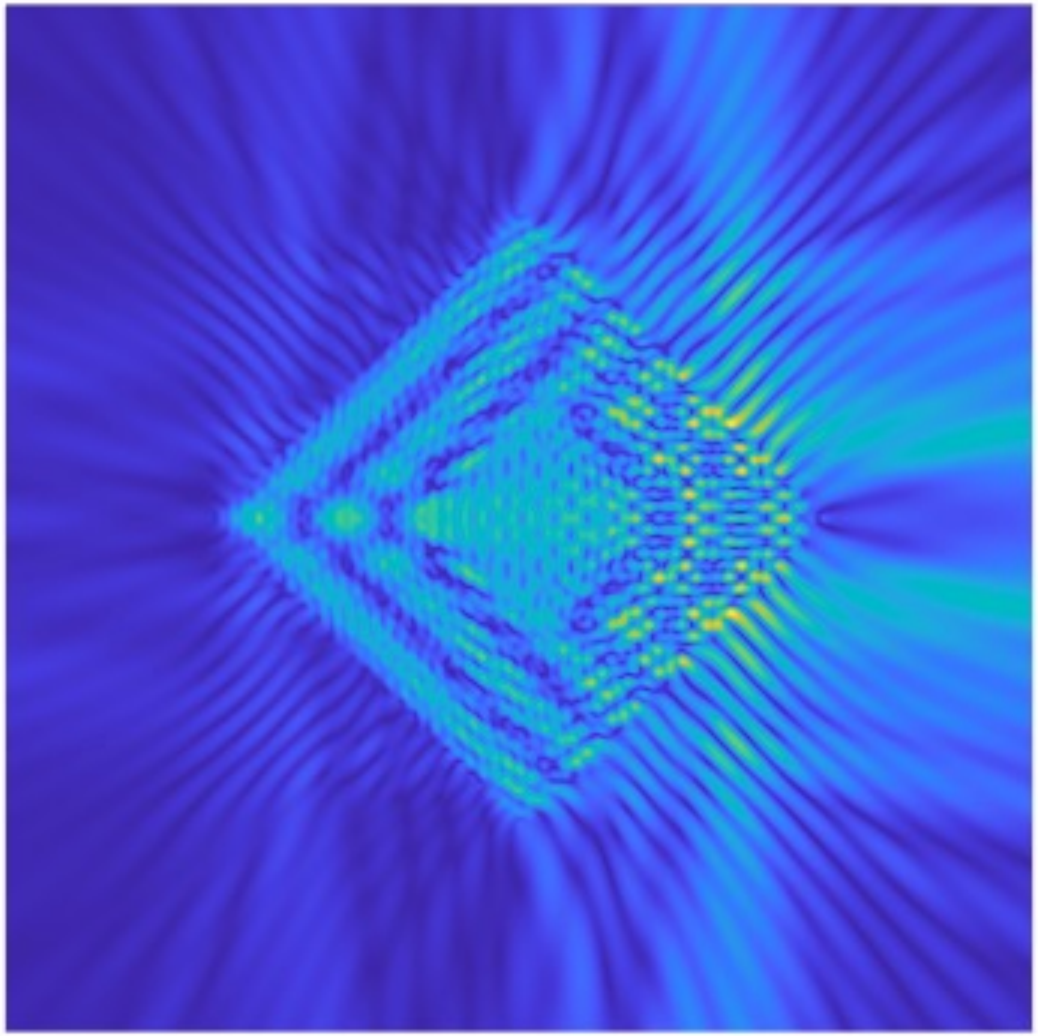

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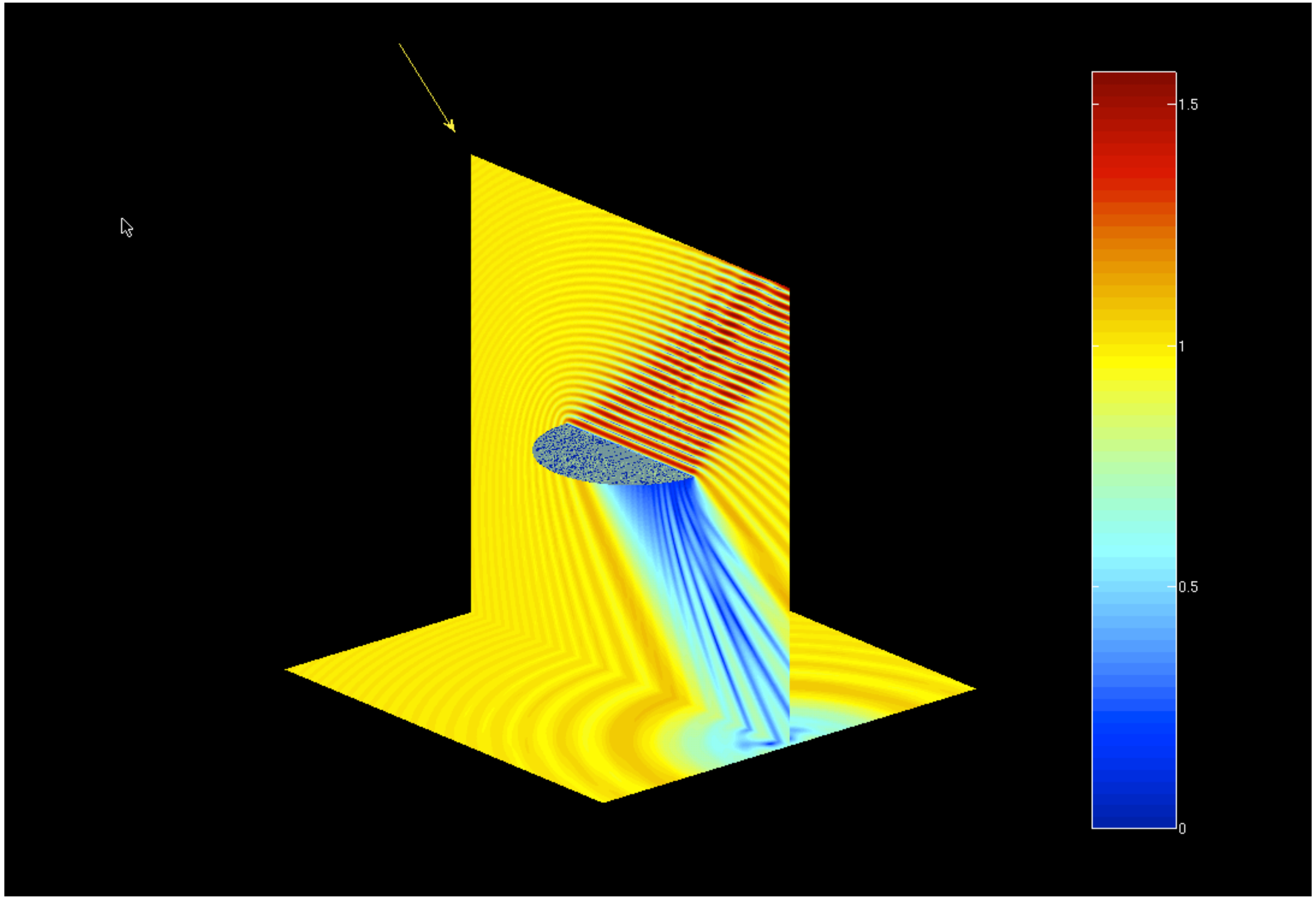

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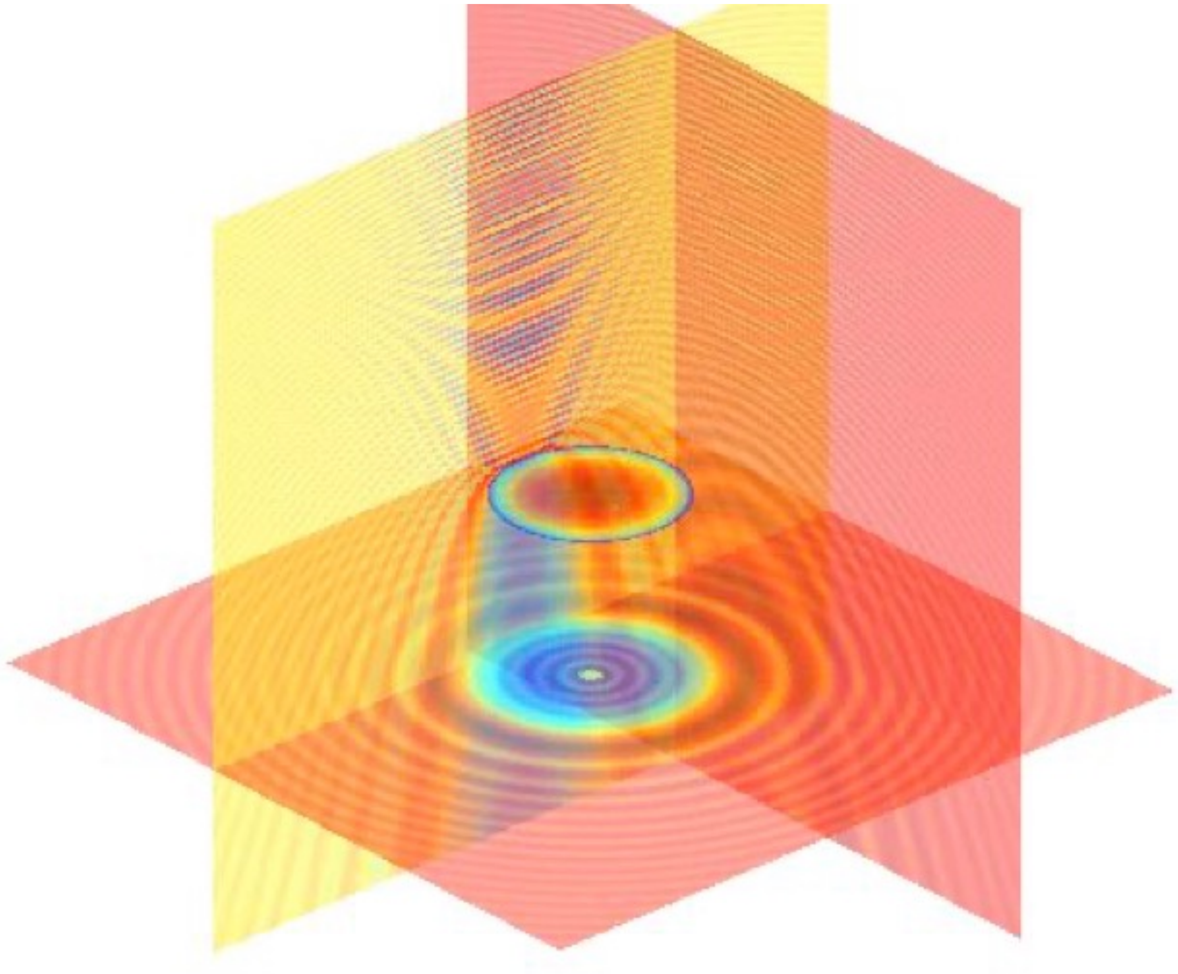

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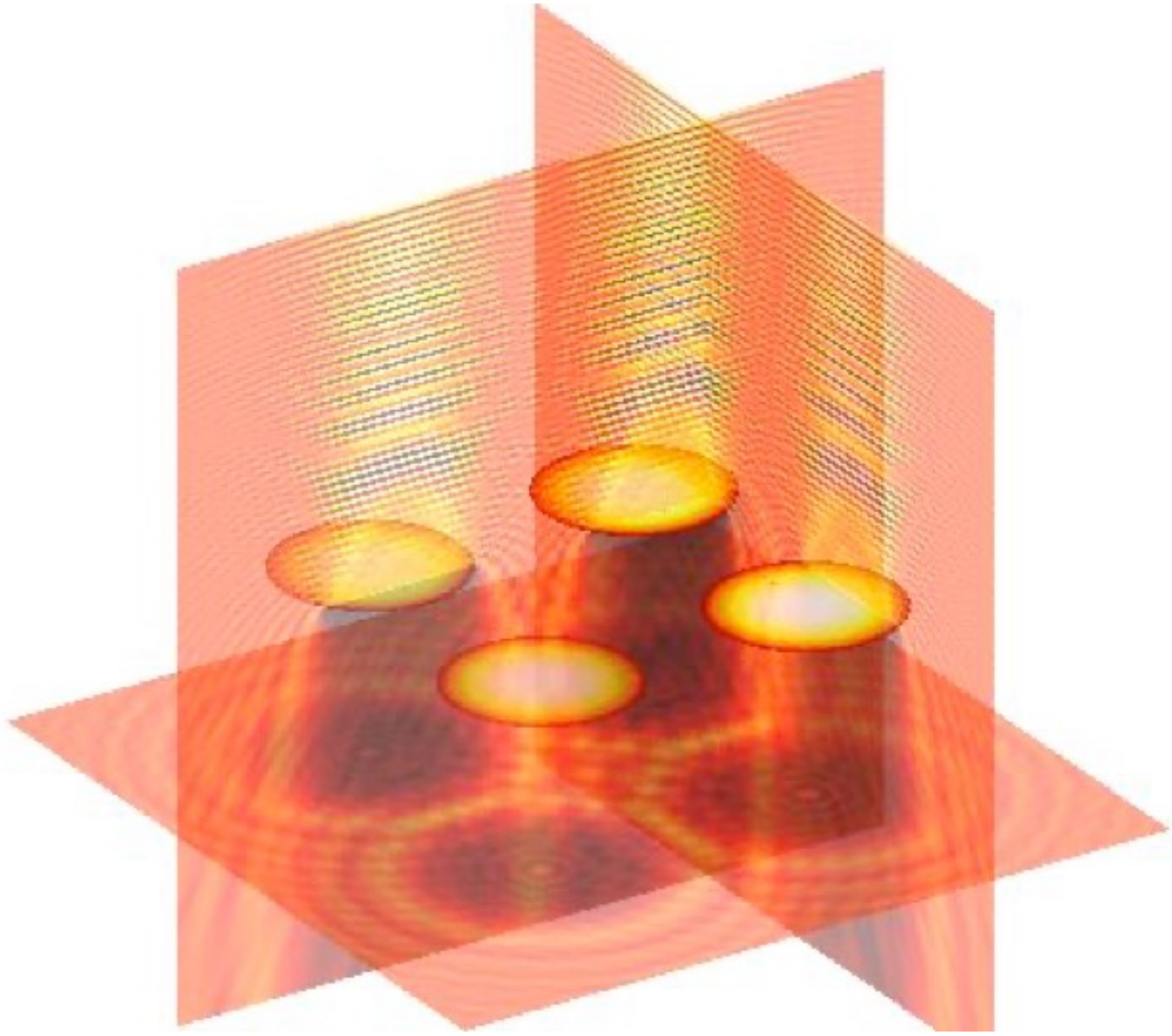

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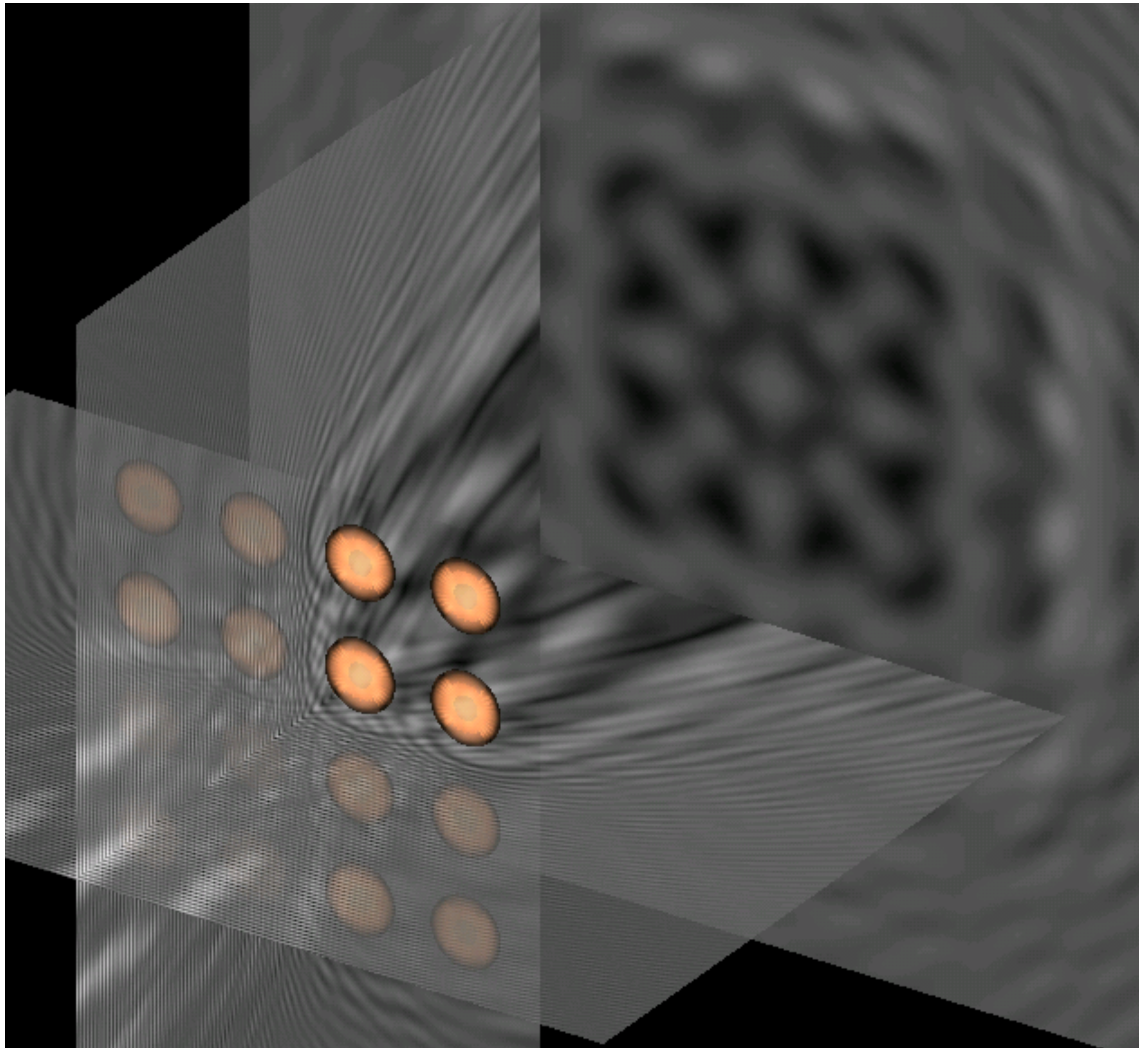

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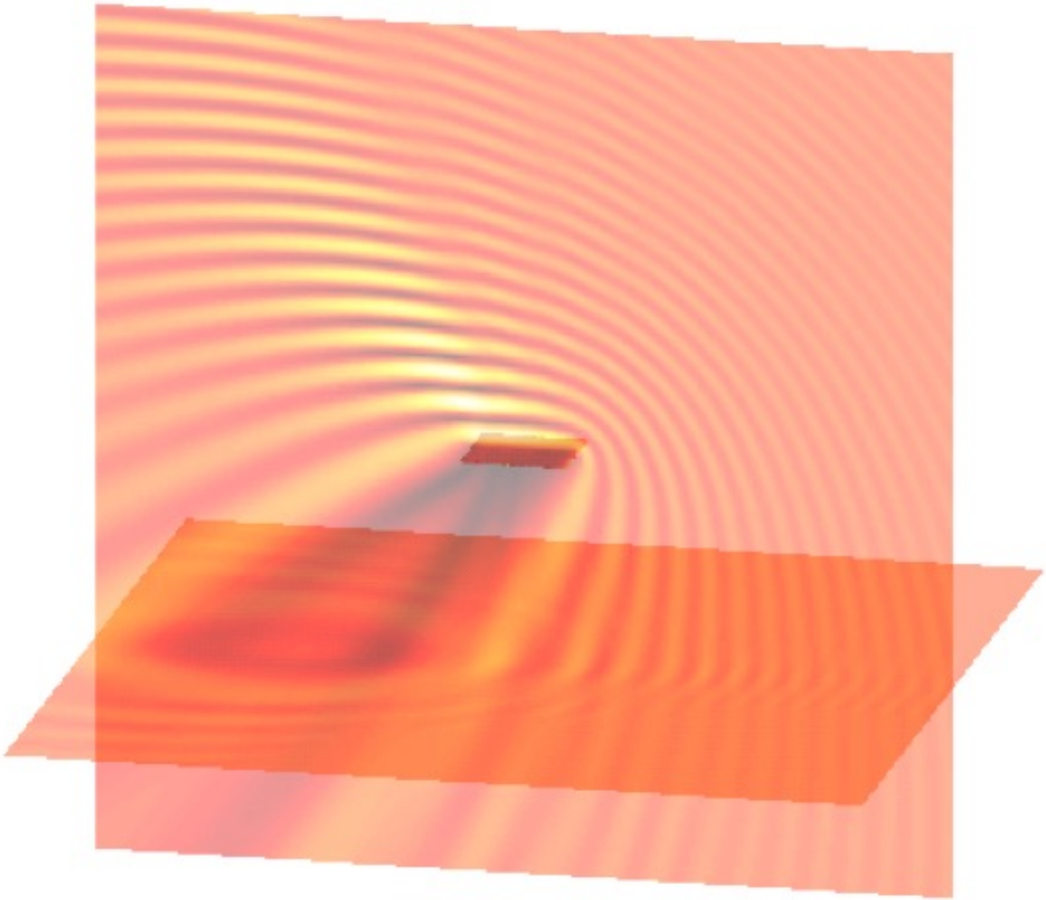

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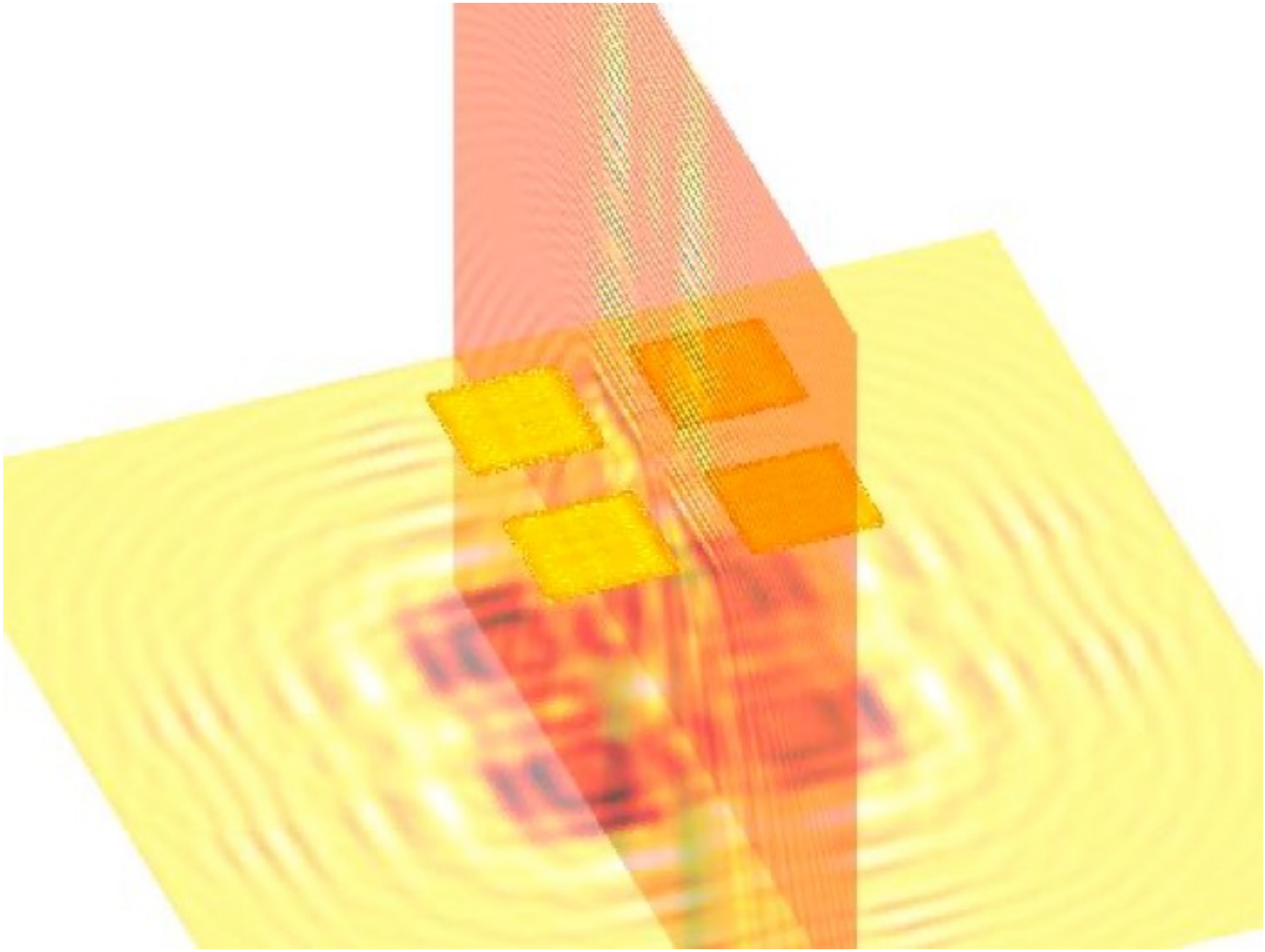

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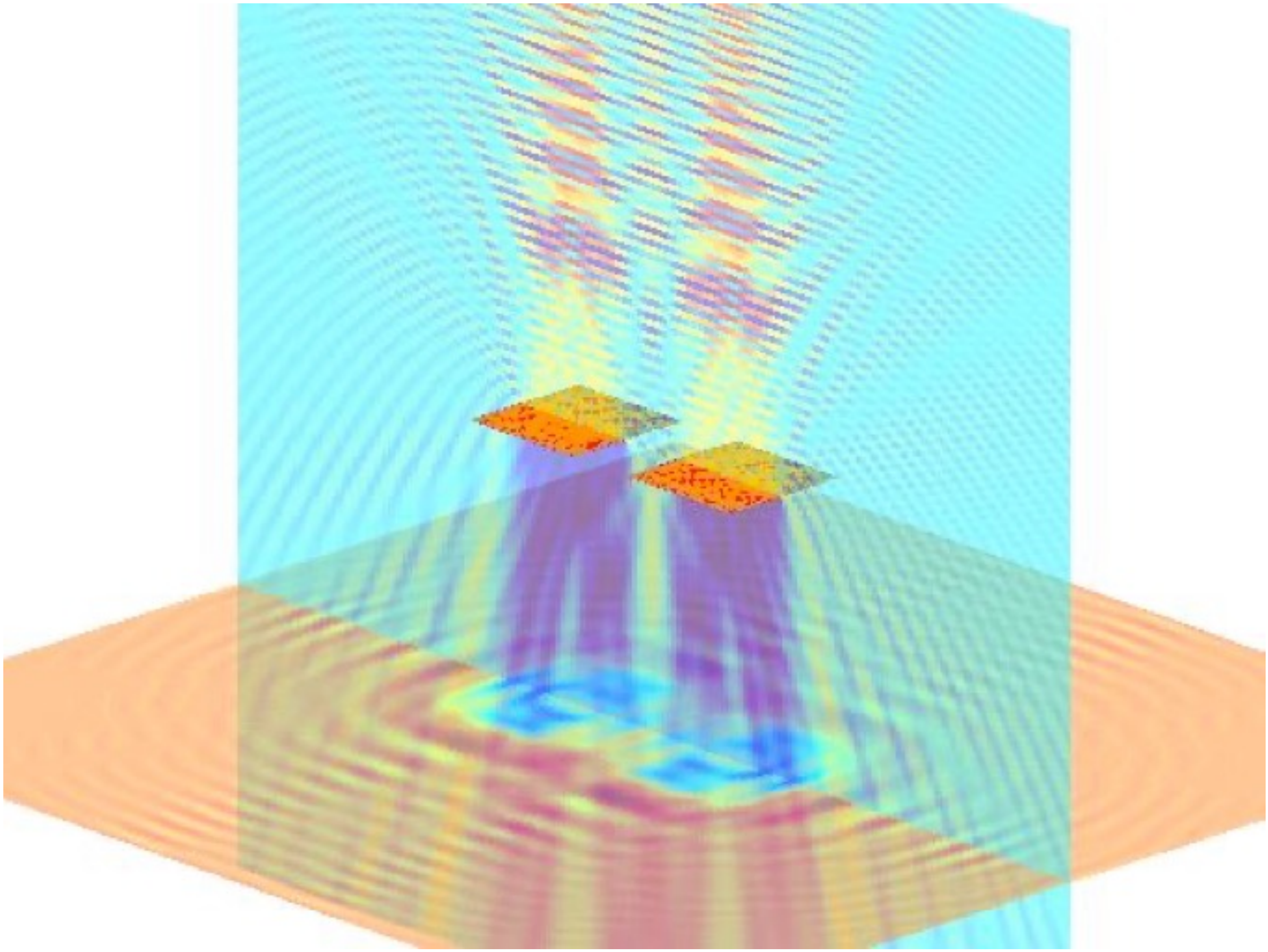

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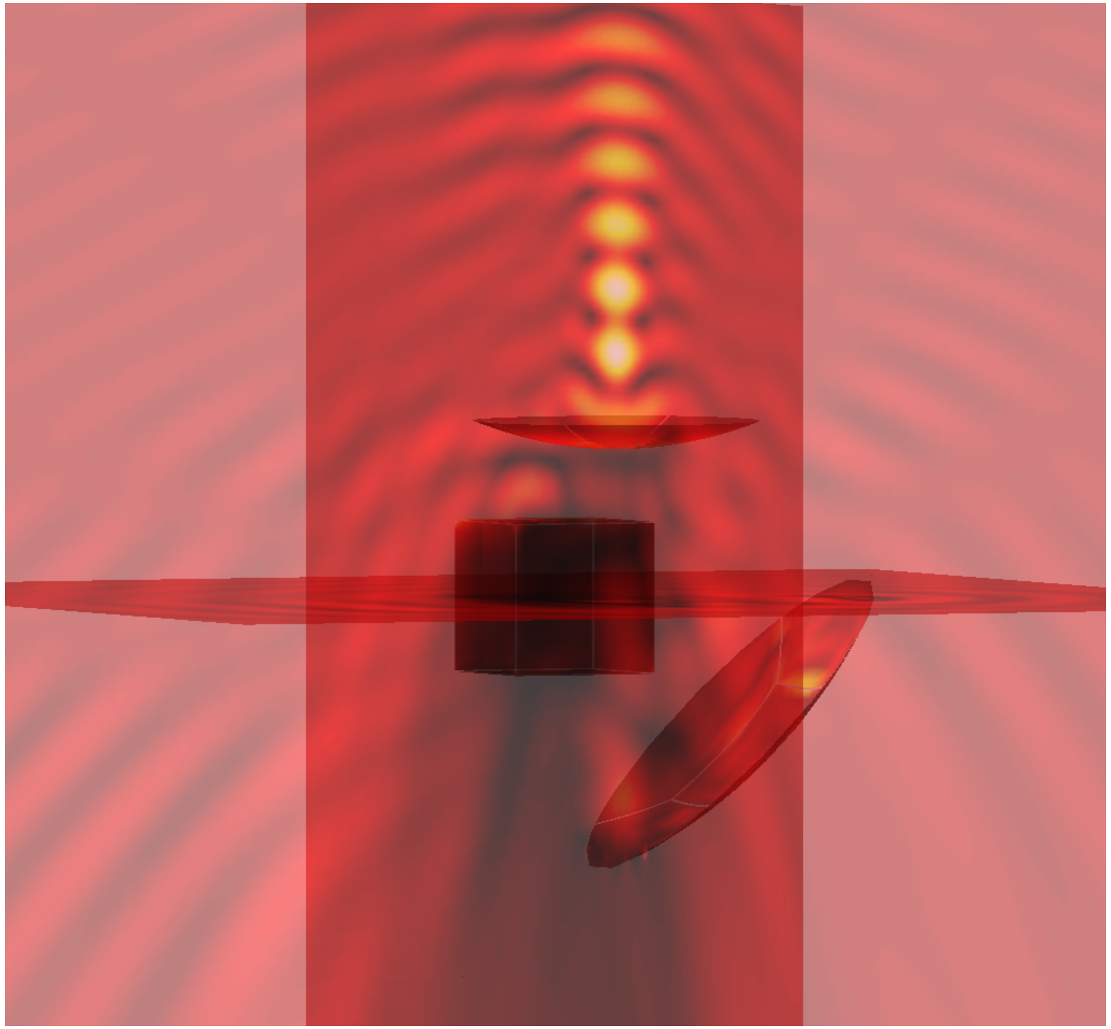

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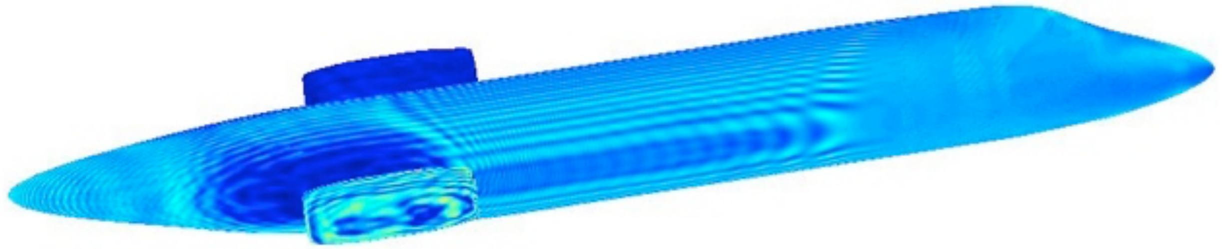

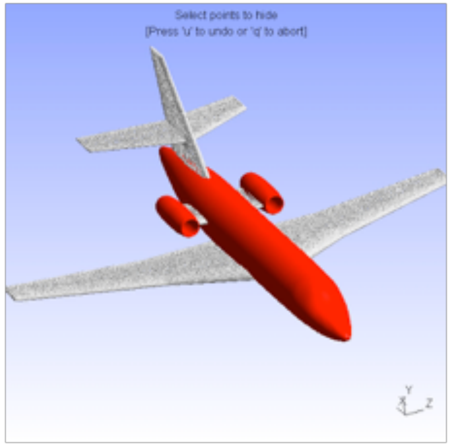

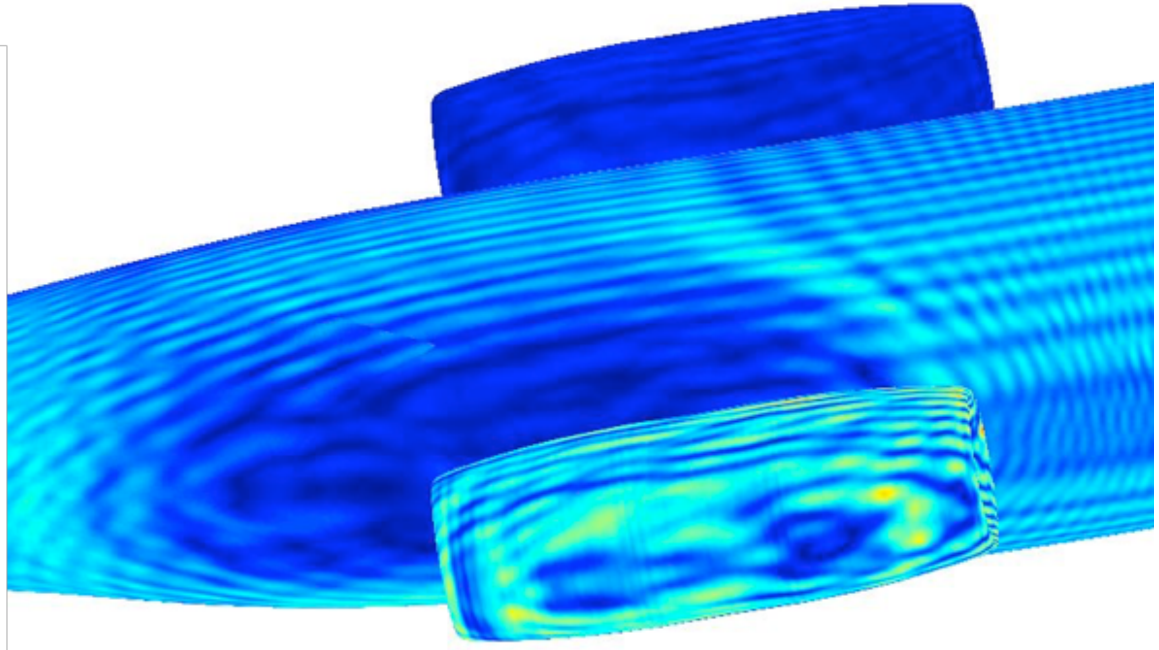

(Akash Anand)

Lesson 5

Integral Equations

5.1 Some solutions of boundary value problems for PDEs via integral equations

5.2 An Introduction

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Integral Equations: An Introduction

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We have already encountered integral equations in this course.

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Recall that the initial value problem

$$y' = f(t, y), y(t_0) = y_0,$$

is equivalent to finding y satisfying

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This is a particular case of the more general Volterra integral equations

$$\int_{t_0}^{t} K\big(t, s, y(s)\big) ds = g(t), \qquad t \geq t_0.$$

where K and g are are given functions.

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The difficulty with these equations, linear or non-linear, is that they are ill-conditioned.

The numerical solution of these equations are closely related to the initial value problem. We will, however, focus on a different type of integral equations known as Fredholm integral equations, in particular, of the second kind.

Integral Equations: An Introduction

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The general form of such an integral equation is

$$u(t) - \int_{\Omega} K(t,s)u(s)ds = f(t), \qquad t \in \Omega.$$

where Ω *is a bounded set in* $\mathbb{R}^m, m \geq 1$.

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Example

Consider solving the problem

$$\Delta u(x) = 0, \qquad x \in \Omega,$$
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where Ω is a bounded region in $\mathbb{R}^3$ with non-empty interior and Γ is the boundary of Ω.

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where Ω is a bounded region in $\mathbb{R}^3$ with non-empty interior and Γ is the boundary of Ω. If we seek u in the form of a single layer potential, that is

$$u(x) = \int_{\Gamma} \frac{1}{|x-y|}\rho(y)dy\,, \qquad x \in \Omega,$$

where $\rho(y)$ is called a single layer density function and it is the unknown in the equation.

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Consider solving the problem

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$$u(x) = \int_\Gamma \frac{\partial}{\partial \nu(y)}\left(\frac{1}{|x-y|}\right)\mu(y)dy\,, \qquad x \in \Omega,$$

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then $\mu(y)$ satisfies a Fredholm integral equation of the second kind. Indeed, the double layer density function $\mu(y)$ satisfies

$$\frac{1}{2}\mu(x) - \int_\Gamma \frac{\partial}{\partial \nu(y)}\left(\frac{1}{|x-y|}\right)\mu(y)dy = -g(x), x \in \Gamma,$$

(another fact from "Potential Theory" know as jump relation for double layer potential).

Integral Equations: An Introduction

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We say that a kernel $K: \Omega\times\Omega \to \mathbb{C}$ is weakly singular if K is defined and continuous for all $x, y \in \Omega \subseteq \mathbb{R}^m$, $x \neq y$, and there exist positive constants M and $\alpha \in (0, m]$ such that

$$|K(x,y)| \leq M|x-y|^{-(m-\alpha)}$$

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Remark

One can show that, the integral operator

$$(Au)(x) = \int_{\Omega} K(x,y)u(y)dy\,,$$

with weakly singular kernel K maps continuous functions to continuous function, that is, the operator $A\colon C(\Omega) \to C(\Omega)$ is well-defined.

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Moreover, it is known that, for such integral operators, the Fredholm alternative holds, that is,

$$(I-A)u = f$$

has a unique solution for every $f \in C(\Omega)$ *if and only if the homogeneous equation* $(I-A)v = 0$ *has only the trivial solution* $v = 0$.

Lesson 5

Integral Equations

5.1 Some solutions of boundary value problems for PDEs via integral equations

5.2 An Introduction

5.3 Numerical Methods

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Integral Equations: An Introduction

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There are three main ideas for numerical solution of the second kind Fredholm integral equation

$$(I - A)u = f$$

with the linear integral operator

$$(Au)(x) = \int_{\Omega} K(x, y)u(y)dy\,.$$

Approximate the integral operator by

- *approximating the kernel $K(x, y)$.*
- *approximating the solution $u(x)$.*
- *approximating the integral $\int_{\Omega} f(y)dy$ by a quadrature.*

Lesson 5

Integral Equations

5.2 An Introduction

5.3 Numerical Methods

- Degenerate Kernel Method

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Integral Equations: Numerical Methods

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The solution of the integral equation of the second kind, $u - Au = f$, is then obtained as

$$u_n(x) - \sum_{j=1}^{n} \langle b_j, u_n \rangle a_j(x) = f(x)\,,$$

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$$u_n = f + \sum_{k=1}^{n} \gamma_k a_k\,,$$

where the coefficients $\gamma_1, \gamma_2, \dots, \gamma_n$ satisfy the linear system

$$\gamma_j - \sum_{k=1}^{n} \langle a_k, b_j \rangle \gamma_k = \langle f, b_j \rangle, \qquad j = 1,2,\dots,n\,.$$

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Theorem

Let X and Y be Banach spaces and let $A: X \to Y$ be a bounded linear operator with a bounded operator $A^{-1}: Y \to X$. Assume the sequence $A_n: X \to Y$ of bounded linear operators to be norm convergent, that is, $\|A_n - A\| \to 0$ as $n \to \infty$. Then, for sufficiently large n, more precisely, for all n with $\|A^{-1}(A_n - A)\| < 1$, the inverse operators $A_n^{-1}: Y \to X$ exist and are bounded by

$$\|A_n^{-1}\| \le \frac{\|A^{-1}\|}{1 - \|A^{-1}(A_n - A)\|}.$$

For all solutions of the equations $A\varphi = f$ and $A_n\varphi_n = f_n$, we have the error estimate

$$\|\varphi_n - \varphi\| \le \frac{\|A^{-1}\|}{1 - \|A^{-1}(A_n - A)\|}\{\|(A_n - A)\varphi\| + \|f_n - f\|\}.$$

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Integral Equations

5.2 An Introduction

5.3 Numerical Methods

- Degenerate Kernel Method

- via interpolation

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Consider the domain to be an interval, that is, $\Omega = (\mathrm{a}, \mathrm{b})$ and let K be continuous.

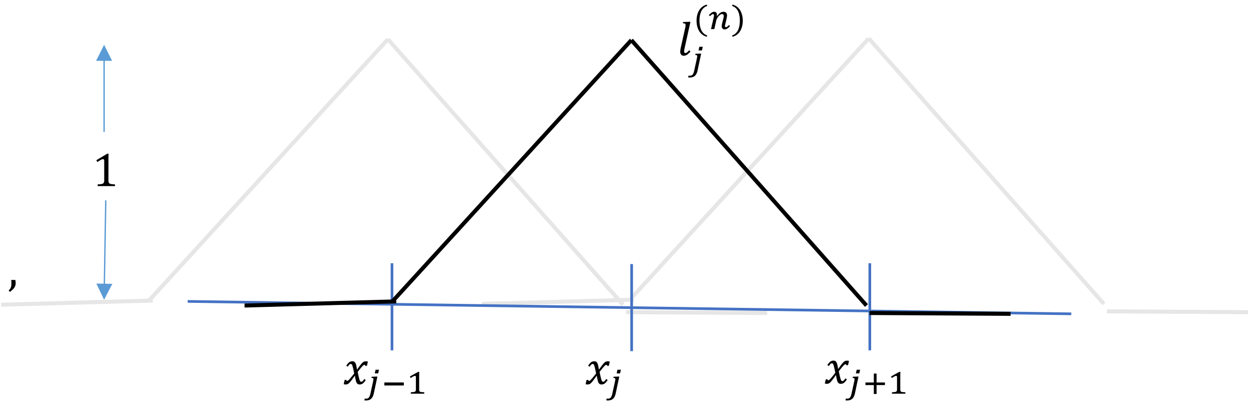
One idea that works when K is continuous is interpolation -- approximate K by interpolating in x with respect to the points $x_1, x_2, \dots, x_n$ in $[a, b]$ for each $y \in [a, b]$, we have

$$K_n(x, y) = \sum_{j=1}^{n} K(x_j, y) l_j^{(n)}(x).$$

Example

Equidistant piecewise linear interpolation:

$$l_j^{(n)}(x) = \begin{cases} (x - x_{j-1})/h, & x \in [x_{j-1}, x_j], j \geq 1, \\ (x_{j+1} - x)/h, & x \in [x_j, x_{j+1}], j \leq n-1, \\ 0, & \text{otherwise.} \end{cases}$$



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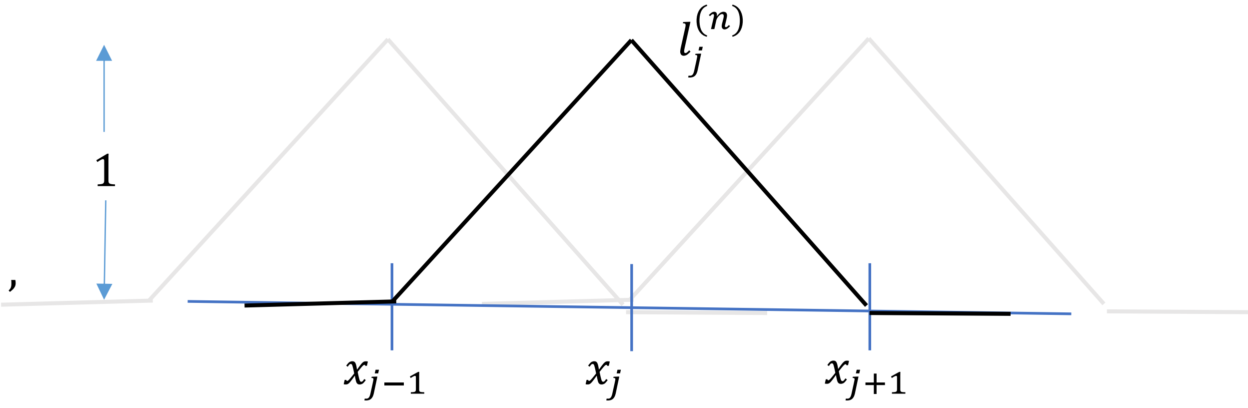

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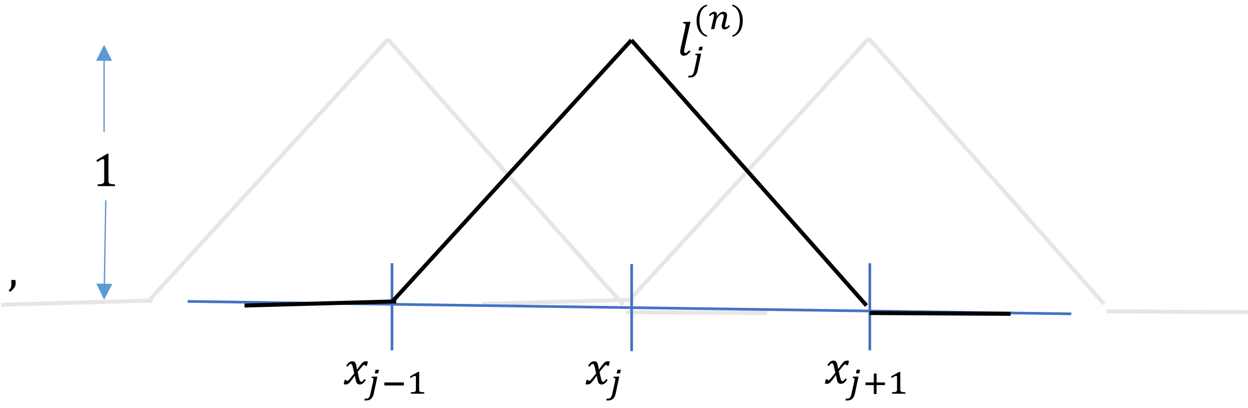

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Integral Equations: Numerical Methods

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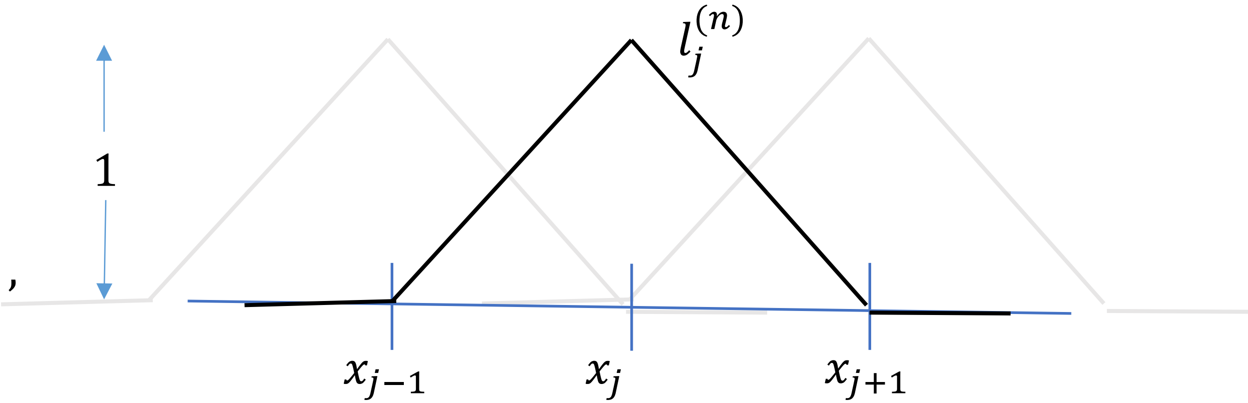

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$$m_{ij} = \langle a_j, b_i \rangle = \int_a^b a_j(x) b_i(x) dx = \int_a^b l_j^{(n)}(x) K(x_i, x) dx \approx \sum_{k=1}^{n} K(x_i, x_k) \int_a^b l_j^{(n)}(x) l_k^{(n)}(x) dx$$

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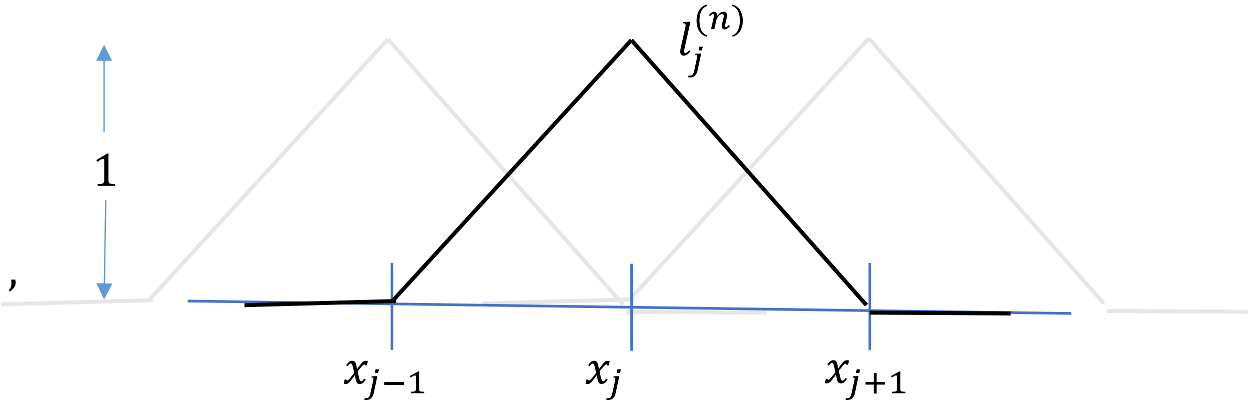

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Integral Equations: Numerical Methods

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Equidistant piecewise linear interpolation:

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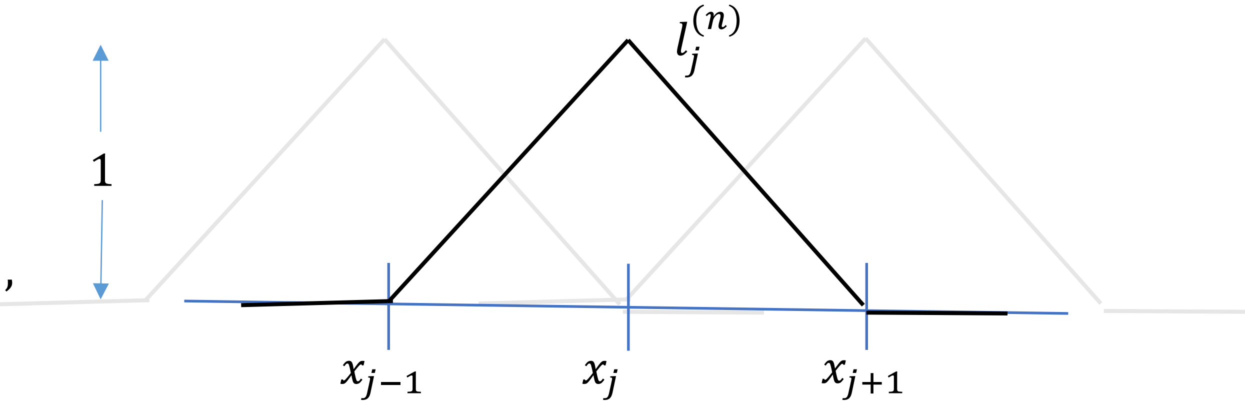

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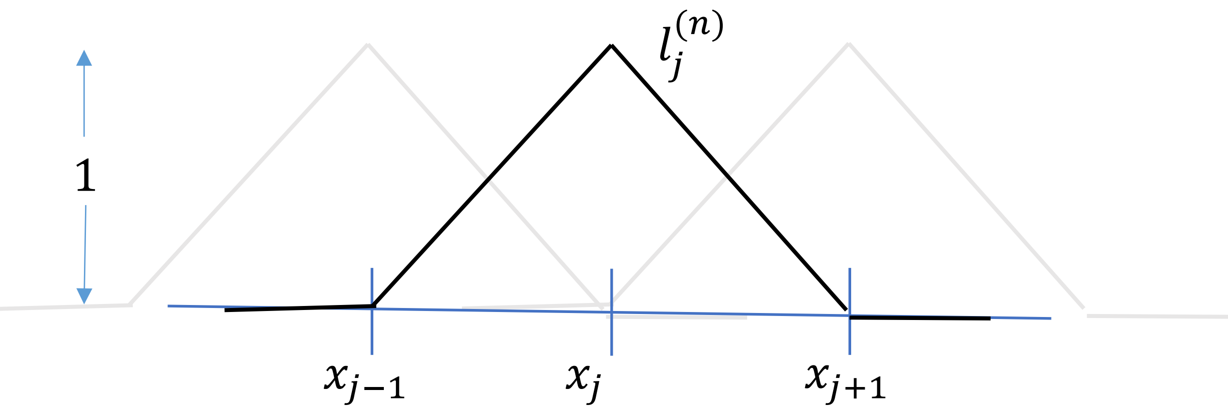

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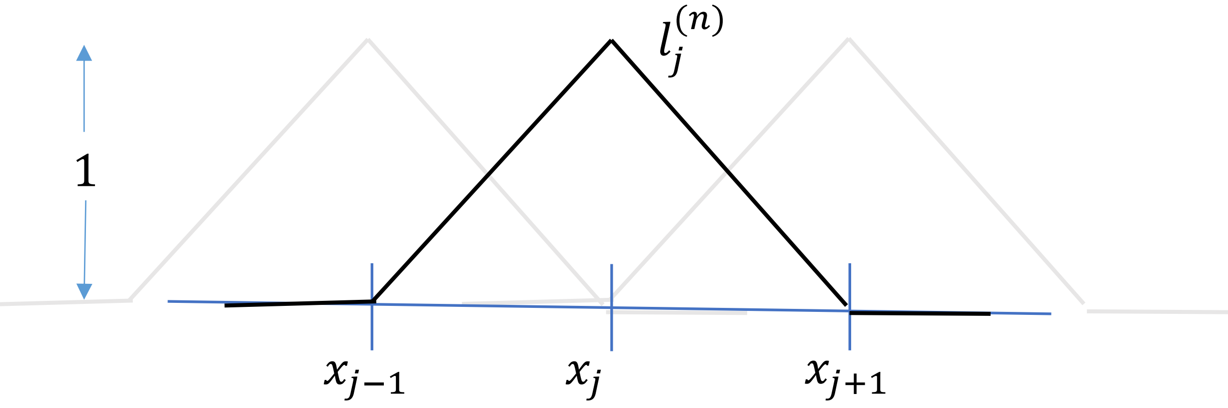

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$$\varphi(x) - \frac{1}{2}\int_0^1 (x+1)e^{-xy}\varphi(y)dy = e^{-x} - \frac{1}{2} + \frac{1}{2}e^{-(x+1)}, \qquad 0 \le x \le 1.$$

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Table showing the error between the approximate and the exact solution

n	x = 0	x = 0.25	x = 0.5	x = 0.75	x = 1
4	0.004808	0.005430	0.006178	0.007128	0.008331
8	0.001199	0.001354	0.001541	0.001778	0.002078
16	0.000300	0.000385	0.000385	0.000444	0.000519
32	0.000075	0.000085	0.000096	0.000111	0.000130

Lesson 5

Integral Equations

5.2 An Introduction

5.3 Numerical Methods

- ***Degenerate Kernel Method***
  - ***via orthogonal expansion***

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$$K_n(x,y) = \sum_{j=1}^{n} u_j(x)\langle K(\cdot,y), u_j\rangle$$

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For this degenerate kernel, the linear system reads

$$\gamma_j - \sum_{k=1}^{n} \gamma_k \int_\Omega u_k(y)\langle K(\cdot,y), u_k\rangle dy = \int_\Omega f(y)\langle K(\cdot,y), u_k\rangle dy\,, j = 1,2,\dots,n.$$

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Note that the setting up the linear system requires a double integration for each coefficient and for each right-hand side.

Example (Expansion with respect to the Chebyshev system)

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Theorem

Let $g\colon[-1,1]\to\mathbb{R}$ be analytic. Then, there exists an ellipse E with foci at -1 and 1 such that g can be extended to a holomorphic and bounded function $g\colon D\to\mathbb{C}$ where D denotes the open interior of E. The orthonormal expansion with respect to the Chebyshev polynomials

$$g=\frac{a_0}{2}T_0+\sum_{n=1}^{\infty}a_nT_n\,,\qquad a_n=\frac{2}{\pi}\int_{-1}^{1}\frac{g(x)T_n(x)}{\sqrt{1-x^2}}\,dx=\frac{2}{\pi}\langle g,T_n\rangle$$

is uniformly convergent with the estimate

$$\left\|g-\frac{a_0}{2}T_0-\sum_{k=1}^{n}a_kT_k\right\|_{\infty}\le\frac{2M}{R-1}R^{-n}.$$

Here R is given through the semi-axis a and b of E by $R=a+b$ and M is a bound on g in D.