Lesson 2

Initial Value Problems

2.4 Implicit method

2.5 Stiffness

2.6 Linear Multistep Methods

- Convergence

Akash Anand
MATH, IIT KANPUR

Convergence

A linear multistep method is convergent if whenever the initial values y_n are chosen such that $\max_{0\le n\le k}\|e_n\| \to 0$, as $h \to 0$, then $\max_{0\le n\le N}\|e_n\| \to 0$.

Example

Does

$$y_{n+1} + 4y_n - 5y_{n-1} = h(4f_n + 2f_{n-1}).$$

this method converge?

Apply the method to the IVP $y' = 0,\ y(0) = 0$. *We have*

$$y_{n+1} = -4y_n + 5y_{n-1}.$$

Solving the difference equation yields

$$y_n = (1 - (-5)^n)y_1/6.$$

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If $h = 1/N$, *then*

$$e_N = y_N - y(1) = (1 - (-5)^N)y_1/6 = \left(1 - (-5)^{1/h}\right)y_1/6$$

For $y_1 = h$, *we have* $e_1 = h$. *We see that even as* $e_1 \to 0$ *as* $h \to 0$, $e_N \nrightarrow 0$. *Thus, the method does not converge.*

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Note that if we take exact starting values $y_0 = y_1 = 0$, then $y_n = 0$ for all n. Thus, a perturbation of size ε in the starting values leads to a difference of size roughly $5^{1/h}\varepsilon$ in the discrete solution. The method is, therefore, not stable.

Module 2

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- Stability

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Stability

A linear $k+1$ step method is stable if for any initial value problem with Lipschitz continuous f and of $\varepsilon > 0$, there exists $\delta, h_0 > 0$ such that if $h \leq h_0$ and two choices of starting values y_j and $\widehat{y}_j$ are chosen satisfying

$$\max_{0 \leq j \leq k} \left\| y_j - \widehat{y}_j \right\| \leq \delta,$$

then the corresponding approximate solutions satisfy

$$\max_{0 \leq j \leq N} \left\| y_j - \widehat{y}_j \right\| \leq \varepsilon.$$

If $y' = 0$, then the general linear multistep method becomes

$$y_{n+1} + \sum_{j=0}^{k} a_j y_{n-j} = 0, \qquad n = k, k+1, \ldots$$

This is an example of a homogeneous linear difference equation of order $k+1$ with constant coefficients.

If a method is stable, it should, in particular, be stable while computing the zero solution of the initial value problem $y' = 0, y(0) = 0$. To investigate this, we need to solve the homogeneous linear difference equation of order $k+1$.

To find the general solution, we first try for a solution of the form $(\lambda^n)_{n=0}^{\infty}$. Substituting this in the difference equation, we see that it is a solution if and only if λ is a root of the characteristic polynomial

$$\rho(t) = t^{k+1} + \sum_{j=0}^{k} a_j t^{k-j}.$$

Initial Value Problems: Linear Multistep Methods

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If there are $k+1$ distinct roots $\lambda_i, i = 0, \ldots, k$, then we have a full basis of $k+1$ linearly independent solutions (note that the solutions form a vector space).

In case of multiple roots, this does not give the complete set of solutions.

If λ is a double root, then $(\lambda^n)_{n=0}^{\infty}$ and $(n\lambda^n)_{n=0}^{\infty}$ both are solutions. Similarly, λ is a root of multiplicity $M > 2$, then $(n^m\lambda^n)_{n=0}^{\infty},\ m = 0,1, \ldots, M-1$, also satisfy the difference equation.

Thus, for $\rho(t) = \prod_{j=1}^{J}\left(t - \lambda_j\right)^{M_j}$ where $\sum_{j=1}^{J} M_j = k+1$, the general solution is $y_n = \sum_{j=1}^{J} \sum_{m=0}^{M_j - 1} c_{jm} n^m \lambda_j^n$.

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$$y_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

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Returning to the homogeneous linear difference equation

$$y_{n+1} + \sum_{j=0}^{k} a_j y_{n-j} = 0, \qquad n = k, k+1, \ldots$$

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is bounded provided …

Initial Value Problems: Linear Multistep Methods

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Definition

A linear multistep method satisfies the root condition if

(1) all roots of the first characteristic polynomial has modulus less that or equal to 1, and

(2) all roots of modulus 1 are simple.

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Note that

$$\begin{bmatrix} y_{n-k+1} \\ y_{n-k+2} \\ \vdots \\ y_n \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_k & -a_{k-1} & -a_{k-2} & \cdots & -a_0 \end{bmatrix} \begin{bmatrix} y_{n-k} \\ y_{n-k+1} \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ h(b_{-1} f_{n+1} + b_0 f_n + \cdots + b_k f_{n-k}) \end{bmatrix}$$

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For solutions y_j and $\hat{y}_j$, let

$$E_n = \begin{bmatrix} y_{n-k} - \hat{y}_{n-k} \\ y_{n-k+1} - \hat{y}_{n-k+1} \\ \vdots \\ y_{n-1} - \hat{y}_{n-1} \\ y_n - \hat{y}_n \end{bmatrix}$$

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to get

$$E_{n+1} = AE_n + Q_n$$

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Note that $|\lambda I - A| = \rho(\lambda)$. (Why?)

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Thus, we have

$$E_{k+n} = A^n E_k + \sum_{j=0}^{n-1} A^{n-1-j} Q_{k+j}.$$

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Thus, using ℓ_∞ norm for vectors and the fact that there is a constant C so that $\|A^m\| \le C$, for all m, we have

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$$Q_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ h\big(b_{-1}(f(t_{n+1}, y_{n+1}) - f(t_{n+1}, \hat{y}_{n+1})) + \cdots + b_k(f(t_{n-k}, y_{n-k}) - f(t_{n-k}, \hat{y}_{n-k}))\big) \end{bmatrix}$$

to get

$$(1 - hC\|b\|_1 L)\|E_{k+n}\| \le C\|E_k\| + 2hC\|b\|_1 L \sum_{j=0}^{n-1} \|E_{k+j}\| .$$

Proof.

For solutions y_j and $\hat{y}_j$, let

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Thus, for $h \le (2C\|b\|_1 L)^{-1}$, we have

$$\|E_{k+n}\| \le \frac{C\|E_k\|}{(1 - hC\|b\|_1 L)} + \frac{2hC\|b\|_1 L\left(\sum_{j=0}^{n-1}\|E_{k+j}\|\right)}{(1 - hC\|b\|_1 L)}$$

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So stability follows.

Module 2

Initial Value Problems

2.4 Implicit method

2.5 Stiffness

2.6 Linear Multistep Methods

- Consistency, stability and convergence

Akash Anand
MATH, IIT KANPUR

Theorem

The linear multistep method is convergent if and only if it is consistent and stable.

Initial Value Problems: Linear Multistep Methods

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Remark

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Proof. (sketch)

Convergence ⇒ Consistency

Convergence ⇒ Stability

Consistency and Stability ⇒ Convergence

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Convergence ⇒ Consistency

Apply the method to $y' = 0, y(0) = 1$ *and* $y' = 1, y(0) = 0$ *for verifying satisfiability of the consistency conditions.*

Convergence ⇒ Stability

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Convergence ⇒ Stability

Apply the method to $y' = 0, y(0) = 0$ *for verifying satisfiability of the root condition.*

Consistency and Stability ⇒ Convergence

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Convergence $\Rightarrow$ Consistency

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Convergence $\Rightarrow$ Stability

Apply the method to $y' = 0, y(0) = 0$ for verifying satisfiability of the root condition.

Consistency and Stability $\Rightarrow$ Convergence

Recall that

$$\begin{bmatrix} y_{n-k+1} \\ y_{n-k+2} \\ \vdots \\ y_n \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_k & -a_{k-1} & -a_{k-2} & \cdots & -a_0 \end{bmatrix} \begin{bmatrix} y_{n-k} \\ y_{n-k+1} \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ h(b_{-1} f_{n+1} + b_0 f_n + \cdots + b_n f_0) \end{bmatrix}$$

We define

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_k & -a_{k-1} & -a_{k-2} & \cdots & -a_0 \end{bmatrix}, \qquad E_n = \begin{bmatrix} y_{n-k} - y(t_{n-k}) \\ y_{n-k+1} - y(t_{n-k+1}) \\ \vdots \\ y_{n-1} - y(t_{n-1}) \\ y_n - y(t_n) \end{bmatrix},$$

where $y(t)$ is the exact solution that satisfies a similar difference equation

$$\begin{bmatrix} y(t_{n-k+1}) \\ y(t_{n-k+2}) \\ \vdots \\ y(t_n) \\ y(t_{n+1}) \end{bmatrix} = A \begin{bmatrix} y(t_{n-k}) \\ y(t_{n-k+1}) \\ \vdots \\ y(t_{n-1}) \\ y(t_n) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ h(b_{-1}f(t_{n+1}, y(t_{n+1})) + b_0 f(t_n, y(t_n)) + \cdots + b_n f(t_0, y(t_0))) - \ell_{n+1}(y, h) \end{bmatrix}.$$

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Then, we have $E_{n+1} = AE_n + Q_n$ where

$$Q_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ h\big(b_{-1}(f_{n+1} - f(t_{n+1}, y(t_{n+1}))) + b_0(f_n - f(t_n, y(t_n))) + \cdots + b_n(f_0 - f(t_0, y(t_0)))\big) + \ell_{n+1}(y,h) \end{bmatrix}.$$

Initial Value Problems: Linear Multistep Methods

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So, there is a constant C such that, for $h \leq (2C\|b\|_1 L)^{-1}$, we have

$$\|E_{k+n}\| \leq C\|E_k\| + C\sum_{j=0}^{n-1}\|Q_{k+j}\| \leq 2C\|E_k\| + 4hC\|b\|_1 L\sum_{j=0}^{n-1}\|E_{k+j}\| + 2nC\max_{0\leq j<n}|\ell_{k+j+1}(y, h)|.$$

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Therefore,

$$\|E_{k+n}\| \le 2C\left(\|E_k\| + (T - t_0)\max_{0\le j<N}\left|\frac{\ell_j(y,h)}{h}\right|\right) e^{4(T-t_0)C\|b\|_1 L} \ldots$$

Remark

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Theorem

The highest order of a stable k-step method is $k + 1$ if k is odd and $k + 2$ if k is even.

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Theorem

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Remark

The Adams method are linear multistep methods with best possible stability properties, namely, the first characteristic polynomial $\rho(t) = t^{k+1} - t^k$ has all its roots at the origin except for the mandatory root at 1.