Problem Set 1

1. Determine Lipschitz constant L given by

$$L = \max_{(t,y)\in D} \left| \frac{\partial f(t,y)}{\partial y} \right|$$

for the following functions:

   (a) $f(t,y) = 2y/t,\ D = \{(t,y) : t \geq 1, y \in \mathbb{R}\}$
   (b) $f(t,y) = \tan^{-1} y,\ D = \mathbb{R}^2$
   (c) $f(t,y) = (t^3 - 2)^{27}/(17t^2 + 4),\ D = \mathbb{R}^2$
   (d) $f(t,y) = t - y^2,\ D = \{(t,y) : t \in \mathbb{R}, |y| \leq 10\}$

2. Write each of the following ODEs with given initial conditions as an equivalent first order system of ODEs:

   (a) $y'' = t + y + y',\ y(0) = 1,\ y'(0) = 1$
   (b) $y''' = ty + y'',\ y(0) = 1,\ y'(0) = 0,\ y''(0) = 1$
   (c) $y''' = y'' - 2y' + y - t + 1,\ y(0) = 1,\ y'(0) = 1,\ y''(0) = 1$
   (d) $y'' = y'(1 - y^2) - y,\ y(0) = 1,\ y'(0) = 0$
   (e) $y''' = -yy'',\ y(0) = 1,\ y'(0) = 0.25,\ y''(0) = 0.5$
   (f) $y_1'' = \alpha y_1/(y_1^2 + y_2^2)^{3/2},\quad y_2'' = \alpha y_2/(y_1^2 + y_2^2)^{3/2},\ y_1(0) = 0.4,\ y_1'(0) = 0,\ y_2(0) = 0,\ y_2'(0) = 2$

3. Let $u(t)$ be the solution, if it exists, to the initial value problem

$$y' = f(t,y), \quad y(t_0) = y_0. \tag{1}$$

By integrating, show that u satisfies

$$u(t) = y_0 + \int_{t_0}^{t} f(s, u(s))\, ds.$$

Conversely, show that if this equation has a continuous solution on the $t_0 \leq t \leq T$, then the initial value problem (1) has the same solution.

4. Show that Euler's method fails to approximate the solution $y(t) = (2t/3)^{3/2}, t \geq 0$, of the problem $y' = y^{1/3}, y(0) = 0$. Explain why.

5. Let $A, B, \eta_0, \ldots, \eta_N$ be non-negative numbers satisfying

$$\eta_{n+1} \leq A\eta_n + B, \quad n = 0, \ldots, N-1.$$

Then, show that

$$\eta_n \leq A^n \eta_0 + \left(\sum_{i=0}^{n-1} A^i \right) B, \quad n = 1, \ldots, N.$$

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6. Let $A_n > 1$ and $B_n \geq 0$ for $n = 0, 1, \ldots, N-1$ and let $\eta_0, \ldots, \eta_N \geq 0$. Suppose that

$$\eta_{n+1} \leq A_n \eta_n + B_n, \quad n = 0, \ldots, N-1.$$

Then, show that

$$\eta_n \leq \left(\prod_{i=0}^{n-1} A_i \right) \eta_0 + \left(\prod_{i=0}^{n-1} A_i - 1 \right) \sup_{0 \leq i \leq n-1} \frac{B_i}{A_i - 1}, \quad n = 1, \ldots, N.$$

7. Consider the initial value problem

$$\begin{aligned} y' &= f(t, y), \quad (t, y) \in [t_0, t^*] \times [a, b] \\ y(t_0) &= y_0, \end{aligned}$$

with a continuous function f that is Lipschitz continuous in y with Lipschitz constant L. Show that, for every $\epsilon > 0$, there exists $\tilde{h}$ such that for any choice of steps $\{h_n = t_{n+1} - t_n\}_{n=0}^{N-1}$ with $t_N = t^*$ satisfying $\max_{0 \leq n \leq N-1} h_n \leq \tilde{h}$, we have that error $e_n = y_n - y(t_n)$ at $t = t_n$ in the Euler's method $y_{n+1} = y_n + h_n f(t_n, y_n), n \geq 1$, satisfies $\|e_n\| \leq \epsilon$ for all $n = 0, \ldots, N$.

Moreover, if the solution $y \in C^2[t_0, t^*]$, $\max_{0 \leq n \leq N} \|e_n\| \leq C\tilde{h}$ where

$$C = \|y''\|_\infty \frac{e^{L|t^* - t_0|} - 1}{2L}.$$

do later

8. Repeat the previous problem for backward Euler's method.

9. Give the general procedure for solving the linear difference equation

$$y_{n+1} = a_0 y_n + a_1 y_{n-1}.$$

Apply this to find the general solution of the following equations.

(a) $y_{n+1} = -\frac{1}{2} y_n + \frac{1}{2} y_{n-1}$

(b) $y_{n+1} = y_n - \frac{1}{4} y_{n-1}$

10. Consider the numerical method

$$y_{n+1} = 4y_n - 3y_{n-1} - 2hf(t_{n-1}, y_{n-1}), \quad n \geq 1.$$

Determine its order.

11. The centered difference approximation

$$y'(t_n) \approx \frac{y_{n+1} - y_{n-1}}{2h}$$

yields the two-step *leap-frog* method

$$y_{n+1} = y_{n-1} + 2hf(t_n, y_n),$$

for solving the ODE $y' = f(t, y)$. Determine the order of accuracy of this method.

12. We have seen different notions of *stability*. The one where a multistep method is stable while solving the differential equation $y' = 0$ (that is, $f(t, y) = 0$) is also known as *zero stability*. In contrast, the study of stability of a method while solving the differential equation $y' = \lambda y$ (that is, $f(t, y) = \lambda y$) is known as *absolute stability*. We have seen that a linear multistep method

$$y_{n+1} = -\sum_{j=0}^{k} a_j y_{n-j} + h \sum_{j=-1}^{k} b_j f_{n-j} \tag{2}$$

is zero-stable if the roots of the first characteristic polynomial $\rho(z)$ as magnitude less that of equal to 1 and the roots that have magnitude equal to one are simple. Applying the (2) to $y' = \lambda y$ yields the difference equation

$$y_{n+1} + \sum_{j=0}^{k} a_j y_{n-j} - h\lambda \sum_{j=-1}^{k} b_j y_{n-j} = 0.$$

Study the absolute stability of the linear multistep in (2) in terms of the roots of the polynomial

$$\pi(z; w) = \rho(z) - w\sigma(z),$$

known as the *stability polynomial* where $\rho(z)$ is the first characteristic polynomial, and

$$\sigma(z) = \sum_{j=-1}^{k} b_j z^{n-j}.$$

13. Show that the two-step method

$$y_{n+1} = -y_n + 2y_{n-1} + h\left(\frac{5}{2}f(t_n, y_n) + \frac{1}{2}f(t_{n-1}, y_{n-1})\right), \quad n \geq 1,$$

is of order 2 and unstable. Also, show directly that it need not converge when solving $y' = f(t, y)$.

14. Find all explicit fourth-order formulas of the form

$$y_{n+1} = a_0 y_n + a_1 y_{n-1} + a_2 y_{n-2} + h\left(b_0 f_n + b_1 f_{n-1} + b_2 f_{n-2}\right), \quad n \geq 2.$$

Show that every such method is unstable.

15. Consider the a linear $(k+1)$ step method of the form

$$y_{n+1} = y_n + h \sum_{j=-1}^{k} b_j f_{n-j}.$$

Derive the coefficients $b_j, j = -1, 0, \ldots .k$ for Adams-Bashford and Adams-Moulton methods of $k = 0, 1, 2$ and 3. Show that the order of a $(k+1)$ step Adams-Bashford method is $k+1$ whereas it is $k+2$ for the corresponding Adams-Moulton scheme.

16. For solving $y' = f(x, y)$, consider the numerical method

$$y_{n+1} = y_n + \frac{h}{2}\left(y_n' + y_{n+1}'\right) + \frac{h^2}{12}\left(y_n'' - y_{n+1}''\right), \quad n \geq 0.$$

Here $y_n' = f(t_, y_n)$ and

$$y_n'' = \frac{\partial f(t_n, y_n)}{\partial t} + f(t_n, y_n))\frac{\partial f(t_n, y_n)}{\partial y}.$$

(a) Show that this is a fourth-order method.
(b) Show that the region of absolute stability contains the entire negative real axis of the complex $h\lambda$-plane.

17. Consider the three stage Runge-Kutta formula

$$\begin{aligned} y_{n+1} &= y_n + h(b_1 k_1 + b_2 k_2 + b_3 k_3), \\ k_1 &= f(t_n, y_n), k_2 = f(t_n + c_2 h, y_n + h a_{21} k_1) \\ k_3 &= f(t_n + c_3 h, y_n + h(a_{31} k_1 + a_{32} k_2)) \end{aligned}$$

Determine the set of equations that the coefficients $\{b_j, c_j, a_{ij}\}$ must satisfy if the formula is to be of order 3. Find a particular solution of these equations.

18. Discuss the uniqueness and existence of the solution to the two-point BVP

$$\begin{aligned} u'' &= -\lambda u, \quad 0 < t < 1, \\ u(0) &= u(1) = 0. \end{aligned}$$

19. Discuss the uniqueness and existence of the solution to the two-point BVP

$$\begin{aligned} u'' &= -\lambda u + g(t), \quad 0 < t < 1, \\ u(0) &= u(1) = 0, \end{aligned}$$

where $g(t)$ is continuous for $0 \leq t \leq 1$.

20. (a) Consider the two-point boundary value problem

$$\begin{aligned} u'' &= f(t, u, u'), \quad a < t < b, \\ u(a) &= \alpha, \quad u(b) = \beta. \end{aligned}$$

To convert this to an equivalent problem with zero boundary conditions, write $u(t) = v(t) + w(t)$, with $w(t)$ a straight line satisfying the following boundary conditions: $w(a) = \alpha, w(b) = \beta$. Derive a new boundary value problem for $v(t)$.

(b) Generalize this procedure to problem

$$\begin{aligned} u'' &= f(t, u, u'), \quad a < t < b, \\ a_0 u(a) - a_1 u'(a) = \gamma_1, &\quad b_0 u(b) + b_1 u'(b) = \gamma_2. \end{aligned}$$

Obtain a new problem with zero boundary conditions: What assumptions, if any, are needed for the coefficients a_0, a_1, b_0, b_1?