Lesson 3

Boundary Value Problems for ODEs

3.1 Well-posedness

3.2 Shooting Method

Akash Anand
MATH, IIT KANPUR

Boundary Value Problems: Shooting Method

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Recall the following discussion in the contest of solvability of two-point BVP

$$y' = f(t, y), \qquad a < t < b,$$

with boundary conditions

$$g(y(a), y(b)) = 0.$$

We noted that if $y(t; x)$ denotes the solution to the IVP $y' = f(t, y),\ y(a) = x,\ x \in \mathbb{R}^n$, then this solution is a solution to the BVP if

$$g(x, y(b; x)) = 0.$$

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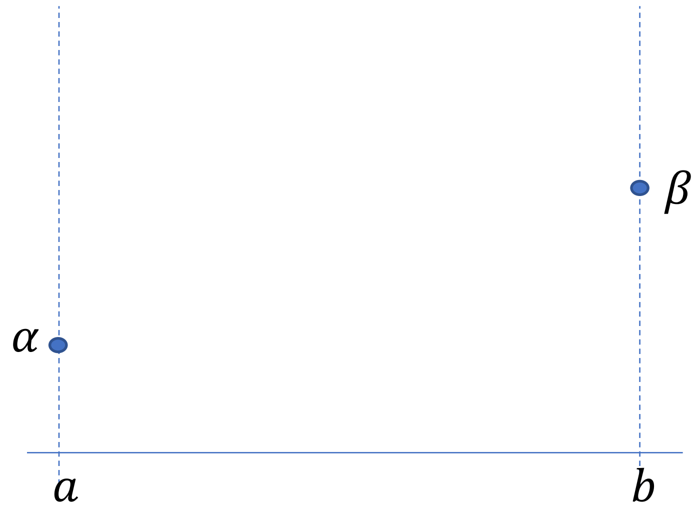

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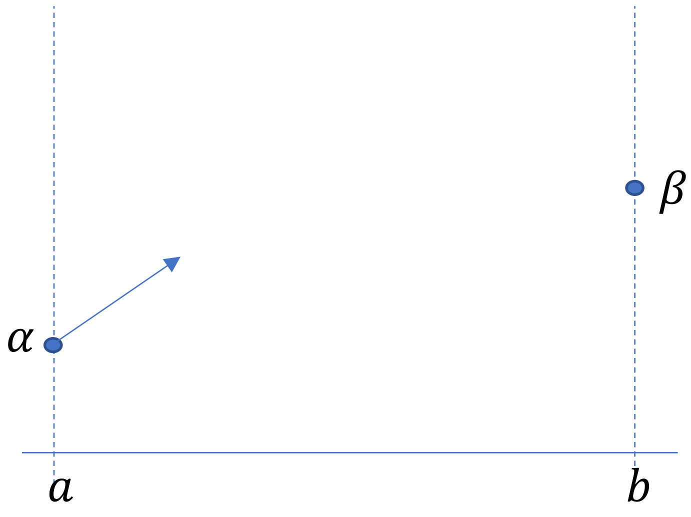

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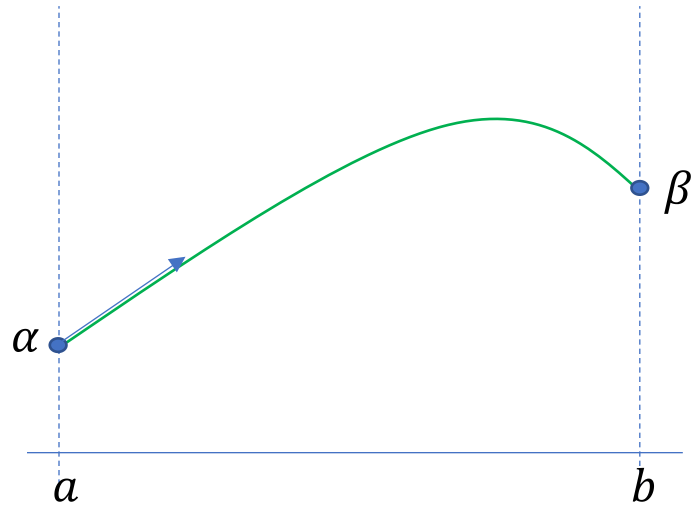

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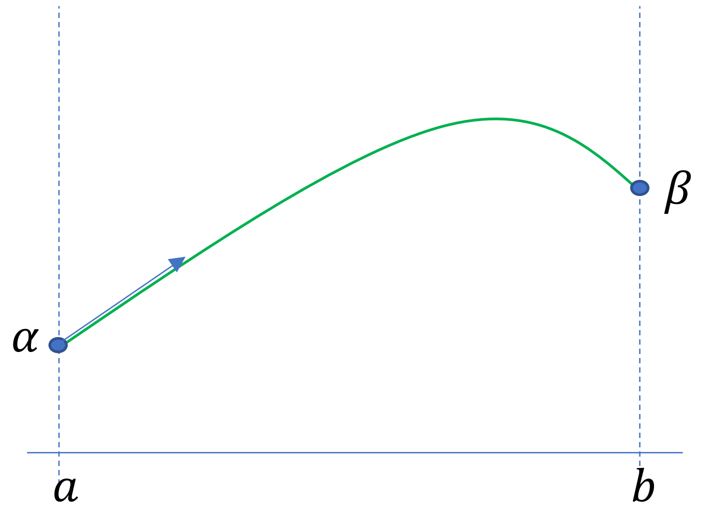

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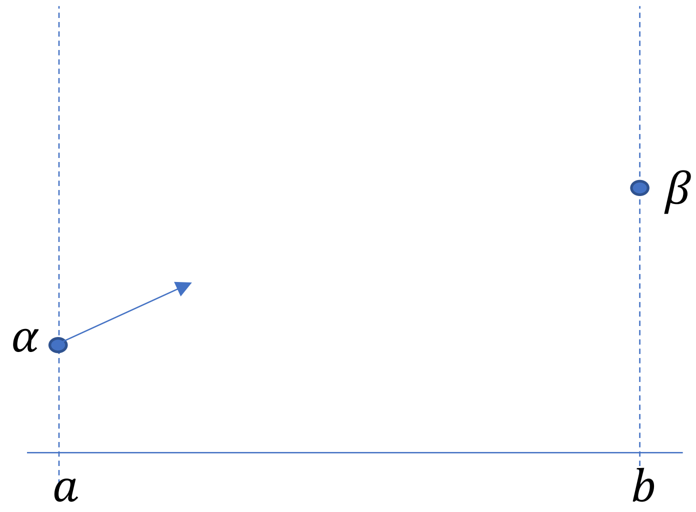

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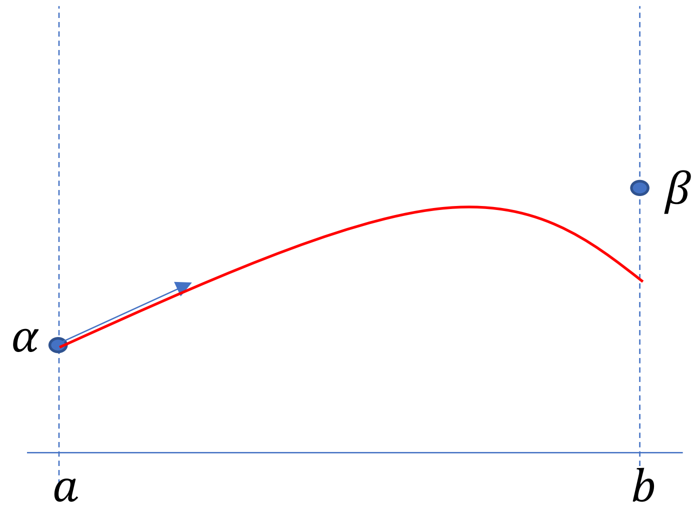

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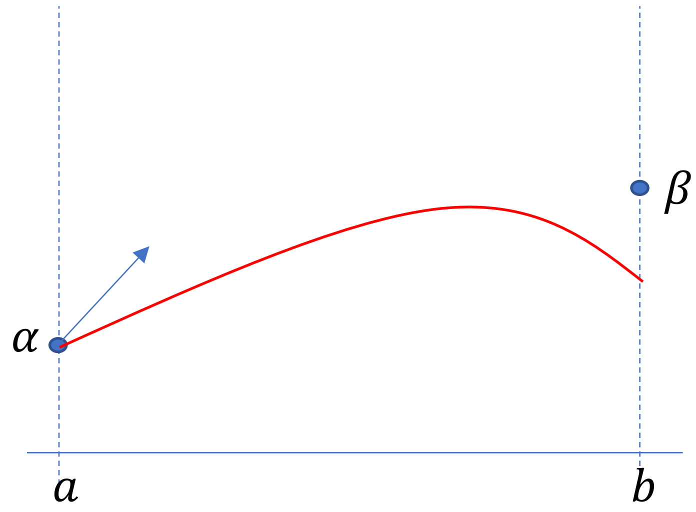

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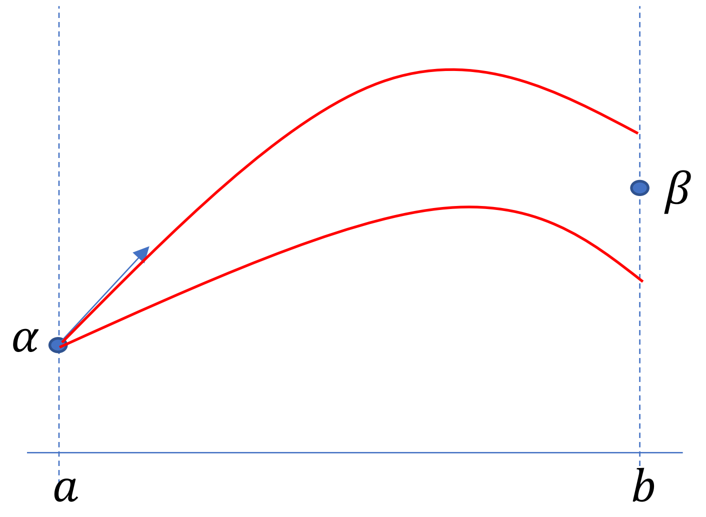

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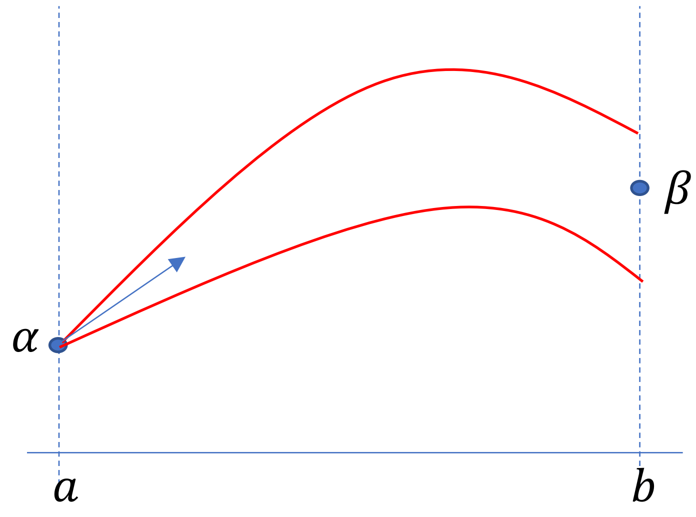

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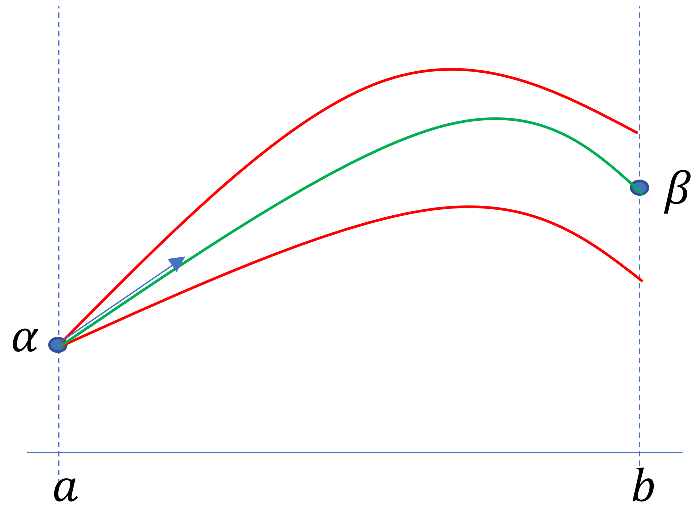

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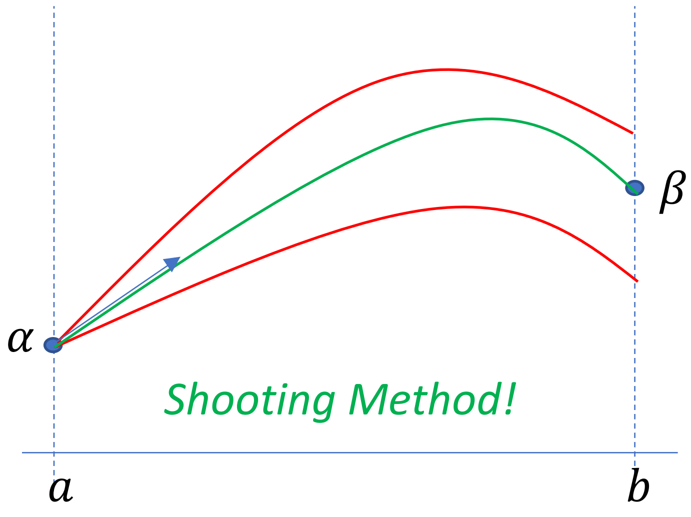

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where the function f is assumed to satisfy the following Lipschitz conditions:

$$|f(t,u_1,v) - f(t,u_2,v)| \le K|u_1 - u_2|,$$
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for all points $(t,u_i,v), \left(t,u,v_j\right) \in R \coloneqq [a,b]\times\mathbb{R}\times\mathbb{R}$. In addition, assume that on R, f satisfies

$$f_u(t,u,v) = \frac{\partial f(t,u,v)}{\partial u} > 0, |f_v(t,u,v)| = \left|\frac{\partial f(t,u,v)}{\partial v}\right| \le M,$$

for some $M > 0$. For the boundary conditions, assume

$$a_0 a_1 \ge 0, \qquad b_0 b_1 \ge 0, \qquad |a_0| + |a_1| \ne 0, \qquad |b_0| + |b_1| \ne 0, \qquad |a_0| + |b_0| \ne 0.$$

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How do we solve the BVP using the shooting method?

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Let $y(t;s)$ be the solution to the IVP

$$y'' = f(t,y,y'), \qquad a < t < b,$$
$$y(a) = a_1 s - c_1 \alpha, \qquad y'(a) = a_0 s - c_0 \alpha$$

depending on the parameter s, where c_0, c_1 are arbitrary constants satisfying $a_1 c_0 - a_0 c_1 = 1$. Note that

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$$a_0 y(a;s) - a_1 y'(a;s) = \alpha.$$

For $y(t;s)$ *to be the solution to the BVP, it must satisfy*

$$h(s) \coloneqq b_0 y(b;s) + b_1 y'(b;s) - \beta = 0.$$

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depending on the parameter s, where c_0, c_1 are arbitrary constants satisfying $a_1 c_0 - a_0 c_1 = 1$. Note that

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For $y(t;s)$ to be the solution to the BVP, it must satisfy

$$h(s) \coloneqq b_0 y(b;s) + b_1 y'(b;s) - \beta = 0.$$

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Boundary Value Problems: Shooting Method

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$$z_s''(t) = f_2\big(t, y(t;s), y'(t;s)\big) z_s(t) + f_3\big(t, y(t;s), y'(t;s)\big) z_s'(t), \qquad z_s(a) = a_1, \qquad z_s'(a) = a_0.$$

The functions f_2 and f_3 denote the partial derivatives of $f(t,u,v)$ with respect to u and v respectively.

Example

Consider the two-point BVP

$$u'' = -u + \frac{2(u')^2}{u}, \qquad -1 < t < 1,$$
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… second-order Runge-Kutta method with stepsize $h = 2/n$ *is used, then*

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$n = 2/h$	$s_* - s_*^h$	Ratio	$E^h = \max_{0 \le i \le n} \lvert u(t_i) - y^h(t_i; s_*^h) \rvert$	Ratio
4	4.01×10^{-3}	–	2.83×10^{-2}	–
8	1.52×10^{-3}	2.64	7.30×10^{-3}	3.88
16	4.64×10^{-4}	3.28	1.82×10^{-3}	4.01
32	1.27×10^{-4}	3.64	4.54×10^{-4}	4.01
64	3.34×10^{-5}	3.82	1.14×10^{-4}	4.00

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$$u'' = p(t)u' + q(t)u + r(t), \qquad u(a) = -\alpha c_1, \qquad u'(a) = -\alpha c_0,$$

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that is (why?)

$$s = \frac{\beta - \big(b_0 u_1(b) + b_1 u_1'(b)\big)}{\big(b_0 u_2(b) + b_1 u_2'(b)\big)}.$$

Example

For approximate solution of the BVP, we form

$$u^h(t_j) = u_1^h(t_j) + s^h\, u_2^h(t_j), \qquad t_j = a + jh, \qquad j = 0,1,\dots,J, \qquad h = (b-a)/J.$$

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However (exercise),

$$s^h - s = -\frac{\left(b_0 e_1(t_J) + b_1 \varepsilon_1(t_J)\right) + s\left(b_0 e_2(t_J) + b_1 \varepsilon_2(t_J)\right)}{\left(b_0 u_2^h(t_J) + b_1 v_2^h(t_J)\right)}.$$

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Let the numerical errors be

$$e_i(t_j) = u_i^h(t_j) - u_i(t_j), \qquad \varepsilon_i(t_j) = v_i^h(t_j) - u_i'(t_j), \qquad i = 1,2.$$

Then

$$e(t_j) = u^h(t_j) - u(t_j) = e_1(t_j) + s^h\, u_2^h(t_j) - s u_2(t_j) = e_1(t_j) + s e_2(t_j) + (s^h - s) u_2^h(t_j).$$

However (exercise),

$$s^h - s = -\frac{\left(b_0 e_1(t_J) + b_1 \varepsilon_1(t_J)\right) + s\left(b_0 e_2(t_J) + b_1 \varepsilon_2(t_J)\right)}{\left(b_0 u_2^h(t_J) + b_1 v_2^h(t_J)\right)}.$$

If $|e_i(t_j)| = O(h^p)$ and $|\varepsilon_i(t_j)| = O(h^p)$, then (why?)

$$|e(t_j)| = O(h^p).$$

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