Module 2

Initial Value Problems

2.5 Stiffness

2.6 Linear Multistep Method

2.7 Non-Linear Methods

Akash Anand
MATH, IIT KANPUR

Initial Value Problems: Non-Linear Methods

Akash Anand
MATH, IIT KANPUR

To implement any implicit method, we need a way to solve, at least approximately, the non-linear algebraic equation that arise at each step.

One common strategy is to solve the equation approximately using a small number of fixed point iterations starting from an initial approximation obtained by an explicit method.

Module 2

Initial Value Problems

2.5 Stiffness

2.6 Linear Multistep Method

2.7 Non-Linear Methods

- Predictor-Corrector Schemes

Akash Anand
MATH, IIT KANPUR

Initial Value Problems: Non-Linear Methods

Akash Anand
MATH, IIT KANPUR

To implement any implicit method, we need a way to solve, at least approximately, the non-linear algebraic equation that arise at each step.

One common strategy is to solve the equation approximately using a small number of fixed point iterations starting from an initial approximation obtained by an explicit method.

A predictor-corrector scheme takes the following form:

1. $p_{n+1} = E(y_n, y_{n-1}, \dots, f_n, f_{n-1}, \dots)$ *(PREDICT)*
2. $f^p_{n+1} = f(t_{n+1}, p_{n+1})$ *(EVALUATE)*
3. $y^{(1)}_{n+1} = I(y_n, y_{n-1}, \dots, f^p_{n+1}, f_n, f_{n-1}, \dots)$ *(CORRECT)*
4. $f^{(1)}_{n+1} = f(t_{n+1}, y^{(1)}_{n+1})$ *(EVALUATE)*
5. $y^{(2)}_{n+1} = I(y_n, y_{n-1}, \dots, f^{(1)}_{n+1}, f_n, f_{n-1}, \dots)$ *(CORRECT)*
6. $f^{(2)}_{n+1} = f(t_{n+1}, y^{(2)}_{n+1})$ *(EVALUATE)*

$$\vdots$$

where $E(y_n, y_{n-1}, \dots, f_n, f_{n-1}, \dots)$ *refers to some explicit scheme and* $I(y_n, y_{n-1}, \dots, f_{n+1}, f_n, f_{n-1}, \dots)$ *stands for an implicit method.*

Initial Value Problems: Non-Linear Methods

Akash Anand
MATH, IIT KANPUR

To implement any implicit method, we need a way to solve, at least approximately, the non-linear algebraic equation that arise at each step.

One common strategy is to solve the equation approximately using a small number of fixed point iterations starting from an initial approximation obtained by an explicit method.

A predictor-corrector scheme takes the following form:

1. $p_{n+1} = E(y_n, y_{n-1}, \dots, f_n, f_{n-1}, \dots)$ *(PREDICT)*
2. $f_{n+1}^{p} = f(t_{n+1}, p_{n+1})$ *(EVALUATE)*
3. $y_{n+1}^{(1)} = I(y_n, y_{n-1}, \dots, f_{n+1}^{p}, f_n, f_{n-1}, \dots)$ *(CORRECT)*
4. $f_{n+1}^{(1)} = f(t_{n+1}, y_{n+1}^{(1)})$ *(EVALUATE)*
5. $y_{n+1}^{(2)} = I(y_n, y_{n-1}, \dots, f_{n+1}^{(1)}, f_n, f_{n-1}, \dots)$ *(CORRECT)*
6. $f_{n+1}^{(2)} = f(t_{n+1}, y_{n+1}^{(2)})$ *(EVALUATE)*

$$\vdots$$

where $E(y_n, y_{n-1}, \dots, f_n, f_{n-1}, \dots)$ refers to some explicit scheme and $I(y_n, y_{n-1}, \dots, f_{n+1}, f_n, f_{n-1}, \dots)$ stands for an implicit method.

It is common to use a fixed number of iterations, but other suitable stopping criterion can also be adopted to stop.

Initial Value Problems: Non-Linear Methods

Akash Anand
MATH, IIT KANPUR

Example

Consider the following predictor-corrector scheme where Euler's method is used as predictor and Trapezoidal method as corrector:

1. $p_{n+1} = y_n + h_k f(t_n, y_n)$ *(PREDICT)*
2. $f^p_{n+1} = f(t_{n+1}, p_{n+1})$ *(EVALUATE)*
3. $y_{n+1} = y_n + h_k\big(f(t_n, y_n) + f^p_{n+1}\big)/2$ *(CORRECT)*

Example

Consider the following predictor-corrector scheme where Euler's method is used as predictor and Trapezoidal method as corrector:

1. $p_{n+1} = y_n + h_k f(t_n, y_n)$ *(PREDICT)*
2. $f_{n+1}^{p} = f(t_{n+1}, p_{n+1})$ *(EVALUATE)*
3. $y_{n+1} = y_n + h_k\big(f(t_n, y_n) + f_{n+1}^{p}\big)/2$ *(CORRECT)*

This can be expressed more concisely as

$$y_{n+1} = y_n + h_k(f(t_n, y_n) + f(t_{n+1}, y_n + h_k f(t_n, y_n)))/2$$

and is commonly known as Heun's method, a non-linear one step method.

Initial Value Problems: Non-Linear Methods

Akash Anand
MATH, IIT KANPUR

Example

Consider the following predictor-corrector scheme where Euler's method is used as predictor and Trapezoidal method as corrector:

1. $p_{n+1} = y_n + h_k f(t_n, y_n)$ *(PREDICT)*
2. $f_{n+1}^p = f(t_{n+1}, p_{n+1})$ *(EVALUATE)*
3. $y_{n+1} = y_n + h_k\big(f(t_n, y_n) + f_{n+1}^p\big)/2$ *(CORRECT)*

This can be expressed more concisely as

$$y_{n+1} = y_n + h_k(f(t_n, y_n) + f(t_{n+1}, y_n + h_k f(t_n, y_n)))/2$$

and is commonly known as Heun's method, a non-linear one step method.

Example

Consider the following PECE scheme with 2-step Adam-Bashford predictor and 2-step Adam-Moulton corrector.

$$p_{n+1} = y_n + h[3f(t_n, y_n) - f(t_{n-1}, y_{n-1})]/2$$
$$y_{n+1} = y_n + h[5f(t_{n+1}, p_{n+1}) + 8f(t_n, y_n) - f(t_{n-1}, y_{n-1})]/12.$$

Example

Consider the following predictor-corrector scheme where Euler's method is used as predictor and Trapezoidal method as corrector:

1. $p_{n+1} = y_n + h_k f(t_n, y_n)$ *(PREDICT)*
2. $f_{n+1}^p = f(t_{n+1}, p_{n+1})$ *(EVALUATE)*
3. $y_{n+1} = y_n + h_k\big(f(t_n, y_n) + f_{n+1}^p\big)/2$ *(CORRECT)*

This can be expressed more concisely as

$$y_{n+1} = y_n + h_k(f(t_n, y_n) + f(t_{n+1}, y_n + h_k f(t_n, y_n)))/2$$

and is commonly known as Heun's method, a non-linear one step method.

Example

Consider the following PECE scheme with 2-step Adam-Bashford predictor and 2-step Adam-Moulton corrector.

$$p_{n+1} = y_n + h[3f(t_n, y_n) - f(t_{n-1}, y_{n-1})]/2$$
$$y_{n+1} = y_n + h[5f(t_{n+1}, p_{n+1}) + 8f(t_n, y_n) - f(t_{n-1}, y_{n-1})]/12.$$

Thus, the resulting method

$$y_{n+1} = y_n + h[5f(t_{n+1}, y_n + h[3f(t_n, y_n) - f(t_{n-1}, y_{n-1})]/2\,) + 8f(t_n, y_n) - f(t_{n-1}, y_{n-1})]/12$$

is a nonlinear 2-step method.

Error Analysis

There are two contributions to the local truncation error, one arising from predictor formula and one arising from corrector formula. Let the predictor be of order p and the corrector be of order q.

Akash Anand
MATH, IIT KANPUR

Error Analysis

There are two contributions to the local truncation error, one arising from predictor formula and one arising from corrector formula. Let the predictor be of order p and the corrector be of order q. Then

$$
\begin{aligned}
&\left|\ell_{n+1}^{PECE}(y,h)\right| \\
&= \left| h b_{-1}^{I} f\left(t_{n+1}, h\sum_{j=-1}^{k} b_j^E y'(t_n - jh) - \sum_{j=0}^{k} a_j^E y(t_n - jh)\right) + h\sum_{j=0}^{k} b_j^I y'(t_n - jh) - \sum_{j=-1}^{k} a_j^I y(t_n - jh)\right|
\end{aligned}
$$

Error Analysis

There are two contributions to the local truncation error, one arising from predictor formula and one arising from corrector formula. Let the predictor be of order p and the corrector be of order q. Then

$$
\begin{aligned}
&\left|\ell_{n+1}^{PECE}(y,h)\right| \\
&= \left| hb_{-1}^{I} f\left(t_{n+1}, h\sum_{j=-1}^{k} b_j^E y'(t_n - jh) - \sum_{j=0}^{k} a_j^E y(t_n - jh) \right) + h\sum_{j=0}^{k} b_j^I y'(t_n - jh) - \sum_{j=-1}^{k} a_j^I y(t_n - jh) \right| \\
&= \left| hb_{-1}^{I} f\left(t_{n+1}, h\sum_{j=-1}^{k} b_j^E y'(t_n - jh) - \sum_{j=0}^{k} a_j^E y(t_n - jh) \right) - hb_{-1}^{I} f\big(t_{n+1}, y(t_n + h)\big) + \ell_{n+1}^{I}(y,h) \right|
\end{aligned}
$$

Error Analysis

There are two contributions to the local truncation error, one arising from predictor formula and one arising from corrector formula. Let the predictor be of order p and the corrector be of order q. Then

$$\begin{aligned}
&\left|\ell_{n+1}^{PECE}(y,h)\right| \\
&= \left|hb_{-1}^{I} f\left(t_{n+1}, h\sum_{j=-1}^{k} b_j^E y'(t_n - jh) - \sum_{j=0}^{k} a_j^E y(t_n - jh)\right) + h\sum_{j=0}^{k} b_j^I y'(t_n - jh) - \sum_{j=-1}^{k} a_j^I y(t_n - jh)\right| \\
&= \left|hb_{-1}^{I} f\left(t_{n+1}, h\sum_{j=-1}^{k} b_j^E y'(t_n - jh) - \sum_{j=0}^{k} a_j^E y(t_n - jh)\right) - hb_{-1}^{I} f\big(t_{n+1}, y(t_n + h)\big) + \ell_{n+1}^{I}(y,h)\right| \\
&\le h\left|b_{-1}^{I}\right| L \left|\left(h\sum_{j=-1}^{k} b_j^E y'(t_n - jh) - \sum_{j=0}^{k} a_j^E y(t_n - jh) - y(t_{n+1})\right)\right| + \left|\ell_{n+1}^{I}(y,h)\right|
\end{aligned}$$

Error Analysis

There are two contributions to the local truncation error, one arising from predictor formula and one arising from corrector formula. Let the predictor be of order p and the corrector be of order q. Then

$$\begin{aligned}
&\left|\ell_{n+1}^{PECE}(y,h)\right| \\
&= \left| h b_{-1}^I f\left(t_{n+1}, h\sum_{j=-1}^{k} b_j^E y'(t_n - jh) - \sum_{j=0}^{k} a_j^E y(t_n - jh)\right) + h\sum_{j=0}^{k} b_j^I y'(t_n - jh) - \sum_{j=-1}^{k} a_j^I y(t_n - jh)\right| \\
&= \left| h b_{-1}^I f\left(t_{n+1}, h\sum_{j=-1}^{k} b_j^E y'(t_n - jh) - \sum_{j=0}^{k} a_j^E y(t_n - jh)\right) - h b_{-1}^I f\big(t_{n+1}, y(t_n + h)\big) + \ell_{n+1}^I(y,h)\right| \\
&\le h\left|b_{-1}^I\right| L \left|\left(h\sum_{j=-1}^{k} b_j^E y'(t_n - jh) - \sum_{j=0}^{k} a_j^E y(t_n - jh) - y(t_{n+1})\right)\right| + \left|\ell_{n+1}^I(y,h)\right| \\
&= h\left|b_{-1}^I\right| L \left|\ell_{n+1}^E(y,h)\right| + \left|\ell_{n+1}^I(y,h)\right|
\end{aligned}$$

Error Analysis

There are two contributions to the local truncation error, one arising from predictor formula and one arising from corrector formula. Let the predictor be of order p and the corrector be of order q. Then

$$
\begin{aligned}
&\left|\ell_{n+1}^{PECE}(y,h)\right| \\
&= \left| h b_{-1}^{I} f\left(t_{n+1}, h \sum_{j=-1}^{k} b_j^E y'(t_n - jh) - \sum_{j=0}^{k} a_j^E y(t_n - jh) \right) + h \sum_{j=0}^{k} b_j^I y'(t_n - jh) - \sum_{j=-1}^{k} a_j^I y(t_n - jh) \right| \\
&= \left| h b_{-1}^{I} f\left(t_{n+1}, h \sum_{j=-1}^{k} b_j^E y'(t_n - jh) - \sum_{j=0}^{k} a_j^E y(t_n - jh) \right) - h b_{-1}^{I} f\big(t_{n+1}, y(t_n + h)\big) + \ell_{n+1}^{I}(y,h) \right| \\
&\le h \left| b_{-1}^{I} \right| L \left| \left(h \sum_{j=-1}^{k} b_j^E y'(t_n - jh) - \sum_{j=0}^{k} a_j^E y(t_n - jh) - y(t_{n+1}) \right) \right| + \left| \ell_{n+1}^{I}(y,h) \right| \\
&= h \left| b_{-1}^{I} \right| L \left| \ell_{n+1}^{E}(y,h) \right| + \left| \ell_{n+1}^{I}(y,h) \right| \le C_1 h \left| b_{-1}^{I} \right| L h^{p+1} + C_2 h^{q+1}
\end{aligned}
$$

Error Analysis

There are two contributions to the local truncation error, one arising from predictor formula and one arising from corrector formula. Let the predictor be of order p *and the corrector be of order* q. *Then*

$$
\begin{aligned}
&\left|\ell_{n+1}^{PECE}(y,h)\right| \\
&= \left| h b_{-1}^{I} f\left(t_{n+1}, h \sum_{j=-1}^{k} b_j^{E} y'(t_n - jh) - \sum_{j=0}^{k} a_j^{E} y(t_n - jh) \right) + h \sum_{j=0}^{k} b_j^{I} y'(t_n - jh) - \sum_{j=-1}^{k} a_j^{I} y(t_n - jh) \right| \\
&= \left| h b_{-1}^{I} f\left(t_{n+1}, h \sum_{j=-1}^{k} b_j^{E} y'(t_n - jh) - \sum_{j=0}^{k} a_j^{E} y(t_n - jh) \right) - h b_{-1}^{I} f\big(t_{n+1}, y(t_n + h)\big) + \ell_{n+1}^{I}(y,h) \right| \\
&\le h \left| b_{-1}^{I} \right| L \left| \left(h \sum_{j=-1}^{k} b_j^{E} y'(t_n - jh) - \sum_{j=0}^{k} a_j^{E} y(t_n - jh) - y(t_{n+1}) \right) \right| + \left| \ell_{n+1}^{I}(y,h) \right| \\
&= h \left| b_{-1}^{I} \right| L \left| \ell_{n+1}^{E}(y,h) \right| + \left| \ell_{n+1}^{I}(y,h) \right| \le C_1 h \left| b_{-1}^{I} \right| L h^{p+1} + C_2 h^{q+1} \le C h^{\min\{p+2, q+1\}}
\end{aligned}
$$

where $C = \max\left\{ \left| b_{-1}^{I} \right| C_1, C_2 \right\}$.

Error Analysis

There are two contributions to the local truncation error, one arising from predictor formula and one arising from corrector formula. Let the predictor be of order p and the corrector be of order q. Then

$$
\begin{aligned}
&\left|\ell_{n+1}^{PECE}(y,h)\right| \\
&= \left| h b_{-1}^{I} f\left(t_{n+1}, h\sum_{j=-1}^{k} b_j^E y'(t_n - jh) - \sum_{j=0}^{k} a_j^E y(t_n - jh) \right) + h\sum_{j=0}^{k} b_j^I y'(t_n - jh) - \sum_{j=-1}^{k} a_j^I y(t_n - jh) \right| \\
&= \left| h b_{-1}^{I} f\left(t_{n+1}, h\sum_{j=-1}^{k} b_j^E y'(t_n - jh) - \sum_{j=0}^{k} a_j^E y(t_n - jh) \right) - h b_{-1}^I f\big(t_{n+1}, y(t_n + h)\big) + \ell_{n+1}^I(y,h) \right| \\
&\le h\left|b_{-1}^I\right| L \left| \left(h\sum_{j=-1}^{k} b_j^E y'(t_n - jh) - \sum_{j=0}^{k} a_j^E y(t_n - jh) - y(t_{n+1}) \right) \right| + \left|\ell_{n+1}^I(y,h)\right| \\
&= h\left|b_{-1}^I\right| L \left|\ell_{n+1}^E(y,h)\right| + \left|\ell_{n+1}^I(y,h)\right| \le C_1 h \left|b_{-1}^I\right| L h^{p+1} + C_2 h^{q+1} \le C h^{\min\{p+2,q+1\}}
\end{aligned}
$$

where $C = \max\{\left|b_{-1}^I\right| C_1, C_2\}$. *Thus, for* $p \ge q-1$, *the local error is* $O(h^{q+1})$.

Error Analysis

There are two contributions to the local truncation error, one arising from predictor formula and one arising from corrector formula. Let the predictor be of order p *and the corrector be of order* q. *Then*

$$
\begin{aligned}
&\left|\ell_{n+1}^{PECE}(y,h)\right| \\
&= \left| h b_{-1}^{I} f\left(t_{n+1}, h\sum_{j=-1}^{k} b_j^E y'(t_n - jh) - \sum_{j=0}^{k} a_j^E y(t_n - jh) \right) + h\sum_{j=0}^{k} b_j^I y'(t_n - jh) - \sum_{j=-1}^{k} a_j^I y(t_n - jh) \right| \\
&= \left| h b_{-1}^{I} f\left(t_{n+1}, h\sum_{j=-1}^{k} b_j^E y'(t_n - jh) - \sum_{j=0}^{k} a_j^E y(t_n - jh) \right) - h b_{-1}^{I} f\big(t_{n+1}, y(t_n + h)\big) + \ell_{n+1}^{I}(y,h) \right| \\
&\leq h\left|b_{-1}^{I}\right| L \left| \left(h\sum_{j=-1}^{k} b_j^E y'(t_n - jh) - \sum_{j=0}^{k} a_j^E y(t_n - jh) - y(t_{n+1}) \right) \right| + \left|\ell_{n+1}^{I}(y,h)\right| \\
&= h\left|b_{-1}^{I}\right| L \left|\ell_{n+1}^{E}(y,h)\right| + \left|\ell_{n+1}^{I}(y,h)\right| \leq C_1 h \left|b_{-1}^{I}\right| L h^{p+1} + C_2 h^{q+1} \leq C h^{\min\{p+2,q+1\}}
\end{aligned}
$$

where $C = \max\left\{\left|b_{-1}^{I}\right| C_1, C_2\right\}$. *Thus, for* $p \geq q-1$, *the local error is* $O(h^{q+1})$. *Most common choice is* $p = q - 1$.

Module 2

Initial Value Problems

2.5 Stiffness

2.6 Linear Multistep Method

2.7 Non-Linear Methods

- One step methods

Akash Anand
MATH, IIT KANPUR

Akash Anand
MATH, IIT KANPUR

One step methods

Are there ways to develop one step methods that are high order?

Initial Value Problems: Non-Linear Methods

One step methods

Are there ways to develop one step methods that are high order? Here is an idea! Consider the Taylor series

$$y(t+h) = y(t) + hy'(t) + \frac{h^2}{2}y''(t) + \frac{h^3}{6}y'''(t) + \cdots$$

Initial Value Problems: Non-Linear Methods

Akash Anand
MATH, IIT KANPUR

One step methods

Are there ways to develop one step methods that are high order? Here is an idea! Consider the Taylor series

$$y(t+h) = y(t) + hy'(t) + \frac{h^2}{2}y''(t) + \frac{h^3}{6}y'''(t) + \cdots$$

Therefore,

$$y(t+h) = y(t) + hy'(t) + \frac{h^2}{2}y''(t)$$

yields a second-order method.

One step methods

Are there ways to develop one step methods that are high order? Here is an idea! Consider the Taylor series

$$y(t+h) = y(t) + hy'(t) + \frac{h^2}{2}y''(t) + \frac{h^3}{6}y'''(t) + \cdots$$

Therefore,

$$y(t+h) = y(t) + hy'(t) + \frac{h^2}{2}y''(t)$$

yields a second-order method. The ODE gives $y'(t) = f(t, y)$ but what about $y''(t)$?

One step methods

Are there ways to develop one step methods that are high order? Here is an idea! Consider the Taylor series

$$y(t+h) = y(t) + hy'(t) + \frac{h^2}{2}y''(t) + \frac{h^3}{6}y'''(t) + \cdots$$

Therefore,

$$y(t+h) = y(t) + hy'(t) + \frac{h^2}{2}y''(t)$$

yields a second-order method. The ODE gives $y'(t) = f(t,y)$ *but what about* $y''(t)$*? Note that*

$$y''(t) = f_t(t,y) + f_y(t,y)y'(t) = f_t(t,y) + f_y(t,y)f(t,y).$$

One step methods

Are there ways to develop one step methods that are high order? Here is an idea! Consider the Taylor series

$$y(t+h) = y(t) + hy'(t) + \frac{h^2}{2}y''(t) + \frac{h^3}{6}y'''(t) + \cdots$$

Therefore,

$$y(t+h) = y(t) + hy'(t) + \frac{h^2}{2}y''(t)$$

yields a second-order method. The ODE gives $y'(t) = f(t,y)$ *but what about* $y''(t)$*? Note that*

$$y''(t) = f_t(t,y) + f_y(t,y)y'(t) = f_t(t,y) + f_y(t,y)f(t,y).$$

Thus the method reads

$$y_{n+1} = y_n + h_n f(t_n, y_n) + \frac{h_n^2}{2}\Big(f_t(t_n, y_n) + f_y(t_n, y_n)f(t_n, y_n)\Big).$$

One step methods

Are there ways to develop one step methods that are high order? Here is an idea! Consider the Taylor series

$$y(t+h) = y(t) + hy'(t) + \frac{h^2}{2}y''(t) + \frac{h^3}{6}y'''(t) + \cdots$$

Therefore,

$$y(t+h) = y(t) + hy'(t) + \frac{h^2}{2}y''(t)$$

yields a second-order method. The ODE gives $y'(t) = f(t,y)$ but what about $y''(t)$? Note that

$$y''(t) = f_t(t,y) + f_y(t,y)y'(t) = f_t(t,y) + f_y(t,y)f(t,y).$$

Thus the method reads

$$y_{n+1} = y_n + h_n f(t_n, y_n) + \frac{h_n^2}{2}\Big(f_t(t_n,y_n) + f_y(t_n,y_n)f(t_n,y_n)\Big).$$

One can use this idea to construct higher order methods. For example, we have

$$y'''(t) = f_{tt}(t,y) + 2f_{ty}(t,y)f(t,y) + f_{yy}(t,y)f^2(t,y) + f_t(t,y)f_y(t,y) + f_y^2(t,y)f(t,y),$$

therefore,

$$y_{n+1} = y_n + h_n f(t_n,y_n) + \frac{h_n^2}{2}\Big(f_t(t_n,y_n) + f_y(t_n,y_n)f(t_n,y_n)\Big) + \frac{h_n^3}{6}y'''(t_n)$$

has order 3.

More generally, by defining the total derivative of f as

$$Df := f_t + ff_y$$

and by further differentiating

$$D^2f := f_{tt} + 2ff_{ty} + f^2 f_{yy} + f_t f_y + ff_y^2,$$

$D^3f = y^{(4)}(t)$, *etc., an order p single step method takes the form*

$$y_{n+1} = y_n + h_n f + \frac{h_n^2}{2} Df_n + \cdots + \frac{h_n^p}{p!} D^{p-1} f_n.$$

Initial Value Problems: Non-Linear Methods

Akash Anand
MATH, IIT KANPUR

More generally, by defining the total derivative of f as

$$Df := f_t + f f_y$$

and by further differentiating

$$D^2 f := f_{tt} + 2 f f_{ty} + f^2 f_{yy} + f_t f_y + f f_y^2,$$

$D^3 f = y^{(4)}(t)$*, etc., an order* p *single step method takes the form*

$$y_{n+1} = y_n + h_n f + \frac{h_n^2}{2} D f_n + \cdots + \frac{h_n^p}{p!} D^{p-1} f_n.$$

Such methods are called Taylor series methods.

More generally, by defining the total derivative of f as

$$Df := f_t + ff_y$$

and by further differentiating

$$D^2 f := f_{tt} + 2ff_{ty} + f^2 f_{yy} + f_t f_y + f f_y^2,$$

$D^3 f = y^{(4)}(t)$, *etc., an order p single step method takes the form*

$$y_{n+1} = y_n + h_n f + \frac{h_n^2}{2} Df_n + \cdots + \frac{h_n^p}{p!} D^{p-1} f_n.$$

Such methods are called Taylor series methods.

Remark

These methods can be implemented in some cases but require evaluation of partial derivatives of f. They are, therefore, not commonly used.

Initial Value Problems: Non-Linear Methods

Akash Anand
MATH, IIT KANPUR

More generally, by defining the total derivative of f as

$$Df := f_t + ff_y$$

and by further differentiating

$$D^2f := f_{tt} + 2ff_{ty} + f^2f_{yy} + f_tf_y + ff_y^2,$$

$D^3f = y^{(4)}(t)$, *etc., an order p single step method takes the form*

$$y_{n+1} = y_n + h_nf + \frac{h_n^2}{2}Df_n + \cdots + \frac{h_n^p}{p!}D^{p-1}f_n.$$

Such methods are called Taylor series methods.

Remark

These methods can be implemented in some cases but require evaluation of partial derivatives of f. They are, therefore, not commonly used.

What else can we do?