Lesson 4

Numerical Solution of PDE

4.1 BVP for 2^{nd} Order Elliptic PDE

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- Finite Difference Method

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Numerical Methods for PDE: 2nd Order Elliptic PDE

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A natural generalization to the two-point BVP

$$u'' = f(t), \qquad a < t < b,$$
$$u(a) = 0, \qquad u(b) = 0,$$

to two dimensions is

$$\Delta u := u_{x_1x_1} + u_{x_2x_2} = f, \qquad \text{in } \Omega,$$
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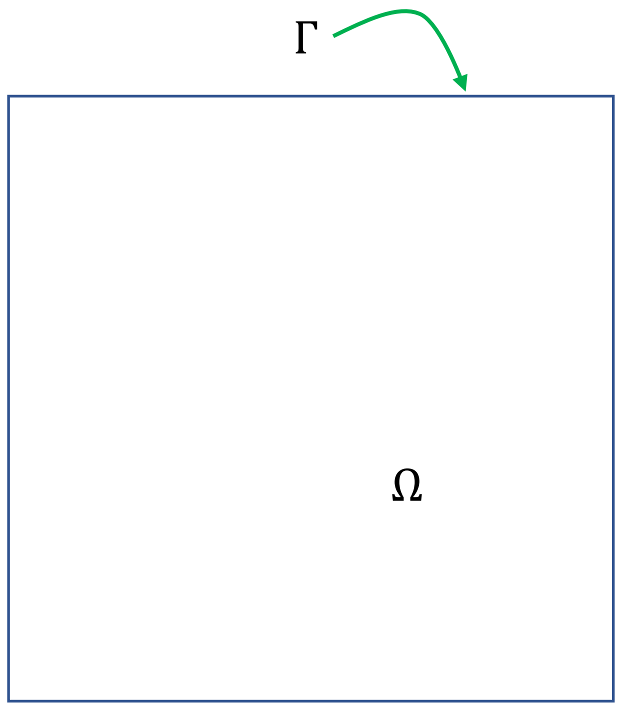

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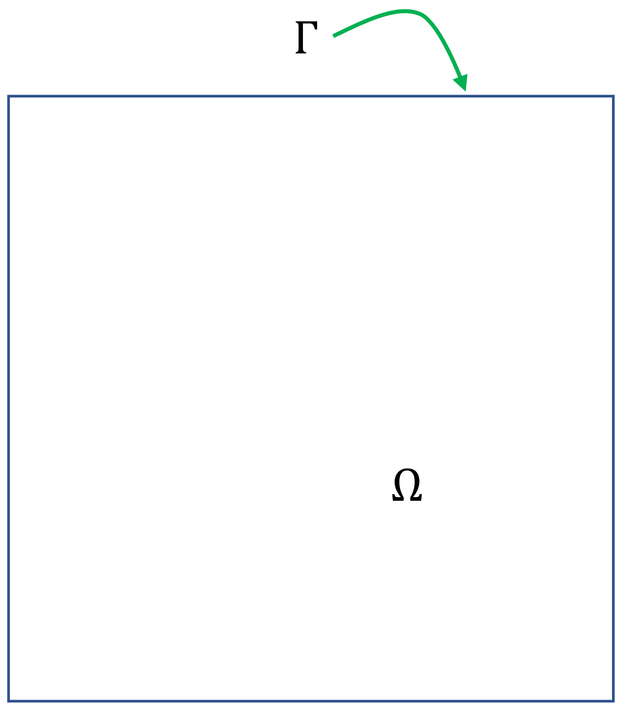

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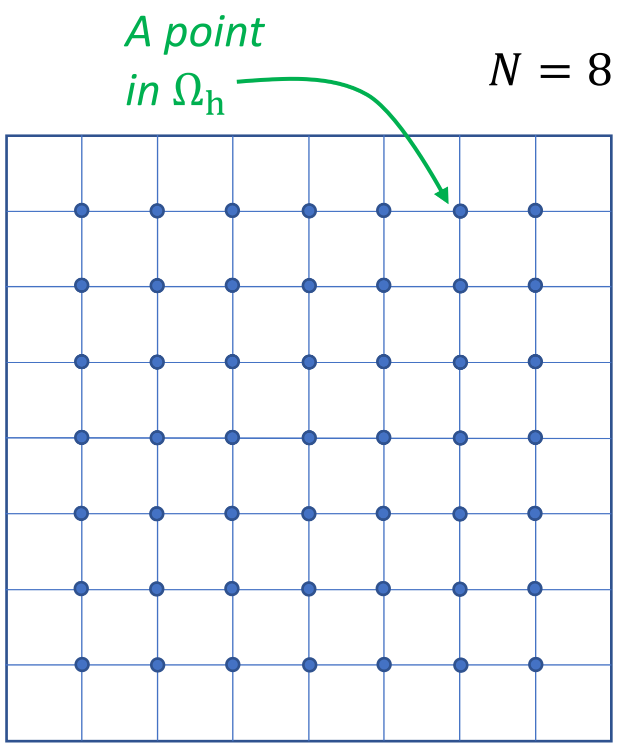

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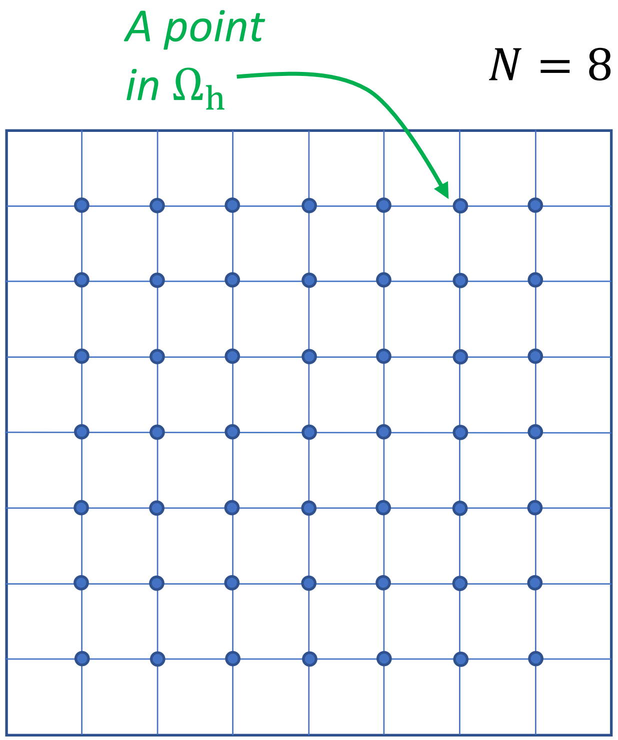

Note that for each $x \in \mathbb{R}_h^2$ *has a set of four nearest neighbors in* $\mathbb{R}_h^2$*, one each to the left, right, above and below. We define* Γ_h *as the set of mesh points in* $\mathbb{R}_h^2$ *which in not in* Ω_h *but has a nearest neighbor in* Ω_h.

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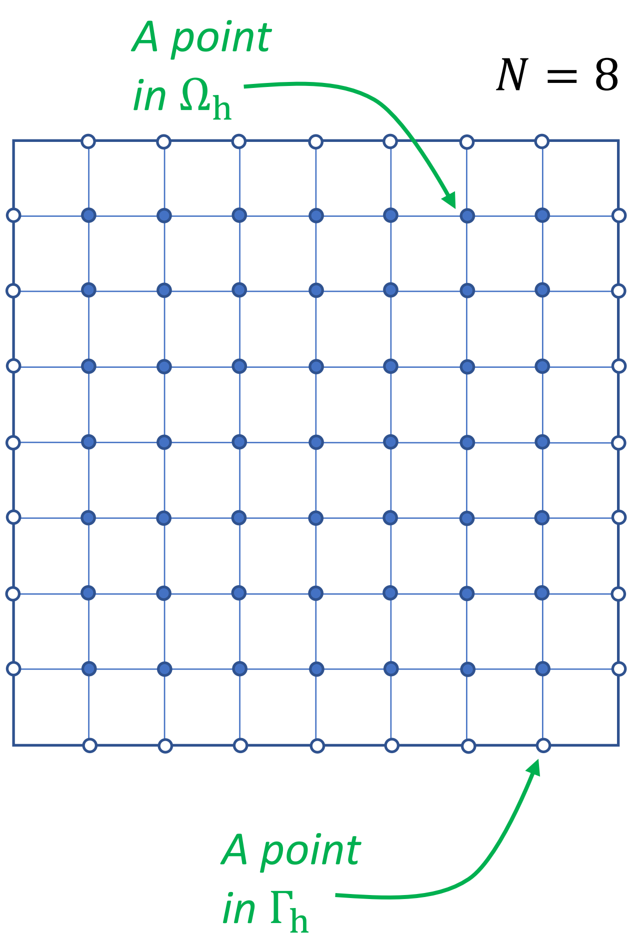

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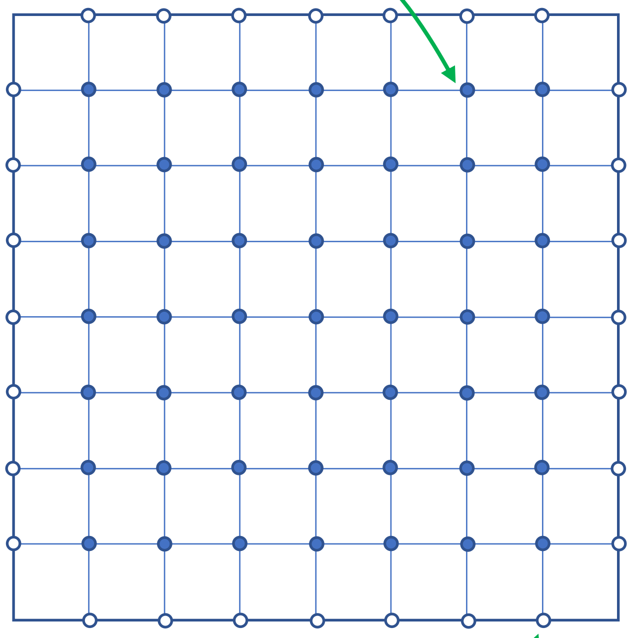

Note that for each $x \in \mathbb{R}^2_h$ *has a set of four nearest neighbors in* $\mathbb{R}^2_h$, *one each to the left, right, above and below. We define* Γ_h *as the set of mesh points in* $\mathbb{R}^2_h$ *which in not in* Ω_h *but has a nearest neighbor in* Ω_h.

Also let $\overline{\Omega}_h = \Omega_h \cup \Gamma_h$.

Numerical Methods for PDE: 2nd Order Elliptic PDE

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To discretize

$$\Delta u = f, \quad \text{in } \Omega,$$
$$u = g, \ \text{on } \Gamma,$$

we seek a function $u_h : \overline{\Omega_h} \to \mathbb{R}$ *satisfying*

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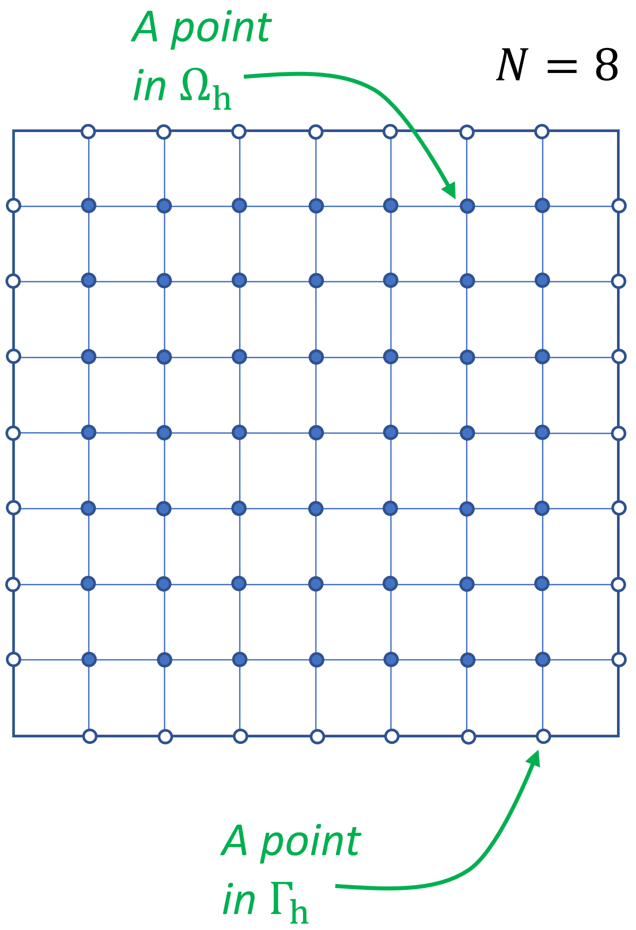
where, writing $v_{m,n} = v(mh, nh)$, *we have the 5-point Laplacian*

$$\Delta_h v(mh, nh) = \frac{v_{m+1,n} - 2v_{m,n} + v_{m-1,n}}{h^2} + \frac{v_{m,n+1} - 2v_{m,n} + v_{m,n-1}}{h^2}$$



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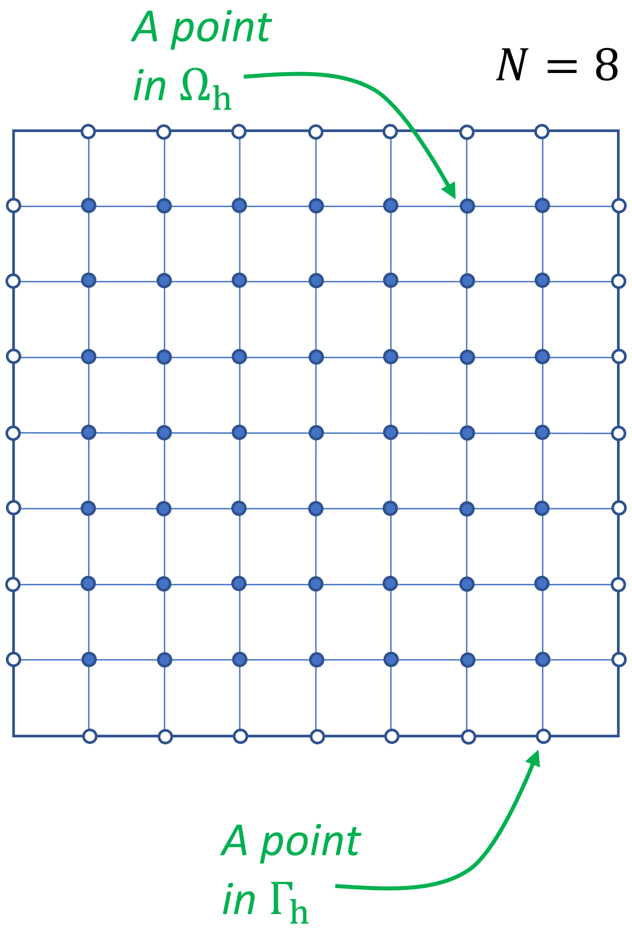

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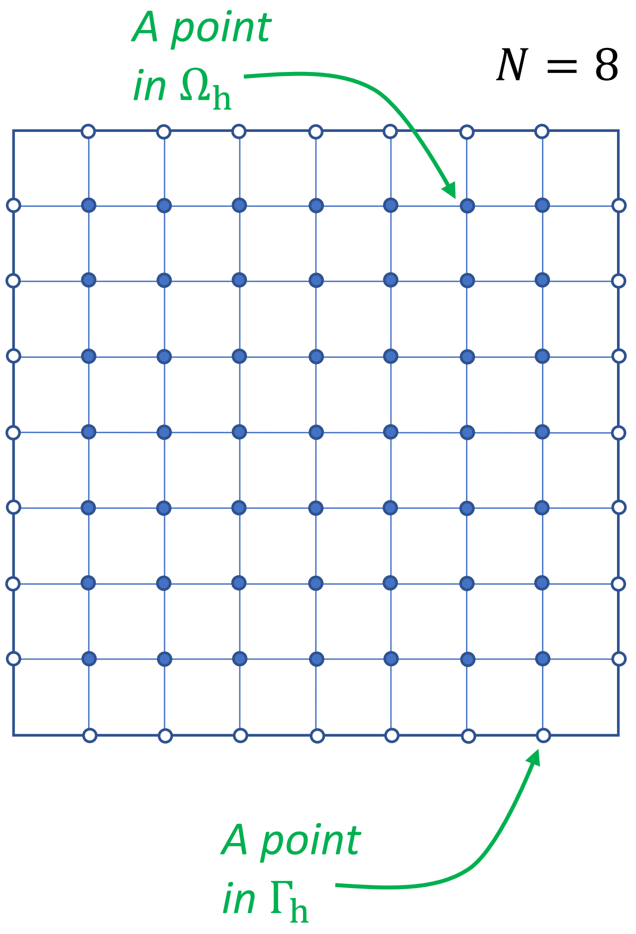
From the error estimate in one-dimensional case, we case easily get that for $v \in C^4(\overline{\Omega})$,

$$\Delta_h v(mh, nh) - \Delta v(mh, nh) = \frac{h^2}{12}\left[\frac{\partial^4 v}{\partial x_1^4}(\xi, nh) + \frac{\partial^4 v}{\partial x_2^4}(mh, \eta)\right]$$

for some ξ, η.



Theorem

If $v \in C^2(\overline{\Omega})$, *then*

$$\lim_{h\to 0} \|\Delta_h v - \Delta v\|_{\infty,\Omega_h} = 0.$$

If $v \in C^4(\overline{\Omega})$, *then*

$$\|\Delta_h v - \Delta v\|_{\infty,\Omega_h} \le \frac{h^2}{6} M_4,$$

where

$$M_4 = \max\left\{ \left\| \frac{\partial^4 v}{\partial x_1^4} \right\|_{\infty,\overline{\Omega}}, \left\| \frac{\partial^4 v}{\partial x_2^4} \right\|_{\infty,\overline{\Omega}} \right\}.$$

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Theorem (Discrete Maximum Principle)

Let v *be a function on* $\overline{\Omega}_h$ *satisfying* $\Delta_h v \geq 0$ *on* Ω_h. *Then* $\max\limits_{\Omega_h} v \leq \max\limits_{\Gamma_h} v$. *Equality holds if and only if* v *is constant.*

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Proof: Exercise. HINT: *Use* $4v(x_0) = \sum_{i=1}^{4} v(x_i) - h^2\,\Delta_h v(x_0)$ *where* x_1, x_2, x_3, x_4 *are neighbors of* x_0.

Theorem (Discrete Minimum Principle)

Let v be a function on $\overline{\Omega}_h$ satisfying $\Delta_h v \leq 0$ on Ω_h. Then $\min_{\Omega_h} v \geq \min_{\Gamma_h} v$. Equality holds if and only if v is constant.

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Theorem

There is a unique solution to the discrete BVP

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The solution u_h to

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$$\|u_h\|_{\infty,\overline{\Omega}_h} \leq \frac{1}{8}\|f\|_{\infty,\Omega_h} + \|g\|_{\infty,\Gamma_h}.$$

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Proof: Exercise. HINT: Use $w(x_1, x_2) = [(x_1 - 1/2)^2 + (x_2 - 1/2)^2]/4$.

Theorem

Let u be the solution to

$$\begin{aligned} \Delta u &= f, && \text{in } \Omega, \\ u &= g, && \text{on } \Gamma, \end{aligned}$$

and u_h be the solution to the corresponding discrete problem

$$\begin{aligned} \Delta_h u_h &= f, && \text{on } \Omega_\mathrm{h}, \\ u_h &= g, && \text{on } \Gamma_\mathrm{h}. \end{aligned}$$

Then,

$$\|u_h - u\|_{\infty,\overline{\Omega}_h} \le \frac{1}{8} \|\Delta u - \Delta_h u\|_{\infty,\overline{\Omega}_h}.$$

Proof: Exercise.

Corollary

If $u \in C^2\left(\overline{\Omega}_h\right)$, then

$$\lim_{h\to 0} \|u_h - u\|_{\infty,\overline{\Omega}_h} = 0.$$

If $u \in C^4\left(\overline{\Omega}_h\right)$, then

$$\|u_h - u\|_{\infty,\overline{\Omega}_h} \le \frac{h^2}{48} M_4, \qquad M_4 = \max\left\{ \left\|\frac{\partial^4 v}{\partial x_1^4}\right\|_{\infty,\overline{\Omega}}, \left\|\frac{\partial^4 v}{\partial x_2^4}\right\|_{\infty,\overline{\Omega}} \right\}.$$

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- More Stability Analysis – Fourier Analysis

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Stability Analysis using Fourier Analysis

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First we consider the one dimensional case. With $h = 1/N$, *let* $I_h = \{h, 2h, \dots, (N-1)h\}$ *and let*

$$L(I_h) = \{u \colon \bar{\mathrm{I}}_h \to \mathbb{R} : u(0) = 0, u(1) = 0\}.$$

Clearly, $L(\mathrm{I_h})$ *is isomorphic to* $\mathbb{R}^{N-1}$.

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$$\langle u, v \rangle_h = h \sum_{k=1}^{N-1} u(kh) v(kh)$$

with the corresponding norm $\|v\|_h$.

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Define $\varphi_m \in L(I_h)$ *by* $\varphi_m(x) = \sin \pi m x$.

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Define $\varphi_m \in L(I_h)$ *by* $\varphi_m(x) = \sin \pi m x$. *Then*

$$D_h^2 \varphi_m(x) = \frac{\sin \pi m(x+h) - 2\sin \pi m x + \sin \pi m(x-h)}{h^2}$$

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Thus,

$$D_h^2 \varphi_m = -\lambda_m \varphi_m, \qquad \lambda_m = \frac{2}{h^2}(1 - \cos \pi m h) = \frac{4}{h^2} \sin^2 \frac{\pi m h}{2}.$$

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Stability Analysis using Fourier Analysis

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Clearly, $L(\mathrm{I_h})$ *is isomorphic to* $\mathbb{R}^{N-1}$. *On* $L(\mathrm{I_h})$, *we define the inner product*

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From this, we obtain the stability result for the one-dimensional Laplacian: if $f = D_h^2 v = -\sum_{m=1}^{N-1} \lambda_m a_m \varphi_m$, *then*

$$\|f\|_h^2 = \sum_{m=1}^{N-1} \lambda_m^2 a_m^2 \|\varphi_m\|_h^2 \geq 8^2 \|v\|_h^2$$

Thus, we have the stability estimate

$$\|v\|_h \leq \frac{1}{8}\|f\|_h.$$

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Let $L(\Omega_\mathrm{h}) = \{u\colon \overline{\Omega}_h \to \mathbb{R} : u(x) = 0, x \in \Gamma_h\}$ *so that* $L(\Omega_\mathrm{h})$ *is isomorphic to* $\mathbb{R}^M, M = (N-1)^2$.

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$$\varphi_{mn}(x_1, x_2) = \varphi_m(x_1)\varphi_n(x_2), \qquad m, n = 1, \dots, N-1.$$

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It is easy to check (exercise) that these $(N-1)^2$ *functions form an orthogonal basis for* $L(\Omega_\mathrm{h})$ *equipped with the inner product*

$$\langle u, v\rangle_h = h^2 \sum_{m=1}^{N-1}\sum_{n=1}^{N-1} u(mh, nh)v(mh, nh)$$

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$$\Delta_\mathrm{h}\varphi_{mn} = -\lambda_{mn}\varphi_{mn}$$

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Theorem

We have $\|v\|_h \leq \frac{1}{16}\|f\|_h$ *as the stability estimate where* v *solves the discrete problem* $\Delta_h v = f, \text{ on } \Omega_\mathrm{h}, v = 0, \text{ on } \Gamma_\mathrm{h}.$

Lesson 4

Numerical Solution of PDE

4.1 BVP for 2nd Order Elliptic PDE

- Finite Difference Method

- More Stability Analysis – Energy Estimate

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Stability Analysis using an energy estimate

Stability Analysis using an energy estimate

Let $v \in L(\Omega_{\mathrm{h}})$ *and define the backward difference operator*

$$\partial_{x_1} v(mh, nh) = \frac{v(mh, nh) - v((m-1)h, nh)}{h}, 1 \le m \le N, 0 \le n \le N.$$

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Stability Analysis using an energy estimate

Let $v \in L(\Omega_\mathrm{h})$ and define the backward difference operator

$$\partial_{x_1} v(mh, nh) = \frac{v(mh, nh) - v((m-1)h, nh)}{h}, 1 \le m \le N, 0 \le n \le N.$$

We denote the inner product

$$\langle u, v \rangle_h = h^2 \sum_{m=1}^{N} \sum_{n=1}^{N} u(mh, nh) v(mh, nh)$$

with the corresponding norm $\|\cdot\|_h$.

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Proof:

For $1 \le m \le N, 0 \le n \le N$,

$$|v(mh, nh)|^2 \le \left(\sum_{k=1}^{m} \left| v(kh, nh) - v\big((k-1)h, nh\big) \right| \right)^2$$

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$$\leq \left(h \sum_{k=1}^{N} \left|\partial_{x_1} v(kh, nh)\right|\right)^2 \leq \left(h \sum_{k=1}^{N} \left|\partial_{x_1} v(kh, nh)\right|^2\right)\left(h \sum_{k=1}^{N} 1^2\right)$$

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Lemma

If $v \in L(\Omega_\mathrm{h})$, *then* $\|v\|_h \le \left\|\partial_{x_1} v\right\|_h$.

Proof:

For $1 \le m \le N, 0 \le n \le N,$

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Therefore,

$$\sum_{m=1}^{N}|v(mh,nh)|^2 \le \sum_{k=1}^{N}\left|\partial_{x_1} v(kh,nh)\right|^2$$

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Therefore,

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and

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Lemma

If $v, w \in L(\Omega_\mathrm{h})$, then

$$-\langle \Delta_\mathrm{h} v, w\rangle_h = \left\langle \partial_{x_1} v, \partial_{x_1} w\right\rangle_h + \left\langle \partial_{x_2} v, \partial_{x_2} w\right\rangle_h.$$

Proof:

For $v_0, v_1, \dots, v_N, w_0, w_1, \dots, w_N \in \mathbb{R}$ with $w_0 = w_N = 0$. Then,

$$\sum_{k=1}^{N}(v_k - v_{k-1})(w_k - w_{k-1}) = \sum_{k=1}^{N} v_k w_k + \sum_{k=1}^{N} v_{k-1} w_{k-1} - \sum_{k=1}^{N} v_{k-1} w_k - \sum_{k=1}^{N} v_k w_{k-1}$$

$$= 2\sum_{k=1}^{N-1} v_k w_k - \sum_{k=1}^{N-1} v_{k-1} w_k - \sum_{k=1}^{N-1} v_{k+1} w_k = -\sum_{k=1}^{N-1}(v_{k+1} - 2v_k + v_{k-1}) w_k.$$

Hence,

$$-h\sum_{k=1}^{N-1}\frac{v\big((k+1)h, nh\big) - 2v(kh, nh) + v\big((k-1)h, nh\big)}{h^2} w(kh, nh) = h\sum_{k=1}^{N} \partial_{x_1} v(kh, nh)\, \partial_{x_1} w(kh, nh)$$

and thus

$$-\left\langle D_{h,x_1}^2 v, w\right\rangle_h = \left\langle \partial_{x_1} v, \partial_{x_1} w\right\rangle_h.$$

Lemma

If $v, w \in L(\Omega_\mathrm{h})$*, then*

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and thus

$$-\left\langle D^2_{h,x_1} v, w\right\rangle_h = \left\langle \partial_{x_1} v, \partial_{x_1} w\right\rangle_h.$$

Similarly

$$-\left\langle D^2_{h,x_2} v, w\right\rangle_h = \left\langle \partial_{x_2} v, \partial_2 w\right\rangle_h.$$

Lemma

If $v, w \in L(\Omega_h)$, then

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Combining the two results, we get the following stability result:

If $v \in L(\Omega_h)$, then

$$\|v\|_h^2 \le \left\|\partial_{x_1} v\right\|_h^2 \le \left\|\partial_{x_1} v\right\|_h^2 + \left\|\partial_{x_2} v\right\|_h^2 = -\langle \Delta_h v, v\rangle_h \le \|\Delta_h v\|_h \|v\|_h,$$

where $\Delta_h v$ is extended to the boundary grid Γ_h as zero while computing $\|\Delta_h v\|_h$.

Lemma

If $v, w \in L(\Omega_\mathrm{h})$, *then*

$$-\langle \Delta_\mathrm{h} v, w\rangle_h = \left\langle \partial_{x_1} v, \partial_{x_1} w\right\rangle_h + \left\langle \partial_{x_2} v, \partial_{x_2} w\right\rangle_h.$$

Proof:

Hence,

$$-h\sum_{k=1}^{N-1} \frac{v\big((k+1)h, nh\big) - 2v(kh, nh) + v\big((k-1)h, nh\big)}{h^2} w(kh, nh) = h\sum_{k=1}^{N} \partial_{x_1} v(kh, nh)\, \partial_{x_1} w(kh, nh)$$

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If $v \in L(\Omega_\mathrm{h})$, *then*

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where $\Delta_\mathrm{h} v$ *is extended to the boundary grid* Γ_h *as zero while computing* $\|\Delta_\mathrm{h} v\|_h$. *Thus,* $\|v\|_h \le \|\Delta_h v\|_h$.

Lesson 4

Numerical Solution of PDE

4.1 BVP for 2nd Order Elliptic PDE

- Finite Difference Method

- General Domains

Akash Anand
MATH, IIT KANPUR

A natural generalization to the two-point BVP

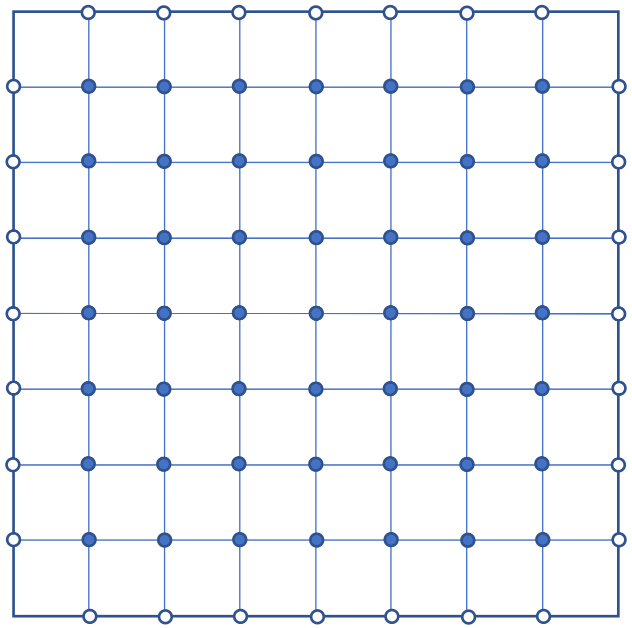
$$u'' = f(t), \qquad a < t < b,$$
$$u(a) = 0, \qquad u(b) = 0,$$

to two dimensions is

$$\Delta u \coloneqq u_{x_1 x_1} + u_{x_2 x_2} = f, \qquad \text{in } \Omega,$$
$$u = g, \qquad \text{on } \Gamma.$$

For simplicity, we will first consider a very simple domain $\Omega = (0,1)\times(0,1)$.

$N = 8$



Numerical Methods for PDE: 2nd Order Elliptic PDE

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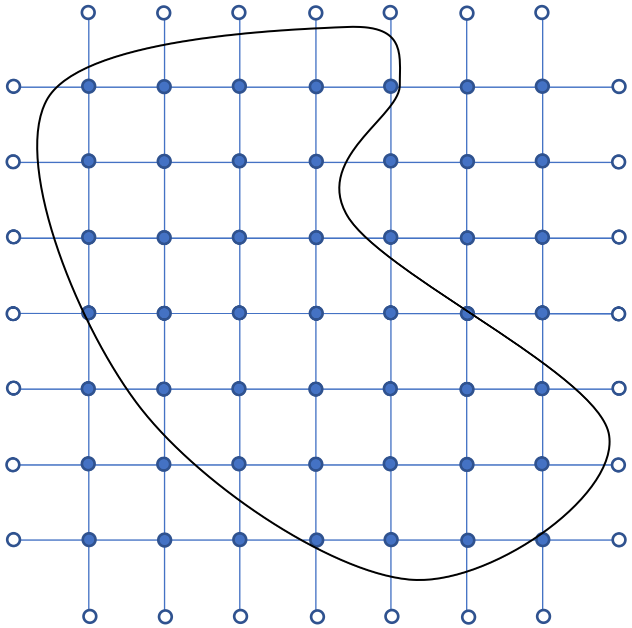

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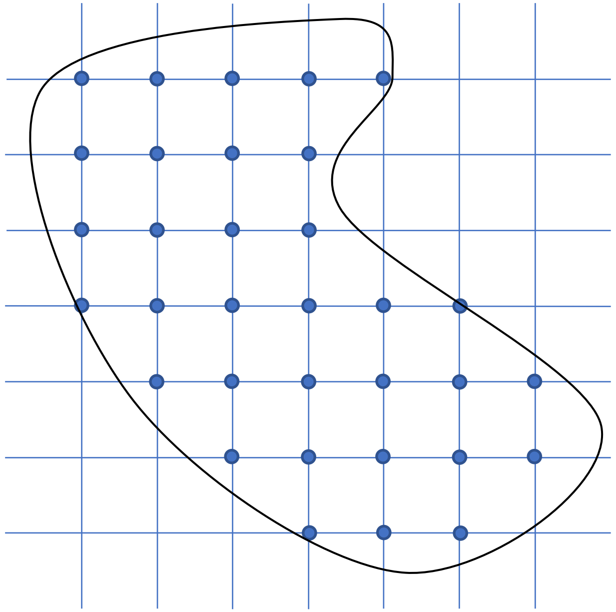

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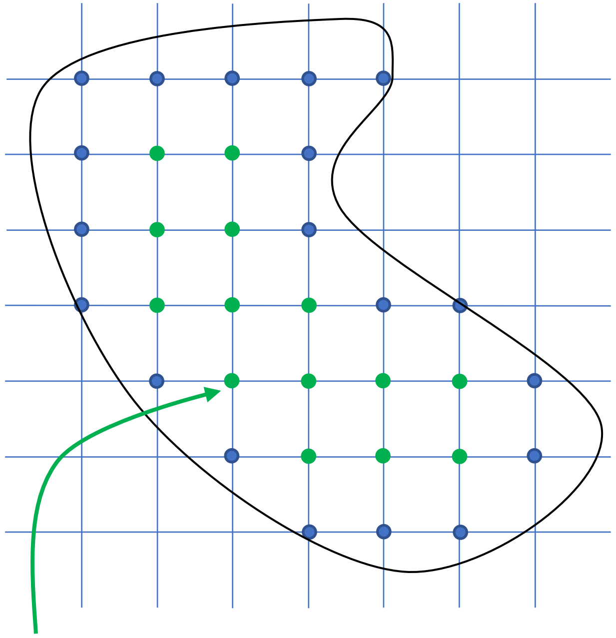

A mesh point with all four nearest neighbors in the interior

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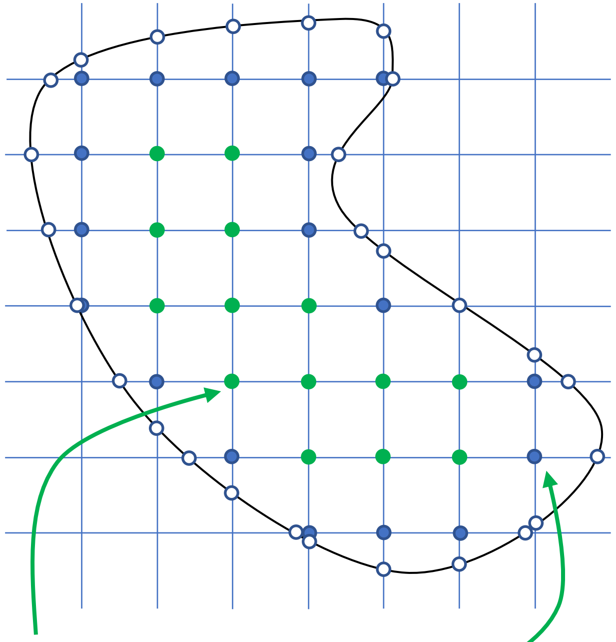

A mesh point with all four nearest neighbors in the interior

A mesh point with at least one nearest neighbor on the boundary

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Also, $\Gamma_\mathrm{h} = \Gamma \cap \mathbb{R}^2_h$ *and* $\overline{\Omega}_h = \Omega \cup \Gamma_h$.

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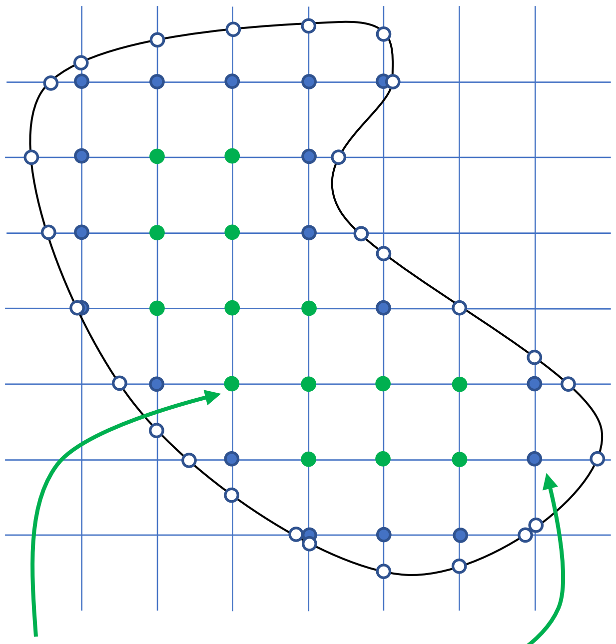

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On Ω_h°, $\Delta_\mathrm{h} v$ *is defined as the usual 5-point Laplacian.*

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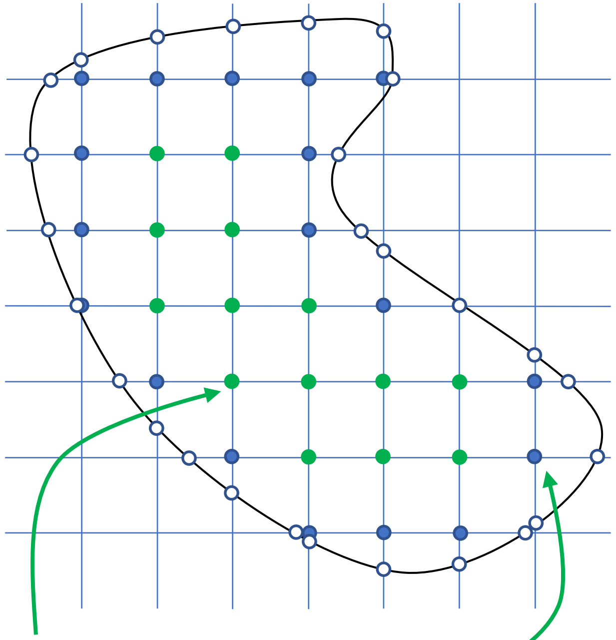

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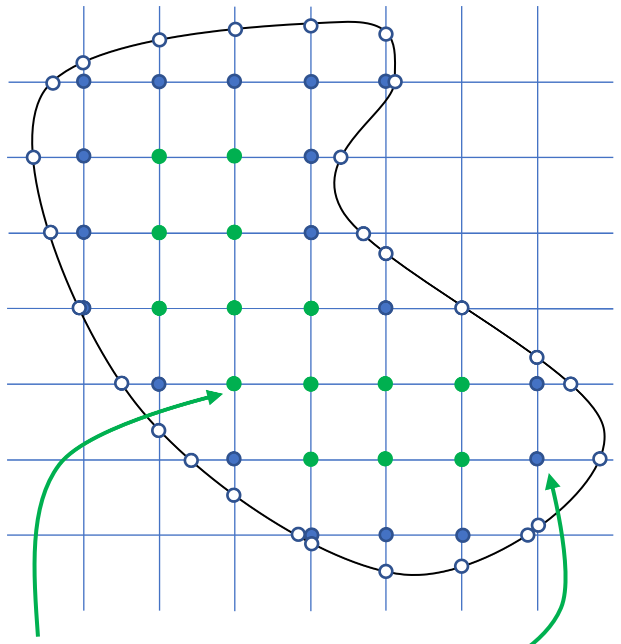
For $(x_1, x_2) \in \Omega_\mathrm{h} \setminus \Omega_h^\circ$, *let* $(x_1 + h_1, x_2)$, $(x_1, x_2 + h_2)$, $(x_1 - h_3, x_2)$, *and* $(x_1, x_2 - h_4)$ *be the nearest neighbors (with* $0 \le h_k \le h$*), and let* v_1, v_2, v_3 *and* v_4 *denote the values of* v *at these four points. And finally, let* $v_0 = v(x_1, x_2)$.

$N = 8$



A mesh point with all four nearest neighbors in the interior

A mesh point with at least one nearest neighbor on the boundary

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On Ω_h°, $\Delta_h v$ is defined as the usual 5-point Laplacian.

For $(x_1, x_2) \in \Omega_h \setminus \Omega_h^\circ$, let $(x_1 + h_1, x_2)$, $(x_1, x_2 + h_2)$, $(x_1 - h_3, x_2)$, and $(x_1, x_2 - h_4)$ be the nearest neighbors (with $0 \le h_k \le h$), and let v_1, v_2, v_3 and v_4 denote the values of v at these four points. And finally, let $v_0 = v(x_1, x_2)$.

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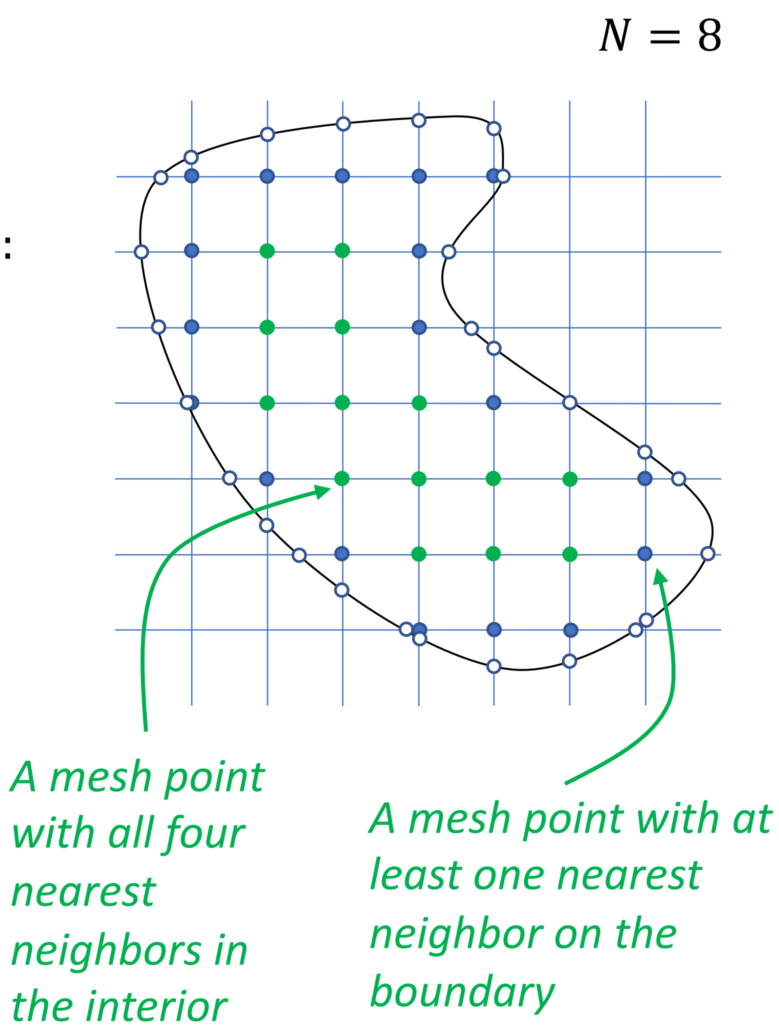

Numerical Methods for PDE: 2nd Order Elliptic PDE

Akash Anand
MATH, IIT KANPUR

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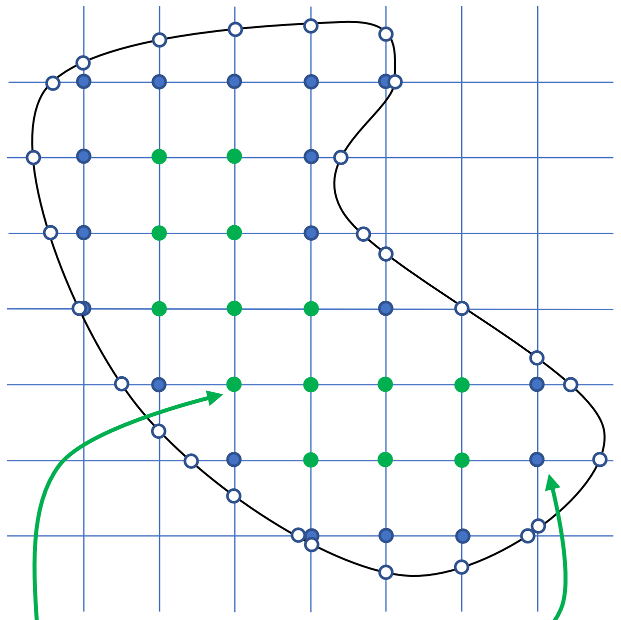

A mesh point with all four nearest neighbors in the interior

A mesh point with at least one nearest neighbor on the boundary

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Thus, to obtain a consistent approximation, we must have

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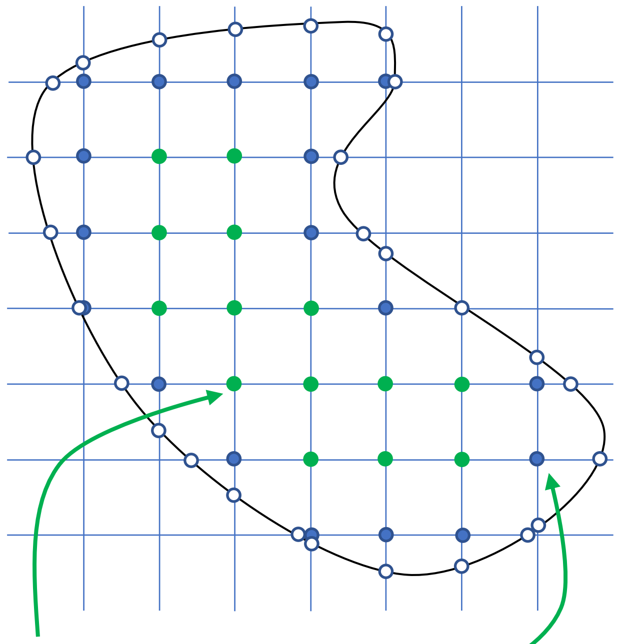

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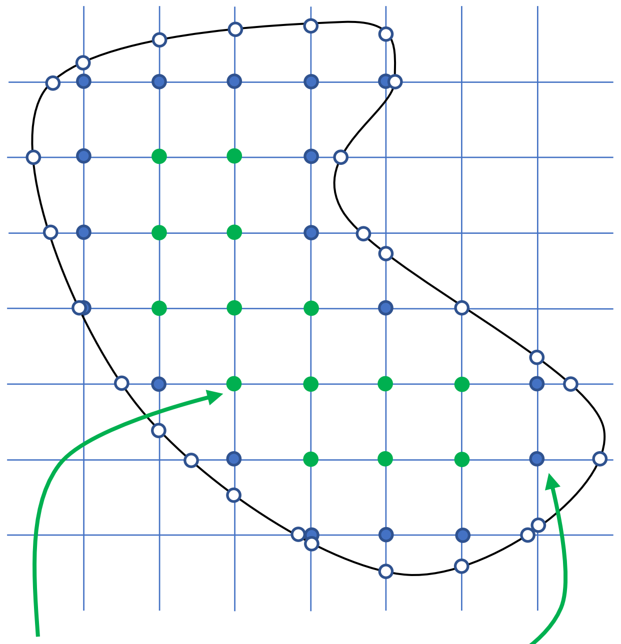

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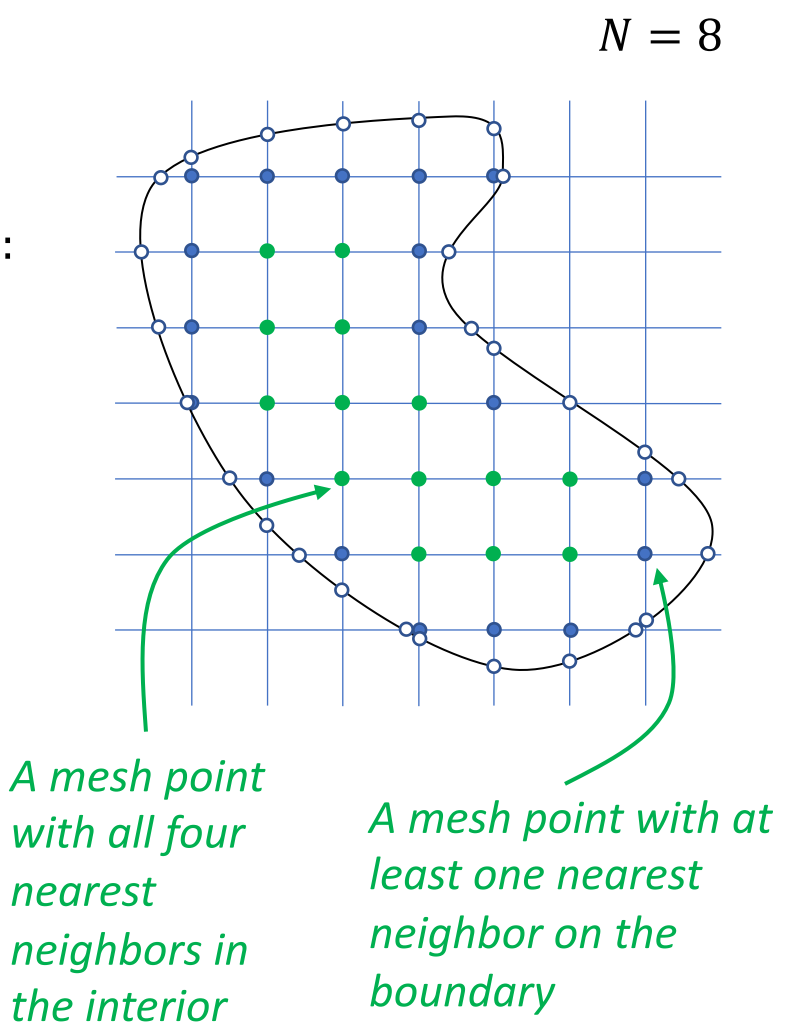
This calculation leads us to the Shortley-Weller formula for $\Delta_\mathrm{h} v$:

$$\Delta_\mathrm{h} v(x_1, x_2) = \frac{2}{h_1(h_1+h_3)} v_1 + \frac{2}{h_2(h_2+h_4)} v_2 + \frac{2}{h_3(h_1+h_3)} v_3 + \frac{2}{h_4(h_2+h_4)} v_4 - \left(\frac{2}{h_1 h_3} + \frac{2}{h_2 h_4}\right) v_0$$



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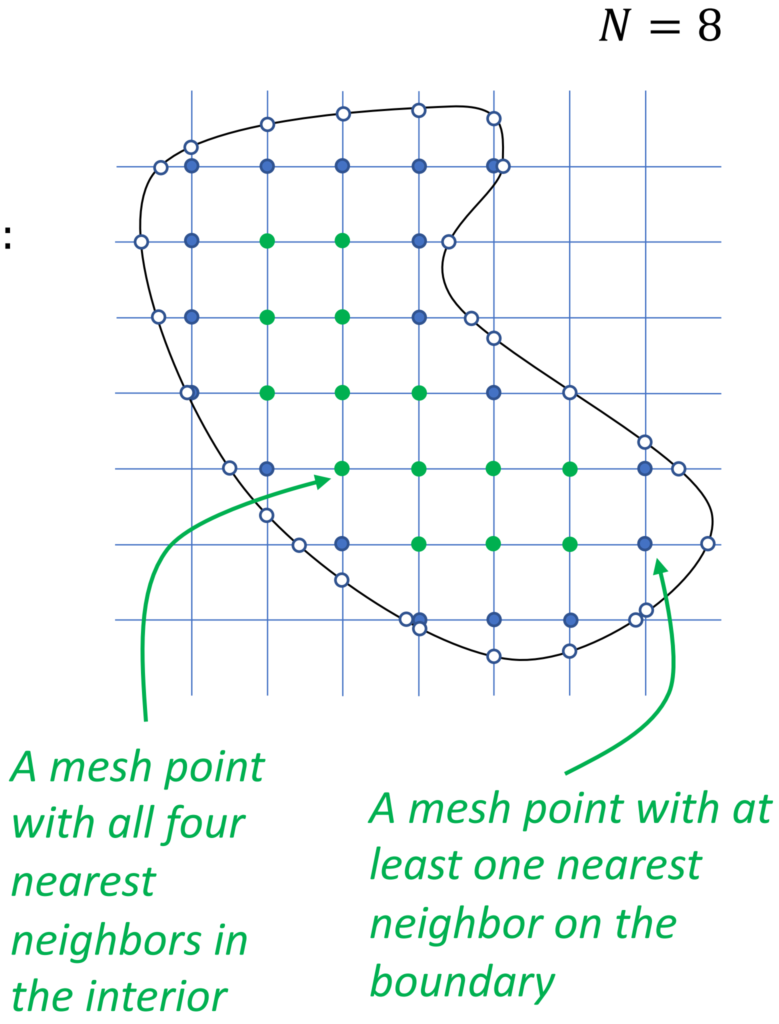
Using Taylor's theorem with remainder, we can easily see that for $v \in C^3(\overline{\Omega})$,

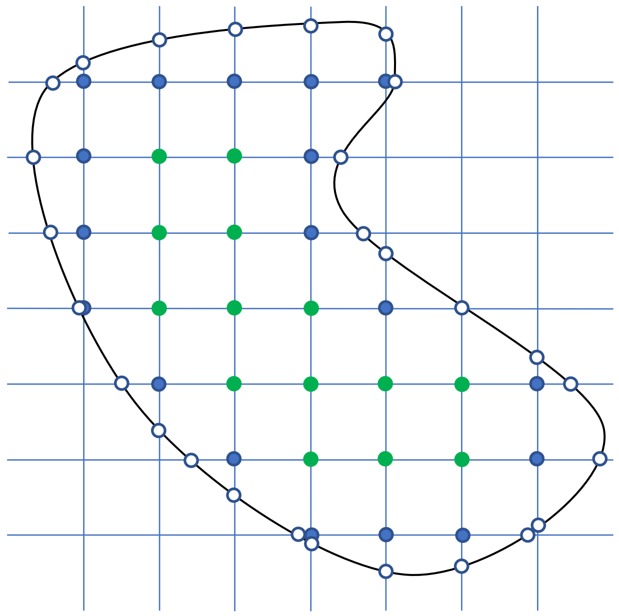
$$\|\Delta v - \Delta_h v\|_{\infty,\Omega_\mathrm{h}} = \frac{2M_3}{3} h,$$

where M_3 is the maximum of the L^∞ norms of the third derivative of v.



We can obtain the discrete maximum/minimum principle with virtually the same proof as for the square domain and then a stability result follows as before (exercise). In this way, we can obtain an $O(h)$ convergence result.

$N = 8$



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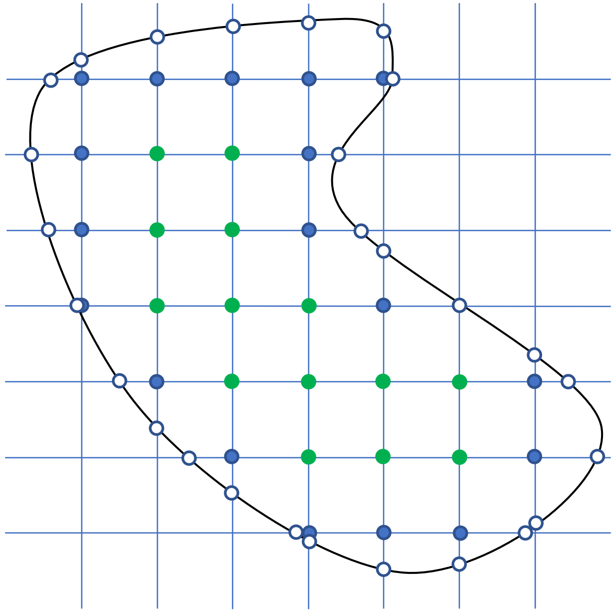

In fact, this result can be improved. Although the truncation error is only $O(h)$, it is $O(h^2)$ all points except those neighboring the boundary and these account for only $O(h^{-1})$ of the total $O(h^{-2})$ points in Ω_h. Moreover, these points are within h of the boundary where the solution is known exactly. For both these reasons, the contribution to the error from these points is smaller than what we observed earlier.

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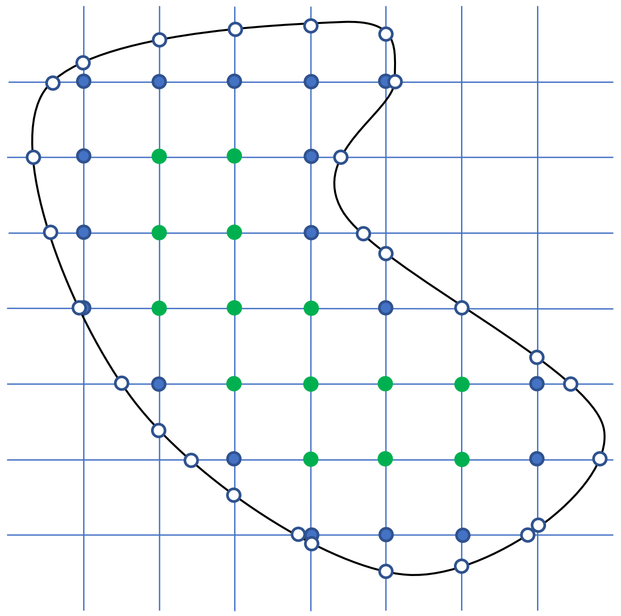

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Theorem

Let u be the solution to

$$\begin{aligned} \Delta u &= f, && \text{in } \Omega, \\ u &= g, && \text{on } \Gamma, \end{aligned}$$

and u_h be the solution to the corresponding discrete problem

$$\begin{aligned} \Delta_h u_h &= f, && \text{on } \Omega_\mathrm{h}, \\ u_h &= g, && \text{on } \Gamma_\mathrm{h}. \end{aligned}$$

Then,

$$\|u_h - u\|_{\infty,\overline{\Omega}_h} \le \frac{M_4 d^2}{96} h^2 + \frac{2M_3}{3} h^3,$$

where d is the diameter of the smallest disc containing Ω and M_k is the maximum of the L^∞ norms of the kth derivative of v.