Module 2

Initial Value Problems

2.5 Stiffness

2.6 Linear Multistep Method

2.7 Non-Linear Methods

- Runge-Kutta methods

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Initial Value Problems: Non-Linear Methods

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Initial Value Problems: Non-Linear Methods

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in terms of the relative increment function Ψ, then for the Heun's method, we have

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More generally,

$$\Psi = b_1 k_1 + b_2 k_2 + \cdots + b_q k_q$$

where $k_i = f(t_n + c_i h, p_i)$ *and*

$$\begin{gathered} p_1 = y_n \\ p_2 = y_n + h(a_{21}k_1) \\ p_3 = y_n + h(a_{31}k_1 + a_{32}k_2) \\ \vdots \\ p_q = y_n + h\big(a_{q1}k_1 + a_{q2}k_2 + \cdots + a_{q,q-1}k_{q-1}\big) \end{gathered}$$

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To specify a particular method of this form we must specify the coefficients b_i, c_i, $1 \le i \le q$, and a_{ij}, $1 \le i \le q$, $1 \le j \le i$. The b_i are called weights, the c_i (or the points $t_n + c_i h$) the nodes, and p_i or, sometimes, the k_i, are called the stages.

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A Runge-Kutta method is often recorded in a tableau of the form

$$\begin{array}{c|c} \boldsymbol{c} & A \\ \hline & \boldsymbol{b}^T \end{array}$$

For example, the tableau for Heun's method is

$$\begin{array}{c|cc} 0 & & \\ 1 & 1 & \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

where we have omitted the zeros in the upper triangle of A. The other well known RK methods are given below:

$$\begin{array}{c|cc} 0 & & \\ \frac{1}{2} & \frac{1}{2} & \\ \hline & 0 & 1 \end{array}$$

Modified Euler method (order 2)

$$\begin{array}{c|ccc} 0 & & & \\ \frac{1}{2} & \frac{1}{2} & & \\ 1 & -1 & 2 & \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array}$$

Heun's 3-stage method (order 3)

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$$

Runge-Kutta-Simpson 4-stage method (the RK method) (order 4)

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1. *We assume that* Ψ *is defined for* $t \in [t_0, T], y \in \mathbb{R}, h \in [0, T - t_0]$, *and is continuous there. Moreover, we assume the uniform Lipschitz condition*

$$|\Psi(f; t, y, h) - \Psi(f; t, \hat{y}, h)| \leq K|y - \hat{y}|,$$

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$$\le |e_n| + Kh|y_n - y(t_n)| + h\left|\frac{\ell_{n+1}(y, h)}{h}\right|$$

Theorem

A single step method is convergent if and only if it is consistent.

Proof.

$(\Longleftarrow)$

Let $e_n = y_n - y(t_n)$. Then

$$|e_{n+1}| = |y_{n+1} - y(t_{n+1})| = |y_n + h\Psi(f;t_n,y_n,h) - y(t_{n+1})| \le |e_n| + \left|h\Psi(f;t_n,y_n,h) - \big(y(t_{n+1}) - y(t_n)\big)\right|$$
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\end{aligned}$$

Now, for all $n \le N$ *where the final time* $T = t_0 + Nh$, *we see that*

$$|e_{n+1}| \le (1 + Kh)|e_n| + \max_{0 \le n < N} |\ell_{n+1}(y,h)|\,.$$

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Now, for all $n \leq N$ *where the final time* $T = t_0 + Nh$, *we see that*

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Therefore, for all $n \leq N$, *we have*

$$|e_n| \leq (1+Kh)^n|e_0| + h\max_{0\leq j<N}\left|\frac{\ell_{j+1}(y,h)}{h}\right|\frac{(1+Kh)^n - 1}{Kh}$$

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Now, for all $n \le N$ *where the final time* $T = t_0 + Nh$, *we see that*

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Thus, we see that consistency implies convergence as $e_0 = 0$ *and the method is consistent!*

Theorem

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We define y^h *by the single step method*

$$y^h(t_{n+1}) = y^h(t_n) + h\Psi(f; t_n, y^h(t_n), h),$$

starting from $y^h(t_0) = y_0$.

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starting from $y^h(t_0) = y_0$.

As $h \to 0$, y^h converges to the initial value problem

$$z'(t) = g\big(t, z(t)\big), \qquad z(t_0) = y_0,$$

where $g(t, y) = \Psi(f; t, y, 0)$.

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$$z(t_{n+1}) = z(t_n) + hz'(\xi_n) = z(t_n) + hg\big(\xi_n, z(\xi_n)\big),$$

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$$e_{n+1} = e_n + h\Big(g\big(\xi_n, z(\xi_n)\big) - \Psi(f; t_n, y^h(t_n), h)\Big).$$

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The term in the brackets may be decomposed as

$$g\big(\xi_n, z(\xi_n)\big) - g\big(t_n, z(t_n)\big) + \Psi(f; t_n, z\,(t_n), 0) - \Psi(f; t_n, z\,(t_n), h) + \Psi(f; t_n, z\,(t_n), h) - \Psi(f; t_n, y^h(t_n), h)$$

where the first two differences tend to 0 with h and the last one is bounded by $K|e_n|$.

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where the first to differences tend to 0 with h and the last one is bounded by $K|e_n|$.

Thus, we have

$$|e_{n+1}| \leq (1 + Kh)|e_n| + \omega(h)$$

where $\lim_{h\to 0} \omega(h) = 0$. Since $e_0 = 0$, it follows that e_n tends to 0 with h, that is, $y^h(t_n) \to z(t_n)$ as $h \to 0$.

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We, therefore, have consistency as me must have $f(t, y) = g(t, y) = \Psi(f; t, y, 0)$.

Remark

Recall that, for RK methods

$$\Psi(f;t,y,h) = b_1 f(t + c_1 h, p_1(f;t,y,h)) + b_2 f(t + c_2 h, p_2(f;t,y,h)) + \cdots + b_q f(t + c_q h, p_q(f;t,y,h))$$

where

$$\begin{aligned} p_1(f;t,y,h) &= y \\ p_2(f;t,y,h) &= y + h(a_{21}k_1) \\ p_3(f;t,y,h) &= y + h(a_{31}k_1 + a_{32}k_2) \\ &\vdots \\ p_q(f;t,y,h) &= y + h\big(a_{q1}k_1 + a_{q2}q_2 + \cdots + a_{q,q-1}k_{q-1}\big) \end{aligned}$$

and $k_i = f(t + c_i h, p_i)$.

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If f is continuous on $[t_0, T] \times \mathbb{R}$ and satisfies uniform Lipschitz condition with respect to y, then Ψ satisfies the assumptions under which the convergence theorem is proved.

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To see this, we have

$$|p_1(f;t,y,h) - p_1(f;t,\hat{y},h)| = |y-\hat{y}|$$
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$$|p_2(f;t,y,h) - p_2(f;t,\hat{y},h)| \le (1 + Lha_{21})|y - \hat{y}|$$

$$|k_2(f;t,y,h) - k_2(f;t,\hat{y},h)| \le L(1 + Lha_{21})|y - \hat{y}|$$

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$$|p_3(f;t,y,h) - p_3(f;t,\hat{y},h)| \le (1 + Lh(a_{31} + a_{32}(1 + Lha_{21})))|y - \hat{y}|$$

and, so on …

Lesson 3

Boundary Value Problems for ODEs

Akash Anand
MATH, IIT KANPUR

Boundary Value Problems: Well-posedness

Akash Anand
MATH, IIT KANPUR

In many practical problems involving ODEs, instead of initial values, specifying additional at more that one points may be more relevant. In such case, we say that the problem is a Boundary Value Problem (BVP) for ODE.

For example, if you want to throw a projectile from location A and want it to hit location B, you would need to solve the equation of motion together with conditions at A and B.

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Boundary Value Problems for ODEs

3.1 Well-posedness

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A general first-order two-point BVP for an ODE has the form

$$y' = f(t, y), \qquad a < t < b,$$

with boundary conditions

$$g\big(y(a), y(b)\big) = 0$$

where $f: \mathbb{R}^{n+1} \to \mathbb{R}^n$ *and* $g: \mathbb{R}^{2n} \to \mathbb{R}^n$.

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The boundary condition is said to be separated if any given component of g involves solution values only at a or b, but not both.

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For example, if you want to throw a projectile from location A and want it to hit location B, you would need to solve the equation of motion together with conditions at A and B.

A general first-order two-point BVP for an ODE has the form

$$y' = f(t, y), \qquad a < t < b,$$

with boundary conditions

$$g\big(y(a), y(b)\big) = 0$$

where $f: \mathbb{R}^{n+1} \to \mathbb{R}^n$ *and* $g: \mathbb{R}^{2n} \to \mathbb{R}^n$.

The boundary condition is said to be separated if any given component of g *involves solution values only at* a *or* b, *but not both.*

The boundary condition is said to be linear if they have the form

$$B_a y(a) + B_b y(b) = c,$$

where $B_a, B_b \in \mathbb{R}^{n\times n}$ *and* $c \in \mathbb{R}^n$.

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The BVP is said to be linear if both the ODE and the boundary conditions are linear.

Example

Consider the two-point BVP for the second-order scalar ODE

$$u'' = f(t, u, u'), \qquad a < t < b,$$

with boundary conditions

$$u(a) = \alpha, \qquad u(b) = \beta.$$

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This problem is equivalent to the first-order system of ODEs

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} y_2 \\ f(t,y_1,y_2) \end{bmatrix}, \qquad a < t < b,$$

with separated linear boundary conditions

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1(a) \\ y_2(a) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_1(b) \\ y_2(b) \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

Boundary Value Problems: Well-posedness

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Existence, Uniqueness and Conditioning

Unlike the IVPs, with the BVP, there is no single point at which complete state information is given, and hence no point at which local existence of a solution can be established.

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Consider the two-point BVP

$$u'' = -u, \qquad 0 < t < b,$$

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The general solution of the ODE satisfying $u(0) = 0$ *is* $u(t) = c \sin t$ *for a constant* c. *If* $b = m\pi, m \in \mathbb{Z}$, *then* $c \sin b = 0$ *for any* c, *so there are infinitely many solutions of the BVP if* $\beta = 0$, *but there is no solution* $\beta \neq 0$.

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Let $y(t; x)$ *denote the solution to the IVP* $y' = f(t, y),\ y(a) = x,\ x \in \mathbb{R}^n$. *This solution is a solution to the BVP if*

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The solvability of the BVP therefore depends on the existence and uniqueness of solutions to the system of nonlinear algebraic equations $h(x) = 0$*, where* $h(x) = g(x, y(b; x))$*. We have seen (in the first course) that this is not always true, and therefore, can not expect a general theorem for existence and uniqueness of solutions for BVP.*

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The solvability of the BVP therefore depends on the existence and uniqueness of solutions to the system of nonlinear algebraic equations $h(x) = 0$, *where* $h(x) = g(x, y(b; x))$. *We have seen (in the first course) that this is not always true, and therefore, can not expect a general theorem for existence and uniqueness of solutions for BVP.*

Such results are available only in certain specialized and simplified conditions.

Lesson 3

Boundary Value Problems for ODEs

3.1 Well-posedness

- Linear two point BVP

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Theorem

Consider the linear two-point BVP

$$y' = A(t)y + r(t), \qquad a < t < b,$$

where $A(t)$ and $b(t)$ are continuous, with boundary conditions

$$B_a y(a) + B_b y(b) = c.$$

The BVP has a unique solution if and only if the matrix

$$Q = B_a Y(a) + B_b Y(b)$$

is non-singular where Y is the fundamental solution matrix for the ODE whose ith column $y_i(t)$ is the solution to the homogeneous ODE $y' = A(t)y$ with initial condition $y(a) = e_i$, where e_i is the ith column of the identity matrix; these columns are called solution modes.

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Proof.

Assume that the matrix Q is invertible.

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Proof.

Assume that the matrix Q is invertible.

Uniqueness of solution follows from the fact that, if $y_1(t)$ and $y_2(t)$ are two solutions to the BVP, then $y(t) = y_1(t) - y_2(t)$ satisfies

$$y'(t) = A(t)y(t), \; B_a y(a) + B_b y(b) = 0,$$

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and hence $y(t)$ must have the form $y(t) = Y(t)d$ for some $d \in \mathbb{R}^n$ satisying, $Qd = 0$.

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Now, one can see that the unique solution to the BVP is given by

$$y(t) = Y(t)Q^{-1}\left(c - B_b Y(b)\int_a^b Y^{-1}(s)r(s)ds\right) + Y(t)\int_a^t Y^{-1}(s)r(s)ds$$

by directly verifying that it satisfies the ODE and the boundary condition.

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$$y'(t) = A(t)Y(t)Q^{-1}\left(c - B_b Y(b)\int_a^b Y^{-1}(s)r(s)ds\right) + A(t)Y(t)\int_a^t Y^{-1}(s)r(s)ds + Y(t)Y^{-1}(t)r(t)$$

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$$y(a) = Q^{-1}\left(c - B_b Y(b)\int_a^b Y^{-1}(s)r(s)ds\right),$$

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Therefore,

$$B_a y(a) + B_b y(b)$$

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Therefore,

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&+ B_b Y(b)\int_a^b Y^{-1}(s)r(s)ds\\
&= (B_a Y(a)Q^{-1} + B_b Y(b)Q^{-1})c - (B_a Y(a)Q^{-1} + B_b Y(b)Q^{-1} - I)B_b Y(b)\int_a^b Y^{-1}(s)r(s)ds
\end{aligned}$$

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Proof.

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\end{aligned}$$

Proof.

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\end{aligned}
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Finally, for the converse, assume that the BVP has a unique solution, say $y_0(t)$, but Q is not invertible.

Proof.

Therefore,

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Then there is $0 \neq d \in \mathbb{R}^n$ such that $Qd = 0$, and therefore, $y(t) = y_0(t) + Yd$ is another solution to the BVP

Proof.

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A contradiction.

Boundary Value Problems: Well-posedness

Akash Anand
MATH, IIT KANPUR

If we define $\Phi(\mathrm{t}) = \mathrm{Y}(\mathrm{t})\mathrm{Q}^{-1}$ *and the Green's function*

$$G(t,s) = \begin{cases} \Phi(t)B_a\Phi(a)\Phi^{-1}(\mathrm{s}), & a \le s \le t, \\ -\Phi(t)B_b\Phi(b)\Phi^{-1}(s), & t < s \le b. \end{cases}$$

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Absolute Condition Number