Lesson 3

Boundary Value Problems for ODEs

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$$u'' = f(t, u, u'), \qquad a < t < b,$$

with boundary conditions

$$u(a) = \alpha, \qquad u(b) = \beta.$$

Boundary Value Problems: Variational Methods

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where $\varphi_i(t)$ are basis functions defined on $[a, b]$ and y is an n-vector of parameters to be determined.

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The residual $r(t, y) \coloneqq v''(t, y) - f(t, v(t, y), v'(t, y))$ measures how well the approximation satisfies the ODE. For an exact approximation, that is, $u(t) = v(t, y)$, we have $r(t, y) = 0$.

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3.2 Shooting Method

3.3 Finite Difference Method

3.4 Variational Methods

- Collocation Method

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that is,

$$\sum_{j=1}^{n} y_j \varphi_j''(t) = f\left(t_i, \sum_{j=1}^{n} y_j \varphi_j(t_i), \sum_{j=1}^{n} y_j \varphi_j'(t_i)\right), \qquad i = 2, \dots, n-1.$$

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In addition, we also enforce the boundary condition at the end-points:

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This yields a system of n algebraic equation in n unknowns. This may be linear or non-linear depending on whether f is linear or non-linear.

In particular, for the approximation space consisting of polynomials of degree $n-1$ *or less (i.e.,* $\wp_{n-1}$*), we have*

$$u(t) \approx v(t, y) = \sum_{i=1}^{n} y_i \ell_i(t) \in \wp_{n-1},$$

where the Lagrange basis, given by

$$\ell_i(t) = \frac{\prod_{k=1, k\neq i}^{n}(t - t_k)}{\prod_{k=1, k\neq i}^{n}(t_i - t_k)} \in \wp_{n-1}$$

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is used. Then the collocation method yields the following system of algebraic equations: $Ay = F(y)$ where

$$Ay = \begin{bmatrix} \ell_1(t_1) & \ell_2(t_1) & \cdots & \ell_n(t_1) \\ \ell_1''(t_2) & \ell_2''(t_2) & & \\ \vdots & \vdots & \ddots & \vdots \\ & & \ell_{n-1}''(t_{n-1}) & \ell_n''(t_{n-1}) \\ \ell_1(t_n) & \cdots & \ell_{n-1}(t_n) & \ell_n(t_n) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix}, F(y) = \begin{bmatrix} \alpha \\ f\left(t_2, \sum_{i=1}^{n} y_i \ell_i(t_2), \sum_{i=1}^{n} y_i \ell_i'(t_2)\right) \\ \vdots \\ f\left(t_{n-1}, \sum_{i=1}^{n} y_i \ell_i(t_{n-1}), \sum_{i=1}^{n} y_i \ell_i'(t_{n-1})\right) \\ \beta \end{bmatrix}.$$

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The Newton iterations take the form $y^{(m+1)} = y^{(m)} - \left(A - F'\left(y^{(m)}\right)\right)^{-1} \left(Ay^{(m)} - F\left(y^{(m)}\right)\right)$.

Example

Consider the two-point BVP

$$u'' = 6t, \qquad 0 < t < 1,$$
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We seek the approximate solution in the space of quadratic polynomials of the form

$$v(t, y) = 2\left(t - \frac{1}{2}\right)(t-1)y_1 - 4t(t-1)y_2 + 2t\left(t - \frac{1}{2}\right)y_3.$$

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Alternately, if we use the monomial basis $\{1, t, t^2\}$ *for the space of quadratic polynomials, we then have approximate solution of the form*

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The system of algebraic equation that we solve is

$$y_1 = 0,$$
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$$y_1 + y_2 + y_3 = 1.$$

Thus, the approximate solution is

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3.2 Shooting Method

3.3 Finite Difference Method

3.4 Variational Methods

- Least Squares Method

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Boundary Value Problems: Variational Methods

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Toward this, consider the functional $F: \mathbb{R}^n \to \mathbb{R}$ given by

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$$=\int_a^b r(t,y)\left(\varphi_i''(t)-f_2\left(t,\sum_{j=1}^n y_j\varphi_j(t),\sum_{j=1}^n y_j\varphi_j'(t)\right)\varphi_i(t)-f_3\left(t,\sum_{j=1}^n y_j\varphi_j(t),\sum_{j=1}^n y_j\varphi_j'(t)\right)\varphi_i'(t)\right)dt.$$

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This yields a system of algebraic equations

$$\sum_{j=1}^n y_j\langle\varphi_j'', \varphi_i''\rangle = \langle f, \varphi_i''\rangle +$$

$$\sum_{j=1}^n y_j\left\langle\varphi_j'', \varphi_i f_2\left(\cdot, \sum_{j=1}^n y_j\varphi_j, \sum_{j=1}^n y_j\varphi_j'\right)\right\rangle - \left\langle f, \varphi_i f_2\left(\cdot, \sum_{j=1}^n y_j\varphi_j, \sum_{j=1}^n y_j\varphi_j'\right)\right\rangle +$$

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$$\int_a^b r(t,y)\left(\varphi_i''(t) - f_2\left(t, \sum_{j=1}^n y_j\varphi_j(t), \sum_{j=1}^n y_j\varphi_j'(t)\right)\varphi_i(t) - f_3\left(t, \sum_{j=1}^n y_j\varphi_j(t), \sum_{j=1}^n y_j\varphi_j'(t)\right)\varphi_i'(t)\right) dt = 0.$$

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In particular, for the linear problem with $f(t,u,v) = f(t)$, we have

$$\sum_{j=1}^n y_j\langle\varphi_j'', \varphi_i''\rangle = \langle f, \varphi_i''\rangle,$$

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Lesson 3

Boundary Value Problems for ODEs

3.2 Shooting Method

3.3 Finite Difference Method

3.4 Variational Methods

- Galerkin Method

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Boundary Value Problems: Variational Methods

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More generally, a variational method works by approximating the differential equation by the equation

$$\langle r, \psi_i \rangle = \int_a^b r(t, y)\psi_i(t)dt = 0$$

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Example

Consider the two-point BVP

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\begin{aligned}
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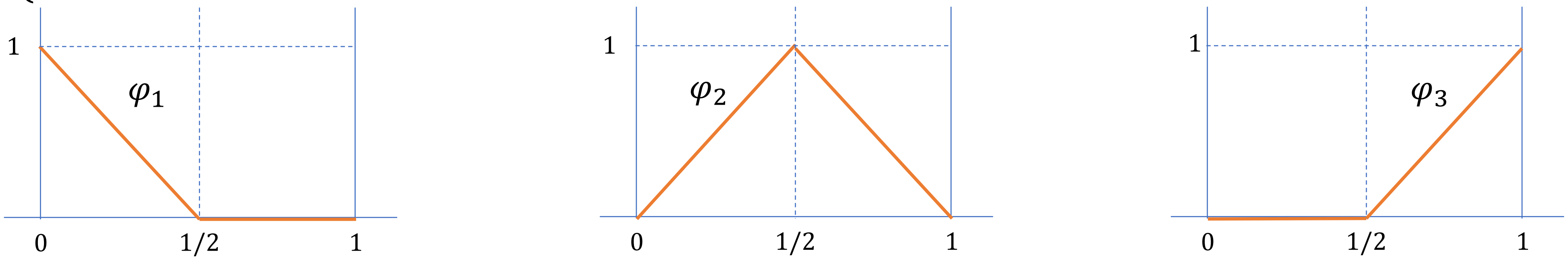

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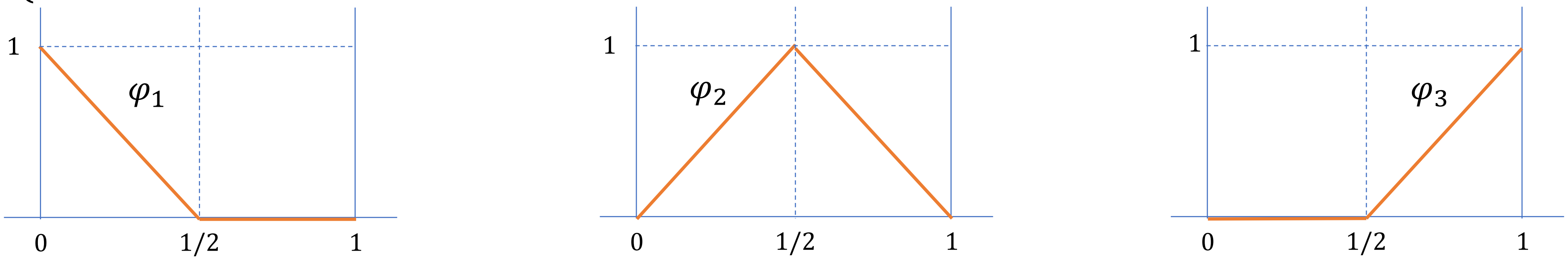

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Remark

In our examples, we have used two kinds of basis functions, one with global support (where the basis function is non-zero over the entire interval, e.g., polynomials, trigonometric polynomials, etc.) and another with local support (e.g., B-splines, etc.).

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Lesson 3

Boundary Value Problems for ODEs

3.2 Shooting Method

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3.4 Variational Methods

- Convergence Analysis for Galerkin Method

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For analyzing the computational error in the Galerkin method, note that the solution u to the boundary value problem

$$u'' = f(t), \qquad a < t < b,$$
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and any function of the form

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Boundary Value Problems: Variational Methods

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$$\int_a^b (v'(t,y) - v'(t,z))v'(t,x)dt = \int_a^b \big(u'(t) - v'(t,z)\big)v'(t,x)dt.$$

Define the H^1 *norm* $\|\cdot\|_{H^1}$ *on* $C_0^1([a,b]) = \{w \in C^1([a,b]) : w(a) = w(b) = 0\}$ *as*

$$\|w\|_{H^1}^2 := \|w\|_{L^2}^2 + \|w'\|_{L^2}^2$$

(*exercise: check that it is a norm*)

Thus, for any $z \in \mathbb{R}^n$*, we have*

$$\int_a^b (v'(t,y) - v'(t,z))v'(t,x)dt = \int_a^b \big(u'(t) - v'(t,z)\big)v'(t,x)dt.$$

Define the H^1 *norm* $\|\cdot\|_{H^1}$ *on* $C_0^1([a,b]) = \{w \in C^1([a,b]) \colon w(a) = w(b) = 0\}$ *as*

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(exercise: check that it is a norm), and check that (exercise), for $w \in C_0^1([a,b])$*, we have*

$$\|w\|_{L^2} \le c(b-a)\|w'\|_{L^2}$$

for some $c > 0$*.*

Thus, for any $z \in \mathbb{R}^n$, we have

$$\int_a^b (v'(t,y) - v'(t,z))v'(t,x)dt = \int_a^b \big(u'(t) - v'(t,z)\big)v'(t,x)dt.$$

Define the H^1 norm $\|\cdot\|_{H^1}$ on $C_0^1([a,b]) = \{w \in C^1([a,b]): w(a) = w(b) = 0\}$ as

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$$\|w\|_{L^2} \le c(b-a)\|w'\|_{L^2}$$

for some $c > 0$. In particular, we have

$$\|v(\cdot,y) - v(\cdot,z)\|_{H^1}^2 \le (1 + c^2(b-a)^2) \int_a^b \big(v'(t,y) - v'(t,z)\big)^2 dt$$

Thus, for any $z \in \mathbb{R}^n$, we have

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Define the H^1 norm $\|\cdot\|_{H^1}$ on $C_0^1([a,b]) = \{w \in C^1([a,b]) : w(a) = w(b) = 0\}$ as

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$$\|v(\cdot,y) - v(\cdot,z)\|_{H^1}^2 \le (1 + c^2(b-a)^2)\int_a^b \big(v'(t,y) - v'(t,z)\big)^2 dt$$

$$= (1 + c^2(b-a)^2)\int_a^b \big(u'(t) - v'(t,z)\big)\big(v'(t,y) - v'(t,z)\big)dt$$

Thus, for any $z \in \mathbb{R}^n$, we have

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Define the H^1 norm $\|\cdot\|_{H^1}$ on $C_0^1([a,b]) = \{w \in C^1([a,b]) : w(a) = w(b) = 0\}$ as

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$$\|v(\cdot,y) - v(\cdot,z)\|_{H^1}^2 \leq (1 + c^2(b-a)^2)\int_a^b \big(v'(t,y) - v'(t,z)\big)^2 dt$$

$$= (1 + c^2(b-a)^2)\int_a^b \big(u'(t) - v'(t,z)\big)\big(v'(t,y) - v'(t,z)\big)dt$$

$$\leq (1 + c^2(b-a)^2)\left(\int_a^b |u'(t) - v'(t,z)|^2 dt\right)^{\frac{1}{2}}\left(\int_a^b |v'(t,y) - |v'(t,z)\Big|^2 dt\right)^{\frac{1}{2}}$$

Boundary Value Problems: Variational Methods

Thus, for any $z \in \mathbb{R}^n$, we have

$$\int_a^b (v'(t,y) - v'(t,z))v'(t,x)dt = \int_a^b \big(u'(t) - v'(t,z)\big)v'(t,x)dt.$$

Define the H^1 norm $\|\cdot\|_{H^1}$ on $C_0^1([a,b]) = \{w \in C^1([a,b]) \colon w(a) = w(b) = 0\}$ as

$$\|w\|_{H^1}^2 := \|w\|_{L^2}^2 + \|w'\|_{L^2}^2$$

(exercise: check that it is a norm), and check that (exercise), for $w \in C_0^1([a,b])$, we have

$$\|w\|_{L^2} \leq c(b-a)\|w'\|_{L^2}$$

for some $c > 0$. In particular, we have

$$\|v(\cdot,y) - v(\cdot,z)\|_{H^1}^2 \leq (1 + c^2(b-a)^2)\int_a^b \big(v'(t,y) - v'(t,z)\big)^2 dt$$

$$= (1 + c^2(b-a)^2)\int_a^b \big(u'(t) - v'(t,z)\big)\big(v'(t,y) - v'(t,z)\big)dt$$

$$\leq (1 + c^2(b-a)^2)\left(\int_a^b |u'(t) - v'(t,z)|^2 dt\right)^{\frac{1}{2}}\left(\int_a^b |v'(t,y) - |v'(t,z)\Big|^2 dt\right)^{\frac{1}{2}}$$

$$\leq (1 + c^2(b-a)^2)\|u - v(\cdot,z)\|_{H^1}\|v(\cdot,y) - v(\cdot,z)\|_{H^1}$$

Thus, for any $z \in \mathbb{R}^n$, we have

$$\int_a^b (v'(t,y) - v'(t,z))v'(t,x)dt = \int_a^b \big(u'(t) - v'(t,z)\big)v'(t,x)dt.$$

Define the H^1 norm $\|\cdot\|_{H^1}$ on $C_0^1([a,b]) = \{w \in C^1([a,b]): w(a) = w(b) = 0\}$ as

$$\|w\|_{H^1}^2 := \|w\|_{L^2}^2 + \|w'\|_{L^2}^2$$

(exercise: check that it is a norm), and check that (exercise), for $w \in C_0^1([a,b])$, we have

$$\|w\|_{L^2} \leq c(b-a)\|w'\|_{L^2}$$

for some $c > 0$. In particular, we have

$$\begin{aligned}
\|v(\cdot,y) - v(\cdot,z)\|_{H^1}^2 &\leq (1 + c^2(b-a)^2)\int_a^b \big(v'(t,y) - v'(t,z)\big)^2 dt \\
&= (1 + c^2(b-a)^2)\int_a^b \big(u'(t) - v'(t,z)\big)\big(v'(t,y) - v'(t,z)\big)dt \\
&\leq (1 + c^2(b-a)^2)\left(\int_a^b |u'(t) - v'(t,z)|^2 dt\right)^{\frac{1}{2}}\left(\int_a^b |v'(t,y) - |v'(t,z)|^2 dt\right)^{\frac{1}{2}} \\
&\leq (1 + c^2(b-a)^2)\|u - v(\cdot,z)\|_{H^1}\|v(\cdot,y) - v(\cdot,z)\|_{H^1}
\end{aligned}$$

Therefore,

$$\|v(\cdot,y) - v(\cdot,z)\|_{H^1} \leq (1 + c^2(b-a)^2)\|u - v(\cdot,z)\|_{H^1}.$$

Therefore,

$$\|v(\cdot,y)-v(\cdot,z)\|_{H^1} \leq (1+c^2(b-a)^2)\|u-v(\cdot,z)\|_{H^1}.$$

Now,

$$\|u-v(\cdot,y)\|_{H^1} \leq \|u-v(\cdot,z)\|_{H^1} + \|v(\cdot,z)-v(\cdot,y)\|_{H^1} \leq (2+c^2(b-a)^2)\|u-v(\cdot,z)\|_{H^1}.$$

As $z \in \mathbb{R}^n$ is arbitrary, we conclude that

$$\|u-v(\cdot,y)\|_{H^1} \leq (2+c^2(b-a)^2)\inf_{z\in\mathbb{R}^n}\|u-v(\cdot,z)\|_{H^1}$$

Therefore,

$$\|v(\cdot,y)-v(\cdot,z)\|_{H^1} \leq (1+c^2(b-a)^2)\|u-v(\cdot,z)\|_{H^1}.$$

Now,

$$\|u-v(\cdot,y)\|_{H^1} \leq \|u-v(\cdot,z)\|_{H^1} + \|v(\cdot,z)-v(\cdot,y)\|_{H^1} \leq (2+c^2(b-a)^2)\|u-v(\cdot,z)\|_{H^1}.$$

As $z \in \mathbb{R}^n$ is arbitrary, we conclude that

$$\|u-v(\cdot,y)\|_{H^1} \leq (2+c^2(b-a)^2) \inf_{z\in\mathbb{R}^n} \|u-v(\cdot,z)\|_{H^1}$$

Best approximation of u by functions in the subspace $\mathrm{span}\{\varphi_1, \ldots, \varphi_n\}$

Therefore,

$$\|v(\cdot,y)-v(\cdot,z)\|_{H^1} \leq (1+c^2(b-a)^2)\|u-v(\cdot,z)\|_{H^1}.$$

Now,

$$\|u-v(\cdot,y)\|_{H^1} \leq \|u-v(\cdot,z)\|_{H^1} + \|v(\cdot,z)-v(\cdot,y)\|_{H^1} \leq (2+c^2(b-a)^2)\|u-v(\cdot,z)\|_{H^1}.$$

As $z \in \mathbb{R}^n$ is arbitrary, we conclude that

$$\|u-v(\cdot,y)\|_{H^1} \leq (2+c^2(b-a)^2)\inf_{z\in\mathbb{R}^n}\|u-v(\cdot,z)\|_{H^1}$$

Best approximation of u by functions in the subspace $\text{span}\{\varphi_1,\ldots,\varphi_n\}$

For piecewise linear elements, we have

$$\inf_{z\in\mathbb{R}^n}\|u-v(\cdot,z)\|_{H^1} \leq Ch\|u''\|_{L^2}$$

Therefore,

$$\|v(\cdot,y)-v(\cdot,z)\|_{H^1} \le (1+c^2(b-a)^2)\|u-v(\cdot,z)\|_{H^1}.$$

Now,

$$\|u-v(\cdot,y)\|_{H^1} \le \|u-v(\cdot,z)\|_{H^1} + \|v(\cdot,z)-v(\cdot,y)\|_{H^1} \le (2+c^2(b-a)^2)\|u-v(\cdot,z)\|_{H^1}.$$

As $z \in \mathbb{R}^n$ is arbitrary, we conclude that

$$\|u-v(\cdot,y)\|_{H^1} \le (2+c^2(b-a)^2)\inf_{z\in\mathbb{R}^n}\|u-v(\cdot,z)\|_{H^1}$$

Best approximation of u by functions in the subspace $\mathrm{span}\{\varphi_1,\dots,\varphi_n\}$

For piecewise linear elements, we have

$$\inf_{z\in\mathbb{R}^n}\|u-v(\cdot,z)\|_{H^1} \le Ch\|u''\|_{L^2}$$

and, more generally, for elements of degree r, we have an estimate

$$\inf_{z\in\mathbb{R}^n}\|u-v(\cdot,z)\|_{H^1} \le Ch^r\|u\|_{H^{r+1}}$$