Numerical Analysis & Scientific Computing II

Module 2

Initial Value Problems

2.4 Implicit method

2.5 Stiffness

2.6 Linear Multistep Methods

Akash Anand
MATH, IIT KANPUR

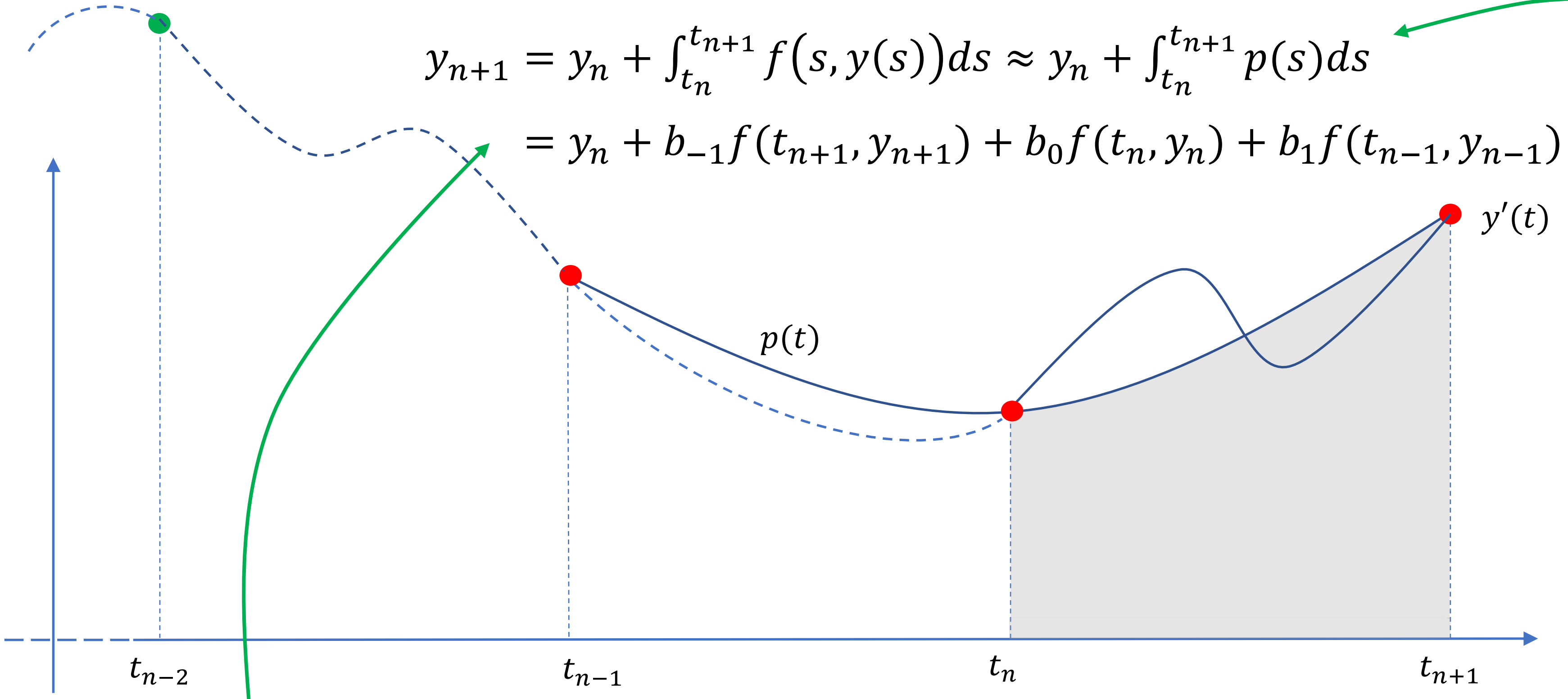
Can we make the method higher order?

Following the idea that we used to derive trapezoidal method, we could use higher order polynomial approximations.

$$y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} f\big(s, y(s)\big)ds \approx y_n + \int_{t_n}^{t_{n+1}} p(s)ds$$

$$= y_n + b_{-1}f(t_{n+1}, y_{n+1}) + b_0 f(t_n, y_n) + b_1 f(t_{n-1}, y_{n-1})$$

Adams—Moulton method



An implicit method

Initial Value Problems: Linear Multistep Methods

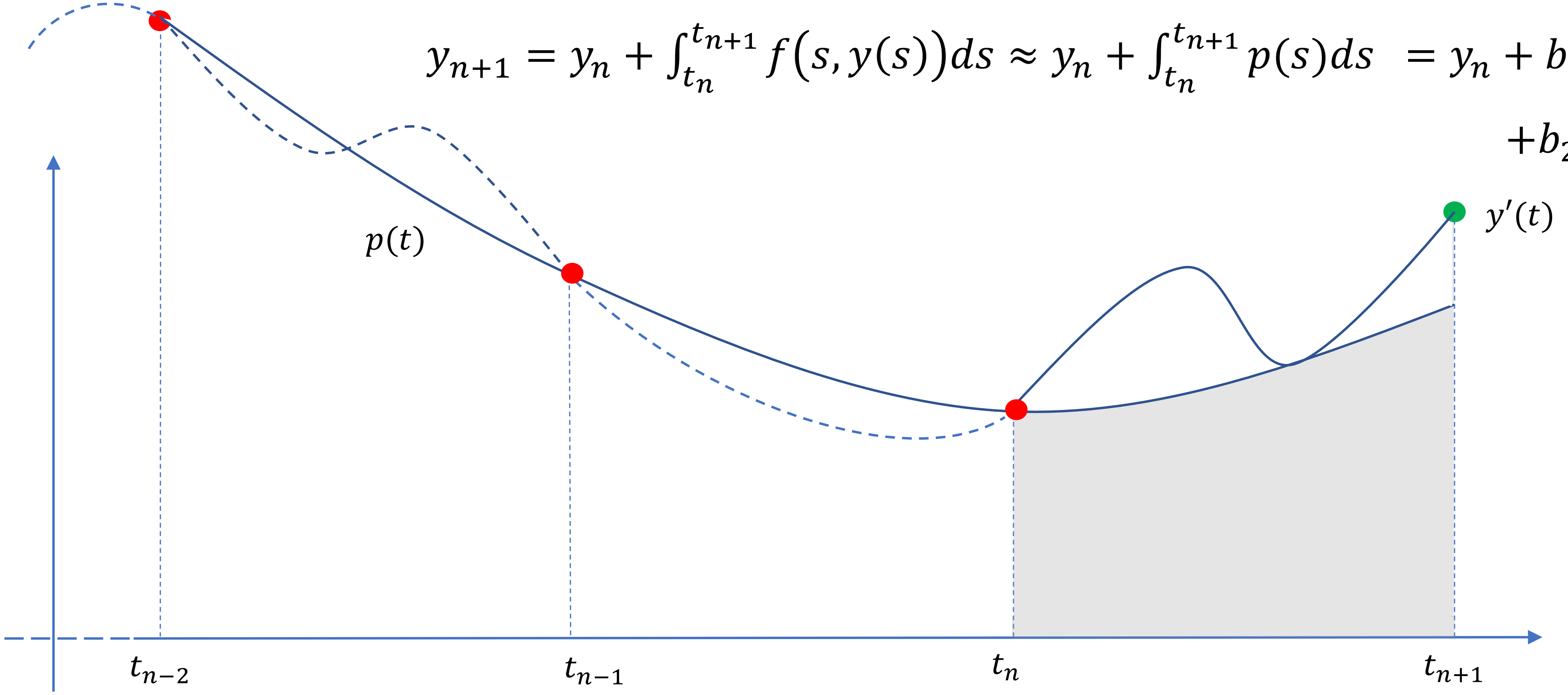
Akash Anand
MATH, IIT KANPUR

Can we make the method higher order?

Following the idea that we used to derive trapezoidal method, we could use higher order polynomial approximations.

$$y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} f\big(s, y(s)\big)ds \approx y_n + \int_{t_n}^{t_{n+1}} p(s)ds \;= y_n + b_0 f(t_n, y_n) + b_1 f(t_{n-1}, y_{n-1}) + b_2 f(t_{n-2}, y_{n-2})$$

Adams-Bashford method



Initial Value Problems: Linear Multistep Methods

Akash Anand
MATH, IIT KANPUR

We consider methods that take constant step size h and determine y_{n+1} using the values from several preceding steps:

$$y_{n+1} = \Phi(f, t_n, y_{n+1}, y_n, y_{n-1}, \dots, y_{n-k}, h).$$

Here y_{n+1} depends on $k+1$ previous values, so this is called a $(k+1)$-step method.

For $k \geq 1$, that is, for a 2 or more step method, how do we start the time marching? Note that we need to know $y_0, \dots, y_k$, to compute y_{k+1} and we typically only know y_0!

… we use some other method such as a single step method.

Improved Euler Method

Examples:

1. $y_{n+1} = y_n + hf(t_n, y_n)$ … *an explicit one step method*
2. $y_{n+1} = y_n + hf(t_{n+1}, y_{n+1})$ … *an implicit one step method*
3. $y_{n+1} = y_n + h\left(f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right)$ … *an implicit one step method*
4. $y_{n+1} = y_n + b_0 f(t_n, y_n) + b_1 f(t_{n-1}, y_{n-1}) + b_2 f(t_{n-2}, y_{n-2})$ … *an explicit three step method*
5. $y_{n+1} = y_n + h\left(f(t_n, y_n) + f(t_{n+1}, y_n + hf(t_n, y_n))\right)/2$ … *an explicit one step method*

Φ Is linear in y_n, $f(t_n, y_n)$, $f(t_{n+1}, y_{n+1})$, etc.

non-linear Φ

Initial Value Problems: Linear Multistep Methods

Akash Anand
MATH, IIT KANPUR

We consider linear multistep methods with constant step size, which by definition, are methods of the form

$$y_{n+1} = -\mathrm{a}_0 \mathrm{y}_\mathrm{n} - \mathrm{a}_1 \mathrm{y}_{\mathrm{n}-1} - \cdots - a_k y_{n-k} + h[b_{-1} f_{n+1} + b_0 f_n + \cdots + b_k f_{n-k}]$$

where f_n denotes $f(t_n, y_n)$ (for brevity) and a_j, b_j are constants which must be given and determine the specific method.

For explicit linear multistep method, $b_{-1} = 0$.

It is also convenient to define $a_{-1} = 1$, so that the method can we written more concisely as

$$\sum_{j=-1}^{k} a_j y_{n-j} = h \sum_{j=-1}^{k} b_j f_{n-j} \,.$$

Theorem

Let $h_0 = 1/(|b_{-1}|L)$ where L is the Lipschitz constant for f. Then for any $h < h_0$ and any $y_n, y_{n-1}, \dots, y_{n-k}$, there is a unique y_{n+1} such that

$$\sum_{j=-1}^{k} a_j y_{n-j} = h \sum_{j=-1}^{k} b_j f(t_{n-j}, y_{n-j}) \,.$$

Theorem

Let $h_0 = 1/(|b_{-1}|L)$ where L is the Lipschitz constant for f. Then for any $h < h_0$ and any $y_n, y_{n-1}, \dots, y_{n-k}$, there is a unique y_{n+1} such that

$$\sum_{j=-1}^{k} a_j y_{n-j} = h \sum_{j=-1}^{k} b_j f(t_{n-j}, y_{n-j}) \,.$$

Proof.

Define

$$F(z) = -\sum_{j=0}^{k} a_j y_{n-j} + h \sum_{j=0}^{k} b_j f(t_{n-j}, y_{n-j}) \; + h b_{-1} f(t_{n+1}, z).$$

Now, $F(z)$ is Lipschitz with Lipschitz constant less than or equal to $h|b_{-1}|L$ (why?). By hypothesis $(h < h_0)$, the Lipschitz constant is strictly less than 1, that is, $F(z)$ is a contraction. The contraction mapping theorem then guarantees a unique fixed point, say y_{n+1}. Thus, we have $y_{n+1} = F(y_{n+1})$, as desired.

Remark

The contraction mapping theorem also implies that the solution can be computed by fixed point iteration as is often done in practice. Moreover, only a fixed (small) number of iterations are made (introducing an additional error).

Module 2

Initial Value Problems

2.4 Implicit method

2.5 Stiffness

2.6 Linear Multistep Methods

- Adams methods

Akash Anand
MATH, IIT KANPUR

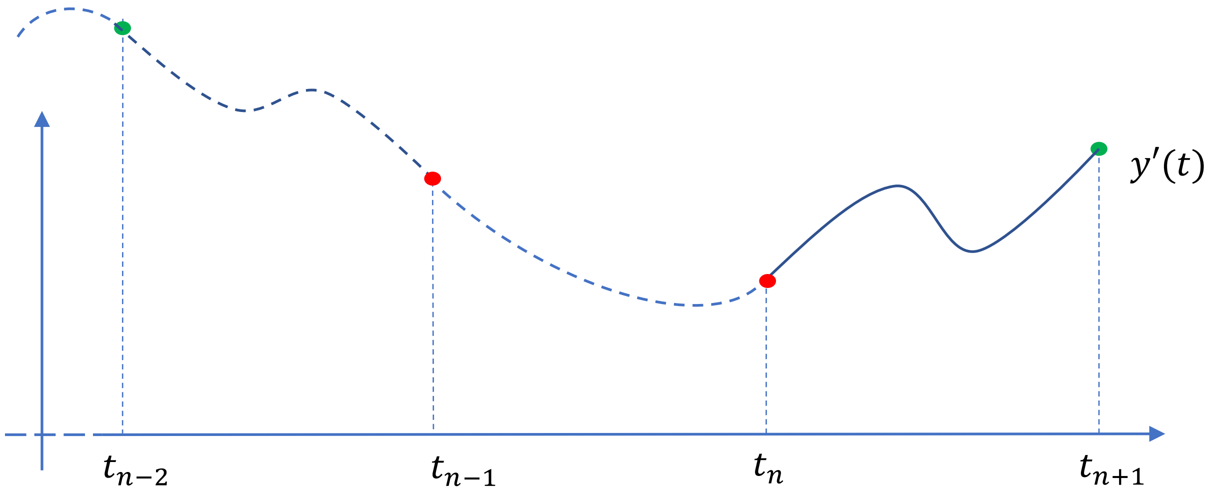
Examples

Adams Bashford methods -

Examples

Adams Bashford methods -

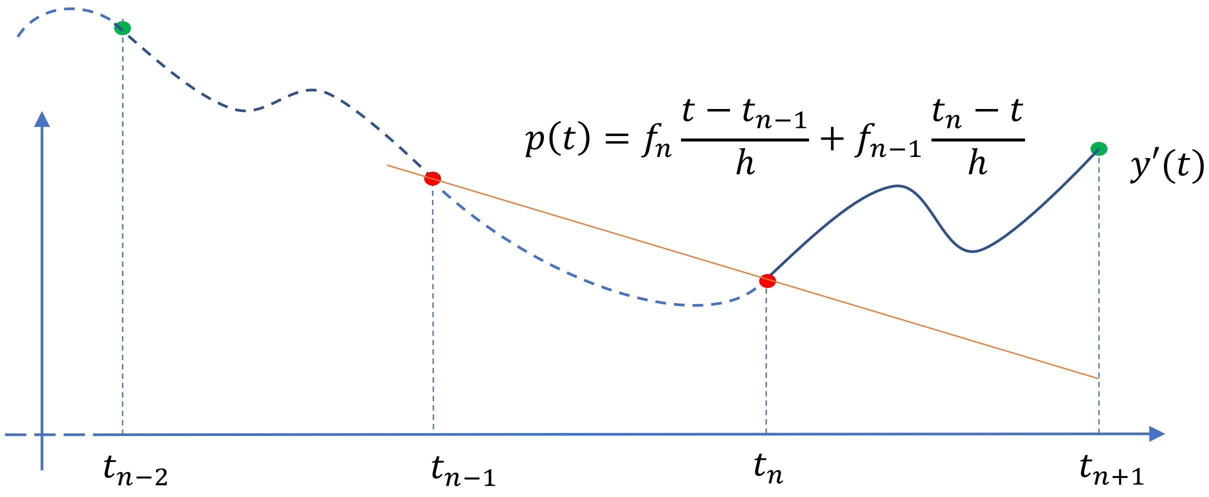
-- 2-step method



Examples

Adams Bashford methods -

-- 2-step method



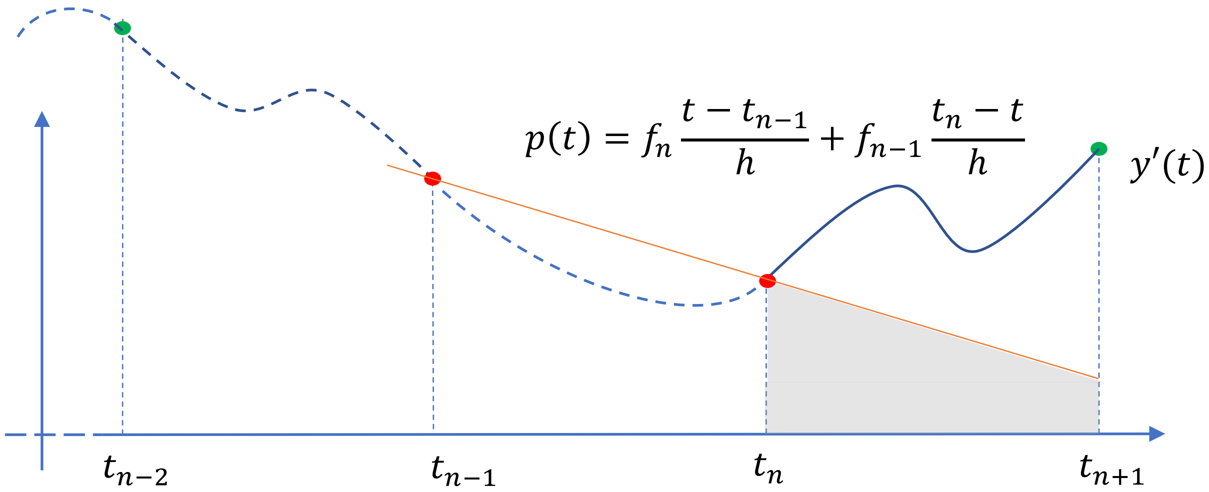
Initial Value Problems: Linear Multistep Methods

Akash Anand
MATH, IIT KANPUR

Examples

Adams Bashford methods -

-- 2-step method

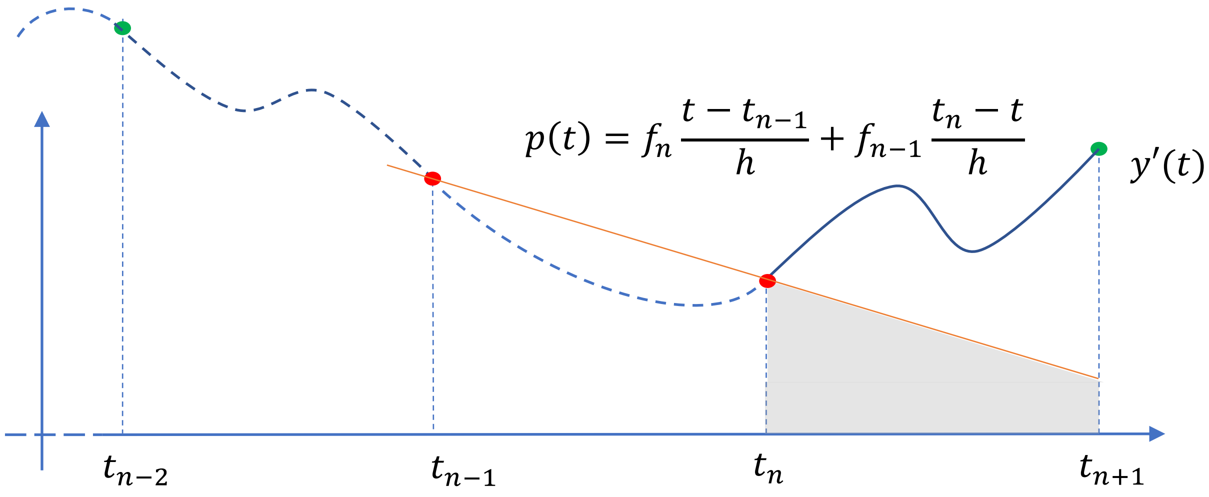


Examples

Adams Bashford methods -

-- 2-step method

$$\int_{t_n}^{t_{n+1}} p(t)dt$$



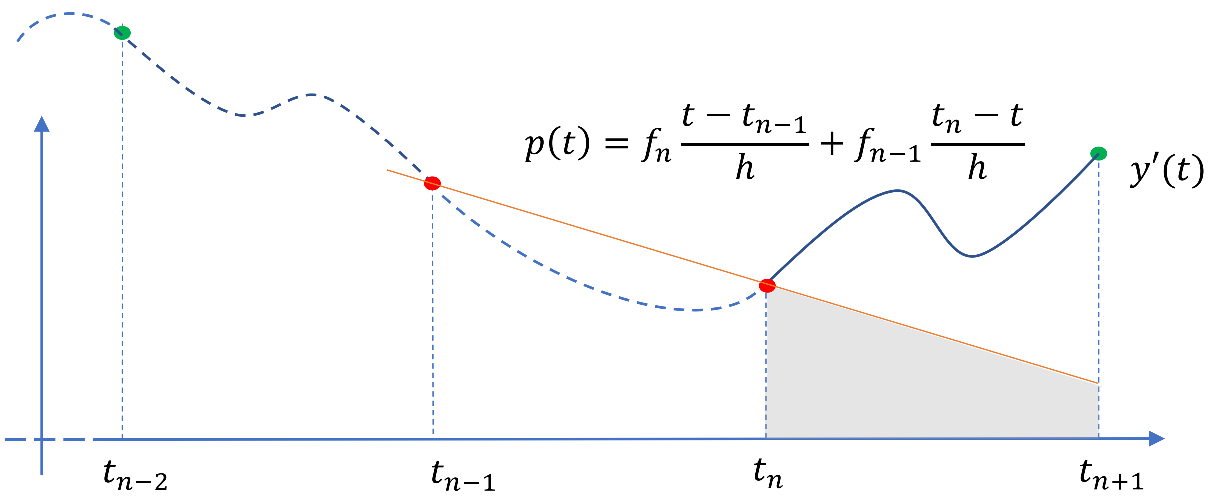
Examples

Adams Bashford methods -

-- 2-step method

$$\int_{t_n}^{t_{n+1}} p(t)dt =$$

$$\int_{t_n}^{t_{n+1}} \left(f_n \frac{t-t_{n-1}}{h} + f_{n-1} \frac{t_n - t}{h}\right) dt$$



Examples

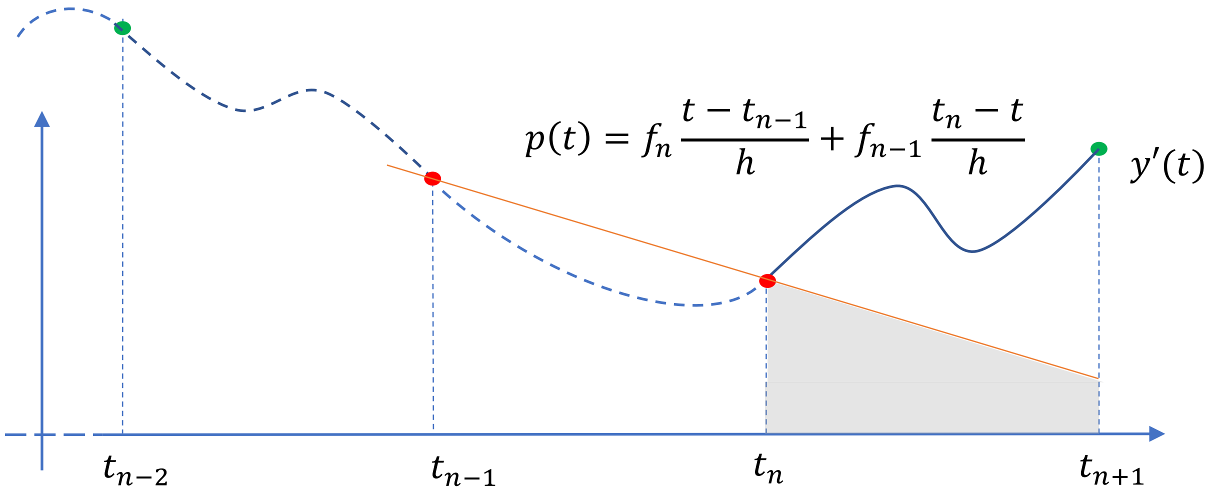
Adams Bashford methods -

-- 2-step method

$$\int_{t_n}^{t_{n+1}} p(t)dt =$$

$$\int_{t_n}^{t_{n+1}} \left(f_n \frac{t-t_{n-1}}{h} + f_{n-1}\frac{t_n - t}{h}\right) dt =$$

$$f_n\left(\frac{3h}{2}\right) + f_{n-1}\left(-\frac{h}{2}\right)$$



Examples

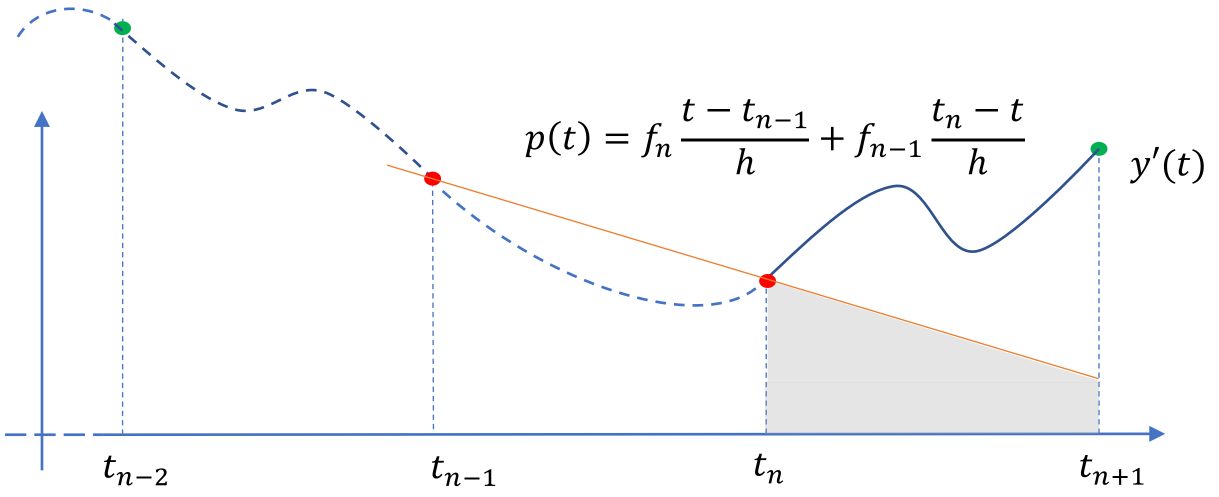
Adams Bashford methods -

-- 2-step method

$$\int_{t_n}^{t_{n+1}} p(t)dt =$$

$$\int_{t_n}^{t_{n+1}} \left(f_n \frac{t-t_{n-1}}{h} + f_{n-1} \frac{t_n - t}{h}\right) dt =$$

$$f_n\left(\frac{3h}{2}\right) + f_{n-1}\left(-\frac{h}{2}\right)$$



Thus,

$$y_{n+1} = y_n + \frac{h}{2}(3f_n - f_{n-1})$$

Initial Value Problems: Linear Multistep Methods

Akash Anand
MATH, IIT KANPUR

Examples

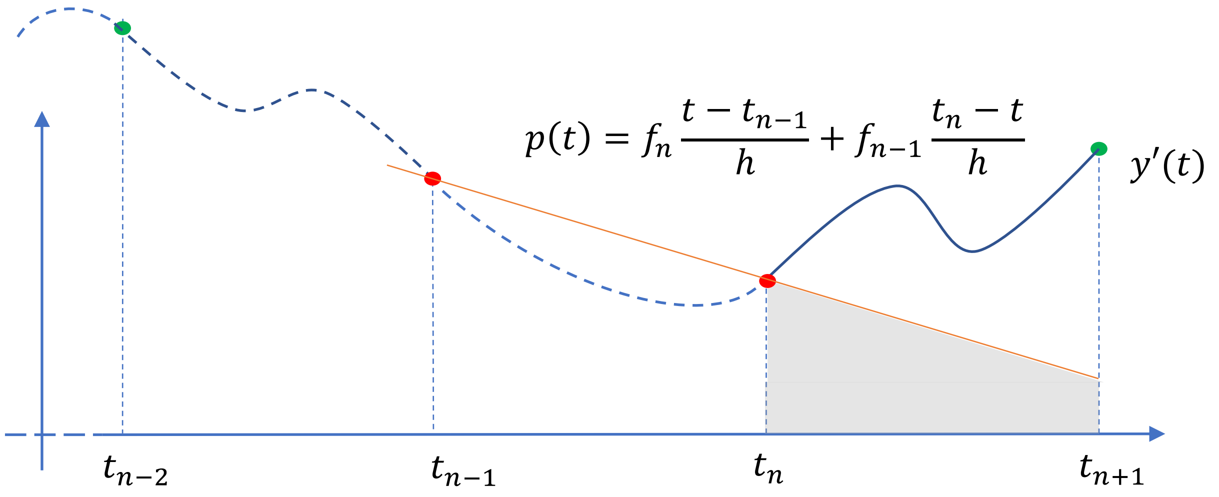
Adams Bashford methods -

-- 2-step method

$$\int_{t_n}^{t_{n+1}} p(t)dt =$$

$$\int_{t_n}^{t_{n+1}} \left(f_n \frac{t-t_{n-1}}{h} + f_{n-1}\frac{t_n - t}{h}\right) dt =$$

$$f_n\left(\frac{3h}{2}\right) + f_{n-1}\left(-\frac{h}{2}\right)$$



Thus,

$$y_{n+1} = y_n + \frac{h}{2}(3f_n - f_{n-1})$$

-- General (k+1) step method

Initial Value Problems: Linear Multistep Methods

Akash Anand
MATH, IIT KANPUR

Examples

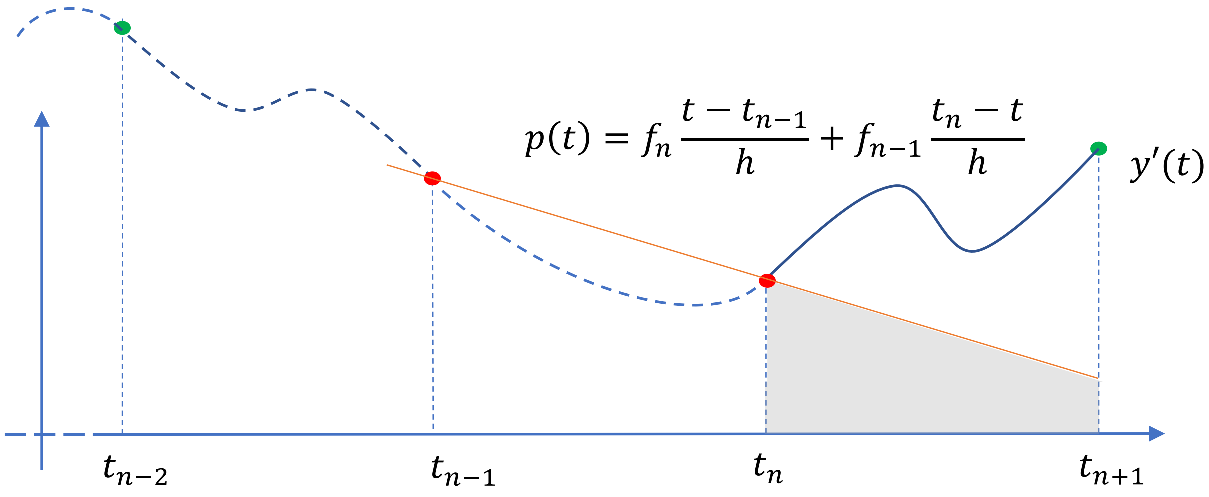
Adams Bashford methods -

-- 2-step method

$$\int_{t_n}^{t_{n+1}} p(t)dt =$$

$$\int_{t_n}^{t_{n+1}} \left(f_n \frac{t-t_{n-1}}{h} + f_{n-1} \frac{t_n-t}{h}\right) dt =$$

$$f_n \left(\frac{3h}{2}\right) + f_{n-1} \left(-\frac{h}{2}\right)$$



Thus,

$$y_{n+1} = y_n + \frac{h}{2}(3f_n - f_{n-1})$$

-- General (k+1) step method

$$p(t) = \sum_{j=0}^{k} l_j^{(k)}(t) f_{n-j}, \qquad \textit{where} \qquad l_j^{(k)}(t) = \prod_{i=0, i\neq j}^{k} (t - t_{n-i})/(t_{n-j} - t_{n-i})$$

Initial Value Problems: Linear Multistep Methods

Akash Anand
MATH, IIT KANPUR

Examples

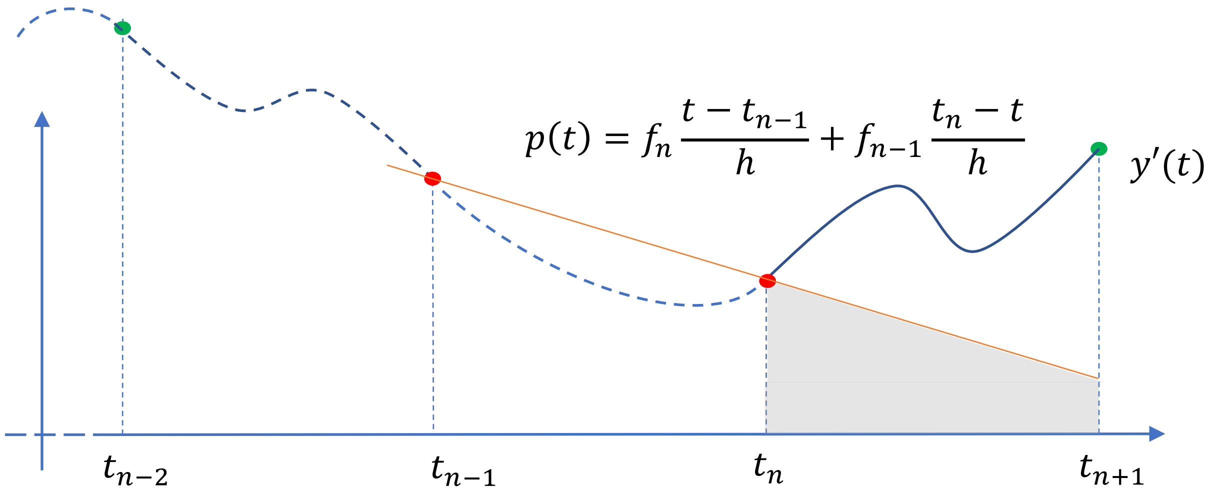
Adams Bashford methods -

-- 2-step method

$$\int_{t_n}^{t_{n+1}} p(t)dt =$$

$$\int_{t_n}^{t_{n+1}} \left(f_n \frac{t-t_{n-1}}{h} + f_{n-1} \frac{t_n - t}{h}\right) dt =$$

$$f_n \left(\frac{3h}{2}\right) + f_{n-1} \left(-\frac{h}{2}\right)$$



Thus,

$$y_{n+1} = y_n + \frac{h}{2}(3f_n - f_{n-1})$$

-- General (k+1) step method

$$p(t) = \sum_{j=0}^{k} l_j^{(k)}(t) f_{n-j}, \quad \text{where} \quad l_j^{(k)}(t) = \prod_{i=0, i\neq j}^{k} (t - t_{n-i})/(t_{n-j} - t_{n-i})$$

Thus,

$$y_{n+1} = y_n + \sum_{j=0}^{k} b_j f_{n-j}, \quad \text{with} \quad b_j = \int_{t_n}^{t_{n+1}} l_j^{(k)}(t)dt.$$

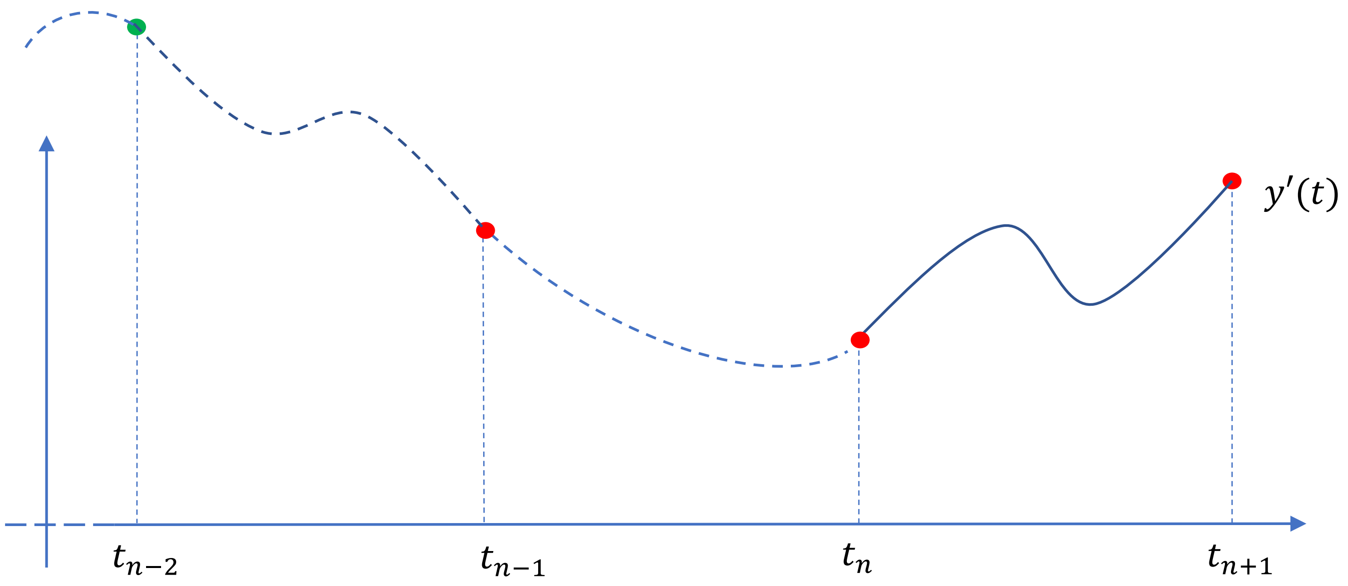
Examples

Adams Moulton methods -

Examples

Adams Moulton methods -

-- 2-step method



Examples

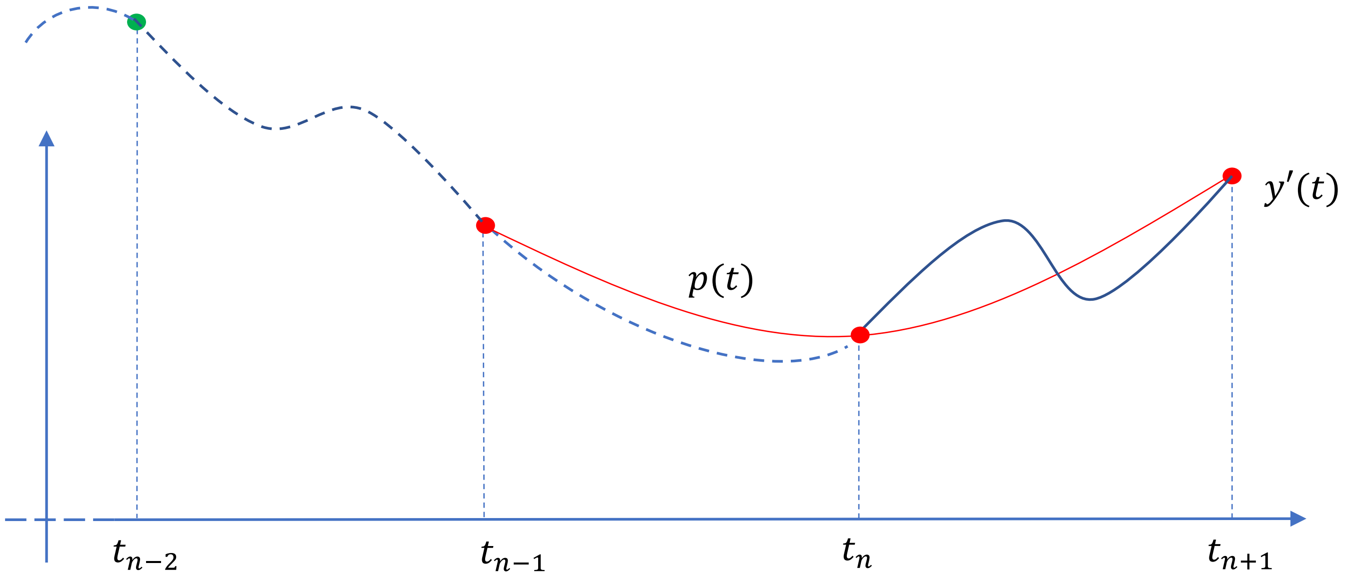
Adams Moulton methods -

-- 2-step method

$$p(t) = f_{n+1}\frac{(t-t_n)(t-t_{n-1})}{2h^2}$$

$$- f_n\frac{(t-t_{n+1})(t-t_{n-1})}{h^2}$$

$$+ f_{n-1}\frac{(t-t_{n+1})(t-t_n)}{2h^2}$$



Examples

Adams Moulton methods -

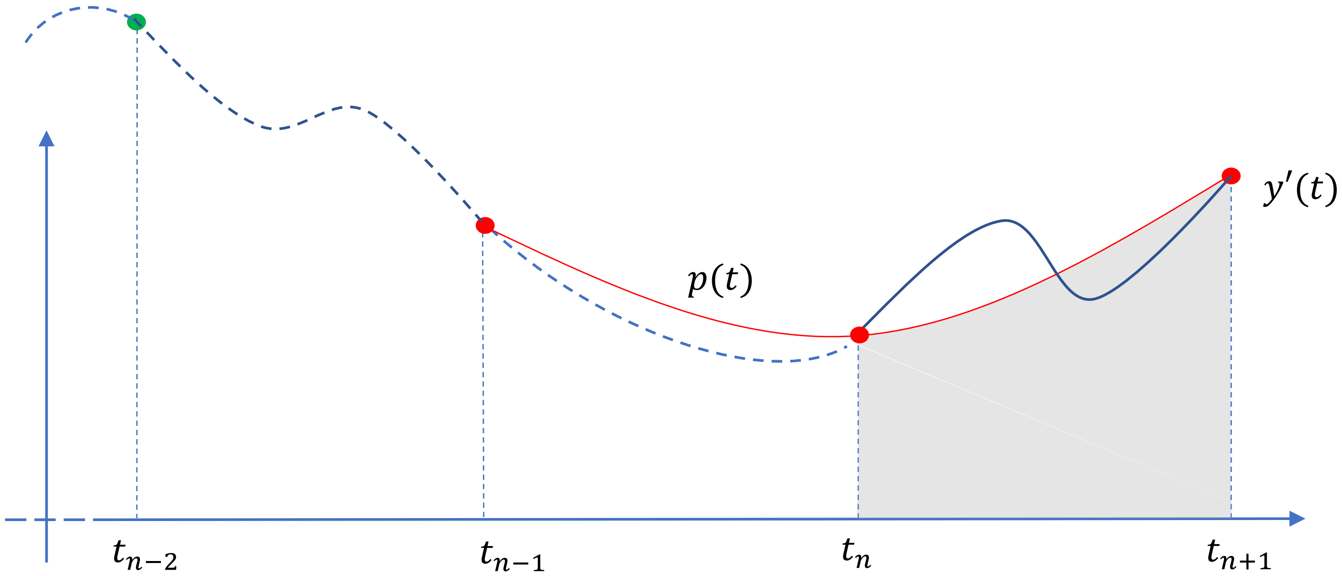
-- 2-step method

$$p(t) = f_{n+1}\frac{(t-t_n)(t-t_{n-1})}{2h^2}$$

$$- f_n\frac{(t-t_{n+1})(t-t_{n-1})}{h^2}$$

$$+ f_{n-1}\frac{(t-t_{n+1})(t-t_n)}{2h^2}$$

$$\int_{t_n}^{t_{n+1}} p(t)dt$$



Akash Anand
MATH, IIT KANPUR

Examples

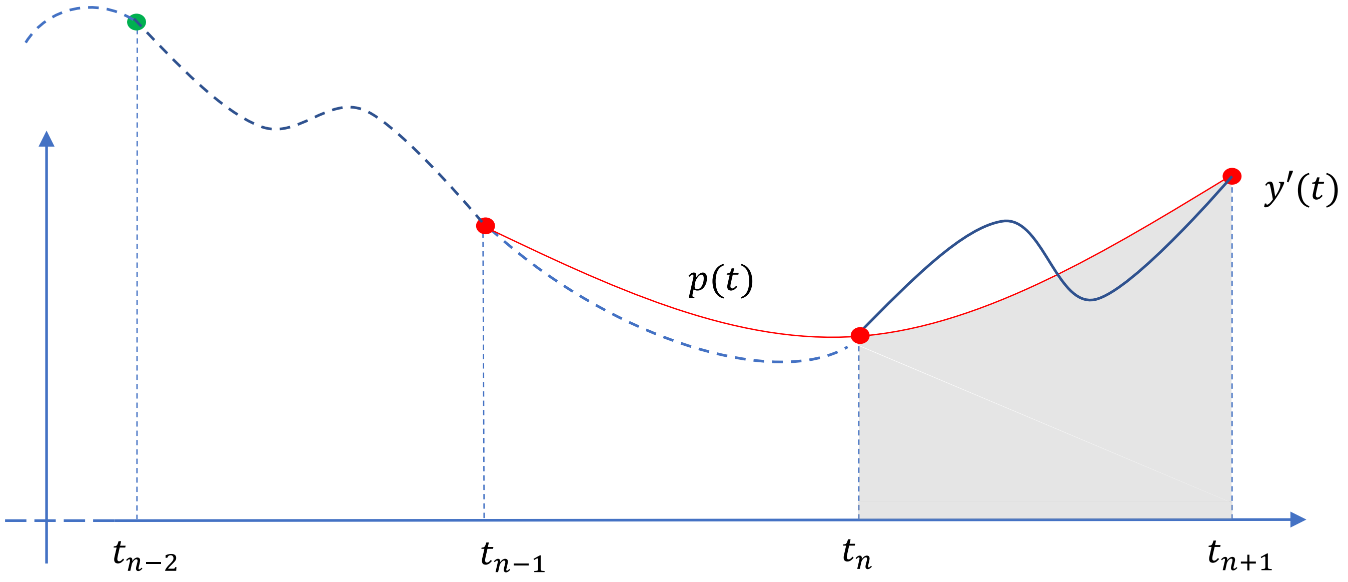
Adams Moulton methods -

-- 2-step method

$$p(t) = f_{n+1}\frac{(t-t_n)(t-t_{n-1})}{2h^2}$$

$$-f_n\frac{(t-t_{n+1})(t-t_{n-1})}{h^2}$$

$$+f_{n-1}\frac{(t-t_{n+1})(t-t_n)}{2h^2}$$



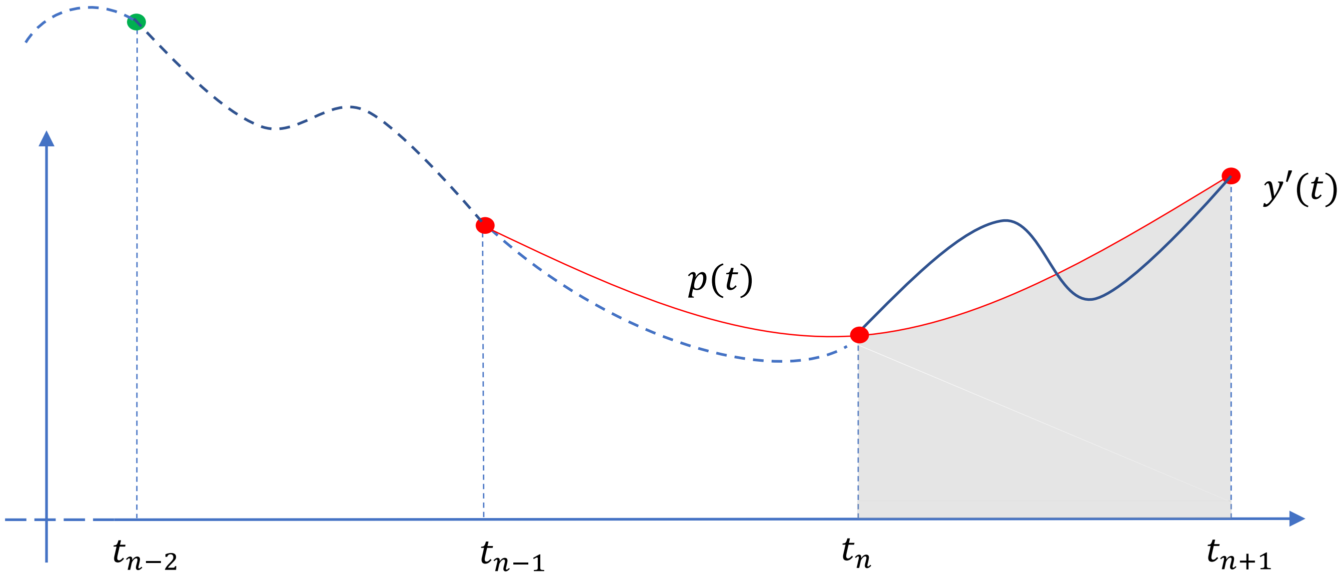
$$\int_{t_n}^{t_{n+1}} p(t)dt = f_{n+1}\left(\frac{5h}{12}\right) - f_n\left(-\frac{2h}{3}\right) + f_{n-1}\left(-\frac{h}{12}\right)$$

Examples

Adams Moulton methods -

-- 2-step method

$$p(t) = f_{n+1}\frac{(t-t_n)(t-t_{n-1})}{2h^2} - f_n\frac{(t-t_{n+1})(t-t_{n-1})}{h^2} + f_{n-1}\frac{(t-t_{n+1})(t-t_n)}{2h^2}$$



$$\int_{t_n}^{t_{n+1}} p(t)dt = f_{n+1}\left(\frac{5h}{12}\right) - f_n\left(-\frac{2h}{3}\right) + f_{n-1}\left(-\frac{h}{12}\right)$$

Thus,

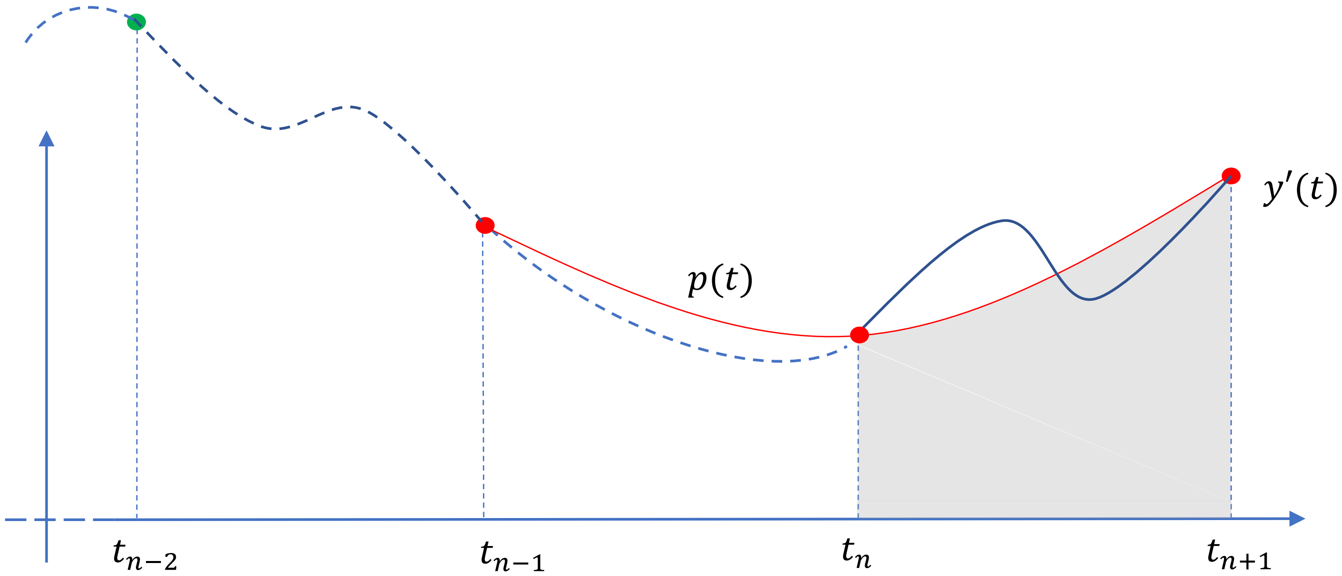
$$y_{n+1} = y_n + \frac{h}{12}(5f_{n+1} + 8f_n - f_{n-1})$$

Examples

Adams Moulton methods -

-- 2-step method

$$y_{n+1} = y_n + \frac{h}{12}(5f_{n+1} + 8f_n - f_{n-1})$$



-- General (k+1)-step method

$$p(t) = \sum_{j=-1}^{k} l_j^{(k)}(t)\, f_{n-j}, \qquad \textit{where} \qquad l_j^{(k)}(t) = \prod_{i=-1, i\neq j}^{k} (t - t_{n-i})/(t_{n-j} - t_{n-i})$$

Thus,

$$y_{n+1} = y_n + \sum_{j=-1}^{k} b_j f_{n-j}, \quad \textit{with} \qquad b_j = \int_{t_n}^{t_{n+1}} l_j^{(k)}(t)dt.$$

Module 2

Initial Value Problems

2.4 Implicit method

2.5 Stiffness

2.6 Linear Multistep Methods

- Consistency and Order

Akash Anand
MATH, IIT KANPUR

Consistency and Order

For the linear multistep method

$$\sum_{j=-1}^{k} a_j y_{n-j} = h \sum_{j=-1}^{k} b_j f(t_{n-j}, y_{n-j})$$

define the local error as

$$\ell_{n+1}(y,h) = h \sum_{j=-1}^{k} b_j y'(t_n - jh) - \sum_{j=-1}^{k} a_j y(t_n - jh)$$

for any $y \in C^1$, and $h > 0$.

Consistency and Order

For the linear multistep method

$$\sum_{j=-1}^{k} a_j y_{n-j} = h \sum_{j=-1}^{k} b_j f(t_{n-j}, y_{n-j})$$

define the local error as

$$\ell_{n+1}(y,h) = h \sum_{j=-1}^{k} b_j y'(t_n - jh) - \sum_{j=-1}^{k} a_j y(t_n - jh)$$

for any $y \in C^1$, *and* $h > 0$.

The linear multistep method is consistent if

$$\lim_{h\to 0} \max_{k\le n<N} \left\| \frac{\ell_{n+1}(y,h)}{h} \right\| = 0$$

for all $y \in C^1$.

Consistency and Order

For the linear multistep method

$$\sum_{j=-1}^{k} a_j y_{n-j} = h \sum_{j=-1}^{k} b_j f(t_{n-j}, y_{n-j})$$

define the local error as

$$\ell_{n+1}(y,h) = h \sum_{j=-1}^{k} b_j y'(t_n - jh) - \sum_{j=-1}^{k} a_j y(t_n - jh)$$

for any $y \in C^1$, and $h > 0$.

The linear multistep method is consistent if

$$\lim_{h\to 0} \max_{k \le n < N} \left\| \frac{\ell_{n+1}(y,h)}{h} \right\| = 0$$

for all $y \in C^1$.

The method has order p is for all $y \in C^{p+1}$ there exists constants $C, h_0 > 0$ such that

$$\max_{k \le n < N} \left\| \frac{\ell_{n+1}(y,h)}{h} \right\| \le C h^p$$

whenever $h < h_0$.

Akash Anand

MATH, IIT KANPUR

Note

It is not true that every method of order p converges with order p. It may not even converge at all!

Note

It is not true that every method of order p converges with order p. It may not even converge at all!

Theorem

A linear multistep is consistent if and only if

$$\sum_{j=-1}^{k} a_j = 0, \qquad \sum_{j=-1}^{k} j a_j + \sum_{j=-1}^{k} b_j = 0\,.$$

The method is of order p if and only if

$$\sum_{j=-1}^{k} (-j)^m a_j - m \sum_{j=-1}^{k} (-j)^{m-1} b_j = 0\,, \qquad m = 0,1,\dots,p.$$

Note

It is not true that every method of order p converges with order p. It may not even converge at all!

Theorem

A linear multistep is consistent if and only if

$$\sum_{j=-1}^{k} a_j = 0, \qquad \sum_{j=-1}^{k} j a_j + \sum_{j=-1}^{k} b_j = 0\,.$$

Consistency conditions

The method is of order p if and only if

$$\sum_{j=-1}^{k} (-j)^m a_j - m \sum_{j=-1}^{k} (-j)^{m-1} b_j = 0\,, \qquad m = 0,1,\ldots,p.$$

Order conditions

Note

It is not true that every method of order p converges with order p. It may not even converge at all!

Theorem

A linear multistep is consistent if and only if

$$\sum_{j=-1}^{k} a_j = 0, \qquad \sum_{j=-1}^{k} j a_j + \sum_{j=-1}^{k} b_j = 0\,.$$

Consistency conditions

The method is of order p if and only if

$$\sum_{j=-1}^{k} (-j)^m a_j - m \sum_{j=-1}^{k} (-j)^{m-1} b_j = 0\,, \qquad m = 0,1,\ldots,p.$$

Order conditions

Proof.

Note

It is not true that every method of order p converges with order p. It may not even converge at all!

Theorem

A linear multistep is consistent if and only if

$$\sum_{j=-1}^{k} a_j = 0, \qquad \sum_{j=-1}^{k} j a_j + \sum_{j=-1}^{k} b_j = 0\,.$$

Consistency conditions

The method is of order p if and only if

$$\sum_{j=-1}^{k} (-j)^m a_j - m \sum_{j=-1}^{k} (-j)^{m-1} b_j = 0\,, \qquad m = 0,1,\ldots,p.$$

Order conditions

Proof.

Using Taylor's theorem, we have $y(t_n - jh) = y(t_n) - jhy'(\xi_j)$ for some $\xi_j \in (t_n - kh, t_n + h), j = -1, \ldots, k$.

Note

It is not true that every method of order p converges with order p. It may not even converge at all!

Theorem

A linear multistep is consistent if and only if

$$\sum_{j=-1}^{k} a_j = 0, \qquad \sum_{j=-1}^{k} j a_j + \sum_{j=-1}^{k} b_j = 0\,.$$

Consistency conditions

The method is of order p if and only if

$$\sum_{j=-1}^{k} (-j)^m a_j - m \sum_{j=-1}^{k} (-j)^{m-1} b_j = 0\,, \qquad m = 0,1,\dots,p.$$

Order conditions

Proof.

Using Taylor's theorem, we have $y(t_n - jh) = y(t_n) - jhy'(\xi_j)$ for some $\xi_j \in (t_n - kh, t_n + h), j = -1, \dots, k$.

Note that $\xi_j \to t_n$ as $h \to 0$.

Note

It is not true that every method of order p converges with order p. It may not even converge at all!

Theorem

A linear multistep is consistent if and only if

$$\sum_{j=-1}^{k} a_j = 0, \qquad \sum_{j=-1}^{k} j a_j + \sum_{j=-1}^{k} b_j = 0\,.$$

Consistency conditions

The method is of order p if and only if

$$\sum_{j=-1}^{k} (-j)^m a_j - m \sum_{j=-1}^{k} (-j)^{m-1} b_j = 0\,, \qquad m = 0,1,\dots,p.$$

Order conditions

Proof.

Using Taylor's theorem, we have $y(t_n - jh) = y(t_n) - jhy'(\xi_j)$ for some $\xi_j \in (t_n - kh, t_n + h), j = -1,\dots,k$.

Note that $\xi_j \to t_n$ as $h \to 0$. Now,

$$\ell_{n+1}(y,h) = h \sum_{j=-1}^{k} b_j y'(t_n - jh) - \sum_{j=-1}^{k} a_j y(t_n - jh)$$

Note

It is not true that every method of order p converges with order p. It may not even converge at all!

Theorem

A linear multistep is consistent if and only if

$$\sum_{j=-1}^{k} a_j = 0, \qquad \sum_{j=-1}^{k} j a_j + \sum_{j=-1}^{k} b_j = 0.$$

Consistency conditions

The method is of order p if and only if

$$\sum_{j=-1}^{k} (-j)^m a_j - m \sum_{j=-1}^{k} (-j)^{m-1} b_j = 0, \qquad m = 0,1,\dots,p.$$

Order conditions

Proof.

Using Taylor's theorem, we have $y(t_n - jh) = y(t_n) - jhy'(\xi_j)$ for some $\xi_j \in (t_n - kh, t_n + h), j = -1,\dots,k$.

Note that $\xi_j \to t_n$ as $h \to 0$. Now,

$$\ell_{n+1}(y,h) = h \sum_{j=-1}^{k} b_j y'(t_n - jh) - \sum_{j=-1}^{k} a_j y(t_n - jh) = h \sum_{j=-1}^{k} b_j y'(t_n - jh) - \sum_{j=-1}^{k} a_j [y(t_n) - jhy'(\xi_j)]$$

Proof.

Using Taylor's theorem, we have $y(t_n - jh) = y(t_n) - jhy'(\xi_j)$ *for some* $\xi_j \in (t_n - kh, t_n + h), j = -1, \dots, k$.

Note that $\xi_j \to t_n$ *as* $h \to 0$. *Now,*

$$\ell_{n+1}(y,h) = h\sum_{j=-1}^{k} b_j y'(t_n - jh) - \sum_{j=-1}^{k} a_j y(t_n - jh) = h\sum_{j=-1}^{k} b_j y'(t_n - jh) - \sum_{j=-1}^{k} a_j [y(t_n) - jhy'(\xi_j)]$$

$$= -y(t_n)C_0 + hy'(t_n)C_1 + R,$$

where

$$C_0 = \sum_{j=-1}^{k} a_j,\ C_1 = \sum_{j=-1}^{k} ja_j + \sum_{j=-1}^{k} b_j\,,$$

$$R = h\sum_{j=-1}^{k} b_j[y'(t_n - jh) - y'(t_n)] + h\sum_{j=-1}^{k} ja_j[y'(\xi_j) - y'(t_n)]$$

Proof.

Using Taylor's theorem, we have $y(t_n - jh) = y(t_n) - jhy'(\xi_j)$ *for some* $\xi_j \in (t_n - kh, t_n + h), j = -1, \dots, k$.

Note that $\xi_j \to t_n$ *as* $h \to 0$. *Now,*

$$\ell_{n+1}(y,h) = h\sum_{j=-1}^{k} b_j y'(t_n - jh) - \sum_{j=-1}^{k} a_j y(t_n - jh) = h\sum_{j=-1}^{k} b_j y'(t_n - jh) - \sum_{j=-1}^{k} a_j [y(t_n) - jhy'(\xi_j)]$$

$$= -y(t_n)C_0 + hy'(t_n)C_1 + R,$$

where

$$C_0 = \sum_{j=-1}^{k} a_j,\ C_1 = \sum_{j=-1}^{k} ja_j + \sum_{j=-1}^{k} b_j\,,$$

$$R = h\sum_{j=-1}^{k} b_j[y'(t_n - jh) - y'(t_n)] + h\sum_{j=-1}^{k} ja_j[y'(\xi_j) - y'(t_n)]$$

By uniform continuity of y', *we see that* $R/h \to 0$ *as* $h \to 0$.

Proof.

Using Taylor's theorem, we have $y(t_n - jh) = y(t_n) - jhy'(\xi_j)$ for some $\xi_j \in (t_n - kh, t_n + h), j = -1, \dots, k$.

Note that $\xi_j \to t_n$ as $h \to 0$. Now,

$$\ell_{n+1}(y,h) = h\sum_{j=-1}^{k} b_j y'(t_n - jh) - \sum_{j=-1}^{k} a_j y(t_n - jh) = h\sum_{j=-1}^{k} b_j y'(t_n - jh) - \sum_{j=-1}^{k} a_j [y(t_n) - jhy'(\xi_j)]$$

$$= -y(t_n)C_0 + hy'(t_n)C_1 + R,$$

where

$$C_0 = \sum_{j=-1}^{k} a_j,\ C_1 = \sum_{j=-1}^{k} j a_j + \sum_{j=-1}^{k} b_j\,,$$

$$R = h\sum_{j=-1}^{k} b_j[y'(t_n - jh) - y'(t_n)] + h\sum_{j=-1}^{k} j a_j[y'(\xi_j) - y'(t_n)]$$

By uniform continuity of y', we see that $R/h \to 0$ as $h \to 0$. Therefore, $\ell_{n+1}(y,h)/h \to 0$ if and only if …

Proof.

Using Taylor's theorem, we have $y(t_n - jh) = y(t_n) - jhy'(\xi_j)$ *for some* $\xi_j \in (t_n - kh, t_n + h), j = -1, \dots, k$.

Note that $\xi_j \to t_n$ *as* $h \to 0$. *Now,*

$$\ell_{n+1}(y,h) = h\sum_{j=-1}^{k} b_j y'(t_n - jh) - \sum_{j=-1}^{k} a_j y(t_n - jh) = h\sum_{j=-1}^{k} b_j y'(t_n - jh) - \sum_{j=-1}^{k} a_j [y(t_n) - jhy'(\xi_j)]$$

$$= -y(t_n)C_0 + hy'(t_n)C_1 + R,$$

where

$$C_0 = \sum_{j=-1}^{k} a_j,\ C_1 = \sum_{j=-1}^{k} ja_j + \sum_{j=-1}^{k} b_j,$$

$$R = h\sum_{j=-1}^{k} b_j[y'(t_n - jh) - y'(t_n)] + h\sum_{j=-1}^{k} ja_j[y'(\xi_j) - y'(t_n)]$$

By uniform continuity of y', *we see that* $R/h \to 0$ *as* $h \to 0$. *Therefore,* $\ell_{n+1}(y,h)/h \to 0$ *if and only if* $C_0 = 0$ *and* $C_1 = 0$, *that is consistency conditions are satisfied.*

Proof. ...

Similarly, if $y \in C^{p+1}$ *we have*

$$y(t_n - jh) = \sum_{m=0}^{p} \frac{(-j)^m}{m!} h^m y^{(m)}(t_n) + \frac{(-j)^{p+1}}{(p+1)!} h^{p+1} y^{(p+1)}(\xi_j)$$

$$y'(t_n - jh) = \sum_{m=1}^{p} \frac{(-j)^{m-1}}{(m-1)!} h^{m-1} y^{(m)}(t_n) + \frac{(-j)^p}{p!} h^p y^{(p+1)}(\zeta_j)$$

for some $\xi_j, \zeta_j \in (t_n - kh, t_n + h), j = -1, \dots, k.$

Proof. ...

Similarly, if $y \in C^{p+1}$ we have

$$y(t_n - jh) = \sum_{m=0}^{p} \frac{(-j)^m}{m!} h^m y^{(m)}(t_n) + \frac{(-j)^{p+1}}{(p+1)!} h^{p+1} y^{(p+1)}(\xi_j)$$

$$y'(t_n - jh) = \sum_{m=1}^{p} \frac{(-j)^{m-1}}{(m-1)!} h^{m-1} y^{(m)}(t_n) + \frac{(-j)^p}{p!} h^p y^{(p+1)}(\zeta_j)$$

for some $\xi_j, \zeta_j \in (t_n - kh, t_n + h), j = -1, \dots, k$. This yields

$$\ell_{n+1}(y, h) = \sum_{m=0}^{p} h^m y^{(m)}(t_n) C_m + R$$

where

$$C_m = \frac{1}{m!}\left[m \sum_{j=-1}^{k} (-j)^{m-1} b_j - \sum_{j=-1}^{k} (-j)^m a_j\right], \qquad R = h^{p+1} \sum_{j=-1}^{k} \left[b_j \frac{(-j)^p}{p!} y^{(p+1)}(\zeta_j) - a_j \frac{(-j)^{p+1}}{(p+1)!} y^{(p+1)}(\xi_j)\right].$$

Proof. ...

Similarly, if $y \in C^{p+1}$ we have

$$y(t_n - jh) = \sum_{m=0}^{p} \frac{(-j)^m}{m!} h^m y^{(m)}(t_n) + \frac{(-j)^{p+1}}{(p+1)!} h^{p+1} y^{(p+1)}(\xi_j)$$

$$y'(t_n - jh) = \sum_{m=1}^{p} \frac{(-j)^{m-1}}{(m-1)!} h^{m-1} y^{(m)}(t_n) + \frac{(-j)^p}{p!} h^p y^{(p+1)}(\zeta_j)$$

for some $\xi_j, \zeta_j \in (t_n - kh, t_n + h), j = -1, \ldots, k$. This yields

$$\ell_{n+1}(y,h) = \sum_{m=0}^{p} h^m y^{(m)}(t_n) C_m + R$$

where

$$C_m = \frac{1}{m!}\left[m \sum_{j=-1}^{k} (-j)^{m-1} b_j - \sum_{j=-1}^{k} (-j)^m a_j\right], \qquad R = h^{p+1} \sum_{j=-1}^{k} \left[b_j \frac{(-j)^p}{p!} y^{(p+1)}(\zeta_j) - a_j \frac{(-j)^{p+1}}{(p+1)!} y^{(p+1)}(\xi_j)\right].$$

Since $R = O(h^{p+1})$, $\ell_{n+1}(y,h)/h = O(h^p)$ if and only if all the C_m vanish.

Theorem

A linear multistep is consistent if and only if

$$\sum_{j=-1}^{k} a_j = 0, \qquad \sum_{j=-1}^{k} j a_j + \sum_{j=-1}^{k} b_j = 0\,.$$

The method is of order p if and only if

$$\sum_{j=-1}^{k} (-j)^m a_j - m \sum_{j=-1}^{k} (-j)^{m-1} b_j = 0\,, \qquad m = 0,1,\ldots,p.$$

Remarks

1. *This theorem is an example of how a complicated analytic condition may sometimes reduce to a simple algebraic criterion.*

Theorem

A linear multistep is consistent if and only if

$$\sum_{j=-1}^{k} a_j = 0, \qquad \sum_{j=-1}^{k} j a_j + \sum_{j=-1}^{k} b_j = 0\,.$$

The method is of order p if and only if

$$\sum_{j=-1}^{k} (-j)^m a_j - m \sum_{j=-1}^{k} (-j)^{m-1} b_j = 0\,, \qquad m = 0,1,\dots,p.$$

Remarks

1. *This theorem is an example of how a complicated analytic condition may sometimes reduce to a simple algebraic criterion.*
2. *Such algebraic criteria for multistep methods can be expressed in terms of characteristic polynomials of the method:*

$$\rho(z) = \sum_{j=-1}^{k} a_j z^{k-j}\,, \qquad \sigma(z) = \sum_{j=-1}^{k} b_j z^{k-j}\,.$$

For example, the consistency conditions are $\rho(1) = 0$ and $\rho'(1) = \sigma(1)$.

Theorem

A linear multistep is consistent if and only if

$$\sum_{j=-1}^{k} a_j = 0, \qquad \sum_{j=-1}^{k} j a_j + \sum_{j=-1}^{k} b_j = 0\,.$$

The method is of order p if and only if

$$\sum_{j=-1}^{k} (-j)^m a_j - m \sum_{j=-1}^{k} (-j)^{m-1} b_j = 0\,, \qquad m = 0,1,\dots,p.$$

Example

Construct the 2-step method of highest order!

Initial Value Problems: Linear Multistep Methods

Akash Anand
MATH, IIT KANPUR

Theorem

A linear multistep is consistent if and only if

$$\sum_{j=-1}^{k} a_j = 0, \qquad \sum_{j=-1}^{k} j a_j + \sum_{j=-1}^{k} b_j = 0\,.$$

The method is of order p if and only if

$$\sum_{j=-1}^{k} (-j)^m a_j - m \sum_{j=-1}^{k} (-j)^{m-1} b_j = 0\,, \qquad m = 0,1,\dots,p.$$

Example

Construct the 2-step method of highest order!

For a 2-step method, there are 5 undetermined coefficients: $a_0, a_1, b_{-1}, b_0, b_1$.

Theorem

A linear multistep is consistent if and only if

$$\sum_{j=-1}^{k} a_j = 0, \qquad \sum_{j=-1}^{k} j a_j + \sum_{j=-1}^{k} b_j = 0\,.$$

The method is of order p if and only if

$$\sum_{j=-1}^{k} (-j)^m a_j - m \sum_{j=-1}^{k} (-j)^{m-1} b_j = 0\,, \qquad m = 0,1,\dots,p.$$

Example

Construct the 2-step method of highest order!

For a 2-step method, there are 5 undetermined coefficients: $a_0, a_1, b_{-1}, b_0, b_1$. The first five order conditions are

$$1 + a_0 + a_1 = 0,$$

Theorem

A linear multistep is consistent if and only if

$$\sum_{j=-1}^{k} a_j = 0, \qquad \sum_{j=-1}^{k} j a_j + \sum_{j=-1}^{k} b_j = 0\,.$$

The method is of order p if and only if

$$\sum_{j=-1}^{k} (-j)^m a_j - m \sum_{j=-1}^{k} (-j)^{m-1} b_j = 0\,, \qquad m = 0,1,\dots,p.$$

Example

Construct the 2-step method of highest order!

For a 2-step method, there are 5 undetermined coefficients: $a_0, a_1, b_{-1}, b_0, b_1$. The first five order conditions are

$$1 + a_0 + a_1 = 0, \quad 1 - a_1 - b_{-1} - b_0 - b_1 = 0,$$

see a0 , for j=0 , m=0

Initial Value Problems: Linear Multistep Methods

Theorem

A linear multistep is consistent if and only if

$$\sum_{j=-1}^{k} a_j = 0, \qquad \sum_{j=-1}^{k} j a_j + \sum_{j=-1}^{k} b_j = 0\,.$$

The method is of order p *if and only if*

$$\sum_{j=-1}^{k} (-j)^m a_j - m \sum_{j=-1}^{k} (-j)^{m-1} b_j = 0\,, \qquad m = 0,1,\dots,p.$$

Example

Construct the 2-step method of highest order!

For a 2-step method, there are 5 undetermined coefficients: $a_0, a_1, b_{-1}, b_0, b_1$. *The first five order conditions are*

$$1 + a_0 + a_1 = 0, \quad 1 - a_1 - b_{-1} - b_0 - b_1 = 0, \quad 1 + a_1 - 2b_{-1} + 2b_1 = 0,$$

Theorem

A linear multistep is consistent if and only if

$$\sum_{j=-1}^{k} a_j = 0, \qquad \sum_{j=-1}^{k} j a_j + \sum_{j=-1}^{k} b_j = 0 .$$

The method is of order p *if and only if*

$$\sum_{j=-1}^{k} (-j)^m a_j - m \sum_{j=-1}^{k} (-j)^{m-1} b_j = 0 , \qquad m = 0,1,\ldots,p.$$

Example

Construct the 2-step method of highest order!

For a 2-step method, there are 5 undetermined coefficients: $a_0, a_1, b_{-1}, b_0, b_1$*. The first five order conditions are*

$$1 + a_0 + a_1 = 0, \quad 1 - a_1 - b_{-1} - b_0 - b_1 = 0, \quad 1 + a_1 - 2b_{-1} + 2b_1 = 0,$$
$$1 - a_1 - 3b_{-1} - 3b_1 = 0,$$

Initial Value Problems: Linear Multistep Methods

Akash Anand
MATH, IIT KANPUR

Theorem

A linear multistep is consistent if and only if

$$\sum_{j=-1}^{k} a_j = 0, \qquad \sum_{j=-1}^{k} j a_j + \sum_{j=-1}^{k} b_j = 0\,.$$

The method is of order p *if and only if*

$$\sum_{j=-1}^{k} (-j)^m a_j - m \sum_{j=-1}^{k} (-j)^{m-1} b_j = 0\,, \qquad m = 0,1,\dots,p.$$

Example

Construct the 2-step method of highest order!

For a 2-step method, there are 5 undetermined coefficients: $a_0, a_1, b_{-1}, b_0, b_1$. *The first five order conditions are*

$$1 + a_0 + a_1 = 0, \quad 1 - a_1 - b_{-1} - b_0 - b_1 = 0, \quad 1 + a_1 - 2b_{-1} + 2b_1 = 0,$$
$$1 - a_1 - 3b_{-1} - 3b_1 = 0, \qquad 1 + a_1 - 4b_{-1} + 4b_1 = 0.$$

Theorem

A linear multistep is consistent if and only if

$$\sum_{j=-1}^{k} a_j = 0, \qquad \sum_{j=-1}^{k} j a_j + \sum_{j=-1}^{k} b_j = 0\,.$$

The method is of order p *if and only if*

$$\sum_{j=-1}^{k} (-j)^m a_j - m \sum_{j=-1}^{k} (-j)^{m-1} b_j = 0\,, \qquad m = 0,1,\ldots,p.$$

Example

Construct the 2-step method of highest order!

For a 2-step method, there are 5 undetermined coefficients: $a_0, a_1, b_{-1}, b_0, b_1$*. The first five order conditions are*

$$1 + a_0 + a_1 = 0, \quad 1 - a_1 - b_{-1} - b_0 - b_1 = 0, \quad 1 + a_1 - 2b_{-1} + 2b_1 = 0,$$
$$1 - a_1 - 3b_{-1} - 3b_1 = 0, \qquad 1 + a_1 - 4b_{-1} + 4b_1 = 0.$$

This system of linear equation has a unique solution

$$a_0 = 0, a_1 = -1, b_{-1} = \frac{1}{3}, b_0 = \frac{4}{3}, b_1 = \frac{1}{3}.$$

Theorem

A linear multistep is consistent if and only if

$$\sum_{j=-1}^{k} a_j = 0, \qquad \sum_{j=-1}^{k} j a_j + \sum_{j=-1}^{k} b_j = 0\,.$$

The method is of order p *if and only if*

$$\sum_{j=-1}^{k} (-j)^m a_j - m \sum_{j=-1}^{k} (-j)^{m-1} b_j = 0\,, \qquad m = 0,1,\ldots,p.$$

Example

Construct the 2-step method of highest order!

For a 2-step method, there are 5 undetermined coefficients: $a_0, a_1, b_{-1}, b_0, b_1$. *The first five order conditions are*

$$1 + a_0 + a_1 = 0, \quad 1 - a_1 - b_{-1} - b_0 - b_1 = 0, \quad 1 + a_1 - 2b_{-1} + 2b_1 = 0,$$
$$1 - a_1 - 3b_{-1} - 3b_1 = 0, \qquad 1 + a_1 - 4b_{-1} + 4b_1 = 0.$$

This system of linear equation has a unique solution

$$a_0 = 0, a_1 = -1, b_{-1} = \frac{1}{3}, b_0 = \frac{4}{3}, b_1 = \frac{1}{3}.$$

This scheme is known as Milne-Simpson method and it is the unique fourth order 2-step method.

Theorem

A linear multistep is consistent if and only if

$$\sum_{j=-1}^{k} a_j = 0, \qquad \sum_{j=-1}^{k} j a_j + \sum_{j=-1}^{k} b_j = 0\,.$$

The method is of order p if and only if

$$\sum_{j=-1}^{k} (-j)^m a_j - m \sum_{j=-1}^{k} (-j)^{m-1} b_j = 0\,, \qquad m = 0,1,\ldots,p.$$

Example

Construct the explicit 2-step method of highest order!

Initial Value Problems: Linear Multistep Methods

Akash Anand
MATH, IIT KANPUR

Theorem

A linear multistep is consistent if and only if

$$\sum_{j=-1}^{k} a_j = 0, \qquad \sum_{j=-1}^{k} j a_j + \sum_{j=-1}^{k} b_j = 0\,.$$

The method is of order p if and only if

$$\sum_{j=-1}^{k} (-j)^m a_j - m \sum_{j=-1}^{k} (-j)^{m-1} b_j = 0\,, \qquad m = 0,1,\ldots,p.$$

Example

Construct the explicit 2-step method of highest order!

There are 4 undetermined coefficients: a_0, a_1, b_0, b_1.

Theorem

A linear multistep is consistent if and only if

$$\sum_{j=-1}^{k} a_j = 0, \qquad \sum_{j=-1}^{k} j a_j + \sum_{j=-1}^{k} b_j = 0\,.$$

The method is of order p *if and only if*

$$\sum_{j=-1}^{k} (-j)^m a_j - m \sum_{j=-1}^{k} (-j)^{m-1} b_j = 0\,, \qquad m = 0,1,\ldots,p.$$

Example

Construct the explicit 2-step method of highest order!

There are 4 undetermined coefficients: a_0, a_1, b_0, b_1. *The first four order conditions are*

$$1 + a_0 + a_1 = 0, \quad 1 - a_1 - b_0 - b_1 = 0,$$
$$1 + a_1 + 2b_1 = 0, \qquad 1 - a_1 - 3b_1 = 0,$$

Theorem

A linear multistep is consistent if and only if

$$\sum_{j=-1}^{k} a_j = 0, \qquad \sum_{j=-1}^{k} j a_j + \sum_{j=-1}^{k} b_j = 0\,.$$

The method is of order p *if and only if*

$$\sum_{j=-1}^{k} (-j)^m a_j - m \sum_{j=-1}^{k} (-j)^{m-1} b_j = 0\,, \qquad m = 0,1,\ldots,p.$$

Example

Construct the explicit 2-step method of highest order!

There are 4 undetermined coefficients: a_0, a_1, b_0, b_1. *The first four order conditions are*

$$1 + a_0 + a_1 = 0, \quad 1 - a_1 - b_0 - b_1 = 0,$$
$$1 + a_1 + 2b_1 = 0, \qquad 1 - a_1 - 3b_1 = 0,$$

which gives

$$a_0 = 4, a_1 = -5, b_0 = 4, b_1 = 2.$$

Theorem

A linear multistep is consistent if and only if

$$\sum_{j=-1}^{k} a_j = 0, \qquad \sum_{j=-1}^{k} j a_j + \sum_{j=-1}^{k} b_j = 0\,.$$

The method is of order p *if and only if*

$$\sum_{j=-1}^{k} (-j)^m a_j - m \sum_{j=-1}^{k} (-j)^{m-1} b_j = 0\,, \qquad m = 0,1,\ldots,p.$$

Example

Construct the explicit 2-step method of highest order!

There are 4 undetermined coefficients: a_0, a_1, b_0, b_1. *The first four order conditions are*

$$1 + a_0 + a_1 = 0, \quad 1 - a_1 - b_0 - b_1 = 0,$$
$$1 + a_1 + 2b_1 = 0, \qquad 1 - a_1 - 3b_1 = 0,$$

which gives

$$a_0 = 4, a_1 = -5, b_0 = 4, b_1 = 2.$$

Thus, the unique explicit 2-step method of order 3 is $\quad y_{n+1} + 4y_n - 5y_{n-1} = h(4f_n + 2f_{n-1})$.

Example

Construct the explicit 2-step method of highest order!

There are 4 undetermined coefficients: a_0, a_1, b_0, b_1. *The first four order conditions are*

$$1 + a_0 + a_1 = 0, \quad 1 - a_1 - b_0 - b_1 = 0,$$
$$1 + a_1 + 2b_1 = 0, \qquad 1 - a_1 - 3b_1 = 0,$$

which gives

$$a_0 = 4, a_1 = -5, b_0 = 4, b_1 = 2.$$

Thus, the unique explicit 2-step method of order 3 is $\quad y_{n+1} + 4y_n - 5y_{n-1} = h(4f_n + 2f_{n-1}).$

Does this method converge?

Lesson 2

Initial Value Problems

2.4 Implicit method

2.5 Stiffness

2.6 Linear Multistep Methods

- Convergence

Akash Anand
MATH, IIT KANPUR

Example

Construct the explicit 2-step method of highest order!

There are 4 undetermined coefficients: a_0, a_1, b_0, b_1. *The first four order conditions are*

$$1 + a_0 + a_1 = 0, \quad 1 - a_1 - b_0 - b_1 = 0,$$
$$1 + a_1 + 2b_1 = 0, \qquad 1 - a_1 - 3b_1 = 0,$$

which gives

$$a_0 = 4, a_1 = -5, b_0 = 4, b_1 = 2.$$

Thus, the unique explicit 2-step method of order 3 is $\quad y_{n+1} + 4y_n - 5y_{n-1} = h(4f_n + 2f_{n-1})$.

Does this method converge?

Convergence

A linear multistep method is convergent if whenever the initial values y_n *are chosen such that* $\max_{0 \le n \le k} \|e_n\| \to 0$, *as* $h \to 0$, *then* $\max_{0 \le n \le N} \|e_n\| \to 0$.

Convergence

A linear multistep method is convergent if whenever the initial values y_n are chosen such that $\max_{0\le n\le k} \|e_n\| \to 0$, as $h \to 0$, then $\max_{0\le n\le N} \|e_n\| \to 0$.

Example

Does

$$y_{n+1} + 4y_n - 5y_{n-1} = h(4f_n + 2f_{n-1}).$$

this method converge?

Convergence

A linear multistep method is convergent if whenever the initial values y_n are chosen such that $\max_{0\le n\le k}\|e_n\| \to 0$, as $h \to 0$, then $\max_{0\le n\le N}\|e_n\| \to 0$.

Example

Does

$$y_{n+1} + 4y_n - 5y_{n-1} = h(4f_n + 2f_{n-1}).$$

this method converge?

Apply the method to the IVP $y' = 0,\ y(0) = 0.$

Convergence

A linear multistep method is convergent if whenever the initial values y_n *are chosen such that* $\max_{0\le n\le k}\|e_n\| \to 0$, *as* $h \to 0$, *then* $\max_{0\le n\le N}\|e_n\| \to 0$.

Example

Does

$$y_{n+1} + 4y_n - 5y_{n-1} = h(4f_n + 2f_{n-1}).$$

this method converge?

Apply the method to the IVP $y' = 0,\ y(0) = 0$. *We have*

$$y_{n+1} = -4y_n + 5y_{n-1}.$$

Convergence

A linear multistep method is convergent if whenever the initial values y_n *are chosen such that* $\max_{0\le n\le k} \|e_n\| \to 0$, *as* $h \to 0$, *then* $\max_{0\le n\le N} \|e_n\| \to 0$.

Example

Does

$$y_{n+1} + 4y_n - 5y_{n-1} = h(4f_n + 2f_{n-1}).$$

this method converge?

Apply the method to the IVP $y' = 0,\ y(0) = 0$. *We have*

$$y_{n+1} = -4y_n + 5y_{n-1}.$$

To solve for y_n, *we need to solve the difference equation.*

Convergence

A linear multistep method is convergent if whenever the initial values y_n are chosen such that $\max_{0\le n\le k} \|e_n\| \to 0$, as $h \to 0$, then $\max_{0\le n\le N} \|e_n\| \to 0$.

Example

Does

$$y_{n+1} + 4y_n - 5y_{n-1} = h(4f_n + 2f_{n-1}).$$

this method converge?

Apply the method to the IVP $y' = 0,\ y(0) = 0$. We have

$$y_{n+1} = -4y_n + 5y_{n-1}.$$

To solve for y_n, we need to solve the difference equation. Toward this, introduce

$$v_n = \begin{bmatrix} y_{n+1} \\ y_n \end{bmatrix}$$

to obtain the equation $v_{n+1} = Av_n$, where

$$A = \begin{bmatrix} -4 & 5 \\ 1 & 0 \end{bmatrix}$$

Convergence

A linear multistep method is convergent if whenever the initial values y_n *are chosen such that* $\max_{0\le n\le k}\|e_n\| \to 0$, *as* $h \to 0$, *then* $\max_{0\le n\le N}\|e_n\| \to 0.$

Example

Does

$$y_{n+1} + 4y_n - 5y_{n-1} = h(4f_n + 2f_{n-1}).$$

this method converge?

Apply the method to the IVP $y' = 0,\ y(0) = 0$. *We have*

$$y_{n+1} = -4y_n + 5y_{n-1}.$$

To solve for y_n, *we need to solve the difference equation. Toward this, introduce*

$$v_n = \begin{bmatrix} y_{n+1} \\ y_n \end{bmatrix}$$

to obtain the equation $v_{n+1} = Av_n$, *where*

$$A = \begin{bmatrix} -4 & 5 \\ 1 & 0 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1 & 5 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -5 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 1 & -1 \end{bmatrix}.$$

Convergence

A linear multistep method is convergent if whenever the initial values y_n are chosen such that $\max\limits_{0\le n\le k}\|e_n\| \to 0$, as $h \to 0$, then $\max\limits_{0\le n\le N}\|e_n\| \to 0$.

Example

Does

$$y_{n+1} + 4y_n - 5y_{n-1} = h(4f_n + 2f_{n-1}).$$

this method converge?

Apply the method to the IVP $y' = 0,\ y(0) = 0$. *We have*

$$y_{n+1} = -4y_n + 5y_{n-1}.$$

To solve for y_n, we need to solve the difference equation. Toward this, introduce

$$v_n = \begin{bmatrix} y_{n+1} \\ y_n \end{bmatrix}$$

to obtain the equation $v_{n+1} = Av_n$, where

$$A = \begin{bmatrix} -4 & 5 \\ 1 & 0 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1 & 5 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -5 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 1 & -1 \end{bmatrix}.$$

Using, $v_n = A^n v_0$ and $y_0 = 0$, we get

$$y_n = (1 - (-5)^n)y_1/6$$

Convergence

A linear multistep method is convergent if whenever the initial values y_n *are chosen such that* $\max_{0\le n\le k} \|e_n\| \to 0$, *as* $h \to 0$, *then* $\max_{0\le n\le N} \|e_n\| \to 0.$

Example

Does

$$y_{n+1} + 4y_n - 5y_{n-1} = h(4f_n + 2f_{n-1}).$$

this method converge?

Apply the method to the IVP $y' = 0,\ y(0) = 0$. *We have*

$$y_{n+1} = -4y_n + 5y_{n-1}.$$

Solving the difference equation yields

$$y_n = (1 - (-5)^n)y_1/6.$$

Convergence

A linear multistep method is convergent if whenever the initial values y_n are chosen such that $\max_{0\le n\le k}\|e_n\| \to 0$, *as* $h \to 0$, *then* $\max_{0\le n\le N}\|e_n\| \to 0.$

Example

Does

$$y_{n+1} + 4y_n - 5y_{n-1} = h(4f_n + 2f_{n-1}).$$

this method converge?

Apply the method to the IVP $y' = 0,\ y(0) = 0$. *We have*

$$y_{n+1} = -4y_n + 5y_{n-1}.$$

Solving the difference equation yields

$$y_n = (1 - (-5)^n)y_1/6.$$

If $h = 1/N$, *then*

$$e_N = y_N - y(1)$$

Convergence

A linear multistep method is convergent if whenever the initial values y_n are chosen such that $\max_{0 \le n \le k} \|e_n\| \to 0$, *as* $h \to 0$, *then* $\max_{0 \le n \le N} \|e_n\| \to 0$.

Example

Does

$$y_{n+1} + 4y_n - 5y_{n-1} = h(4f_n + 2f_{n-1}).$$

this method converge?

Apply the method to the IVP $y' = 0$, $y(0) = 0$. *We have*

$$y_{n+1} = -4y_n + 5y_{n-1}.$$

Solving the difference equation yields

$$y_n = (1 - (-5)^n)y_1/6.$$

If $h = 1/N$, *then*

$$e_N = y_N - y(1) = (1 - (-5)^N)y_1/6$$

Convergence

A linear multistep method is convergent if whenever the initial values y_n *are chosen such that* $\max_{0\le n\le k}\|e_n\| \to 0$, *as* $h \to 0$, *then* $\max_{0\le n\le N}\|e_n\| \to 0.$

Example

Does

$$y_{n+1} + 4y_n - 5y_{n-1} = h(4f_n + 2f_{n-1}).$$

this method converge?

Apply the method to the IVP $y' = 0,\ y(0) = 0$. *We have*

$$y_{n+1} = -4y_n + 5y_{n-1}.$$

Solving the difference equation yields

$$y_n = (1 - (-5)^n)y_1/6.$$

If $h = 1/N$, *then*

$$e_N = y_N - y(1) = (1 - (-5)^N)y_1/6 = \left(1 - (-5)^{1/h}\right)y_1/6$$

Convergence

A linear multistep method is convergent if whenever the initial values y_n are chosen such that $\max_{0\le n\le k}\|e_n\| \to 0$, as $h \to 0$, then $\max_{0\le n\le N}\|e_n\| \to 0$.

Example

Does

$$y_{n+1} + 4y_n - 5y_{n-1} = h(4f_n + 2f_{n-1}).$$

this method converge?

Apply the method to the IVP $y' = 0,\ y(0) = 0$. We have

$$y_{n+1} = -4y_n + 5y_{n-1}.$$

Solving the difference equation yields

$$y_n = (1 - (-5)^n)y_1/6.$$

If $h = 1/N$, then

$$e_N = y_N - y(1) = (1 - (-5)^N)y_1/6 = \left(1 - (-5)^{1/h}\right)y_1/6$$

For $y_1 = h$, we have $e_1 = h$. We see that even as $e_1 \to 0$ as $h \to 0$, $e_N \nrightarrow 0$.

Convergence

A linear multistep method is convergent if whenever the initial values y_n are chosen such that $\max_{0\le n\le k}\|e_n\| \to 0$, as $h \to 0$, then $\max_{0\le n\le N}\|e_n\| \to 0$.

Example

Does

$$y_{n+1} + 4y_n - 5y_{n-1} = h(4f_n + 2f_{n-1}).$$

this method converge?

Apply the method to the IVP $y' = 0,\ y(0) = 0$. We have

$$y_{n+1} = -4y_n + 5y_{n-1}.$$

Solving the difference equation yields

$$y_n = (1 - (-5)^n)y_1/6.$$

If $h = 1/N$, then

$$e_N = y_N - y(1) = (1 - (-5)^N)y_1/6 = \left(1 - (-5)^{1/h}\right)y_1/6$$

For $y_1 = h$, we have $e_1 = h$. We see that even as $e_1 \to 0$ as $h \to 0$, $e_N \nrightarrow 0$. Thus, the method does not converge.

Convergence

A linear multistep method is convergent if whenever the initial values y_n are chosen such that $\max\limits_{0\le n\le k}\|e_n\| \to 0$, *as* $h \to 0$, *then* $\max\limits_{0\le n\le N}\|e_n\| \to 0$.

Example

Does

$$y_{n+1} + 4y_n - 5y_{n-1} = h(4f_n + 2f_{n-1}).$$

this method converge?

Apply the method to the IVP $y' = 0$, $y(0) = 0$. *We have*

$$y_{n+1} = -4y_n + 5y_{n-1}.$$

Solving the difference equation yields

$$y_n = (1 - (-5)^n)y_1/6.$$

If $h = 1/N$, *then*

$$e_N = y_N - y(1) = (1 - (-5)^N)y_1/6 = \left(1 - (-5)^{1/h}\right)y_1/6$$

For $y_1 = h$, *we have* $e_1 = h$. *We see that even as* $e_1 \to 0$ *as* $h \to 0$, $e_N \nrightarrow 0$. *Thus, the method does not converge.*

Note that if we take exact starting values $y_0 = y_1 = 0$, *then* $y_n = 0$ *for all* n.

Convergence

A linear multistep method is convergent if whenever the initial values y_n are chosen such that $\max_{0\le n\le k}\|e_n\| \to 0$, as $h \to 0$, then $\max_{0\le n\le N}\|e_n\| \to 0$.

Example

Does

$$y_{n+1} + 4y_n - 5y_{n-1} = h(4f_n + 2f_{n-1}).$$

this method converge?

Apply the method to the IVP $y' = 0,\; y(0) = 0$. We have

$$y_{n+1} = -4y_n + 5y_{n-1}.$$

Solving the difference equation yields

$$y_n = (1 - (-5)^n)y_1/6.$$

If $h = 1/N$, then

$$e_N = y_N - y(1) = (1 - (-5)^N)y_1/6 = \left(1 - (-5)^{1/h}\right)y_1/6$$

For $y_1 = h$, we have $e_1 = h$. We see that even as $e_1 \to 0$ as $h \to 0$, $e_N \nrightarrow 0$. Thus, the method does not converge.

Note that if we take exact starting values $y_0 = y_1 = 0$, then $y_n = 0$ for all n. Thus, a perturbation of size ε in the starting values leads to a difference of size roughly $5^{1/h}\varepsilon$ in the discrete solution.

Convergence

A linear multistep method is convergent if whenever the initial values y_n are chosen such that $\max_{0 \le n \le k} \|e_n\| \to 0$, as $h \to 0$, then $\max_{0 \le n \le N} \|e_n\| \to 0$.

Example

Does

$$y_{n+1} + 4y_n - 5y_{n-1} = h(4f_n + 2f_{n-1}).$$

this method converge?

Apply the method to the IVP $y' = 0$, $y(0) = 0$. We have

$$y_{n+1} = -4y_n + 5y_{n-1}.$$

Solving the difference equation yields

$$y_n = (1 - (-5)^n)y_1/6.$$

If $h = 1/N$, then

$$e_N = y_N - y(1) = (1 - (-5)^N)y_1/6 = \left(1 - (-5)^{1/h}\right)y_1/6$$

For $y_1 = h$, we have $e_1 = h$. We see that even as $e_1 \to 0$ as $h \to 0$, $e_N \nrightarrow 0$. Thus, the method does not converge.

Note that if we take exact starting values $y_0 = y_1 = 0$, then $y_n = 0$ for all n. Thus, a perturbation of size ε in the starting values leads to a difference of size roughly $5^{1/h}\varepsilon$ in the discrete solution. The method is, therefore, not stable.

Module 2

Initial Value Problems

2.4 Implicit method

2.5 Stiffness

2.6 Linear Multistep Methods

- Stability

Akash Anand
MATH, IIT KANPUR

Stability

A linear $k+1$ step method is stable if for any initial value problem with Lipschitz continuous f and of $\varepsilon > 0$, there exists $\delta, h_0 > 0$ such that if $h \le h_0$ and two choices of starting values y_j and $\hat{y}_j$ are chosen satisfying

$$\max_{0 \le j \le k} \left\| y_j - \hat{y}_j \right\| \le \delta,$$

then the corresponding approximate solutions satisfy

$$\max_{0 \le j \le N} \left\| y_j - \hat{y}_j \right\| \le \varepsilon.$$

Stability

A linear $k+1$ step method is stable if for any initial value problem with Lipschitz continuous f and of $\varepsilon > 0$, there exists $\delta, h_0 > 0$ such that if $h \le h_0$ and two choices of starting values y_j and $\hat{y}_j$ are chosen satisfying

$$\max_{0 \le j \le k} \|y_j - \hat{y}_j\| \le \delta,$$

then the corresponding approximate solutions satisfy

$$\max_{0 \le j \le N} \|y_j - \hat{y}_j\| \le \varepsilon.$$

If $y' = 0$, then the general linear multistep method becomes

$$y_{n+1} + \sum_{j=0}^{k} a_j y_{n-j} = 0, \qquad n = k, k+1, \dots$$

This is an example of a homogeneous linear difference equation of order $k+1$ with constant coefficients.

Stability

A linear $k+1$ step method is stable if for any initial value problem with Lipschitz continuous f and of $\varepsilon > 0$, there exists $\delta, h_0 > 0$ such that if $h \le h_0$ and two choices of starting values y_j and $\hat{y}_j$ are chosen satisfying

$$\max_{0\le j\le k} \|y_j - \hat{y}_j\| \le \delta,$$

then the corresponding approximate solutions satisfy

$$\max_{0\le j\le N} \|y_j - \hat{y}_j\| \le \varepsilon.$$

If $y' = 0$, then the general linear multistep method becomes

$$y_{n+1} + \sum_{j=0}^{k} a_j y_{n-j} = 0, \qquad n = k, k+1, \ldots$$

This is an example of a homogeneous linear difference equation of order $k+1$ with constant coefficients.

If a method is stable, it should, in particular, be stable while computing the zero solution of the initial value problem $y' = 0, y(0) = 0$. To investigate this, we need to solve the homogeneous linear difference equation of order $k+1$.

Stability

A linear $k+1$ step method is stable if for any initial value problem with Lipschitz continuous f and of $\varepsilon > 0$, there exists $\delta, h_0 > 0$ such that if $h \leq h_0$ and two choices of starting values y_j and $\hat{y}_j$ are chosen satisfying

$$\max_{0\leq j\leq k} \left\| y_j - \hat{y}_j \right\| \leq \delta,$$

then the corresponding approximate solutions satisfy

$$\max_{0\leq j\leq N} \left\| y_j - \hat{y}_j \right\| \leq \varepsilon.$$

If $y' = 0$, then the general linear multistep method becomes

$$y_{n+1} + \sum_{j=0}^{k} a_j y_{n-j} = 0, \qquad n = k, k+1, \ldots$$

This is an example of a homogeneous linear difference equation of order $k+1$ with constant coefficients.

If a method is stable, it should, in particular, be stable while computing the zero solution of the initial value problem $y' = 0, y(0) = 0$. To investigate this, we need to solve the homogeneous linear difference equation of order $k+1$.

To find the general solution, we first try for a solution of the form $(\lambda^n)_{n=0}^{\infty}$. Substituting this in the difference equation, we see that it is a solution if and only if λ is a root of the characteristic polynomial

$$\rho(t) = t^{k+1} + \sum_{j=0}^{k} a_j t^{k-j}.$$

If $y' = 0$, then the general linear multistep method becomes

$$y_{n+1} + \sum_{j=0}^{k} a_j y_{n-j} = 0, \qquad n = k, k+1, \ldots$$

This is an example of a homogeneous linear difference equation of order $k+1$ with constant coefficients.

If a method is stable, it should, in particular, be stable while computing the zero solution of the initial value problem $y' = 0, y(0) = 0$. To investigate this, we need to solve the homogeneous linear difference equation of order $k+1$.

To find the general solution, we first try for a solution of the form $(\lambda^n)_{n=0}^{\infty}$. Substituting this in the difference equation, we see that it is a solution if and only if λ is a root of the characteristic polynomial

$$\rho(t) = t^{k+1} + \sum_{j=0}^{k} a_j t^{k-j}.$$

If there are $k+1$ distinct roots $\lambda_i, i = 0, \ldots, k$, then we have a full basis of $k+1$ linearly independent solutions (note that the solutions form a vector space).

If $y' = 0$, then the general linear multistep method becomes

$$y_{n+1} + \sum_{j=0}^{k} a_j y_{n-j} = 0, \qquad n = k, k+1, \ldots$$

This is an example of a homogeneous linear difference equation of order $k+1$ with constant coefficients.

If a method is stable, it should, in particular, be stable while computing the zero solution of the initial value problem $y' = 0, y(0) = 0$. To investigate this, we need to solve the homogeneous linear difference equation of order $k+1$.

To find the general solution, we first try for a solution of the form $(\lambda^n)_{n=0}^{\infty}$. Substituting this in the difference equation, we see that it is a solution if and only if λ is a root of the characteristic polynomial

$$\rho(t) = t^{k+1} + \sum_{j=0}^{k} a_j t^{k-j}.$$

If there are $k+1$ distinct roots $\lambda_i, i = 0, \ldots, k$, then we have a full basis of $k+1$ linearly independent solutions (note that the solutions form a vector space).

In case of multiple roots, this does not give the complete set of solutions.

If $y' = 0$, then the general linear multistep method becomes

$$y_{n+1} + \sum_{j=0}^{k} a_j y_{n-j} = 0, \qquad n = k, k+1, \ldots$$

This is an example of a homogeneous linear difference equation of order $k+1$ with constant coefficients.

If a method is stable, it should, in particular, be stable while computing the zero solution of the initial value problem $y' = 0, y(0) = 0$. To investigate this, we need to solve the homogeneous linear difference equation of order $k+1$.

To find the general solution, we first try for a solution of the form $(\lambda^n)_{n=0}^{\infty}$. Substituting this in the difference equation, we see that it is a solution if and only if λ is a root of the characteristic polynomial

$$\rho(t) = t^{k+1} + \sum_{j=0}^{k} a_j t^{k-j}\,.$$

If there are $k+1$ distinct roots $\lambda_i, i = 0, \ldots, k$, then we have a full basis of $k+1$ linearly independent solutions (note that the solutions form a vector space).

In case of multiple roots, this does not give the complete set of solutions.

If λ is a double root, then $(\lambda^n)_{n=0}^{\infty}$ and $(n\lambda^n)_{n=0}^{\infty}$ both are solutions.

Initial Value Problems: Linear Multistep Methods

Akash Anand
MATH, IIT KANPUR

If $y' = 0$, then the general linear multistep method becomes

$$y_{n+1} + \sum_{j=0}^{k} a_j y_{n-j} = 0, \qquad n = k, k+1, \ldots$$

This is an example of a homogeneous linear difference equation of order $k+1$ with constant coefficients.

If a method is stable, it should, in particular, be stable while computing the zero solution of the initial value problem $y' = 0, y(0) = 0$. To investigate this, we need to solve the homogeneous linear difference equation of order $k+1$.

To find the general solution, we first try for a solution of the form $(\lambda^n)_{n=0}^{\infty}$. Substituting this in the difference equation, we see that it is a solution if and only if λ is a root of the characteristic polynomial

$$\rho(t) = t^{k+1} + \sum_{j=0}^{k} a_j t^{k-j}.$$

If there are $k+1$ distinct roots $\lambda_i, i = 0, \ldots, k$, then we have a full basis of $k+1$ linearly independent solutions (note that the solutions form a vector space).

In case of multiple roots, this does not give the complete set of solutions.

If λ is a double root, then $(\lambda^n)_{n=0}^{\infty}$ and $(n\lambda^n)_{n=0}^{\infty}$ both are solutions. Similarly, λ is a root of multiplicity $M > 2$, then $(n^m\lambda^n)_{n=0}^{\infty}$, $m = 0,1, \ldots, M-1$, also satisfy the difference equation.

If $y' = 0$, then the general linear multistep method becomes

$$y_{n+1} + \sum_{j=0}^{k} a_j y_{n-j} = 0, \qquad n = k, k+1, \ldots$$

This is an example of a homogeneous linear difference equation of order $k+1$ with constant coefficients.

If a method is stable, it should, in particular, be stable while computing the zero solution of the initial value problem $y' = 0, y(0) = 0$. To investigate this, we need to solve the homogeneous linear difference equation of order $k+1$.

To find the general solution, we first try for a solution of the form $(\lambda^n)_{n=0}^{\infty}$. Substituting this in the difference equation, we see that it is a solution if and only if λ is a root of the characteristic polynomial

$$\rho(t) = t^{k+1} + \sum_{j=0}^{k} a_j t^{k-j}\,.$$

If there are $k+1$ distinct roots $\lambda_i, i = 0, \ldots, k$, then we have a full basis of $k+1$ linearly independent solutions (note that the solutions form a vector space).

In case of multiple roots, this does not give the complete set of solutions.

If λ is a double root, then $(\lambda^n)_{n=0}^{\infty}$ and $(n\lambda^n)_{n=0}^{\infty}$ both are solutions. Similarly, λ is a root of multiplicity $M > 2$, then $(n^m \lambda^n)_{n=0}^{\infty},\ m = 0,1,\ldots, M-1$, also satisfy the difference equation.

Thus, for $\rho(t) = \prod_{j=1}^{J} (t - \lambda_j)^{M_j}$ where $\sum_{j=1}^{J} M_j = k+1$, the general solution is $y_n = \sum_{j=1}^{J} \sum_{m=0}^{M_j - 1} c_{jm} n^m \lambda_j^n$.