Lesson 3

Boundary Value Problems for ODEs

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Boundary Value Problems: Finite Difference Method

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In the shooting method, we start by approximately satisfying the ODE (using an IVP solver) and iterate until the boundary conditions are satisfied.

Alternatively, we can start by satisfying the boundary conditions and iterate until the ODE is satisfied approximately.

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$$u'' = f(t, u, u'), \qquad a < t < b,$$

with boundary conditions

$$u(a) = \alpha, \qquad u(b) = \beta.$$

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We replace the derivatives appearing in the ODE by finite difference approximations

$$u'(t_i) \approx \frac{u_{i+1} - u_{i-1}}{2h}, \quad u''(t_i) \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}.$$

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This yields a system of algebraic equations

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = f\left(t_i, u_i, \frac{u_{i+1} - u_{i-1}}{2h}\right), i = 1, \dots, n,$$

to be solved for the unknowns u_i, $i = 1, \dots, n$.

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In the matrix form, we have

$$\frac{1}{h^2}\begin{bmatrix} -2 & 1 & 0 & & & \cdots & 0 \\ 1 & -2 & 1 & & & & \\ \vdots & \vdots & & \ddots & \vdots & & \vdots \\ & & & & 1 & -2 & 1 \\ 0 & \cdots & & & 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} f\left(t_1, u_1, \frac{u_2 - \alpha}{2h}\right) \\ f\left(t_2, u_2, \frac{u_3 - u_1}{2h}\right) \\ \vdots \\ f\left(t_n, u_n, \frac{\beta - u_n}{2h}\right) \end{bmatrix}$$

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which is denoted as

$$\frac{1}{h^2} Au = F(u) + g.$$

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Thus, the Newton's method for solving the system of algebraic equations is given by

$$u^{(m+1)} = u^{(m)} - \left[\frac{1}{h^2}A - F'\left(u^{(m)}\right)\right]^{-1}\left[\frac{1}{h^2}Au^{(m)} - F\left(u^{(m)}\right) - g\right]$$

where the Jacobian matrix is given by $[F(u)]_{ij} = [\partial f(t_i, u_i, (u_{i+1} - u_{i-1})/(2h))/\partial u_j]$.

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where the Jacobian matrix is given by $[F'(u)]_{ij} = [\partial f(t_i, u_i, (u_{i+1} - u_{i-1})/(2h))/\partial u_j]$.

In particular,

$$[F'(u)]_{ii} = f_2\left(t_i, u_i, \frac{u_{i+1} - u_{i-1}}{2h}\right), \qquad 1 \le i \le n,$$

$$[F'(u)]_{i,i-1} = -\frac{1}{2h}f_3\left(t_i, u_i, \frac{u_{i+1} - u_{i-1}}{2h}\right), \qquad 2 \le i \le n,$$

$$[F'(u)]_{i,i+1} = \frac{1}{2h}f_3\left(t_i, u_i, \frac{u_{i+1} - u_{i-1}}{2h}\right), \qquad 1 \le i \le n-1,$$

where all other entries are $F'(u)$ *are* 0 *and* $f_2(t,u,v)$, $f_3(t,u,v)$ *denote the partial derivatives of* f *with respect to* u *and* v *respectively.*

Example

Consider the two-point BVP

$$u'' = -u + \frac{2(u')^2}{u}, \qquad -1 < t < 1,$$
$$u(-1) = u(1) = (e + e^{-1})^{-1}.$$

The iterative solution via Newton's method satisfies

$$u^{(m+1)} = u^{(m)} - \left[\frac{1}{h^2}A - F'\left(u^{(m)}\right)\right]^{-1}\left[\frac{1}{h^2}Au^{(m)} - F\left(u^{(m)}\right) - g\right],$$

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$$[F'(u)]_{ii} = -1 - 2\left(\frac{u_{i+1} - u_{i-1}}{2hu_i}\right)^2, [F'(u)]_{i,i-1} =$$

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and

$$g = \frac{-1}{h^2}\begin{bmatrix}(e + e^{-1})^{-1} \\ 0 \\ \vdots \\ 0 \\ (e + e^{-1})^{-1}\end{bmatrix}.$$

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3.1 Well-posedness

3.2 Shooting Method

3.3 Finite Difference Method

- Error Analysis

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We define

$$D_h^2 v(t) = \frac{v(t+h) - 2v(t) + v(t-h)}{h^2}.$$

Then, the local (truncation) error

$$\ell(t_i, u) = D_h^2 u(t_i) - f(t_i)$$

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Now, if $u \in C^4([a,b])$, then

$$u(t_i + h) = u(t_i) + hu'(t_i) + \frac{h^2}{2}u''(t_i) + \frac{h^3}{6}u'''(t_i) + \frac{h^4}{24}u^{(4)}(\xi_1), \quad \xi_1 \in (t_i, t_i + h),$$

$$u(t_i - h) = u(t_i) - hu'(t_i) + \frac{h^2}{2}u''(t_i) - \frac{h^3}{6}u'''(t_i) + \frac{h^4}{24}u^{(4)}(\xi_2), \quad \xi_2 \in (t_i - h, t_i).$$

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Now, if $u \in C^4([a,b])$, *then*

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Thus,

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Thus,

$$\ell(t_i, u) = \frac{h^2}{24}\left(u^{(4)}(\xi_1) + u^{(4)}(\xi_2)\right) = \frac{h^2}{12}u^{(4)}(\xi), \qquad \xi \in (t_i - h, t_i + h).$$

Theorem

If $v \in C^2([a,b])$, *then*

$$\lim_{h\to 0}\left\|D_h^2 v - v''\right\|_{\infty,h} = 0.$$

If $v \in C^4([a,b])$, *then*

$$\left\|D_h^2 v - v''\right\|_{\infty,h} \le \frac{h^2}{12}\left\|v^{(4)}\right\|_{\infty,[a,b]}.$$

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$$\left\| D_h^2 v - v'' \right\|_{\infty,h} \leq \frac{h^2}{12} \left\| v^{(4)} \right\|_{\infty,[a,b]}.$$

Proof:

We have already seen the proof for the second part. For the first part, we use

$$v(t_i + h) = v(t_i) + hv'(t_i) + \frac{h^2}{2} v''(\xi_1), \quad \xi_1 \in (t_i, t_i + h),$$

$$v(t_i - h) = v(t_i) - hv'(t_i) + \frac{h^2}{2} v''(\xi_2), \quad \xi_2 \in (t_i - h, t_i),$$

yielding

$$D_h^2 v(t_i) - v''(t_i) = \frac{v''(\xi_1) + v''(\xi_2)}{2} - v''(t_i) = v''(\xi) - v''(t_i), \qquad \xi \in (t_i - h, t_i + h).$$

The result follows!

Theorem (Discrete Maximum Principle)

Let v be a function on $[a, b]$ satisfying $D_h^2 v \geq 0$ on $t_i, i = 1, \ldots, n$. Then $\max_{1 \leq i \leq n} v(t_i) \leq \max\{v(t_0), v(t_{n+1})\}$. Equality holds if and only if v is constant.

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Boundary Value Problems: Finite Difference Method

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Theorem

There is a unique solution to the discrete BVP

$$\begin{aligned} D_h^2 u_h(t_i) &= f(t_i), \; t_i, i = 1, \ldots, n, \\ u_h(a) = \alpha, &\qquad u_h(b) = \beta. \end{aligned}$$

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It is sufficient to show that, if $D_h^2 u_h(t_i) = 0$ *for* $t_i, i = 1, \dots, n$, *and* $u_h(a) = 0, \; u_h(b) = 0$, *then* $u_h \equiv 0$.

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Boundary Value Problems: Finite Difference Method

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Boundary Value Problems: Finite Difference Method

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A similar argument applies to $-u_h$ *giving the theorem.*

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The error estimate follows from the previous theorem applied to the discrete problem

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Corollary

If $u \in C^2([a, b])$, then

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If $u \in C^2([a,b])$, then

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If $u \in C^4([a,b])$, then

$$\|u_h - u\|_{\infty,h} \le \frac{h^2(b-a)^2}{96} \left\| u^{(4)} \right\|_{\infty,[a,b]}.$$

Remark

The quantity $\left\|f - D_h^2 u\right\| = \left\|u'' - D_h^2 u\right\|$ *is the consistency error of the discretization and the statement* $\lim\limits_{h\to 0}\left\|u'' - D_h^2 u\right\| = 0$ *means that the discretization is consistent.*

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As a final remark, the finite difference method helped us find the solution values at the mesh points, but the solution at non-mesh points are not readily available from the method. If needed, one can obtain the solution at non-mesh points through interpolation or try other approximation approaches …