Lesson 4

Numerical Solution of PDE

4.2 Parabolic PDE

- Semi-discretization
- Full finite difference discretization
- **Fourier Analysis**

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Another useful way to analyze is to use Fourier analysis.

Recall that, on $L(\mathrm{I_h})$*, we define the inner product*

$$\langle u, v\rangle_h = h\sum_{k=1}^{N-1} u(kh)v(kh)$$

with the corresponding norm $\|v\|_h$ *and* $\varphi_m(x) = \sin \pi m x, m = 1, \dots, N-1,$ *form an orthogonal basis.*

Also,

$$D_h^2\varphi_m = -\lambda_m\varphi_m, \qquad \lambda_m = \frac{2}{h^2}(\cos \pi m h - 1) = \frac{4}{h^2}\sin^2\frac{\pi m h}{2},$$

where the eigenvalues satisfy

$$8 < \lambda_1 < \lambda_2 < \cdots < \lambda_{N-1} < \frac{4}{h^2}.$$

We write the semi-discrete solution $u_h(t,\cdot) = \sum_{m=1}^{N-1} a_m^h \varphi_m$*. Then,*

$$\frac{\partial u_h}{\partial t} = \sum_{m=1}^{N-1}\frac{da_m^h}{dt}\varphi_m, \qquad D_h^2 u_h = -\sum_{m=1}^{N-1} a_m^h\lambda_m\varphi_m\,.$$

Thus,

$$\frac{da_m^h}{dt} = -c a_m^h\lambda_m \qquad so \qquad a_m^h(t) = a_m^h(0)e^{-c\lambda_m t}.$$

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Thus,

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The numbers $a_m^h(0)$ are the coefficients in the discrete Fourier sine transform of the initial data:

$$u_0(x) = \sum_{m=1}^{N-1} a_m^h(0)\varphi_m(x), \ \ x \in \bar{I}_h.$$

Thus, the solution of the semi-discrete system may be written as

$$u(x,t) = \sum_{m=1}^{N-1} a_m^h(0) e^{-c\lambda_m t}\varphi_m(x), \quad x \in \bar{I}_h.$$

For the fully discrete forward-centered scheme, we write the solution as

$$u^j(x) = \sum_{m=1}^{N-1} A_m^j \varphi_m(x).$$

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For the fully discrete forward-centered scheme, we write the solution at time $t = jk$ as

$$u^j(x) = \sum_{m=1}^{N-1} A_m^j \varphi_m(x).$$

The difference equation then gives

$$\sum_{m=1}^{N-1} A_m^{j+1} \varphi_m(x) = \sum_{m=1}^{N-1} A_m^j \varphi_m(x) - ck \sum_{m=1}^{N-1} A_m^j \lambda_m \varphi_m(x)$$

yielding

$$A_m^{j+1} = (1 - ck\lambda_m) A_m^j.$$

It follows that

$$A_m^j = (1 - ck\lambda_m)^j a_m^h(0), \qquad u^j = \sum_{m=1}^{N-1} (1 - ck\lambda_m)^j a_m^h(0) \varphi_m\,.$$

If we assume $ck/h^2 \le 1/2$, then $ck\lambda_m \le ck(4/h^2) \le 2$ and hence $|1 - ck\lambda_m| \le 1$ for all m and the solution remains bounded. On the other hand, if $|1 - ck\lambda_m| > 1$ for some m, the initial data will increase exponentially.

This idea can be used for rigorous stability analysis.

In the matrix form, we write

$$u^{j+1} = \left(I + ckD_h^2\right)u^j + kf^j, \qquad j = 0,1,\dots,M-1,$$

where $\left(I + ckD_h^2\right)$ *is a symmetric operator on* $L(I_h)$*. Thus,*

$$\left\|\left(I + ckD_h^2\right)v\right\|_h \le \left(\max_m |1 - ck\lambda_m|\right) \|v\|_h$$

and, therefore, we have

$$\left\|\left(I + ckD_h^2\right)\right\|_h \le \max_m |1 - ck\lambda_m|.$$

Finally,

$$\left\|u^j\right\|_h \le \max_m |1 - ck\lambda_m| \left\|u^{j-1}\right\|_h + k\left\|f^{j-1}\right\|_h \le \left(\max_m |1 - ck\lambda_m|\right)^j \|u^0\|_h + Mk \max_j \left\|f^j\right\|_h.$$

Because of the condition $ck/h^2 \le 1/2$*, which we know is not only sufficient but necessary for stability, forward-centered difference method is called conditionally stable.*

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From stability, we obtain the convergence result in the same way as earlier. Let $e_n^j = u_n^j - u(nh, jk)$, *we have*

$$\frac{e_n^{j+1} - e_n^j}{k} = c\,\frac{e_{n+1}^j - 2e_n^j + e_{n-1}^j}{h^2} - \ell_n^j, \qquad 0 < n < N, j = 0,1,\dots,M-1,$$
$$e_0^j = e_N^j = 0, \qquad j = 0,1,\dots,M-1,$$
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The stability result then yields

$$\left\|e^j\right\|_h \le \left(\max_m |1 - ck\lambda_m|\right)^j \|e^0\|_h + Mk \max_j \left\|\ell^j\right\|_h \le T \max_j \left\|\ell^j\right\|_h.$$

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- *Semi-discretization*
- *Full finite difference discretization*
- *Fourier Analysis*
- ***Unconditional Stability***

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Can we design a scheme that is unconditionally stable?

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Try backward Euler method in time!

Numerical Methods for PDE: Parabolic PDE

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The backward-centered difference method for the heat equation leads to

$$\frac{u_n^{j+1} - u_n^j}{k} = c\,\frac{u_{n+1}^{j+1} - 2u_n^{j+1} + u_{n-1}^{j+1}}{h^2} + f_n^{j+1}, \qquad 0 < n < N, j = 0,1,\dots,M-1,$$
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Thus, u_n^{j+1} *can be obtained by solving a tridiagonal system*

$$-\lambda u_{n+1}^{j+1}+(1+2\lambda)u_n^{j+1}-\lambda u_{n-1}^{j+1} = u_n^j + k f_n^{j+1}, \qquad u_0^{j+1}=u_N^{j+1}=0.$$

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We again use the Fourier analysis to determine the stability of the scheme. In operator form, the method reads

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The operator $I-ckD_h^2$ *has eigenvalues* $1+ck\lambda_m$ *which are all greater that one, so* $\left\|\left(I-ckD_h^2\right)^{-1}v\right\|_h \le \|v\|_h$.

We, therefore, have

$$\|u^{j+1}\|_h = \left\|\left(I - ckD_h^2\right)^{-1}\left(u^j + kf^{j+1}\right)\right\|_h \le \|u^j\|_h + k\|f^j\|_h, \qquad j = 0,1,\dots,M-1.$$

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Combining the stability estimate with the local truncation error, we obtain the error estimate

$$\max_{1\le j\le M}\left\|e^j\right\|_h = O(k + h^2)$$

for the method.

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If we use trapezoidal rule to discretize in time, we get the Crank-Nicolson method:

$$\frac{u^{j+1} - u^j}{k} = \frac{c}{2}\left(D_h^2 u^j + D_h^2 u^{j+1}\right) + \frac{1}{2}\left(f^j + f^{j+1}\right).$$

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Using the Taylor expansion about the point $(nh, (j+1/2)k)$, it is straightforward to show that the local truncation error is $O(k^2 + h^2)$, so the Crank-Nicholson method is second order in both space and time.

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that is

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The eigenvalues of $\left(I - \frac{1}{2} ck D_h^2\right)^{-1}\left(I + \frac{1}{2} ck D_h^2\right)$ is $(1 - ck\lambda_m/2)/(1 + ck\lambda_m/2)$ which is less than 1. Therefore, we get unconditional stability. The Crank-Nicolson method converges with $O(k^2+h^2)$.

Lesson 4

Numerical Solution of PDE

4.3 Hyperbolic PDE

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Numerical Methods for PDE: Hyperbolic PDE

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For the first example, we take the advection equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0,$$

where c *is a constant. We consider an initial value problem so that the function* $u = u(x, t)$ *is given when* $t = 0$ *and is to be found for* $t > 0$.

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where c is a constant. We consider an initial value problem so that the function $u = u(x,t)$ is given when $t = 0$ and is to be found for $t > 0$.

The spatial domain can be an interval in which case we will need to impose boundary conditions to obtain a well posed initial value-boundary value problem, or the entire real line (no boundaries) for which the pure initial value problem is well-posed.

Numerical Methods for PDE: Hyperbolic PDE

For the first example, we take the advection equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0,$$

where c is a constant. We consider an initial value problem so that the function $u = u(x, t)$ is given when $t = 0$ and is to be found for $t > 0$.

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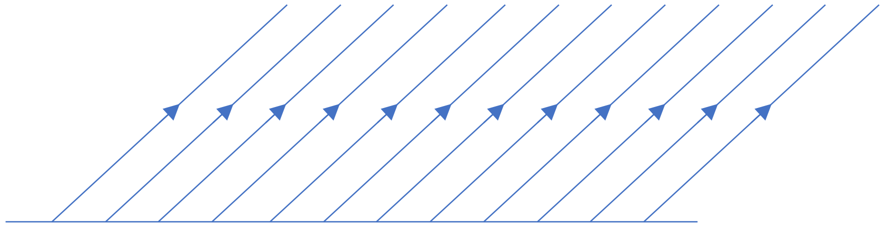

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MATH, IIT KANPUR

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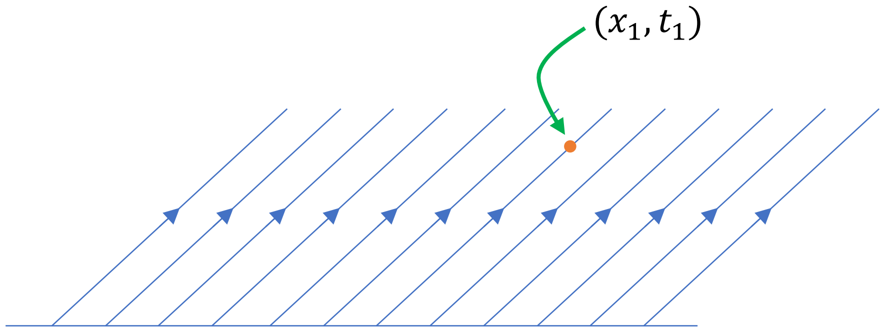

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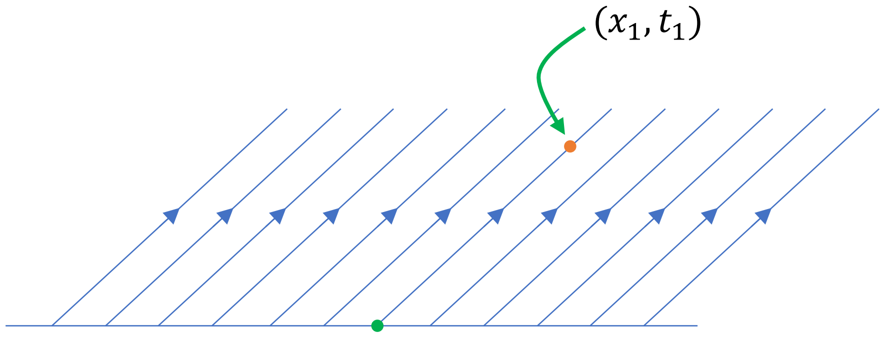

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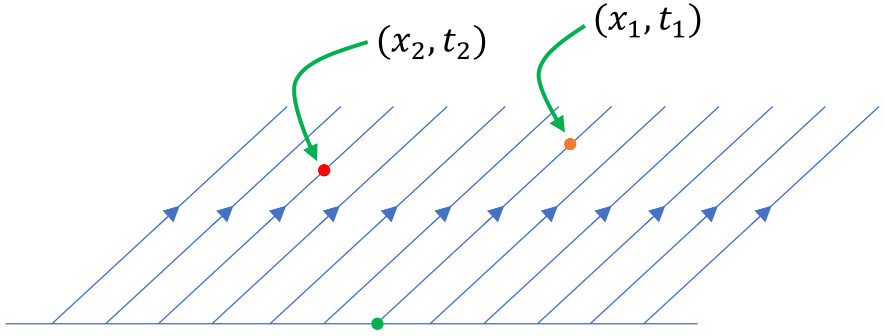

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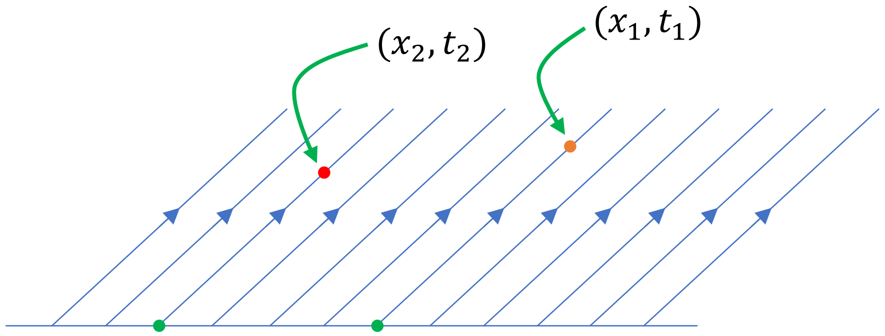

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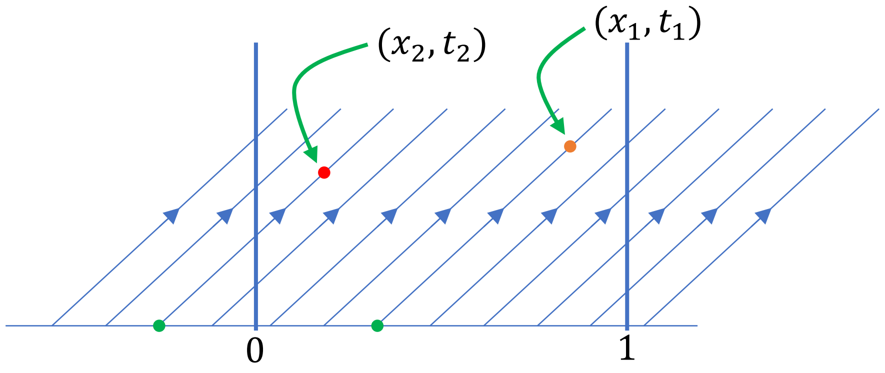
Suppose that the spatial domain is an interval, say $(0,1)$*.*



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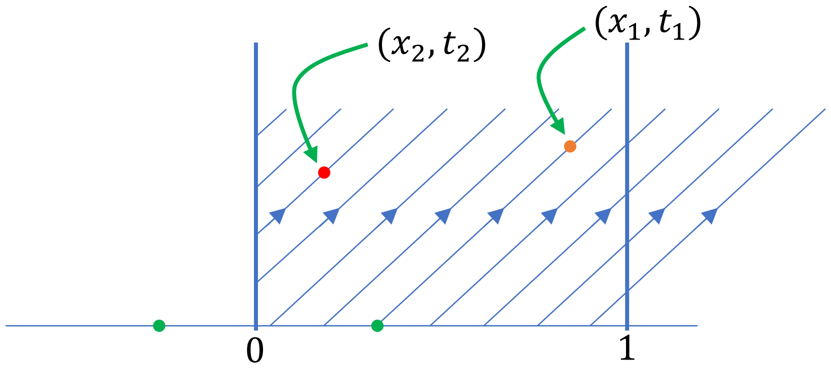

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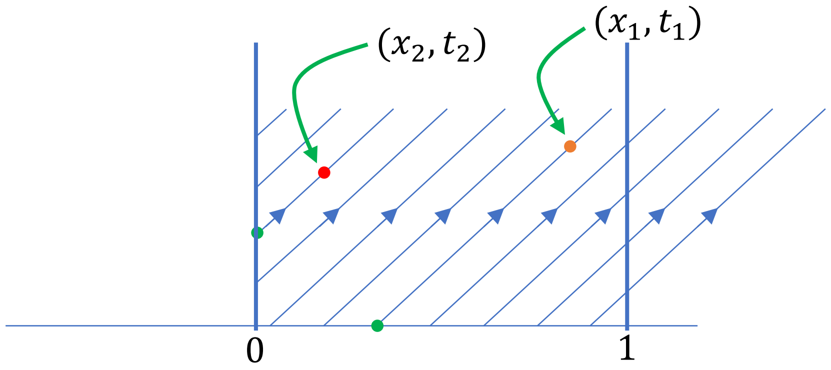

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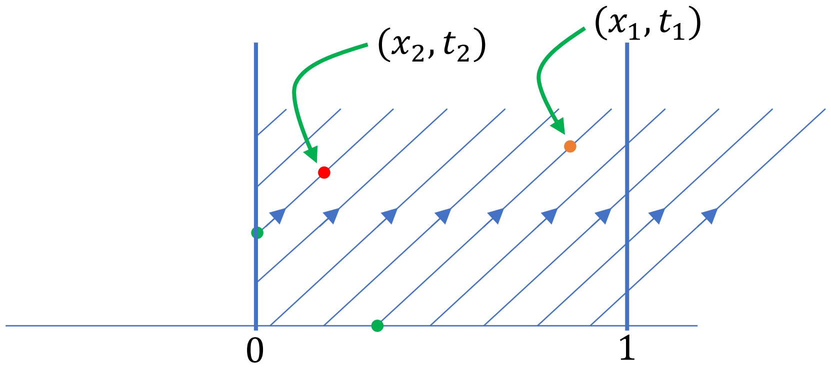

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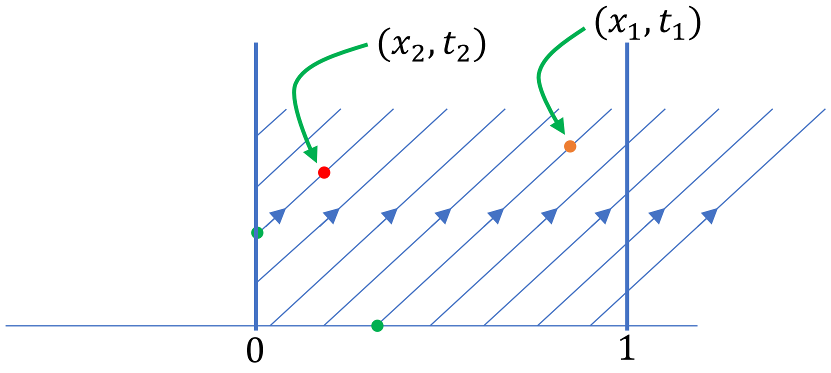

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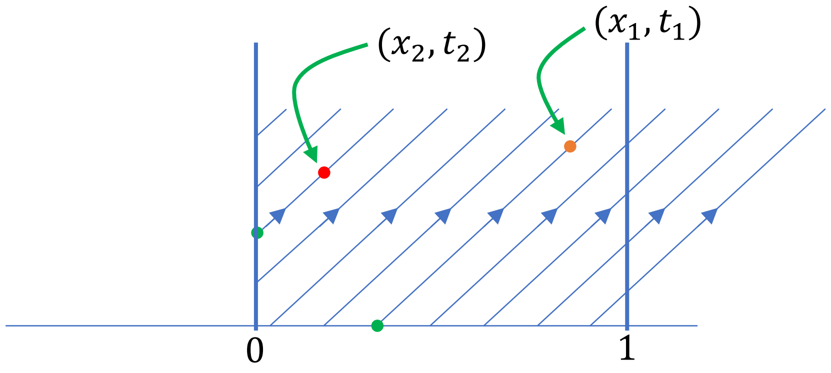

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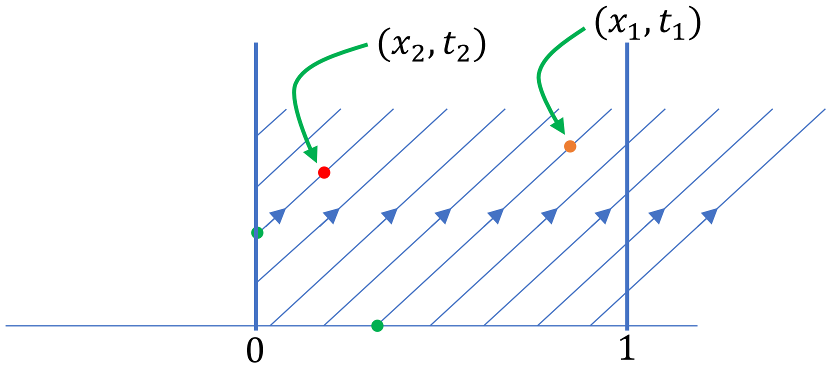

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Now consider a system of the form

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Consider the wave equation

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2}.$$

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Finally, we find the solution (u_1, u_2) *from* $u_1 + u_2$ *which is a wave moving from right to left and from* $u_1 - u_2$, *a wave moving to the right.*