Model-Checking of Real-Time Systems: A Telecommunications Application

Experience Report

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ABSTRACT

We describe the application of model checking tools to analyze a real-time software challenge in the design of Lucent Technologies' 5ESS telephone switching system. We use two tools: COSPAN for checking real-time properties, and TPWB for checking probabilistic specifications. We report on the feedback given by the tools, and based on our experience, discuss the advantages and the limitations of the approach used.

Keywords

Telecommunications software, Formal methods, Model checking, Real-time verification, Probabilistic verification.

INTRODUCTION

The design of concurrent real-time systems is a notoriously difficult problem. In particular, the interaction of concurrent processes in large real-time systems often leads to subtle bugs that are extremely difficult to discover using the conventional techniques of simulation and testing. Automated verification based on model-checking promises a more effective way of discovering design errors. In this approach, a high-level model of the system under design is described in a formal notation, and the verifier automatically checks whether the given model satisfies correctness properties such as absence of deadlocks or unreachability of "bad" states. This check is performed, unlike simulation, by exploring all possible interactions of the concurrent components. If the property is violated, the tool reports a scenario as evidence of the violation. In recent years, there has been steady progress in the capabilities of tools for automated verification. Current tools incorporate a variety of algorithmic heuristics, and case-studies, usually in the hardware domain, suggest that these tools can handle real-world applications (see [8] for an introduction to model checking).

Lucent Technologies' 5ESS®telephone switching system is a concurrent real-time system comprised of several million lines of C code. The 5ESS software consists of thousands of interacting concurrent processes, all with stringent real-time constraints. One component of the 5ESS software is responsible for detecting malfunctions on the hardware connections between switches. In a recent 5ESS software release, the hardware facilities were substantially upgraded, and consequently the possible rate of incoming alarms increased dramatically. It was discovered during testing that, under certain conditions in which the incoming alarm rate was high, the design violated some correctness conditions such as absence of process abortion and buffer overflow. After much simulation, the software was redesigned to perform its computations significantly faster by reducing the number of required database operations. A battery of tests seemed to indicate that the redesigned software prevents the undesirable conditions. As part of a collaboration between research and development, we are studying the application of model checking technology to this 5ESS software.

Since the correctness of the design crucially depends on the real-time constraints, we need a model checker that supports real-time verification. One such verifier, developed at Bell Laboratories, is called COSPAN [3]. We constructed a high-level model of the relevant functional and real-time aspects of the design, and used COSPAN to detect scenarios leading to conditions like queue overflow or process abortion. As expected, the model of the earlier version of the 5ESS software violates these properties. However, to our surprise, we also discovered that in the model of the new version of the 5ESS software, the queue may still overflow under certain conditions

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when the incoming rate is high. COSPAN has generated scenarios under which this problem may arise, and we are currently running these scenarios as test cases in the 5ESS lab environment.

To verify the intuition that the redesigned software was an improvement over the old one, we also constructed high-level probabilistic models of the two versions. The use of probabilities, while not supported by COSPAN, seems useful in capturing scheduling assumptions more realistically. The verification of models with real-time and probabilities is supported by a tool called TPWB [11, 13]. While computational requirements forced us to significantly scale down many of the parameters (such as the number of incoming alarms), we still could obtain useful feedback from the tool.

In addition to generating potentially useful test cases, we believe that the use of model checking aids in the design and redesign of real-time software, since the model can be easily modified to reflect changes in the functional and real-time requirements. For example, one can verify whether a decrease in the potential incoming alarm rate or a speedup in computation would avoid these queue and timing difficulties.

The paper is organized as follows. An overview of the 5ESS software and this application is given in the following section. The next section describes the COSPAN verifier, our COSPAN model and the analysis; the following section describes the TPWB tool, our TPWB model and the analysis. Our observations, along with the limitations of these approaches, are reported in the last two sections.

THE 5ESS SWITCHING SYSTEM

The 5ESS Software

Lucent Technologies' 5ESS telephone switching system [20] is a highly reliable, concurrent real-time system which provides telecommunications services, including the connection of telephone calls.

The 5ESS software is comprised of several million lines of C code, and consists of thousands of interacting concurrent processes residing on numerous processors. Each process has a fixed priority; scheduling is non-preemptive and respects process priority. In particular, on each processor, the operating system indicates to the scheduler when processes are ready to be scheduled – for example, when messages are sent to a process or when a timeout expires. The scheduler then passes control to a process in the highest-priority class ready to be scheduled: if more than one such process exists in this class, the choice is made according to the order in which the processes gave up control. The scheduled process then voluntarily indicates to the scheduler when it is ready to relinquish control, and the cycle is repeated.

5ESS processes all have stringent real-time constraints. For example, every process, when scheduled, must voluntarily relinquish control within a bounded amount of time. There are two types of time-slices: a *recommended time-slice* in which design guidelines suggest that a process voluntarily relinquish control, and a longer *maximum allocated time-slice*. If a process exceeds its maximum allocated time slice, it is killed by the operating system; clearly an undesirable consequence. In addition, most processes have real-time requirements on the rate at which they process their inputs. Together, these requirements contribute to the overall reliability of the 5ESS switching system.

Carrier Group Alarms

In switches, long-distance telephone calls are typically routed through a network of hardware, referred to as *carrier groups*. Status changes on carrier groups – such as malfunctions or recoveries from malfunctions – appear as inputs to the switch. In response to such inputs, the switch removes/restores the associated carrier groups and routes new calls over functioning carrier groups.

One component of the 5ESS software is responsible for handling alarms that signal malfunctions on carrier groups [14]. This component consists of several interacting concurrent processes divided among the *5ESS Peripheral Control* software and the *5ESS Carrier Group Alarms* software. In particular, at fixed intervals, a high-priority process in the Peripheral Control software samples the carrier group alarms and places them in a bounded-size queue. A lower-priority process in the Peripheral Control software removes alarms from this queue and calls a Carrier Group Alarms software function to perform database lookups and to send information about each alarm to a Carrier Group Alarms software process. Clearly, in order to avoid the loss of alarm information, the queue must not overflow. As described above, the design should also guarantee that no process ever exceeds its time-slice.

A Real-Time Difficulty

In a recent 5ESS software release, the hardware facilities were substantially upgraded, and consequently the possible rate of incoming carrier group alarms increased dramatically. It was discovered during testing that, under certain conditions in which the incoming alarm rate was high, the alarm queue would sometimes overflow and the lower-priority Peripheral Control process would sometimes exceed its time-slice and be killed. In the context of the new software release, these difficulties arose from the subtle interactions between the Peripheral Control software and the Carrier Group Alarms software.

After a significant amount of simulation, the Periph-

eral Control software and the Carrier Group Alarms software were redesigned to perform their computations significantly faster by reducing the number of required database operations. A battery of tests seemed to indicate that the redesigned software prevents queue overflow and process abortion.

The following two sections describe our application of two automatic verification tools to these software designs.

REAL-TIME VERIFICATION

Model checking is a method of automatically verifying concurrent systems in which a finite-state model of a system is compared with a correctness requirement. The early model checkers used a temporal logic called Computation Tree Logic (CTL) for specifying correctness requirements, and employed enumerative algorithms for verification [7, 23]. The applicability of model checking paradigm was greatly enhanced due to the introduction of symbolic algorithms for verification [21]. Research in recent years has led to the development of model checkers that can analyze real-time systems [2, 3, 10, 15, 18].

Overview of COSPAN

We consider the tool COSPAN developed at Bell Laboratories. In the following, we briefly review the relevant aspects of COSPAN (see [17] for the underlying theory). We first describe the formalism for modeling state machines, and next consider the extension with real-time constraints.

Modeling Language

The system to be verified is modeled as a collection of coordinating processes described in the language S/R. Each process has two types of variables: *state* variables and *selection* (or output) variables. A state of a process P is an assignment of values of its state variables, and a selection of P is an assignment of values to its selection variables. A *global state* consists of states of all the processes and a *global selection* consists of selections of all the processes. The declaration of a process P contains the type declaration of all its variables, a set of possible initial states, and a transition relation specified as a set of update commands for all of the variables of process P. The update command for a selection variable specifies the set of possible values as a function of the current state of P and selections of some other processes. The update command for a state variable specifies the set of possible next states as a function of the current state of P and the current global selection. Thus, selection variables are used for communication among processes. The execution of an S/R model proceeds in rounds. In round 0, initial values of all the state variables are chosen. In each subsequent round, first the global selection is determined (which depends on the current global state), and then the next global state is determined (which depends on the current global state and the global selection). Thus, an execution of the model M produces an infinite sequence r of the form $s_0 \stackrel{\sigma_0}{\rightarrow} s_1 \stackrel{\sigma_1}{\rightarrow} s_2 \stackrel{\sigma_2}{\rightarrow} \cdots$ of global states s_i and global selections σ_i. *Acceptance conditions* are used to rule out uninteresting executions, for instance, to enforce fairness in resolving a nondeterministic choice. With the model M, we associate the set $L(M)$ consisting of all the sequences that satisfy the acceptance conditions.

Verification

Once the system under design is described as an S/R model M, it can be checked for different correctness conditions. The property to be checked is described as another process T whose acceptance conditions classify the executions of M in sets "good" and "bad," and all the good executions are removed from the language. Let N be the new model that contains the original model M together with the property T (technically, N is the product automaton $M \otimes T$). The model M satisfies the property T if $L(N)$ is empty. For instance, the user can check whether all states appearing in all executions of the model satisfy a state-predicate φ, that is, whether φ is an invariant of the model. The language S/R provides pre-defined macros for many of the commonly used properties such as invariants, absence of deadlocks, and liveness (e.g. whether a request is always followed by a response).

As just described, the verification problem is phrased as a language-emptiness question. Given an S/R model N, COSPAN checks whether $L(N)$ is empty using efficient algorithms. The algorithms supported include on-the-fly enumerative search and symbolic search using binary decision diagrams (BDDs). If the language is non-empty, then an execution is reported as evidence. For instance, if we are trying to verify whether φ is an invariant of the model, then COSPAN reports either an affirmation or an execution that leads to a state that violates φ.

Real-time Constraints

The S/R model is synchronous: the computation of all processes occurs in lock-step and is modeled by rounds. However, asynchrony can be modeled using delays and non-determinism. Intuitively, whenever an asynchronous process changes its state, it waits a nondeterministic number of rounds before proceeding. Every asynchronous process P has an additional state variable called *the pause-bit*. Whenever the state of P changes, its pause-bit is set to 1. In each subsequent round, the pause-bit may stay unchanged, or may be reset to 0. The selection of P, and hence, the next state, depends on the current value of the pause-bit. Pause-bits of different processes may stay set for different numbers of rounds, thereby introducing asynchrony.

In S/R, real-time constraints are expressed by associating lower and upper bounds with local states of asynchronous processes. These bounds limit the duration for which the pause-bit stays set in a given local state of a process. An execution is *timing-consistent* if all the global states can be assigned real-valued time-stamps that satisfy all the specified bounds. The semantics of a timed S/R model M with a table B of bounds is then the set $L(M, B)$ of its timing-consistent executions (see [3] for a formal definition). For instance, consider two asynchronous processes P and P', and suppose both simultaneously transition to their respective local states s and s'. Suppose the upper bound associated with s is less than the lower bound associated with s', then the pause-bit of P is reset *before* the pause-bit of P' is reset for the execution to be timing-consistent.

To check the correctness conditions of a timed model, an additional process T that removes good behaviors is introduced as before. The timing verification problem corresponds to checking emptiness of the language $L(N, B)$, where N is the product $M \otimes T$. Note that $L(N, B)$ can be empty while $L(N)$ is not. For instance, a model M may not satisfy an invariant φ, but with an appropriate choice of upper and lower bounds, all timing-consistent executions may satisfy the invariant φ. The property T itself can have real-time constraints, and thus, it is possible to check correctness conditions such as bounded-response (eg. whether a request is followed by a response within a specified period).

Real-time Verification
For real-time verification, COSPAN checks emptiness of the language $L(N, B)$ by constructing another automaton A_B, also as a S/R process. This process, when composed with the original model, rules out behaviors that do not satisfy the timing constraints: $L(N \otimes A_B)$ equals $L(N, B)$. The existence of such a finite-state constraining automaton A_B follows from the so-called region construction for timed automata [2]. The implementation in COSPAN supports a variety of heuristics to improve efficiency of this method. The states of the constraining automaton correspond to sets of values for timers, and may be either regions or convex unions of regions called zones. Once the definition of the constraining automaton has been generated, the original verification engine that tests emptiness of an S/R model can be used as a black-box, and the actual search can be done either enumeratively or symbolically using BDDs. These choices are coupled with an iterative solution which involves generating successive approximations to the constraining automaton. Furthermore, the underlying continuous semantics of time can be approximated in a conservative way by the integers, and this also provides a heuristic simplification of the timing analysis. An overview of the timing verification in COSPAN appears in [3].

Specification of the Design
The first step towards analysis of the software is the description of the design in S/R. We have constructed two models, one capturing the old design and one capturing the upgraded design.

Each model consists of several interacting processes: the PC process that models the peripheral control software, the INP process that models the input buffer, the SCHED process modeling the scheduler, and finally, the ENV process that issues the alarms. A brief description of these processes follows.

- The ENV process models the environment that issues the alarms. The alarms are issued periodically; the duration of the period is fixed. The number of alarms issued in each period varies, and is chosen nondeterministically from a given range.
- The process INP models the buffer used to store the alarms. It interacts with ENV and with PC: whenever ENV issues alarms, the number of pending alarms is increased, and whenever PC removes alarms, the number of pending alarms is decreased. The buffer has a fixed size. While this buffer is implemented as a queue in the switch software, for the chosen level of abstraction for us, the actual entries in the buffer are irrelevant. So we model the buffer by a single (bounded) integer variable that tracks the number of pending alarms.
- The main process, called PC, is the software that samples the alarms in INP and stores them in the output queue. The PC process has a boolean selection variable READY that indicates whether it needs to be scheduled. READY is true if there are any pending alarms in INP and false otherwise. Once READY is set and PC receives the appropriate signal from SCHED, it is activated. After initialization, it chooses a set of alarms to process from the buffer INP. The number of alarms chosen is determined by a (simple) algorithm that depends on the current number of pending alarms. The next step is the computation phase. It should be noted that we describe the design at a very high level of abstraction. For instance, all the computation required to sample the hardware alarms is abstracted, retaining only the time required for this computation. The time required for computation depends upon the number of alarms processed, and is different in the old and the new models. Once the computation is finished, INP is decremented according to the number of alarms that have been processed, and the PC process signals to SCHED to give up its time-slice. In the switch software the processed alarms are stored in an output queue, but we do not model the output queue explicitly.

- The scheduler process SCHED controls the invocation of the PC process, and models the real-time scheduling assumptions in the switch. We assume that the PC process has the highest priority; hence, every time the scheduler allocates the next time-slot, PC gets the priority if READY is true. Since the PC process has high priority, it is not considered a design violation for it to retain control past its recommended time-slice; however, if it exceeds its maximum allocated time-slice, it will be killed by the scheduler process. We assume all other processes in the switch are well-behaved, and thus, will relinquish control within their recommended time-slice. Observe that the modeling of the scheduler abstracts the interaction of the PC with the rest of the switch.

We check the models for two properties: buffer overflow and process abortion. The buffer overflow condition occurs when the input buffer to the PC process exceeds its size. The process abortion condition occurs when the time taken by the PC to finish its computation in one invocation exceeds the maximum allowed time-slice. Both these conditions can be specified as simple invariants in S/R.

Analysis

As explained earlier, COSPAN uses the language-emptiness test to detect executions of the model that violate the property. If such an execution is detected then it is reported as a counter-example. For the properties that we checked, the counter-example is a sequence of global states, starting with an initial state, leading to a state in which the buffer overflows or the process gets aborted.

As expected, the model capturing the old design does not satisfy either property, and the verifier reported a counter-example as an evidence. This is consistent with the problems discovered during testing. In this case, both buffer overflow and process abortion are possible.

For the upgraded design, while the process abortion condition does not occur, the verifier reported that the buffer *can* overflow, and produced a counter-example involving several thousand steps of the model. Thus, to our surprise, the new design also does not satisfy the correctness requirements. We are currently working with the software developers to reproduce these conditions in testing of the switch software in the lab environment, and to analyze possible solutions.

It is easy to change parameters of the model such as the size of the buffer, the maximum number of alarms per interval, and the duration of the interval between successive arrivals, and to test which combinations satisfy the properties and which do not. This can be useful feedback to the designers. For instance, for the new model, we found the maximum number of alarms that can be permitted without leading to the buffer overflow condition; this computation was performed by iteratively changing the model and repeating the analysis.

For this example, among the various options available in COSPAN, the use of regions for the constraining automaton and the symbolic search using binary decision diagrams is computationally least expensive. The computational requirements vary depending upon the choice of the parameters: the number of states explored ranges between 10^4 to 10^6, the amount of memory required ranges between 1MB to 20MB, and the CPU time required ranges between 5 to 5000 seconds on an SGI machine with 12 150MHz IP19 processors and 1280 MB of main memory.

PROBABILISTIC VERIFICATION

The models of the previous section assume that 5ESS processes relinquish control within their recommended time-slice. Under this assumption, functional and real-time properties can be proved of the PC software using verification tools such as COSPAN. In the context of a large system such as the 5ESS switch, this parallels the best form of reasoning feasible by individual developers: namely, that their software behaves correctly under the assumption that software developed by others behaves well.[1]

In reality, however, all other processes do not necessarily relinquish control within the recommended time-slice. For example, the PC process itself does not! (Note however, in the new design, it does relinquish control within its maximum allocated time-slice and hence is not aborted.) This behavior is intentional since the PC process has high priority; in the case of other processes, such behavior may be unintentional and hence undocumented. Thus, in order to prove properties about the reliability of the 5ESS software, it would also be desirable to take into account the *probability* that other software processes violate their recommended time-slices.

Probabilities arise in two other ways in our application. First, since our COSPAN analysis showed that both the old and new software designs can result in queue overflow, it would be desirable to show that the new software design is in fact *better than* the old design in the sense that it is *less likely* to result in queue overflow. Second, it would also be desirable to model the probability distribution of the incoming rate of alarms.

[1] Typically, individual software components in the 5ESS software exceed their real-time requirements only when the switch is experiencing overload conditions. The architecture of the switch software has been designed to ensure overall reliability of the switch even under such conditions: for example, some processes are throttled during peak calling hours.

Criteria for a Probabilistic Real-Time Model
In recent years, there has been growing research on integrating probabilistic and real-time verification, and numerous probabilistic real-time process algebras, automata, temporal logics, and algorithms have been developed (e.g. [1, 19, 9]). In particular, most of these approaches extend labeled transition systems and automata by allowing probability distributions to be placed on the transitions, corresponding to the probability that the given transition is selected from its parent state. The associated logics give the probabilities with which temporal properties are satisfied by the specifications. However, most of this work has been theoretical, and few tools have been developed to support these formalisms.

Since COSPAN does not support probabilistic reasoning, we were interested in an approach that would allow us to add probabilities to our real-time models. The following criteria are important for such an approach in the context of our application:

- Verification should support temporal logic extended with real-time and probabilistic information, and *the theory should be supported by a toolset* so that practical applications can be verified. This has implications for our choice of the real-time model. In particular, we are not aware of the existence of any tools that support continuous-time and probabilities, and that satisfy all the following criteria. Hence, we decided to limit ourselves to models in which time is assumed to pass discretely.

- There must be a way to compositionally specify concurrent processes, so that the model reflects the architecture of the software application. For example, it must be possible to specify the PC process and the scheduler process independently.

- There must be a way to model the passage of time independently from the rest of the specification, for example using clocks. In particular, time should *not* be modeled as one time unit elapsing per transition, as different events typically take different amounts of time. For example, the time needed to retrieve data from a database may vary depending on the structure of the data.

- It must not be necessary to associate probabilities with individual events, since it is difficult to estimate the probabilities with which individual events occur.

- It must be possible to assign probabilities to timing delays, so that we can specify, for example, the probability that a process exceeds its recommended time-slice by n milliseconds.

- It must be possible to specify geometric probability distributions, since the inter-arrival time between alarms is conveniently modeled as a geometric distribution (assuming independence of alarms).

TPWB: Timing and Probabilities Workbench
We discovered that the Timing and Probabilities Workbench (TPWB) [11, 13] satisfies all of our above criteria. The modeling language in TPWB is TPCCS – a version of the Calculus for Communicating Systems (CCS) [22], and the logic in TPWB is TPCTL – a version of Computation Tree Logic (CTL) [7]. Both the modeling language and logic extend their respective theories with real-time and probabilistic information.

Like CCS, TPCCS is a process-algebra describing labeled transition systems. Unlike standard labeled transition systems, however, labeled transition systems in TPCCS have two kinds of states: reactive states and probabilistic states. Transitions from reactive states are labeled only with events. Transitions from probabilistic states are labeled only with probabilities; these specify the probability that the given transition is traversed from its parent state. Finally, the passage of (discrete) time is modeled as a distinguished event in TPCCS; hence time-labeled transitions can only emanate from reactive states.

Like CTL, TPCTL is a branching time logic for describing the behavior of some or all paths in a labeled transition system specification. For example, the property that "in all paths of the system, the queue does not overflow" can be described as a CTL property. In addition, TPCTL allows the quantification of such properties over real-time: for example, the property that "in all paths of the system, the queue does not overflow within 20 milliseconds from the initial state of the system" can be described in TPCTL. Finally, TPCTL extends properties with probabilistic information. For example, in TPCTL one can state the property that "in all paths of the system, with 95 percent probability, the queue does not overflow within 20 milliseconds from the initial state of the system." Thus, properties regarding the reliability of systems can be formulated in TPCTL.

The TPWB supports automatic verification of TPCTL formulas on specifications described in TPCCS. It also supports bisimulation equivalence checking of TPCCS specifications, but we have not used this facility for our application.

Specification of the Design
We first wrote two models in TPCCS: one describing the old 5ESS design and the other describing the upgraded design. These models are very similar to their S/R counterparts, the main extension being that probabilistic information is added to the incoming alarm rate

and the scheduler. The other main difference is that computational requirements of the TPWB forced us to significantly scale down many of the parameters, such as the number of incoming alarms, the size of the INP buffer, and the computation time of the PC process.

The TPCCS models consist of the following interacting processes; we describe below the differences between these processes and those in our S/R models.

- The ENV process models the environment that issues the alarms. The important difference from its S/R counterpart is that the number of alarms issued in each period varies within a given range according to a uniform probability distribution.
- The process INP models the buffer. It is essentially the same as its S/R counterpart, the only differences being in some minor implementation details. For example, since the version of TPCCS we used does not support value-passing, we modeled the buffer as a concurrent TPCCS process with increment/decrement/overflow events, rather than as a single integer variable as in S/R.
- The main process, called PC, is the software that samples the alarms in INP and stores them in the output queue. Again, it is essentially the same as its S/R counterpart, and the time required for its computation is different in the old and the new models.
- The scheduler process SCHED controls the invocation of the PC process, and models the real-time assumptions in the switch. The important difference from its S/R counterpart is that other processes are not necessarily assumed to relinquish control within their recommended time-slice, but only within their maximum allocated time-slice (after which they will get killed by the scheduler anyway). Furthermore, the time taken by other processes is specified by a uniform probability distribution.

We note that in our TPCCS models, one time instant represents one millisecond of elapsed time.

We first checked some sanity conditions of our models to ensure that their functional and timing behavior corresponds to some of our expectations:

- With probability 1, the PC process is allowed to run for its maximum allocated time-slice without getting killed by the scheduler.
- With probability 1, the PC process gets killed by the scheduler if it does not relinquish control within its maximum allocated time-slice.

Time (millisec)	Old Model (probability)	New Model (probability)
20	1	1
50	1	1
59	1	1
60	.9998	1
70	.9992	1
80	.9985	1
90	.9957	1
100	.9943	1
150	.9796	1
200	.9631	1

Table 1: Avoidance of PC Process Abortion

- With probability 1, the PC process is only scheduled if it is ready (i.e. if alarms exist).
- With probability 1, if an alarm arrives then the PC is eventually scheduled, or is currently running and either voluntarily relinquishes control or is killed.
- An alarm will eventually arrive. (This is due to the alarm inter-arrival time being modeled as a geometric distribution.)

As with our COSPAN application, we then checked both models for the two main properties: buffer overflow and process abortion. In contrast to COSPAN, however, we also computed the likelihood that each of two models will satisfy the properties within particular time bounds.

Analysis

As mentioned earlier, we had to significantly scale many of the parameters of the models in order to effectively use the TPWB. This scaling resulted in non-integer values for some parameters, which consequently had to be rounded off. This compromises the accuracy of the model with respect to the actual design. However, we have still obtained some useful feedback from our analysis.

After scaling, our model of the old 5ESS software design has approximately 8000 states, while the model of the new design has approximately 6000 states. We used TPWB to show that both models are deadlock-free and have no "non-Zeno" behavior; that is, both models only permit finite computations within bounded amounts of time. Furthermore, the TPWB shows that both models satisfy all of the sanity conditions above.

We also used the TPWB to show that the old model can result in process abortion and queue overflow, as expected. Furthermore, as with its S/R counterpart, the new model is guaranteed to prevent process abortion, but still may result in queue overflow. In addition to checking the possibility of these violations, we also

Time (millisec)	Old Model (probability)	New Model (probability)
20	1	1
30	1	1
39	1	1
40	.9999	.99997
50	.9992	.99995
60	.9982	.99994
70	.9970	.99991
80	.9955	.99991
90	.9937	.99988
100	.9919	.99986
150	.9808	.99981
200	.9683	.99970

Table 2: Avoidance of Queue Overflow

used the TPWB to compute the likelihood of these conditions being satisfied by the models within certain time bounds. Our results are given in Table 1, which lists the probabilities for the avoidance of process abortion, and Table 2, which lists the probabilities for the avoidance of queue overflow. The information in the tables should be interpreted as follows. Suppose for a given model and given time t, p is the probability listed in a table. Then, on all paths of the model, the property is satisfied for t milliseconds with probability p. For example, Table 1 specifies that, on all paths of the old model, with probability .9998 the PC process will not be aborted within 60 milliseconds. Since the new model is guaranteed not to have process abortion, it satisfies the property with probability 1 on all paths for all finite lengths of time.

The tables show that, for at least up to 200 milliseconds, the new model satisfies both the process abortion avoidance and queue overflow avoidance properties with greater probability than the old model. Thus, in a precise formal sense, the new design can be regarded as an improvement over the old one.

Ideally, we would have continued this analysis for much longer real-time segments. However, even for the 200 millisecond analysis, we required upwards of 24 hours of CPU time and 200 MB memory on a lightly loaded SGI Challenge XL with 4 MIPS R4400 150MHz processors and 512 MB of main memory size. Thus, more detailed analyses seem infeasible at the current time.

Similar limitations of the TPWB tool prevent us from reducing the significant scaling we did on our TPCCS models. We note that the modeling of probability distributions over timers also seems to significantly affect the state space of the model, and hence the performance of the TPWB tool. We had originally specified uniform probability distributions over timers as labeled transition systems with no branching after the root node. For example, a timer that can fire between 0 and 3 time instants with uniform probability was modeled as a labeled transition system whose root contains four outgoing transitions, each labeled with probability 1/4. The first transition is succeeded by a path (with no subsequent branching) modeling a timer that fires immediately, the second transition is succeeded by a path (with no subsequent branching) that models a timer that fires after 1 time instant, and so forth. Thus, the choice of the timer duration is made at the root.

We later changed our model of probability distributions over timers to be binary-branching labeled transition systems, where the choice of timer duration is made after every time instant, rather than at the root node. The above example is then modeled as a labeled transition system with two outgoing transitions, one labeled with probability 1/4 and the other with probability 3/4. The first transition is succeeded by a timer that fires immediately. The second transition is succeeded by time-passing transition; this in turn leads to a binary-branching node whose first transition is labeled with probability 1/3 and second transition is labeled with probability 2/3. The first transition again is succeeded by a timer that fires immediately; the second transition again by a time-passing transition and then by a binary-branching node, and so forth. Hence, the timer duration can be between 0 and 3 time instants with uniform probability; however, the choice of remaining duration is made after every time instant.

Our experiments indicated that the binary-branching technique significantly reduced the size (of the state spaces) of TPCCS processes, and hence improved the performance of the TPWB tool. We believe that this optimization is correct for our purposes for the following reasons. First, we conjecture that the modified model has the same linear traces as our original model (i.e. the models are language-equivalent). Second, all of the properties we verified seem to be contained in a linear-time subset of TPCTL. In the usual untimed setting, linear-time properties – even when expressed in a branching-time logic – respect language equivalence. We conjecture that this result generalizes to TPCTL/TPCCS as well, and hence that both models satisfy the same subset of our properties.

Performance Analysis

The TPWB also provides an interface to the Generalized Timed Petri Net Analyzer (GTPNA) [16], which supports performance analysis on a stochastic extension of Petri Nets. Some interesting properties that can be proved using GTPNA include average throughput and resource utilization, both of which are interesting in our application.

The TPWB transforms a restricted subset of TPCCS

into equivalent Generalized Timed Petri Nets. Using the TPWB and GTPNA together, one can thus prove both worst-case temporal properties and average-case performance properties on a single model written in TPCCS. Unfortunately, due to the non-determinism in the incoming alarm rate, our TPCCS models do not fall into the class currently supported by the TPWB tool, and hence we could not use the GTPNA tool in conjunction with our TPCCS models.

BENEFITS

Based on our experience in using COSPAN and TPWB in the context of this 5ESS application, we believe that these tools have many potential benefits in software development. From our experience in using other automatic verification techniques, we are convinced that these benefits also hold for most current verification techniques and tools.

- *Automatic verification can be used to find subtle errors in software designs.*

  Automatic verification performs an exhaustive state-space exploration and considers *all* possible interactions of the concurrent processes in large real-time systems; these interactions often lead to subtle bugs that are extremely difficult to discover using the conventional techniques of simulation and testing.

  We note that verification of software designs does *not* necessarily prove the *absence* of bugs in the target software, since the design may abstract away important details. This is not necessarily a limitation of automatic verification, since "finding the last bug in software" is a well-known myth [6].

  Automatic verification is, however, a very powerful tool in the *detection* of bugs, and can be regarded as a sophisticated form of testing. Furthermore, it provides a form of regression testing: verification tools can automatically check that new versions of the software designs continue to satisfy the desired properties that held of the original design.

- *Errors can be caught earlier in the software development cycle.*

  In contrast to the conventional techniques of simulation and testing, automatic verification finds bugs in software designs as opposed to actual implementations. Hence, bugs can be found at design-time, an early stage of software development, rather than at the late development stage of testing. As is well-known, the cost of correcting software errors is a function of the phase in which corrections are made [5]. In particular, it is far cheaper to fix a design error at design-time rather than after implementation has been completed and testing has begun. Fixing the error at this late stage may entail significant re-implementation and re-testing of the software. For example, significant effort was needed to correct the actual 5ESS application in the context of the upgraded carrier group hardware.

- *Formalisms used in automatic verification support the modeling of designs at a high-level of abstraction.*

  Formal specification languages can be used to describe software designs at a high-level of abstraction, allowing the omission of low-level information. This can result in designs that are easier to understand and maintain. For example, in our models, we abstracted all the computation required to sample the hardware alarms and retained only the time required for this computation. This was sufficient for our purposes since only the timing requirements, and not the details of the computation, led to the real-time difficulty in the software.

- *Automatic verification can be used to effectively generate test cases.*

  Upon discovery that the software design violates a desired property, typical automatic verification tools based on model-checking provide scenarios as evidence of the violation. By mapping from the abstract events in the model to the actual events in the implementation, these scenarios can be used as test-cases on the actual software implementation. Since these scenarios can often be quite subtle – comprised of several hundreds or thousands of state-changes – the corresponding test-cases are extremely difficult to devise by hand. For example, the scenario generated by COSPAN consists of several thousand states.

- *Automatic verification can aid in the maintenance of real-time software.*

  Empirical observations have shown that a significant percentage of errors can occur in redesigns during software maintenance [12]. Upgrades to hardware or processors during the maintenance of real-time systems may introduce real-time difficulties, the correction of which often require the execution of some concurrent processes to be speeded up. Two problems immediately arise, as with our 5ESS application. First, the original software has typically been engineered to be quite efficient in the first place. Thus, even speeding it up by a very small amount can be extremely difficult and may entail significant restructuring of the software architecture. Second, the amount of speedup required to resolve the real-time difficulty is not known prior to testing. Therefore, a coarse estimation of the required speedup needs to be made while modifying

the software. Testing may reveal that the speedup was not sufficient, requiring further modifications of the software and further iterations of testing.

In contrast, formal specifications can be parameterized with respect to the real-time behavior of processes, and automatic verification can be easily repeated on models with different instantiations of these parameters. For example, in our COSPAN models, we changed parameters such as the size of the buffer, the maximum number of alarms per interval, and the duration of the interval between successive arrivals, and tested which combinations satisfy the desired properties and which do not.

- *Automatic verification can support reasoning about the reliability of real-time systems.*

  Using the combination of probabilistic and real-time verification, properties can be proved about the reliability of real-time systems. For example, we were able to show in our TPCCS models that the new software design satisfies the desired properties with higher probability than the older software design, and hence is more reliable.

- *The verification of temporal properties and average-case performance analysis can be combined within a single formalism and tool-set.*

  Typically, automatic verification techniques together with temporal logic support reasoning about worst-case behavior of programs. For example, we proved properties regarding the possibility of process abortion and queue overflow in our S/R models, and the possibility and likelihood of these events in our TPCCS models.

  In addition, average-case properties generally supported by performance analysis tools are also interesting in the context of real-time systems. For example, it would be useful to calculate the average throughput and resource utilization for our application.

  At present, specifications must typically be described in different formalisms in order to use both automatic verification tools and performance analysis tools. The drawback is that there is generally no formal connection between these formalisms, and hence no guarantee that the specifications are equivalent in any rigorous sense. Recently, there has been some promising work on combining these approaches [11, 13, 4].

LIMITATIONS

During the course of our application, we have observed several limitations of the COSPAN and TPWB tools. From our experience in using other automatic verification techniques, we strongly believe that these limitations hold for most current software[2] verification techniques and tools.

- *Expertise with the specification formalisms and the verification algorithms is needed in order to use the tools effectively.*

  We found that formulating suitable abstractions of this application in S/R and TPCCS was subtle. If the synchronization operations and modeling of time passage in these formalisms are not used with care, the models can have subtle errors. Furthermore, knowledge of the model-checking algorithms can greatly affect the performance of the tools. For example, the use of regions for the constraining automaton and the symbolic search using binary decision diagrams is computationally least expensive for this COSPAN application. Concurrent timers must also be used with care to avoid state-space explosion in COSPAN. Similarly, the modeling of uniform distributions using binary branching trees significantly reduced the state space of our TPCCS models.

- *The specification may not be faithful to the software it is intended to model.*

  By definition, the specification formalisms are intended to abstract low-level information from the target software. Even if the specification formalism is used correctly, the abstractions themselves may or may not be correct. Thus, the verification of a specification does not necessarily guarantee that the target software is correct.

  Similarly, the specified probability distributions may not reflect the actual system or environment. For example, we have assumed a uniform probability distribution for both the incoming alarm rate and the process scheduling delays, and a geometric probability distribution for the alarm inter-arrival rate. This is our best guess of the behavior of the actual carrier group alarm hardware and the 5ESS scheduler process. Since our analysis is highly dependent on these distributions, even slightly incorrect modeling of the actual distribution renders the analysis incorrect. This is a limitation of performance analysis tools in general.

- *Verification techniques are computationally expensive and hence use a great deal of space and time.*

  The state-space explosion problem is well-known in the verification community. This problem is even

[2] Some of these limitations are in contrast to hardware verification, where the verification formalisms more closely correspond to the actual hardware.

more acute in real-time verification due to the use of concurrent timers. COSPAN performed efficiently on our particular model; however, in the future, upgraded hardware may again significantly increase the possible rate of incoming 5ESS alarms. Our expectation – based on our other experiences using COSPAN – is that such significant upward scaling would result in state-space explosion problems.

For our TPWB model, we had to significantly scale our design; even with this scaling, we could not prove many potentially interesting timing properties of our models due to space and time limitations of the tool. More research is needed to develop efficient algorithms and heuristics for probabilistic verification.

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The Algorithmic Analysis of Hybrid Systems*

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Abstract

We present a general framework for the formal specification and algorithmic analysis of hybrid systems. A hybrid system consists of a discrete program with an analog environment. We model hybrid systems as finite automata equipped with variables that evolve continuously with time according to dynamical laws. For verification purposes, we restrict ourselves to linear hybrid systems, where all variables follow piecewise-linear trajectories. We provide decidability and undecidability results for classes of linear hybrid systems, and we show that standard program-analysis techniques can be adapted to linear hybrid systems. In particular, we consider symbolic model-checking and minimization procedures that are based on the reachability analysis of an infinite state space. The procedures iteratively compute state sets that are definable as unions of convex polyhedra in multidimensional real space. We also present approximation techniques for dealing with systems for which the iterative procedures do not converge.

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1 Introduction

A hybrid system consists of a discrete program with an analog environment. We assume that a run of a hybrid system is a sequence of steps. Within each step the system state evolves continuously according to a dynamical law until a transition occurs. Transitions are instantaneous state changes that separate continuous state evolutions.

We model a hybrid system as a finite automaton that is equipped with a set of variables. The control locations of the automaton are labeled with evolution laws. At a location the values of the variables change continuously with time according to the associated law. The transitions of the automaton are labeled with guarded sets of assignments. A transition is enabled when the associated guard is true, and its execution modifies the values of the variables according to the assignments. Each location is also labeled with an invariant condition that must hold when the control resides at the location. This model for hybrid systems is inspired by the phase transition systems of [MMP92, NSY93], and can be viewed as a generalization of timed safety automata [AD94, HNSY94].

The purpose of this paper is to demonstrate that standard program-analysis techniques can be adapted to hybrid systems. For verification purposes we restrict ourselves to linear hybrid systems. In a linear hybrid system, for each variable the rate of change is constant—though this constant may vary from location to location—and the terms involved in the invariants, guards, and assignments are required to be linear. An interesting special case of a linear hybrid system is a timed automaton [AD94]. In a timed automaton each continuously changing variable is an accurate clock whose rate of change with time is always 1. Furthermore, in a timed automaton all terms involved in assignments are constants, and all invariants and guards only involve comparisons of clock values with constants. Even though the reachability problem for linear hybrid systems is undecidable, it can be solved for timed automata. In this paper, we provide new decidability and undecidability results for classes of linear hybrid systems, and we show that some algorithms for the analysis of timed automata can be extended to linear hybrid systems to obtain semidecision procedures for various verification problems.

In particular, we consider the symbolic model-checking method for timed automata presented in [HNSY94], and the minimization procedure for timed automata presented in [ACD+92]. Both methods perform a reachability analysis over an infinite state space. The procedures compute state sets by iterative approximation such that each intermediate result is definable by a linear formula; that is, each computed state set is a finite union of convex polyhedra in multidimensional real space. The termination of the procedures, however, is not guaranteed for linear hybrid systems. To cope with this problem, approximate analysis techniques are used to enforce the convergence of iterations by computing upper approximations of state sets. Approximate techniques yield either necessary or sufficient verification conditions.

The paper is essentially a synthesis of the results presented in [ACHH93, NOSY93, HPR94]. Section 2 presents a general model for hybrid systems. Section 3 defines linear hybrid systems, and presents decidability and undecidability results for the reachability problem of subclasses of linear hybrid systems. The verification methods are presented in Section 4. Some paradigmatic examples are specified and verified to illustrate the application of our results. These examples are analyzed using the KRONOS tool [NSY92, NOSY93] (available from Grenoble) and the HYTECH tool [AHH93, HH94] (available from Cornell), two symbolic model checkers for timed and hybrid systems.

2 A Model for Hybrid Systems

We specify hybrid systems by graphs whose edges represent discrete transitions and whose vertices represent continuous activities.

A *hybrid system* $H = (Loc, Var, Lab, Edg, Act, Inv)$ consists of six components:

- A finite set Loc of vertices called *locations*.
- A finite set Var of real-valued *variables*. A *valuation* ν for the variables is a function that assigns a real-value $\nu(x) \in \mathsf{R}$ to each variable $x \in Var$. We write V for the set of valuations.

  A *state* is a pair (ℓ, ν) consisting of a location $\ell \in Loc$ and a valuation $\nu \in V$. We write Σ for the set of states.
- A finite set Lab of *synchronization labels* that contains the *stutter label* $\tau \in Lab$.
- A finite set Edg of edges called *transitions*. Each transition $e = (\ell, a, \mu, \ell')$ consists of a *source* location $\ell \in Loc$, a *target* location $\ell' \in Loc$, a synchronization label $a \in Lab$, and a *transition relation* $\mu \subseteq V^2$. We require that for each location $\ell \in Loc$, there is a set $Con \subseteq Var$ of *controlled variables* and a *stutter transition* of the form $(\ell, \tau, Id_{Con}, \ell)$, where $(\nu, \nu') \in Id_{Con}$ iff for all variables $x \in Var$, either $x \notin Con$ or $\nu(x) = \nu'(x)$.

  The transition e is *enabled* in a state (ℓ, ν) if for some valuation $\nu' \in V$, $(\nu, \nu') \in \mu$. The state (ℓ', ν'), then, is a *transition successor* of the state (ℓ, ν).
- A labeling function Act that assigns to each location $\ell \in Loc$ a set of *activities*. Each activity is a function from the nonnegative reals $\mathsf{R}^{\geq 0}$ to V. We require that the activities of each location are *time-invariant*: for all locations $\ell \in Loc$, activities $f \in Act(\ell)$, and nonnegative reals $t \in \mathsf{R}^{\geq 0}$, also $(f + t) \in Act(\ell)$, where $(f + t)(t') = f(t + t')$ for all $t' \in \mathsf{R}^{\geq 0}$.

  For all locations $\ell \in Loc$, activities $f \in Act(\ell)$, and variables $x \in Var$, we write f^x the function from $\mathsf{R}^{\geq 0}$ to R such that $f^x(t) = f(t)(x)$.
- A labeling function Inv that assigns to each location $\ell \in Loc$ an *invariant* $Inv(\ell) \subseteq V$.

The hybrid system H is *time-deterministic* if for every location $\ell \in Loc$ and every valuation $\nu \in V$, there is at most one activity $f \in Act(\ell)$ with $f(0) = \nu$. The activity f, then, is denoted by $\varphi_\ell[\nu]$.

The runs of a hybrid system

At any time instant, the state of a hybrid system is given by a control location and values for all variables. The state can change in two ways:

- By a *discrete* and *instantaneous* transition that changes both the control location and the values of the variables according to the transition relation;
- By a *time delay* that changes only the values of the variables according to the activities of the current location.

The system may stay at a location only if the location invariant is true; that is, some discrete transition must be taken before the invariant becomes false.

A *run* of the hybrid system H, then, is a finite or infinite sequence

$$\rho: \quad \sigma_0 \mapsto^{t_0}_{f_0} \sigma_1 \mapsto^{t_1}_{f_1} \sigma_2 \mapsto^{t_2}_{f_2} \cdots$$

of states $\sigma_i = (\ell_i, \nu_i) \in \Sigma$, nonnegative reals $t_i \in \mathsf{R}^{\geq 0}$, and activities $f_i \in Act(\ell_i)$, such that for all $i \geq 0$,

1. $f_i(0) = \nu_i$,
2. for all $0 \leq t \leq t_i$, $f_i(t) \in Inv(\ell_i)$,
3. the state σ_{i+1} is a transition successor of the state $\sigma'_i = (\ell_i, f_i(t_i))$.

The state σ'_i is called a *time successor* of the state σ_i; the state σ_{i+1}, a *successor* of σ_i. We write $[H]$ for the set of runs of the hybrid system H.

Notice that if we require all activities to be smooth functions, then the run ρ can be described by a piecewise smooth function whose values at the points of higher-order discontinuity are sequences of discrete state changes. Also notice that for time-deterministic systems, we can omit the subscripts f_i from the *next relation* $\mapsto$.

The run ρ *diverges* if ρ is infinite and the infinite sum $\sum_{i \geq 0} t_i$ diverges. The hybrid system H is *nonzeno* if every finite run of H is a prefix of some divergent run of H. Nonzeno systems can be executed [AH94].

Hybrid systems as transition systems

With the hybrid system H, we associate the labeled transition system $\mathcal{T}_H = (\Sigma, Lab \cup \mathsf{R}^{\geq 0}, \rightarrow)$, where the *step relation* $\rightarrow$ is the union of the *transition-step relations* $\rightarrow^a$, for $a \in Lab$,

$$\frac{(\ell, a, \mu, \ell') \in Edg \qquad (\nu, \nu') \in \mu \qquad \nu, \nu' \in Inv(\ell)}{(\ell, \nu) \rightarrow^a (\ell', \nu')}$$

and the *time-step relations* $\rightarrow^t$, for $t \in \mathsf{R}^{\geq 0}$,

$$\frac{f \in Act(\ell) \qquad f(0) = \nu \qquad \forall 0 \leq t' \leq t.\ f(t') \in Inv(\ell)}{(\ell, \nu) \rightarrow^t (\ell, f(t))}$$

Notice that the stutter transitions ensure that the transition system $\mathcal{T}_H$ is reflexive.

There is a natural correspondence between the runs of the hybrid system H and the paths through the transition system $\mathcal{T}_H$: for all states $\sigma, \sigma' \in \Sigma$, where $\sigma = (\ell, \nu)$, and for all $t \in \mathsf{R}^{\geq 0}$,

$$\exists f \in Act(\ell), \sigma \mapsto^t_f \sigma' \quad \text{iff} \quad \exists \sigma'' \in \Sigma, a \in Lab.\ \sigma \rightarrow^t \sigma'' \rightarrow^a \sigma'.$$

It follows that for every hybrid system, the set of runs is closed under prefixes, suffixes, stuttering, and fusion [HNSY94].

For time-deterministic hybrid systems, the rule for the time-step relation can be simplified. *Time can progress* by the amount $t \in \mathsf{R}^{\geq 0}$ from the state (ℓ, ν) if this is permitted by the invariant of location ℓ; that is,

$$\mathsf{tcp}_\ell[\nu](t) \quad \text{iff} \quad \forall 0 \leq t' \leq t.\ \varphi_\ell[\nu](t') \in Inv(\ell).$$

Now we can rewrite the time-step rule for time-deterministic systems as

$$\frac{\mathsf{tcp}_\ell[\nu](t)}{(\ell, \nu) \rightarrow^t (\ell, \varphi_\ell[\nu](t))}$$

Example: thermostat

The temperature of a room is controlled through a thermostat, which continuously senses the temperature and turns a heater on and off. The temperature is governed by differential equations. When the heater is off, the temperature, denoted by the variable x, decreases according to the exponential function $x(t) = \theta e^{-Kt}$, where t is the time, θ is the initial temperature, and K is a constant determined by the room; when the heater is on, the temperature follows the function $x(t) = \theta e^{-Kt} + h(1 - e^{-Kt})$, where h is a constant that depends on the power of the heater. We wish to keep the temperature between m and M degrees and turn the heater on and off accordingly.

The resulting time-deterministic hybrid system is shown in Figure 1. The system has two locations: in location ℓ_0, the heater is turned off; in location ℓ_1, the heater is on. The transition relations are specified by guarded commands; the activities, by differential equations; and the location invariants, by logical formulas.

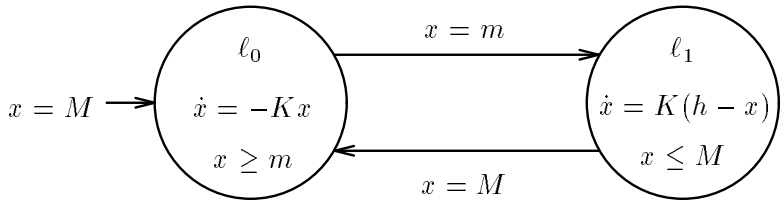


Figure 1: Thermostat

The parallel composition of hybrid systems

Let $H_1 = (Loc_1, Var, Lab_1, Edg_1, Act_1, Inv_1)$ and $H_2 = (Loc_2, Var, Lab_2, Edg_2, Act_2, Inv_2)$ be two hybrid systems over a common set Var of variables. The two hybrid systems synchronize on the common set $Lab_1 \cap Lab_2$ of synchronization labels; that is, whenever H_1 performs a discrete transition with the synchronization label $a \in Lab_1 \cap Lab_2$, then so does H_2.

The *product* $H_1 \times H_2$ is the hybrid system $(Loc_1 \times Loc_2, Var, Lab_1 \cup Lab_2, Edg, Act, Inv)$ such that

- $((\ell_1, \ell_2), a, \mu, (\ell_1', \ell_2')) \in Edg$ iff

  (1) $(\ell_1, a_1, \mu_1, \ell_1') \in Edg_1$ and $(\ell_2, a_2, \mu_2, \ell_2') \in Edg_2$,
  (2) either $a_1 = a_2 = a$, or $a_1 \notin Lab_2$ and $a_2 = \tau$, or $a_1 = \tau$ and $a_2 \notin Lab_1$,
  (3) $\mu = \mu_1 \cap \mu_2$;

- $Act(\ell_1, \ell_2) = Act_1(\ell_1) \cap Act_2(\ell_2)$;

- $Inv(\ell_1, \ell_2) = Inv_1(\ell_1) \cap Inv_2(\ell_2)$.

It follows that all runs of the product system are runs of both component systems:

$$[H_1 \times H_2]_{Loc_1} \subseteq [H_1] \quad \text{and} \quad [H_1 \times H_2]_{Loc_2} \subseteq [H_2]$$

where $[H_1 \times H_2]_{Loc_i}$ is the projection of $[H_1 \times H_2]$ on Loc_i.

Notice also that the product of two time-deterministic hybrid systems is again time-deterministic.

3 Linear Hybrid Systems

A *linear term* over the set *Var* of variables is a linear combination of the variables in *Var* with integer coefficients. A *linear formula* over *Var* is a boolean combination of inequalities between linear terms over *Var*.

The time-deterministic hybrid system $H = (Loc, Var, Lab, Edg, Act, Inv)$ is *linear* if its activities, invariants, and transition relations can be defined by linear expressions over the set *Var* of variables:

1. For all locations $\ell \in Loc$, the activities $Act(\ell)$ are defined by a set of differential equations of the form $\dot{x} = k_x$, one for each variable $x \in Var$, where $k_x \in \mathsf{Z}$ is an integer constant: for all valuations $\nu \in V$, variables $x \in Var$, and nonnegative reals $t \in \mathsf{R}^{\geq 0}$,

$$\varphi_\ell^x[\nu](t) \;=\; \nu(x) + k_x \cdot t.$$

   We write $Act(\ell, x) = k_x$ to refer to the *rate* of the variable x at location ℓ.

2. For all locations $\ell \in Loc$, the invariant $Inv(\ell)$ is defined by a linear formula ψ over *Var*:

$$\nu \in Inv(\ell) \quad \text{iff} \quad \nu(\psi).$$

3. For all transitions $e \in Edg$, the transition relation μ is defined by a guarded set of nondeterministic assignments

$$\psi \;\Rightarrow\; \{x := [\alpha_x, \beta_x] \mid x \in Var\},$$

   where the guard ψ is a linear formula and for each variable $x \in Var$, both interval boundaries α_x and β_x are linear terms:

$$(\nu, \nu') \in \mu \quad \text{iff} \quad \nu(\psi) \wedge \forall x \in Var.\, \nu(\alpha_x) \leq \nu'(x) \leq \nu(\beta_x).$$

   If $\alpha_x = \beta_x$, we write $\mu(e, x) = \alpha_x$ to refer to the updated value of the variable x after the transition e.

Notice that every run of a linear hybrid system can be described by a piecewise linear function whose values at the points of first-order discontinuity are finite sequences of discrete state changes.

Special cases of linear hybrid systems

Various special cases of linear hybrid systems are of particular interest:

- If $Act(\ell, x) = 0$ for each location $\ell \in Loc$, then x is a *discrete variable*. Thus, a discrete variable changes only when the control location changes. A *discrete system* is a linear hybrid system all of whose variables are discrete.

- A discrete variable x is a *proposition* if $\mu(e, x) \in \{0, 1\}$ for each transition $e \in Edg$. A *finite-state system* is a linear hybrid system all of whose variables are propositions.

- If $Act(\ell, x) = 1$ for each location ℓ and $\mu(e, x) \in \{0, x\}$ for each transition e, then x is a *clock*. Thus, (1) the value of a clock increases uniformly with time, and (2) a discrete transition either resets a clock to 0, or leaves it unchanged. A *timed automaton* [AD94] is a linear hybrid system all of whose variables are propositions or clocks, and the linear expressions are boolean combinations of inequalities of the form $x \# c$ or $x - y \# c$ where c is a nonnegative integer and $\# \in \{<, \leq, =, >, \geq\}$.

- If there is a nonzero integer constant $k \in \mathsf{Z}$ such that $Act(\ell, x) = k$ for each location ℓ and $\mu(e, x) \in \{0, x\}$ for each transition e, then x is a *skewed clock*. Thus, a skewed clock is similar to a clock except that it changes with time at some fixed rate different from 1. A *multirate timed system* is a linear hybrid system all of whose variables are propositions and skewed clocks. An *n-rate timed system* is a multirate timed system whose skewed clocks proceed at n different rates.

- If $Act(\ell, x) \in \{0, 1\}$ for each location ℓ and $\mu(e, x) \in \{0, x\}$ for each transition e, then x is an *integrator*. Thus, an integrator is a clock that can be stopped and restarted; it is typically used to measure accumulated durations. An *integrator system* is a linear hybrid system all of whose variables are propositions and integrators.

- A discrete variable x is a *parameter* if $\mu(e, x) = x$ for each transition $e \in Edg$. Thus, a parameter is a symbolic constant. For each of the subclasses of linear hybrid systems listed above, we obtain *parameterized* versions by admitting parameters.

Notice that linear hybrid systems, and all of the subclasses of linear hybrid systems listed above, are closed under parallel composition.

3.1 Examples of Linear Hybrid Systems

A water-level monitor

The water level in a tank is controlled through a monitor, which continuously senses the water level and turns a pump on and off. The water level changes as a piecewise-linear function over time. When the pump is off, the water level, denoted by the variable y, falls by 2 inches per second; when the pump is on, the water level rises by 1 inch per second. Suppose that initially the water level is 1 inch and the pump is turned on. We wish to keep the water level between 1 and 12 inches. But from the time that the monitor signals to change the status of the pump to the time that the change becomes effective, there is a delay of 2 seconds. Thus the monitor must signal to turn the pump on before the water level falls to 1 inch, and it must signal to turn the pump off before the water level reaches 12 inches.

The linear hybrid system of Figure 2 describes a water-level monitor that signals whenever the water level passes 5 and 10 inches, respectively. The system has four locations: in locations 0 and 1, the pump is turned on; in locations 2 and 3, the pump is off. The clock x is used to specify the delays: whenever the control is in location 1 or 3, the signal to switch the pump off or on, respectively, was sent x seconds ago. In the next section, we will prove that the monitor indeed keeps the water level between 1 and 12 inches.

A mutual-exclusion protocol

This example describes a parameterized multirate timed system. We present a timing-based algorithm that implements mutual exclusion for a distributed system with skewed clocks. Consider an asynchronous shared-memory system that consists of two processes P_1 and P_2 with atomic read and write operations. Each process has a critical section and at each time instant, at most one of the two processes is allowed to be in its critical section. Mutual exclusion is ensured by a version of Fischer's protocol [Lam87], which we describe first in pseudocode. For each process P_i, where $i = 1, 2$:

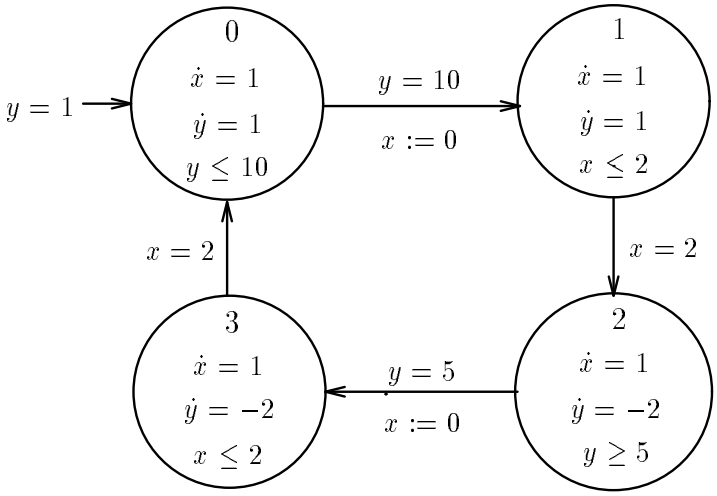


Figure 2: Water-level monitor

repeat
  repeat
    await $k = 0$
    $k := i$
    delay b
  until $k = i$
  Critical section
  $k := 0$
forever

The two processes P_1 and P_2 share a variable k and process P_i is allowed to be in its critical section iff $k = i$. Each process has a private clock. The instruction **delay** b delays a process for at least b time units as measured by the process's local clock. Furthermore, each process takes at most a time units, as measured by the process's clock, for a single write access to the shared memory (i.e., for the assignment $k := i$). The values of a and b are the only information we have about the timing behavior of instructions. Clearly, the protocol ensures mutual exclusion only for certain values of a and b. If both private processor clocks proceed at precisely the same rate, then mutual exclusion is guaranteed iff $a < b$.

To make the example more interesting, we assume that the two private clocks of the processes P_1 and P_2 proceed at different rates, namely, the local clock of P_2 is 1.1 times faster than the clock of P_1. The resulting system can be modeled by the product of the two hybrid systems presented in Figure 3.

Each of the two graphs models one process, with the two critical sections being represented by the locations 4 and D. The private clocks of the processes P_1 and P_2 determine the rate of change of the two skewed-clock variables x and y, respectively.

A leaking gas burner

Now we consider an integrator system. In [CHR91], the duration calculus is used to prove that a gas burner does not leak excessively. It is assumed that (1) any leakage can be detected and

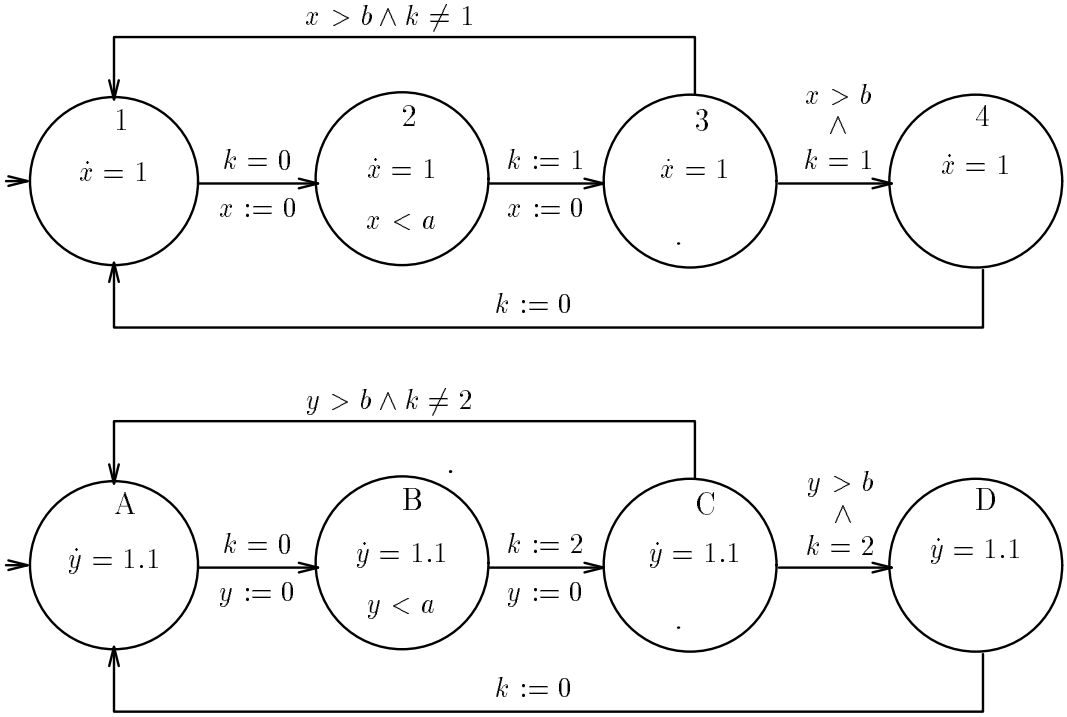


Figure 3: Mutual-exclusion protocol

stopped within 1 second and (2) the gas burner will not leak for 30 seconds after a leakage has been stopped. We wish to prove that the accumulated time of leakage is at most one twentieth of the time in any interval of at least 60 seconds. The system is modeled by the hybrid system of Figure 4. The system has two locations: in location 1, the gas burner leaks; location 2 is the nonleaking location. The integrator z records the cumulative leakage time; that is, the accumulated amount of time that the system has spent in location 1. The clock x records the time the system has spent in the current location; it is used to specify the properties (1) and (2). The clock y records the total elapsed time. In the next section, we will prove that $y \geq 60 \;\Rightarrow\; 20z \leq y$ is an invariant of the system.

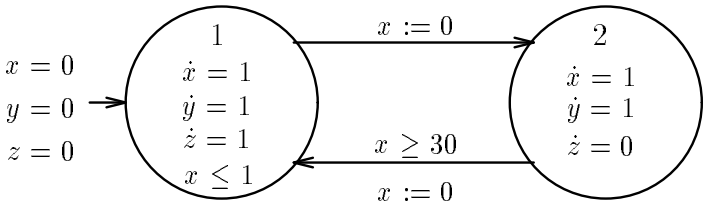


Figure 4: Leaking gas burner

A temperature control system

This example appears in [JLHM91]. The system controls the coolant temperature in a reactor tank by moving two independent control rods. The goal is to maintain the coolant between the temperatures θ_m and θ_M. When the temperature reaches its maximum value θ_M, the tank must be refrigerated with one of the rods. The temperature rises at a rate v_r and decreases at rates v_1 and v_2 depending on which rod is being used. A rod can be moved again only if T time units have elapsed since the end of its previous movement. If the temperature of the coolant cannot decrease because there is no available rod, a complete shutdown is required. Figure 5 shows the hybrid system of this example: variable θ measures the temperature, and the values of clocks x_1 and x_2 represent the times elapsed since the last use of rod 1 and rod 2, respectively.

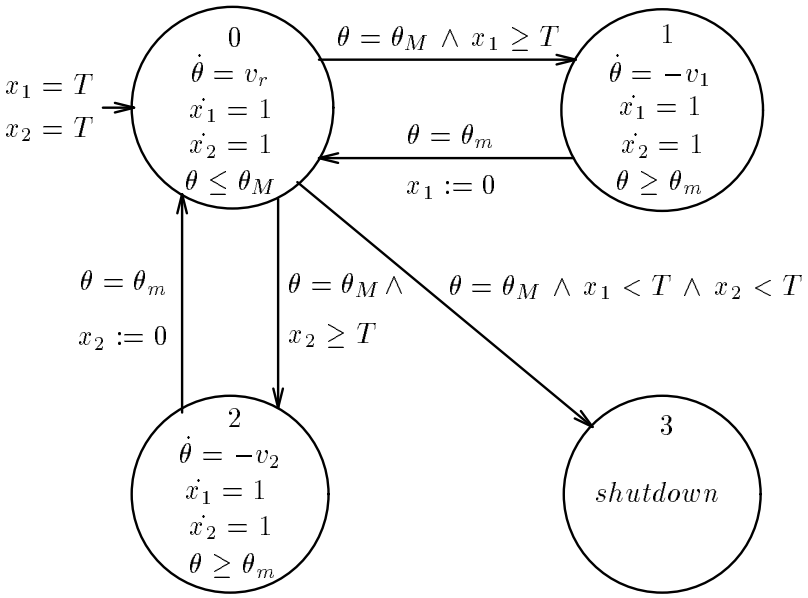


Figure 5: Temperature control system

A game of billiards

Consider a billiard table of dimensions l and h, with a grey ball and a white ball (Figure 6).

Initially, the balls are placed at positions $b_g = (x_g, y_g)$ and $b_w = (x_w, y_w)$. The grey ball is knocked and starts moving with constant velocity v. If the ball reaches a vertical side then it rebounds, i.e., the sign of the horizontal velocity component v_x changes. The same occurs with the vertical velocity component v_y when the ball reaches a horizontal side. The combination of signs of velocity components gives four different directions of movement.

The hybrid system shown in Figure 7 describes the movement of the grey ball for the billiards game. Each possible combination of directions is represented by a location. The rebounds correspond to the execution of transitions between locations.

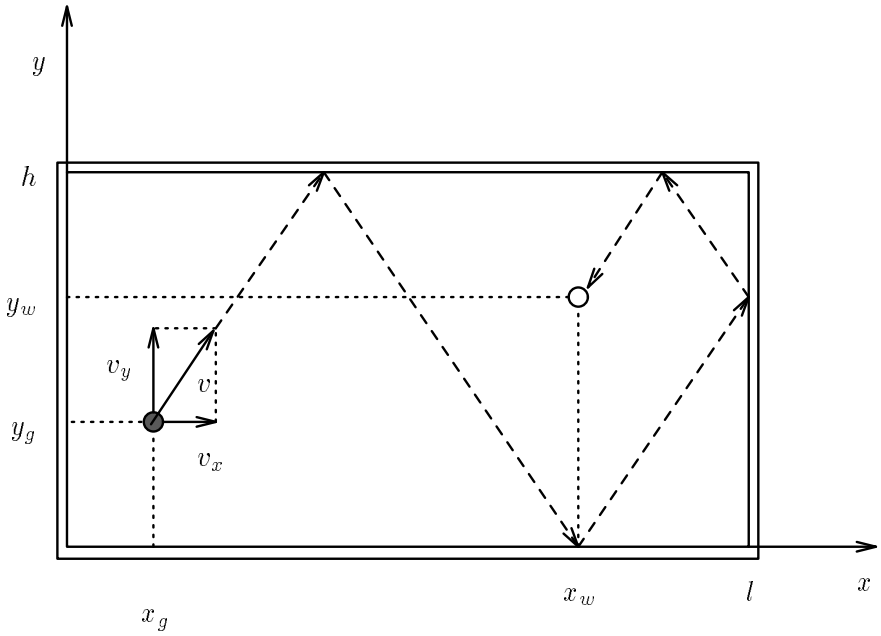


Figure 6: Billiards game

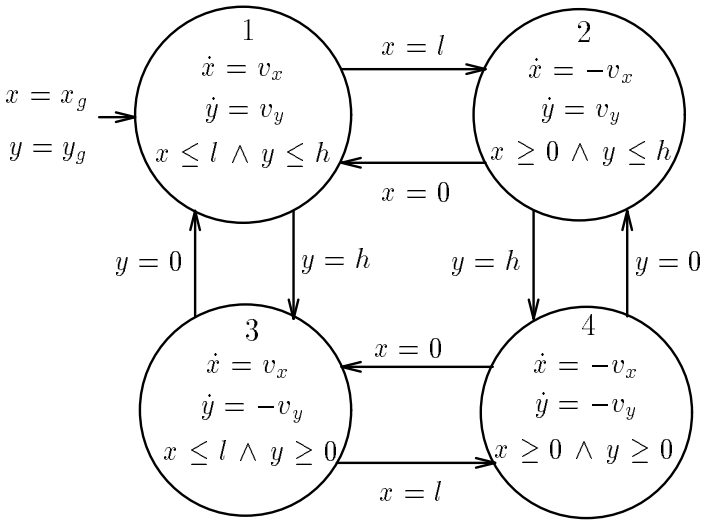


Figure 7: Movement of the grey ball

3.2 The Reachability Problem for Linear Hybrid Systems

Let σ and σ' be two states of a hybrid system H. The state σ' is *reachable* from the state σ, written $\sigma \mapsto^* \sigma'$, if there is a run of H that starts in σ and ends in σ'. The *reachability question* asks, then, if $\sigma \mapsto^* \sigma'$ for two given states σ and σ' of a hybrid system H.

The reachability problem is central to the verification of hybrid systems. In particular, the verification of invariance properties is equivalent to the reachability question: a set $R \subseteq \Sigma$ of states is an invariant of the hybrid system H iff no state in $\Sigma - R$ is reachable from an initial state of H.

A decidability result

A linear hybrid system is *simple* if all linear atoms in location invariants and transition guards are of the form $x \leq k$ or $k \leq x$, for a variable $x \in \mathit{Var}$ and an integer constant $k \in \mathsf{Z}$. In particular, for multirate timed systems the simplicity condition prohibits the comparison of skewed clocks with different rates.

Theorem 3.1 *The reachability problem is decidable for simple multirate timed systems.*

Proof. Let H be a simple multirate timed system. We translate H into a timed automaton $sc(H)$: (1) adjust the rates of all skewed clocks to 1, and (2) replace all occurrences of each skewed clock x in location invariants and transition guards with $k_x \cdot x$. Given a valuation ν of H, let the valuation $sc(\nu)$ be such that $sc(\nu)(x) = k_x \cdot \nu(x)$ for all skewed clocks x and $sc(\nu)(p) = \nu(p)$ for all propositions p; moreover, $sc(\ell, \nu) = (\ell, sc(\nu))$. It is not difficult to check that there is a run of H from σ to σ' iff there is a run of $sc(H)$ from $sc(\sigma)$ to $sc(\sigma')$. The reachability problem for timed automata is solved in [ACD93]. ■

Two undecidability results

Theorem 3.2 *The reachability problem is undecidable for 2-rate timed systems.*

Proof. The theorem follows from the undecidability of the halting problem for nondeterministic 2-counter machines. Given any two distinct clock rates, a 2-rate timed system can encode the computations of the given 2-counter machine M. For the 2-rate timed system H, we use "accurate" clocks of rate 1 and skewed clocks of rate 2. We use an accurate clock y to mark intervals of length 1: the clock y is zero initially, and is reset whenever it reaches 1. The i-th configuration of the machine M is encoded by the state of H at time i. The location of H encodes the program counter of M, and the values of two accurate clocks x_1 and x_2 encode the counter values: the counter value n is encoded by the clock value $1/2^n$.

Encoding the program counter, setting up the initial configuration, and testing a counter for being 0, is straightforward. Hence it remains to be shown how to update the counter values. Suppose at time i the value of an accurate clock x is $1/2^n$, that is, suppose that the clock x is reset to 0 at time $i - 1/2^n$. Suppose the value of the counter encoded by x stays unchanged. Then simply reset x to 0 when its value reaches 1 (that is, at time $(i + 1 - 1/2^n)$); the value of x at time $i + 1$ will then be $1/2^n$. To increment the counter represented by x, reset an accurate clock z when the value of x reaches 1, then nondeterministically reset both x and a skewed clock z' in the interval $(i + 1 - 1/2^n, i + 1)$ and test $z = z'$ at time $i + 1$. The equality test ensures that the value of the skewed clock z' is $1/2^n$ at time $i + 1$, and hence, the value of x is $1/2^{n+1}$ at time $i + 1$. To decrement the counter represented by x, nondeterministically reset an accurate clock z in the interval $(i - 1, i - 1/2^n)$, reset a skewed clock z' simultaneously with x at time $i - 1/2^n$, and test

the condition $z = z'$ at time i. This ensures that the value of z at time i is $1/2^{n-1}$. Then resetting the clock x when the value of z reaches 1 ensures that the value of x is $1/2^{n-1}$ at time $i+1$.

Thus, the runs of H encode the runs of M, and the halting problem for M is reduced to a reachability problem for H. ■

Theorem 3.3 *The reachability problem is undecidable for simple integrator systems.*

Proof. This is proved in [Čer92]. ■

4 The Verification of Linear Hybrid Systems

We present a methodology for analyzing linear hybrid systems that is based on predicate transformers for computing the step predecessors and the step successors of a given set of states. Throughout this section, let $H = (Loc, Var, Lab, Edg, Act, Inv)$ be a linear hybrid system.

4.1 Forward Analysis

Given a location $\ell \in Loc$ and a set of valuations $P \subseteq V$, the *forward time closure* $\langle P \rangle_\ell^\nearrow$ of P at ℓ is the set of valuations that are reachable from some valuation $\nu \in P$ by letting time progress:

$$\nu' \in \langle P \rangle_\ell^\nearrow \quad \text{iff} \quad \exists \nu \in V, t \in \mathsf{R}^{\geq 0}.\ \nu \in P \wedge \mathsf{tcp}_\ell[\nu](t) \wedge \nu' = \varphi_\ell[\nu](t).$$

Thus, for all valuations $\nu' \in \langle P \rangle_\ell^\nearrow$, there exist a valuation $\nu \in P$ and a nonnegative real $t \in \mathsf{R}^{\geq 0}$ such that $(\ell, \nu) \rightarrow^t (\ell, \nu')$.

Given a transition $e = (\ell, a, \mu, \ell')$ and a set of valuations $P \subseteq V$, the *postcondition* $\mathsf{post}_e[P]$ of P with respect to e is the set of valuations that are reachable from some valuation $\nu \in P$ by executing the transition e:

$$\nu' \in \mathsf{post}_e[P] \quad \text{iff} \quad \exists \nu \in V.\ \nu \in P \wedge (\nu, \nu') \in \mu.$$

Thus, for all valuations $\nu' \in \mathsf{post}_e[P]$, there exists a valuation $\nu \in P$ such that $(\ell, \nu) \rightarrow^a (\ell', \nu')$.

A set of states is called a *region*. Given a set $P \subseteq V$ of valuations, by (ℓ, P) we denote the region $\{(\ell, \nu) \mid \nu \in P\}$; we write $(\ell, \nu) \in (\ell, P)$ iff $\nu \in P$. The forward time closure and the postcondition can be naturally extended to regions: for $R = \bigcup_{\ell \in Loc} (\ell, R_\ell)$,

$$\begin{aligned} \langle R \rangle^\nearrow &= \bigcup_{\ell \in Loc} (\ell, \langle R_\ell \rangle_\ell^\nearrow) \\ \mathsf{post}[R] &= \bigcup_{e=(\ell,\ell') \in Edg} (\ell', \mathsf{post}_e[R_\ell]) \end{aligned}$$

A *symbolic run* of the linear hybrid system H is a finite or infinite sequence

$$\varrho: \quad (\ell_0, P_0)\,(\ell_1, P_1) \ldots (\ell_i, P_i) \ldots$$

of regions such that for all $i \geq 0$, there exists a transition e_i from ℓ_i to ℓ_{i+1} and

$$P_{i+1} = \mathsf{post}_{e_i}[\langle P_i \rangle_{\ell_i}^\nearrow];$$

that is, the region (ℓ_{i+1}, P_{i+1}) is the set of states that are reachable from a state $(\ell_0, \nu_0) \in (\ell_0, P_0)$ after executing the sequence $e_0, \ldots, e_i$ of transitions. There is a natural correspondence between

the runs and the symbolic runs of the linear hybrid system H. The symbolic run ϱ represents the set of all runs of the form

$$(\ell_0, \nu_0) \mapsto^{t_0} (\ell_1, \nu_1) \mapsto^{t_1} \cdots$$

such that $(\ell_i, \nu_i) \in (\ell_i, P_i)$ for all $i \geq 0$. Besides, every run of H is represented by some symbolic run of H.

Given a region $I \subseteq \Sigma$, the *reachable region* $(I \mapsto^*) \subseteq \Sigma$ of I is the set of all states that are reachable from states in I:

$$\sigma \in (I \mapsto^*) \quad \text{iff} \quad \exists \sigma' \in I.\, \sigma' \mapsto^* \sigma.$$

Notice that $I \subseteq (I \mapsto^*)$.

The following proposition suggests a method for computing the reachable region $(I \mapsto^*)$ of I.

Proposition 4.1 *Let $I = \bigcup_{\ell \in Loc}(\ell, I_\ell)$ be a region of the linear hybrid system H. The reachable region $(I \mapsto^*) = \bigcup_{\ell \in Loc}(\ell, R_\ell)$ is the least fixpoint of the equation*

$$X \quad = \quad \langle I \cup \mathsf{post}[X] \rangle^{\nearrow}$$

or, equivalently, for all locations $\ell \in Loc$, the set R_ℓ of valuations is the least fixpoint of the set of equations

$$X_\ell \quad = \quad \langle I_\ell \cup \bigcup_{e=(\ell',\ell) \in Edg} \mathsf{post}_e[X_{\ell'}] \rangle^{\nearrow}_{\ell}.$$

Let ψ be a linear formula over *Var*. By $[\![\psi]\!]$ we denote the set of valuations that satisfy ψ. A set $P \subseteq V$ of valuations is *linear* if P is definable by a linear formula; that is, $P = [\![\psi]\!]$ for some linear formula ψ. If *Var* contains n variables, then a linear set of valuations can be thought of as a union of polyhedra in n-dimensional space.

Lemma 4.1 *For all linear hybrid systems H, if $P \subseteq V$ is a linear set of valuations, then for all locations $\ell \in Loc$ and transitions $e \in Edg$, both $\langle P \rangle^{\nearrow}_{\ell}$ and $\mathsf{post}_e[P]$ are linear sets of valuations.*

Given a linear formula ψ, we write $\langle \psi \rangle^{\nearrow}_{\ell}$ and $\mathsf{post}_e[\psi]$ for the linear formulas that define the sets of valuations $\langle [\![\psi]\!] \rangle^{\nearrow}_{\ell}$ and $\mathsf{post}_e[[\![\psi]\!]]$, respectively.

Let $pc \notin Var$ be a *control variable* that ranges over the set *Loc* of locations and let $R = \bigcup_{\ell \in Loc}(\ell, R_\ell)$ be a region. The region R is *linear* if for every location $\ell \in Loc$, the set R_ℓ of valuations is linear. If the sets R_ℓ are defined by the linear formulas ψ_ℓ, then the region R is defined by the linear formula

$$\psi \quad = \quad \bigvee_{\ell \in Loc} (pc = \ell \wedge \psi_\ell);$$

that is, $[\![\psi]\!] = R$. Hence, by Lemma 4.1, for all linear hybrid systems, if R is a linear region, then so are both $\langle R \rangle^{\nearrow}$ and $\mathsf{post}[R]$.

Using Proposition 4.1, we compute the reachable region $(I \mapsto^*)$ of a region I by successive approximation. Lemma 4.1 ensures that all regions computed in the process are linear. Since the reachability problem for linear hybrid systems is undecidable, the successive-approximation procedure does not terminate in general. The procedure does terminate for simple multirate timed systems (Theorem 3.1) and for the following example.

Example: the leaking gas burner

Let I be the set of initial states defined by the linear formula

$$\psi_I \;=\; (pc = 1 \wedge x = y = z = 0).$$

The set $(I \mapsto^*)$ of reachable states is characterized by the least fixpoint of the two equations

$$\begin{aligned} \psi_1 &= \langle x = y = z = 0 \vee \mathsf{post}_{(2,1)}[\psi_2] \rangle_1^{\nearrow} \\ \psi_2 &= \langle \mathit{false} \vee \mathsf{post}_{(1,2)}[\psi_1] \rangle_2^{\nearrow} \end{aligned}$$

which can be iteratively computed as

$$\begin{aligned} \psi_{1,i} &= \psi_{1,i-1} \vee \langle \mathsf{post}_{(2,1)}[\psi_{2,i-1}] \rangle_1^{\nearrow} \\ \psi_{2,i} &= \psi_{2,i-1} \vee \langle \mathsf{post}_{(1,2)}[\psi_{1,i-1}] \rangle_2^{\nearrow} \end{aligned}$$

where $\psi_{1,0} = \langle x = y = z = 0 \rangle_1^{\nearrow} = (x \leq 1 \wedge y = x = z)$ and $\psi_{2,0} = \mathit{false}$. For $i = 1$, we have

$$\begin{aligned} \psi_{1,1} &= \psi_{1,0} \vee \langle \mathsf{post}_{(2,1)}[\psi_{2,0}] \rangle_1^{\nearrow} \\ &= \psi_{1,0} \\ \psi_{2,1} &= \psi_{2,0} \vee \langle \mathsf{post}_{(1,2)}[\psi_{1,1}] \rangle_2^{\nearrow} \\ &= \langle \mathsf{post}_{(1,2)}[x \leq 1 \wedge y = x = z = 0] \rangle_2^{\nearrow} \\ &= \langle (x = 0 \wedge y \leq 1 \wedge z = y) \rangle_2^{\nearrow} \\ &= (z \leq 1 \wedge y = z + x) \end{aligned}$$

Now, it is easy to show by induction that for all $i \geq 2$,

$$\psi_{1,i} = \psi_{1,i-1} \vee x \leq 1 \wedge 0 \leq z - x \leq i \wedge 30i + z \leq y$$

and

$$\psi_{2,i} = \psi_{2,i-1} \vee y \leq i + 1 \wedge 30i + x + z \leq y$$

Hence, the least solution of the equations above is the linear formula

$$\psi_R \;=\; (pc = 1 \wedge \psi_1) \vee (pc = 2 \wedge \psi_2)$$

where

$$\begin{aligned} \psi_1 &= x \leq 1 \wedge x = y = z \vee \exists i \geq 1.\, (x \leq 1 \wedge 0 \leq z - x \leq i \wedge 30i + z \leq y) \\ &= (x \leq 1 \wedge x = y = z) \vee (x \leq 1 \wedge x \leq z \wedge y + 30x \geq 31z) \\ \psi_2 &= z \leq 1 \wedge y = x + z \wedge x \geq 0 \vee \exists i \geq 1.\, (z \leq i + 1 \wedge 30i + x + z \leq y) \\ &= (z \leq 1 \wedge y = x + z \wedge x \geq 0) \vee y \geq x + 31z - 30 \end{aligned}$$

This characterization of the reachable states can be used to verify invariance properties of the gas burner system (ψ_R is the strongest invariant of the system). For instance, the formula ψ_R implies the design requirement $y \geq 60 \;\Rightarrow\; 20z \leq y$.

4.2 Backward Analysis

The forward time closure and the postcondition define the successor of a region R. Dually, we can compute the predecessor of R.

Given a location $\ell \in Loc$ and a set of valuations $P \subseteq V$, the *backward time closure* of P at ℓ is the set of valuations from which it is possible to reach some valuation $\nu \in P$ by letting time progress:

$$\nu' \in \langle P \rangle_\ell^{\swarrow} \quad \text{iff} \quad \exists \nu \in V, t \in \mathsf{R}^{\geq 0}.\, \nu = \varphi_\ell[\nu'](t) \wedge \nu \in P \wedge \mathsf{tcp}_\ell[\nu'](t).$$

Thus, for all valuations $\nu' \in \langle P \rangle_\ell^{\swarrow}$, there exist a valuation $\nu \in P$ and a nonnegative real $t \in \mathsf{R}^{\geq 0}$ such that $(\ell, \nu') \rightarrow^t (\ell, \nu)$.

Given a transition $e = (\ell, a, \mu, \ell')$ and a set of valuations $P \subseteq V$, the *precondition* $\mathsf{pre}_e[P]$ of P with respect to e is the set of valuations from which it is possible to reach a valuation $\nu \in P$ by executing the transition e:

$$\nu' \in \mathsf{pre}_e[P] \quad \text{iff} \quad \exists \nu \in V.\, \nu \in P \wedge (\nu', \nu) \in \mu.$$

Thus, for all valuations $\nu' \in \mathsf{pre}_e[P]$, there exists a valuation $\nu \in P$ such that $(\ell, \nu') \rightarrow^a (\ell', \nu)$.

The backward time closure and the precondition can be naturally extended to regions: for $R = \bigcup_{\ell \in Loc} (\ell, R_\ell)$,

$$\begin{aligned} \langle R \rangle^{\swarrow} &= \bigcup_{\ell \in Loc} (\ell, \langle R_\ell \rangle_\ell^{\swarrow}) \\ \mathsf{pre}[R] &= \bigcup_{e=(\ell',\ell) \in Edg} (\ell', \mathsf{pre}_e[R_\ell]) \end{aligned}$$

Given a region $R \subseteq \Sigma$, the *initial region* $(\mapsto^* R) \subseteq \Sigma$ of R is the set of all states from which a state in R is reachable:

$$\sigma \in (\mapsto^* R) \quad \text{iff} \quad \exists \sigma' \in R.\, \sigma \mapsto^* \sigma'.$$

Notice that $R \subseteq (\mapsto^* R)$.

The following proposition suggests a method for computing the initial region $(\mapsto^* R)$ of R.

Proposition 4.2 *Let $R = \bigcup_{\ell \in Loc} (\ell, R_\ell)$ be a region of the linear hybrid system H. The initial region $I = \bigcup_{\ell \in Loc} (\ell, I_\ell)$ is the least fixpoint of the equation*

$$X = \langle R \cup \mathsf{pre}[X] \rangle^{\swarrow}$$

or, equivalently, for all locations $\ell \in Loc$, the set I_ℓ of valuations is the least fixpoint of the set

$$X_\ell = \langle R_\ell \cup \bigcup_{e=(\ell,\ell') \in Edg} \mathsf{pre}_e[X_{\ell'}] \rangle_\ell^{\swarrow}$$

of equations.

Lemma 4.2 *For all linear hybrid systems H, if $P \subseteq V$ is a linear set of valuations, then for all locations $\ell \in Loc$ and transitions $e \in Edg$, both $\langle P \rangle_\ell^{\swarrow}$ and $\mathsf{pre}_e[P]$ are linear sets of valuations.*

It follows that for all linear hybrid systems, if R is a linear region, then so are both $\langle R \rangle^{\swarrow}$ and $\mathsf{pre}[R]$. Given a linear formula ψ, we write $\langle \psi \rangle_\ell^{\swarrow}$ and $\mathsf{pre}_e[\psi]$ for the linear formulas that define the sets of valuations $\langle [\![\psi]\!] \rangle_\ell^{\swarrow}$ and $\mathsf{pre}_e[[\![\psi]\!]]$, respectively.

Example: the leaking gas burner

We apply backward analysis to prove that the design requirement $y \geq 60 \;\Rightarrow\; 20z \leq y$ is an invariant of the gas burner system; that is, the region R defined by the linear formula

$$\psi_R \;=\; (y \geq 60 \wedge 20z > y)$$

is not reachable from the set I of initial states defined by the linear formula

$$\psi_I \;=\; (pc = 1 \wedge x = y = z = 0).$$

The set $(\mapsto^* R)$ of states from which it is possible to reach a state in R is characterized by the least fixpoint of the two equations

$$\begin{aligned}\psi_1 &= \langle (y \geq 60 \wedge 20z > y) \vee \mathsf{pre}_{(1,2)}[\psi_2]\rangle_1^{\swarrow} \\ \psi_2 &= \langle (y \geq 60 \wedge 20z > y) \vee \mathsf{pre}_{(2,1)}[\psi_1]\rangle_2^{\swarrow}\end{aligned}$$

which can be iteratively computed as

$$\begin{aligned}\psi_{1,i} &= \langle \mathsf{pre}_{(1,2)}[\psi_{2,i-1}]\rangle_1^{\swarrow} \\ \psi_{2,i} &= \langle \mathsf{pre}_{(2,1)}[\psi_{1,i-1}]\rangle_2^{\swarrow}\end{aligned}$$

where $\psi_{1,0} = \langle (y \geq 60 \wedge 20z > y)\rangle_1^{\swarrow}$ and $\psi_{2,0} = \langle (y \geq 60 \wedge 20z > y)\rangle_2^{\swarrow}$. Then,

$$\begin{aligned}\psi_{1,0} &= (-19 < 20\,z - 19\,x - y \wedge 59 < -x + y \wedge x \leq 1), \\ \psi_{2,0} &= (0 < 20\,z + x - y \wedge 0 < 20\,z - y \wedge 3 < z), \\ \psi_{1,1} &= (-19 < 20\,z - y - 19\,x \wedge 2 < z - x \wedge x \leq 1), \\ \psi_{2,1} &= (-19 < 20\,z - y \wedge 2 < z \wedge 11 < 20\,z + x - y), \\ \psi_{1,2} &= (-8 < 20\,z - 19\,x - y \wedge 1 < z - x \wedge x \leq 1), \\ \psi_{2,2} &= (-19 < 20\,z - y \wedge 2 < z \wedge 11 < 20\,z + x - y), \\ \psi_{1,3} &= (-8 < 20\,z - 19\,x - y \wedge 1 < z - x \wedge x \leq 1), \\ \psi_{2,3} &= (-8 < 20\,z - y \wedge 1 < z \wedge 22 < 20\,z + x - y), \\ \psi_{1,4} &= (3 < 20\,z - 19\,x - y \wedge 0 < z - x \wedge x \leq 1), \\ \psi_{2,4} &= (-8 < 20\,z - y \wedge 1 < z \wedge 22 < 20\,z + x - y), \\ \psi_{1,5} &= (3 < 20\,z - 19\,x - y \wedge 0 < z - x \wedge x \leq 1), \\ \psi_{2,5} &= (3 < 20\,z - y \wedge 0 < z \wedge 33 < 20\,z + x - y), \\ \psi_{1,6} &= (14 < 20\,z - 19\,x - y \wedge -1 < z - x \wedge x \leq 1), \text{ and} \\ \psi_{2,6} &= (3 < 20\,z - y \wedge 0 < z \wedge 33 < 20\,z + x - y).\end{aligned}$$

Since $\psi_{1,7} \;\Rightarrow\; \psi_{1,6}$ and $\psi_{2,7} \;\Rightarrow\; \psi_{2,6}$, the solution ψ is

$$\bigvee_{0 \leq i \leq 6} (pc = 1 \wedge \psi_{1,i}) \vee (pc = 2 \wedge \psi_{2,i}),$$

which contains no initial states; that is, $\psi_I \wedge \psi = \mathit{false}$. It follows that the design requirement is an invariant.

4.3 Approximate Analysis

In this section, we briefly present an approximate technique for dealing with systems where the (forward or backward) iterative procedure does not converge. For more details, see [HH94, HPR94]. We will compute *upper approximations* of the sets

- $(I \mapsto^*)$ of states which are reachable from the initial states I (forward analysis)
- $(\mapsto^* R)$ of states from which the region R is reachable (backward analysis)

We focus on forward analysis, backward analysis is similar. Let us come back to the system of fixpoint equations whose least solution gives, for each location ℓ, the set X_ℓ of reachable states at location ℓ:

$$X_\ell \quad = \quad \langle I_\ell \cup \bigcup_{e=(\ell',\ell)\in Edg} \mathsf{post}_e[X_{\ell'}] \rangle_\ell^\nearrow$$

Two problems arise in the practical resolution of such a system:

- Handling disjunctions of systems of linear inequalities; for instance there is no easy way for deciding if a union of polyhedra is included into another.
- The fixpoint computation may involve infinite iteration.

An approximate solution to these problems is provided by abstract interpretation techniques [CC77, CH78].

First, union of polyhedra is approximated by their *convex hull*, i.e., the least convex polyhedron containing the operands of the union. Let $\sqcup$ denote the convex hull operator:

$$P \sqcup P' = \{\lambda x + (1-\lambda)x' \mid x \in P, x' \in P', \lambda \in [0,1]\}$$

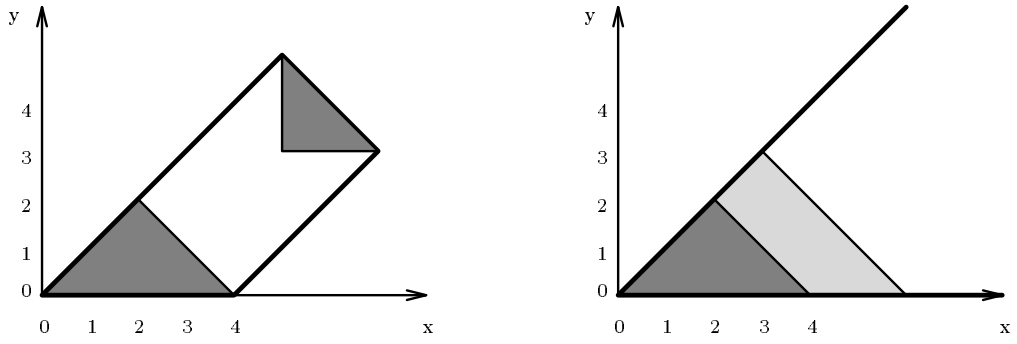
Fig. 8.a shows an example of convex hull. See [CH78, LeV92] for efficient algorithms to compute the convex hull. The system of equations becomes:

$$X_\ell \quad = \quad \langle I_\ell \sqcup \bigsqcup_{e=(\ell',\ell)\in Edg} \mathsf{post}_e[X_{\ell'}] \rangle_\ell^\nearrow$$

To enforce the convergence of iterations, we apply Cousot's "widening technique" [CC77, CH78]. The idea is to extrapolate the limit of a sequence of polyhedra, in such a way that an upper approximation of the limit be always reached in a finite number of iterations. We define a *widening operator*, noted ∇, on polyhedra, such that

- For each pair (P, P') of polyhedra, $P \sqcup P' \subseteq P \nabla P'$
- For each infinite increasing sequence $(P_0, P_1, \ldots, P_n, \ldots)$ of polyhedra, the sequence defined by $Q_0 = P_0$, $Q_{n+1} = Q_n \nabla P_{n+1}$ is not strictly increasing (i.e., remains constant after a finite number of terms).

A widening operator on polyhedra has been defined in [CH78, Hal93]. Intuitively, the system of linear constraints of $P \nabla P'$ is made of exactly those constraints of P which are also satisfied by P'. So it is built by removing constraints from P and since we cannot remove infinitely many constraints, the finiteness property follows. Fig. 8.b illustrates the widening operation. Now, this operator is used as follows: Choose, in each loop of the graph of the hybrid system, at least



$\{0 \leq y \leq x \leq 4 - y\}$
$\sqcup\{x \geq 5 \wedge y \geq 3 \wedge x + y \leq 10\}$
$= \{0 \leq y \leq x \leq y + 4 \wedge x + y \leq 10\}$

(a). Convex hull

$\{0 \leq y \leq x \leq 4 - y\}$
$\nabla\{0 \leq y \leq x \leq 6 - y\}$
$= \{0 \leq y \leq x\}$

(b). Widening

Figure 8: Approximation operators

one location, and call them "*widening locations*" (So, removing these locations would cut each loop in the graph). Let $X_\ell^{(n)} = F(X^{(n-1)})$ be the n-th step computation at location ℓ; that is, $F(X^{(n-1)}) = \langle I_\ell \sqcup \bigsqcup_{e=(\ell',\ell)\in Edg} \mathsf{post}_e[X_{\ell'}^{(n-1)}]\rangle_\ell^\nearrow$. Instead, for each widening location ℓ and each step $n \geq 1$, compute $X_\ell^{(n)} = X_\ell^{(n-1)} \nabla F(X^{(n-1)})$. Then, the new iterative computation converges after a finite number of steps toward an upper approximation of the least solution of the original system.

Example: the leaking gas burner

With I defined by $\psi_I = (pc = 1 \wedge x = y = z = 0)$, we have $(I \mapsto^*) = X_1 \cup X_2$, with $X_i = \lim X_i^{(n)}, (i = 1, 2)$ and (choosing location 1 as the only widening location)

$$\begin{array}{lcl} X_1^{(n)} & = & X_1^{(n-1)} \nabla \langle (x = y = z = 0) \sqcup \mathsf{post}_{(2,1)}[X_2^{(n-1)}]\rangle_1^\nearrow \\ X_2^{(n)} & = & \langle \mathsf{post}_{(1,2)}[X_1^{(n)}]\rangle_2^\nearrow \end{array}$$

The successive iterations are as follows:

Step 1:

$$\begin{array}{lcl} X_1^{(1)} & = & x = y = z \wedge 0 \leq x \leq 1 \\ X_2^{(1)} & = & y = x + z \wedge 0 \leq x \wedge 0 \leq z \leq 1 \end{array}$$

Step 2:

$$\begin{array}{lcl} X_1^{(2)} & = & 31z \leq 30x + y \wedge x \leq z \wedge 0 \leq x \leq 1 \\ X_2^{(2)} & = & x + z \leq y \wedge 0 \leq x \wedge 0 \leq z \wedge x + 31z \leq y + 30 \end{array}$$

and Step 3 shows the convergence:

$$X_1^{(3)} = X_1^{(2)} \quad , \quad X_2^{(3)} = X_2^{(2)}$$

So the final results are:

$$\begin{aligned} X_1 &= 0 \le x \le 1 \wedge x \le z \wedge 31z \le y + 30x \\ X_2 &= 0 \le x \wedge 0 \le z \wedge x + z \le y \wedge x + 31z \le y + 30 \end{aligned}$$

These results are obtained in 0.2 sec. on SUN 4 Sparc Station. Notice that, in this case, the results are almost exact, and have been obtained automatically, without the induction step used in §4.1.

Other examples

Water-level monitor. Choosing location 0 as the only widening location, we get (in 0.4 sec.) the following results:

$$\begin{aligned} X_0 &= 1 \le y \le 10 \\ X_1 &= y = x + 10 \wedge 0 \le x \le 2 \\ X_2 &= 2x + y = 16 \wedge 4 \le 2x \le 11 \\ X_3 &= 2x + y = 5 \wedge 0 \le x \le 2 \end{aligned}$$

We can easily check that X_i implies $1 \le y \le 12$ for $0 \le i \le 3$. So, the water level is kept between 1 and 2 inches as required.

Fischer's mutual-exclusion protocol. In this example, we can consider delays a and b as symbolic constants, letting the analysis *discover* sufficient conditions for the algorithm to work. With two processes, the results (obtained in 0.3 sec.) show that the locations where the mutual exclusion is violated can only be reached when $a \ge b$ (resp., $11a \ge 10b$ when P_2's local clock runs 1.1 faster than P_1's).

4.4 Minimization

We extend the next relation $\mapsto$ to regions: for all regions R and R', we write $R \mapsto R'$ if some state $\sigma' \in R'$ is a successor of some state $\sigma \in R$, that is

$$R \mapsto R' \quad \text{iff} \quad \exists \sigma \in R, \sigma' \in R'.\, \sigma \mapsto \sigma'.$$

We write $\mapsto^*$ for the reflexive-transitive closure of $\mapsto$.

Let π be a partition of the state space Σ. A region $R \in \pi$ is *stable* if for all $R' \in \pi$,

$$R \mapsto R' \quad \text{implies} \quad \forall \sigma \in R.\, \{\sigma\} \mapsto R'$$

or, equivalently,

$$R \cap \mathsf{pre}[\langle R' \rangle^{\swarrow}] \neq \emptyset \quad \text{implies} \quad R \subseteq \mathsf{pre}[\langle R' \rangle^{\swarrow}].$$

The partition π is a *bisimulation* if every region $R \in \pi$ is stable. The partition π *respects* the region R_F if for every region $R \in \pi$, either $R \subseteq R_F$ or $R \cap R_F = \emptyset$.

If a partition π that respects the region R_F is a bisimulation, then it can be used to compute the initial region $(\mapsto^* R_F)$: for all regions $R \in \pi$, if $R \mapsto^* R_F$ then $R \subseteq (\mapsto^* R_F)$, otherwise $R \cap (\mapsto^* R_F) = \emptyset$. Thus, our objective is to construct the coarsest bisimulation that respects a given region R_F, provided there is a finite bisimulation that respects R_F.

If we are given, in addition to R_F, an initial region I that restricts our interest to the reachable region ($I \mapsto^*$), then it is best to use an algorithm that performs a simultaneous reachability and minimization analysis of transition systems [BFH90, LY92].

The minimization procedure of [BFH90] is given below. Starting from the initial partition $\{R_F, \Sigma - R_F\}$ that respects R_F, the procedure selects a region R and checks if R is stable with respect to the current partition; if not, then R is split into smaller sets. Additional book-keeping is needed to record which regions are reachable from the initial region I. In the following procedure, π is the current partition, $\alpha \subseteq \pi$ contains the regions R that have been found reachable from I, and $\beta \subseteq \pi$ contains the regions R that have been found stable with respect to π. The function $\mathsf{split}[\pi](R)$ splits the region $R \in \pi$ into subsets that are "more" stable with respect to π:

$$\mathsf{split}[\pi](R) := \begin{cases} \{R', R - R'\} & \text{if} \quad \exists R'' \in \pi.\ R' = \mathsf{pre}[\langle R'' \rangle^{\swarrow}] \cap R \wedge R' \subset R, \\ \{R\} & \text{otherwise.} \end{cases}$$

The minimization procedure returns YES iff $I \mapsto^* R_F$.

State-space minimization:

$\pi := \{R_F, \Sigma - R_F\}$; $\alpha := \{R \mid R \cap I \neq \emptyset\}$; $\beta := \emptyset$
while $\alpha \neq \beta$ **do**
  choose $R \in (\alpha - \beta)$
  let $\alpha' := \mathsf{split}[\pi](R)$
  if $\alpha' = \{R\}$ **then**
    $\beta := \beta \cup \{R\}$
    $\alpha := \alpha \cup \{R' \in \pi \mid R \mapsto R'\}$
  else
    $\alpha := \alpha - \{R\}$
    if $\exists R' \in \alpha'$ such that $R' \cap I \neq \emptyset$ **then** $\alpha := \alpha \cup \{R'\}$ **fi**
    $\beta := \beta - \{R' \in \pi \mid R' \mapsto R\}$
    $\pi := (\pi - \{R\}) \cup \alpha'$
  fi
od
return there is $R \in \alpha$ such that $R \subseteq R_F$.

If the regions R_F and I are linear, from Lemma 4.2 it follows that all regions that are constructed by the minimization procedure are linear. The minimization procedure terminates if the coarsest bisimulation has only a finite number of equivalence classes. An alternative minimization procedure is presented in [LY92], which can also be implemented using the primitives $\langle\rangle^{\swarrow}$ and pre.

Example: the water-level monitor

Let H be the hybrid automaton defined in Figure 2. We use the minimization procedure to prove that the formula $1 \leq y \leq 12$ is an invariant of H. It follows that the water-level monitor keeps the water level between 1 and 12 inches.

Let the set I of initial states be so defined by the linear formula

$$\psi_I \quad = \quad (pc = 0 \wedge x = 0 \wedge y = 1)$$

and let the set R_F of "bad" states be defined by the linear formula

$$\psi_f \;=\; (y < 1 \vee y > 12).$$

The initial partition is $\pi_1 = \{$

$$\begin{array}{llll}
\psi_{00} = & (pc = 0 \wedge 1 \leq y \leq 12), & \psi_{01} = & (pc = 0 \wedge (y < 1 \vee y > 12)),\\
\psi_{10} = & (pc = 1 \wedge 1 \leq y \leq 12), & \psi_{11} = & (pc = 1 \wedge (y < 1 \vee y > 12)),\\
\psi_{20} = & (pc = 2 \wedge 1 \leq y \leq 12), & \psi_{21} = & (pc = 2 \wedge (y < 1 \vee y > 12)),\\
\psi_{30} = & (pc = 3 \wedge 1 \leq y \leq 12), & \psi_{31} = & (pc = 3 \wedge (y < 1 \vee y > 12))\}.
\end{array}$$

The bad states are represented by ψ_{i1}, for $i \in \{0,1,2,3\}$. Since the set I of initial states is contained in ψ_{00}, that is $\psi_I \;\Rightarrow\; \psi_{00}$, let $\alpha = \{\psi_{00}\}$. Considering $\psi = \psi_{00} \in \alpha$, we find that $\mathsf{split}[\pi_1](\psi_{00}) = \{$

$$\begin{array}{lll}
\psi_{000} & = & (pc = 0 \wedge 1 \leq y \leq 10),\\
\psi_{001} & = & (pc = 0 \wedge 10 < y \leq 12)\}.
\end{array}$$

Therefore, $\pi_2 = \{\psi_{000}, \psi_{001}, \psi_{01}, \psi_{10}, \psi_{11}, \psi_{20}, \psi_{21}, \psi_{30}, \psi_{31}\}$. Now $\psi_I \;\Rightarrow\; \psi_{000}$, so take $\alpha = \{\psi_{000}\}$ and $\beta = \emptyset$. Considering $\psi = \psi_{000}$, we find that it is stable with respect to π_2. Thus $\alpha = \alpha \cup \{R' \in \pi \mid R \mapsto R'\} = \{\psi_{000}, \psi_{001}, \psi_{10}\}$ and $\beta = \{\psi_{000}\}$. Since $\psi = \psi_{001}$ is also stable in π_2 and is not reaching any new states not in α, α remains the same and $\beta = \{\psi_{000}, \psi_{001}\}$. However, considering $\psi = \psi_{10}$, we obtain $\mathsf{split}[\pi_2](\psi_{10}) = \{$

$$\begin{array}{lll}
\psi_{100} & = & (pc = 1 \wedge 0 \leq x \leq 2 \wedge 1 \leq y \leq 12),\\
\psi_{101} & = & (pc = 1 \wedge x > 2 \wedge 1 \leq y \leq 12)\}.
\end{array}$$

Now, ψ_{100} and ψ_{101} together with π_2, except for ψ_{10}, constitute π_3. The new β is obtained by removing $\{R' \in \pi \mid R' \mapsto R\} = \psi_{000}$ from the old β. The new α becomes $\{\psi_{000}, \psi_{001}\}$. Now $\psi = \psi_{000}$ is stable in π_3. Hence $\alpha = \{\psi_{000}, \psi_{001}, \psi_{100}\}$ and $\beta = \{\psi_{000}, \psi_{001}\}$. Since $\psi = \psi_{100}$ is stable in π_3, we have $\alpha = \{\psi_{000}, \psi_{001}, \psi_{100}, \psi_{101}, \psi_{20}\}$ and $\beta = \{\psi_{000}, \psi_{001}, \psi_{100}\}$. $\psi = \psi_{101}$ is also stable in π_3, so $\beta = \{\psi_{000}, \psi_{001}, \psi_{100}, \psi_{101}\}$ and α remains unchanged. Considering $\psi = \psi_{20}$, we obtain $\mathsf{split}[\pi_3](\psi_{20}) = \{$

$$\begin{array}{lll}
\psi_{200} & = & (pc = 2 \wedge 5 \leq y \leq 12),\\
\psi_{201} & = & (pc = 2 \wedge 1 \leq y < 5)\}.
\end{array}$$

Now π_4 contains ψ_{200} and ψ_{201}, and thus ψ_{100} must be reconsidered. It is split into $\mathsf{split}[\pi_4](\psi_{100}) = \{$

$$\begin{array}{lll}
\psi_{1000} & = & (pc = 1 \wedge 0 \leq x \leq 2 \wedge 3 \leq y \leq 12 \wedge 3 \leq y - x \leq 12),\\
\psi_{1001} & = & (pc = 1 \wedge 0 \leq x \leq 2 \wedge 1 \leq y < 3 \wedge 1 \leq y - x < 3)\}.
\end{array}$$

Thus π_5 contains ψ_{1000} and ψ_{1001}. After finding that ψ_{000}, ψ_{1000} and ψ_{200} all are stable, we finally have $\alpha = \{\psi_{000}, \psi_{001}, \psi_{1000}, \psi_{200}, \psi_{201}, \psi_{30}\}$ and $\beta = \{\psi_{000}, \psi_{001}, \psi_{1000}, \psi_{200}\}$. So let $\psi = \psi_{201}$. It is stable, so $\beta = \beta \cup \{\psi_{200}\}$ and α does not change. Then $\psi = \psi_{30}$ is partitioned into $\{$

$$\begin{array}{lll}
\psi_{300} & = & (pc = 3 \wedge 0 \leq x \leq 2 \wedge 1 \leq y \leq 12),\\
\psi_{301} & = & (pc = 3 \wedge x > 2 \wedge 1 \leq y \leq 12)\}.
\end{array}$$

ψ_{200} has to be considered again. It is stable with respect to the current partition. Then $\psi = \psi_{300}$ is considered and $\mathsf{split}[\pi_6](\psi_{300}) = \{$

$$\begin{array}{lll}
\psi_{3000} & = & (pc = 3 \wedge 0 \leq x \leq 2 \wedge 5 \leq y \leq 12 \wedge 5 \leq y + 2x \leq 14),\\
\psi_{3001} & = & (pc = 3 \wedge 0 \leq x \leq 2 \wedge 1 \leq y < 5 \wedge 1 \leq y + 2x < 5)\}.
\end{array}$$

We must consider ψ_{200} again. It turns out that it is still stable. After considering $\psi = \psi_{3000}$, we have $\beta = \{\psi_{000}, \psi_{001}, \psi_{1000}, \psi_{200}, \psi_{201}, \psi_{3000}\}$ and $\alpha = \alpha \cup \{\psi_{000}\}$. Now the partition is

$$\pi_7 \;=\; \{\psi_{000}, \psi_{001}, \psi_{01}, \psi_{1000}, \psi_{1001}, \psi_{101}, \psi_{11}, \psi_{200}, \psi_{201}, \psi_{21}, \psi_{3000}, \psi_{3001}, \psi_{301}, \psi_{31}\}.$$

Since ψ_{000} is stable in π_7, we have $\alpha = \beta = \{\psi_{000}, \psi_{001}, \psi_{1000}, \psi_{200}, \psi_{201}, \psi_{3000}\}$. Notice that α contains no bad states from R_F, that is $\psi \wedge \psi_f = \mathit{false}$ for all $\psi \in \alpha$. Therefore, the invariant property has been verified.

4.5 Model Checking

Previously, we presented three semidecision procedures for the reachability problem of linear hybrid systems. Now we address the more general problem of whether the given linear hybrid system H satisfies a requirement that is expressed in the real-time temporal logic TCTL [ACD93].

Timed computation tree logic

Let C be a set of clocks not in Var; that is, $C \cap \mathit{Var} = \emptyset$. A *state predicate* is a linear formula over the set $\mathit{Var} \cup C$ of variables.

The formulas of TCTL are built from the state predicates by boolean connectives, the two temporal operators $\exists\mathcal{U}$ and $\forall\mathcal{U}$, and the reset quantifier for the clocks in C. The formulas of TCTL, then, are defined by the grammar

$$\phi \;::=\; \psi \mid \neg\phi \mid \phi_1 \vee \phi_2 \mid z.\,\phi \mid \phi_1 \exists\mathcal{U} \phi_2 \mid \phi_1 \forall\mathcal{U} \phi_2$$

where ψ is a state predicate and $z \in C$. The formula ϕ is *closed* if all occurrences of a clock $z \in C$ are within the scope of a reset quantifier z.

The closed formulas of TCTL are interpreted over the state space Σ of the linear hybrid system H. Intuitively, a state σ satisfies the TCTL-formula $\phi_1 \exists\mathcal{U} \phi_2$ if there exists a run of H from σ to a state σ' satisfying ϕ_2 such that $\phi_1 \vee \phi_2$ continuously holds along the run. Dually, the state σ satisfies the TCTL-formula $\phi_1 \forall\mathcal{U} \phi_2$ if every divergent run from σ leads to a state σ' satisfying ϕ_2 such that $\phi_1 \vee \phi_2$ continuously holds along from σ to σ'. Clocks can be used to express timing constraints. For instance, the TCTL-formula $z.\,(\mathit{true} \exists\mathcal{U} (\phi \wedge z \leq 5))$ asserts that there is a run on which ϕ is satisfied within 5 time units.

We use the standard abbreviations such as $\forall\Diamond\phi$ for $\mathit{true}\forall\mathcal{U}\phi$, $\exists\Diamond\phi$ for $\mathit{true}\exists\mathcal{U}\phi$, $\exists\Box\phi$ for $\neg\forall\Diamond\neg\phi$, and $\forall\Box\phi$ for $\neg\exists\Diamond\neg\phi$. We also put timing constraints as subscripts on the temporal operators. For example, the formula $z.\,\exists\Diamond(\phi \wedge z < 5)$ is abbreviated to $\exists\Diamond_{<5}\phi$.

Let $\rho = \sigma_0 \mapsto^{t_0} \sigma_1 \mapsto^{t_1} \ldots$ be a run of the linear hybrid system H, with $\sigma_i = (\ell_i, \nu_i)$ for all $i \geq 0$. A *position* π of ρ is a pair (i, t) consisting of a nonnegative integer i and a nonnegative real $t \leq t_i$. The positions of ρ are ordered lexicographically; that is, $(i, t) \leq (j, t')$ iff $i < j$, or $i = j$ and $t \leq t'$. For all positions $\pi = (i, t)$ of ρ,

- the state $\rho(\pi)$ at the position π of ρ is $(\ell_i, \varphi_{\ell_i}[\nu_i](t))$, and
- the time $\delta_\rho(\pi)$ at the position π of ρ is $t + \sum_{j<i} t_j$.

A *clock valuation* ξ is a function from C to $\mathsf{R}^{\geq 0}$. For any nonnegative real $t \in \mathsf{R}^{\geq 0}$, by $\xi + t$ we denote the clock valuation ξ' such that $\xi'(z) = \xi(z) + t$ for all clocks $z \in C$. For any clock $z \in C$, by $\xi[z := 0]$ we denote the valuation ξ' such that $\xi'(z) = 0$ and $\xi'(z') = \xi(z')$ for all clocks $z' \neq z$.

An *extended state* (σ, ξ) consists of a state $\sigma \in \Sigma$ and a clock valuation ξ. The extended state (σ, ξ) *satisfies* the TCTL-formula ϕ, denoted $(\sigma, \xi) \models \phi$, if

$(\sigma,\xi) \models \psi$ iff $(\sigma,\xi)(\psi)$;

$(\sigma,\xi) \models \neg\phi$ iff $(\sigma,\xi) \not\models \phi$;

$(\sigma,\xi) \models \phi_1 \vee \phi_2$ iff $(\sigma,\xi) \models \phi_1$ or $(\sigma,\xi) \models \phi_2$;

$(\sigma,\xi) \models z.\,\phi_1$ iff $(\sigma,\xi[z := 0]) \models \phi_1$;

$(\sigma,\xi) \models \phi_1 \exists\mathcal{U} \phi_2$ iff there is a run ρ of H with $\rho(0,0) = \sigma$, and a position π of ρ such that (1) $(\rho(\pi), \xi + \delta_\rho(\pi)) \models \phi_2$, and (2) for all positions $\pi' \leq \pi$ of ρ, $(\rho(\pi'), \xi + \delta_\rho(\pi')) \models \phi_1 \vee \phi_2$;

$(\sigma,\xi) \models \phi_1 \forall\mathcal{U} \phi_2$ iff for all divergent runs ρ of H with $\rho(0,0) = \sigma$ there is a position π of ρ such that (1) $(\rho(\pi), \xi + \delta_\rho(\pi)) \models \phi_2$, and (2) for all positions $\pi' \leq \pi$ of ρ, $(\rho(\pi'), \xi + \delta_\rho(\pi')) \models \phi_1 \vee \phi_2$.

Let ϕ be a closed formula of TCTL. A state $\sigma \in \Sigma$ satisfies ϕ, denoted $\sigma \models \phi$, if $(\sigma,\xi) \models \phi$ for all clock valuations ξ. The linear hybrid system H satisfies ϕ, denoted $H \models \phi$, if all states of H satisfy ϕ. The *characteristic set* $[\![\phi]\!] \subseteq \Sigma$ of ϕ is the set of states that satisfy ϕ.

The model-checking algorithm

Given a closed TCTL-formula ϕ, a model-checking algorithm computes the characteristic set $[\![\phi]\!]$. We present the symbolic model-checking algorithm for timed automata [HNSY94], which is a semidecision procedure for model checking TCTL-formulas over linear hybrid systems.

The procedure is based on fixpoint characterizations of the TCTL-modalities in terms of a binary next operator $\triangleright$. Given two regions $R, R' \subseteq \Sigma$, the region $R \triangleright R'$ is the set of states σ that have a successor $\sigma' \in R'$ such that all states between σ and σ' are contained in $R \cup R'$: $(\ell,\nu) \in (R \triangleright R')$ iff

$$\exists(\ell',\nu') \in R', t \in \mathsf{R}^{\geq 0}.\ ((\ell,\nu) \mapsto^t (\ell',\nu') \ \wedge\ \forall 0 \leq t' \leq t.\ (\ell,\nu+t') \in (R \cup R'));$$

that is, the $\triangleright$ operator is a "single-step until" operator.

To define the $\triangleright$ operator syntactically, we introduce some notation. For a linear formula ψ, we extend the tcp operator such that

$$\mathsf{tcp}_\ell[\psi][\nu](t) \quad \text{iff} \quad \forall 0 \leq t' \leq t.\ \varphi_\ell[\nu](t') \in (Inv(\ell) \cap [\![\psi]\!]);$$

that is, all valuations along the evolution by time t from the state (ℓ,ν) satisfy not only the invariant of location ℓ but also ψ. For a state $\sigma = (\ell,\nu) \in \Sigma$ we write $\varphi[\sigma]$ for the function $\varphi_\ell[\nu]$, and for a region $R = \bigcup_{\ell \in Loc}(\ell, R_\ell)$ we write

$$\mathsf{tcp}[R][\sigma](t) \quad \text{iff} \quad \mathsf{tcp}_\ell[R_\ell][\nu](t).$$

Now, for two regions $R, R' \subseteq \Sigma$, we define the region $R \triangleright R'$ as

$$\sigma \in (R \triangleright R') \quad \text{iff} \quad \exists t \in \mathsf{R}^{\geq 0}.\ (\varphi[\sigma](t) \in \mathsf{pre}[R'] \wedge \mathsf{tcp}[R \cup R'][\sigma](t)).$$

Lemma 4.3 *For all linear hybrid systems H, if R and R' are two linear regions of H, then so is $R \triangleright R'$.*

In [HNSY94] it is shown that for nonzeno timed automata, the meaning of both TCTL-modalities $\exists\mathcal{U}$ and $\forall\mathcal{U}$ can be computed iteratively as fixpoints, using the $\triangleright$ operator. While for multirate timed systems, the iterative fixpoint computation always terminates, this is no longer the case for linear hybrid systems in general. Lemma 4.3, however, ensures that all regions that are computed by the process are linear and each step of the procedure is, therefore, effective.

Here, we present the method for some important classes of TCTL-formulas:

- Let R and R' be the characteristic sets of the two TCTL-formulas ϕ and ϕ', respectively. The characteristic set of the formula $\phi \exists \mathcal{U} \phi'$ can be iteratively computed as $\bigcup_i R_i$ with
  - $R_0 = R'$, and
  - for all $i \geq 0$, $R_{i+1} = R_i \cup (R \triangleright R_i)$.
- To check if the TCTL-formula ϕ is an invariant of H, we check if the set of initial states is contained in the characteristic set of the formula $\forall \Box \phi$. This characteristic set can be iteratively computed as $\bigcap_i R_i$ with
  - $R_0 = [\![\phi]\!]$, and
  - for all $i \geq 0$, $R_{i+1} = R_i \cap \neg(true \triangleright \neg R_i)$.
- The real-time response property asserting that a given event occurs within a certain time bound is expressed in TCTL by a formula of the form $\forall \Diamond_{\leq c} \phi$, whose characteristic set can be iteratively computed as $\neg \bigcup_i R_i[z := 0]$ with
  - $R_0 = [\![z > c]\!]$, and
  - for all $i \geq 0$, $R_{i+1} = R_i \cup ((\neg R) \triangleright R_i)$,

  where $R = [\![\phi]\!]$ and $z \in C$.

Example: the temperature control system

The goal is to maintain the temperature of the coolant between lower and upper bounds θ_m and θ_M. If the temperature rises to its maximum θ_M and it cannot decrease because no rod is available, a complete shutdown is required.

Now, let $\Delta\theta = \theta_M - \theta_m$. Clearly, the time the coolant needs to increase its temperature from θ_m to θ_M is $\tau_r = \frac{\Delta\theta}{v_r}$, and the refrigeration times for rod 1 and rod 2 are $\tau_1 = \frac{\Delta\theta}{v_1}$ and $\tau_2 = \frac{\Delta\theta}{v_2}$, respectively.

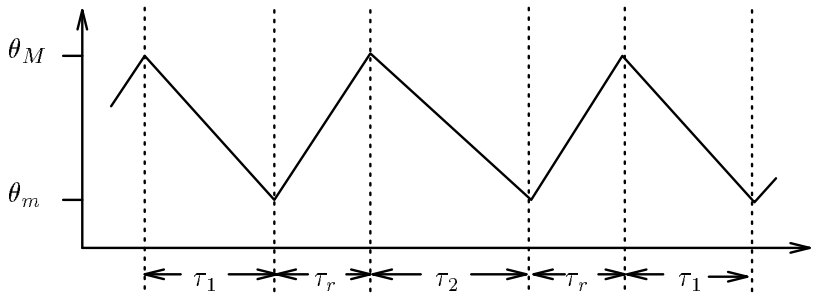


Figure 9: Refrigeration times

The question is whether the system will ever reach the shutdown state. Clearly, if temperature rises at a rate slower than the time of recovery for the rods, i.e., $\tau_r \geq T$, shutdown is unreachable. Moreover, it can be seen that $2\tau_r + \tau_1 \geq T \wedge 2\tau_r + \tau_2 \geq T$ is a necessary and sufficient condition for never reaching the shutdown state (see Fig. 9).

The property stating that state 3 (shutdown) is always unreachable corresponds to the following TCTL formula:

$$(pc = 0 \wedge \theta \leq \theta_M \wedge x_1 \geq T \wedge x_2 \geq T) \Rightarrow \forall\Box\neg(pc = 3)$$

or equivalently,

$$(pc = 0 \wedge \theta \leq \theta_M \wedge x_1 \geq T \wedge x_2 \geq T) \Rightarrow \neg\exists\Diamond(pc = 3)$$

- Let $v_r = 6$, $v_1 = 4$, $v_2 = 3$, $\theta_m = 3$, $\theta_M = 15$ and $T = 6$. In this case the condition $2\tau_r + \tau_1 \geq T \wedge 2\tau_r + \tau_2 \geq T$ holds. Using KRONOS, we compute the characteristic set of $\exists\Diamond pc = 3$. The results obtained at each iteration are shown below, where each ψ_i has been computed according to the method described above:

$$\begin{aligned}
\psi_0 &= pc = 3 \\
\psi_1 &= (pc = 0 \wedge \theta \leq 15 \wedge 6x_1 < \theta + 21 \wedge 6x_2 < \theta + 21) \vee pc = 3 \\
\psi_2 &= (pc = 0 \wedge \theta \leq 15 \wedge 6x_1 < \theta + 21 \wedge 6x_2 < \theta + 21) \vee \\
&\quad (pc = 1 \wedge 3 \leq \theta \leq 15 \wedge 4x_2 + \theta < 19) \vee \\
&\quad (pc = 2 \wedge 3 \leq \theta \leq 15 \wedge 3x_1 + \theta < 15) \vee pc = 3 \\
\psi_3 &= (pc = 0 \wedge \theta \leq 15 \wedge (6x_1 < \theta + 21 \wedge 6x_2 < \theta + 21 \vee 6x_2 + 3 < \theta)) \vee \\
&\quad (pc = 1 \wedge 3 \leq \theta \leq 15 \wedge 4x_2 + \theta < 19) \vee \\
&\quad (pc = 2 \wedge 3 \leq \theta \leq 15 \wedge 3x_1 + \theta < 15) \vee pc = 3 \\
\psi_4 &= \psi_3
\end{aligned}$$

  The state predicate $\neg \bigvee_{i=0}^{3} \psi_i[z := 0]$ representing the meaning of $\neg\exists\Diamond(pc = 3)$ is

$$\begin{aligned}
&pc = 0 \wedge \theta \leq 15 \wedge (\theta + 21 \leq 6x_1 \wedge \theta \leq 6x_2 + 3 \vee \theta + 21 \leq 6x_2) \vee \\
&pc = 1 \wedge 3 \leq \theta \leq 15 \wedge 19 \leq 4x_2 + \theta \vee \\
&pc = 2 \wedge 3 \leq \theta \leq 15 \wedge 15 \leq 3x_1 + \theta
\end{aligned}$$

  Since the state predicate $pc = 0 \wedge \theta \leq 15 \wedge x_1 \geq 6 \wedge x_2 \geq 6$ characterizing the set of initial states implies the predicate above, the system satisfies the invariant as required.

- Suppose that we change the time of recovery to $T = 8$. Now, the condition $2\tau_r + \tau_1 \geq T \wedge 2\tau_r + \tau_2 \geq T$ is no longer satisfied. Again, we compute using KRONOS the characteristic set of $\exists\Diamond pc = 3$. The results obtained at each iteration are the following:

$$\begin{aligned}
\psi_0 &= pc = 3 \\
\psi_1 &= (pc = 0 \wedge \theta \leq 15 \wedge 6x_1 < \theta + 33 \wedge 6x_2 < \theta + 33) \vee pc = 3 \\
\psi_2 &= (pc = 0 \wedge \theta \leq 15 \wedge 6x_1 < \theta + 33 \wedge 6x_2 < \theta + 33) \vee \\
&\quad (pc = 1 \wedge 3 \leq \theta \leq 15 \wedge 4x_2 + \theta < 27) \vee \\
&\quad (pc = 2 \wedge 3 \leq \theta \leq 15 \wedge 3x_1 + \theta < 21) \vee pc = 3 \\
\psi_3 &= (pc = 0 \wedge \theta \leq 15 \wedge (6x_1 + 3 < \theta \vee 6x_2 < \theta + 3 \vee \\
&\quad (6x_1 < \theta + 33 \wedge 6x_2 < \theta + 33))) \vee \\
&\quad (pc = 1 \wedge 3 \leq \theta \leq 15 \wedge 4x_2 + \theta < 27) \vee
\end{aligned}$$

parameters						*number of iterations*	*running times*
θ_m	θ_M	v_r	v_1	v_2	T		
3	15	6	4	3	6	4	0.033
3	15	6	4	3	8	4	0.033
10	190	45	30	18	20	6	0.083
250	1100	34	25	10	80	4	0.033

Table 1: Performances for the temperature control system

$$
\begin{aligned}
& (pc = 2 \wedge 3 \leq \theta \leq 15 \wedge 3x_1 + \theta < 21) \vee\ pc = 3 \\
\psi_4 \;=\; & (pc = 0 \wedge \theta \leq 15 \wedge (6x_1 + 3 < \theta \vee\ 6x_2 < \theta + 3 \vee \\
& (6x_1 < \theta + 33 \wedge\ 6x_2 < \theta + 33))) \vee \\
& (pc = 1 \wedge 3 \leq \theta \leq 15 \wedge 4x_2 + \theta < 27) \vee \\
& (pc = 2 \wedge 3 \leq \theta \leq 15) \vee\ pc = 3 \\
\psi_5 \;=\; & (pc = 0 \wedge \theta \leq 15 \wedge (\theta + 33 \leq 6x_2 \vee\ 6x_1 < \theta + 33 \vee\ 6x_2 < \theta + 3)) \vee \\
& (pc = 1 \wedge 3 \leq \theta \leq 15 \wedge 4x_2 + \theta < 27) \vee\ (pc = 2 \wedge 3 \leq \theta \leq 15) \vee \\
& pc = 3 \\
\psi_6 \;=\; & (pc = 0 \wedge \theta \leq 15 \wedge (\theta + 33 \leq 6x_2 \vee\ 6x_1 < \theta + 33 \vee\ 6x_2 < \theta + 3)) \vee \\
& (pc = 1 \wedge 3 \leq \theta \leq 15) \vee\ (pc = 2 \wedge 3 \leq \theta \leq 15) \vee\ pc = 3 \\
\psi_7 \;=\; & (pc = 0 \wedge \theta \leq 15) \vee\ (pc = 1 \wedge 3 \leq \theta \leq 15) \vee\ (pc = 2 \wedge 3 \leq \theta \leq 15) \vee \\
& pc = 3 \\
\psi_8 \;=\; & \psi_7
\end{aligned}
$$

The state predicate $\neg \bigvee_{i=0}^{7} \psi_i[z := 0]$ representing the meaning of $\neg\exists\Diamond(pc = 3)$ is

$$
\begin{aligned}
& pc = 0 \wedge \theta > 15 \vee \\
& pc = 1 \wedge (\theta < 3 \vee\ \theta > 15) \vee \\
& pc = 2 \wedge (\theta < 3 \vee\ \theta > 15)
\end{aligned}
$$

and since the state predicate $pc = 0 \wedge \theta \leq 15 \wedge x_1 \geq 6 \wedge x_2 \geq 6$ characterizing the set of initial states does not imply the predicate above we have that shutdown is reachable.

Table 1 shows the number of iterations and the running times (measured in seconds) obtained with KRONOS on a SUN 4 Sparc Station for verifying the formula on the system for different values of the parameters. (Performance figures for HYTECH can be found in [AHH93, HH94].)

Example: the billiards game

Consider the movement of the grey ball on the billard table. It is possible that the grey ball returns to the initial position with the initial direction. In this case the movement is periodic. A sufficient condition for the periodicity is that l, h, v_x and v_y are integers. The period T is calculated as follows:

$$T = \mathsf{lcm}\left(\frac{2l}{v_x}\ ,\ \frac{2h}{v_y}\right)$$

parameters								*formula*	*number of iterations*	*running times*
l	h	v_x	v_y	x_g	y_g	x_w	y_w			
13	10	2	1	0	0	10	8	$[periodT]$	55	7.77
								$[touch]$	55	6.69
								$[touchT]$	55	8.17
4	2	5	1	0	0	1	1	$[periodT]$	24	1.97
								$[touch]$	24	1.58
								$[touchT]$	24	1.90
3	8	1	2	0	0	1	6	$[periodT]$	10	0.56
								$[touch]$	10	0.40
								$[touchT]$	10	0.48

Table 2: Performances for the billards game

Now, since the movement of the grey ball has period T, the first collision with the white ball, if it takes place, will occur before time T. We can express this property in TCTL as follows:[1]

$$[periodT] \qquad \neg(\neg(x = x_w \wedge y = y_w)\exists \mathcal{U}_{>T}(x = x_w \wedge y = y_w))$$

We would like to characterize also all the positions where the grey ball may be placed in order to be able to touch the white ball. This set of points is characterized by the formula:

$$[touch] \qquad \exists\Diamond(x = x_w \wedge y = y_w)$$

Since the movement of the grey ball has period T, this property can also be specified by the formula

$$[touchT] \qquad \exists\Diamond_{\leq T}(x = x_w \wedge y = y_w)$$

Table 2 shows the number of iterations and the running times (measured in seconds) obtained with KRONOS on a SUN 4 Sparc Station for verifying the formulas $[periodT]$, $[touch]$ and $[touchT]$ on the billiards game for different values of the parameters.

5 Conclusion

We showed that the verification problem for hybrid systems is intrinsically difficult even under severe restrictions. Then we identified linear hybrid systems as a class of hybrid systems for which algorithmic analysis techniques exist and perform reasonably well. For general hybrid systems our analysis methods can be applied modulo limitations that concern the effective computation of boolean operations, time closures, preconditions, and postconditions of state sets.

Future work is necessary to improve both the cost and the scope of our approach. The cost can be improved by designing efficient algorithms for representing, comparing, manipulating, and approximating state sets. The scope can be improved by identifying other classes of hybrid systems to which semidecision procedures based on reachability analysis apply. For example, our results have recently been extended to a more general model, where the rates of variables are not constant in each location, but vary arbitrarily between given constant lower and upper bounds [AHH93, OSY94].

[1] If T is not an integer, but is a rational $\frac{p}{q}$, we have to multiply l, h, x_g, y_g, x_w and y_w by q to make it an integer.

In that case the state sets that are computed by the verification procedures are also definable by linear formulas. The more general case is interesting for the approximation of nonlinear hybrid systems.

We did not discuss any analysis techniques that cannot be formulated within the framework of reachability analysis. Most of these techniques are based on digitization methods that reduce verification problems for hybrid systems to verification problems for discrete systems, which are decidable [KPSY93, PV94].

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