ℓ-ADIC ÉTALE COHOMOLOGY OF THE MODULI OF ELLIPTIC & HYPERELLIPTIC FIBRATIONS

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ABSTRACT. We determine ℓ-adic étale cohomology with Frobenius weights for the moduli stack $\mathcal{L}_{1,12n} := \mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$ of stable elliptic fibrations over $\mathbb{P}^1$ with a marked Weierstrass section over $\overline{\mathbb{F}}_q$ with $\mathrm{char}(\overline{\mathbb{F}}_q) \neq 2, 3$. Also for the moduli stacks $\mathcal{L}^{[\Gamma]}_{1,12n} := \mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma])$ of generalized elliptic curves over $\mathbb{P}^1$ with $[\Gamma]$-level structure over $\overline{\mathbb{F}}_q$ with $\mathrm{char}(\overline{\mathbb{F}}_q) \neq 2$ for $[\Gamma] = [\Gamma_1(2)], [\Gamma_1(4)], [\Gamma(2)]$ and $\neq 3$ for $[\Gamma] = [\Gamma_1(3)]$.

1. Introduction

Fix a basefield K, and define for the weight $\vec{\lambda} = (\lambda_0, \dots, \lambda_N)$ with positive weights $\lambda_i \in \mathbb{N}$, the N-dimensional weighted projective stack $\mathcal{P}(\vec{\lambda}) := [(\mathbb{A}^{N+1}_{x_0,\dots,x_N} \setminus 0)/\mathbb{G}_m]$ where $\zeta \in \mathbb{G}_m$ acts by $\zeta \cdot (x_0, \dots, x_N) = (\zeta^{\lambda_0} x_0, \dots, \zeta^{\lambda_N} x_N)$. In this paper, we find ℓ-adic étale cohomology and eigenvalues of geometric Frobenius map for the Hom stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ of degree $n \in \mathbb{N}$ rational curves on $\mathcal{P}(\vec{\lambda})$ which serves as a 'model' *moduli of fibrations* since $\mathcal{P}(\vec{\lambda})$ is isomorphic to proper Deligne–Mumford fine moduli stacks of diverse kinds of elliptic and hyperelliptic curves (see Example 2.3) which in turn makes $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ into a moduli stack of (stable) elliptic and hyperelliptic fibrations over $\mathbb{P}^1$ formulated and studied in [HP, PS, HP2]. In short, understanding Hom stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ is meaningful as $\mathcal{P}(\vec{\lambda})$ is a versatile representing object parameterizing various families of elliptic and hyperelliptic curves.

By a careful analysis of the flat evaluation morphism $\mathrm{ev}_\infty : \mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda})) \to \mathcal{P}(\vec{\lambda})$ (see Proposition 2.12) showing the cohomology of the fibers are fixed together with the fact that the base $\mathcal{P}(\vec{\lambda})$ is simply connected, we show that the associated ℓ-adic Leray spectral sequence is actually computable as Serre spectral sequence leading to the ℓ-adic étale Betti numbers $\dim_{\mathbb{Q}_\ell} \left(H^i_{\acute{e}t}(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell) \right)$. Consequently, we have the Main Theorem through the Grothendieck-Lefschetz trace formula (see Theorem 2.9) applied to the weighted $\mathbb{F}_q$-point count of the Hom stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ (see Theorem 2.6) showing the *geometric aspect* as follows.

Theorem 1.1. *Hom stack* $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ *with the weight* $\vec{\lambda} = (\lambda_0, \dots, \lambda_N)$ *parameterizing degree* $n \in \mathbb{N}$ *morphisms* $f : \mathbb{P}^1 \to \mathcal{P}(\vec{\lambda})$ *over* $\overline{\mathbb{F}}_q$ *with* $\mathrm{char}(\overline{\mathbb{F}}_q)$ *not dividing* $\lambda_i \in \mathbb{N}$ *for every* i *has the following compactly supported* ℓ*-adic étale cohomology and Galois representations of mixed Tate type. Let* $|\vec{\lambda}| := \sum_{i=0}^{N} \lambda_i$.

For $N = 1$ *case with e.g.,* $\overline{\mathcal{M}}_{1,1} \cong \mathcal{P}(4,6)$ *or* $\overline{\mathcal{M}}_{1,1}[\Gamma_1(2)] \cong \mathcal{P}(2,4)$*, we have*

$$H^i_{et,c}\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\lambda_0, \lambda_1))_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell\right) \cong \begin{cases} \mathbb{Q}_\ell\left(-(|\vec{\lambda}|n+1)\right) & i = 2|\vec{\lambda}|n+2, \\ \mathbb{Q}_\ell\left(-(|\vec{\lambda}|n-1)\right) & i = 2|\vec{\lambda}|n-1, \\ 0 & \textit{else.} \end{cases}$$

For $N = 3$ *case with e.g.,* $\overline{\mathcal{M}}_{1,3}(2) \cong \mathcal{P}(1,2,2,3)$ *or* $\mathcal{H}_4[3] \cong \mathcal{P}(4,6,8,10)$*, we have*

$$H^i_{\acute{e}t,c}\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\lambda_0,\lambda_1,\lambda_2,\lambda_3))_{/\mathbb{F}_q}; \mathbb{Q}_\ell\right) \cong \begin{cases} \mathbb{Q}_\ell\left(-(|\vec{\lambda}|n+3)\right) & i = 2|\vec{\lambda}|n+6, \\ \mathbb{Q}_\ell\left(-(|\vec{\lambda}|n+2)\right) & i = 2|\vec{\lambda}|n+4, \\ \mathbb{Q}_\ell\left(-(|\vec{\lambda}|n+1)\right) & i = 2|\vec{\lambda}|n+2, \\ \mathbb{Q}_\ell\left(-(|\vec{\lambda}|n-1)\right) & i = 2|\vec{\lambda}|n-1, \\ \mathbb{Q}_\ell\left(-(|\vec{\lambda}|n-2)\right) & i = 2|\vec{\lambda}|n-3, \\ \mathbb{Q}_\ell\left(-(|\vec{\lambda}|n-3)\right) & i = 2|\vec{\lambda}|n-5, \\ 0 & \textit{else.} \end{cases}$$

Corresponding ℓ-adic étale cohomology for a given N-dimensional $\mathcal{P}(\vec{\lambda})$ can be similarly worked out by combining Proposition 2.13 with Theorem 2.6 as in the Proof of Theorem 1.1.

2. Étale cohomology with eigenvalues of geometric Frobenius

In this section, we will introduce some definitions, background materials on schemes, algebraic stacks and discuss properties of étale cohomology that will be needed in the proof. For further details on the material presented here, the reader is referred to [Liu, Olsson2, DIS, Behrend].

We first recall the definition of a weighted projective stack $\mathcal{P}(\vec{\lambda})$.

Definition 2.1. Fix a tuple of nondecreasing positive integers $\vec{\lambda} = (\lambda_0, \dots, \lambda_N)$. The N-dimensional weighted projective stack $\mathcal{P}(\vec{\lambda}) = \mathcal{P}(\lambda_0, \dots, \lambda_N)$ with the weight $\vec{\lambda}$ is defined as a quotient stack

$$\mathcal{P}(\vec{\lambda}) := \left[(\mathbb{A}^{N+1}_{x_0,\dots,x_N} \setminus 0)/\mathbb{G}_m\right]$$

where $\zeta \in \mathbb{G}_m$ acts by $\zeta \cdot (x_0, \dots, x_N) = (\zeta^{\lambda_0} x_0, \dots, \zeta^{\lambda_N} x_N)$. In this case, the degree of x_i's are λ_i's respectively. A line bundle $\mathcal{O}_{\mathcal{P}(\vec{\lambda})}(m)$ is defined to be the line bundle associated to the sheaf of degree m homogeneous rational functions without poles on $\mathbb{A}^{N+1}_{x_0,\dots,x_N} \setminus 0$. We say that a map $\mathbb{P}^1 \to \mathcal{P}(\vec{\lambda})$ has degree n if $f^*\mathcal{O}_{\mathcal{P}(\vec{\lambda})}(1) \cong \mathcal{O}_{\mathbb{P}^1}(n)$.

Note that $\mathcal{P}(\vec{\lambda})$ is not an (effective) orbifold when $\gcd(\lambda_0, \dots, \lambda_N) \neq 1$. In this case, the cyclic isotropy group scheme $\mu_{\gcd(\lambda_0,\dots,\lambda_N)}$ is the generic stabilizer of $\mathcal{P}(\vec{\lambda})$. Nevertheless, the following proposition shows that it behaves well in most characteristics as a tame Deligne–Mumford stack:

Proposition 2.2. *The N-dimensional weighted projective stack $\mathcal{P}(\vec{\lambda}) = \mathcal{P}(\lambda_0, \dots, \lambda_N)$ is a tame Deligne–Mumford stack over a basefield K if $\mathrm{char}(K)$ does not divide $\lambda_i \in \mathbb{N}$ for every i.*

Proof. For any algebraically closed field extension $\overline{K}$ of K, any point $y \in \mathcal{P}(\vec{\lambda})(\overline{K})$ is represented by the coordinates $(y_0, \dots, y_N) \in \mathbb{A}^{N+1}_{\overline{K}}$ with its stabilizer group as the subgroup of $\mathbb{G}_m$ fixing $(y_0, \dots, y_N)$. Hence, any stabilizer group of such $\overline{K}$-points is $\mathbb{Z}/u\mathbb{Z}$ where u divides λ_i for some i. Since the characteristic of K does not divide the orders of $\mathbb{Z}/\lambda_i\mathbb{Z}$ for any i, the stabilizer group of y is $\overline{K}$-linearly reductive. Hence, $\mathcal{P}(\vec{\lambda})$ is tame by [AOV, Theorem 3.2]. Note that the stabilizer groups constitute fibers of the diagonal $\Delta : \mathcal{P}(\vec{\lambda}) \to \mathcal{P}(\vec{\lambda}) \times_K \mathcal{P}(\vec{\lambda})$. Since $\mathcal{P}(\vec{\lambda})$ is of finite type and $\mathbb{Z}/u\mathbb{Z}$'s are unramified over K whenever u does not divide λ_i for some i, Δ is unramified as well. Therefore, $\mathcal{P}(\vec{\lambda})$ is also Deligne–Mumford by [Olsson2, Theorem 8.3.3]. ■

The tameness is analogous to flatness for stacks in positive/mixed characteristic as it is preserved under base change by [AOV, Corollary 3.4]. Moreover, if a stack $\mathcal{X}$ is tame and Deligne–Mumford, then the formation of the coarse moduli space $c : \mathcal{X} \to X$ commutes with base change as well by [AOV, Corollary 3.3].

Example 2.3. As mentioned in the Introduction, there is a whole array of moduli stacks of curves that are isomorphic to $\mathcal{P}(\vec{\lambda})$ for various weights $\vec{\lambda} = (\lambda_0, \ldots, \lambda_N)$ over a field K with $\mathrm{char}(K)$ does not divide λ_i for every i.

When $\mathrm{char}(K) \neq 2, 3$, [Hassett, Proposition 3.6] shows that one example is given by the proper Deligne–Mumford stack of stable elliptic curves $(\overline{\mathcal{M}}_{1,1})_K \cong [(\mathrm{Spec}\ K[a_4, a_6] - (0,0))/\mathbb{G}_m] = \mathcal{P}_K(4,6)$ by using the short Weierstrass equation $y^2 = x^3 + a_4x + a_6x$, where $\zeta \cdot a_i = \zeta^i a_i$ for $\zeta \in \mathbb{G}_m$ and $i = 4, 6$. This is no longer true when $\mathrm{char}(K) \in \{2, 3\}$, as the Weierstrass equations are more complicated.

Similarly, one could consider $\overline{\mathcal{M}}_{1,1}[\Gamma]$ of generalized elliptic curves with $[\Gamma]$-level structure by the work of Deligne and Rapoport [DR] (summarized in [Conrad, §2] and also in [Niles, §2]) such as $\overline{\mathcal{M}}_{1,1}[\Gamma_1(2)] \cong \mathcal{P}(2,4)$ [Behrens, §1.3], $\overline{\mathcal{M}}_{1,1}[\Gamma_1(3)] \cong \mathcal{P}(1,3)$ [HMe, Proposition 4.5], $\overline{\mathcal{M}}_{1,1}[\Gamma_1(4)] \cong \mathcal{P}(1,2)$ [Meier, Examples 2.1] and $\overline{\mathcal{M}}_{1,1}[\Gamma(2)] \cong \mathcal{P}(2,2)$ [Stojanoska, Proposition 7.1].

Also, one could consider $\overline{\mathcal{M}}_{1,m}(m-1)$ of m-marked $(m-1)$-stable curves of arithmetic genus one formulated originally by the works of [Smyth, Smyth2] such as $\overline{\mathcal{M}}_{1,2}(1) \cong \mathcal{P}(2,3,4)$, $\overline{\mathcal{M}}_{1,3}(2) \cong \mathcal{P}(1,2,2,3)$, $\overline{\mathcal{M}}_{1,4}(3) \cong \mathcal{P}(1,1,1,2,2)$ and $\overline{\mathcal{M}}_{1,5}(4) \cong \mathbb{P}(1,1,1,1,1,1) \cong \mathbb{P}^5$ as shown by [LP, Theorem 1.5.7.].

We refer to [HP, PS] and [HP2, Appendix] for further arithmetic geometric considerations.

For higher genus $g \geq 2$, the moduli stack $\mathcal{H}_{2g}[2g-1]$ of quasi-admissible hyperelliptic genus g curves originally introduced by [Fedorchuk] is a tame Deligne–Mumford stack isomorphic to $\mathcal{P}(4,6,8,\ldots,4g+2)$ if $\mathrm{char}(K) = 0$ [Fedorchuk, Proposition 4.2(1)] or $> 2g+1$ by [HP2, Theorem 4.9]. The case with $g = 2$ by $\mathcal{H}_4[3] \cong \mathcal{P}(4,6,8,10)$ is of special interest due to [HP2, Theorem 4.11] which allowed the enumeration of principally polarized Abelian surfaces as in [HP2, Theorem 1.6] by the enumeration of stable genus 2 curves as in [HP2, Theorem 1.15].

We now generalize the Hom stack formulation to $\mathcal{P}(\vec{\lambda})$ as follows:

Definition 2.4. The stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ with the weight $\vec{\lambda} = (\lambda_0, \ldots, \lambda_N)$ is defined to be the Hom stack of degree $n \in \mathbb{N}$ morphisms $f : \mathbb{P}^1 \to \mathcal{P}(\vec{\lambda})$ with $f^*\mathcal{O}_{\mathcal{P}(\vec{\lambda})}(1) \simeq \mathcal{O}_{\mathbb{P}^1}(n)$. Concretely,

$$\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda})) = [T/\mathbb{G}_m]$$

where $T \subset \left(\bigoplus_{i=0}^{N} H^0(\mathcal{O}_{\mathbb{P}^1}(\lambda_i \cdot n)) \right) \backslash 0$ is the open subset of tuples of nonzero polynomials $(\widetilde{u}_0(z,t), \ldots, \widetilde{u}_N(z,t))$ homogeneous of degrees $\deg u_i = \lambda_i \cdot n$ for every i, sharing no common factor, and $\zeta \in \mathbb{G}_m$ acts by $\zeta \cdot (\widetilde{u}_0(z,t), \ldots, \widetilde{u}_N(z,t)) := (\widetilde{u}_0(\zeta z, \zeta t), \ldots, \widetilde{u}_N(\zeta z, \zeta t)) = (\zeta^{\lambda_0} \cdot \widetilde{u}_0(z,t), \ldots, \zeta^{\lambda_N} \cdot \widetilde{u}_N(z,t))$.

Proposition 2.5. *Over a basefield K with* $\mathrm{char}(K)$ *not dividing $\lambda_i \in \mathbb{N}$ for every i, the Hom stack* $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ *parameterizing degree $n \in \mathbb{N}$ morphisms $f : \mathbb{P}^1 \to \mathcal{P}(\vec{\lambda})$ is a smooth separated tame Deligne–Mumford stack of finite type with dimension equal to $|\vec{\lambda}|n + N$ where $|\vec{\lambda}| := \sum_{i=0}^{N} \lambda_i$.*

Proof. $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda})) = [T/\mathbb{G}_m]$ is a smooth Deligne–Mumford stack by [Olsson, Theorem 1.1], admitting T as a smooth schematic cover. Note that

$$\dim T = \sum_{i=0}^{n} h^0(\mathcal{O}_{\mathbb{P}^1}(\lambda_i \cdot n)) = \sum_{i=0}^{n} (\lambda_i + 1) = |\vec{\lambda}| + N + 1,$$

implying that $\dim \mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda})) = |\vec{\lambda}| + N$ since $\dim \mathbb{G}_m = 1$. As $\mathbb{G}_m$ acts on T properly with positive weights $\lambda_i \in \mathbb{N}$ for every i the quotient stack $[T/\mathbb{G}_m]$ is separated. It is tame as in [AOV, Theorem 3.2] since $\mathrm{char}(K)$ does not divide λ_i for every i . ■

As a polynomial in $H^0(\mathcal{O}_{\mathbb{P}^1}(\lambda_i \cdot n)) \setminus 0$ is determined by its restriction away from $\infty = [0 : 1]$ (i.e., by evaluating $t = 1$), we may consider T as the set of not-necessarily-monic tuples of nonzero coprime polynomials $(u_0, \ldots, u_N)$ such that for a fixed $j \in \{0, \ldots, N\}$, $\deg u_j = \lambda_j \cdot n$ and the rests have $0 \le \deg u_i \le \lambda_i \cdot n$ for every $i \ne j$. Here the degree conditions are precisely ensuring that $\widetilde{u}_0, \ldots, \widetilde{u}_N$ do not share a common zero at ∞.

We recall the *arithmetic aspect* of $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ as follows.

Theorem 2.6 (Theorem 1.11 & Corollary 1.13 of [HP2])**.** *Fix the weight $\vec{\lambda} = (\lambda_0, \ldots, \lambda_N)$ with $|\vec{\lambda}| := \sum_{i=0}^{N} \lambda_i$. Suppose that $\mathrm{char}(\mathbb{F}_q)$ does not divide $\lambda_i \in \mathbb{N}$ for every i, then the weighted point count of the Hom stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ over $\mathbb{F}_q$ is*

$$\begin{aligned}\#_q\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))\right) &= \left(\sum_{i=0}^{N} q^i\right) \cdot \left(q^{|\vec{\lambda}|n} - q^{|\vec{\lambda}|n-N}\right) \\ &= q^{|\vec{\lambda}|n} \cdot \left(q^N + q^{N-1} + \cdots + q^2 + q^1 - q^{-1} - q^{-2} - \cdots - q^{-N+1} - q^{-N}\right) \\ &= q^{|\vec{\lambda}|n+N} + \cdots + q^{|\vec{\lambda}|n+1} - q^{|\vec{\lambda}|n-1} - \cdots - q^{|\vec{\lambda}|n-N}\end{aligned}$$

$\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ has the stratification by $\mathrm{Poly}_1^{(d_1,\ldots,d_m)}$ as in [HP, §4.1] and [HP2, §3.1].

Definition 2.7. Fix $m \in \mathbb{Z}_{>0}$ and $d_1, \ldots, d_m \in \mathbb{N}$, define

$$\mathrm{Poly}_1^{(d_1,\ldots,d_m)} := \{(f_1(z), \ldots, f_m(z)) : f_i(z) \text{ are monic and coprime of degrees } d_i\}.$$

Equivalently, $\mathrm{Poly}_1^{(d_1,\ldots,d_m)}$ is the complement of the resultant hypersurface $\mathcal{R}^{(d_1,\ldots,d_m)} \subset \prod_{i=1}^{m} \mathrm{Sym}^{d_i} \mathbb{A}^1 = \prod_{i=1}^{m} \mathbb{A}^{d_i}$ of tuples of monic polynomials $(f_1(z), \ldots, f_m(z))$ of degrees d_i which share a factor.

Arithmetic of $\mathrm{Poly}_1^{(d_1,\ldots,d_m)}$ was studied in [HP, PS, HP2] which was inspired by the work of [Segal, FW]. As the resultant hypersurface $\mathcal{R}^{(d_1,\ldots,d_m)}$ is defined over $\mathbb{Z}$, it makes sense to define $\mathrm{Poly}_1^{d_1,\ldots,d_m} = (\prod_{i=1}^{m} \mathbb{A}^{d_i}) \setminus \mathcal{R}^{(d_1,\ldots,d_m)}$ as a variety over $\mathbb{Z}$.

Generalizing the proof of [FW, Theorem 1.2] and [HP, Proposition 18] with the correction from [PS, Proposition 3.1.], we find the motive of $\mathrm{Poly}_1^{(d_1,\ldots,d_m)}$ leading to its point count over finite fields $\mathbb{F}_q$ as follows:

Proposition 2.8 (Proposition 3.4. of [HP2])**.** *Fix* $0 \le d_1 \le d_2 \le \ldots \le d_m$*. Then for any prime power* q *:*

$$|\mathrm{Poly}_1^{(d_1,\ldots,d_m)}(\mathbb{F}_q)| = \begin{cases} q^{d_1+\ldots+d_m} - q^{d_1+\ldots+d_m-m+1}, & \text{if } d_1 \neq 0 \\ q^{d_1+\ldots+d_m}, & \text{if } d_1 = 0 \end{cases}$$

Since the absolute Galois group of finite fields $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ acting on the ℓ-adic étale cohomology is a procyclic group that is topologically generated by the geometric Frobenius, the task of finding the eigenvalues of Frobenius map can be achieved through the trace formula where the cardinality of the fixed set of $\mathrm{Frob}_q : \mathfrak{X}(\overline{\mathbb{F}}_q) \to \mathfrak{X}(\overline{\mathbb{F}}_q)$ coincides with $\#_q(\mathfrak{X})$ the weighted $\mathbb{F}_q$–point count. We recall the *Grothendieck-Lefschetz trace formula* [Sun, Behrend] for Artin stacks of finite type over finite fields.

Theorem 2.9 (Theorem 1.1. of [Sun])**.** *Let* $\mathfrak{X}$ *be an Artin stack of finite type over* $\mathbb{F}_q$*. Let* Frob_q *be the geometric Frobenius on* $\mathfrak{X}$*. Let* ℓ *be a prime number different from the characteristic of* $\mathbb{F}_q$*, and let* $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$ *be an isomorphism of fields. For an integer* i*, let* $H^i_c(\mathfrak{X}_{/\overline{\mathbb{F}}_q}; \overline{\mathbb{Q}}_\ell)$ *be the cohomology with compact support of the constant sheaf* $\overline{\mathbb{Q}}_\ell$ *on* $\mathfrak{X}$ *as in* [LO]*. Then the infinite sum regarded as a complex series via* ι

$$\sum_{i\in\mathbb{Z}} (-1)^i tr\big(\mathrm{Frob}_q^* : H^i_c(\mathfrak{X}_{/\overline{\mathbb{F}}_q}; \overline{\mathbb{Q}}_\ell) \to H^i_c(\mathfrak{X}_{/\overline{\mathbb{F}}_q}; \overline{\mathbb{Q}}_\ell)\big) \tag{1}$$

is absolutely convergent to $\#_q(\mathfrak{X})$ *over* $\mathbb{F}_q$*. And its limit is the number of* $\mathbb{F}_q$*–points on stacks that are counted with weights, where a point with its stabilizer group* G *contributes a weight* $\frac{1}{|G|}$*.*

As $\mathfrak{X} = \mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ is a separated Deligne–Mumford stack of finite type over $\mathbb{F}_q$ (see Proposition 2.5), the corresponding (compactly supported) ℓ-adic étale cohomology for prime number ℓ invertible in $\mathbb{F}_q$ is finite dimensional as a $\mathbb{Q}_\ell$-algebra, making the trace formula hold in $\mathbb{Q}_\ell$-coefficients. Also by the smoothness, the Poincaré duality $H^i_{\acute{e}t,c}(\mathfrak{X}_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell) \cong H^{2dim(\mathfrak{X})-i}_{\acute{e}t}(\mathfrak{X}_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell(-dim(\mathfrak{X})))^\vee$ hold for the dual of ordinary étale cohomology as in [LO, 7.3.1. Theorem]. From now on, we identify the dual spaces with the ordinary étale cohomology as finite dimensional $\mathbb{Q}_\ell$-vector spaces. We denote by $\mathbb{Q}_\ell(-i)$ the $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-representation of rank 1 on which the *geometric Frobenius* acts by q^{-i} and the *arithmetic Frobenius* acts by q^i. Also, recall that the étale cohomology of a stack $\mathfrak{X}$ is *pure* if the absolute values of the eigenvalues of Frob_q on $H^i_{\acute{e}t}$ are all $q^{i/2}$ and if otherwise it is *mixed.* The group $H^i_{\acute{e}t}(\mathfrak{X}_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$ is of *Tate type* if the eigenvalues of Frob_q are all equal to integer powers of q.

We are now ready to prove the Theorem 1.1.

2.1. **Proof of Theorem 1.1.**

Proof. We first recall the étale cohomology of the smooth proper base $\mathcal{P}(\vec{\lambda})$.

Proposition 2.10. *There are isomorphisms of Galois representations*

$$H^i_{\acute{e}t}(\mathcal{P}(\vec{\lambda})_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell) \cong \begin{cases} \mathbb{Q}_\ell(-\frac{i}{2}) & i \in \{0, 2, \ldots, 2N\} \\ 0 & else \end{cases}$$

Proof. This follows from [Kawasaki, Theorem 1.] where we use the well-known fact that the cohomology of a smooth Deligne–Mumford stack $\mathfrak{X}$ with projective coarse moduli space X is pure.

■

Next, we need the compactly supported étale cohomology of $\mathrm{Hom}_n^*(\mathbb{P}^1, \mathcal{P}(\vec{\lambda})) \cong \mathrm{Poly}_1^{(\lambda_0 \cdot n, \dots, \lambda_N \cdot n)}$ which follows from [FW, HP, HP2]. Let $|\vec{\lambda}| := \sum_{i=0}^{N} \lambda_i$.

Proposition 2.11. *There are isomorphisms of Galois representations*

$$H^i_{\acute{e}t,c}(\mathrm{Poly}_1^{(\lambda_0 \cdot n, \dots, \lambda_N \cdot n)}{}_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell) \cong \begin{cases} \mathbb{Q}_\ell(-(|\vec{\lambda}|n)) & i = 2(|\vec{\lambda}|n) \\ \mathbb{Q}_\ell(-(|\vec{\lambda}|n - N)) & i = 2(|\vec{\lambda}|n) - N \\ 0 & else \end{cases}$$

Proof. We can express $\mathrm{Poly}_1^{(\lambda_0 \cdot n, \dots, \lambda_N \cdot n)}$ as the quotient of a finite group $S_{\lambda_0 \cdot n} \times \cdots \times S_{\lambda_N \cdot n}$ acting on the complement of a hyperplane arrangement in $\mathbb{A}^{|\vec{\lambda}|n}$. It is well known by [Lehrer, Shapiro, Kim] that the geometric Frobenius Frob_q^* acts by q^i on H^i_{et} of the complement of a hyperplane arrangement. By transfer and Poincaré duality, we see that the geometric Frobenius acts by $q^{-(|\vec{\lambda}|n)+i}$ on $H^i_{\acute{e}t,c}(\mathrm{Poly}_1^{(\lambda_0 \cdot n, \dots, \lambda_N \cdot n)}{}_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$. Applying the *Grothendieck-Lefschetz trace formula* for compactly supported étale cohomology, we see that

$$\left|\mathrm{Poly}_1^{(\lambda_0 \cdot n, \dots, \lambda_N \cdot n)}(\mathbb{F}_q)\right| = \sum_{i=0}^{2(|\vec{\lambda}|n)} (-1)^i \cdot q^{-(|\vec{\lambda}|n)+i} \cdot \dim_{\mathbb{Q}_\ell} \left(H^i_{\acute{e}t,c}(\mathrm{Poly}_1^{(\lambda_0 \cdot n, \dots, \lambda_N \cdot n)}{}_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell) \right)$$

and we use the exact point count of Proposition 2.8 to conclude the above. ∎

We now consider the ℓ-adic Leray spectral sequence as in [Behrend, Theorem 1.2.5] with respect to the evaluation morphism $\mathrm{ev}_\infty : \mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda})) \to \mathcal{P}(\vec{\lambda})$ to determine the ℓ-adic étale Betti numbers of $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$.

Proposition 2.12. *Evaluation morphism* $\mathrm{ev}_\infty : \mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda})) \to \mathcal{P}(\vec{\lambda})$ *mapping an equivalence class of degree* $n \in \mathbb{N}$ *unbased morphism* $f : \mathbb{P}^1 \to \mathcal{P}(\vec{\lambda})$ *with the chosen basepoint* $\infty \in \mathbb{P}^1$ *to the basepoint of the target* $f(\infty) \in \mathcal{P}(\vec{\lambda})$ *is a flat morphism where* $H^j_{\acute{e}t,c}(\mathrm{ev}_\infty^{-1}(p)_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$ *the compactly supported ℓ-adic étale cohomology of the fiber at each geometric point* $p \in \mathcal{P}(\vec{\lambda})$ *is isomorphic to* $H^j_{\acute{e}t,c}(\mathrm{Poly}_1^{(\lambda_0 \cdot n, \dots, \lambda_N \cdot n)}{}_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$ *the compactly supported ℓ-adic étale cohomology of the space* $\mathrm{Poly}_1^{(\lambda_0 \cdot n, \dots, \lambda_N \cdot n)} \cong \mathrm{Hom}_n^*(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ *of degree* $n \in \mathbb{N}$ *based morphisms.*

Proof. We have a $\mathbb{G}_m$-equivariant map $\widetilde{\mathrm{ev}}_\infty : T \to \mathbb{A}^{N+1} \backslash 0$, which induces $\mathrm{ev}_\infty : \mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda})) = [T/\mathbb{G}_m] \to \mathcal{P}(\vec{\lambda}) = [(\mathbb{A}^{N+1} \backslash 0)/\mathbb{G}_m]$. Note that for such maps and a closed subscheme $(x_0, \dots, x_N) \in \mathbb{A}^{N+1} \setminus 0$ on which $\mathbb{G}_m$ acts, $\mathrm{ev}_\infty^{-1}([x_0 : \dots : x_N]) \cong \widetilde{\mathrm{ev}}_\infty^{-1}((x_0, \dots, x_N))/\mathbb{G}_m$. This implies in particular that the special fiber at $[x_0 : \dots : x_N]$ with at least one of the $x_i = 0$ (but not all zero) is the complement of the hyperplane arrangement by the same principle that the generic fiber such as over at $[1 : \dots : 1]$ is the complement of the hyperplane arrangement.

The fiber at the generic stacky points $[x_0 : \dots : x_N] \in \mathcal{P}(\vec{\lambda})$ such that $x_i \neq 0$ for every i with $\mu_{\gcd(\lambda_0, \dots, \lambda_N)}$ stabilizer can be accounted for by looking at $\mathrm{ev}_\infty^{-1}([1 : \dots : 1])$ where we have $\mathrm{ev}_\infty^{-1}([1 : \dots : 1]) \cong \mathrm{Poly}_1^{(\lambda_0 \cdot n, \dots, \lambda_N \cdot n)}$. This is because a tuple of monic coprime polynomials $(f_0, \dots, f_N)$ with $\deg(f_0, \dots, f_N) = (\lambda_0 \cdot n, \dots, \lambda_N \cdot n)$ determines, and is determined by, a morphism $f : \mathbb{P}^1 \to \mathcal{P}(\vec{\lambda})$ such that $f^* \mathcal{O}_{\mathcal{P}(\vec{\lambda})}(1) \simeq \mathcal{O}_{\mathbb{P}^1}(n)$ and $f(\infty) = [1 : \dots : 1]$ via $f(z) := [f_0(z), \dots, f_N(z)]$ under the weighted homogeneous coordinate of the target stack $\mathcal{P}(\vec{\lambda})$. By the $\mathbb{G}_m$ action, this implies that the compactly supported cohomology $H^j_{\acute{e}t,c}(\mathrm{ev}_\infty^{-1}(p)_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)$ away from the special orbifold points is isomorphic to $H^j_{\acute{e}t,c}(\mathrm{Poly}_1^{(\lambda_0\cdot n,\dots,\lambda_N\cdot n)}{}_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)$.

For the compactly supported cohomology of the special fibers at $[x_0 : \dots : x_N]$ with at least one of the $x_i = 0$, without loss of generality, we have the fiber at $[1 : 0 : \dots : 0]$ with $\mu_{\gcd(\lambda_1,\dots,\lambda_N)}$ stabilizer that has the stratification $\mathrm{ev}_\infty^{-1}([1 : 0 : \dots : 0]) = \left(\coprod_{k=1}^{\lambda_0\cdot n} \mathbb{G}_m \times \mathrm{Poly}_1^{(\lambda_0\cdot n-k,\lambda_1\cdot n,\dots,\lambda_N\cdot n)}\right)$ as it must map to $[1 : 0 : \dots : 0]$ by ev_∞. By applying the Grothendieck relation [Ekedahl, §1] on the stratification as in [HP, HP2], we have $[\mathbb{G}_m \times \mathrm{Poly}_1^{(\lambda_0\cdot n-1,\lambda_1\cdot n,\dots,\lambda_N\cdot n)}] + [\mathbb{G}_m \times \mathrm{Poly}_1^{(\lambda_0\cdot n-2,\lambda_1\cdot n,\dots,\lambda_N\cdot n)}] + \dots + [\mathbb{G}_m \times \mathrm{Poly}_1^{(0,\lambda_1\cdot n,\dots,\lambda_N\cdot n)}] = \mathbb{L}^{|\vec{\lambda}|n} - \mathbb{L}^{|\vec{\lambda}|n-N} = [\mathrm{Poly}_1^{(\lambda_0\cdot n,\dots,\lambda_N\cdot n)}]$ which implies that the special fibers have the same $\mathbb{F}_q$–point counts as $\mathrm{Poly}_1^{(\lambda_0\cdot n,\dots,\lambda_N\cdot n)}$ by [PS]. We apply the Proposition 2.11 to conclude that the compactly supported ℓ-adic étale cohomology of the special fibers are isomorphic to $H^j_{\acute{e}t,c}(\mathrm{ev}_\infty^{-1}(p)_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell) \cong H^j_{\acute{e}t,c}(\mathrm{Poly}_1^{(\lambda_0\cdot n,\dots,\lambda_N\cdot n)}{}_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)$ as well.

Flatness follows from the fibers having the constant dimension of $|\vec{\lambda}|n$. ■

Since the base $\mathcal{P}(\vec{\lambda})$ is simply-connected (c.f. [Noohi, Example 9.2]), the stalk of the locally constant sheaves with compact support $(\mathrm{R}^j\,\mathrm{ev}_{\infty !}\,\mathbb{Q}_\ell)_p$ along a flat morphism $\mathrm{ev}_\infty : \mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda})) \to \mathcal{P}(\vec{\lambda})$ over a geometric point $p \in \mathcal{P}(\vec{\lambda})$ can be identified with $H^j_{\acute{e}t,c}(\mathrm{ev}_\infty^{-1}(p)_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)$ by the base change for the cohomology with compact support as in [DIS, Section 6.3] (see also [Petersen, §2]). Working fiberwise, we have $\mathrm{R}^j\,\mathrm{ev}_{\infty !}\,\mathbb{Q}_\ell = H^j_{\acute{e}t,c}(\mathrm{ev}_\infty^{-1}{}_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell) = H^j_{\acute{e}t,c}(\mathrm{Poly}_1^{(\lambda_0\cdot n,\dots,\lambda_N\cdot n)}{}_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)$ by the Proposition 2.12 which implies that the ℓ-adic Leray spectral sequence has the following E_2 page through the Poincaré duality.

$$E_2^{i,j} = \begin{cases} H^i_{\acute{e}t}(\mathcal{P}(\vec{\lambda})_{/\overline{\mathbb{F}}_q}; H^j_{\acute{e}t}(\mathrm{Poly}_1^{(\lambda_0\cdot n,\dots,\lambda_N\cdot n)}{}_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)) & \text{if } i,j \geq 0 \\ 0 & \text{else} \end{cases}$$

and the spectral sequence converges to $H^{i+j}_{\acute{e}t}(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)$.

Proposition 2.13. *Hom stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ with $n \in \mathbb{N}$ over $\overline{\mathbb{F}}_q$ with $\mathrm{char}(\overline{\mathbb{F}}_q)$ not dividing $\lambda_i \in \mathbb{N}$ for every i has the ℓ-adic étale cohomology $H^i_{\acute{e}t}(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)$ as $\mathbb{Q}_\ell$-vector spaces equal to one-dimensional for every $i = (0, 2N-1) \otimes (0, 2, \dots, 2N)$ with the middle-degree cohomology at $i = (2N-1,\ 2N)$ equal to zero. By the Poincaré duality, we have $H^i_{\acute{e}t,c}(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)$ one-dimensional for $i = 2(a+b)n + 2$, $i = 2(a+b)n - 1$ and vanishes for all other i.*

Proof. Consider the coarse moduli map $c : \mathcal{P}(a,b) \to \mathbb{P}^1$ where c can be identified with $c([x : y]) = [x^{\mathrm{lcm}(a,b)/a} : y^{\mathrm{lcm}(a,b)/b}] \in \mathbb{P}^1$ for any $[x : y] \in \mathcal{P}(a,b) \cong [(\mathbb{A}^2_{x,y} \setminus 0)/\mathbb{G}_m]$. Since each coordinate function of $\mathbb{P}^1$ lifts to degree $\mathrm{lcm}(a,b)$ functions on $\mathcal{P}(a,b)$, we conclude that $c^*\mathcal{O}_{\mathbb{P}^1}(1) \cong \mathcal{O}_{\mathcal{P}(a,b)}(\mathrm{lcm}(a,b))$. This implies that $\deg(c \circ \varphi_f) = \mathrm{lcm}(a,b) \cdot \deg \varphi_f$ where $\deg \varphi_f := \deg \varphi_f^* \mathcal{O}_{\mathcal{P}(a,b)}(1) = n$. Then the ℓ-adic étale cohomology of the fine moduli stack $\mathcal{P}(a,b)$ and its coarse moduli space $\mathbb{P}^1$ are isomorphic through the induced map $c^* : H^i_{\acute{e}t}(\mathbb{P}^1{}_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell) \cong H^i_{\acute{e}t}(\mathcal{P}(a,b)_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)$ by Lemma refFineCoarseCohomology. By the fuctoriality of the spectral sequence with regard to c^*, once we determine the transgression differential $d_2^{(2,0)}$ is an isomorphism for the case of $\mathrm{Hom}_{(\mathrm{lcm}(a,b)\cdot n)}(\mathbb{P}^1, \mathbb{P}^1)$ then we have the transgression differential $d_2^{(2,0)}$ is an isomorphism for the case of $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))$ as well. By the comparison as in [FW, Theorem 1.2.(3)], we consider $H^i_{\mathrm{sing}}(\mathrm{Hom}_{(\mathrm{lcm}(a,b)\cdot n)}(\mathbb{P}^1, \mathbb{P}^1)(\mathbb{C});\mathbb{C})$ where we have [KS, Proposition 2.2] showing that the transgression differential $d_2^{(2,0)}$ is an isomorphism for $\mathrm{Hom}_{(\mathrm{lcm}(a,b)\cdot n)}(\mathbb{P}^1, \mathbb{P}^1)$. As the transgression differential $d_2^{(2,0)}$ is also an isomorphism for

$\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))$, this implies that the ℓ-adic étale Betti numbers $\dim_{\mathbb{Q}_\ell}\left(H^i_{\acute{e}t}(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)\right)$ are equal to one for $i = 0$, $i = 3$ and vanishes for all other i. ■

As $H^i_{\acute{e}t,c}(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$ are all one dimensional, the trace of Frob_q on each of these $\mathbb{Q}_\ell$-vector spaces is just the corresponding eigenvalue λ_i of Frob_q.

When $i = 2(|\vec{\lambda}|n) + 2$, the connectedness of $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ together with the Poincaré duality implies that $\lambda_{2(|\vec{\lambda}|n)+2N} = q^{|\vec{\lambda}|n+N}$

Plugging all of the above into the *Grothendieck-Lefschetz trace formula* (2.9) :

$$\#_q(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))) = q^{|\vec{\lambda}|n+N} + \cdots + q^{|\vec{\lambda}|n+1} - q^{|\vec{\lambda}|n-1} - \cdots - q^{|\vec{\lambda}|n-N}$$

$$= \lambda_{2(|\vec{\lambda}|n+N)} + \cdots + \lambda_{2(|\vec{\lambda}|n+1)} - \lambda_{2(|\vec{\lambda}|n-1)} - \cdots - \lambda_{2(|\vec{\lambda}|n-N)}$$

which implies that $\lambda_{2(|\vec{\lambda}|n+i)} = q^{|\vec{\lambda}|n+i}$ for every $i \in \{N, \ldots, 1, -1, \ldots, -N\}$ as claimed.

This finishes the proof of Theorem 1.1.

■

Acknowledgments. It is a pleasure to thank Dori Bejleri, Benson Farb, Changho Han, Aaron Landesman, Behrang Noohi, Hunter Spink, Craig Westerland and Jesse Wolfson for many helpful conversations. The author was supported by IBS-R003-D1, Institute for Basic Science in Korea.

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