

# ENUMERATING ALGEBRAIC CURVES AND ABELIAN VARIETIES OVER GLOBAL FUNCTION FIELDS WITH LOWER ORDER TERMS

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ABSTRACT. Given asymptotic counts in number theory, a question of Venkatesh asks what is the topological nature of lower order terms. We consider the arithmetic aspect of the inertia stack of an algebraic stack over finite fields to answer this question. Subsequently, we enumerate algebraic curves and abelian varieties with precise lower order terms over  $\mathbb{F}_q(t)$  which renders new heuristics over  $\mathbb{Q}$  through the global fields analogy.

## 1. INTRODUCTION

In [GGW, Problem 5], Akshay Venkatesh asks the following question:

*What is the topological meaning of secondary terms  
appearing in asymptotic counts in number theory?*

In many interesting number theory problems (e.g., counting number fields, arithmetic curves or abelian varieties over a number field) one has not only a main term in the asymptotic count, but a secondary term or more. For example, the number of cubic fields of discriminant up to  $\mathcal{B}$  is

$$a\mathcal{B} + b\mathcal{B}^{5/6} + \text{lower order terms}$$

We have very little understanding of these lower order terms. They are not just of theoretical interest: when one tries to verify the conjectures numerically, one finds that the secondary terms are dominant in the computational range. Following the framework of the Weil conjectures, they do not correspond to stable cohomology classes (these are the main terms), but to some kind of slightly weaker structure, which is still much better behaved than cohomology near middle degree.

Note that the moduli functors we wish to enumerate are often represented by algebraic stacks rather than by schemes (or algebraic spaces) due to the presence of non-trivial automorphisms of the objects we wish to parameterize. If we consider a finite field analogue, the traditional approaches to count the number of rational points on the moduli spaces do not render the lower order terms. This is because the Grothendieck-Lefschetz trace formula (relating point counts and  $\ell$ -adic cohomologies) for algebraic stacks as in [Behrend, Sun] counts the rational points with weights (given a rational point  $x$ , its weight is  $\frac{1}{|\text{Aut}(x)|}$ ). Instead, we must acquire the number  $|\mathcal{X}(\mathbb{F}_q)/\sim|$  of  $\mathbb{F}_q$ -isomorphism classes of  $\mathbb{F}_q$ -points of the algebraic stack  $\mathcal{X}$ , i.e., the non-weighted point count of  $\mathcal{X}$  over  $\mathbb{F}_q$ . In this regard, the coarse moduli space  $c: \mathcal{X} \rightarrow X$  is insufficient as  $|X(\mathbb{F}_q)| \neq |\mathcal{X}(\mathbb{F}_q)/\sim|$ .

This discrepancy naturally raises the following question:

*What geometric object has the non-weighted point count  $|\mathcal{X}(\mathbb{F}_q)/\sim|$   
of the algebraic stack  $\mathcal{X}$  over  $\mathbb{F}_q$  as its natural arithmetic invariant?*

Given an algebraic stack of interest  $\mathcal{X}$  over  $\mathbb{F}_q$ , we clarify the arithmetic role of its *inertia stack*  $\mathcal{I}(\mathcal{X})$  which parameterizes pairs  $(x, \xi)$ , where  $x \in \mathcal{X}$  and  $\xi$  is the conjugate class of  $g \in \text{Aut}(x)$ .

**Theorem 1.1.** *Let  $\mathcal{X}$  be an algebraic stack over  $\mathbb{F}_q$  of finite type with quasi-separated finite type diagonal and let  $\mathcal{I}(\mathcal{X})$  be the inertia stack of  $\mathcal{X}$ . Then,*

$$\#_q(\mathcal{I}(\mathcal{X})) = |\mathcal{X}(\mathbb{F}_q)/\sim|$$

*where  $\#_q(\mathcal{I}(\mathcal{X}))$  is the weighted point count of the inertia stack  $\mathcal{I}(\mathcal{X})$  over  $\mathbb{F}_q$  and  $|\mathcal{X}(\mathbb{F}_q)/\sim|$  is the non-weighted point count of the algebraic stack  $\mathcal{X}$  over  $\mathbb{F}_q$ .*

It is difficult in general to compute  $\#_q(\mathcal{I}(\mathcal{X}))$  for an arbitrary algebraic stack  $\mathcal{X}$ . To overcome this, we focus on the quotient stacks  $\mathcal{X} \cong [U/G]$  where we have an explicit presentation of the inertia stack as  $\mathcal{I}(\mathcal{X}) \cong [R_\Delta/G]$  for some  $R_\Delta$  determined by  $G$ -scheme  $U$  (see Corollary 2.4). As we have a natural stratification of  $R_\Delta$  in certain cases, we use the idea of cut-and-paste by Grothendieck in  $K_0(\text{Stck}_K)$  to acquire the motive of the inertia stack as  $\{\mathcal{I}(\mathcal{X})\} = \sum_r \{\mathcal{I}_{\mu_r}(\mathcal{X})\}$ .

As a canonical example, we consider the weighted projective stack  $\mathcal{P}(\vec{\lambda})$  with the Hom stack  $\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$  parameterizing degree  $n \in \mathbb{N}$  morphisms  $f : \mathbb{P}^1 \rightarrow \mathcal{P}(\vec{\lambda})$  with  $f^* \mathcal{O}_{\mathcal{P}(\vec{\lambda})}(1) \simeq \mathcal{O}_{\mathbb{P}^1}(n)$ . We prove that the exact weighted point count  $\#_q \left( \mathcal{I} \left( \text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda})) \right) \right)$  of the inertia stack over  $\mathbb{F}_q$  can be acquired to be a polynomial in  $q$  which in turn provides the non-weighted point count  $\left| \text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda})) / \sim \right|$  of the Hom stack over  $\mathbb{F}_q$  by Theorem 1.1. However, efforts to acquire the arithmetic invariants of  $\mathcal{I}(\text{Hom}(\mathbb{P}^1, \overline{\mathcal{M}}_g))$  comes up against hard obstructions due to the global geometry of the Deligne–Mumford moduli stack  $\overline{\mathcal{M}}_g$  of stable genus  $g$  curves formulated in [DM]. For example, the coarse moduli space  $\overline{M}_g$  is of general type for  $g \geq 24$  by the fundamental works of Harris, Mumford and Eisenbud in [HM, EH] which in turn makes the study of (rational) curves on  $\overline{\mathcal{M}}_{g \geq 24}$  ineffective for counting stable curves of sufficiently high genus over  $\mathbb{P}_{\mathbb{F}_q}^1$ .

On the other hand, all smooth genus 2 curves are hyperelliptic, so  $\mathcal{M}_2 \cong \mathcal{H}_2$ . Therefore, we are naturally led to count hyperelliptic genus  $g \geq 2$  curves. Recall that an *odd degree* hyperelliptic curve has a marked rational Weierstrass point at  $\infty$ . In this paper, we will concentrate on the moduli substack  $\mathcal{H}_{g,1} \subset \mathcal{M}_{g,1}$  of hyperelliptic genus  $g \geq 2$  curves with 1 marked Weierstrass point (which has the same dimension as  $\mathcal{H}_g$ ) as we focus on counting odd degree hyperelliptic genus  $g \geq 2$  curves. Since  $\mathcal{H}_{g,1}$  is not proper, we consider the proper moduli stack  $\overline{\mathcal{H}}_{g,1} := \overline{\mathcal{H}_{g,1}} \subset \overline{\mathcal{M}}_{g,1}$  (meaning the reduced closure) of stable odd degree hyperelliptic curves. Extracting the exact arithmetic invariants of  $\text{Hom}_n(\mathbb{P}^1, \overline{\mathcal{H}}_{g,1})$  is challenging as it is difficult to write down a general member of  $K$ -points of  $\text{Hom}_n(\mathbb{P}^1, \overline{\mathcal{H}}_{g,1})$  explicitly as an equation  $y^2 = f(x, t)$  where  $t$  is a parameter of  $\mathbb{A}_K^1 \subset \mathbb{P}_K^1$  and  $f \in K(t)[x]$  of  $x$ -degree  $2g + 1$ . Therefore, we are led to impose conditions on characteristic of  $\mathbb{F}_q$ , then consider a different extension of smooth odd degree hyperelliptic curves, originally introduced as a special case of [Fedorchuk, Definition 2.5]:

**Definition 1.2.** Fix an integral reduced  $K$ -scheme  $B$ , where  $\text{char}(K) \neq 2$ . A flat family  $u : C \rightarrow B$  of genus  $g \geq 2$  curves is *quasi-admissible* if every geometric fiber has at worst  $A_{2g-1}$ -singularity (i.e., étale locally defined by  $x^2 + y^m$  for some  $0 < m \leq 2g$ ), and factors through a separable morphism  $\phi : C \rightarrow H$  of degree 2 where  $H$  is a  $\mathbb{P}^1$ -bundle over  $B$  with a distinguished section (often called  $\infty$ ) which is a connected component of the branch locus of  $u$ .

For example, if  $\text{char}(K) > 2g + 1$  or 0, then a quasi-admissible curve over any  $K$ -scheme  $B$  can be written as

$$(1) \quad y^2 = f(x) = x^{2g+1} + a_4 x^{2g-1} + a_6 x^{2g-2} + a_8 x^{2g-3} + \cdots + a_{4g+2},$$

where  $a_i$ 's are appropriate sections of suitable line bundles on  $B$  where not all of them simultaneously vanish at anywhere on  $B$ . This equation should be thought of as a generalized Weierstrass equation. Here, we identify the section at  $\infty$  as the locus missed by the above affine equation. This identification is a consequence of Proposition 5.9, where we show that the Deligne–Mumford moduli stack  $\mathcal{H}_{2g}[2g-1]$  of quasi-admissible curves of genus  $g$  is isomorphic to the weighted projective stack  $\mathcal{P}(\vec{\lambda}_g)$  for  $\vec{\lambda}_g := (4, 6, 8, \dots, 4g+2)$  over base field  $K$  with  $\text{char}(K) = 0$  or  $> 2g + 1$ . To effectively count the non-weighted  $\mathbb{F}_q$ -points of the Hom stack  $\text{Hom}(\mathbb{P}^1, \mathcal{H}_{2g}[2g-1])$ , we need to impose a notion of *bounded height* on those  $\mathbb{F}_q$ -points. Thanks to the works of Lockhart and

Liu, we have a natural definition (see Definition 5.16) of a hyperelliptic discriminant  $\Delta_g$  of quasi-admissible curves as in [Lockhart, Liu2]. In fact, it is a homogeneous polynomial of degree  $4g(2g+1)$  on variables  $a_i$ 's, where each  $a_i$  has degree  $i$  ( $a_i$ 's are as in equation (1) where  $B = \mathbb{P}_{\mathbb{F}_q}^1$  in this case). Moreover, since  $\mathcal{P}(4, 6, 8, \dots, 4g+2)$  is a weighted projective stack (Definition 3.1) carrying a primitive ample line bundle  $\mathcal{O}_{\mathcal{P}(4,6,8,\dots,4g+2)}(1)$ , the degree of the discriminant  $\Delta_g$  of a given quasi-admissible fibration  $f : \mathbb{P}^1 \rightarrow \mathcal{H}_{2g}[2g-1] \cong \mathcal{P}(4, 6, 8, \dots, 4g+2)$  is equal to  $4g(2g+1)n$  where  $f^*\mathcal{O}_{\mathcal{P}(4,6,8,\dots,4g+2)}(1) \cong \mathcal{O}_{\mathbb{P}^1}(n)$ . Therefore, the Hom stack  $\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(4, 6, 8, \dots, 4g+2))$  parameterizing such morphisms is the moduli stack of quasi-admissible genus  $g \geq 2$  fibrations of a fixed discriminant degree  $|\Delta_g| \cdot n = 4g(2g+1)n$ . Consequently, we acquire the exact arithmetic invariant of the inertia stack  $\mathcal{I}(\mathcal{L}_{g,|\Delta_g| \cdot n})$  of the moduli stack  $\mathcal{L}_{g,|\Delta_g| \cdot n} \cong \text{Hom}_n(\mathbb{P}^1, \mathcal{H}_{2g}[2g-1])$ .

**Theorem 1.3.** *If  $\text{char}(\mathbb{F}_q) > 2g+1$ , the number  $|\mathcal{L}_{g,|\Delta_g| \cdot n}(\mathbb{F}_q)/\sim|$  of  $\mathbb{F}_q$ -isomorphism classes of quasi-admissible hyperelliptic genus  $g = 2, 3, 4$  fibrations over  $\mathbb{P}_{\mathbb{F}_q}^1$  with a marked Weierstrass point and a hyperelliptic discriminant of degree  $|\Delta_g| \cdot n = 4g(2g+1)n$  is equal to*

$$|\mathcal{L}_{2,40n}(\mathbb{F}_q)/\sim| = 2 \cdot q^{28n} \cdot (q^3 + q^2 + q^1 - q^{-1} - q^{-2} - q^{-3}) + \delta(4, q-1) \cdot 2 \cdot q^{12n} \cdot (q^1 - q^{-1})$$

$$\begin{aligned} |\mathcal{L}_{3,84n}(\mathbb{F}_q)/\sim| &= 2 \cdot q^{54n} \cdot (q^5 + \dots + q^1 - q^{-1} - \dots - q^{-5}) \\ &\quad + \delta(4, q-1) \cdot 2 \cdot q^{24n} \cdot (q^2 + q^1 - q^{-1} - q^{-2}) + \delta(6, q-1) \cdot 4 \cdot q^{18n} \cdot (q^1 - q^{-1}) \end{aligned}$$

$$\begin{aligned} |\mathcal{L}_{4,144n}(\mathbb{F}_q)/\sim| &= 2 \cdot q^{88n} \cdot (q^7 + \dots + q^1 - q^{-1} - \dots - q^{-7}) \\ &\quad + \delta(4, q-1) \cdot 2 \cdot q^{40n} \cdot (q^3 + q^2 + q^1 - q^{-1} - q^{-2} - q^{-3}) \\ &\quad + \delta(6, q-1) \cdot 4 \cdot q^{36n} \cdot (q^2 + q^1 - q^{-1} - q^{-2}) + \delta(8, q-1) \cdot 4 \cdot q^{24n} \cdot (q^1 - q^{-1}) \end{aligned}$$

where

$$\delta(a, b) := \begin{cases} 1 & \text{if } a \text{ divides } b, \\ 0 & \text{otherwise.} \end{cases}$$

For genus  $g \geq 5$ , the corresponding exact non-weighted point count  $|\mathcal{L}_{g,|\Delta_g| \cdot n}(\mathbb{F}_q)/\sim|$  of the moduli stack  $\mathcal{L}_{g,|\Delta_g| \cdot n}$  over  $\mathbb{F}_q$  can be similarly worked out.

Given a stable odd hyperelliptic genus  $g \geq 2$  curve  $X$  over  $\mathbb{P}_{\mathbb{F}_q}^1$  with  $\text{char}(\mathbb{F}_q) > 2g+1$ , define the *height* of the hyperelliptic discriminant  $\Delta_g(X)$  to be  $ht(\Delta_g(X)) := q^{\deg \Delta_g(X)} = q^{4g(2g+1)n}$  (see Definition 6.5). Then, for a positive real number  $\mathcal{B}$ , we define the counting function  $\mathcal{Z}_{g, \mathbb{F}_q(t)}(\mathcal{B})$ .

$$\mathcal{Z}_{g, \mathbb{F}_q(t)}(\mathcal{B}) := |\{\text{Stable odd hyperelliptic genus } g \geq 2 \text{ curves over } \mathbb{P}_{\mathbb{F}_q}^1 \text{ with } 0 < ht(\Delta_g(X)) \leq \mathcal{B}\}|$$

We relate in §5 the arithmetic invariants of  $\text{Hom}_n(\mathbb{P}^1, \overline{\mathcal{H}}_{g,1})$  to the arithmetic invariants of  $\text{Hom}_n(\mathbb{P}^1, \mathcal{H}_{2g}[2g-1])$ ; through the birational geometry between the family of stable odd degree hyperelliptic curves with generically smooth fibers and the family of quasi-admissible curves. Specifically, as a consequence of Theorem 5.12 over a field  $K$  with  $\text{char}(K) = 0$  or  $> 2g+1$ , we show that any given stable odd hyperelliptic model over  $\mathbb{P}_K^1$  admits a birational transformation into a unique quasi-admissible curve over  $\mathbb{P}_K^1$ , and no two non-isomorphic stable odd hyperelliptic models correspond to the isomorphic quasi-admissible curve. *Therefore, counting the number  $\mathcal{Z}'_{g, \mathbb{F}_q(t)}(\mathcal{B})$  of quasi-admissible genus  $g \geq 2$  curves over  $\mathbb{P}_{\mathbb{F}_q}^1$  renders an upper bound for counting the number  $\mathcal{Z}_{g, \mathbb{F}_q(t)}(\mathcal{B})$  of stable odd hyperelliptic genus  $g \geq 2$  curves over  $\mathbb{P}_{\mathbb{F}_q}^1$  with generically smooth fibers.*

In §6, we finally prove the following enumerations.

**Theorem 1.4** (Estimate on  $\mathcal{Z}_{g,\mathbb{F}_q(t)}(\mathcal{B})$ ). *If  $\text{char}(\mathbb{F}_q) > 2g + 1$ , then the function  $\mathcal{Z}_{g,\mathbb{F}_q(t)}(\mathcal{B})$ , which counts the number of stable hyperelliptic genus  $g \geq 2$  curves  $X$  with a marked Weierstrass point over  $\mathbb{P}_{\mathbb{F}_q}^1$  ordered by  $0 < \text{ht}(\Delta_g(X)) = q^{4g(2g+1)n} \leq \mathcal{B}$ , satisfies:*

$$\mathcal{Z}_{2,\mathbb{F}_q(t)}(\mathcal{B}) \leq a_{2,2} \cdot \mathcal{B}^{\frac{7}{10}} + a_{2,4} \cdot \mathcal{B}^{\frac{3}{10}} + b_{2,4}$$

$$\mathcal{Z}_{3,\mathbb{F}_q(t)}(\mathcal{B}) \leq a_{3,2} \cdot \mathcal{B}^{\frac{9}{14}} + a_{3,4} \cdot \mathcal{B}^{\frac{2}{7}} + a_{3,6} \cdot \mathcal{B}^{\frac{3}{14}} + b_{3,6}$$

$$\mathcal{Z}_{4,\mathbb{F}_q(t)}(\mathcal{B}) \leq a_{4,2} \cdot \mathcal{B}^{\frac{11}{18}} + a_{4,4} \cdot \mathcal{B}^{\frac{5}{18}} + a_{4,6} \cdot \mathcal{B}^{\frac{1}{4}} + a_{4,8} \cdot \mathcal{B}^{\frac{1}{6}} + b_{4,8}$$

where  $\delta(a, b)$  is as in Theorem 1.3, and for each  $g, m \in \mathbb{N}_{\geq 2}$ ,  $a_{g,2}(q), a_{g,2m}(q, \delta(2m, q-1))$ , and  $b_{g,2m}(q, \delta(4, q-1), \delta(6, q-1), \dots, \delta(2m, q-1))$  are explicit rational functions as in Theorem 6.6.

For higher genus  $g \geq 5$ , the corresponding upper bound on  $\mathcal{Z}_{g,\mathbb{F}_q(t)}(\mathcal{B})$  rendering a closed-form formula with precise lower order terms can be similarly worked out through Theorem 6.6.

**Remark 1.5.** The topological meaning of the weighted point count  $\#_q(\mathcal{I}(\mathcal{X}))$  of the inertia stack  $\mathcal{I}(\mathcal{X})$  over  $\mathbb{F}_q$  is the compactly-supported,  $\ell$ -adic étale cohomology  $H_{\text{ét},c}^* \left( \mathcal{I}(\mathcal{X})_{/\mathbb{F}_q}; \mathbb{Q}_\ell \right)$  with Frobenius weights by the *Grothendieck-Lefschetz trace formula* [Behrend, Sun] for Artin stacks of finite type over  $\mathbb{F}_q$  since the inertia stack  $\mathcal{I}(\mathcal{X})$  is also an algebraic stack of finite type over  $\mathbb{F}_q$  as long as  $\mathcal{X}$  is (see Lemma 2.2). Generally, however, the inertia stack  $\mathcal{I}(\mathcal{X})$  is not connected and the irreducible components could have varying dimensions.

Todo: special case, some components come from substack of  $\mathcal{X}$  with higher order stabilizers.

In the context of our Hom stacks, we recognize that the terms with division function  $\delta(r, q-1)$  factors (originating from the higher order automorphisms in relation with the residue field; explained in Proposition 4.10) in the weighted point count  $\#_q(\mathcal{I}(\mathcal{L}_{g,|\Delta_g| \cdot n}))$  of the inertia stack over  $\mathbb{F}_q$  are precisely what contribute to the lower order terms in the subsequent enumeration of Theorem 1.4. In this regard, we see that the compactly-supported,  $\ell$ -adic étale cohomology  $H_{\text{ét},c}^i \left( \mathcal{I}(\mathcal{L}_{g,|\Delta_g| \cdot n})_{/\mathbb{F}_q}; \mathbb{Q}_\ell \right)$  of lower-degree ( $i < 2 \dim(\mathcal{L}_{g,|\Delta_g| \cdot n})$ ) is the topological meaning of the lower-order terms (i.e., “slightly weaker structure”; see [Park] for an actual computation) we were originally looking for when we extend the framework of the Weil conjectures from the moduli spaces to the inertia stacks of the moduli stacks. Specifically, if we just have the motive  $\{\mathcal{L}_{g,|\Delta_g| \cdot n}\}$  of the fine moduli stack  $\mathcal{L}_{g,|\Delta_g| \cdot n}$  leading to its weighted point count  $\#_q(\mathcal{L}_{g,|\Delta_g| \cdot n})$  over  $\mathbb{F}_q$  as in Corollary 5.11 (which is very close to the point count  $|\mathcal{L}_{g,|\Delta_g| \cdot n}(\mathbb{F}_q)|$  of the coarse moduli space  $c : \mathcal{L}_{g,|\Delta_g| \cdot n} \rightarrow L_{g,|\Delta_g| \cdot n}$  over  $\mathbb{F}_q$  via the usual isomorphisms for the étale cohomology in  $\mathbb{Q}_\ell$ -coefficients  $c^* : H_{\text{ét}}^i(L_{g,|\Delta_g| \cdot n}/\mathbb{F}_q; \mathbb{Q}_\ell) \cong H_{\text{ét}}^i(\mathcal{L}_{g,|\Delta_g| \cdot n}/\mathbb{F}_q; \mathbb{Q}_\ell)$  for a smooth separated tame Deligne–Mumford stack of finite type) then all we acquire in the subsequent enumeration is the order of the leading term (i.e., the main term) which corresponds to the stable cohomology classes of the fine moduli stack  $\mathcal{L}_{g,|\Delta_g| \cdot n}$ . In light of all this, we make an observation that *the arithmetic geometry & topology of the inertia stack  $\mathcal{I}(\mathcal{X})$  is fundamentally more intricate than that of the algebraic stack  $\mathcal{X}$  itself.*

From the perspective of arithmetic, once we have found an interesting moduli stack  $\mathcal{X}$  then we have the presentation of its inertia stack  $\mathcal{I}(\mathcal{X}) \cong [R_\Delta/G]$  from Corollary 2.4. The question then leads to how well can we stratify  $R_\Delta$  similar to Proposition 3.8 to study its arithmetic & cohomology in detail as in Proposition 4.5 into Proposition 4.10. It would be significant (and very difficult) if we could systematically understand the inertia stacks for diverse fine moduli algebraic

stacks  $\mathcal{X}$  relevant in number theory which are non-cyclotomic in nature whereby the objects we wish to parameterize & enumerate have various non-abelian automorphisms. ■

And we naturally have an important consequence to consider regarding the enumeration of abelian varieties of dimension 2, i.e., *abelian surfaces* over global function fields. By the local (i.e., infinitesimal) Torelli theorem in [OS, Theorem 2.6 and 2.7] and [Milne, Theorem 12.1], the Torelli map  $\tau_2 : \mathcal{M}_2 \hookrightarrow \mathcal{A}_2$ , which sends a smooth projective genus 2 curve  $X$  defined over a field  $K$  to its principally polarized Jacobian  $(\text{Jac}(X), \lambda_\theta)/K$  (where  $\lambda_\theta$  is the theta divisor of  $\text{Jac}(X)$ ), is an open immersion. Furthermore, it is shown in [OU, 4. Theorem] (see also [Weil, Satz 2]) that given a principally polarized abelian surface  $(A, \lambda)$  over a field  $K$ , after a finite extension of scalars, is isomorphic to the canonically principally polarized (generalized) Jacobian variety  $(\text{Jac}(X), \lambda_\theta)$  of a stable genus 2 curve  $X$ . Recall that if a curve  $X$  has good reduction at a place  $v \in S$  then so does its Jacobian  $\text{Jac}(X)$ . We have the following enumeration.

**Corollary 1.6** (Estimate on  $\mathcal{N}_{2, \mathbb{F}_q(t)}(\mathcal{B})$ ). *If  $\text{char}(\mathbb{F}_q) \neq 2, 3, 5$ , then the function  $\mathcal{N}_{2, \mathbb{F}_q(t)}(\mathcal{B})$ , which counts the number of principally polarized abelian surfaces  $A = \text{Jac}(X)$  where  $X$  is a stable genus 2 curve with a marked Weierstrass point over  $\mathbb{P}_{\mathbb{F}_q}^1$  ordered by  $0 < ht(\Delta_2(X)) = q^{40n} \leq \mathcal{B}$ , satisfies:*

$$\mathcal{N}_{2, \mathbb{F}_q(t)}(\mathcal{B}) \leq a_{2,2} \cdot \mathcal{B}^{\frac{7}{10}} + a_{2,4} \cdot \mathcal{B}^{\frac{3}{10}} + b_{2,4}$$

$$a_{2,2}(q) = 2 \cdot \frac{(q^{31} + q^{30} + q^{29} - q^{27} - q^{26} - q^{25})}{(q^{28} - 1)}$$

$$a_{2,4}(q) = \delta(4, q-1) \cdot 2 \cdot \frac{(q^{13} - q^{11})}{(q^{12} - 1)}$$

$$b_{2,4}(q) = -2 \cdot \frac{(q^{31} + q^{30} + q^{29} - q^{27} - q^{26} - q^{25})}{(q^{28} - 1)} - \delta(4, q-1) \cdot 2 \cdot \frac{(q^{13} - q^{11})}{(q^{12} - 1)}$$

where

$$\delta(4, q-1) := \begin{cases} 1 & \text{if 4 divides } q-1, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Theorem 1.4 provides an explicit upper bound with precise lower order terms on the number of stable genus 2 curves with a marked Weierstrass point over  $\mathbb{P}_{\mathbb{F}_q}^1$  with  $\text{char}(\mathbb{F}_q) \neq 2, 3, 5$ . The upper bound follows from the properties of the Torelli map  $\tau_2$  as all principally polarized abelian surfaces are isomorphic to Jacobians of genus 2 curves of compact type (i.e., genus 2 curve with dual graph equal to tree) (c.f., [OS, Theorem 2.6 and 2.7] & [OU, 4. Theorem]). ■

In the end, we formulate “strong asymptotic” heuristics (see **Conventions**) for the analogous asymptotic counts over  $\mathbb{Q}$  by passing the above upper bounds through the global fields analogy where the bounded height  $ht(\Delta_g)$  of the discriminant on the number field side is the cardinality of the ring of functions on  $\text{Spec}(\mathbb{Z}/(\Delta_g))$ .

**Conjecture 1.7** (Heuristic on  $\mathcal{N}_{2, \mathbb{Q}}(\mathcal{B})$ ). The function  $\mathcal{N}_{2, \mathbb{Q}}(\mathcal{B})$ , which counts the number of principally polarized abelian surfaces  $A = \text{Jac}(X)$  where  $X$  is a stable genus 2 curve over  $\mathbb{Z}$  with  $0 < ht(\Delta_2(X)) \leq \mathcal{B}$ , satisfies:

$$\mathcal{N}_{2, \mathbb{Q}}(\mathcal{B}) \leq a \cdot \mathcal{B}^{\frac{7}{10}} + b \cdot \mathcal{B}^{\frac{3}{10}} + c$$

having the leading term of order  $\mathcal{O}\left(\mathcal{B}^{\frac{7}{10}}\right)$  with lower order terms of orders  $\mathcal{O}\left(\mathcal{B}^{\frac{3}{10}}\right)$  and  $\mathcal{O}(1)$ .

And similarly for the higher genus  $g \geq 3$  hyperelliptic curves we formulate the following:

**Conjecture 1.8** (Heuristic on  $\mathcal{Z}_{g,\mathbb{Q}}(\mathcal{B})$ ). The function  $\mathcal{Z}_{g,\mathbb{Q}}(\mathcal{B})$ , which counts the number of stable hyperelliptic genus  $g \geq 3$  curves with a marked rational Weierstrass point over  $\mathbb{Z}$  with  $0 < ht(\Delta_g) \leq \mathcal{B}$ , satisfies:

$$\mathcal{Z}_{3,\mathbb{Q}}(\mathcal{B}) \leq a \cdot \mathcal{B}^{\frac{9}{14}} + b \cdot \mathcal{B}^{\frac{2}{7}} + c \cdot \mathcal{B}^{\frac{3}{14}} + d$$

$$\mathcal{Z}_{4,\mathbb{Q}}(\mathcal{B}) \leq a \cdot \mathcal{B}^{\frac{11}{18}} + b \cdot \mathcal{B}^{\frac{5}{18}} + c \cdot \mathcal{B}^{\frac{1}{4}} + d \cdot \mathcal{B}^{\frac{1}{6}} + e$$

For higher genus  $g \geq 5$ , the corresponding orders of the leading term and the lower order terms can be similarly worked out through Theorem 1.4.

In the Appendix, we determine the sharp estimate for counting elliptic curves with prescribed level structure or multiple marked points over  $\mathbb{P}_{\mathbb{F}_q}^1$  and formulate the following Heuristics.

**Conjecture 1.9** (Heuristic on  $\mathcal{Z}_{1,\mathbb{Q}}^{[\Gamma]}(\mathcal{B})$ ). The function  $\mathcal{Z}_{1,\mathbb{Q}}^{[\Gamma]}(\mathcal{B})$ , which counts the number of generalized elliptic curves with  $[\Gamma]$ -structures over  $\mathbb{Z}$  ordered by  $0 < ht(\Delta_1) \leq \mathcal{B}$ , satisfies:

$$\mathcal{Z}_{1,\mathbb{Q}}^{[\Gamma_1(2)]}(\mathcal{B}) \leq a \cdot \mathcal{B}^{\frac{1}{2}} + b$$

$$\mathcal{Z}_{1,\mathbb{Q}}^{[\Gamma_1(3)]}(\mathcal{B}) \leq a \cdot \mathcal{B}^{\frac{1}{3}} + b$$

$$\mathcal{Z}_{1,\mathbb{Q}}^{[\Gamma_1(4)]}(\mathcal{B}) \leq a \cdot \mathcal{B}^{\frac{1}{4}} + b$$

$$\mathcal{Z}_{1,\mathbb{Q}}^{[\Gamma(2)]}(\mathcal{B}) \leq a \cdot \mathcal{B}^{\frac{1}{3}} + b$$

**Conjecture 1.10** (Heuristic on  $\mathcal{Z}_{1,m,\mathbb{Q}}(\mathcal{B})$ ). The function  $\mathcal{Z}_{1,m,\mathbb{Q}}(\mathcal{B})$ , which counts the number of  $m$ -marked  $(m-1)$ -stable genus one curves for  $2 \leq m \leq 5$  over  $\mathbb{Z}$  ordered by  $0 < ht(\Delta_1) \leq \mathcal{B}$ , satisfies:

$$\mathcal{Z}_{1,2,\mathbb{Q}}(\mathcal{B}) \leq a \cdot \mathcal{B}^{\frac{3}{4}} + b \cdot \mathcal{B}^{\frac{1}{2}} + c$$

$$\mathcal{Z}_{1,3,\mathbb{Q}}(\mathcal{B}) \leq a \cdot \mathcal{B}^{\frac{2}{3}} + b \cdot \mathcal{B}^{\frac{1}{3}} + c$$

$$\mathcal{Z}_{1,4,\mathbb{Q}}(\mathcal{B}) \leq a \cdot \mathcal{B}^{\frac{7}{12}} + b \cdot \mathcal{B}^{\frac{1}{3}} + c$$

$$\mathcal{Z}_{1,5,\mathbb{Q}}(\mathcal{B}) \leq a \cdot \mathcal{B}^{\frac{1}{2}} + b$$

**Remark 1.11.** On a related note, the classical [Ogg, Saito] formula gives the *Artin conductor* = (order of the) *discriminant* equality for elliptic curves which implies that counting the elliptic curves by discriminant  $\Delta_1$  is the same as counting the elliptic curves by conductor  $\mathcal{N}_1$ . We recall that the number of discriminants  $\Delta_1$  of an elliptic curve over  $\mathbb{Z}$  with smooth generic fiber such

that  $\Delta_1 \leq \mathcal{B}$  is estimated to be asymptotic to  $\mathcal{O}\left(\mathcal{B}^{\frac{5}{6}}\right)$  by [BMc]. The lower order term of order  $\mathcal{O}\left(\mathcal{B}^{(7-\frac{5}{27}+\epsilon)/12}\right)$  for counting the stable elliptic curves over  $\mathbb{Q}$  by the bounded height of squarefree  $\Delta_1$  was suggested by the work of [Baier] improving upon their previous error term in [BB] (see [Hortsch] for asymptotic with bounded Faltings height). In fact, Baier proved his asymptotic under the assumption of the generalized Riemann hypothesis with the twelfth root of the naïve height function on elliptic curves which gives the prediction above. For global function fields  $\mathbb{F}_q(t)$ , by considering the moduli of semistable elliptic surfaces and finding its motive/point count, we acquire the sharp estimate [HP, Theorem 3] on  $\mathcal{Z}_{1,\mathbb{F}_q(t)}$  for counting the semistable elliptic curves by the bounded height of  $\Delta_1(X)$  over  $\mathbb{P}_{\mathbb{F}_q}^1$  with  $\text{char}(\mathbb{F}_q) \neq 2, 3$  giving the leading term of order  $\mathcal{O}_q\left(\mathcal{B}^{\frac{5}{6}}\right)$  and the lower order term of zeroth order  $\mathcal{O}_q(1)$ . The arithmetic invariant which leads to the above counting also has been established in the past via different method by the seminal work of [de Jong].

For genus 2 curves, the work of [Liu3] proves  $\mathcal{N}_2 \leq \nu(\Delta_2)$  where he also shows that equality can fail to hold. And for hyperelliptic genus  $g \geq 3$  curves, we have  $\mathcal{N}_g \leq \nu(\Delta_g)$  proven recently by the work of [Srinivasan, OSr]. Consequently, our enumeration  $\mathcal{Z}_{g,\mathbb{F}_q(t)}(\mathcal{B})$  of stable hyperelliptic genus  $g \geq 2$  curves with a marked Weierstrass point over  $\mathbb{P}_{\mathbb{F}_q}^1$  with  $\text{char}(\mathbb{F}_q) > 2g + 1$  by the hyperelliptic discriminant  $\Delta_g(X)$  bounds above the parallel counting by the conductor  $\mathcal{N}_g$ . Since the qualitative finiteness shown by the classical works of [Parshin, Oort] (see [BG] for counting by naïve height and [Känel] for partly explicit upper bound), we currently do not have an explicit estimate with precise lower order terms for the number of stable genus 2 curves over  $\mathbb{Q}$  ordered by the bounded height of  $\Delta_2$  (and similarly for the number of principally polarized abelian surfaces).

We note that counting curves over a global field  $K$  by height of discriminant  $ht(\Delta_g)$  is more difficult than counting by *naïve height* (see [Brumer, p. 446]). In the case of counting elliptic curves ordered by height of discriminant  $ht(\Delta_1)$ , the region of lattice points with bounded discriminant has *cusps*, meaning that there are *unbounded points* with large  $a_4, a_6 \in \mathcal{O}_K$  (i.e., elliptic curves with large naïve height) and small discriminant  $\Delta_1$ . As explained in [Hortsch], controlling these cusps is difficult, even if the ABC conjecture is assumed. ■

**Conventions.** In the present paper, schemes/stacks are assumed to be defined over a field  $K$ , if  $K$  is not mentioned explicitly or if such scheme is obviously not defined over any field (e.g.,  $\text{Spec } \mathbb{Z}$ ). We identify the Weil divisors and the associated divisorial sheaves implicitly (e.g., if  $X$  is a Cohen-Macaulay scheme, then the canonical divisor  $K_X$  corresponds to the dualizing sheaf  $\omega_X \cong \mathcal{O}(K_X)$  of  $X$ ). Given a point  $x$  of a scheme/stack,  $\kappa(g)$  means the field of definition of  $x$  (i.e., the residue field). Here, we use the convention in [Olsson2, §8] that the diagonal of an algebraic stack is representable (by algebraic spaces). For any  $T$ -point  $x$  of a stack  $\mathcal{X}$ ,  $\text{Aut}(x)$  is the group of automorphisms of  $x \in \mathcal{X}(T)$  (defined over  $T$ ).

Given a finite morphism  $f : X \rightarrow Y$  of reduced equidimensional schemes, a branch divisor of  $f$  on  $Y$  means the pushforward of the ramification divisor of  $f$  on  $X$ . Given a morphism  $f : X \rightarrow Y$  of schemes with an isolated subset  $Z \subset Y$  (i.e.,  $Y$  as a topological space is  $Z \sqcup (Y \setminus Z)$  under the Zariski topology), the preimage of  $Z$  in  $X$  refers to the components of  $X$  with their image supported on  $Z$ .

Given a function  $f : \mathbb{N} \rightarrow \mathbb{R}$  with coefficients depending on an integer  $q$ ,  $f \sim \mathcal{O}_q(x^\alpha)$  means that  $f \sim c(q)x^\alpha$  for some function  $c(q)$  of  $q$  (independent of the input variable  $x$ ). Additionally, given two functions  $f, g$  each with a leading and lower order terms; they are asymptotically tight by  $h$  if  $(f - g) \in \mathcal{O}(h)$ . This implies that  $f$  has the “strong asymptotic” configuration  $x^{e_1}, x^{e_2}, \dots, x^{e_k}$  upto  $x^{e_{k+1}}$  as there are coefficients  $c_i$ ’s such that  $f$  and  $\sum c_i x^{e_i}$  are asymptotically tight by  $x^{e_{k+1}}$ . Finally, a “sharp estimate” of a function  $f$  (with coefficients depending on  $q$ ) means an upper bound  $g$  (with same properties) that coincide with  $f$  on infinitely many values of  $x \in \mathbb{N}$ .

## CONTENTS

1. Introduction	1
Conventions	7
2. Arithmetic geometry of the inertia stack $\mathcal{I}(\mathcal{X})$ of an algebraic stack $\mathcal{X}$	8
3. Hom stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ of rational curves on a weighted projective stack	12
4. Motive/Point count of Hom and inertia stacks	15
4.1. Motive of Hom stack	16
4.2. Point count of Hom stack	19
4.3. Point count of Inertia of Hom stack	20
5. Moduli stack $\mathcal{L}_{g, \Delta_g  \cdot n}$ of quasi-admissible hyperelliptic genus $g$ fibrations over $\mathbb{P}^1$	21
5.1. Hyperelliptic discriminant $\Delta_g$ of quasi-admissible hyperelliptic genus $g$ fibration	26
6. Counting stable odd degree hyperelliptic curves over global fields by $\Delta_g$	28
7. Appendix - Arithmetic of the moduli of elliptic curves with a prescribed level structure or multiple marked points	32
7.1. Arithmetic of the moduli of generalized elliptic curves over $\mathbb{P}^1$ with level structures	34
7.2. Arithmetic of the moduli of $m$ -marked genus one fibrations over $\mathbb{P}^1$	37
References	40

## 2. ARITHMETIC GEOMETRY OF THE INERTIA STACK $\mathcal{I}(\mathcal{X})$ OF AN ALGEBRAIC STACK $\mathcal{X}$

In this section, we describe the geometry of inertia stacks associated to algebraic stacks; particularly, we first recall various key properties of inertia stacks. By using these properties, we prove Theorem 1.1. Then we describe the groupoid structure of inertia stacks, in particular, inertia stacks of quotient stacks. For general reference on algebraic stacks, we refer the reader to [Olsson2, Stacks].

Recall that given an algebraic stack  $\mathcal{X}$  defined over a field  $K$ , its inertia stack  $\mathcal{I}(\mathcal{X})$  is defined by:

- (1) objects:  $(x, \alpha)$  where  $x \in \mathcal{X}(T)$  for some scheme  $T$  (i.e.,  $x : T \rightarrow \mathcal{X}$ ) and  $\alpha \in \mathrm{Aut}(x)$
- (2) morphisms:  $\psi : (x, \alpha) \rightarrow (y, \beta)$  is given by  $\phi : x \rightarrow y$  in  $\mathrm{Mor}(\mathcal{X})(T)$  such that  $\phi \circ \alpha = \beta \circ \phi$ , i.e.,  $\beta = \phi \circ \alpha \circ \phi^{-1}$

Also,  $\mathcal{I}(\mathcal{X})$  is characterized by the following Cartesian diagram (by [Olsson2, Definition 8.1.17]):

$$(2) \quad \begin{array}{ccc} \mathcal{I}(\mathcal{X}) & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \Delta \\ \mathcal{X} & \xrightarrow{\Delta} & \mathcal{X} \times \mathcal{X} \end{array}$$

Note that if the representable morphism  $\Delta$  satisfies a property (such as finite type, quasi-separated, etc.), then this property is also satisfied for the representable morphism  $\mathcal{I}(\mathcal{X}) \rightarrow \mathcal{X}$ . In particular,  $\mathcal{I}(\mathcal{X})$  is a  $\mathcal{X}$ -algebraic space, i.e.,  $\mathcal{I}(\mathcal{X}) \times_{\mathcal{X}} T$  is an algebraic space for any morphism  $T \rightarrow \mathcal{X}$  from a scheme  $T$ .

To understand  $\mathcal{I}(\mathcal{X}) \rightarrow \mathcal{X}$ , we first pay attention to  $\Delta$ . Given an object  $x : T \rightarrow \mathcal{X}$  of  $\mathcal{X}$  from a scheme  $T$ , recall that the automorphism space  $\underline{\mathrm{Aut}}_x$  of  $x$  is defined to be the fiber product  $\mathcal{X} \times_{\Delta \times_{\mathcal{X}} \mathcal{X}} T$ . This means that  $S$ -points of  $\underline{\mathrm{Aut}}_x$  are characterized by pairs  $(s, \alpha)$  of maps  $s : S \rightarrow T$  and automorphisms  $\alpha : s^*x \rightarrow s^*x$  in the groupoid  $\mathcal{X}(T)$ . Since  $x \times x$  factors through  $\Delta$ ,  $\underline{\mathrm{Aut}}_x$  fits



into the following Cartesian diagram:

$$(3) \quad \begin{array}{ccc} \underline{\mathrm{Aut}}_x & \longrightarrow & T \\ \downarrow & & \downarrow x \\ \mathcal{I}(\mathcal{X}) & \longrightarrow & \mathcal{X} \end{array}$$

As before, representability of  $\Delta$  implies that  $\underline{\mathrm{Aut}}_x \rightarrow T$  is a morphism of algebraic spaces, and the group algebraic space structure on  $\underline{\mathrm{Aut}}_x$  lift, realizing  $\mathcal{I}(\mathcal{X})$  as a group algebraic space over  $\mathcal{X}$ .

Before proving Theorem 1.1, we recall the definition of a weighted point count of an algebraic stack  $\mathcal{X}$  over  $\mathbb{F}_q$ :

**Definition 2.1.** The weighted point count of  $\mathcal{X}$  over  $\mathbb{F}_q$  is defined as a sum:

$$\#_q(\mathcal{X}) := \sum_{x \in \mathcal{X}(\mathbb{F}_q)/\sim} \frac{1}{|\mathrm{Aut}(x)|},$$

where  $\mathcal{X}(\mathbb{F}_q)/\sim$  is the set of  $\mathbb{F}_q$ -isomorphism classes of  $\mathbb{F}_q$ -points of  $\mathcal{X}$  (i.e., the set of non-weighted points of  $\mathcal{X}$  over  $\mathbb{F}_q$ ), and we take  $\frac{1}{|\mathrm{Aut}(x)|}$  to be 0 when  $|\mathrm{Aut}(x)| = \infty$ .

A priori, the weighted point count can be  $\infty$ , but when  $\mathcal{X}$  is of finite type, then the stratification of  $\mathcal{X}$  by schemes as in [Behrend, Proof of Lemma 3.2.2] implies that  $\mathcal{X}(\mathbb{F}_q)/\sim$  is a finite set, so that  $\#_q(\mathcal{X}) < \infty$ .

Now we are ready to prove Theorem 1.1:

*Proof of Theorem 1.1.* Choose any  $x \in \mathcal{X}(\mathbb{F}_q)/\sim$ . Then the morphism  $x : \mathbb{F}_q \rightarrow \mathcal{X}$  factors through a representable morphism  $\bar{x} : [\mathrm{Spec}(\mathbb{F}_q)/\underline{\mathrm{Aut}}_x] \rightarrow \mathcal{X}$ . Note that for any  $\mathbb{F}_q$ -scheme  $T$  and any  $y, z \in \mathcal{X}(T)$  such that  $y \sim x_T$  and  $z \sim x_T$  in  $\mathcal{X}(T)$ , then  $y, z$  factors through  $x$  and  $\underline{\mathrm{Isom}}_{\mathcal{X}}(y, z) \cong \underline{\mathrm{Isom}}_{[\mathrm{Spec}(\mathbb{F}_q)/\underline{\mathrm{Aut}}_x]}(y', z')$ , where  $y = \bar{x} \circ y'$  and  $z = \bar{x} \circ z'$ . Thus,  $[\mathrm{Spec}(\mathbb{F}_q)/\underline{\mathrm{Aut}}_x]$  is a substack of  $\mathcal{X}$  via  $\bar{x}$ .

Now consider  $\mathcal{I}(\mathcal{X})_{\bar{x}}$  defined by the following Cartesian square:

$$\begin{array}{ccc} \mathcal{I}(\mathcal{X})_{\bar{x}} & \longrightarrow & \mathcal{I}(\mathcal{X}) \\ \downarrow & & \downarrow \\ [\mathrm{Spec}(\mathbb{F}_q)/\underline{\mathrm{Aut}}_x] & \xrightarrow{\bar{x}} & \mathcal{X} \end{array}$$

This is a substack of  $\mathcal{I}(\mathcal{X})$ , and  $(y, \beta) \in (\mathcal{I}(\mathcal{X})_{\bar{x}})(\mathbb{F}_q)$  iff  $y \sim x$  in  $\mathcal{X}(\mathbb{F}_q)$ . Since  $x$  contributes 1 on the unweighted point count  $|\mathcal{X}(\mathbb{F}_q)/\sim|$ , it suffices to show that  $\#_q(\mathcal{I}(\mathcal{X})_{\bar{x}}) = 1$ .

Observe that two points  $(x, \alpha)$  and  $(x, \beta)$  in  $\mathcal{I}(\mathcal{X})_{\bar{x}}$  are equivalent iff  $\beta = \phi \circ \alpha \circ \phi^{-1}$  for some  $\phi \in \mathrm{Aut}(x)$ . This holds in general if we replace  $x$  by  $y : U \rightarrow \mathcal{X}$  that factors thru  $x$ . Thus,  $\mathcal{I}(\mathcal{X})_{\bar{x}} \cong [\underline{\mathrm{Aut}}_x/\underline{\mathrm{Aut}}_x]$ , where the group space action is the conjugation. Since the diagonal of  $\mathcal{X}$  is quasi-separated and of finite type,  $\underline{\mathrm{Aut}}_x$  is a quasi-separated group algebraic space of finite type over  $\mathbb{F}_q$  by Diagram (3); henceforth,  $\mathrm{Aut}(x) = \underline{\mathrm{Aut}}_x(\mathbb{F}_q)$  is a finite group since  $\underline{\mathrm{Aut}}_x$  admits a finite stratification by schemes of finite type by [Knutson, II.6.6]. Moreover,  $\mathrm{Aut}(x, \alpha)$  is the finite centralizer subgroup  $C_{\mathrm{Aut}(x)}(\alpha) \subset \mathrm{Aut}(x)$ , and the set  $(\mathcal{I}(\mathcal{X})_{\bar{x}})(\mathbb{F}_q)/\sim$  is exactly the set  $Cl(\mathrm{Aut}(x))$  of orbits of  $\mathrm{Aut}(x)$  under the conjugation. Then, the Orbit-Stabilizer Theorem implies that as a set,

$$\mathrm{Aut}(x) \cong \bigsqcup_{\alpha \in Cl(\mathrm{Aut}(x))} \mathrm{Aut}(x)/C_{\mathrm{Aut}(x)}(\alpha).$$

Finally, we can divide the cardinality of both sides by the finite number  $|\mathrm{Aut}(x)|$ ; then right hand side becomes  $\#_q(\mathcal{I}(\mathcal{X})_{\bar{x}})$ , proving the statement.  $\blacksquare$

The following Lemma shows that certain nice property of  $\mathcal{X}$  carries over to  $\mathcal{I}(\mathcal{X})$  as well.

**Lemma 2.2.** *If  $\mathcal{X}$  is an algebraic stack of finite type with affine finite type diagonal, then so is  $\mathcal{I}(\mathcal{X})$ .*

*Proof.* Since  $\mathcal{X}$  is of finite type with finite type diagonal,  $\mathcal{I}(\mathcal{X})$  must be of finite type as well by Diagram (2). It remains to show that  $\mathcal{I}(\mathcal{X})$  has an affine diagonal. This is equivalent to showing that for any scheme  $T$  and any pairs  $(x, \alpha), (y, \beta) \in \mathcal{I}(\mathcal{X})(T)$ , the Isom space  $\underline{\text{Isom}}_{\mathcal{I}(\mathcal{X})}((x, \alpha), (y, \beta))$  is an affine  $T$ -scheme by the following Cartesian diagram:

$$\begin{array}{ccc} \underline{\text{Isom}}_{\mathcal{I}(\mathcal{X})}((x, \alpha), (y, \beta)) & \longrightarrow & T \\ \downarrow & & \downarrow (x, \alpha) \times (y, \beta) \\ \mathcal{I}(\mathcal{X}) & \xrightarrow{\Delta} & \mathcal{I}(\mathcal{X}) \times \mathcal{I}(\mathcal{X}) \end{array}$$

To see the structure of  $\underline{\text{Isom}}_{\mathcal{I}(\mathcal{X})}((x, \alpha), (y, \beta)) \rightarrow T$ , observe that  $\underline{\text{Isom}}_{\mathcal{X}}(x, y) \rightarrow T$  and  $\underline{\text{Aut}}_x \rightarrow T$  are affine morphisms of finite type by the conditions on the diagonal of  $\mathcal{X}$ . Then  $\underline{\text{Isom}}_{\mathcal{I}(\mathcal{X})}((x, \alpha), (y, \beta))$  is the preimage under the closed subscheme  $1_x \in \underline{\text{Aut}}_x$  of a morphism between affine  $T$ -schemes:

$$\begin{array}{c} \underline{\text{Isom}}_{\mathcal{X}}(x, y) \rightarrow \underline{\text{Aut}}_x \\ \phi \mapsto \phi \circ \alpha \circ \phi^{-1} \circ \beta^{-1} \end{array}$$

Therefore,  $\underline{\text{Isom}}_{\mathcal{I}(\mathcal{X})}((x, \alpha), (y, \beta))$  is an affine  $T$ -scheme as well. ■

In practice, an algebraic stack  $\mathcal{X}$  can be characterized by its smooth cover  $U \rightarrow \mathcal{X}$  by an algebraic space  $U$  (most of the time,  $U$  is assumed to be a scheme) with the space of equivalence relations  $R$ , i.e.,  $R$  is defined via the following Cartesian diagram

$$\begin{array}{ccc} R & \xrightarrow{t} & U \\ \downarrow s & & \downarrow \\ U & \longrightarrow & \mathcal{X} \end{array}$$

where  $s(r) = x$  and  $t(r) = y$  for any directed equivalence relation  $r : x \rightarrow y$  of  $x, y \in U$  via  $r \in R$ . In this case,  $\mathcal{X} \cong [U/R]$  (technically,  $[U/R]$  is the sheafification of the presheaf  $U/R$  of groupoids over  $\text{Sch}$ ). Note that the following Cartesian square

$$\begin{array}{ccc} R & \xrightarrow{s \times t} & U \times U \\ \downarrow & & \downarrow \\ \mathcal{X} & \xrightarrow{\Delta} & \mathcal{X} \times \mathcal{X} \end{array}$$

implies that  $R$  is an algebraic space as well, since  $\Delta : \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$  is representable. Given this presentation, we obtain the following presentation of  $\mathcal{I}(\mathcal{X})$ :

**Proposition 2.3.**  $\mathcal{I}(\mathcal{X}) \cong [R_{\Delta}/(R_{\Delta} \times_s R)]$ , where  $R_{\Delta}$  is defined by the following Cartesian square:

$$\begin{array}{ccc} R_{\Delta} & \longrightarrow & R \\ \downarrow & & \downarrow \\ U & \xrightarrow{\Delta} & U \times U \end{array}$$

*Proof.* By [Stacks, Tag 06PR],  $R_{\Delta}$  (denoted  $G$  in loc.cit.) is a smooth cover of  $\mathcal{I}(\mathcal{X})$ . To see that  $R_{\Delta} \times_{\mathcal{I}(\mathcal{X})} R_{\Delta}$  is isomorphic to  $R_{\Delta} \times_s R$ , it suffices to compare their  $T$ -points for any scheme  $T$ . Recall by the Cartesian diagram above that any  $T$ -point of  $R_{\Delta}$  is characterized by a pair  $(u, r) \in U \times R$  where  $r : u \rightarrow u$ . Then, given any  $((u_1, r_1), (u_2, r_2)) \in (R_{\Delta} \times_{\mathcal{I}(\mathcal{X})} R_{\Delta})(T)$ , then there

is  $\tau : u_1 \rightarrow u_2$  in  $R(T)$  such that  $\tau \circ r_1 = r_2 \circ \tau$ . This gives an element  $((u_1, r_1), \tau) \in (R_\Delta \times_s R)(T)$ . The converse can be recovered, as  $u_2$  is the target of  $r_1$  and  $r_2 = \tau \circ r_1 \circ \tau^{-1}$ . This establishes the bijection between  $T$ -points of  $R_\Delta \times_{\mathcal{I}(\mathcal{X})} R_\Delta$  and  $R_\Delta \times_s R$ .  $\blacksquare$

Sometimes, we will denote  $R(\mathcal{X})$  (resp,  $R_\Delta(\mathcal{X})$ ) instead of  $R$  (resp,  $R_\Delta$ ) when we need to emphasize the algebraic stack  $\mathcal{X}$  in question.

Recall that a quotient stack, denoted  $[U/G]$ , corresponds to  $U$  a scheme with the action of a group scheme  $G$ . In this case,  $R = U \times G$  with  $s$  being the first projection and  $t$  being the  $G$ -action map  $t : (u, g) \mapsto g \cdot u$ . Then  $R_\Delta \subset U \times G$  consists of  $(u, g)$  with  $t(u, g) = g \cdot u = u$ .

**Corollary 2.4.** *If  $\mathcal{X} \cong [U/G]$  is a quotient stack, then  $\mathcal{I}(\mathcal{X})$  is also a quotient stack  $[R_\Delta/G]$ , where  $R \cong U \times G$  and  $G$  acts on  $R_\Delta \subset R$  by  $g \cdot (u, h) = (g \cdot u, ghg^{-1})$ .*

*Proof.* By the proof of Proposition 2.3, it suffices to show that  $R_\Delta \times_s R \cong R_\Delta \times G$  and that the action map

$$\begin{aligned} R_\Delta \times_s R &\rightarrow R_\Delta \\ ((u, r), \tau) &\mapsto (t(r), \tau \circ r \circ \tau^{-1}), \end{aligned}$$

which coincides with the second projection of  $R_\Delta \times_{\mathcal{I}(\mathcal{X})} R_\Delta$ , coincides with the conjugate  $G$ -action described above. The isomorphism  $R_\Delta \times_s R \cong R_\Delta \times G$  is given by:

$$\begin{aligned} R_\Delta \times G &\rightarrow R_\Delta \times_s R \\ ((u, g), h) &\mapsto ((u, g), (g \cdot u = u, h)) \end{aligned}$$

By the description of the action map above,  $G$  acts on  $R_\Delta$  by the conjugation.  $\blacksquare$

Now assume that a quotient stack  $\mathcal{X} \cong [U/G]$  of finite type has the affine diagonal. Then,  $R_\Delta$  is not irreducible in general; in fact, not even connected. Since the image of the second projection  $\pi_2 : U \times G \supset R_\Delta \rightarrow G$  can have many irreducible components  $G_i$ , we have the decomposition  $R_\Delta = \cup \pi_2^{-1}(G \cdot G_i)$  (where  $G$  acts on itself by conjugation). When  $\pi_2(R_\Delta)$  is disconnected, so is  $R_\Delta$ . Thus, assume furthermore that  $\mathcal{X}$  is a Deligne-Mumford (DM) stack. Since the diagonal of  $\mathcal{X}$  is affine (by previous assumption) and formally unramified (by DM), the diagonal must be finite; this implies that  $\pi_2(R_\Delta)$  lies in torsion subset of  $G$ . Instead of taking the same stratification as above, Abramovich-Graber-Vistoli in [AGV, Definition 3.1.5] stratify  $\mathcal{I}(\mathcal{X})$  by looking at orders of automorphism elements: in our language, this induces a coarser stratification of  $R_\Delta$ :

$$(4) \quad \begin{aligned} R_\Delta(\mathcal{X}) &= \bigsqcup_{r \in \mathbb{Z}_{>0}} R_{\Delta, \mu_r}(\mathcal{X}) \\ \mathcal{I}(\mathcal{X}) &= \bigsqcup_{r \in \mathbb{Z}_{>0}} \mathcal{I}_{\mu_r}(\mathcal{X}) = \bigsqcup_{r \in \mathbb{Z}_{>0}} [R_{\Delta, \mu_r}(\mathcal{X})/G] \end{aligned}$$

where  $R_{\Delta, \mu_r}(\mathcal{X})$  is the preimage under  $\pi_2$  of the subscheme of order  $r$  elements of  $G$ . However,  $R_{\Delta, \mu_r}(\mathcal{X})$  can still be disconnected with many components of different dimensions.

Instead, assume that we have chosen a nice presentation of  $\mathcal{X}$  into a quotient stack  $[U/G]$  such that the support of  $\pi_2(R_\Delta)$  consists of finitely many closed points of  $G$ . Let's use our initial decomposition of  $R_\Delta$  as above by  $G$ -orbits of connected components of  $\pi_2(R_\Delta)$ . Then for each closed point  $g$  in the image of  $\pi_2(R)$ ,  $G \cdot g = Cl(g)$  the set of conjugates class of  $g$  (i.e., the conjugacy class of  $g$ ). This induces the following stratification by taking the preimage  $R_{\Delta, \alpha}(\mathcal{X})$  under  $\pi_2$  of conjugate classes  $\alpha \in Cl(G)$ :

**Definition 2.5.** Let  $\mathcal{X}$  be a Deligne-Mumford stack of finite type with affine diagonal such that there exists a quotient stack presentation  $\mathcal{X} = [U/G]$  with a finite set  $Cl(G)$  of conjugacy classes. We define

$$(5) \quad \begin{aligned} R_{\Delta}(\mathcal{X}) &= \bigsqcup_{\alpha \in Cl(G)} R_{\Delta, \alpha}(\mathcal{X}) \\ \mathcal{I}(\mathcal{X}) &= \bigsqcup_{\alpha \in Cl(G)} \mathcal{I}_{\alpha}(\mathcal{X}) = \bigsqcup_{\alpha \in Cl(G)} [R_{\Delta, \alpha}(\mathcal{X})/G] \end{aligned}$$

and we name it the *decomposition of the inertia stack  $\mathcal{I}(\mathcal{X})$  via the conjugacy classes  $\alpha \in Cl(G)$* .

Note that,  $R_{\Delta, \alpha} = \sqcup_{g \in \alpha} R_{\Delta, g}$  where  $R_{\Delta, g} := \pi_2^{-1}(g)$ , a subscheme of  $U$  which is the fixed locus of  $g \in G$  (i.e., every point is fixed under the action of  $g$ ). Observe that  $R_{\Delta, hgh^{-1}} = h \cdot R_{\Delta, g}$ , which is itself  $R_{\Delta, g}$  (then  $h \in C_G(g)$ ) or is disjoint from  $R_{\Delta, g}$ . Therefore,  $\mathcal{I}_{\alpha}(\mathcal{X}) = [R_{\Delta, g}/C_G(g)]$  for any fixed  $g \in \alpha$ .

As a summary, the decomposition in (5) is finer than (4) when it exists, but we had to assume the finiteness of subset  $\pi_2(R_{\Delta}) \subset G$ .

### 3. HOM STACK $\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ OF RATIONAL CURVES ON A WEIGHTED PROJECTIVE STACK

In this section, we formulate the Hom stack  $\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$  over a basefield  $K$ .

First, we recall the definition of a weighted projective stack  $\mathcal{P}(\vec{\lambda})$  with the weight  $\vec{\lambda}$  over  $K$ .

**Definition 3.1.** Fix a tuple of nondecreasing positive integers  $\vec{\lambda} = (\lambda_0, \dots, \lambda_N)$ . The  $N$ -dimensional weighted projective stack  $\mathcal{P}(\vec{\lambda}) = \mathcal{P}(\lambda_0, \dots, \lambda_N)$  with the weight  $\vec{\lambda}$  is defined as a quotient stack

$$\mathcal{P}(\vec{\lambda}) := [(\mathbb{A}_{x_0, \dots, x_N}^{N+1} \setminus 0)/\mathbb{G}_m]$$

where  $\zeta \in \mathbb{G}_m$  acts by  $\zeta \cdot (x_0, \dots, x_N) = (\zeta^{\lambda_0} x_0, \dots, \zeta^{\lambda_N} x_N)$ . In this case, the degree of  $x_i$ 's are  $\lambda_i$ 's respectively. A line bundle  $\mathcal{O}_{\mathcal{P}(\vec{\lambda})}(m)$  is defined to be the line bundle associated with the sheaf of degree  $m$  homogeneous rational functions without poles on  $\mathbb{A}_{x_0, \dots, x_N}^{N+1} \setminus 0$ .

Note that  $\mathcal{P}(\vec{\lambda})$  is not an (effective) orbifold when  $\gcd(\lambda_0, \dots, \lambda_N) \neq 1$ . In this case, the finite cyclic group scheme  $\mu_{\gcd(\lambda_0, \dots, \lambda_N)}$  is the generic stabilizer of  $\mathcal{P}(\vec{\lambda})$ . When we need to emphasize the field  $K$  of definition of  $\mathcal{P}(\vec{\lambda})$ , we instead use the notation  $\mathcal{P}_K(\vec{\lambda})$ .

**Lemma 3.2.** *The  $N$ -dimensional weighted projective stack  $\mathcal{P}(\vec{\lambda}) = \mathcal{P}(\lambda_0, \dots, \lambda_N)$  over any field  $K$  is of finite type with finite type affine diagonal.*

*Proof.* Since the smooth schematic cover  $\mathbb{A}_{x_0, \dots, x_N}^{N+1} \setminus 0$  of  $\mathcal{P}(\vec{\lambda})$  is of finite type over  $K$ ,  $\mathcal{P}(\vec{\lambda})$  is of finite type over  $K$  as well. It remains to prove the properties of the diagonal of  $\mathcal{P}(\vec{\lambda})$ . Choose any  $T$ -point  $x = (x_0, \dots, x_N)$  of  $U := \mathbb{A}_K^{N+1} \setminus 0$ . The fiber over  $x$  of  $R_{\Delta} \rightarrow U$  as in Corollary 2.4 is a proper subgroup scheme of  $\mathbb{G}_m$  (over  $T$ ), which is always affine of finite type over  $T$ . Henceforth, the diagonal of  $\mathcal{P}(\vec{\lambda})$  satisfies the desired properties.  $\blacksquare$

However, when  $K = \mathbb{F}_p$  for some prime  $p$ ,  $\mathcal{P}(1, p)$  is not Deligne-Mumford, as  $\underline{\text{Aut}}_{[0:1]} \cong \mu_p$ , which is not formally unramified over  $\mathbb{F}_p$ . Nevertheless, the following proposition shows that any  $\mathcal{P}(\vec{\lambda})$  behaves well in most characteristics as a tame Deligne-Mumford stack:

**Proposition 3.3.** *The weighted projective stack  $\mathcal{P}(\vec{\lambda}) = \mathcal{P}(\lambda_0, \dots, \lambda_N)$  is a tame Deligne-Mumford stack over  $K$  if  $\text{char}(K)$  does not divide  $\lambda_i \in \mathbb{N}$  for every  $i$ .*

*Proof.* For any algebraically closed field extension  $\overline{K}$  of  $K$ , any point  $y \in \mathcal{P}(\vec{\lambda})(\overline{K})$  is represented by the coordinates  $(y_0, \dots, y_N) \in \mathbb{A}_{\overline{K}}^{N+1}$  with its stabilizer group as the subgroup of  $\mathbb{G}_m$  fixing  $(y_0, \dots, y_N)$ . Hence, any stabilizer group of such  $\overline{K}$ -points is  $\mathbb{Z}/u\mathbb{Z}$  where  $u$  divides  $\lambda_i$  for some  $i$ . Since the characteristic of  $K$  does not divide the orders of  $\mathbb{Z}/\lambda_i\mathbb{Z}$  for any  $i$ , the stabilizer group of  $y$  is  $\overline{K}$ -linearly reductive. Hence,  $\mathcal{P}(\vec{\lambda})$  is tame by [AOV, Theorem 3.2]. Note that the stabilizer groups constitute fibers of the diagonal  $\Delta : \mathcal{P}(\vec{\lambda}) \rightarrow \mathcal{P}(\vec{\lambda}) \times_K \mathcal{P}(\vec{\lambda})$ . Since  $\mathcal{P}(\vec{\lambda})$  is of finite type and  $\mathbb{Z}/u\mathbb{Z}$ 's are unramified over  $K$  whenever  $u$  does not divide  $\lambda_i$  for some  $i$ ,  $\Delta$  is unramified as well. Therefore,  $\mathcal{P}(\vec{\lambda})$  is also Deligne–Mumford by [Olsson2, Theorem 8.3.3].  $\blacksquare$

The tameness is analogous to flatness for stacks in positive/mixed characteristic as it is preserved under base change by [AOV, Corollary 3.4]. Moreover, if a stack  $\mathcal{X}$  is tame and Deligne–Mumford, then the formation of the coarse moduli space  $c : \mathcal{X} \rightarrow X$  commutes with base change as well by [AOV, Corollary 3.3].

**Example 3.4.** When the characteristic of the field  $K$  is not equal to 2 or 3, [Hassett2, Proposition 3.6] shows that one example is given by the proper Deligne–Mumford stack of stable elliptic curves  $(\overline{\mathcal{M}}_{1,1})_K \cong [(\text{Spec } K[a_4, a_6] - (0, 0))/\mathbb{G}_m] = \mathcal{P}_K(4, 6)$  by using the short Weierstrass equation  $y^2 = x^3 + a_4x + a_6x$ , where  $\zeta \cdot a_i = \zeta^i \cdot a_i$  for  $\zeta \in \mathbb{G}_m$  and  $i = 4, 6$ . Thus,  $a_i$ 's have degree  $i$ 's respectively. Note that this is no longer true if characteristic of  $K$  is 2 or 3, as the Weierstrass equations are more complicated.

In the proof of Lemma 3.2, we have shown that  $R_\Delta \rightarrow U$  is proper, implying that  $\pi_2(R_\Delta) \subset \mathbb{G}_m$  is a proper subgroup scheme, i.e., supported on finitely many closed points. Thus, we can apply the decomposition in Equation (5) to the inertia stack  $\mathcal{I}(\mathcal{P}(\vec{\lambda}))$ :

**Proposition 3.5.** *For any  $N$ -dimensional weighted projective stack  $\mathcal{P}_K(\vec{\lambda})$ , The decomposition in Equations (5) describes connected components of  $\mathcal{P}_K(\vec{\lambda})$ :*

$$\mathcal{I}(\mathcal{P}_K(\vec{\lambda})) \cong \bigsqcup_{g \in |(\mathbb{G}_m)_K|} \mathcal{P}_{\kappa(g)}(\vec{\lambda}_{I_g})$$

where  $|(\mathbb{G}_m)_K|$  is set of closed points of  $(\mathbb{G}_m)_K$ ,  $I_g$  is the largest subset of  $\{0, \dots, N\}$  such that  $\text{ord}(g)$  divides  $\gcd_{i \in I_g}(\lambda_i)$ , and  $\vec{\lambda}_{I_g}$  is the subtuple of  $\vec{\lambda}$  indexed by  $I_g \subset \{0, \dots, N\}$ .

Note that  $I_g = I_{g'}$  when  $\text{ord}(g) = \text{ord}(g')$ , as any subgroup of  $\mathbb{G}_m$  is cyclic. Also, when  $|I_g| = 0$ , then  $\mathcal{P}(\vec{\lambda}_{I_g}) = \emptyset$  vacuously.

*Proof of Proposition 3.5.* It suffices to show that  $R_{\Delta, g}$  is the subspace

$$\{(x_0, \dots, x_N, g) \in (\mathbb{A}^{N+1} \setminus 0) \times \mathbb{G}_m \mid x_i = 0 \text{ if } \text{ord}(g) \text{ does not divide } \lambda_i\}$$

as commutativity of  $\mathbb{G}_m$  implies that  $C_G(g) = \mathbb{G}_m$  for any  $g \in \mathbb{G}_m$  (here,  $g$  as a closed point of  $\mathbb{G}_m$  in above coordinates is equivalent to taking a Galois orbit of a representative of  $g$  as a  $\kappa(g)$ -point of  $\mathbb{G}_m$ ). Note that this space is a  $\kappa(g)$ -variety as its projection onto  $\mathbb{G}_m$  maps to  $g \in \mathbb{G}_m$ . For any  $g \in |(\mathbb{G}_m)_K|$ ,  $x = (x_0, \dots, x_N) \in \mathbb{A}^{N+1}$  is fixed by  $g$  iff  $g^{\lambda_i} x_i = x_i$  for all  $i$ . Whenever  $g^{\lambda_i} \neq 1$ ,  $x_i$  must be zero. Thus,  $x$  lies in the closed subscheme  $\{x_i = 0 : \forall i, g^{\lambda_i} \neq 1\}$ , which is exactly the desired subspace.  $\blacksquare$

We now generalize the Hom stack formulation to  $\mathcal{P}(\vec{\lambda})$  as follows:

**Proposition 3.6.** *Hom stack  $\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$  with weight  $\vec{\lambda} = (\lambda_0, \dots, \lambda_N)$ , which parameterize degree  $n \in \mathbb{N}$   $K$ -morphisms  $f : \mathbb{P}^1 \rightarrow \mathcal{P}(\vec{\lambda})$  with  $f^* \mathcal{O}_{\mathcal{P}(\vec{\lambda})}(1) \cong \mathcal{O}_{\mathbb{P}^1}(n)$  over a basefield  $K$  with*

$\text{char}(K)$  not dividing  $\lambda_i \in \mathbb{N}$  for every  $i$ , is a smooth separated tame Deligne–Mumford stack of finite type with  $\dim_K \left( \text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda})) \right) = |\vec{\lambda}|n + N$  where  $|\vec{\lambda}| := \sum_{i=0}^N \lambda_i$ .

*Proof.*  $\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$  is a smooth Deligne–Mumford stack by [Olsson, Theorem 1.1]. It is isomorphic to the quotient stack  $[T/\mathbb{G}_m]$ , admitting a smooth schematic cover  $T \subset \left( \bigoplus_{i=0}^N H^0(\mathcal{O}_{\mathbb{P}^1}(\lambda_i \cdot n)) \right) \setminus 0$ , parameterizing the set of tuples  $(u_0, \dots, u_N)$  of sections with no common zero (here, we interpret  $H^0(\mathcal{O}_{\mathbb{P}^1}(\lambda_i \cdot n))$  as an affine space over  $K$  of appropriate dimension, induced by its  $K$ -vector space structure). The  $\mathbb{G}_m$  action on  $T$  is given by  $\zeta \cdot (u_0, \dots, u_N) = (\zeta^{\lambda_0} u_0, \dots, \zeta^{\lambda_N} u_N)$ . Note that

$$\dim T = \sum_{i=0}^n h^0(\mathcal{O}_{\mathbb{P}^1}(\lambda_i \cdot n)) = \sum_{i=0}^n (\lambda_i + 1) = |\vec{\lambda}| + N + 1,$$

implying that  $\dim \text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda})) = |\vec{\lambda}| + N$  since  $\dim \mathbb{G}_m = 1$ .

As  $\mathbb{G}_m$  acts on  $T$  properly with positive weights  $\lambda_i \in \mathbb{N}$  for every  $i$ , the quotient stack  $[T/\mathbb{G}_m]$  is separated. It is tame as in [AOV, Theorem 3.2] since  $\text{char}(K)$  does not divide  $\lambda_i$  for every  $i$ . ■

**Remark 3.7.** In the proof of Proposition 3.6, we showed that  $\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda})) \cong [T/\mathbb{G}_m]$  where  $T$  is an open dense  $\mathbb{G}_m$ -invariant subscheme of  $\bigoplus H^0(\mathcal{O}_{\mathbb{P}^1}(\lambda_i \cdot n))$  not containing zero, where for each  $i$ ,  $\mathbb{G}_m$  acts on  $H^0(\mathcal{O}_{\mathbb{P}^1}(\lambda_i \cdot n))$  with weight  $\lambda_i$ . In fact, this remains true even when the characteristic assumption fails, as the arguments still follow. Since  $h^0(\mathcal{O}_{\mathbb{P}^1}(\lambda_i \cdot n)) = n\lambda_i + 1$ ,  $\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$  is an open substack of  $\mathcal{P}(\vec{\Lambda})$  where

$$\vec{\Lambda} := (\underbrace{\lambda_0, \dots, \lambda_0}_{n\lambda_0+1 \text{ times}}, \dots, \underbrace{\lambda_N, \dots, \lambda_N}_{n\lambda_N+1 \text{ times}}).$$

Furthermore,  $(u_0, \dots, u_N) \in \bigoplus H^0(\mathcal{O}_{\mathbb{P}^1}(\lambda_i \cdot n))$  lies in  $T$  iff  $u_i$ 's have no common zero on  $\mathbb{P}^1$ . ■

Similar to  $\mathcal{P}(\vec{\lambda})$ , the inertia stack  $\mathcal{I}(\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda})))$  also admits a clear decomposition (i.e., each summand is the Hom stack  $\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}_{I_g}))$ ) that will play a crucial role.

**Proposition 3.8.** *The inertia stack of the Hom stack  $\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$  admits the following decomposition into connected components as in Equation (5), where  $I_g$  and  $\vec{\lambda}_{I_g}$  are the same as in Proposition 3.5:*

$$\mathcal{I}(\text{Hom}_n(\mathbb{P}_K^1, \mathcal{P}_K(\vec{\lambda}))) \cong \bigsqcup_{g \in |(\mathbb{G}_m)_K|} \text{Hom}_n(\mathbb{P}_{\kappa(g)}^1, \mathcal{P}_{\kappa(g)}(\vec{\lambda}_{I_g}))$$

Note that  $\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}_{I_g})) = \emptyset$  whenever  $|I_g| \leq 1$ , as there are no maps from  $\mathbb{P}^1$  to  $\mathcal{P}(\vec{\lambda}_{I_g})$  where the pullback of  $\mathcal{O}(1)$  to  $\mathbb{P}^1$  has degree  $n$ .

*Proof of Proposition 3.8.* By Remark 3.7,  $\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$  is an open substack of  $\mathcal{P}(\vec{\Lambda})$ . Restricting the decomposition of  $\mathcal{I}(\mathcal{P}(\vec{\Lambda}))$  as in Proposition 3.5 to  $\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ ,

$$\mathcal{I}(\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))) \cong \bigsqcup_{g \in |(\mathbb{G}_m)_K|} \left[ T \cap \{ \vec{u} \in \bigoplus H^0(\mathcal{O}_{\mathbb{P}^1}(\lambda_i \cdot n)) : u_i = 0 \text{ if } g^{\lambda_i} \neq 1 \} / \mathbb{G}_m \right]_{\kappa(g)}$$

Since  $\vec{u} \in T$  iff  $u_i$ 's have no common zeroes, each summand is isomorphic to  $\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}_{I_g}))$ . ■

#### 4. MOTIVE/POINT COUNT OF Hom AND INERTIA STACKS

In this section, we use the idea of cut-and-paste by Grothendieck and acquire the motive  $\{\mathcal{X}\} \in K_0(\text{Stck}_K)$  of moduli stack  $\mathcal{X}$  in the Grothendieck ring of  $K$ -stacks; in fact, we show that  $\{\mathcal{X}\}$  is a polynomial in the Lefschetz motive  $\mathbb{L} := \{\mathbb{A}_K^1\}$ . Particularly, we acquire the motive  $\{\text{Poly}_1^{(d_1, \dots, d_m)}\}$  of the space of monic coprime polynomials through filtration which in turn provides the motive  $\{\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))\}$  of the Hom stack through stratification. In the end, we acquire the weighted point count of the inertia stack  $\left\{ \mathcal{I} \left( \text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda})) \right) \right\}$  of the Hom stack over  $\mathbb{F}_q$  through the decomposition of Proposition 3.8.

First, we recall the definition of the Grothendieck ring of algebraic stacks following [Ekedahl].

**Definition 4.1.** [Ekedahl, §1] Fix a field  $K$ . Then the *Grothendieck ring*  $K_0(\text{Stck}_K)$  of algebraic stacks of finite type over  $K$  all of whose stabilizer group schemes are affine is an abelian group generated by isomorphism classes of  $K$ -stacks  $\{\mathcal{X}\}$  of finite type, modulo relations:

- $\{\mathcal{X}\} = \{\mathcal{Z}\} + \{\mathcal{X} \setminus \mathcal{Z}\}$  for  $\mathcal{Z} \subset \mathcal{X}$  a closed substack,
- $\{\mathcal{E}\} = \{\mathcal{X} \times \mathbb{A}^n\}$  for  $\mathcal{E}$  a vector bundle of rank  $n$  on  $\mathcal{X}$ .

Multiplication on  $K_0(\text{Stck}_K)$  is induced by  $\{\mathcal{X}\}\{\mathcal{Y}\} := \{\mathcal{X} \times_K \mathcal{Y}\}$ . There is a distinguished element  $\mathbb{L} := \{\mathbb{A}^1\} \in K_0(\text{Stck}_K)$ , called the *Lefschetz motive*.

Given an algebraic  $K$ -stack  $\mathcal{X}$  of finite type with affine diagonal, the *motive* of  $\mathcal{X}$  refers to  $\{\mathcal{X}\} \in K_0(\text{Stck}_K)$ .

As the Grothendieck ring  $K_0(\text{Stck}_K)$  is the universal object for *additive invariants*, it is easy to see that when  $K = \mathbb{F}_q$ , the assignment  $\{X\} \mapsto \#_q(X)$  gives a well-defined ring homomorphism  $\#_q : K_0(\text{Stck}_{\mathbb{F}_q}) \rightarrow \mathbb{Q}$  (c.f. [Ekedahl, §2]) rendering the weighted point count of a stack  $\mathcal{X}$  over  $\mathbb{F}_q$ . Note that  $\#_q(\mathcal{X}) < \infty$  when  $\mathcal{X}$  is of finite type (see discussion right below Definition 2.1).

Since many algebraic stacks can be written locally as a quotient of a scheme by an algebraic group  $\mathbb{G}_m$ , the following lemma (a special case of [Ekedahl, §1]) is very useful:

**Lemma 4.2.** [HP, Lemma 15] *For any  $\mathbb{G}_m$ -torsor  $\mathcal{X} \rightarrow \mathcal{Y}$  of finite type algebraic stacks, we have  $\{\mathcal{Y}\} = \{\mathcal{X}\}\{\mathbb{G}_m\}^{-1}$ .*

The subsequent proofs involves the following variety of its own interest (a slight generalization of [FW, Definition 1.1]) :

**Definition 4.3.** Fix  $m \in \mathbb{Z}_{>0}$  and  $d_1, \dots, d_m \geq 0$ . Define  $\text{Poly}_1^{(d_1, \dots, d_m)}$  as the set of tuples  $(f_1, \dots, f_m)$  of monic polynomials in  $K[z]$  so that

- (1)  $\deg f_i = d_i$  for each  $i$ , and
- (2)  $f_1, \dots, f_m$  have no common roots in  $\overline{K}$ .

Since the set  $\text{Poly}_1^{(d_1, \dots, d_m)}$  is open inside the affine space (complement of the resultant hypersurface) parameterizing the tuples of monic coprime polynomials of degrees  $(d_1, \dots, d_m)$ , we can endow  $\text{Poly}_1^{(d_1, \dots, d_m)}$  with a structure of affine variety defined over  $\mathbb{Z}$ .

Generalizing the proof of [FW, Theorem 1.2] with the correction from [PS, Proposition 3.1.], we find the motive of  $\text{Poly}_1^{(d_1, \dots, d_m)}$  :

**Proposition 4.4** (Motive of the Poly space  $\text{Poly}_1^{(d_1, \dots, d_m)}$  over  $K$ ). *Fix  $0 \leq d_1 \leq d_2 \leq \dots \leq d_m$ . Then,*

$$\left\{ \text{Poly}_1^{(d_1, \dots, d_m)} \right\} = \begin{cases} \mathbb{L}^{d_1 + \dots + d_m} - \mathbb{L}^{d_1 + \dots + d_m - m + 1} & \text{if } d_1 \neq 0 \\ \mathbb{L}^{d_1 + \dots + d_m} & \text{if } d_1 = 0 \end{cases}$$

*Proof.* The proof is analogous to [FW, Theorem 1.2 (1)], with the correction from [PS, Proposition 3.1.], and is a direct generalization of [HP, Proposition 18]. Here, we recall the differences to the work in [FW, HP, PS].

**Step 1:** The space of  $(f_1, \dots, f_m)$  monic polynomials of degree  $d_1, \dots, d_m$  is instead the quotient  $\mathbb{A}^{d_1} \times \dots \times \mathbb{A}^{d_m} / (S_{d_1} \times \dots \times S_{d_m}) \cong \mathbb{A}^{d_1 + \dots + d_m}$ . We have the same filtration of  $\mathbb{A}^{\sum d_i}$  by  $R_{1,k}^{(d_1, \dots, d_m)}$ : the space of monic polynomials  $(f_1, \dots, f_m)$  of degree  $d_1, \dots, d_m$  respectively for which there exists a monic  $h \in K[z]$  with  $\deg(h) \geq k$  and monic polynomials  $g_i \in K[z]$  so that  $f_i = g_i h$  for any  $i$ . The rest of the arguments follow analogously, keeping in mind that the group action is via  $S_{d_1} \times \dots \times S_{d_m}$ .

**Step 2:** Here, we prove that  $\{R_{1,k}^{(d_1, \dots, d_m)} - R_{1,k+1}^{(d_1, \dots, d_m)}\} = \{\text{Poly}_1^{(d_1-k, \dots, d_m-k)} \times \mathbb{A}^k\}$ . Just as in [FW], the base case of  $k = 0$  follows from the definition (in fact, loc.cit. shows that the two schemes are indeed isomorphic). For  $k \geq 1$ , [PS, Proposition 3.1] proves that the map

$$\Psi : \text{Poly}_1^{(d_1-k, \dots, d_m-k)} \times \mathbb{A}^k \rightarrow R_{1,k}^{(d_1, \dots, d_m)} \setminus R_{1,k+1}^{(d_1, \dots, d_m)}$$

induces a piecewise isomorphism (where each piece is a locally closed subset, see [PS, Proposition 3.1] for more details); this immediately implies the claim by the definition of the Grothendieck Ring.

**Step 3:** By combining Step 1 and 2 as in [FW], we obtain

$$\left\{ \text{Poly}_1^{(d_1, \dots, d_m)} \right\} = \mathbb{L}^{d_1 + \dots + d_m} - \sum_{k \geq 1} \left\{ \text{Poly}_1^{(d_1-k, \dots, d_m-k)} \right\} \mathbb{L}^k$$

For the induction on the class  $\left\{ \text{Poly}_1^{(d_1, \dots, d_m)} \right\}$ , we use lexicographic induction on the pair  $(d_1, \dots, d_m)$ . For the base case, consider when  $d_1 = 0$ . Here the monic polynomial of degree 0 is nowhere vanishing, so that any tuple of polynomials of degree  $d_i$  for  $i > 1$  constitutes a member of  $\text{Poly}_1^{(0, d_2, \dots, d_m)}$ , so that  $\text{Poly}_1^{(0, d_2, \dots, d_m)} \cong \mathbb{A}^{d_2 + \dots + d_m}$ .

Now assume that  $d_1 > 0$ . Then, we obtain

$$\begin{aligned} & \left\{ \text{Poly}_1^{(d_1, \dots, d_m)} \right\} \\ &= \mathbb{L}^{d_1 + \dots + d_m} - \sum_{k \geq 1} \left\{ \text{Poly}_1^{(d_1-k, \dots, d_m-k)} \right\} \mathbb{L}^k \\ &= \mathbb{L}^{d_1 + \dots + d_m} - \left( \sum_{k=1}^{d_1-1} (\mathbb{L}^{(d_1-k) + \dots + (d_m-k)} - \mathbb{L}^{(d_1-k) + \dots + (d_m-k) - m + 1}) \mathbb{L}^k + \mathbb{L}^{(d_2-d_1) + \dots + (d_m-d_1)} \mathbb{L}^{d_1} \right) \\ &= \mathbb{L}^{d_1 + \dots + d_m} - \left( \sum_{k=1}^{d_1-1} (\mathbb{L}^{d_1 + \dots + d_m - (m-1)k} - \mathbb{L}^{d_1 + \dots + d_m - (m-1)(k+1)}) + \mathbb{L}^{d_1 + \dots + d_m - (m-1)d_1} \right) \\ &= \mathbb{L}^{d_1 + \dots + d_m} - \mathbb{L}^{d_1 + \dots + d_m - m + 1} \end{aligned}$$

■

**4.1. Motive of Hom stack.** Now we are ready to find the class in Grothendieck ring of the Hom stack  $\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ :

**Proposition 4.5.** *Fix the weight  $\vec{\lambda} = (\lambda_0, \dots, \lambda_N)$  with  $|\vec{\lambda}| := \sum_{i=0}^N \lambda_i$ . Then the motive of the Hom stack  $\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$  in the Grothendieck ring of  $K$ -stacks  $K_0(\text{Stck}_K)$  is equivalent to*



$$\left\{ \text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda})) \right\} = \left( \sum_{i=0}^N \mathbb{L}^i \right) \cdot \left( \mathbb{L}^{|\vec{\lambda}|n} - \mathbb{L}^{|\vec{\lambda}|n-N} \right)$$

$$= \mathbb{L}^{|\vec{\lambda}|n-N} \cdot (\mathbb{L}^{2N} + \dots + \mathbb{L}^{N+1} - \mathbb{L}^{N-1} - \dots - 1)$$

where  $\mathbb{L}^1 := \{\mathbb{A}_K^1\}$  is the Lefschetz motive.

*Proof.* Let  $\vec{\lambda} = (\lambda_0, \dots, \lambda_N)$  and  $\lambda_i \in \mathbb{N}$  for every  $i$  with  $|\vec{\lambda}| := \sum_{i=0}^N \lambda_i$ . Then the Hom stack

$\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda})) \cong [T/\mathbb{G}_m]$  is the quotient stack by the proof of Proposition 3.6. By Lemma 4.2, we have  $\{\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))\} = (\mathbb{L} - 1)^{-1}\{T\}$ . Henceforth, it suffices to find the motive  $\{T\}$ , and not worry about the original  $\mathbb{G}_m$ -action on  $T$ . To do so, we need to reinterpret  $T$  as follows.

Fix a chart  $\mathbb{A}^1 \hookrightarrow \mathbb{P}^1$  with  $x \mapsto [1 : x]$ , and call  $0 = [1 : 0]$  and  $\infty = [0 : 1]$ . It comes from a homogeneous chart of  $\mathbb{P}^1$  by  $[Y : X]$  with  $x := X/Y$  away from  $\infty$ . Then for any  $u \in H^0(\mathcal{O}_{\mathbb{P}^1}(d))$  with  $d \geq 0$ ,  $u$  is a homogeneous polynomial of degree  $d$  in  $X$  and  $Y$ . By substituting in  $Y = 1$ , we obtain a representation of  $u$  as a polynomial in  $x$  with degree at most  $d$ . For instance,  $\deg u < d$  as a polynomial in  $x$  if and only if  $u(X, Y)$  is divisible by  $Y$  (i.e.,  $u$  vanishes at  $\infty$ ). From now on,  $\deg u$  means the degree of  $u$  as a polynomial in  $x$ . Conventionally, set  $\deg 0 := -\infty$ .

Therefore,  $T$  parameterizes a  $N$ -tuple  $(f_0, \dots, f_N)$  of polynomials in  $K[x]$  with no common roots in  $\overline{K}$ , where  $\deg f_i \leq n\lambda_i$  for each  $i$  with equality for some  $i$ . We use this interpretation to construct  $\Phi : T \rightarrow \mathbb{A}^{N+1} \setminus 0$ ,  $\Phi(f_0, \dots, f_N) = (a_0, \dots, a_N)$ , where  $a_i$  is the coefficient of degree  $n\lambda_i$  term of  $f_i$ .

Now, we stratify  $T$  by taking preimages under  $\Phi$  of a stratification of  $\mathbb{A}^{N+1} \setminus 0$  by  $\sqcup E_J$ , where  $J$  is any proper subset of  $\{0, \dots, N\}$  and

$$E_J = \{(a_0, \dots, a_N) \mid a_j = 0 \ \forall j \in J\} \cong \mathbb{G}_m^{N+1-|J|}$$

Observe that  $E_J$  has the natural free  $\mathbb{G}_m^{N+1-|J|}$ -action, which lifts to  $\Phi^{-1}(E_J)$  via multiplication on  $\mathbb{G}_m$ -scalars on  $f_i$  for  $i \notin J$ . The action is free on  $\Phi^{-1}(E_J)$  as well, so that  $\Phi|_{\Phi^{-1}(E_J)}$  is a Zariski-locally trivial fibration with base  $E_J$ . Each fiber is isomorphic to  $F_J(n\vec{\lambda})$  defined below:

**Definition 4.6.** Fix  $m \in \mathbb{N}$  and  $\vec{d} := (d_0, \dots, d_N) \in \mathbb{Z}_{\geq 0}^{N+1}$ . Given  $J \subsetneq \{0, \dots, N\}$ ,  $F_J(\vec{d})$  is defined as a variety consisting of tuples  $(f_0, \dots, f_N)$  of  $K$ -polynomials without common roots such that

- for any  $j \notin J$ , then  $f_j$  is monic of degree  $n\lambda_j$ , and
- for any  $j \in J$ , then  $\deg f_j < n\lambda_j$  ( $f_j$  is not necessarily monic).

If instead  $J = \{0, \dots, N\}$ , then we define  $F_J(\vec{d}) := \emptyset$

This implies that  $\{\Phi^{-1}(E_J)\} = \{E_J\}\{F_J(n\vec{\lambda})\} = (\mathbb{L} - 1)^{N+1-|J|}\{F_J(n\vec{\lambda})\}$ . Since

$$(6) \quad \{T\} = \sum_{J \subsetneq \{0, \dots, N\}} \{\Phi^{-1}(E_J)\} = \sum_{J \subsetneq \{0, \dots, N\}} \{E_J\}\{F_J(n\vec{\lambda})\},$$

it suffices to find  $\{F_J(n\vec{\lambda})\}$  as a polynomial of  $\mathbb{L}$ .

**Proposition 4.7.**  $\{F_J(n\vec{\lambda})\} = \{\text{Poly}_1^{(n\lambda_0, \dots, n\lambda_N)}\} = (\mathbb{L}^{|\vec{\lambda}|n} - \mathbb{L}^{|\vec{\lambda}|n-N})$ , where  $|\vec{\lambda}| := \sum_i \lambda_i$ . In other words,  $\{F_J(n\vec{\lambda})\}$  only depends on  $n\vec{\lambda}$ .

*Proof.* Set  $d_i := n\lambda_i > 0$  for the notational convention. Upto  $S_{N+1}$ -action on  $\{0, \dots, N\}$  (forgetting that  $\lambda_0 \leq \dots \leq \lambda_N$ ), consider instead  $F_{\langle m \rangle}(\vec{d})$  with  $\langle m \rangle = \{0, \dots, m-1\}$  and  $\vec{d} = (d_0, \dots, d_N)$

with  $|\vec{d}| := \sum_{i=0}^N d_i$ . We now want to show that

$$(7) \quad \{F_{\langle m \rangle}(\vec{d})\} = \{\text{Poly}_1^{(d_0, \dots, d_N)}\} = \left(\mathbb{L}^{|\vec{d}|} - \mathbb{L}^{|\vec{d}| - N}\right).$$

In order to prove this, we first check that if we set  $d_i = 0$  for some  $i \geq m$ , then

$$\{F_{\langle m \rangle}(\vec{d})\} = \{\text{Poly}_1^{(d_0, \dots, d_N)}\} = \mathbb{L}^{|\vec{d}|}.$$

To see this, note that  $i \notin \langle m \rangle$ , so that  $f_i$  is monic of degree  $d_i = 0$  for any  $(f_0, \dots, f_N) \in F_{\langle m \rangle}(\vec{d})$ ; so  $f_i = 1$ . Therefore, the common root condition from Definition 4.6 is vacuous, so that  $\{F_{\langle m \rangle}(\vec{d})\} = \mathbb{L}^{|\vec{d}|}$  (as the space of monic polynomials of degree  $d$  is isomorphic to  $\mathbb{A}^d$  and so is the space of polynomials of degree  $< d$ ).

We prove equation (7) by lexicographical induction on the ordered pairs  $(N, m)$  such that  $N > 0$  and  $0 \leq m < N + 1$ . There are two base cases to consider:

- (1) If  $m = 0$ , then  $\langle 0 \rangle = \emptyset$ , so that  $F_{\emptyset}(\vec{d}) \cong \text{Poly}_1^{(d_0, \dots, d_N)} =: \text{Poly}_1^{\vec{d}}$  by Definition 4.3.
- (2) If  $N = 1$ , then  $m$  is 0 or 1.  $m = 0$  follows from above. Now assume  $m = 1$ . Then  $(f_0, f_1) \in F_{\langle 1 \rangle}(\vec{d})$  if and only if  $\deg f_0 < d_0$  and  $\deg f_1 = d_1 > 0$  with  $f_1$  monic. Observe that  $f_0$  cannot be 0, otherwise  $f_1$  has no roots while having positive degree, which is a contradiction. Since  $f_0$  can be written as  $a_0 g_0$  for  $g_0$  monic of degree  $\deg f_0$  and  $a_0 \in \mathbb{G}_m$ ,  $F_{\langle 1 \rangle}(\vec{d})$  decomposes into the following locally closed subsets:

$$F_{\langle 1 \rangle}(\vec{d}) = \bigsqcup_{l=0}^{d_0-1} \mathbb{G}_m \times F_{\emptyset}(l, d_1) = \mathbb{G}_m \times \bigsqcup_{l=0}^{d_0-1} \text{Poly}_1^{(l, d_1)}.$$

Therefore,

$$\begin{aligned} \{F_{\langle 1 \rangle}(\vec{d})\} &= \{\mathbb{G}_m\} \sum_{l=0}^{d_0-1} \left\{ \text{Poly}_1^{(l, d_1)} \right\} = (\mathbb{L} - 1) \left( \mathbb{L}^{d_1} + \sum_{l=1}^{d_0-1} (\mathbb{L}^{l+d_1} - \mathbb{L}^{l+d_1-1}) \right) \\ &= (\mathbb{L} - 1)(\mathbb{L}^{d_1} + \mathbb{L}^{d_0+d_1-1} - \mathbb{L}^{d_1}) = (\mathbb{L} - 1)\mathbb{L}^{d_0+d_1-1} \\ &= \mathbb{L}^{d_0+d_1} - \mathbb{L}^{d_0+d_1-1} \end{aligned}$$

In general, assume that the statement is true for any  $(N', m')$  whenever  $N' < N$  or  $N' = N$  and  $m' \leq m$ . If  $m + 1 < N + 1$ , then we want to prove the assertion for  $(N, m + 1)$ . We can take the similar decomposition as the base case (1, 1), except that we vary the degree of  $f_m$ , which is the  $(m + 1)$ -st term of  $(f_0, \dots, f_N) \in F_{\langle m+1 \rangle}(\vec{d})$ , and  $f_m$  can be 0. If  $f_m = 0$ , then  $(f_0, \dots, \widehat{f_m}, \dots, f_N)$  have no common roots, so that  $(f_0, \dots, \widehat{f_m}, \dots, f_N) \in F_{\langle m \rangle}(d_0, \dots, \widehat{d_m}, \dots, d_N)$  (and vice versa). Henceforth, as a set,

$$\begin{aligned} F_{\langle m+1 \rangle}(\vec{d}) &= F_{\langle m \rangle}(d_0, \dots, \widehat{d_m}, \dots, d_N) \bigsqcup (\mathbb{G}_m \times F_{\langle m \rangle}(d_0, \dots, 0, \dots, d_N)) \\ &\bigsqcup \left( \mathbb{G}_m \times \bigsqcup_{\ell=1}^{d_m-1} F_{\langle m \rangle}(d_0, \dots, \ell, \dots, d_N) \right). \end{aligned}$$

By induction,

$$\{F_{\langle m+1 \rangle}(\vec{d})\} = \{F_{\langle m \rangle}(d_0, \dots, \widehat{d_m}, \dots, d_N)\} + (\mathbb{L} - 1) \{F_{\langle m \rangle}(d_0, \dots, 0, \dots, d_N)\}$$

$$\begin{aligned}
& + (\mathbb{L} - 1) \sum_{\ell=0}^{d_m-1} \{F_{\langle m \rangle}(d_0, \dots, \ell, \dots, d_N)\} \\
& = \mathbb{L}^{|\vec{d}|-d_m} - \mathbb{L}^{\vec{d}-d_m-N+1} + (\mathbb{L} - 1) \cdot \mathbb{L}^{|\vec{d}|-d_m} \\
& \quad + (\mathbb{L} - 1) \sum_{\ell=1}^{d_m-1} \left( \mathbb{L}^{|\vec{d}|-d_m+\ell} - \mathbb{L}^{|\vec{d}|-d_m+\ell-N} \right) \\
& = \mathbb{L}^{|\vec{d}|-d_m} - \mathbb{L}^{|\vec{d}|-d_m-N+1} + \mathbb{L}^{|\vec{d}|-d_m+1} - \mathbb{L}^{|\vec{d}|-d_m} \\
& \quad + (\mathbb{L} - 1) \mathbb{L}(\mathbb{L}^{|\vec{d}|-d_m} - \mathbb{L}^{|\vec{d}|-d_m-N})(1 + \mathbb{L} + \dots + \mathbb{L}^{d_m-2}) \\
& = \mathbb{L}^{|\vec{d}|-d_m+1} - \mathbb{L}^{|\vec{d}|-d_m-N+1} + \mathbb{L}(\mathbb{L}^{|\vec{d}|-d_m} - \mathbb{L}^{|\vec{d}|-d_m-N})(\mathbb{L}^{d_m-1} - 1) \\
& = \mathbb{L}^{|\vec{d}|-d_m+1} - \mathbb{L}^{|\vec{d}|-d_m-N+1} + \mathbb{L}^{|\vec{d}|} - \mathbb{L}^{|\vec{d}|-d_m+1} - \mathbb{L}^{|\vec{d}|-N} + \mathbb{L}^{|\vec{d}|-d_m-N+1} \\
& = \mathbb{L}^{|\vec{d}|} - \mathbb{L}^{|\vec{d}|-N}
\end{aligned}$$

■

Combining (6) and Proposition 4.7 with  $\sum_{J \subsetneq \{0, \dots, N\}} E_J = (\mathbb{A}^{N+1} \setminus 0)$ , we finally acquire

$$\begin{aligned}
\{\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))\} &= \{\mathbb{G}_m\}^{-1}\{T\} = (\mathbb{L} - 1)^{-1} \sum_{J \subsetneq \{0, \dots, N\}} \{E_J\} \{\text{Poly}_1^{(n\vec{\lambda})}\} \\
&= (\mathbb{L} - 1)^{-1} (\mathbb{L}^{N+1} - 1) \{\text{Poly}_1^{(n\vec{\lambda})}\} = \left( \sum_{i=0}^N \mathbb{L}^i \right) \cdot (\mathbb{L}^{|\vec{\lambda}| \cdot n} - \mathbb{L}^{|\vec{\lambda}| \cdot n - N})
\end{aligned}$$

This finishes the proof of Proposition 4.5.

■

**4.2. Point count of Hom stack.** Using Proposition 4.5, we immediately obtain the weighted point count  $\#_q(\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda})))$ :

**Corollary 4.8.** *Fix the weight  $\vec{\lambda} = (\lambda_0, \dots, \lambda_N)$  with  $|\vec{\lambda}| := \sum_{i=0}^N \lambda_i$ . Then the weighted point count of the Hom stack  $\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$  over  $\mathbb{F}_q$  is*

$$\begin{aligned}
\#_q(\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))) &= \left( \sum_{i=0}^N q^i \right) \cdot (q^{|\vec{\lambda}|n} - q^{|\vec{\lambda}|n-N}) \\
&= q^{|\vec{\lambda}|n-N} \cdot (q^{2N} + \dots + q^{N+1} - q^{N-1} - \dots - 1)
\end{aligned}$$

Denote  $\delta := \gcd(\lambda_0, \dots, \lambda_N)$  and  $\omega := \max \gcd(\lambda_i, \lambda_j)$  for  $0 \leq i, j \leq N$ . Then the number  $|\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))(\mathbb{F}_q)/\sim|$  of  $\mathbb{F}_q$ -isomorphism classes of  $\mathbb{F}_q$ -points (i.e., the non-weighted point count over  $\mathbb{F}_q$ ) of  $\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$  satisfies

$$\delta \cdot \#_q(\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))) \leq \left| \text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))(\mathbb{F}_q)/\sim \right| \leq \omega \cdot \#_q(\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))) .$$

*Proof.* The first part of the Corollary follows as  $\#_q : K_0(\text{Stck}_{\mathbb{F}_q}) \rightarrow \mathbb{Q}$  is a ring homomorphism with  $\#_q(\mathbb{L}) = q$  as  $\mathbb{L} = \{\mathbb{A}_{\mathbb{F}_q}^1\}$ . For the second part, notice that for each  $\varphi \in \text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))(\mathbb{F}_q)/\sim$ , it contributes 1 towards  $|\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))(\mathbb{F}_q)/\sim|$  instead of  $\frac{1}{|\text{Aut}(\varphi)|}$  for  $\#_q(\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda})))$ .

Thus, we need to check that for any  $\varphi \in \text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))(\mathbb{F}_q)$  with  $\delta := \gcd(\lambda_0, \dots, \lambda_N)$  and  $\omega := \max \gcd(\lambda_i, \lambda_j)$  for  $0 \leq i, j \leq N$ , the automorphism group satisfies the following :

$$\delta \leq |\text{Aut}(\varphi)| \leq \omega .$$

By Proposition 3.6, we can represent  $\varphi$  as a tuple  $(f_0, \dots, f_N)$  of sections  $f_i \in H^0(\mathcal{O}_{\mathbb{P}^1_{\mathbb{F}_q}}(n\lambda_i))$ , with equivalence relation given by a  $\mathbb{G}_m$ -action. Since the automorphism group of  $\varphi$  is identified with the subgroup of  $\mathbb{G}_m$  fixing  $(f_0, \dots, f_N)$ ,  $\text{Aut}(\varphi)$  consists of  $u \in \mathbb{G}_m(\mathbb{F}_q)$  such that  $u^{\lambda_i} f_i = f_i$  for any  $i$ . Since  $f_i$ 's have no common root and the degree of the morphism  $\varphi$  is  $n \in \mathbb{N}$ , at least two of those are nonzero; call  $I$  to be the set of  $i$ 's with  $f_i \neq 0$ . Then,  $u^{\lambda_i} = 1$  for any  $i \in I$ , so that  $u$  is a  $\gcd(\lambda_i : i \in I)^{\text{th}}$  root of unity. This shows that  $\text{Aut}(\varphi)$  is a finite cyclic group of order  $\gcd(\lambda_i : i \in I)$ , proving the second part of the Corollary. ■

Above proof shows that computing automorphism groups of  $\mathbb{F}_q$ -points of  $\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$  is the key ingredient for comparing between weighted and non-weighted point counts over  $\mathbb{F}_q$ . Instead, we bypass this issue by using properties of the inertia stack  $\mathcal{I}(\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda})))$ , particularly by using Theorem 1.1 and Proposition 3.8.

**4.3. Point count of Inertia of Hom stack.** We compute the class  $\mathcal{I}(\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda})))$  to be a polynomial in  $\mathbb{L} := \{\mathbb{A}_K^1\}$  the Lefschetz motive. This ultimately renders the non-weighted point count of the moduli stack  $\mathcal{L}_{g, |\Delta_g| \cdot n}$  over  $\mathbb{F}_q$ .

**Proposition 4.9.** *Take the same notation as in Proposition 3.5. Then,*

$$\left\{ \mathcal{I}(\text{Hom}_n(\mathbb{P}_K^1, \mathcal{P}_K(\vec{\lambda}))) \right\} = \sum_{g \in |(\mathbb{G}_m)_K|} \left\{ \text{Hom}_n(\mathbb{P}_{\kappa(g)}^1, \mathcal{P}_{\kappa(g)}(\vec{\lambda}_{\kappa(g)})) \right\}$$

*Proof.* This directly follows from Proposition 3.8 and the definition of the Grothendieck Ring. ■

Above definition combined with the proof of Corollary 4.8 gives an algorithm for computing  $|\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))(\mathbb{F}_q)/\sim|$ :

**Proposition 4.10.** *Take the same notation as in Proposition 3.8 for  $K = \mathbb{F}_q$ , and let  $R$  be the set of positive integers  $r$  (including  $r = 1$ ) that divides  $q - 1$ . Then,*

$$|\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))(\mathbb{F}_q)/\sim| = \sum_{r \in R} \varphi(r) \cdot \#_q(\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}_r)))$$

where  $\varphi$  is the Euler  $\varphi$ -function and  $\vec{\lambda}_r := \vec{\lambda}_{I_{\zeta_r}}$  for some primitive  $r^{\text{th}}$  root of unity  $\zeta_r$  in  $\mathbb{G}_m$ .

*Proof.* Recall that the multiplicative group  $\mathbb{F}_q^*$  of a finite field  $\mathbb{F}_q$  is a cyclic group of order  $q - 1$ . By the primitive root condition, we see that  $\zeta_r \in \mathbb{G}_m(\mathbb{F}_q)$  iff  $r|q - 1$ . This implies that the substack  $\text{Hom}_n(\mathbb{P}_{\kappa(\zeta_r)}^1, \mathcal{P}_{\kappa(\zeta_r)}(\vec{\lambda}_r))$  contribute  $\mathbb{F}_q$ -rational points iff  $r$  divides  $q - 1$ , hence the definition of the set  $R$ . As  $\vec{\lambda}_r$  is independent of the choice of a primitive  $r^{\text{th}}$  root of unity and there are  $\varphi(r)$  number of them, simplifying  $\#_q \left\{ \mathcal{I}(\text{Hom}_n(\mathbb{P}_K^1, \mathcal{P}_K(\vec{\lambda}))) \right\}$  gives the desired formula. ■

**Remark 4.11.** Note that writing a closed-form formula for  $|\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))(\mathbb{F}_q)/\sim|$  is difficult in general, as Euler  $\varphi$ -function is used, the sum is over all possible positive factors of  $q - 1$ , and the length of  $\vec{\lambda}_r$  can vary. Nevertheless, it is possible to obtain a closed-form formula by hand for special cases with mild assumptions on  $q$  (Theorem 1.3 is a good example).

## 5. MODULI STACK $\mathcal{L}_{g,|\Delta_g| \cdot n}$ OF QUASI-ADMISSIBLE HYPERELLIPTIC GENUS $g$ FIBRATIONS OVER $\mathbb{P}^1$

In this section, we first define a rational fibration with a marked section, which allows us to define a hyperelliptic genus  $g$  fibration with a marked Weierstrass section as a double cover fibration. Subsequently, we focus on a quasi-admissible hyperelliptic genus  $g$  fibration over  $\mathbb{P}^1$  with a marked Weierstrass section which extends a family of odd degree hyperelliptic genus  $g$  curves over  $\mathbb{F}_q(t)$  with a marked Weierstrass point. For detailed references on hyperelliptic fibrations or fibered algebraic surfaces (over an algebraically closed field), we refer the reader to [Liu, Liedtke].

Recall that a hyperelliptic curve  $C$  is a separable morphism  $\phi : C \rightarrow \mathbb{P}^1$  of degree 2. In order to extend the notion of hyperelliptic curve  $C$  into family, we first generalize the notion of rational curve  $\mathbb{P}^1$  into family.

**Definition 5.1.** A *rational fibration with a marked section* is given by a flat proper morphism  $h : H \rightarrow \mathbb{P}^1$  of pure relative dimension 1 with a marked section  $s' : \mathbb{P}^1 \rightarrow H$  such that

- (1) any geometric fiber  $h^{-1}(c)$  is a connected rational curve (so that arithmetic genus is 0),
- (2)  $s'(\mathbb{P}^1)$  is away from the non-reduced locus of any geometric fiber, and
- (3)  $s'(\mathbb{P}^1)$  is away from the singular locus of  $H$ .

If the geometric generic fiber of  $h$  is a smooth rational curve, then we call  $(H, h, s')$  a  $\mathbb{P}^1$ -fibration.

We will occasionally call  $(H, h, s')$  a *rational fibration* when there is no ambiguity on the marked section  $s'$ . Note that we allow a rational fibration  $H$  to be reducible (when generic fiber is a nodal chain), and the total space of a  $\mathbb{P}^1$ -fibration can be singular. Certain double cover of the rational fibration gives us the hyperelliptic genus  $g$  fibration with a marked Weierstrass section.

**Definition 5.2.** A *hyperelliptic genus  $g$  fibration with a marked Weierstrass section* consists of a tuple  $(X, H, h, f, s, s')$  of a rational fibration  $h : H \rightarrow \mathbb{P}^1$ , a flat proper morphism  $f : X \rightarrow H$  of degree 2 with  $X$  connected and reduced, and sections  $s : \mathbb{P}^1 \rightarrow X$  and  $s' : \mathbb{P}^1 \rightarrow H$  such that

- (1) Each geometric fiber  $(h \circ f)^{-1}(c)$  is a connected 1-dimensional scheme of arithmetic genus  $g$ ,
- (2)  $s(\mathbb{P}^1)$  is contained in the smooth locus of  $h \circ f$  and is away from the non-reduced locus of any geometric fiber,
- (3)  $s' = f \circ s$  and  $s(\mathbb{P}^1)$  is a connected component of the ramification locus of  $f$  (i.e.,  $s'(\mathbb{P}^1)$  is a connected component of the branch locus of  $f$ ), and
- (4) if  $p$  is a node of a geometric fiber  $h^{-1}(c)$ , then any  $q \in f^{-1}(p)$  is a node of the fiber  $(h \circ f)^{-1}(c)$ ,
- (5) if the branch divisor of  $f$  contains a node  $e$  of a fiber  $h^{-1}(t)$  with  $t$  a closed geometric point of  $\mathbb{P}^1$ , then the branch divisor contains either an irreducible component of  $h^{-1}(t)$  containing  $e$  or an irreducible component of the singular locus of  $H$  containing  $e$ .

The *underlying genus  $g$  fibration* is a tuple  $(\pi := h \circ f, s)$  with  $\pi : X \rightarrow \mathbb{P}^1$  a flat proper morphism with geometric fibers of arithmetic genus  $g$  with a marked Weierstrass point given by  $s$ .

**Note 5.3.** An isomorphism between hyperelliptic genus  $g$  fibrations  $(X_1, H_1, h_1, f_1, s_1, s'_1)$  and  $(X_2, H_2, h_2, f_2, s_2, s'_2)$  is given by a pair of isomorphisms  $\alpha : X_1 \rightarrow X_2$  and  $\beta : H_1 \rightarrow H_2$  such that

- (1)  $h_2 \circ \beta = h_1$  and  $f_2 \circ \alpha = \beta \circ f_1$  ( $\mathbb{P}^1$ -isomorphism criteria), and
- (2)  $\beta \circ s = s'$  (compatibility with sections).

From now on, we only consider non-isotrivial hyperelliptic fibrations, i.e., the underlying genus  $g$  fibrations must be non-isotrivial. Thus, non-isotriviality will be assumed on every statement and discussions below.

Recall that a fibration with a section is said to be *stable* if all of its fibers are stable pointed curves. This leads to the following definition in the hyperelliptic case:

**Definition 5.4.** A *stable hyperelliptic genus  $g$  fibration with a marked Weierstrass section* is a hyperelliptic genus  $g$  fibration  $(X, H, h, f, s, s')$  with  $K_X + s(\mathbb{P}^1)$  is  $\pi$ -ample. We assume that  $X$  is not isotrivial, i.e., the trivial hyperelliptic fiber bundle over  $\mathbb{P}^1$  with no singular fibers.

Moreover, if the geometric generic fiber is smooth, then  $(X, H, h, f, s, s')$  is called a *stable odd hyperelliptic genus  $g$  model over  $\mathbb{P}^1$* .

Conditions in the above definition implies that  $(X, s(\mathbb{P}^1))/\mathbb{P}^1$  is log canonical. In classical language, this means that there are no smooth rational curves of self-intersection  $-1$  and  $-2$  in a fiber without meeting  $s(\mathbb{P}^1)$ .

**Example 5.5.** Suppose that  $(X, H, h, f, s, s')$  is a stable odd hyperelliptic genus  $g$  model with a marked Weierstrass section. Then, it is possible that  $f : X \rightarrow H$  in a étale local neighborhood of  $p \in H$  is the map  $\mathbb{A}_{x,y}^2 \rightarrow \mathbb{A}_{x,y}^2/\mu_2$ , where  $\mu_2$  acts on  $\mathbb{A}_{x,y}^2$  by  $(x, y) \mapsto (-x, -y)$ . In this case,  $\pi$  can be given by  $\mathbb{A}_{x,y}^2 \rightarrow \mathbb{A}_z^1$  by  $z = xy$ . Note that  $H$  admits an  $A_1$ -singularity at  $p$ ,  $f^{-1}(p)$  is a node of a fiber of  $\pi$ , but  $X$  is nonsingular. In general,  $X$  and  $H$  admit at worst  $A_l$ -singularities for some  $l$  (because geometric fibers of  $X$  are nodal curves), where  $A_u$ -singularities of surfaces are étale locally given by  $w^2 + x^2 + y^{u+1} = 0$ . This follows from the fact that 1-parameter deformation of nodes create such singularities. Note that on the neighborhood of such an isolated singular point of  $H$ , the branch locus of  $f$  is concentrated at the point if it contains the point, which only appears possibly at singular points of the fibers of  $h : H \rightarrow \mathbb{P}^1$ .

**Example 5.6.** Suppose that  $(X, H, h, f, s, s')$  is a stable odd hyperelliptic genus  $g$  model with a marked Weierstrass section over a field  $K$ . The goal is to classify singularities of the branch divisor of  $f$ . By the definition, the branch divisor decomposes into  $B \sqcup s'(\mathbb{P}_K^1)$ , which is contained in the smooth locus of  $H$  by the definition. First, consider a geometric point  $c$  in the intersection  $B \cap H_t$ , where  $t$  is a geometric point of  $\mathbb{P}_K^1$  and  $H_t$  is the fiber  $h^{-1}(t)$ . Since the corresponding double cover  $X_t$  (which is a fiber over  $t$  of  $h \circ f$ ) only admits nodes as singularities, the multiplicity  $m$  of  $B \cap H_t$  at  $c$  is at most 2, as  $f_t : X_t \rightarrow Y_t$  étale locally near  $c$  is given by the equation

$$\text{Spec}(\overline{K}[y, z]/(z^2 - y^m)) \rightarrow \text{Spec}(\overline{K}[y]), \text{ where } y \text{ is the uniformizer of } c \in H_t.$$

Since  $B$  does not contain any irreducible component of geometric fibers of  $h$  (as any geometric fiber of  $h \circ f$  is reduced), above implies that the multiplicity of  $B$  at any geometric point is at most 2. Thus, the support of  $B$  possibly admits plane double point curve singularities, étale locally of the form  $y^2 - x^m = 0$  with  $m \in \mathbb{N}_{\geq 2}$ , on the geometrically reduced locus of  $B$ , and is smooth elsewhere. Those singularities are in fact  $A_{m-1}$  (curve) singularities.

Example 5.5 and 5.6 illustrate that a general stable odd hyperelliptic genus  $g$  model often gives a mildly singular  $\mathbb{P}^1$ -fibration and mildly singular branch divisor on it. On the other hand, we could instead consider the hyperelliptic fibrations with smooth  $\mathbb{P}^1$ -bundle  $H$ , but with  $X$  and the branch divisor having worse singularities. Then, each fiber of  $X$  is irreducible and is a double cover of  $\mathbb{P}^1$  branched over  $2g + 2$  number of points, where many of these points could collide. For instance, if  $l$  branch points collide, then the preimage has  $A_{l-1}$ -singularity on the fiber, given étale locally by an equation  $y^2 - x^l = 0$ . Such a curve is called the quasi-admissible hyperelliptic curve, defined in Definition 1.2. Quasi-admissible hyperelliptic curves over  $\mathbb{P}_K^1$  (which are non-isotrivial) are equivalent to the following:

**Definition 5.7.** A hyperelliptic fibration  $(X, H, h, f, s, s')$  is *quasi-admissible* if for every geometric point  $c \in C$ ,  $f$  restricted to the fibers of  $X$  and  $H$  is quasi-admissible. We assume that  $X$  is not isotrivial over  $\mathbb{P}^1$ , i.e., all geometric fibers are isomorphic.

**Remark 5.8.** Observe that the Definitions 5.1, 5.2, 5.4, and 5.7 should be interpreted as rational / hyperelliptic / stable / quasi-admissible curves over  $\mathbb{P}_K^1$ , instead of a point  $\text{Spec } K$  (just as in

Definition 1.2). Thus, these definitions can be extended to corresponding curves over a general scheme  $T$ , assuming that any geometric point  $t$  of  $T$  has the property that the characteristic of the residue field is 0 or larger than  $2g + 1$  (when instead  $g = 1$ , the standard definition of semistable over  $T$  is more delicate whenever the characteristic of geometric point is 2 or 3, and is not analogous to the definitions proposed in this paper).

In particular, a quasi-admissible hyperelliptic fibration  $(X, H, h, f, s, s')$  has the property that  $H$  is a  $\mathbb{P}^1$ -bundle, and on each geometric fiber of  $H$ , each point of the branch divisor away from  $s'$  has the multiplicity at most  $2g$ . Moreover,  $X$  is the double cover of  $H$  branched along the branch divisor (which coincides with the branch locus).

To parameterize such fibrations, we first consider the moduli stack  $\mathcal{H}_{2g}[2g-1]$  of quasi-admissible hyperelliptic genus  $g$  curves characterized by [Fedorchuk, Proposition 4.2(1)] :

**Proposition 5.9.** *If  $p := \text{char}(K)$  is 0 or  $> 2g + 1$ , then the moduli stack  $\mathcal{H}_{2g}[2g-1]$  of quasi-admissible hyperelliptic genus  $g$  curves is a tame Deligne–Mumford stack isomorphic to  $\mathcal{P}(4, 6, 8, \dots, 4g+2)$ , where a point  $(a_4, a_6, a_8, \dots, a_{4g+2})$  of  $\mathcal{P}(4, 6, 8, \dots, 4g+2)$  corresponds to the quasi-admissible hyperelliptic genus  $g$  curve with the Weierstrass equation*

$$(8) \quad y^2 = x^{2g+1} + a_4x^{2g-1} + a_6x^{2g-2} + a_8x^{2g-3} + \dots + a_{4g+2}$$

*Proof.* Proof of [Fedorchuk, Proposition 4.2(1)] is originally done when  $p = 0$ , so it suffices to show that the proof in loc.cit. extends to the case when  $p > 2g + 1$ .

When  $p = 0$ , the proof of loc.cit. shows that the quasi-admissible hyperelliptic curves are characterized by the base  $\mathbb{P}^1$  with the branch locus of degree  $2g + 1$  on  $\mathbb{A}^1 = \mathbb{P}^1 \setminus \infty$ , of the form

$$x^{2g+1} + a_2x^{2g} + a_4x^{2g-1} + a_6x^{2g-2} + a_8x^{2g-3} + \dots + a_{4g+2} = 0$$

where  $a_2 = 0$  and not all of the rest of  $a_i$ 's vanish. When  $p > 2g + 1$ , any monic polynomial of degree  $2g + 1$  with not all roots being identical can be written in the same way (via same method) by replacing  $x$  by  $x - \frac{a_2}{(2g+1)}$  (this is allowed as  $2g + 1 < p$  is invertible). Hence, the moduli stack is indeed isomorphic to  $\mathcal{P}(4, 6, 8, \dots, 4g+2)$ , with  $a_{2i}$ 's referring to the standard coordinates of  $\mathcal{P}(4, 6, 8, \dots, 4g+2)$  of degree  $2i$ .

Since  $p > 2g + 1$ ,  $\mathcal{P}(4, 6, 8, \dots, 4g+2)$  is tame Deligne–Mumford by Proposition 3.3. ■

Assigning  $\mathcal{H}_{2g}[2g-1]$  as the target stack, we can now formulate the moduli stack  $\mathcal{L}_g$  of quasi-admissible hyperelliptic genus  $g$  fibrations with a marked Weierstrass section as the following :

**Proposition 5.10.** *Assume  $\text{char}(K) = 0$  or  $> 2g+1$ . Then, the moduli stack  $\mathcal{L}_g$  of quasi-admissible hyperelliptic genus  $g$  fibrations over  $\mathbb{P}^1$  with a marked Weierstrass section is the tame Deligne–Mumford stack  $\text{Hom}_{>0}(\mathbb{P}^1, \mathcal{H}_{2g}[2g-1])$  parameterizing the  $K$ -morphisms  $f : \mathbb{P}^1 \rightarrow \mathcal{H}_{2g}[2g-1]$  with  $\deg f^*\mathcal{O}_{\mathcal{H}_{2g}[2g-1]}(1) > 0$ .*

*Proof.* By the definition of the universal family  $p$ , any quasi-admissible hyperelliptic genus  $g$  fibration  $f : Y \rightarrow \mathbb{P}^1$  comes from a morphism  $\varphi_f : \mathbb{P}^1 \rightarrow \mathcal{H}_{2g}[2g-1]$  and vice versa. As this correspondence also works in families, the moduli stack  $\mathcal{L}_g$  is a substack of  $\text{Hom}(\mathbb{P}^1, \mathcal{H}_{2g}[2g-1])$ . As  $\mathcal{H}_{2g}[2g-1]$  is tame Deligne–Mumford by Proposition 5.9, the Hom stack  $\text{Hom}(\mathbb{P}^1, \mathcal{H}_{2g}[2g-1])$  is Deligne–Mumford by [Olsson]. Tameness follows from [AOV], as  $\mathcal{H}_{2g}[2g-1]$  itself is tame. Thus,  $\mathcal{L}_g$  is tame Deligne–Mumford as well.

Since any quasi-admissible hyperelliptic genus  $g$  fibration  $f$  is not isotrivial,  $\varphi_f$  must be a non-trivial morphism, i.e., the image of  $f$  in  $\mathcal{H}_{2g}[2g-1]$  is 1-dimensional. Since non-trivialness of a morphism is an clopen condition, the corresponding clopen locus (consisting of the union of connected components)  $\text{Hom}_{>0}(\mathbb{P}^1, \mathcal{H}_{2g}[2g-1])$  is indeed isomorphic to  $\mathcal{L}_g$ . ■

We now have the following arithmetic invariant of the moduli stack  $\mathcal{L}_{g,|\Delta_g|n}$  over  $\mathbb{F}_q$ .

**Corollary 5.11** (Motive and weighted point count of  $\mathcal{L}_{g,|\Delta_g| \cdot n}$  over  $\mathbb{F}_q$ ). *If  $\text{char}(K) = 0$  or  $> 2g+1$ , then the motive of  $\mathcal{L}_{g,|\Delta_g| \cdot n}$  in the Grothendieck ring of  $K$ -stacks  $K_0(\text{Stck}_K)$  is equivalent to*

$$\begin{aligned} [\mathcal{L}_{g,|\Delta_g| \cdot n}] &= \left( \sum_{i=0}^{2g-1} \mathbb{L}^i \right) \cdot \left( \mathbb{L}^{|\vec{\lambda}_g|^n} - \mathbb{L}^{|\vec{\lambda}_g|^n - 2g+1} \right) \\ &= \mathbb{L}^{2g(2g+3)n} \cdot (\mathbb{L}^{2g-1} + \mathbb{L}^{2g-2} + \dots + \mathbb{L}^2 + \mathbb{L}^1 - \mathbb{L}^{-1} - \mathbb{L}^{-2} - \dots - \mathbb{L}^{-2g+2} - \mathbb{L}^{-2g+1}). \end{aligned}$$

If  $K = \mathbb{F}_q$  with  $\text{char}(\mathbb{F}_q) > 2g+1$ , then

$$\#_q(\mathcal{L}_{g,|\Delta_g| \cdot n}) = q^{2g(2g+3)n} \cdot (q^{2g-1} + q^{2g-2} + \dots + q^2 + q^1 - q^{-1} - q^{-2} - \dots - q^{-2g+2} - q^{-2g+1}).$$

*Proof.* This follows from combining Proposition 5.9 and Proposition 5.10 with Corollary 4.8.  $\blacksquare$

By working out the birational geometry of surfaces over  $\text{char}(K) = 0$  as well as  $\text{char}(K) > 2g+1$ , we construct a geometric transformation from  $\mathcal{S}_g(K)$  the  $K$ -points of the moduli functor  $\mathcal{S}_g$  of the stable odd hyperelliptic genus  $g \geq 2$  models (see Definition 5.4) over  $\mathbb{P}^1$  with a marked Weierstrass point to  $\mathcal{L}_g(K)$  the  $K$ -points of the moduli functor  $\mathcal{L}_g$ . In fact, this transformation is injective:

**Theorem 5.12.** *If  $\text{char}(K) = 0$  or  $\text{char}(K) > 2g+1$ , then there is a canonical fully faithful functor of groupoids  $\mathcal{F} : \mathcal{S}_g(K) \rightarrow \mathcal{L}_g(K)$ .*

*Proof.* The key idea of proof is to construct  $\mathcal{F}$  by using relative canonical model, a particular birational transformation from the subject of relative minimal model program. We prove this in a few steps, beginning with a preliminary step. We construct and verify properties of  $\mathcal{F}$  in the other steps:

**Step 1. Log canonical singularities and log canonical models.** The main reference here is [Fujino] when  $\text{char}(K) = 0$ , and [Tanaka, §5–6] when  $\text{char}(K) \neq 0$ , noting that both references deal with algebraically closed fields instead.

First, we need the following definition for types of singularities of a pair  $(S, D)$  of a normal  $\overline{K}$ -surface  $S$  and an effective  $\mathbb{R}$ -divisor  $D$  on  $S$ :

**Definition 5.13.** ([Fujino, §2.4], [Tanaka, Definition 5.1]) A pair  $(S, D)$  is log canonical if

- (1) the log canonical divisor  $K_S + D$  is  $\mathbb{R}$ -Cartier,
- (2) for any proper birational morphism  $\pi : W \rightarrow S$  and the divisor  $D_W$  defined by

$$K_W + D_W = \pi^*(K_S + D),$$

then  $D_W \leq 1$ , i.e., when writing  $D_W = \sum_i a_i E_i$  as a sum of distinct irreducible divisors  $E_i$ ,  $a_i \leq 1$  for every  $i$ .

Moreover, if a pair  $(S, D)$  is defined over a non-algebraically closed field  $K$ , then it is called log canonical if its base-change to  $\overline{K}$  is.

For instance, if  $S$  is smooth and  $D$  is a reduced simple normal crossing divisor, then  $(S, D)$  is log canonical. Similarly, if  $w \in \mathbb{R} \cap [0, 1]$ , then  $(S, wD)$  is log canonical under the same assumptions. Note that we cannot consider  $w > 1$  under the same assumptions, as the weight on each irreducible component of  $D$  must be at most 1.

For example, consider a stable odd hyperelliptic genus  $g$  model  $(X, H, h, f, s, s')$  over  $K$ , consider the pair  $(H_{\overline{K}}, wB_{\overline{K}} + (s'(\mathbb{P}_{\overline{K}}^1))_{\overline{K}})$  defined over  $\overline{K}$  where the branch divisor of  $h$  decomposes as  $B \sqcup s'(\mathbb{P}_{\overline{K}}^1)$  and  $w \in \mathbb{R} \cap (0, 1/2]$  is a weight (since  $B$  can have components of multiplicity 2 by Example 5.6, we consider weights at most  $1/2$ ). To claim that this pair is log canonical under additional condition on  $w$ , it suffices to consider neighborhoods of singular points of  $H_{\overline{K}}$  and support of  $B_{\overline{K}}$  by the above observation.



First, recall that the isolated singularities of  $H_{\overline{K}}$  away from the support of  $wB_{\overline{K}} + (s'(\mathbb{P}_K^1))_{\overline{K}}$  is of type  $A_{l'}$  for some  $l'$  by Example 5.5. Hence, the pair is log canonical at neighborhoods of such points (in fact, those points are called canonical singular points of  $H_{\overline{K}}$ ). Also, at a singular point  $c$  of the support of  $B_{\overline{K}}$ ,  $H_{\overline{K}}$  is smooth and  $B_{\overline{K}}$  is reduced at  $c$  but  $B_{\overline{K}}$  admits  $A_l$ -singularities by Example 5.6. Therefore, the pair has log canonical singularities whenever  $w \leq \frac{1}{2} + \frac{1}{l+1}$  by [Järvilehto] (summarized in [GHM, Introduction], where the log canonical threshold is the supremum of values  $w$  that makes the pair log canonical).

To construct a log canonical model, consider a pair  $(S, D)$  as the beginning of this step with projective  $\overline{K}$ -morphism  $f : S \rightarrow C$  into a  $\overline{K}$ -variety  $C$ , and assume that  $D$  is  $\mathbb{Q}$ -divisor and  $S$  is  $\mathbb{Q}$ -factorial. If  $(S, D)$  is log canonical with  $K_S + D$  not  $f$ -antef, then [HP, Pages 1750–1751] uses key birational geometry results from [Fujino, Tanaka] to construct the  $f$ -log canonical model, defined below. In fact, analogous arguments from [HP, Proof of Proposition 11] implies that the same procedure can be applied to  $f : (S, D) \rightarrow C$  over a field  $K$ , leading to the following definition:

**Definition 5.14.** Suppose that  $(S, D)$  is a log canonical pair over a field  $K$  where  $S$  is a normal projective  $\mathbb{Q}$ -factorial surface and  $D$  is a  $\mathbb{Q}$ -divisor. Assume that  $f : S \rightarrow C$  is a projective morphism into a  $K$ -variety  $C$  with  $K_S + D$  not  $f$ -antef. If  $K$  is algebraically closed, then the  $f$ -log canonical model is a pair  $(S', D')$  with a projective morphism  $f' : S' \rightarrow C$ , where

$$S' := \underline{\text{Proj}} \bigoplus_{m \in \mathbb{N}} f_* \mathcal{O}_S(m(K_S + D))$$

and  $D' := \phi_* D$  where  $\phi : S \rightarrow S'$  is the induced birational morphism.

If  $K$  is not algebraically closed, then the  $f$ -log canonical model is the  $\text{Gal}(\overline{K}/K)$ -fixed locus of the  $f_{\overline{K}}$ -log canonical model of  $(S_{\overline{K}}, D_{\overline{K}})$ .

**Step 2. Construction of faithful  $\mathcal{F} : \mathcal{S}_g(K) \rightarrow \mathcal{L}_g(K)$ .** Fix any member of  $\mathcal{S}_g(K)$ , i.e., a stable odd hyperelliptic genus  $g$  model  $(X, H, h, f, s, s')$ . Denote  $B \sqcup s'(\mathbb{P}_K^1)$  to be the divisorial part of the branch locus of  $f : X \rightarrow H$  ( $B$  is also called branch divisor in literature). Notice that  $h$  restricted to  $B$  has degree  $2g+1$ . By Step 1 above,  $(H, \frac{1}{2g}B + s'(C))$  is log canonical. Take  $h$ -log canonical model of  $(H, \frac{1}{2g}B + s'(C))/\mathbb{P}_K^1$  to obtain a birational  $\mathbb{P}_K^1$ -morphism  $\varphi : (H, \frac{1}{2g}B + s'(\mathbb{P}_K^1)) \rightarrow (H', D')$  where  $H'$  is a rational fibration over  $K$  and  $D'$  is a  $\mathbb{R}$ -divisor of  $H'$  defined over  $K$  (c.f. Definition 5.14). Since the only canonical rational curve, defined over an algebraically closed field with  $\frac{1}{2g}$  weights on  $(2g+1)$  points and weight 1 on another point, is a smooth rational curve where the point of weight 1 is distinct from the other points (of weight  $\frac{1}{2g}$ ),  $H'$  is a  $\mathbb{P}^1$ -bundle (given by  $h' : H' \rightarrow \mathbb{P}_K^1$ ). This description shows that  $D'$  decomposes into  $\frac{1}{2g}A' + T'$  where  $A'$  is a divisor of  $H'$  and  $T'$  consists of weight 1 points on each geometric fiber of  $H'/\mathbb{P}_K^1$ . Thus,  $T'$  comes from a section  $t'$  of  $h'$ . We will show that  $H'$  is the  $\mathbb{P}^1$ -fibration associated to the desired quasi-admissible hyperelliptic genus  $g$  fibration.

To finish the construction of the quasi-admissible fibration, take Stein factorization on  $\varphi \circ f$ . This gives a finite morphism  $f' : X' \rightarrow H'$  and a morphism  $\psi : X \rightarrow X'$  with geometrically connected fibers such that  $\varphi \circ f = f' \circ \psi$ . Since  $f$  is finite of degree 2 and  $\varphi$  is birational,  $f'$  is finite of degree 2 and  $\psi$  is birational. Moreover,  $B' := A' + T'$  is the branch locus of  $f'$ . By calling  $t$  to be the unique lift of  $t'$  on  $h' \circ f'$ ,  $(X', H', h', f', t, t')$  is the desired quasi-admissible hyperelliptic fibration. Define  $\mathcal{F}(X, H, h, f, s, s') := (X', H', h', f', t, t')$ .

To see that  $\mathcal{F}$  is faithful, suppose that there are two isomorphisms

$$(\alpha_i, \beta_i) : (X_1, H_1, h_1, f_1, s_1, s'_1) \rightarrow (X_2, H_2, h_2, f_2, s_2, s'_2)$$

between stable odd hyperelliptic genus  $g$  models that induce the same isomorphism under  $\mathcal{F}$  :

$$(\alpha', \beta') : \mathcal{F}(X_1, H_1, h_1, f_1, s_1, s'_1) \rightarrow \mathcal{F}(X_2, H_2, h_2, f_2, s_2, s'_2)$$

Denote  $(X'_j, H'_j, h'_j, f'_j, t_j, t'_j) = \mathcal{F}(X_j, H_j, h_j, f_j, s_j, s'_j)$  for  $j = 1, 2$ . From the construction of  $\mathcal{F}$  shown above, induced morphisms  $X_j \rightarrow X'_j$  and  $H_j \rightarrow H'_j$  are birational for each  $j$ . Since they are separated varieties over  $K$ ,  $(\alpha_1, \beta_1)$  must be equal to  $(\alpha_2, \beta_2)$ , hence  $\mathcal{F}$  is faithful.

**Step 3. Fullness of  $\mathcal{F}$ .** Given any isomorphism  $\psi$  between  $(X'_i, H'_i, h'_i, f'_i, t_i, t'_i)$ 's in  $\mathcal{L}_g(K)$  as images of  $(X_i, H_i, h_i, f_i, s_i, s'_i) \in \mathcal{S}_g(K)$  under  $\mathcal{F}$ , notice that  $h'_i$ 's and  $h'_i \circ f'_i$ 's have smooth geometric generic fibers for  $i = 1, 2$  and  $\psi$  comes in pairs of isomorphisms  $\psi_1 : X'_1 \rightarrow X'_2$  and  $\psi_2 : H'_1 \rightarrow H'_2$  (so denote  $\psi = (\psi_1, \psi_2)$ ). Then,  $\psi$  lifts to a pair of birational maps  $\bar{\psi} = (\bar{\psi}_1, \bar{\psi}_2)$  between  $X_i$ 's and  $H_i$ 's which induce isomorphisms on geometric generic fibers and irreducible components of any geometric fiber meeting the sections  $s_i$ 's or  $s'_i$ 's. To claim that those extend to isomorphisms that induce  $\psi_i$ 's, it suffices to understand geometric properties of related moduli stacks, as we claim that  $\psi_i$ 's can be interpreted as an element of  $Isom$  spaces of such stacks.

Observe first that for each  $i = 1, 2$ ,  $H_i$  is a  $\mathbb{Z}/2\mathbb{Z}$ -quotient of  $X_i$ , and  $K_{X_i} + s_i(\mathbb{P}_K^1)$  is ample over  $\mathbb{P}_K^1$  by the definition. Since the branch divisor of  $f_i$  is  $B_i \sqcup s'_i(\mathbb{P}_K^1)$ , the log canonical divisor  $K_{H_i} + \frac{1}{2}B_i + s'_i(\mathbb{P}_K^1)$  is also ample over  $\mathbb{P}_K^1$  as  $f_i^*(K_{H_i} + \frac{1}{2}B_i + s'_i(\mathbb{P}_K^1)) = K_{X_i} + s_i(\mathbb{P}_K^1)$ . Since  $X_i$  admits nodes as the only singularities of geometric fibers,  $B_i$  on each fiber has multiplicity at most 2 at any  $\bar{K}$ -points in the support. Therefore, fibers of the pair  $(H_i, \frac{1}{2}B_i + s'_i(\mathbb{P}_K^1))$  are  $((\frac{1}{2}, 2g+1), (1, 1))$ -stable curves of genus 0 in the sense of [Hassett, §2.1.3], meaning that  $H_i$  for each  $i$  is a family of such curves over  $\mathbb{P}_K^1$ . Note that the moduli stack  $\overline{\mathcal{M}}_{0, ((\frac{1}{2}, 2g+1), (1, 1))}$  of  $((\frac{1}{2}, 2g+1), (1, 1))$ -stable curves of genus 0 is a proper (so separated) Deligne–Mumford stack (it easily follows from loc.cit. and [Hassett, Theorem 2.1]), and  $H_i$  is realized as  $\alpha_i : \mathbb{P}_K^1 \rightarrow \overline{\mathcal{M}}_{0, ((\frac{1}{2}, 2g+1), (1, 1))}$ . Since there is a nonempty open subset  $U \subset \mathbb{P}_K^1$  such that  $\psi_2$  induces an isomorphism between  $h_i^{-1}(U)$ 's,  $\bar{\psi}_2$  is an element of  $Isom_{\overline{\mathcal{M}}_{0, ((\frac{1}{2}, 2g+1), (1, 1))}}(\alpha_1, \alpha_2)(U)$ . Then, separatedness of  $\overline{\mathcal{M}}_{0, ((\frac{1}{2}, 2g+1), (1, 1))}$  implies that  $\bar{\psi}_2$  extends to an isomorphism between  $H_i$ 's.

Similar argument shows that  $\psi_1$  also extends to an isomorphism between  $X_i$ 's (as  $\overline{H}_{g, \frac{1}{2}} \subset \overline{\mathcal{M}}_{2, \frac{1}{2}}$  is a separated Deligne–Mumford stack by [Knudsen]), so it suffices to show that  $\bar{\psi}_i$ 's commute with  $f_i$ 's and induce  $\psi$ . The commutativity of  $\bar{\psi}_i$ 's follows because  $H_i$ 's are  $\mathbb{Z}/2\mathbb{Z}$ -quotients of  $X_i$ 's and any isomorphism between families of stable hyperelliptic curves with marked Weierstrass points commute with  $\mathbb{Z}/2\mathbb{Z}$ -actions. Because the birational morphisms  $X_i \rightarrow X'_i$  and  $H_i \rightarrow H'_i$  for any  $i$  contract all but irreducible components of fibers over  $\mathbb{P}_K^1$  meeting marked sections,  $\bar{\psi} := (\bar{\psi}_1, \bar{\psi}_2)$  induce  $\psi$  as well. Henceforth,  $\bar{\psi}$  maps to  $\psi$  under  $\mathcal{F}$ , proving that  $\mathcal{F}$  is full. ■

**Remark 5.15.** Due to log abundance being a conjecture for higher dimensions, which is a key ingredient of the existence of log canonical models (c.f. [HP, Remark 13]), it is unclear whether  $\mathcal{F}$  in the proof above extends to a functor from the moduli of stable odd hyperelliptic genus  $g$  models to  $\mathcal{L}_g$ . If it extends, we expect the functor to be still fully faithful, as opposed to [HP, Remark 13] for birational transformations between semistable elliptic surfaces and stable elliptic curves over  $\mathbb{P}^1$ . The key obstruction on [HP, Remark 13], assuming that the functor discussed in loc.cit. (which is an equivalence) extends, is that the essential surjectivity may not hold on the extension, whereas the functor from Theorem 5.12 is not even essentially surjective to begin with.

**5.1. Hyperelliptic discriminant  $\Delta_g$  of quasi-admissible hyperelliptic genus  $g$  fibration.** As we consider the algebraic surfaces  $X$  as fibrations in genus  $g$  curves over  $\mathbb{P}^1$ , the discriminant  $\Delta_g(X)$  is the basic invariant of  $X$ . For the quasi-admissible hyperelliptic genus  $g$  fibrations over  $\mathbb{P}^1$ , we have the work of [Lockhart, Liu2] which describes the hyperelliptic discriminant  $\Delta_g(X)$ .

$r$	$\varphi(r)$	$g = 2$	$g = 3$	$g = 4$
1, 2	1	(4, 6, 8, 10)	(4, 6, 8, 10, 12, 14)	(4, 6, 8, 10, 12, 14, 16, 18)
3	2	–	(6, 12)	(6, 12, 18)
4	2	(4, 8)	(4, 8, 12)	(4, 8, 12, 16)
6	2	–	(6, 12)	(6, 12, 18)
8	4	–	–	(8, 16)

TABLE 1. Table of all tuples  $(\vec{\lambda}_g)_r$  of length at least two for low genus  $g = 2, 3, 4$ . Entry has – when  $(\vec{\lambda}_g)_r$  has length zero or one.

**Definition 5.16.** [Lockhart, Definition 1.6, Proposition 1.10] The hyperelliptic discriminant  $\Delta_g$  of the monic odd degree Weierstrass equation  $y^2 = x^{2g+1} + a_4x^{2g-1} + a_6x^{2g-2} + a_8x^{2g-3} + \cdots + a_{4g+2}$  over a basefield  $K$  with  $\text{char}(K) \neq 2$  is

$$\Delta_g = 2^{4g} \cdot \text{Disc}(x^{2g+1} + a_4x^{2g-1} + a_6x^{2g-2} + \cdots + a_{4g+2})$$

which has  $\deg(\Delta_g) := |\Delta_g| = 4g(2g+1)$  formally when we associate each variable  $a_i$  with degree  $i$ .

Note that when  $g = 1$ , the discriminant  $\Delta_1$  of the short Weierstrass equation  $y^2 = x^3 + a_4x + a_6$  coincides with the usual discriminant  $-16(4a_4^3 - 27a_6^2)$  of an elliptic curve. We can now formulate the moduli stack  $\mathcal{L}_{g,|\Delta_g| \cdot n}$  of quasi-admissible fibration over  $\mathbb{P}^1$  with a fixed discriminant degree  $|\Delta_g| \cdot n = 4g(2g+1)n$  and a marked Weierstrass point :

**Proposition 5.17.** Assume  $\text{char}(K) = 0$  or  $> 2g+1$ . Then, the moduli stack  $\mathcal{L}_{g,|\Delta_g| \cdot n}$  of quasi-admissible hyperelliptic genus  $g$  fibrations over  $\mathbb{P}_K^1$  with a marked Weierstrass point and a hyperelliptic discriminant of degree  $|\Delta_g| \cdot n = 4g(2g+1)n$  over a basefield  $K$  is the tame Deligne–Mumford Hom stack  $\text{Hom}_n(\mathbb{P}^1, \mathcal{H}_{2g}[2g-1])$  parameterizing the  $K$ -morphisms  $f : \mathbb{P}^1 \rightarrow \mathcal{H}_{2g}[2g-1]$  with  $\mathcal{H}_{2g}[2g-1] \cong \mathcal{P}(\vec{\lambda}_g) = \mathcal{P}(4, 6, 8, \dots, 4g+2)$  such that  $f^*\mathcal{O}_{\mathcal{P}(\vec{\lambda}_g)}(1) \cong \mathcal{O}_{\mathbb{P}^1}(n)$  with  $n \in \mathbb{N}$ .

*Proof.* Since  $\deg f^*\mathcal{O}_{\mathcal{P}(\vec{\lambda}_g)}(1) = n$  is an open condition,  $\text{Hom}_n(\mathbb{P}^1, \mathcal{H}_{2g}[2g-1])$  is an open substack of  $\text{Hom}(\mathbb{P}^1, \mathcal{H}_{2g}[2g-1])$ . Now, it suffices to show that  $\deg f = n$  (i.e.,  $\deg f^*\mathcal{O}_{\mathcal{P}(\vec{\lambda}_g)}(1) = n$ ) if and only if the discriminant degree of the corresponding quasi-admissible fibration is  $4g(2g+1)n$ . Note that  $\deg f = n$  if and only if the quasi-admissible fibration is given by the Weierstrass equation

$$y^2 = x^{2g+1} + a_4x^{2g-1} + a_6x^{2g-2} + \cdots + a_{4g+2}$$

where  $a_i$ 's are sections of  $\mathcal{O}(in)$ , since  $a_i$ 's represent the coordinates of  $\mathcal{P}(4, 6, \dots, 4g+2)$ . Then by Definition 5.16, it is straightforward to check that  $\Delta_g$  has the discriminant degree  $4g(2g+1)n$ . ■

Now we are ready to count the number  $|\mathcal{L}_{g,|\Delta_g| \cdot n}(\mathbb{F}_q)/\sim|$  of  $\mathbb{F}_q$ -isomorphism classes of quasi-admissible genus  $g$  fibrations over  $\mathbb{P}_{\mathbb{F}_q}^1$ :

*Proof of Theorem 1.3.* By Proposition 4.10, for a fixed  $g$ , it suffices to understand when a connected component  $\text{Hom}_n(\mathbb{P}^1, \mathcal{P}((\vec{\lambda}_g)_r))$  (indexed by  $r$ ) of  $\mathcal{I}(\mathcal{L}_{g,|\Delta_g| \cdot n})$  is nonempty for  $\vec{\lambda}_g = (4, 6, \dots, 4g+2)$ ; this is equivalent to finding  $r$  such that the subtuple  $(\vec{\lambda}_g)_r$  has length at least two. Table 1 describes all such possible  $r$ 's for given low values of  $g = 2, 3, 4$ :

Summing the weighted point counts of Hom stacks from Proposition 4.5 into Proposition 4.10 gives the desired formula, where we use the division function  $\delta(r, q-1)$  (defined in Theorem 1.3)

to indicate that we take the contribution of  $\#_q(\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}_r)))$  whenever  $r \in R$  (i.e.,  $r$  divides  $q - 1$ ).

The same method directly applies when  $g \geq 5$ . ■

## 6. COUNTING STABLE ODD DEGREE HYPERELLIPTIC CURVES OVER GLOBAL FIELDS BY $\Delta_g$

We first recall the definition of a global field. Let  $S$  be the set of *places* (i.e., the set of equivalence classes of nontrivial absolute values) of a field  $K$ .

**Definition 6.1.** A field  $K$  is a global field if all completions  $K_v$  of  $K$  at each place  $v \in S$  is a local field, and  $K$  satisfies the product formula  $\prod_v |\alpha|_v = 1$  for all  $\alpha \in K^*$  and over all places  $v \in S$  where  $|\cdot|_v$  denotes the normalized absolute value for each place  $v \in S$ .

And we have the fundamental Artin-Whaples Theorem proven in 1945 [AW, AW2] which emphasized the close analogy between the theory of algebraic number fields and the theory of function fields of algebraic curves over finite fields. The axiomatic method used in these papers unified the two global fields from a valuation theoretic perspective by clarifying the role of the product formula.

**Theorem 6.2** (Artin-Whaples Theorem). *Every global field is a finite extension of  $\mathbb{Q}$  or  $\mathbb{F}_q(t)$ .*

Focusing upon the global fields  $\mathbb{F}_q(t)$  or  $\mathbb{Q}$ , in order to draw the analogy, we need to fix an affine chart  $\mathbb{A}_{\mathbb{F}_q}^1 \subset \mathbb{P}_{\mathbb{F}_q}^1$  and its corresponding ring of functions  $\mathbb{F}_q[t]$  interpreted as the ring of integers of the field of fractions  $\mathbb{F}_q(t)$  of  $\mathbb{P}_{\mathbb{F}_q}^1$ . This is necessary since  $\mathbb{F}_q[t]$  could come from any affine chart of  $\mathbb{P}_{\mathbb{F}_q}^1$ , whereas the ring of integers  $\mathcal{O}_K$  for the number field  $K$  is canonically determined. We denote  $\infty \in \mathbb{P}_{\mathbb{F}_q}^1$  to be the unique point not in the chosen affine chart.

Note that for a maximal ideal  $\mathfrak{p}$  in  $\mathcal{O}_K$ , the residue field  $\mathcal{O}_K/\mathfrak{p}$  is finite for both of our global fields. One could think of  $\mathfrak{p}$  as a point in  $\text{Spec } \mathcal{O}_K$  and define the *height* of a point  $\mathfrak{p}$ .

**Definition 6.3.** Define the height of a point  $\mathfrak{p}$  to be  $ht(\mathfrak{p}) := |\mathcal{O}_K/\mathfrak{p}|$  the cardinality of the residue field  $\mathcal{O}_K/\mathfrak{p}$ .

We recall the notion of *bad reduction* & *good reduction*:

**Definition 6.4.** Let  $C$  be an odd degree hyperelliptic genus  $g$  curve over  $K$  given by the odd degree Weierstrass equation

$$y^2 = x^{2g+1} + a_4 x^{2g-1} + a_6 x^{2g-2} + \cdots + a_{4g+2},$$

with  $a_{2i+2} \in \mathcal{O}_K$  for every  $1 \leq i \leq 2g$ . Then  $C$  has bad reduction at  $\mathfrak{p}$  if the fiber  $C_{\mathfrak{p}}$  over  $\mathfrak{p}$  is a singular curve of degree  $2g + 1$ . The prime  $\mathfrak{p}$  is said to be of good reduction if  $C_{\mathfrak{p}}$  is a smooth hyperelliptic genus  $g$  curve.

Consider the case when  $K = \mathbb{F}_q(t)$ , and a quasi-admissible model  $f : X \rightarrow \mathbb{P}_{\mathbb{F}_q}^1$  (a quasi-admissible fibration with smooth geometric generic fiber). For simplicity, assume that  $X$  does not have a singular fiber over  $\infty \in \mathbb{P}_{\mathbb{F}_q}^1$ . Note that the primes  $\mathfrak{p}_i$  of bad reductions of  $f$  are precisely points of the discriminant divisor  $\Delta_g(X) = \sum k_i \cdot \mathfrak{p}_i$ , as the fiber  $X_{\mathfrak{p}_i}$  is singular over  $\Delta_g(X)$ . When  $K = \mathbb{F}_q(t)$  the global function field, we have  $\Delta_g(X) \in H^0(\mathbb{P}^1, \mathcal{O}(4g(2g+1)n))$  by the proof of Proposition 5.17. We can define the height of  $\Delta_g(X)$  as follows:

**Definition 6.5.** The height  $ht(\Delta_g(X))$  of a discriminant divisor  $\Delta_g(X)$  in  $\mathbb{P}_{\mathbb{F}_q}^1$  is  $q^{\deg \Delta_g(X)}$ . As a convention, if a divisor  $\Delta_g(X)$  is given as a zero section of any line bundle, then set  $ht(\Delta_g(X)) = \infty$ .

In general, the height of a hyperelliptic discriminant  $\Delta_g(X)$  of any  $X$  (without nonsingular fiber assumption over  $\infty$ ) is defined as  $q^{|\Delta_g(X)|}$  where  $\deg(\Delta_g(X)) := |\Delta_g(X)|$  is equal to  $4g(2g+1)n$ .

As the hyperelliptic discriminant divisor  $\Delta_g(X)$  is an invariant of a quasi-admissible model  $f : X \rightarrow \mathbb{P}^1$ , we count the number of  $\mathbb{F}_q$ -isomorphism classes of quasi-admissible hyperelliptic genus  $g$  fibrations on the function field  $\mathbb{F}_q(t)$  by the bounded height of  $\Delta_g(X)$  :

$$\mathcal{Z}'_{g,\mathbb{F}_q(t)}(\mathcal{B}) := |\{\text{Quasi-admissible hyperelliptic genus } g \text{ curves over } \mathbb{P}_{\mathbb{F}_q}^1 \text{ with } 0 < ht(\Delta_g(X)) \leq \mathcal{B}\}|$$

A very important consequence of Theorem 5.12 is that the number  $\mathcal{Z}'_{g,\mathbb{F}_q(t)}(\mathcal{B})$  of quasi-admissible genus  $g \geq 2$  curves over  $\mathbb{P}_{\mathbb{F}_q}^1$  ordered by height of hyperelliptic discriminant  $ht(\Delta_g(X))$  gives an upper bound for the number  $\mathcal{Z}_{g,\mathbb{F}_q(t)}(\mathcal{B})$  of stable odd hyperelliptic genus  $g \geq 2$  models over  $\mathbb{P}_{\mathbb{F}_q}^1$  ordered by height of hyperelliptic discriminant  $ht(\Delta_g(X))$ :

$$(9) \quad \mathcal{Z}_{g,\mathbb{F}_q(t)}(\mathcal{B}) \leq \mathcal{Z}'_{g,\mathbb{F}_q(t)}(\mathcal{B})$$

We now prove the sharp estimates on  $\mathcal{Z}'_{g,\mathbb{F}_q(t)}(\mathcal{B})$  which induces the upper bounds on  $\mathcal{Z}_{g,\mathbb{F}_q(t)}(\mathcal{B})$

**Theorem 6.6** (Estimate on  $\mathcal{Z}_{g,\mathbb{F}_q(t)}(\mathcal{B})$ ). *If  $\text{char}(\mathbb{F}_q) > 2g+1$ , then the function  $\mathcal{Z}_{g,\mathbb{F}_q(t)}(\mathcal{B})$ , which counts the number of stable hyperelliptic genus  $g \geq 2$  curves  $X$  with a marked Weierstrass point over  $\mathbb{P}_{\mathbb{F}_q}^1$  ordered by  $0 < ht(\Delta_g(X)) = q^{4g(2g+1)n} \leq \mathcal{B}$ , satisfies:*

$$\mathcal{Z}_{2,\mathbb{F}_q(t)}(\mathcal{B}) \leq 2 \frac{(q^{31} + q^{30} + q^{29} - q^{27} - q^{26} - q^{25})}{(q^{28} - 1)} \cdot (\mathcal{B}_{10}^7 - 1) + 2\delta(4, q-1) \frac{(q^{13} - q^{11})}{(q^{12} - 1)} \cdot (\mathcal{B}_{10}^3 - 1)$$

$$\begin{aligned} \mathcal{Z}_{3,\mathbb{F}_q(t)}(\mathcal{B}) &\leq 2 \frac{(q^{59} + q^{58} + \dots + q^{55} - q^{53} - q^{52} - \dots - q^{49})}{(q^{54} - 1)} \cdot (\mathcal{B}_{14}^9 - 1) \\ &\quad + 2\delta(4, q-1) \frac{(q^{26} + q^{25} - q^{23} - q^{22})}{(q^{24} - 1)} \cdot (\mathcal{B}_7^2 - 1) + 4\delta(6, q-1) \frac{(q^{19} - q^{17})}{(q^{18} - 1)} \cdot (\mathcal{B}_{14}^3 - 1) \end{aligned}$$

$$\begin{aligned} \mathcal{Z}_{4,\mathbb{F}_q(t)}(\mathcal{B}) &\leq 2 \frac{(q^{95} + q^{94} + \dots + q^{89} - q^{87} - q^{52} - \dots - q^{81})}{(q^{88} - 1)} \cdot (\mathcal{B}_{18}^{11} - 1) \\ &\quad + 2\delta(4, q-1) \frac{(q^{43} + q^{42} + q^{41} - q^{39} - q^{38} - q^{37})}{(q^{40} - 1)} \cdot (\mathcal{B}_{18}^5 - 1) \\ &\quad + 4\delta(6, q-1) \frac{(q^{38} + q^{37} - q^{35} - q^{34})}{(q^{36} - 1)} \cdot (\mathcal{B}_4^1 - 1) + 4\delta(8, q-1) \frac{(q^{25} - q^{23})}{(q^{24} - 1)} \cdot (\mathcal{B}_6^1 - 1) \end{aligned}$$

*Proof of Theorem 1.4 for  $g = 2$ .* Knowing the number of  $\mathbb{F}_q$ -isomorphism classes of quasi-admissible genus 2 fibrations of discriminant degree  $40n$  over  $\mathbb{F}_q$  is

$$|\mathcal{L}_{2,40n}(\mathbb{F}_q)/\sim| = 2q^{28n} \cdot (q^3 + q^2 + q^1 - q^{-1} - q^{-2} - q^{-3}) + 2\delta(4, q-1)q^{12n} \cdot (q^1 - q^{-1})$$

by Theorem 1.3, we explicitly compute the upper bound for  $\mathcal{Z}'_{2,\mathbb{F}_q(t)}(\mathcal{B})$  as the following,

$$\mathcal{Z}'_{2,\mathbb{F}_q(t)}(\mathcal{B}) = \sum_{n=1}^{\lfloor \frac{\log_q \mathcal{B}}{40} \rfloor} |\mathcal{L}_{2,40n}(\mathbb{F}_q)/\sim|$$

$$\begin{aligned}
&= \sum_{n=1}^{\left\lfloor \frac{\log_q \mathcal{B}}{40} \right\rfloor} 2 \cdot q^{28n} \cdot (q^3 + q^2 + q^1 - q^{-1} - q^{-2} - q^{-3}) + 2\delta(4, q-1)q^{12n} \cdot (q^1 - q^{-1}) \\
&= 2 \cdot (q^3 + q^2 + q^1 - q^{-1} - q^{-2} - q^{-3}) \sum_{n=1}^{\left\lfloor \frac{\log_q \mathcal{B}}{40} \right\rfloor} q^{28n} + 2\delta(4, q-1)(q^1 - q^{-1}) \sum_{n=1}^{\left\lfloor \frac{\log_q \mathcal{B}}{40} \right\rfloor} q^{12n} \\
&\leq 2 \cdot (q^3 + q^2 + q^1 - q^{-1} - q^{-2} - q^{-3}) \left( q^{28} + \dots + q^{28 \cdot \left(\frac{\log_q \mathcal{B}}{40}\right)} \right) \\
&\quad + 2\delta(4, q-1)(q^1 - q^{-1}) \left( q^{12} + \dots + q^{12 \cdot \left(\frac{\log_q \mathcal{B}}{40}\right)} \right) \\
&= 2 \cdot (q^3 + q^2 + q^1 - q^{-1} - q^{-2} - q^{-3}) \left( \frac{q^{28} \cdot (\mathcal{B}_{10}^{\frac{7}{10}} - 1)}{(q^{28} - 1)} \right) \\
&\quad + 2\delta(4, q-1)(q^1 - q^{-1}) \left( \frac{q^{12} \cdot (\mathcal{B}_{10}^{\frac{3}{10}} - 1)}{(q^{12} - 1)} \right) \\
&= 2 \frac{(q^{31} + q^{30} + q^{29} - q^{27} - q^{26} - q^{25})}{(q^{28} - 1)} \cdot (\mathcal{B}_{10}^{\frac{7}{10}} - 1) + 2\delta(4, q-1) \frac{(q^{13} - q^{11})}{(q^{12} - 1)} \cdot (\mathcal{B}_{10}^{\frac{3}{10}} - 1)
\end{aligned}$$

On the fourth line of the equations above, inequality becomes an equality if and only if  $n := \frac{\log_q \mathcal{B}}{40} \in \mathbb{N}$ , i.e.,  $\mathcal{B} = q^{40n}$  with  $n \in \mathbb{N}$ . This implies that the acquired upper bound on  $\mathcal{Z}'_{2, \mathbb{F}_q(t)}(\mathcal{B})$  is a sharp estimate of order  $\mathcal{O}_q(\mathcal{B}_{10}^{\frac{7}{10}})$  with the lower order terms of orders  $\mathcal{O}_q(\mathcal{B}_{10}^{\frac{3}{10}})$  and  $\mathcal{O}_q(1)$  where  $\mathcal{O}_q$ -constant is an explicit rational function of  $q$ . ■

As there are non-hyperelliptic curves for higher genus  $g \geq 3$  curves,  $\mathcal{Z}'_{g \geq 3, \mathbb{F}_q(t)}(\mathcal{B})$  counts the quasi-admissible hyperelliptic genus  $g \geq 3$  curves over  $\mathbb{P}_{\mathbb{F}_q}^1$  only. We determine  $\mathcal{Z}'_{3, \mathbb{F}_q(t)}(\mathcal{B})$  explicitly thereby counting the quasi-admissible hyperelliptic genus 3 curves over  $\mathbb{P}_{\mathbb{F}_q}^1$ .

*Proof of Theorem 1.4 for  $g = 3$ .* Knowing the number of  $\mathbb{F}_q$ -isomorphism classes of quasi-admissible hyperelliptic genus 3 fibrations of discriminant degree  $84n$  over  $\mathbb{F}_q$  is  $|\mathcal{L}_{3, 84n}(\mathbb{F}_q)/\sim| = 2 \cdot q^{54n} \cdot (q^5 + \dots + q^1 - q^{-1} - \dots - q^{-5}) + 2\delta(4, q-1)q^{24n} \cdot (q^2 + q^1 - q^{-1} - q^{-2}) + 4\delta(6, q-1)q^{18n} \cdot (q^1 - q^{-1})$  by Theorem 1.3, we explicitly compute the upper bound for  $\mathcal{Z}'_{3, \mathbb{F}_q(t)}(\mathcal{B})$  similarly as genus 2 case. ■

We conclude with  $\mathcal{Z}'_{4, \mathbb{F}_q(t)}(\mathcal{B})$  counting the quasi-admissible hyperelliptic genus 4 curves over  $\mathbb{P}_{\mathbb{F}_q}^1$ .

*Proof of Theorem 1.4 for  $g = 4$ .* Knowing the number of  $\mathbb{F}_q$ -isomorphism classes of quasi-admissible hyperelliptic genus 4 fibrations of discriminant degree  $144n$  over  $\mathbb{F}_q$  is  $|\mathcal{L}_{4, 144n}(\mathbb{F}_q)/\sim| = 2 \cdot q^{88n} \cdot (q^7 + \dots + q^1 - q^{-1} - \dots - q^{-7}) + 2\delta(4, q-1)q^{40n} \cdot (q^3 + q^2 + q^1 - q^{-1} - q^{-2} - q^{-3}) + 4\delta(6, q-1)q^{36n} \cdot (q^2 + q^1 - q^{-1} - q^{-2}) + 4\delta(8, q-1)q^{24n} \cdot (q^1 - q^{-1})$  by Theorem 1.3, we explicitly compute the upper bound for  $\mathcal{Z}'_{4, \mathbb{F}_q(t)}(\mathcal{B})$  similarly as genus 2 case. ■

The computation of explicit upper bounds with precise lower order terms for the higher genus cases  $\mathcal{Z}'_{g, \mathbb{F}_q(t)}(\mathcal{B})$  can be done similarly after working out  $|\mathcal{L}_{g, |\Delta_g| \cdot n}(\mathbb{F}_q)/\sim|$  by the Theorem 1.3. While the lower order terms vary, the order of the leading term can be found by the following. Below, we prove the following closed-form upper bound for  $\mathcal{Z}_{g, \mathbb{F}_q(t)}(\mathcal{B})$ :

**Proposition 6.7** (Estimate on  $\mathcal{Z}_{g, \mathbb{F}_q(t)}(\mathcal{B})$ ). *If  $\text{char}(\mathbb{F}_q) > 2g + 1$ , then the function  $\mathcal{Z}_{g, \mathbb{F}_q(t)}(\mathcal{B})$ , which counts the number of stable hyperelliptic genus  $g \geq 2$  curves  $X$  with a marked Weierstrass*

point over  $\mathbb{P}_{\mathbb{F}_q}^1$  ordered by  $0 < ht(\Delta_g(X)) = q^{4g(2g+1)n} \leq \mathcal{B}$ , satisfies:

$$\mathcal{Z}_{g, \mathbb{F}_q(t)}(\mathcal{B}) \leq 2g \cdot \frac{q^{4g(g+1)+1} \cdot (q^{2g-1} - 1)(q^{2g} - 1)}{(q - 1)(q^{2g(2g+3)} - 1)} \cdot \left( \mathcal{B}^{\frac{2g+3}{4g+2}} - 1 \right)$$

*Proof.* Note that the automorphism group of minimum order of  $\varphi_g$  is the generic stabilizer group  $\mu_\delta = \mu_2$  of  $\mathcal{P}(\vec{\lambda}_g)$  and the automorphism group of maximum order of  $\varphi_g$  is  $\mu_\omega = \mu_{2g}$  as  $2g$  is the maximum value of GCD for all possible pairs among  $\vec{\lambda}_g = (4, 6, 8, \dots, 4g + 2)$ . By Corollary 4.8 we know that the number of  $\mathbb{F}_q$ -isomorphism classes of quasi-admissible hyperelliptic genus  $g$  fibrations of hyperelliptic discriminant degree  $|\Delta_g| \cdot n = 4g(2g + 1)n$  over  $\mathbb{F}_q$  is  $2 \cdot \#_q(\mathcal{L}_{g, |\Delta_g| \cdot n}) \leq |\mathcal{L}_{g, |\Delta_g| \cdot n}(\mathbb{F}_q) / \sim| \leq 2g \cdot \#_q(\mathcal{L}_{g, |\Delta_g| \cdot n})$ , we can explicitly compute the upper bound for  $\mathcal{Z}'_{g, \mathbb{F}_q(t)}(\mathcal{B})$  as the following,

$$\begin{aligned} \mathcal{Z}'_{g, \mathbb{F}_q(t)}(\mathcal{B}) &= \sum_{n=1}^{\left\lfloor \frac{\log_q \mathcal{B}}{4g(2g+1)} \right\rfloor} |\mathcal{L}_{g, |\Delta_g| \cdot n}(\mathbb{F}_q) / \sim| \\ &\leq \sum_{n=1}^{\left\lfloor \frac{\log_q \mathcal{B}}{4g(2g+1)} \right\rfloor} 2g \cdot q^{2g(2g+3)n} \cdot \frac{(q^{2g-1} - 1)(q - q^{-2g+1})}{q - 1} \\ &= 2g \cdot \frac{(q^{2g-1} - 1)(q - q^{-2g+1})}{q - 1} \sum_{n=1}^{\left\lfloor \frac{\log_q \mathcal{B}}{4g(2g+1)} \right\rfloor} q^{2g(2g+3)n} \\ &= 2g \cdot \frac{(q^{2g-1} - 1)(q - q^{-2g+1})}{q - 1} \left( q^{2g(2g+3)} + \dots + q^{2g(2g+3) \cdot \left( \frac{\log_q \mathcal{B}}{4g(2g+1)} \right)} \right) \\ &= 2g \cdot \frac{(q^{2g-1} - 1)(q - q^{-2g+1})}{q - 1} \cdot \left( \frac{q^{2g(2g+3)} \cdot (\mathcal{B}^{\frac{2g+3}{4g+2}} - 1)}{q^{2g(2g+3)} - 1} \right) \\ &= 2g \cdot \frac{q^{4g(g+1)+1} \cdot (q^{2g-1} - 1)(q^{2g} - 1)}{(q - 1)(q^{2g(2g+3)} - 1)} \cdot \left( \mathcal{B}^{\frac{2g+3}{4g+2}} - 1 \right) \end{aligned}$$

This implies that the sharp estimate (which can be found explicitly from  $|\mathcal{L}_{g, |\Delta_g| \cdot n}(\mathbb{F}_q) / \sim|$  by Theorem 1.3) has the leading term of order  $\mathcal{O}_q \left( \mathcal{B}^{\frac{2g+3}{4g+2}} \right)$  where  $\mathcal{O}_q$ -constant is an explicit rational function of  $q$  with the corresponding lower order terms for each genus  $g \geq 2$ .  $\blacksquare$

For higher genus  $g \geq 3$ , we count the Jacobians of hyperelliptic curves.

**Corollary 6.8** (Estimate on  $\mathcal{N}_{g, \mathbb{F}_q(t)}(\mathcal{B})$ ). *If  $\text{char}(\mathbb{F}_q) > 2g + 1$ , then the function  $\mathcal{N}_{g, \mathbb{F}_q(t)}(\mathcal{B})$ , which counts the number of principally polarized hyperelliptic Jacobians  $A = \text{Jac}(X)$  where  $X$  is a stable hyperelliptic genus  $g \geq 3$  curves  $X$  with a marked Weierstrass point over  $\mathbb{P}_{\mathbb{F}_q}^1$  ordered by  $0 < ht(\Delta_g(X)) = q^{4g(2g+1)n} \leq \mathcal{B}$ , satisfies:*

$$\mathcal{N}_{g, \mathbb{F}_q(t)}(\mathcal{B}) \leq 2g \cdot \frac{q^{4g(g+1)+1} \cdot (q^{2g-1} - 1)(q^{2g} - 1)}{(q - 1)(q^{2g(2g+3)} - 1)} \cdot \left( \mathcal{B}^{\frac{2g+3}{4g+2}} - 1 \right)$$

*Proof.* Proposition 6.7 provides an upper bound on the number of stable hyperelliptic genus  $g \geq 3$  curves with a marked Weierstrass point over  $\mathbb{P}_{\mathbb{F}_q}^1$  with  $\text{char}(\mathbb{F}_q) > 2g + 1$ . The upper bound follows from the open immersion property of the Torelli map  $\tau_g$  restricted to the hyperelliptic locus (c.f. [Landesman, Theorem 1.2]). Note that one could sharpen the upper bound through the Theorem 1.3 to acquire the precise lower order terms for each genus  $g \geq 5$ .  $\blacksquare$

Switching to the number field with  $K = \mathbb{Q}$  and  $\mathcal{O}_K = \mathbb{Z}$ , one could choose the minimal integral Weierstrass model of a stable odd degree hyperelliptic curve with the given hyperelliptic discriminant divisor  $\Delta_g$  which is already a number. This renders the following Conjecture 1.7 and 1.8, stated in the Introduction.

## 7. APPENDIX - ARITHMETIC OF THE MODULI OF ELLIPTIC CURVES WITH A PRESCRIBED LEVEL STRUCTURE OR MULTIPLE MARKED POINTS

In this appendix, we extend the sharp estimate on the number of semistable elliptic curves over  $\mathbb{P}_{\mathbb{F}_q}^1$  from [HP, Theorem 3] by using the above methods (i.e., Proposition 4.5 into Proposition 4.10).

Specifically, we explicitly estimate the sharp bound on the number of elliptic curves over global function fields  $\mathbb{F}_q(t)$  with level structures  $[\Gamma_1(n)]$  for  $2 \leq n \leq 4$  or  $[\Gamma(2)]$ . Recall that a level structure  $[\Gamma_1(n)]$  on an elliptic curve  $E$  is a choice of point  $P \in E$  of exact order  $n$  in the smooth part of  $E$  such that over every geometric point of the base scheme every irreducible component of  $E$  contains a multiple of  $P$  (see [KM, §1.4]). And a level structure  $[\Gamma(2)]$  on an elliptic curve  $E$  is a choice of isomorphism  $\phi : \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \rightarrow E(2)$  where  $E(2)$  is the scheme of 2-torsion Weierstrass points (i.e., kernel of the multiplication-by-2 map  $[2] : E \rightarrow E$ ) (see [DR, II.1.18 & IV.2.3]).

Additionally, we consider curves of arithmetic genus one over  $\mathbb{F}_q(t)$  with  $m$ -marked rational points for  $2 \leq m \leq 5$  by instead acquiring sharp estimate on the number of  $(m-1)$ -stable  $m$ -marked curves of arithmetic genus one formulated originally by the works of [Smyth, Smyth2].

To estimate the number of certain elliptic curves over global function fields  $\mathbb{F}_q(t)$  with level structures  $[\Gamma(n)]$  or  $[\Gamma_1(n)]$  as in Theorem 1.4, we need to extend the notion of (nonsingular) elliptic curves (semistable in the case of [HP]) that admits desired level structures. By the work of Deligne and Rapoport [DR] (summarized in [Niles, §2]), we consider the generalized elliptic curves over  $\mathbb{P}_K^1$  with  $[\Gamma]$ -structures (where  $\Gamma$  is  $\Gamma(n)$  or  $\Gamma_1(n)$ ) over a field  $K$  (focusing on  $K = \mathbb{F}_q$ ). Roughly, a generalized elliptic curve  $X$  over  $\mathbb{P}_K^1$  can be thought of as a flat family of semistable elliptic curves admitting a group structure, such that a finite group scheme  $\mathcal{G} \rightarrow \mathbb{P}_K^1$  (determined by  $\Gamma$ ) embeds into  $X$  and its image meets every irreducible component of every geometric fibers of  $X$ . Again, we only consider the *non-isotrivial* generalized elliptic curves. If  $X$  is as above, then  $\Delta(X)$  is the discriminant of a generalized elliptic curve and if  $K = \mathbb{F}_q$ , then  $0 < ht(\Delta(X)) := q^{\deg \Delta(X)}$ .

Now, define  $\mathcal{Z}_{1, \mathbb{F}_q(t)}^{[\Gamma]}(\mathcal{B})$  as follows:

$$\mathcal{Z}_{1, \mathbb{F}_q(t)}^{[\Gamma]}(\mathcal{B}) := |\{\text{Generalized elliptic curves over } \mathbb{P}_{\mathbb{F}_q}^1 \text{ with } [\Gamma]\text{-structures and } 0 < ht(\Delta(X)) \leq \mathcal{B}\}|$$

Then, we acquire the following descriptions of  $\mathcal{Z}_{1, \mathbb{F}_q(t)}^{[\Gamma]}(\mathcal{B})$ :

**Theorem 7.1** (Sharp estimate of  $\mathcal{Z}_{1, \mathbb{F}_q(t)}^{[\Gamma]}(\mathcal{B})$ ). *If  $\text{char}(\mathbb{F}_q) \neq 2$ , then the function  $\mathcal{Z}_{1, \mathbb{F}_q(t)}^{[\Gamma]}(\mathcal{B})$  ( $\text{char}(\mathbb{F}_q) \neq 3$  for  $\mathcal{Z}_{1, \mathbb{F}_q(t)}^{[\Gamma_1(3)]}(\mathcal{B})$ ), which counts the number of generalized elliptic curves with  $[\Gamma]$ -structures over  $\mathbb{P}_{\mathbb{F}_q}^1$  ordered by  $0 < ht(\Delta(X)) = q^{12n} \leq \mathcal{B}$ , satisfies:*

$$\mathcal{Z}_{1, \mathbb{F}_q(t)}^{[\Gamma_1(2)]}(\mathcal{B}) \leq 2 \cdot \frac{(q^7 - q^5)}{(q^6 - 1)} \cdot \mathcal{B}^{\frac{1}{2}} - 2 \cdot \frac{(q^7 - q^5)}{(q^6 - 1)}$$

$$\mathcal{Z}_{1, \mathbb{F}_q(t)}^{[\Gamma_1(3)]}(\mathcal{B}) \leq \frac{(q^5 - q^3)}{(q^4 - 1)} \cdot \mathcal{B}^{\frac{1}{3}} - \frac{(q^5 - q^3)}{(q^4 - 1)}$$



$$\mathcal{Z}_{1, \mathbb{F}_q(t)}^{[\Gamma_1(4)]}(\mathcal{B}) \leq \frac{(q^4 - q^2)}{(q^3 - 1)} \cdot \mathcal{B}^{\frac{1}{4}} - \frac{(q^4 - q^2)}{(q^3 - 1)}$$

$$\mathcal{Z}_{1, \mathbb{F}_q(t)}^{[\Gamma_1(2)]}(\mathcal{B}) \leq 2 \cdot \frac{(q^5 - q^3)}{(q^4 - 1)} \cdot \mathcal{B}^{\frac{1}{3}} - 2 \cdot \frac{(q^5 - q^3)}{(q^4 - 1)}$$

which is an equality when  $\mathcal{B} = q^{12n}$  with  $n \in \mathbb{N}$  implying that the upper bound is a sharp estimate.

*Proof.* The proof is at the end of §7.1. ■

Through the global fields analogy, we formulate a heuristic for the analogous asymptotic counts over  $\mathbb{Q}$  by passing the above sharp estimates where the bounded height  $ht(\Delta)$  of the discriminant on the number field side is the cardinality of the ring of functions on  $\text{Spec}(\mathbb{Z}/(\Delta))$ .

**Conjecture 7.2** (Heuristic on  $\mathcal{Z}_{1, \mathbb{Q}}^{[\Gamma]}(\mathcal{B})$ ). The function  $\mathcal{Z}_{1, \mathbb{Q}}^{[\Gamma]}(\mathcal{B})$ , which counts the number of generalized elliptic curves with  $[\Gamma]$ -structures over  $\mathbb{Z}$  ordered by  $0 < ht(\Delta) \leq \mathcal{B}$ , satisfies:

$$\mathcal{Z}_{1, \mathbb{Q}}^{[\Gamma_1(2)]}(\mathcal{B}) \leq a \cdot \mathcal{B}^{\frac{1}{2}} + b$$

$$\mathcal{Z}_{1, \mathbb{Q}}^{[\Gamma_1(3)]}(\mathcal{B}) \leq a \cdot \mathcal{B}^{\frac{1}{3}} + b$$

$$\mathcal{Z}_{1, \mathbb{Q}}^{[\Gamma_1(4)]}(\mathcal{B}) \leq a \cdot \mathcal{B}^{\frac{1}{4}} + b$$

$$\mathcal{Z}_{1, \mathbb{Q}}^{[\Gamma(2)]}(\mathcal{B}) \leq a \cdot \mathcal{B}^{\frac{1}{3}} + b$$

The leading order term of the acquired sharp estimates over  $\mathbb{F}_q(t)$  matches the analogous asymptotic counts by Harron and Snowden in [HS, Theorem 1.2] over  $\mathbb{Q}$ , but uses instead naïve height function of underlying elliptic curves (see also [Duke, Grant]). The lower order term over global function fields  $\mathbb{F}_q(t)$  being a constant is new and would be interesting to prove or disprove over a number field  $\mathbb{Q}$ . It also remains a worthwhile problem to count the remaining ten possibilities (classified by the fundamental [Mazur, Theorem 8]) of the torsion subgroups with  $|G| > 4$  over  $\mathbb{F}_q(t)$  by height of discriminant and compare with analogous counting over  $\mathbb{Q}$ .

Now, let's consider instead elliptic curves with  $m$ -marked rational points. To count the number of certain curves of arithmetic genus one over global function fields  $\mathbb{F}_q(t)$  with  $m$ -markings as in Theorem 1.4, we need to again extend the notion of (nonsingular) elliptic curves that admits desired  $m$ -markings. Here, we consider the  $(m - 1)$ -stable  $m$ -marked curves of arithmetic genus one (defined by Smyth in [Smyth, §1.1] for characteristic  $\neq 2, 3$ , extended to lower characteristic with mild conditions by [LP, Definition 1.5.3]), see Definition 7.10 for a precise definition. Note that if  $\text{char}(\mathbb{F}_q) > 3$  and  $m = 1$ , then 0-stable 1-marked curves are exactly stable elliptic curves as in [DM]. We now consider the following definition:

**Definition 7.3.** Fix an integral reduced  $K$ -scheme  $B$ , where  $K$  is a field. Then a non-isotrivial flat morphism  $\pi : X \rightarrow B$  is a  $m$ -marked  $(m - 1)$ -stable genus one fibration over  $B$  if any fiber of  $\pi$  is a  $(m - 1)$ -stable  $m$ -marked curves of arithmetic genus one.

Observe that if  $\text{char}(K) = 0$  or  $> 3$ , then a  $m$ -marked  $(m - 1)$ -stable genus one fibration  $X \rightarrow \mathbb{P}_K^1$  has a discriminant  $\Delta(X) \subset \mathbb{P}_K^1$ , and if  $K = \mathbb{F}_q$ , then  $0 < ht(\Delta(X)) := q^{\deg \Delta(X)}$ .

Now, define  $\mathcal{Z}_{1,m,\mathbb{F}_q(t)}(\mathcal{B})$  as follows:

$$\mathcal{Z}_{1,m,\mathbb{F}_q(t)}(\mathcal{B}) := |\{m\text{-marked } (m-1)\text{-stable genus one fibrations over } \mathbb{P}_{\mathbb{F}_q}^1 \text{ with } 0 < ht(\Delta(X)) \leq \mathcal{B}\}|$$

Note that when  $m = 1$ ,  $\mathcal{Z}_{1,m,\mathbb{F}_q(t)}(\mathcal{B})$  counts stable elliptic fibrations, which is described in [HP, Theorem 3] (by identifying stable elliptic fibrations with nonsingular semistable elliptic surfaces, see [HP, Proposition 11]). When  $2 \leq m \leq 5$ , we acquire the following sharp estimate of  $\mathcal{Z}_{1,m,\mathbb{F}_q(t)}(\mathcal{B})$ :

**Theorem 7.4** (Sharp estimate of  $\mathcal{Z}_{1,m,\mathbb{F}_q(t)}(\mathcal{B})$ ). *If  $\text{char}(\mathbb{F}_q) \neq 2, 3$ , then the function  $\mathcal{Z}_{1,m,\mathbb{F}_q(t)}(\mathcal{B})$ , which counts the number of  $m$ -marked  $(m-1)$ -stable genus one fibration over  $\mathbb{P}_{\mathbb{F}_q}^1$  for  $2 \leq m \leq 5$  ordered by  $0 < ht(\Delta(X)) = q^{12n} \leq \mathcal{B}$ , satisfies:*

$$\mathcal{Z}_{1,2,\mathbb{F}_q(t)}(\mathcal{B}) \leq \frac{(q^{11} + q^{10} - q^8 - q^7)}{(q^9 - 1)} \cdot (\mathcal{B}^{\frac{3}{4}} - 1) + \frac{(q^7 - q^5)}{(q^6 - 1)} \cdot (\mathcal{B}^{\frac{1}{2}} - 1)$$

$$\mathcal{Z}_{1,3,\mathbb{F}_q(t)}(\mathcal{B}) \leq \frac{(q^{11} + q^{10} + q^9 - q^7 - q^6 - q^5)}{(q^8 - 1)} \cdot (\mathcal{B}^{\frac{2}{3}} - 1) + \frac{(q^5 - q^3)}{(q^4 - 1)} \cdot (\mathcal{B}^{\frac{1}{3}} - 1)$$

$$\mathcal{Z}_{1,4,\mathbb{F}_q(t)}(\mathcal{B}) \leq \frac{(q^{11} + q^{10} + q^9 + q^8 - q^6 - q^5 - q^4 - q^3)}{(q^7 - 1)} \cdot (\mathcal{B}^{\frac{7}{12}} - 1) + \frac{(q^5 - q^3)}{(q^4 - 1)} \cdot (\mathcal{B}^{\frac{1}{3}} - 1)$$

$$\mathcal{Z}_{1,5,\mathbb{F}_q(t)}(\mathcal{B}) \leq \frac{(q^{11} + q^{10} + q^9 + q^8 + q^7 - q^5 - q^4 - q^3 - q^2 - q^1)}{(q^6 - 1)} \cdot (\mathcal{B}^{\frac{1}{2}} - 1)$$

which is an equality when  $\mathcal{B} = q^{12n}$  with  $n \in \mathbb{N}$  implying that the upper bound is a sharp estimate.

*Proof.* The proof is at the end of §7.2. ■

Through the global fields analogy, we formulate a heuristic for the analogous asymptotic counts over  $\mathbb{Q}$  by passing the above sharp estimates where the bounded height  $ht(\Delta)$  of the discriminant on the number field side is the cardinality of the ring of functions on  $\text{Spec}(\mathbb{Z}/(\Delta))$ .

**Conjecture 7.5** (Heuristic on  $\mathcal{Z}_{1,m,\mathbb{Q}}(\mathcal{B})$ ). The function  $\mathcal{Z}_{1,m,\mathbb{Q}}(\mathcal{B})$ , which counts the number of  $m$ -marked  $(m-1)$ -stable genus one curves for  $2 \leq m \leq 5$  over  $\mathbb{Z}$  ordered by  $0 < ht(\Delta) \leq \mathcal{B}$ , satisfies:

$$\mathcal{Z}_{1,2,\mathbb{Q}}(\mathcal{B}) \leq a \cdot \mathcal{B}^{\frac{3}{4}} + b \cdot \mathcal{B}^{\frac{1}{2}} + c$$

$$\mathcal{Z}_{1,3,\mathbb{Q}}(\mathcal{B}) \leq a \cdot \mathcal{B}^{\frac{2}{3}} + b \cdot \mathcal{B}^{\frac{1}{3}} + c$$

$$\mathcal{Z}_{1,4,\mathbb{Q}}(\mathcal{B}) \leq a \cdot \mathcal{B}^{\frac{7}{12}} + b \cdot \mathcal{B}^{\frac{1}{3}} + c$$

$$\mathcal{Z}_{1,5,\mathbb{Q}}(\mathcal{B}) \leq a \cdot \mathcal{B}^{\frac{1}{2}} + b$$

### 7.1. Arithmetic of the moduli of generalized elliptic curves over $\mathbb{P}^1$ with level structures.

The essential geometrical idea in acquiring the sharp estimate is to consider the moduli stack of rational curves on a compactified modular curve as in [HP]. The various compactified modular curves  $\overline{\mathcal{M}}_{1,1}[\Gamma]$  are isomorphic to the 1-dimensional weighted projective stacks  $\mathcal{P}(a, b)$ .

**Proposition 7.6.** *The moduli stack  $\overline{\mathcal{M}}_{1,1}[\Gamma]$  of generalized elliptic curves with  $[\Gamma]$ -structures is isomorphic to the following when over a field  $K$ :*

- (1) *if  $\text{char}(K) \neq 2$ , the tame Deligne–Mumford moduli stack of generalized elliptic curves with  $[\Gamma_1(2)]$ -structures is isomorphic to*

$$(\overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])_K \cong [(\text{Spec } K[a_2, a_4] - (0, 0))/\mathbb{G}_m] = \mathcal{P}_K(2, 4),$$

- (2) *if  $\text{char}(K) \neq 3$ , the tame Deligne–Mumford moduli stack of generalized elliptic curves with  $[\Gamma_1(3)]$ -structures is isomorphic to*

$$(\overline{\mathcal{M}}_{1,1}[\Gamma_1(3)])_K \cong [(\text{Spec } K[a_1, a_3] - (0, 0))/\mathbb{G}_m] = \mathcal{P}_K(1, 3),$$

- (3) *if  $\text{char}(K) \neq 2$ , the tame Deligne–Mumford moduli stack of generalized elliptic curves with  $[\Gamma_1(4)]$ -structures is isomorphic to*

$$(\overline{\mathcal{M}}_{1,1}[\Gamma_1(4)])_K \cong [(\text{Spec } K[a_1, a_2] - (0, 0))/\mathbb{G}_m] = \mathcal{P}_K(1, 2),$$

- (4) *if  $\text{char}(K) \neq 2$ , the tame Deligne–Mumford moduli stack of generalized elliptic curves with  $[\Gamma(2)]$ -structures is isomorphic to*

$$(\overline{\mathcal{M}}_{1,1}[\Gamma(2)])_K \cong [(\text{Spec } K[a_2, a_2] - (0, 0))/\mathbb{G}_m] = \mathcal{P}_K(2, 2),$$

where  $\lambda \cdot a_i = \lambda^i a_i$  for  $\lambda \in \mathbb{G}_m$  and  $i = 1, 2, 3, 4$ . Thus, the  $a_i$ 's have degree  $i$  respectively. Moreover, the discriminant divisors of  $(\overline{\mathcal{M}}_{1,1}[\Gamma])_K \cong \mathcal{P}_K(i, j)$  as above have degree 12.

*Proof.* Proof of the first, second, third and fourth equivalence can be found in [Behrens, §1.3], [HMe, Proposition 4.5], [Meier, Examples 2.1] and [Stojanoska, Proposition 7.1] respectively. By Proposition 3.3, the weighted projective stacks are tame Deligne–Mumford as well.

For the degree of the discriminant, it suffices to find the weight of the  $\mathbb{G}_m$ -action. First, the four papers cited above explicitly construct universal families of elliptic curves over the schematic covers  $(\text{Spec } K[a_i, a_j] - (0, 0)) \rightarrow \mathcal{P}_K(i, j)$  of the corresponding moduli stacks. The explicit defining equation of the respective universal family implies that the  $\lambda \in \mathbb{G}_m$  also acts on the discriminant of the universal family by multiplying  $\lambda^{12}$ . Therefore, the discriminant has degree 12. ■

We now consider the moduli stack  $\mathcal{L}_{1,12n}^{[\Gamma]} := \text{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma])$  of generalized elliptic curves over  $\mathbb{P}^1$  with  $[\Gamma]$ -structures.

**Proposition 7.7.** *Assume  $\text{char}(K) = 0$  or  $\neq 2$  for  $[\Gamma] = [\Gamma_1(2)], [\Gamma_1(4)], [\Gamma(2)]$ , and  $\text{char}(K) \neq 3$  for  $[\Gamma] = [\Gamma_1(3)]$ . Then, the moduli stack  $\mathcal{L}_{1,12n}^{[\Gamma]}$  of generalized elliptic curves over  $\mathbb{P}^1$  with discriminant degree  $12n > 0$  and  $[\Gamma]$ -structures is the tame Deligne–Mumford stack  $\text{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma])$  parameterizing the  $K$ -morphisms  $f : \mathbb{P}^1 \rightarrow \overline{\mathcal{M}}_{1,1}[\Gamma]$  such that  $f^* \mathcal{O}_{\overline{\mathcal{M}}_{1,1}[\Gamma]}(1) \cong \mathcal{O}_{\mathbb{P}^1}(n)$ .*

*Proof.* Without the loss of generality, we prove the  $\text{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])$  case over a field  $K$  with  $\text{char}(K) \neq 2$ . The proof for the other cases are analogous. By the definition of the universal family  $p$ , any generalized elliptic curves  $\pi : Y \rightarrow \mathbb{P}^1$  with  $[\Gamma_1(2)]$ -structures comes from a morphism  $f : \mathbb{P}^1 \rightarrow \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)]$  and vice versa. As this correspondence also works in families, the moduli stack of generalized elliptic curves over  $\mathbb{P}^1$  with  $[\Gamma_1(2)]$ -structures is isomorphic to  $\text{Hom}(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])$ .

Since the discriminant degree of  $f$  is  $12 \deg f^* \mathcal{O}_{\overline{\mathcal{M}}_{1,1}[\Gamma_1(2)]}(1)$  by Proposition 7.6, the substack  $\text{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])$  parametrizing such  $f$ 's with  $\deg f^* \mathcal{O}_{\overline{\mathcal{M}}_{1,1}[\Gamma_1(2)]}(1) = n$  is the desired moduli stack. Since  $\deg f^* \mathcal{O}_{\overline{\mathcal{M}}_{1,1}[\Gamma_1(2)]}(1) = n$  is an open condition,  $\text{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])$  is an open substack of  $\text{Hom}(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])$ , which is tame Deligne–Mumford by Proposition 3.6 as  $\overline{\mathcal{M}}_{1,1}[\Gamma_1(2)]$  itself is tame Deligne–Mumford by Proposition 7.6. This shows that  $\text{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])$  satisfies the desired properties as well. ■

We now acquire the exact number  $|\mathcal{L}_{1,12n}^{[\Gamma]}(\mathbb{F}_q)/\sim|$  of  $\mathbb{F}_q$ -isomorphism classes of  $\mathbb{F}_q$ -points (i.e., the non-weighted point count) of the moduli stack  $\mathcal{L}_{1,12n}^{[\Gamma]} \cong \text{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma])$  of generalized elliptic curves over  $\mathbb{P}^1$  with discriminant degree  $12n > 0$  and  $[\Gamma]$ -structures.

**Proposition 7.8.** *If  $\text{char}(\mathbb{F}_q) \neq 2$ , then*

$$|\mathcal{L}_{1,12n}^{[\Gamma_1(2)]}(\mathbb{F}_q)/\sim| = 2 \cdot \#_q(\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(2, 4))) = 2(q^{6n+1} - q^{6n-1})$$

$$|\mathcal{L}_{1,12n}^{[\Gamma_1(4)]}(\mathbb{F}_q)/\sim| = \#_q(\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(1, 2))) = q^{3n+1} - q^{3n-1}$$

$$|\mathcal{L}_{1,12n}^{[\Gamma(2)]}(\mathbb{F}_q)/\sim| = 2 \cdot \#_q(\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(2, 2))) = 2(q^{4n+1} - q^{4n-1})$$

*If  $\text{char}(\mathbb{F}_q) \neq 3$ , then*

$$|\mathcal{L}_{1,12n}^{[\Gamma_1(3)]}(\mathbb{F}_q)/\sim| = \#_q(\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(1, 3))) = q^{4n+1} - q^{4n-1}$$

*Proof.* Fix  $n > 0$ . Since any  $\varphi_g \in \text{Hom}_n(\mathbb{P}^1, \mathcal{P}(a, b))$  is surjective, the generic stabilizer group  $\mu_{\gcd(a,b)}$  of  $\mathcal{P}(a, b)$  is the automorphism group of  $\varphi_g$ . Using the identification from Proposition 7.7 and then summing the weighted point counts of Hom stacks from Proposition 4.5 into Proposition 4.10 gives the desired formula.  $\blacksquare$

**Remark 7.9.** For weighted projective lines  $\mathcal{P}(a, b)$  as in the cases of  $\mathcal{L}_{1,12n}^{[\Gamma]}$ ,  $\{\mathcal{I}(\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(a, b)))\}$  is a sum of  $\{\text{Hom}_n(\mathbb{P}_{\kappa(g)}^1, \mathcal{P}_{\kappa(g)}(a, b))\}$  for each closed point  $g \in \mathbb{G}_m$  with  $\text{ord}(g) \mid \gcd(a, b)$ , as the only possible generic stabilizer of positive dimensional substacks of  $\mathcal{P}(a, b)$ . On the other hand, the terms with division function  $\delta(r, q-1)$  do not occur in  $|\mathcal{L}_{1,12n}^{[\Gamma]}(\mathbb{F}_q)/\sim|$  as the characteristic condition required to identify  $\overline{\mathcal{M}}_{1,1}[\Gamma]$  as a weighted projective line implies that  $\gcd(a, b) \mid q-1$ .

We now finally prove the Theorem 7.1 using the above arithmetic invariants as follows:

*Proof of Theorem 7.1.* Without the loss of the generality, we prove the  $[\Gamma_1(2)]$ -structures case over  $\text{char}(\mathbb{F}_q) \neq 2$ . The proof for the other cases are analogous. By Proposition 7.7 and Proposition 7.8, we know the number of  $\mathbb{F}_q$ -isomorphism classes of generalized elliptic curves of discriminant degree  $12n$  with  $[\Gamma_1(2)]$ -structures over  $\mathbb{P}_{\mathbb{F}_q}^1$  is  $|\mathcal{L}_{1,12n}^{[\Gamma_1(2)]}(\mathbb{F}_q)/\sim| = 2 \cdot (q^{6n+1} - q^{6n-1})$ . Using this, we can explicitly compute the sharp bound on  $\mathcal{Z}_{1, \mathbb{F}_q(t)}^{[\Gamma_1(2)]}(\mathcal{B})$  as the following,

$$\begin{aligned} \mathcal{Z}_{1, \mathbb{F}_q(t)}^{[\Gamma_1(2)]}(\mathcal{B}) &= \sum_{n=1}^{\lfloor \frac{\log_q \mathcal{B}}{12} \rfloor} |\mathcal{L}_{1,12n}^{[\Gamma_1(2)]}(\mathbb{F}_q)/\sim| = \sum_{n=1}^{\lfloor \frac{\log_q \mathcal{B}}{12} \rfloor} 2 \cdot (q^{6n+1} - q^{6n-1}) \\ &= 2 \cdot (q^1 - q^{-1}) \sum_{n=1}^{\lfloor \frac{\log_q \mathcal{B}}{12} \rfloor} q^{6n} \leq 2 \cdot (q^1 - q^{-1}) \left( q^6 + \dots + q^{6 \cdot (\frac{\log_q \mathcal{B}}{12})} \right) \\ &= 2 \cdot (q^1 - q^{-1}) \frac{q^6(\mathcal{B}^{\frac{1}{2}} - 1)}{(q^6 - 1)} = 2 \cdot \frac{(q^7 - q^5)}{(q^6 - 1)} \cdot (\mathcal{B}^{\frac{1}{2}} - 1) \end{aligned}$$

On the second line of the equations above, inequality becomes an equality if and only if  $n := \frac{\log_q \mathcal{B}}{12} \in \mathbb{N}$ , i.e.,  $\mathcal{B} = q^{12n}$  with  $n \in \mathbb{N}$ . This implies that the acquired upper bound on  $\mathcal{Z}_{1, \mathbb{F}_q(t)}^{[\Gamma_1(2)]}(\mathcal{B})$  is a sharp estimate.  $\blacksquare$

**7.2. Arithmetic of the moduli of  $m$ -marked genus one fibrations over  $\mathbb{P}^1$ .** We proceed to estimate the sharp bound on the number of  $m$ -marked  $(m-1)$ -stable genus one fibrations over  $\mathbb{P}_{\mathbb{F}_q}^1$  for  $2 \leq m \leq 5$ . First, we state the definition of  $m$ -marked  $(m-1)$ -stability from [LP, Definition 1.5.3], which is a modification of the Deligne–Mumford stability [DM]:

**Definition 7.10.** Let  $K$  be a field and  $m$  be a positive integer. Then, a tuple  $(C, p_1, \dots, p_m)$ , of a geometrically connected, geometrically reduced, and proper  $K$ -curve  $C$  of arithmetic genus one with  $m$  distinct  $K$ -rational points  $p_i$  in the smooth locus of  $C$ , is a  $(m-1)$ -stable  $m$ -marked curve of arithmetic genus one if the curve  $C_{\overline{K}} := C \times_K \overline{K}$  and the divisor  $\Sigma := \{p_1, \dots, p_m\}$  satisfy the following properties, where  $\overline{K}$  is the algebraic closure of  $K$ :

- (1)  $C_{\overline{K}}$  has only nodes and elliptic  $u$ -fold points as singularities (see below), where  $u < m$ ,
- (2)  $C_{\overline{K}}$  has no disconnecting nodes, and
- (3) every irreducible component of  $C_{\overline{K}}$  contains at least one marked point.

**Remark 7.11.** A singular point of a curve over  $\overline{K}$  is an elliptic  $u$ -fold singular point if it is Gorenstein and étale locally isomorphic to a union of  $u$  general lines in  $\mathbb{P}_{\overline{K}}^{u-1}$  passing through a common point.

Note that the name “ $(m-1)$ -stability” comes from [Smyth, §1.1], which is defined when  $\text{char}(K) \neq 2, 3$ . By [LP, Proposition 1.5.4], the above definition (by [LP, Definition 1.5.3]) coincides with that of Smyth when  $\text{char}(K) \neq 2, 3$ , hence we adapt Smyth’s naming convention on Lekili and Polishchuk’s definition. Regardless, we focus on the case when  $\text{char}(K) \neq 2, 3$ , so that the moduli stack of such curves behaves reasonably.

By the work of Smyth [Smyth, Theorem 3.8], we are able to formulate the moduli stack of  $(m-1)$ -stable  $m$ -marked curves of arithmetic genus one over any field of characteristic  $\neq 2, 3$ :

**Theorem 7.12.** *There exists a proper irreducible Deligne–Mumford moduli stack  $\overline{\mathcal{M}}_{1,m}(m-1)$  of  $(m-1)$ -stable  $m$ -marked curves arithmetic genus one over  $\text{Spec}(\mathbb{Z}[1/6])$*

Note that when  $m = 1$ ,  $\overline{\mathcal{M}}_{1,1}(0) \cong \overline{\mathcal{M}}_{1,1}$  is the Deligne–Mumford moduli stack of stable elliptic curves.

In fact, the construction of  $\overline{\mathcal{M}}_{1,m}(m-1)$  extends to  $\text{Spec } \mathbb{Z}$  by [LP, Theorem 1.5.7] (called  $\overline{\mathcal{M}}_{1,m}^\infty$  in loc.cit.) as an algebraic stack, which is proper over  $\text{Spec } \mathbb{Z}[1/N]$  where  $N$  depends on  $m$ :

- if  $m \geq 3$ , then  $N = 1$ ,
- if  $m = 2$ , then  $N = 2$ , and
- if  $m = 1$ , then  $N = 6$ .

However, even with those assumptions above,  $\overline{\mathcal{M}}_{1,m}(m-1)$  is not necessarily Deligne–Mumford. Nevertheless, by [LP, Theorem 1.5.7.], we obtain the explicit descriptions of  $\overline{\mathcal{M}}_{1,m}(m-1)$ :

**Proposition 7.13.** *The moduli stack  $\overline{\mathcal{M}}_{1,m}(m-1)$  of  $m$ -marked  $(m-1)$ -stable curves of arithmetic genus one for  $2 \leq m \leq 5$  is isomorphic to the following, for a field  $K$ :*

- (1) *if  $\text{char}(K) \neq 2, 3$ , the tame Deligne–Mumford moduli stack of 2-marked 1-stable curves of arithmetic genus one is isomorphic to*

$$(\overline{\mathcal{M}}_{1,2}(1))_K \cong [(\text{Spec } K[a_2, a_3, a_4] - 0)/\mathbb{G}_m] = \mathcal{P}_K(2, 3, 4),$$

- (2) *if  $\text{char}(K) \neq 2, 3$ , the tame Deligne–Mumford moduli stack of 3-marked 2-stable curves of arithmetic genus one is isomorphic to*

$$(\overline{\mathcal{M}}_{1,3}(2))_K \cong [(\text{Spec } K[a_1, a_2, a_3] - 0)/\mathbb{G}_m] = \mathcal{P}_K(1, 2, 2, 3),$$

(3) if  $\text{char}(K) \neq 2$ , the tame Deligne–Mumford moduli stack of 4-marked 3-stable curves of arithmetic genus one is isomorphic to

$$(\overline{\mathcal{M}}_{1,4}(3))_K \cong [(\text{Spec } K[a_1, a_1, a_1, a_2, a_2] - 0)/\mathbb{G}_m] = \mathcal{P}_K(1, 1, 1, 2, 2),$$

(4) the moduli stack of 5-marked 4-stable curves of arithmetic genus one is isomorphic to a scheme

$$(\overline{\mathcal{M}}_{1,5}(4))_K \cong [(\text{Spec } K[a_1, a_1, a_1, a_1, a_1, a_1] - 0)/\mathbb{G}_m] = \mathbb{P}_K(1, 1, 1, 1, 1, 1) \cong \mathbb{P}_K^5,$$

where  $\lambda \cdot a_i = \lambda^i a_i$  for  $\lambda \in \mathbb{G}_m$  and  $i = 1, 2, 3, 4$ . Thus, the  $a_i$ 's have degree  $i$  respectively. Furthermore, if  $\text{char}(K) \neq 2, 3$ , then the discriminant divisors of such  $\overline{\mathcal{M}}_{1,m}(m-1)$  have degree 12.

*Proof.* Proof of [LP, Theorem 1.5.7.] gives the corresponding isomorphisms  $\overline{\mathcal{M}}_{1,m}(m-1) \cong \mathcal{P}(\vec{\lambda})$ . By Proposition 3.3, the weighted projective stacks are tame Deligne–Mumford as well, and in fact, smooth.

For the degree of the discriminant when  $\text{char}(K) \neq 2, 3$ , it suffices to describe the discriminant divisor, the locus of singular curves in  $\overline{\mathcal{M}}_{1,m}(m-1)$ . First, [LP, Theorem 1.5.7.] shows that in the above case, where  $\overline{\mathcal{M}}_{1,m}(m-1) \cong \mathcal{P}(\vec{\lambda})$ , the line bundle  $\mathcal{O}_{\mathcal{P}(\vec{\lambda})}(1)$  of degree one is isomorphic to  $\lambda := \pi_* \omega_\pi$ , where  $\pi : \overline{\mathcal{C}}_{1,m}(m-1) \rightarrow \overline{\mathcal{M}}_{1,m}(m-1)$  is the universal family of  $(m-1)$ -stable  $m$ -marked curves of arithmetic genus one. Since  $\overline{\mathcal{M}}_{1,m}(m-1)$  is smooth and the Picard rank is one (generated by  $\lambda$ ), the discriminant divisor is Cartier. In fact, by [Smyth2, §3.1], it coincides with the locus  $\Delta_{\text{irr}}$  of curves with non-disconnecting nodes or non-nodal singular points. Then [Smyth2, Remark 3.3] (which assumes  $\text{char}(K) \neq 2, 3$ ) implies that  $\Delta_{\text{irr}} \sim 12\lambda$ , thus the discriminant divisor has degree 12. ■

We now consider the moduli stacks of  $m$ -marked  $(m-1)$ -stable genus one fibrations over  $\mathbb{P}_K^1$  for any field  $K$  of  $\text{char}(K) = 0$  or  $> 3$ :

**Proposition 7.14.** *Assume  $\text{char}(K) = 0$  or  $> 3$ . If  $2 \leq m \leq 5$ , then the moduli stack  $\mathcal{L}_{1,m,12n}$  of  $m$ -marked  $(m-1)$ -stable genus one fibrations over  $\mathbb{P}_K^1$  with discriminant degree  $12n$  is the tame Deligne–Mumford stack  $\text{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,m}(m-1))$  parameterizing the  $K$ -morphisms  $f : \mathbb{P}^1 \rightarrow \overline{\mathcal{M}}_{1,m}(m-1)$  such that  $f^* \mathcal{O}_{\mathcal{P}(\vec{\lambda})}(1) \cong \mathcal{O}_{\mathbb{P}^1}(n)$ .*

*Proof.* Without the loss of the generality, we prove the 2-marked 1-stable curves of arithmetic genus one case  $\text{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,2}(1))$  over  $\text{char}(\mathbb{F}_q) \neq 2, 3$ . The proof for the other cases are analogous. By the definition of the universal family  $p$ , any 2-marked 1-stable arithmetic genus one curves  $\pi : Y \rightarrow \mathbb{P}^1$  with discriminant degree  $12n$  comes from a morphism  $f : \mathbb{P}^1 \rightarrow \overline{\mathcal{M}}_{1,2}(1)$  and vice versa. As this correspondence also works in families, the moduli stack of 2-marked 1-stable curves of arithmetic genus one over  $\mathbb{P}_K^1$  is isomorphic to  $\text{Hom}(\mathbb{P}^1, \overline{\mathcal{M}}_{1,2}(1))$ .

Since the discriminant degree of  $f$  is  $12 \deg f^* \mathcal{O}_{\overline{\mathcal{M}}_{1,2}(1)}(1)$  by Proposition 7.13, the substack  $\text{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,2}(1))$  parametrizing such  $f$ 's with  $\deg f^* \mathcal{O}_{\overline{\mathcal{M}}_{1,2}(1)}(1) = n$  is the desired moduli stack. Since  $\deg f^* \mathcal{O}_{\overline{\mathcal{M}}_{1,2}(1)}(1) = n$  is an open condition,  $\text{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,2}(1))$  is an open substack of  $\text{Hom}(\mathbb{P}^1, \overline{\mathcal{M}}_{1,2}(1))$ , which is tame Deligne–Mumford by Proposition 3.6 as  $\overline{\mathcal{M}}_{1,2}(1)$  itself is tame Deligne–Mumford by Proposition 7.13. This shows that  $\text{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,2}(1))$  satisfies the desired properties as well. ■

We now acquire the exact number  $|\mathcal{L}_{1,m,12n}(\mathbb{F}_q)| \sim$  of  $\mathbb{F}_q$ -isomorphism classes of  $\mathbb{F}_q$ -points (i.e., the non-weighted point count) of the moduli stack  $\mathcal{L}_{1,m,12n} \cong \text{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,m}(m-1))$  of  $m$ -marked  $(m-1)$ -stable genus one fibrations over  $\mathbb{P}^1$  with discriminant degree  $12n > 0$ .

**Proposition 7.15.** *If  $\text{char}(\mathbb{F}_q) \neq 2, 3$ , then*

$$\begin{aligned} |\mathcal{L}_{1,2,12n}(\mathbb{F}_q)/\sim| &= \#_q(\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(2, 3, 4))) + \#_q(\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(2, 4))) \\ &= (q^{9n+2} + q^{9n+1} - q^{9n-1} - q^{9n-2}) + (q^{6n+1} - q^{6n-1}) \end{aligned}$$

$$\begin{aligned} |\mathcal{L}_{1,3,12n}(\mathbb{F}_q)/\sim| &= \#_q(\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(1, 2, 2, 3))) + \#_q(\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(2, 2))) \\ &= (q^{8n+3} + q^{8n+2} + q^{8n+1} - q^{8n-1} - q^{8n-2} - q^{8n-3}) + (q^{4n+1} - q^{4n-1}) \end{aligned}$$

$$\begin{aligned} |\mathcal{L}_{1,4,12n}(\mathbb{F}_q)/\sim| &= \#_q(\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(1, 1, 1, 2, 2))) + \#_q(\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(2, 2))) \\ &= (q^{7n+4} + q^{7n+3} + q^{7n+2} + q^{7n+1} - q^{7n-1} - q^{7n-2} - q^{7n-3} - q^{7n-4}) \\ &\quad + (q^{4n+1} - q^{4n-1}) \end{aligned}$$

$$\begin{aligned} |\mathcal{L}_{1,5,12n}(\mathbb{F}_q)/\sim| &= \#_q(\text{Hom}_n(\mathbb{P}^1, \mathbb{P}(1, 1, 1, 1, 1) \cong \mathbb{P}^5)) \\ &= q^{6n+5} + q^{6n+4} + q^{6n+3} + q^{6n+2} + q^{6n+1} - q^{6n-1} - q^{6n-2} - q^{6n-3} - q^{6n-4} - q^{6n-5} \end{aligned}$$

*Proof.* Note that  $\overline{\mathcal{M}}_{1,2}(1) \cong \mathcal{P}(2, 3, 4)$  has the substack  $\mathcal{P}(2, 4)$  with the generic stabilizer of order 2. This implies that the number of isomorphism classes of  $\mathbb{F}_q$ -points of  $\mathcal{L}_{1,2,12n}$  with discriminant degree  $12n$  is  $|\mathcal{L}_{1,2,12n}(\mathbb{F}_q)/\sim| = (q^{9n+2} + q^{9n+1} - q^{9n-1} - q^{9n-2}) + (q^{6n+1} - q^{6n-1})$  by summing the weighted point counts of Hom stacks from Proposition 4.5 into Proposition 4.10. Similarly,  $\overline{\mathcal{M}}_{1,3}(2) \cong \mathcal{P}(1, 2, 2, 3)$  and  $\overline{\mathcal{M}}_{1,4}(3) \cong \mathcal{P}(1, 1, 1, 2, 2)$  has the substack  $\mathcal{P}(2, 2)$  with the generic stabilizer of order 2. This implies that adding  $(q^{4n+1} - q^{4n-1})$  to the corresponding weighted points count gives the desired non-weighted point counts. Finally,  $\overline{\mathcal{M}}_{1,5}(4) \cong \mathbb{P}^5$ , so that the non-weighted point count coincides with the weighted point count.  $\blacksquare$

We now finally prove the Theorem 7.4 using the above arithmetic invariants as follows:

*Proof.* Without the loss of the generality, we prove the 2-marked 1-stable curves of arithmetic genus one case  $\text{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,2}(1) \cong \mathcal{P}(2, 3, 4))$  over  $\text{char}(\mathbb{F}_q) \neq 2, 3$ . The proof for the other cases are analogous. Knowing the number of  $\mathbb{F}_q$ -isomorphism classes of 1-stable arithmetic genus one curves over  $\mathbb{P}^1$  with discriminant degree  $12n$  and 2-marked Weierstrass sections over  $\mathbb{F}_q$  is  $|\mathcal{L}_{1,2,12n}(\mathbb{F}_q)/\sim| = (q^{9n+2} + q^{9n+1} - q^{9n-1} - q^{9n-2}) + (q^{6n+1} - q^{6n-1})$  by Proposition 7.15, we can explicitly compute the sharp bound on  $\mathcal{Z}_{1,2,\mathbb{F}_q(t)}(\mathcal{B})$  as the following,

$$\begin{aligned} \mathcal{Z}_{1,2,\mathbb{F}_q(t)}(\mathcal{B}) &= \sum_{n=1}^{\lfloor \frac{\log_q \mathcal{B}}{12} \rfloor} |\mathcal{L}_{1,2,12n}(\mathbb{F}_q)/\sim| = \sum_{n=1}^{\lfloor \frac{\log_q \mathcal{B}}{12} \rfloor} (q^{9n+2} + q^{9n+1} - q^{9n-1} - q^{9n-2}) + (q^{6n+1} - q^{6n-1}) \\ &= (q^2 + q^1 - q^{-1} - q^{-2}) \sum_{n=1}^{\lfloor \frac{\log_q \mathcal{B}}{12} \rfloor} q^{9n} + (q^1 - q^{-1}) \sum_{n=1}^{\lfloor \frac{\log_q \mathcal{B}}{12} \rfloor} q^{6n} \\ &\leq (q^2 + q^1 - q^{-1} - q^{-2}) \left( q^9 + \dots + q^{9 \cdot (\frac{\log_q \mathcal{B}}{12})} \right) + (q^1 - q^{-1}) \left( q^6 + \dots + q^{6 \cdot (\frac{\log_q \mathcal{B}}{12})} \right) \\ &= (q^2 + q^1 - q^{-1} - q^{-2}) \cdot \frac{q^9 (\mathcal{B}^{\frac{3}{4}} - 1)}{(q^9 - 1)} + (q^1 - q^{-1}) \frac{q^6 (\mathcal{B}^{\frac{1}{2}} - 1)}{(q^6 - 1)} \\ &= \frac{(q^{11} + q^{10} - q^8 - q^7)}{(q^9 - 1)} \cdot (\mathcal{B}^{\frac{3}{4}} - 1) + \frac{(q^7 - q^5)}{(q^6 - 1)} \cdot (\mathcal{B}^{\frac{1}{2}} - 1) \end{aligned}$$

On the third line of the equations above, inequality becomes an equality if and only if  $n := \frac{\log_q \mathcal{B}}{12} \in \mathbb{N}$ , i.e.,  $\mathcal{B} = q^{12n}$  with  $n \in \mathbb{N}$ . This implies that the acquired upper bound on  $\mathcal{Z}_{1,2,\mathbb{F}_q(t)}(\mathcal{B})$  is a sharp estimate. ■

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