ℓ-ADIC ÉTALE COHOMOLOGY OF THE MODULI OF ELLIPTIC AND HYPERELLIPTIC FIBRATIONS

JUN–YONG PARK

Abstract. We determine ℓ-adic étale cohomology with Frobenius weights for the Hom stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ of degree n rational curves on a weighted projective stack. By various isomorphisms of $\mathcal{P}(\vec{\lambda})$ to moduli stacks of elliptic & hyperelliptic curves, we acquire Galois representations of mixed Tate type for the moduli stacks of elliptic & hyperelliptic fibrations over $\mathbb{P}^1$.

1. Introduction

Fix a base field K, and define for the weight $\vec{\lambda} = (\lambda_0, \ldots, \lambda_N)$ with $|\vec{\lambda}|$ as total sum of positive weights $\lambda_i \in \mathbb{N}$, the N-dimensional weighted projective stack $\mathcal{P}(\vec{\lambda}) := [(\mathbb{A}^{N+1}_{x_0,\ldots,x_N} \setminus 0)/\mathbb{G}_m]$ where $\zeta \in \mathbb{G}_m$ acts by $\zeta \cdot (x_0, \ldots, x_N) = (\zeta^{\lambda_0} x_0, \ldots, \zeta^{\lambda_N} x_N)$. In this paper, we find ℓ-adic étale cohomology as Galois representations for the Hom stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ of degree $n \in \mathbb{N}$ rational curves on $\mathcal{P}(\vec{\lambda})$. In essence, understanding the Hom stack is significant as $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ serves as a 'model' of *fine moduli stack of fibrations over* $\mathbb{P}^1_K$ meaning that as $\mathcal{P}(\vec{\lambda})$ is isomorphic to diverse kinds of smooth, proper, irreducible Deligne–Mumford moduli stack of (stable) elliptic & hyperelliptic curves (see Example 2.3 for various isomorphisms of $\mathcal{P}(\vec{\lambda})$) which in turn transforms $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ into a smooth Deligne–Mumford moduli stack of elliptic & hyperelliptic fibrations over $\mathbb{P}^1_K$. We acquire the following *arithmetic geometric & topological aspects* of $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$.

Theorem 1.1. *Hom stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ parameterizing degree $n \in \mathbb{N}$ morphisms $f : \mathbb{P}^1 \to \mathcal{P}(\vec{\lambda})$ with $f^*\mathcal{O}_{\mathcal{P}(\vec{\lambda})}(1) \simeq \mathcal{O}_{\mathbb{P}^1}(n)$ over $\overline{\mathbb{F}}_q$ with $\mathrm{char}(\overline{\mathbb{F}}_q)$ not dividing $\lambda_i \in \mathbb{N}$ has the following compactly supported ℓ-adic étale cohomology as Galois representations of mixed Tate type.*

$$H^i_{et,c}\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell\right) \cong \begin{cases} \mathbb{Q}_\ell\left(-(|\vec{\lambda}|n + (N))\right) & i = 2(|\vec{\lambda}|n + N), \\ \mathbb{Q}_\ell\left(-(|\vec{\lambda}|n + (N-1))\right) & i = 2(|\vec{\lambda}|n + N) - (2), \\ \vdots & \vdots \\ \mathbb{Q}_\ell\left(-(|\vec{\lambda}|n + 2)\right) & i = 2(|\vec{\lambda}|n + N) - (2N-4), \\ \mathbb{Q}_\ell\left(-(|\vec{\lambda}|n + 1)\right) & i = 2(|\vec{\lambda}|n + N) - (2N-2), \\ \mathbb{Q}_\ell\left(-(|\vec{\lambda}|n - 1)\right) & i = 2(|\vec{\lambda}|n + N) - (2N+1), \\ \mathbb{Q}_\ell\left(-(|\vec{\lambda}|n - 2)\right) & i = 2(|\vec{\lambda}|n + N) - (2N+3), \\ \vdots & \vdots \\ \mathbb{Q}_\ell\left(-(|\vec{\lambda}|n - (N-1))\right) & i = 2(|\vec{\lambda}|n + N) - (4N-3), \\ \mathbb{Q}_\ell\left(-(|\vec{\lambda}|n - (N))\right) & i = 2(|\vec{\lambda}|n + N) - (4N-1), \\ 0 & \textit{else}. \end{cases}$$

Our approach is to first perform a careful analysis of the flat evaluation morphism $\mathrm{ev}_\infty : \mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda})) \to \mathcal{P}(\vec{\lambda})$ showing the cohomology of the fibers are fixed as in Proposition 2.12. Since the base $\mathcal{P}(\vec{\lambda})$ is simply-connected, we show the associated ℓ-adic Leray spectral sequence is actually computable as Serre spectral sequence leading to ℓ-adic étale Betti numbers $\dim_{\mathbb{Q}_\ell} \left(H^i_{\acute{e}t}(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell) \right)$ as in Proposition 2.13. Consequently, we have the Main Theorem through the Grothendieck-Lefschetz trace formula (see Theorem 2.9) applied to the motive/weighted $\mathbb{F}_q$-point count of the Hom stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ as in Theorem 2.6 acquired from [HP, PS, HP2].

Lastly, we note that the moduli stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ is an algebraic stack generalization of the famous moduli space $\mathrm{Hom}_n(\mathbb{P}^1, \mathbb{P}^N)$ of degree $n \in \mathbb{N}$ rational curves to a projective space $\mathbb{P}^N$ studied in depth by [Segal, CCMM, Silverman, KS] and many others in various perspectives.

2. Étale cohomology with eigenvalues of geometric Frobenius

In this section, we will introduce some definitions, background materials on schemes, algebraic stacks and discuss properties of étale cohomology that will be needed in the proof. For further details on the material presented here, the reader is referred to [Liu, Olsson2, DIS, Behrend].

We first recall the definition of a weighted projective stack $\mathcal{P}(\vec{\lambda})$.

Definition 2.1. Fix a tuple of nondecreasing positive integers $\vec{\lambda} = (\lambda_0, \dots, \lambda_N)$. The N-dimensional weighted projective stack $\mathcal{P}(\vec{\lambda}) = \mathcal{P}(\lambda_0, \dots, \lambda_N)$ with the weight $\vec{\lambda}$ is defined as a quotient stack

$$\mathcal{P}(\vec{\lambda}) := \left[(\mathbb{A}^{N+1}_{x_0, \dots, x_N} \setminus 0) / \mathbb{G}_m \right]$$

where $\zeta \in \mathbb{G}_m$ acts by $\zeta \cdot (x_0, \dots, x_N) = (\zeta^{\lambda_0} x_0, \dots, \zeta^{\lambda_N} x_N)$. In this case, the degree of x_i's are λ_i's respectively. A line bundle $\mathcal{O}_{\mathcal{P}(\vec{\lambda})}(m)$ is defined to be the line bundle associated to the sheaf of degree m homogeneous rational functions without poles on $\mathbb{A}^{N+1}_{x_0, \dots, x_N} \setminus 0$.

Note that $\mathcal{P}(\vec{\lambda})$ is not an (effective) orbifold when $\gcd(\lambda_0, \dots, \lambda_N) \neq 1$. In this case, the cyclic isotropy group scheme $\mu_{\gcd(\lambda_0, \dots, \lambda_N)}$ is the generic stabilizer of $\mathcal{P}(\vec{\lambda})$. Nevertheless, the following proposition shows that it behaves well in most characteristics as a tame Deligne–Mumford stack:

Proposition 2.2. *The N-dimensional weighted projective stack $\mathcal{P}(\vec{\lambda}) = \mathcal{P}(\lambda_0, \dots, \lambda_N)$ is a tame Deligne–Mumford stack over a base field K if* $\mathrm{char}(K)$ *does not divide $\lambda_i \in \mathbb{N}$ for every i.*

Proof. For any algebraically closed field extension $\overline{K}$ of K, any point $y \in \mathcal{P}(\vec{\lambda})(\overline{K})$ is represented by the coordinates $(y_0, \dots, y_N) \in \mathbb{A}^{N+1}_{\overline{K}}$ with its stabilizer group as the subgroup of $\mathbb{G}_m$ fixing $(y_0, \dots, y_N)$. Hence, any stabilizer group of such $\overline{K}$-points is $\mathbb{Z}/u\mathbb{Z}$ where u divides λ_i for some i. Since the characteristic of K does not divide the orders of $\mathbb{Z}/\lambda_i\mathbb{Z}$ for any i, the stabilizer group of y is $\overline{K}$-linearly reductive. Hence, $\mathcal{P}(\vec{\lambda})$ is tame by [AOV, Theorem 3.2]. Note that the stabilizer groups constitute fibers of the diagonal $\Delta : \mathcal{P}(\vec{\lambda}) \to \mathcal{P}(\vec{\lambda}) \times_K \mathcal{P}(\vec{\lambda})$. Since $\mathcal{P}(\vec{\lambda})$ is of finite type and $\mathbb{Z}/u\mathbb{Z}$'s are unramified over K whenever u does not divide λ_i for some i, Δ is unramified as well. Therefore, $\mathcal{P}(\vec{\lambda})$ is also Deligne–Mumford by [Olsson2, Theorem 8.3.3]. ∎

The tameness is analogous to flatness for stacks in positive/mixed characteristic as it is preserved under base change by [AOV, Corollary 3.4]. Moreover, if a stack $\mathcal{X}$ is tame and Deligne–Mumford, then the formation of the coarse moduli space $c : \mathcal{X} \to X$ commutes with base change as well by [AOV, Corollary 3.3].

Example 2.3. There is a whole array of moduli stacks of curves that are isomorphic to $\mathcal{P}(\vec{\lambda})$ for various weights $\vec{\lambda} = (\lambda_0, \ldots, \lambda_N)$ over a field K with $\mathrm{char}(K)$ does not divide $\lambda_i \in \mathbb{N}$ for every i.

When $\mathrm{char}(K) \neq 2, 3$, [Hassett, Proposition 3.6] shows that one example is given by the proper Deligne–Mumford stack of stable elliptic curves $(\overline{\mathcal{M}}_{1,1})_K \cong [(\mathrm{Spec}\ K[a_4, a_6] - (0,0))/\mathbb{G}_m] = \mathcal{P}_K(4,6)$ by using the short Weierstrass equation $y^2 = x^3 + a_4 x + a_6 x$, where $\zeta \cdot a_i = \zeta^i a_i$ for $\zeta \in \mathbb{G}_m$ and $i = 4, 6$. This is no longer true when $\mathrm{char}(K) \in \{2, 3\}$, as the Weierstrass equations are more complicated.

Similarly, one could consider $\overline{\mathcal{M}}_{1,1}[\Gamma]$ of generalized elliptic curves with $[\Gamma]$-level structure by the work of Deligne and Rapoport [DR] (summarized in [Conrad, §2] and also in [Niles, §2]) such as $\overline{\mathcal{M}}_{1,1}[\Gamma_1(2)] \cong \mathcal{P}(2,4)$ [Behrens, §1.3], $\overline{\mathcal{M}}_{1,1}[\Gamma_1(3)] \cong \mathcal{P}(1,3)$ [HMe, Proposition 4.5], $\overline{\mathcal{M}}_{1,1}[\Gamma_1(4)] \cong \mathcal{P}(1,2)$ [Meier, Examples 2.1] and $\overline{\mathcal{M}}_{1,1}[\Gamma(2)] \cong \mathcal{P}(2,2)$ [Stojanoska, Proposition 7.1].

Also, one could consider $\overline{\mathcal{M}}_{1,m}(m-1)$ of m-marked $(m-1)$-stable curves of arithmetic genus one formulated originally by the works of [Smyth, Smyth2] such as $\overline{\mathcal{M}}_{1,2}(1) \cong \mathcal{P}(2,3,4)$, $\overline{\mathcal{M}}_{1,3}(2) \cong \mathcal{P}(1,2,2,3)$, $\overline{\mathcal{M}}_{1,4}(3) \cong \mathcal{P}(1,1,1,2,2)$ and $\overline{\mathcal{M}}_{1,5}(4) \cong \mathbb{P}(1,1,1,1,1,1) \cong \mathbb{P}^5$ as shown by [LP, Theorem 1.5.7.].

For higher genus $g \geq 2$, the moduli stack $\mathcal{H}_{2g}[2g-1]$ of quasi-admissible hyperelliptic genus g curves originally introduced by [Fedorchuk] is a Deligne–Mumford stack isomorphic to $\mathcal{P}(4, 6, 8, \ldots, 4g+2)$ if $\mathrm{char}(K) = 0$ [Fedorchuk, Proposition 4.2(1)] or $> 2g+1$ by [HP2, Proposition 4.9].

We refer interested readers to [HP, PS, HP2] for *arithmetic & birational geometric* considerations.

We now generalize the Hom stack formulation to $\mathcal{P}(\vec{\lambda})$ as follows:

Definition 2.4. The stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ with the weight $\vec{\lambda} = (\lambda_0, \ldots, \lambda_N)$ is defined to be the Hom stack of degree $n \in \mathbb{N}$ morphisms $f : \mathbb{P}^1 \to \mathcal{P}(\vec{\lambda})$ with $f^*\mathcal{O}_{\mathcal{P}(\vec{\lambda})}(1) \simeq \mathcal{O}_{\mathbb{P}^1}(n)$. Concretely,

$$\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda})) = [T/\mathbb{G}_m]$$

where $T \subset \left(\bigoplus_{i=0}^{N} H^0(\mathcal{O}_{\mathbb{P}^1}(\lambda_i \cdot n)) \right) \setminus 0$ is the open subset of tuples of nonzero polynomials $(\widetilde{u}_0(z,t), \ldots, \widetilde{u}_N(z,t))$ homogeneous of degrees $\deg u_i = \lambda_i \cdot n$ for every i, sharing no common factor, and $\zeta \in \mathbb{G}_m$ acts by $\zeta \cdot (\widetilde{u}_0(z,t), \ldots, \widetilde{u}_N(z,t)) := (\widetilde{u}_0(\zeta z, \zeta t), \ldots, \widetilde{u}_N(\zeta z, \zeta t)) = (\zeta^{\lambda_0} \cdot \widetilde{u}_0(z,t), \ldots, \zeta^{\lambda_N} \cdot \widetilde{u}_N(z,t))$.

Proposition 2.5. *Over a base field K with $\mathrm{char}(K)$ not dividing $\lambda_i \in \mathbb{N}$ for every i, the Hom stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ parameterizing degree $n \in \mathbb{N}$ morphisms $f : \mathbb{P}^1 \to \mathcal{P}(\vec{\lambda})$ is a smooth separated tame Deligne–Mumford stack of finite type with dimension equal to $|\vec{\lambda}|n + N$ where $|\vec{\lambda}| := \sum_{i=0}^{N} \lambda_i$.*

Proof. $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda})) = [T/\mathbb{G}_m]$ is a smooth Deligne–Mumford stack by [Olsson, Theorem 1.1], admitting T as a smooth schematic cover. Note that

$$\dim T = \sum_{i=0}^{n} h^0(\mathcal{O}_{\mathbb{P}^1}(\lambda_i \cdot n)) = \sum_{i=0}^{n} (\lambda_i + 1) = |\vec{\lambda}| + N + 1,$$

implying that $\dim \mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda})) = |\vec{\lambda}| + N$ since $\dim \mathbb{G}_m = 1$. As $\mathbb{G}_m$ acts on T properly with positive weights $\lambda_i \in \mathbb{N}$ for every i the quotient stack $[T/\mathbb{G}_m]$ is separated. It is tame as in [AOV, Theorem 3.2] since $\mathrm{char}(K)$ does not divide λ_i for every i . ■

As a polynomial in $H^0(\mathcal{O}_{\mathbb{P}^1}(\lambda_i \cdot n)) \setminus 0$ is determined by its restriction away from $\infty = [0:1]$ (i.e., by evaluating $t = 1$), we may consider T as the set of not-necessarily-monic tuples of nonzero coprime polynomials $(u_0, \ldots, u_N)$ such that for a fixed $j \in \{0, \ldots, N\}$, $\deg u_j = \lambda_j \cdot n$ and the rests have $0 \leq \deg u_i \leq \lambda_i \cdot n$ for every $i \neq j$. Here the degree conditions are precisely ensuring that $\widetilde{u}_0, \ldots, \widetilde{u}_N$ do not share a common zero at ∞.

We recall the *arithmetic aspect* (i.e., motive/weighted $\mathbb{F}_q$-point count) of $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$.

Theorem 2.6 (Theorem 1.11 & Corollary 1.13 of [HP2]). *Fix the weight $\vec{\lambda} = (\lambda_0, \ldots, \lambda_N)$ with $|\vec{\lambda}| := \sum_{i=0}^{N} \lambda_i$. Suppose that $\mathrm{char}(\mathbb{F}_q)$ does not divide $\lambda_i \in \mathbb{N}$ for every i, then the weighted point count of the Hom stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ over $\mathbb{F}_q$ is*

$$
\begin{aligned}
\#_q \left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))\right) &= \left(\sum_{i=0}^{N} q^i\right) \cdot \left(q^{|\vec{\lambda}|n} - q^{|\vec{\lambda}|n-N}\right) \\
&= q^{|\vec{\lambda}|n} \cdot \left(q^N + q^{N-1} + \cdots + q^2 + q^1 - q^{-1} - q^{-2} - \cdots - q^{-(N-1)} - q^{-N}\right) \\
&= q^{|\vec{\lambda}|n+N} + \cdots + q^{|\vec{\lambda}|n+1} - q^{|\vec{\lambda}|n-1} - \cdots - q^{|\vec{\lambda}|n-N}
\end{aligned}
$$

$\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ has the stratification by $\mathrm{Poly}_1^{(d_1,\ldots,d_m)}$ as in [HP, §4.1] and [HP2, §3.1].

Definition 2.7. Fix $m \in \mathbb{Z}_{>0}$ and $d_1, \ldots, d_m \in \mathbb{N}$, define

$$\mathrm{Poly}_1^{(d_1,\ldots,d_m)} := \{(f_1(z), \ldots, f_m(z)) : f_i(z) \text{ are monic and coprime of degrees } d_i\}.$$

Equivalently, $\mathrm{Poly}_1^{(d_1,\ldots,d_m)}$ is the complement of the resultant hypersurface $\mathcal{R}^{(d_1,\ldots,d_m)} \subset \prod_{i=1}^{m} \mathrm{Sym}^{d_i} \mathbb{A}^1 = \prod_{i=1}^{m} \mathbb{A}^{d_i}$ of tuples of monic polynomials $(f_1(z), \ldots, f_m(z))$ of degrees d_i which share a factor.

Arithmetic of $\mathrm{Poly}_1^{(d_1,\ldots,d_m)}$ was studied in [HP, PS, HP2] which was inspired by the work of [Segal, FW]. As the resultant hypersurface $\mathcal{R}^{(d_1,\ldots,d_m)}$ is defined over $\mathbb{Z}$, it makes sense to define $\mathrm{Poly}_1^{d_1,\ldots,d_m} = (\prod_{i=1}^{m} \mathbb{A}^{d_i}) \setminus \mathcal{R}^{(d_1,\ldots,d_m)}$ as a variety over $\mathbb{Z}$.

Generalizing the proof of [FW, Theorem 1.2] and [HP, Proposition 18] with the correction from [PS, Proposition 3.1.], we find the motive/weighted $\mathbb{F}_q$-point count of $\mathrm{Poly}_1^{(d_1,\ldots,d_m)}$.

Proposition 2.8 (Proposition 3.4. of [HP2]). *Fix $0 \leq d_1 \leq d_2 \leq \ldots \leq d_m$. Then for any prime power q :*

$$|\mathrm{Poly}_1^{(d_1,\ldots,d_m)}(\mathbb{F}_q)| = \begin{cases} q^{d_1+\ldots+d_m} - q^{d_1+\ldots+d_m-m+1}, & \text{if } d_1 \neq 0 \\ q^{d_1+\ldots+d_m}, & \text{if } d_1 = 0 \end{cases}$$

Since the absolute Galois group of finite fields $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ acting on the ℓ-adic étale cohomology is a procyclic group that is topologically generated by the geometric Frobenius, the task of finding the eigenvalues of Frobenius map can be achieved through the trace formula where the cardinality of the fixed set of $\mathrm{Frob}_q : \mathfrak{X}(\overline{\mathbb{F}}_q) \to \mathfrak{X}(\overline{\mathbb{F}}_q)$ coincides with $\#_q(\mathfrak{X})$ the weighted $\mathbb{F}_q$–point count. We recall the *Grothendieck-Lefschetz trace formula* [Sun, Behrend] for Artin stacks of finite type over finite fields.

Theorem 2.9 (Theorem 1.1. of [Sun])**.** *Let $\mathfrak{X}$ be an Artin stack of finite type over $\mathbb{F}_q$. Let Frob_q be the geometric Frobenius on $\mathfrak{X}$. Let ℓ be a prime number different from the characteristic of $\mathbb{F}_q$, and let $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$ be an isomorphism of fields. For an integer i, let $H^i_c(\mathfrak{X}_{/\overline{\mathbb{F}}_q}; \overline{\mathbb{Q}}_\ell)$ be the cohomology with compact support of the constant sheaf $\overline{\mathbb{Q}}_\ell$ on $\mathfrak{X}$ as in* [LO]*. Then the infinite sum regarded as a complex series via ι*

$$\sum_{i\in\mathbb{Z}}(-1)^i tr\big(\mathrm{Frob}_q^* : H^i_c(\mathfrak{X}_{/\overline{\mathbb{F}}_q}; \overline{\mathbb{Q}}_\ell) \to H^i_c(\mathfrak{X}_{/\overline{\mathbb{F}}_q}; \overline{\mathbb{Q}}_\ell)\big) \tag{1}$$

is absolutely convergent to $\#_q(\mathfrak{X})$ over $\mathbb{F}_q$. And its limit is the number of $\mathbb{F}_q$–points on stacks that are counted with weights, where a point with its stabilizer group G contributes a weight $\frac{1}{|G|}$.

As $\mathfrak{X} = \mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ is a separated Deligne–Mumford stack of finite type over $\mathbb{F}_q$ (see Proposition 2.5), the corresponding (compactly supported) ℓ-adic étale cohomology for prime number ℓ invertible in $\mathbb{F}_q$ is finite dimensional as a $\mathbb{Q}_\ell$-algebra, making the trace formula hold in $\mathbb{Q}_\ell$-coefficients. Also by the smoothness, the Poincaré duality $H^i_{\acute{e}t,c}(\mathfrak{X}_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell) \cong H^{2dim(\mathfrak{X})-i}_{\acute{e}t}(\mathfrak{X}_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell(-dim(\mathfrak{X})))^\vee$ hold for the dual of ordinary étale cohomology as in [LO, 7.3.1. Theorem]. From now on, we identify the dual spaces with the ordinary étale cohomology as finite dimensional $\mathbb{Q}_\ell$-vector spaces. We denote by $\mathbb{Q}_\ell(-i)$ the $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-representation of rank 1 on which the *geometric Frobenius* acts by q^{-i} and the *arithmetic Frobenius* acts by q^i. Also, recall that the étale cohomology of a stack $\mathfrak{X}$ is *pure* if the absolute values of the eigenvalues of Frob_q on $H^i_{\acute{e}t}$ are all $q^{i/2}$ and if otherwise it is *mixed*. The group $H^i_{\acute{e}t}(\mathfrak{X}_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$ is of *Tate type* if the eigenvalues of Frob_q are all equal to integer powers of q.

We are now ready to prove the Theorem 1.1.

2.1. **Proof of Theorem 1.1.**

Proof. We first recall the étale cohomology of the smooth proper base $\mathcal{P}(\vec{\lambda})$.

Proposition 2.10. *There are isomorphisms of Galois representations*

$$H^i_{\acute{e}t}\left(\mathcal{P}(\vec{\lambda})_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell\right) \cong \begin{cases} \mathbb{Q}_\ell\left(-\frac{i}{2}\right) & i \in \{0, 2, \dots, 2N\} \\ 0 & else \end{cases}$$

Proof. This follows from [Kawasaki, Theorem 1.] where we use the well-known fact that the cohomology of a smooth Deligne–Mumford stack $\mathfrak{X}$ with projective coarse moduli space X is pure. ∎

Next, we need the compactly supported étale cohomology of $\mathrm{Hom}^*_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda})) \cong \mathrm{Poly}_1^{(\lambda_0\cdot n,\dots,\lambda_N\cdot n)}$ which follows from [FW, HP, HP2].

Proposition 2.11. *There are isomorphisms of Galois representations*

$$H^i_{\acute{e}t,c}\left(\mathrm{Poly}_1^{(\lambda_0\cdot n,\dots,\lambda_N\cdot n)}{}_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell\right) \cong \begin{cases} \mathbb{Q}_\ell\left(-(|\vec{\lambda}|n)\right) & i = 2|\vec{\lambda}|n \\ \mathbb{Q}_\ell\left(-(|\vec{\lambda}|n-N)\right) & i = 2|\vec{\lambda}|n-(2N-1) \\ 0 & else \end{cases}$$

Proof. We can express $\mathrm{Poly}_1^{(\lambda_0 \cdot n, \dots, \lambda_N \cdot n)}$ as the quotient of a finite group $S_{\lambda_0 \cdot n} \times \cdots \times S_{\lambda_N \cdot n}$ acting on the complement of a hyperplane arrangement in $\mathbb{A}^{|\vec{\lambda}|n}$. It is well known by [Lehrer, Shapiro, Kim] that the geometric Frobenius Frob_q^* acts by q^i on $H^i_{\acute{e}t}$ of the complement of a hyperplane arrangement. By transfer and Poincaré duality, we see that the geometric Frobenius acts by $q^{-(|\vec{\lambda}|n)+i}$ on $H^i_{\acute{e}t,c}(\mathrm{Poly}_1^{(\lambda_0 \cdot n, \dots, \lambda_N \cdot n)}{}_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$. Applying the *Grothendieck-Lefschetz trace formula* for compactly supported étale cohomology, we see that

$$\left|\mathrm{Poly}_1^{(\lambda_0 \cdot n, \dots, \lambda_N \cdot n)}(\mathbb{F}_q)\right| = \sum_{i=0}^{2|\vec{\lambda}|n} (-1)^i \cdot q^{-(|\vec{\lambda}|n)+i} \cdot \dim_{\mathbb{Q}_\ell} \left(H^i_{\acute{e}t,c}(\mathrm{Poly}_1^{(\lambda_0 \cdot n, \dots, \lambda_N \cdot n)}{}_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell) \right)$$

and we use the exact point count of Proposition 2.8 to conclude the above. ∎

We now consider the ℓ-adic Leray spectral sequence as in [Behrend, Theorem 1.2.5] with respect to the evaluation morphism $\mathrm{ev}_\infty : \mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda})) \to \mathcal{P}(\vec{\lambda})$ to determine the ℓ-adic étale Betti numbers of $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$.

Proposition 2.12. *Evaluation morphism* $\mathrm{ev}_\infty : \mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda})) \to \mathcal{P}(\vec{\lambda})$ *mapping an equivalence class of degree* $n \in \mathbb{N}$ *unbased morphism* $f : \mathbb{P}^1 \to \mathcal{P}(\vec{\lambda})$ *with the chosen basepoint* $\infty \in \mathbb{P}^1$ *to the basepoint of the target* $f(\infty) \in \mathcal{P}(\vec{\lambda})$ *is a flat morphism where* $H^j_{\acute{e}t,c}(\mathrm{ev}_\infty^{-1}(p)_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$ *the compactly supported* ℓ*-adic étale cohomology of the fiber at each geometric point* $p \in \mathcal{P}(\vec{\lambda})$ *is isomorphic to* $H^j_{\acute{e}t,c}(\mathrm{Poly}_1^{(\lambda_0 \cdot n, \dots, \lambda_N \cdot n)}{}_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$ *the compactly supported* ℓ*-adic étale cohomology of the space* $\mathrm{Poly}_1^{(\lambda_0 \cdot n, \dots, \lambda_N \cdot n)} \cong \mathrm{Hom}_n^*(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ *of degree* $n \in \mathbb{N}$ *based morphisms.*

Proof. We have a $\mathbb{G}_m$-equivariant map $\widetilde{\mathrm{ev}}_\infty : T \to \mathbb{A}^{N+1} \backslash 0$, which induces $\mathrm{ev}_\infty : \mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda})) = [T/\mathbb{G}_m] \to \mathcal{P}(\vec{\lambda}) = [(\mathbb{A}^{N+1} \backslash 0)/\mathbb{G}_m]$. Note that for such maps and a closed subscheme $(x_0, \dots, x_N) \in \mathbb{A}^{N+1} \setminus 0$ on which $\mathbb{G}_m$ acts, $\mathrm{ev}_\infty^{-1}([x_0 : \dots : x_N]) \cong \widetilde{\mathrm{ev}}_\infty^{-1}((x_0, \dots, x_N))/\mathbb{G}_m$. This implies in particular that the special fiber at $[x_0 : \dots : x_N]$ with at least one of the $x_i = 0$ (but not all zero) is the complement of the hyperplane arrangement by the same principle that the generic fiber such as over at $[1 : \dots : 1]$ is the complement of the hyperplane arrangement.

The fiber at the generic stacky points $[x_0 : \dots : x_N] \in \mathcal{P}(\vec{\lambda})$ such that $x_i \neq 0$ for every i with $\mu_{\gcd(\lambda_0, \dots, \lambda_N)}$ stabilizer can be accounted for by looking at $\mathrm{ev}_\infty^{-1}([1 : \dots : 1])$ where we have $\mathrm{ev}_\infty^{-1}([1 : \dots : 1]) \cong \mathrm{Poly}_1^{(\lambda_0 \cdot n, \dots, \lambda_N \cdot n)}$. This is because a tuple of monic coprime polynomials $(f_0, \dots, f_N)$ with $\deg(f_0, \dots, f_N) = (\lambda_0 \cdot n, \dots, \lambda_N \cdot n)$ determines, and is determined by, a morphism $f : \mathbb{P}^1 \to \mathcal{P}(\vec{\lambda})$ such that $f^*\mathcal{O}_{\mathcal{P}(\vec{\lambda})}(1) \simeq \mathcal{O}_{\mathbb{P}^1}(n)$ and $f(\infty) = [1 : \dots : 1]$ via $f(z) := [f_0(z), \dots, f_N(z)]$ under the weighted homogeneous coordinate of the target stack $\mathcal{P}(\vec{\lambda})$. By the $\mathbb{G}_m$ action, this implies that the compactly supported cohomology $H^j_{\acute{e}t,c}(\mathrm{ev}_\infty^{-1}(p)_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$ away from the special orbifold points is isomorphic to $H^j_{\acute{e}t,c}(\mathrm{Poly}_1^{(\lambda_0 \cdot n, \dots, \lambda_N \cdot n)}{}_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$.

For the compactly supported cohomology of the special fibers at $[x_0 : \dots : x_N]$ with at least one of the $x_i = 0$, without loss of generality, we have the fiber at $[0 : 1 : \dots : 1]$ with $\mu_{\gcd(\lambda_1, \dots, \lambda_N)}$ stabilizer that has the stratification $\mathrm{ev}_\infty^{-1}([0 : 1 : \dots : 1]) = \left(\coprod_{k=1}^{\lambda_0 \cdot n} \mathbb{G}_m \times \mathrm{Poly}_1^{(\lambda_0 \cdot n - k, \lambda_1 \cdot n, \dots, \lambda_N \cdot n)} \coprod \mathrm{Poly}_1^{(\lambda_1 \cdot n, \lambda_2 \cdot n, \dots, \lambda_N \cdot n)} \right)$ as it must map to $[0 : 1 : \dots : 1]$ by ev_∞. Note that we need a strata $\mathrm{Poly}_1^{(\lambda_1 \cdot n, \lambda_2 \cdot n, \dots, \lambda_N \cdot n)}$ which accounts for the $(0, f_1(z), \dots, f_N(z))$ cases as well as a strata $\mathbb{G}_m \times \mathrm{Poly}_1^{(0, \lambda_1 \cdot n, \dots, \lambda_N \cdot n)}$ which accounts for the $(\mathbb{G}_m, f_1(z), \dots, f_N(z))$ cases both of which maps to $[0 : 1 : \dots : 1]$ by ev_∞. For $N = 1$ case, we have $\mathrm{Poly}_1^{(\lambda_1 \cdot n)} = \emptyset$. By applying the Grothendieck relation [Ekedahl, §1] on the stratification as in [HP, HP2], we have $[\mathbb{G}_m \times \mathrm{Poly}_1^{(\lambda_0\cdot n-1,\lambda_1\cdot n,\dots,\lambda_N\cdot n)}] + [\mathbb{G}_m \times \mathrm{Poly}_1^{(\lambda_0\cdot n-2,\lambda_1\cdot n,\dots,\lambda_N\cdot n)}] + \cdots + [\mathbb{G}_m \times \mathrm{Poly}_1^{(0,\lambda_1\cdot n,\dots,\lambda_N\cdot n)}] + [\mathrm{Poly}_1^{(\lambda_1\cdot n,\dots,\lambda_N\cdot n)}] = \mathbb{L}^{|\vec{\lambda}|n} - \mathbb{L}^{|\vec{\lambda}|n-N} = [\mathrm{Poly}_1^{(\lambda_0\cdot n,\dots,\lambda_N\cdot n)}]$ which implies that the special fibers have the same $\mathbb{F}_q$–point counts as $\mathrm{Poly}_1^{(\lambda_0\cdot n,\dots,\lambda_N\cdot n)}$ by [PS, Corollary 1.2.]. We apply the Proposition 2.11 to conclude that the compactly supported ℓ-adic étale cohomology of the special fibers are isomorphic to $H^j_{ét,c}(\mathrm{ev}_\infty^{-1}(p)_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell) \cong H^j_{ét,c}(\mathrm{Poly}_1^{(\lambda_0\cdot n,\dots,\lambda_N\cdot n)}{}_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)$ as well.

Flatness follows from the fibers having the constant dimension of $|\vec{\lambda}|n$. ■

Since the base $\mathcal{P}(\vec{\lambda})$ is simply-connected (c.f. [Noohi, Example 9.2]), the stalk of the locally constant sheaves with compact support $(\mathrm{R}^j\,\mathrm{ev}_{\infty !}\,\mathbb{Q}_\ell)_p$ along a flat morphism $\mathrm{ev}_\infty : \mathrm{Hom}_n(\mathbb{P}^1,\mathcal{P}(\vec{\lambda})) \to \mathcal{P}(\vec{\lambda})$ over a geometric point $p \in \mathcal{P}(\vec{\lambda})$ can be identified with $H^j_{ét,c}(\mathrm{ev}_\infty^{-1}(p)_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)$ by the base change for the cohomology with compact support as in [DIS, Section 6.3] (see also [Petersen, §2]). Working fiberwise, we have $\mathrm{R}^j\,\mathrm{ev}_{\infty !}\,\mathbb{Q}_\ell = H^j_{ét,c}(\mathrm{ev}_\infty^{-1}{}_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell) = H^j_{ét,c}(\mathrm{Poly}_1^{(\lambda_0\cdot n,\dots,\lambda_N\cdot n)}{}_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)$ by the Proposition 2.12 which implies that the ℓ-adic Leray spectral sequence has the following E_2 page through the Poincaré duality.

$$E_2^{i,j} = \begin{cases} H^i_{ét}(\mathcal{P}(\vec{\lambda})_{/\overline{\mathbb{F}}_q}; H^j_{ét}(\mathrm{Poly}_1^{(\lambda_0\cdot n,\dots,\lambda_N\cdot n)}{}_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)) & \text{if } i,j \geq 0 \\ 0 & \text{else} \end{cases}$$

and the spectral sequence converges to $H^{i+j}_{ét}(\mathrm{Hom}_n(\mathbb{P}^1,\mathcal{P}(\vec{\lambda}))_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)$.

Proposition 2.13. *Hom stack $\mathrm{Hom}_n(\mathbb{P}^1,\mathcal{P}(\vec{\lambda}))$ has the ℓ-adic étale cohomology $H^i_{ét}(\mathrm{Hom}_n(\mathbb{P}^1,\mathcal{P}(\vec{\lambda}))_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)$ that is one-dimensional in even degree for $i \in \{0,2,\dots,2N-4,2N-2\}$, in odd degree for $i \in \{2N+1,2N+3,\dots,4N-3,4N-1\}$ and vanishes for all other i showing the cohomology is independent of degree $n \in \mathbb{N}$. By the Poincaré duality, we have $H^j_{ét,c}(\mathrm{Hom}_n(\mathbb{P}^1,\mathcal{P}(\vec{\lambda}))_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)$ one-dimensional in $j = 2(|\vec{\lambda}|n+N)-(i)$ for every degree i above and vanishes for all other j.*

Proof. It suffices to show that the transgression differential is an isomorphism as we only have one possible non-zero differential $d_{2N}^{(2N,0)} : H^{2N}_{ét}\left(\mathcal{P}(\vec{\lambda})_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell\right) \to H^{2N-1}_{ét}\left(\mathrm{Poly}_1^{(\lambda_0\cdot n,\dots,\lambda_N\cdot n)}{}_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell\right)$ on the E_{2N} page. Adopting the computation of [KS, Proposition 2.2] to $\mathrm{Hom}_n(\mathbb{P}^1,\mathcal{P}(\vec{\lambda}))$, we see that $d_{2N}^{(2N,0)}([\mathcal{P}(\vec{\lambda})]) = \chi(\mathcal{P}(\vec{\lambda}))\cdot n\cdot[\mathrm{Hom}^*_n(\mathbb{P}^1,\mathcal{P}(\vec{\lambda}))]$ where $\chi(\mathcal{P}(\vec{\lambda})) = |\vec{\lambda}|$ is the orbifold Euler characteristic of smooth Deligne–Mumford stack $\mathcal{P}(\vec{\lambda})$ as in [AJY, Example 3.18], $[\mathcal{P}(\vec{\lambda})]$ and $[\mathrm{Hom}^*_n(\mathbb{P}^1,\mathcal{P}(\vec{\lambda}))]$ are the fundamental classes of $H^{2N}_{ét}\left(\mathcal{P}(\vec{\lambda})_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell\right)$ and $H^{2N-1}_{ét}\left(\mathrm{Poly}_1^{(\lambda_0\cdot n,\dots,\lambda_N\cdot n)}{}_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell\right)$ respectively. This implies that for a non-zero degree $n \in \mathbb{N}$ morphism, $d_{2N}^{(2N,0)} \neq 0$ is an isomorphism in $\mathbb{Q}_\ell$-coefficient leading to the middle degree cohomology equal to zero in degree $i = (2N-1,2N)$ while the rest of the ℓ-adic étale Betti numbers $\dim_{\mathbb{Q}_\ell}\left(H^i_{ét}(\mathrm{Hom}_n(\mathbb{P}^1,\mathcal{P}(a,b))_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)\right)$ are equal to one in degree $i \in \{0,2,\dots,2N-4,2N-2,2N+1,2N+3,\dots,4N-3,4N-1\}$ and vanishes for all other i.

■

As $H^i_{ét,c}(\mathrm{Hom}_n(\mathbb{P}^1,\mathcal{P}(\vec{\lambda}))_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)$ are all one dimensional, the trace of Frob_q on each of these $\mathbb{Q}_\ell$-vector spaces is just the corresponding eigenvalue w_i of Frob_q.

Plugging all of the above into the *Grothendieck-Lefschetz trace formula* (2.9) :

$$\#_q(\mathrm{Hom}_n(\mathbb{P}^1,\mathcal{P}(\vec{\lambda}))) = q^{|\vec{\lambda}|n+N} + \cdots + q^{|\vec{\lambda}|n+1} - q^{|\vec{\lambda}|n-1} - \cdots - q^{|\vec{\lambda}|n-N}$$

$$= w_{2(|\vec{\lambda}|n+N)} + \cdots + w_{2(|\vec{\lambda}|n+N)-(2N-2)} - w_{2(|\vec{\lambda}|n+N)-(2N+1)} - \cdots - w_{2(|\vec{\lambda}|n+N)-(4N-1)}$$

For a fixed N, one can easily work out the non-zero degree of the compactly supported ℓ-adic étale cohomology $H^i_{et,c}(\mathrm{Hom}_n(\mathbb{P}^1,\mathcal{P}(\vec{\lambda}))_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)$ as in Proposition 2.13 and assign $w_i = q^{|\vec{\lambda}|n+j}$ for each $j \in \{N,\dots,1,-1,\dots,-N\}$ in sequence which follows from an upper bound for the weights of the cohomology groups of stacks as in [Sun, Theorem 1.4.].

This finishes the proof of Theorem 1.1. ■

Acknowledgments. It is a pleasure to thank Dori Bejleri, Benson Farb, Changho Han, Aaron Landesman, Behrang Noohi, Hunter Spink, Craig Westerland and Jesse Wolfson for many helpful conversations. The author was supported by IBS-R003-D1, Institute for Basic Science in Korea.

References

[AJY] A. Alejandro, L. Johann and R. Yongbin, *Orbifolds and stringy topology*, Cambridge Tracts in Mathematics, **171**, Cambridge University Press, Cambridge, (2007).

[AOV] D. Abramovich, M. Olsson and A. Vistoli, *Tame stacks in positive characteristic*, Annales de l'Institut Fourier, **58**, No. 4, (2008): 1057–1091.

[Behrend] K. A. Behrend, *The Lefschetz trace formula for algebraic stacks*, Inventiones mathematicae, **112**, No. 1, (1993): 127–149.

[Behrens] M. Behrens, *A modular description of the $K(2)$-local sphere at the prime 3*, Topology, **45**, No. 2, (2006): 343–402.

[CCMM] F. R. Cohen, R. L. Cohen, B. M. Mann, and R. J. Milgram, *The topology of rational functions and divisors of surfaces*, Acta Mathematica, **166**, No. 1, (1991): 163–221.

[Conrad] B. Conrad, *Arithmetic moduli of generalized elliptic curves*, Journal of the Institute of Mathematics of Jussieu, **6**, No. 2, (2007): 209–278.

[DIS] V. I. Danilov, V. A. Iskovskikh and I.R. Shafarevich, *Algebraic Geometry II : Cohomology of Algebraic Varieties, Algebraic Surfaces*, Encyclopaedia of Mathematical Sciences, **35**, Springer–Verlag Berlin Heidelberg (1996).

[DR] P. Deligne and M. Rapoport, *Les Schémas de Modules de Courbes Elliptiques*, Modular Functions of One Variable II. Lecture Notes in Mathematics, **349**, (1973): 143–316.

[Ekedahl] T. Ekedahl, *The Grothendieck group of algebraic stacks*, arXiv:0903.3143, (2009).

[Fedorchuk] M. Fedorchuk, *Moduli spaces of hyperelliptic curves with A and D singularities*, Mathematische Zeitschrift, **276**, No. 1-2, (2014): 299–328.

[FW] B. Farb and J. Wolfson, *Topology and arithmetic of resultants, I*, New York Journal of Mathematics, **22**, (2016): 801–821.

[Hassett] B. Hassett, *Classical and minimal models of the moduli space of curves of genus two*, Geometric Methods in Algebra and Number Theory, Progress in Mathematics, **235**, Birkhäuser Boston (2005): 169—192.

[HMe] M. Hill and L. Meier, *The C_2-spectrum $Tmf_1(3)$ and its invertible modules*, Algebraic & Geometric Topology, **17**, No. 4, (2017): 1953–2011.

[HP] C. Han and J. Park, *Arithmetic of the moduli of semistable elliptic surfaces*, Mathematische Annalen, **375**, No. 3–4, (2019): 1745–1760.

[HP2] C. Han and J. Park, *Enumerating algebraic curves and Abelian varieties over global fields with lower order terms*, arXiv:2002.00563, (2020).

[Kawasaki] T. Kawasaki, *Cohomology of twisted projective spaces and lens complexes*, Mathematische Annalen, **206**, No. 3, (1973): 243–248.

[Kim] M. Kim, *Weights in cohomology groups arising from hyperplane arrangements*, Proceedings of the American Mathematical Society, **120**, No. 3, (1994): 697–703.

[KM] N. Katz and B. Mazur, *Arithmetic Moduli of Elliptic Curves*, Annals of Mathematics Studies, **108**, Princeton University Press (1985).

[KS] S. Kallel and P. Salvatore, *Rational maps and string topology*, Geometry & Topology, **10**, No. 3, (2006): 1579–1606.

[Lehrer] G. I. Lehrer, *The ℓ-adic cohomology of hyperplane complements*, Bulletin of the London Mathematical Society, **24**, No. 1, (1992): 76–82.

[Liu] Q. Liu, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics, **6**, Oxford University Press (2002).

[LO] Y. Laszlo and M. Olsson *The six operations for sheaves on Artin stacks II: Adic coefficients*, Publications Mathématiques de l'I.H.É.S., **107**, (2008): 169–210.

[LP] Y. Lekili and A. Polishchuk, *A modular compactification of $\mathcal{M}_{1,n}$ from $\mathcal{A}_\infty$-structures*, Journal für die reine und angewandte Mathematik, **755**, (2019): 151–189.

[Meier] L. Meier, *Additive decompositions for rings of modular forms*, arXiv:1710.03461, (2017).

[Niles] A. Niles, *Moduli of elliptic curves via twisted stable maps*, Algebra and Number Theory, **7**, No. 9, (2013): 2141–2202.

[Noohi] B. Noohi, *Fundamental groups of algebraic stacks*, Journal of the Institute of Mathematics of Jussieu, **3**, No. 1, (2004): 69–103.

[Olsson] M. Olsson, *$\underline{Hom}$-stacks and restriction of scalars*, Duke Mathematical Journal, **134**, No. 1, (2006): 139–164.

[Olsson2] M. Olsson, *Algebraic Spaces and Stacks*, Colloquium Publications, **62**, American Mathematical Society (2016).

[Petersen] D. Petersen, *The structure of the tautological ring in genus one*, Duke Mathematical Journal, **163**, No. 4, (2014): 777–793.

[PS] J. Park and H. Spink, *Motive of the moduli stack of rational curves on a weighted projective stack*, Research in the Mathematical Sciences, Special Issue of PIMS 2019 Workshop on Arithmetic Topology, **8**, No. 1, (2021).

[Segal] G. Segal, *The topology of spaces of rational functions*, Acta Mathematica, **143**, (1979): 39–72.

[Shapiro] B. Z. Shapiro, *The mixed Hodge structure of the complement to an arbitrary arrangement of affine complex hyperplanes is pure*, Proceedings of the American Mathematical Society, **117**, No. 4, (1993): 931–933.

[Silverman] J. H. Silverman, *The space of rational maps on $\mathbb{P}^1$*, Duke Mathematical Journal, **94**, No. 1, (1998): 41–77.

[Smyth] D. I. Smyth, *Modular compactifications of the space of pointed elliptic curves I*, Compositio Mathematica, **147**, No. 3, (2011): 877–913.

[Smyth2] D. I. Smyth, *Modular compactifications of the space of pointed elliptic curves II*, Compositio Mathematica, **147**, No. 6, (2011): 1843–1884.

[Stojanoska] V. Stojanoska, *Duality for topological modular forms*, Documenta Mathematica, **17**, (2012): 271–311.

[Sun] S. Sun, *L-series of Artin stacks over finite fields*, Algebra & Number Theory, **6**, No. 1, (2012): 47–122.

Center for Geometry and Physics, Institute for Basic Science (IBS), Pohang 37673, Korea
Email address: `junepark@ibs.re.kr`