A Counting elliptic curves with prescribed structures

In this appendix, we determine the sharp estimate on the number of elliptic curves over $\mathbb{P}^1_{\mathbb{F}_q}$ with prescribed level structures or multiple marked points by extending the method as in [HP, Theorem 3] regarding the number of semistable elliptic curves over $\mathbb{P}^1_{\mathbb{F}_q}$.

Specifically, we explicitly estimate the sharp bound on the number of elliptic curves over global function fields $\mathbb{F}_q(t)$ with level structures $[\Gamma_1(n)]$ for $2 \le n \le 4$ or $[\Gamma(2)]$. Recall that a level structure $[\Gamma_1(n)]$ on an elliptic curve E is a choice of point $P \in E$ of exact order n in the smooth part of E such that over every geometric point of the base scheme every irreducible component of E contains a multiple of P (see [KM, §1.4]). And a level structure $[\Gamma(2)]$ on an elliptic curve E is a choice of isomorphism $\phi : \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to E(2)$ where $E(2)$ is the scheme of 2-torsion Weierstrass points (i.e., kernel of the multiplication-by-2 map $[2] : E \to E$) (see [DR, II.1.18 & IV.2.3]).

Additionally, we consider curves of arithmetic genus one over $\mathbb{F}_q(t)$ with m-marked rational points for $2 \le m \le 5$ by acquiring sharp estimate on the number of $(m-1)$-stable m-marked curves of arithmetic genus one formulated originally by the works of [Smyth, Smyth2].

To estimate the number of certain elliptic curves over global function fields $\mathbb{F}_q(t)$ with level structures $[\Gamma(n)]$ or $[\Gamma_1(n)]$, we need to first extend the notion of (nonsingular) elliptic curves (semistable in the case of [HP]) that admits desired level structures. By the work of Deligne and Rapoport [DR] (summarized in [Niles, §2]), we consider the generalized elliptic curves over $\mathbb{P}^1_K$ with $[\Gamma]$–structures (where Γ is $\Gamma(n)$ or $\Gamma_1(n)$) over a field K (focusing on $K = \mathbb{F}_q$). Roughly, a generalized elliptic curve X over $\mathbb{P}^1_K$ can be thought of as a flat family of semistable elliptic curves admitting a group structure, such that a finite group scheme $\mathcal{G} \to \mathbb{P}^1_K$ (determined by Γ) embeds into X and its image meets every irreducible component of every geometric fibers of X. Again, we only consider the *non-isotrivial* generalized elliptic curves. If X is as above, then Δ is the discriminant of a generalized elliptic curve and if $K = \mathbb{F}_q$, then $0 < ht(\Delta) := q^{\deg \Delta}$.

Now, define $\mathcal{Z}^{[\Gamma]}_{\mathbb{F}_q(t)}(\mathcal{B})$ as follows:

$$\mathcal{Z}^{[\Gamma]}_{\mathbb{F}_q(t)}(\mathcal{B}) := |\{\text{Generalized elliptic curves over } \mathbb{P}^1_{\mathbb{F}_q} \text{ with } [\Gamma]\text{–structures and } 0 < ht(\Delta) \le \mathcal{B}\}|$$

Then, we acquire the following descriptions of $\mathcal{Z}^{[\Gamma]}_{\mathbb{F}_q(t)}(\mathcal{B})$:

Theorem A.1 (Sharp estimate of $\mathcal{Z}^{[\Gamma]}_{\mathbb{F}_q(t)}(\mathcal{B})$)**.** *The function $\mathcal{Z}^{[\Gamma]}_{\mathbb{F}_q(t)}(\mathcal{B})$, which counts the number of generalized elliptic curves with $[\Gamma]$-structures over $\mathbb{P}^1_{\mathbb{F}_q}$ with char$(\mathbb{F}_q) \neq 2$ $\big($char$(\mathbb{F}_q) \neq 3$ for $\mathcal{Z}^{[\Gamma_1(3)]}_{\mathbb{F}_q(t)}(\mathcal{B})\big)$ ordered by $0 < ht(\Delta) = q^{12n} \le \mathcal{B}$, satisfies:*

$$\mathcal{Z}^{[\Gamma_1(2)]}_{\mathbb{F}_q(t)}(\mathcal{B}) \le 2 \cdot \frac{(q^7 - q^5)}{(q^6 - 1)} \cdot \Big(\mathcal{B}^{\frac{1}{2}} - 1\Big)$$

$$\mathcal{Z}^{[\Gamma_1(3)]}_{\mathbb{F}_q(t)}(\mathcal{B}) \le \frac{(q^5 - q^3)}{(q^4 - 1)} \cdot \Big(\mathcal{B}^{\frac{1}{3}} - 1\Big)$$

$$\mathcal{Z}^{[\Gamma_1(4)]}_{\mathbb{F}_q(t)}(\mathcal{B}) \le \frac{(q^4-q^2)}{(q^3-1)} \cdot \left(\mathcal{B}^{\frac{1}{4}} - 1\right)$$

$$\mathcal{Z}^{[\Gamma(2)]}_{\mathbb{F}_q(t)}(\mathcal{B}) \le 2 \cdot \frac{(q^5-q^3)}{(q^4-1)} \cdot \left(\mathcal{B}^{\frac{1}{3}} - 1\right)$$

which is an equality when $\mathcal{B} = q^{12n}$ with $n \in \mathbb{Z}_{\ge 1}$ implying that the upper bound is a sharp estimate, i.e., the upper bound is equal to the function at infinitely many values of $\mathcal{B} \in \mathbb{N}$.

Proof. The proof is at the end of §A.1. ■

The main leading term of the acquired sharp estimates over $\mathbb{F}_q(t)$ matches the analogous asymptotic counts ordered by bounded naïve height of underlying elliptic curves over $\mathbb{Q}$ by Harron and Snowden in [HS, Theorem 1.2] (see also [Duke, Grant]). The lower order term over global function fields $\mathbb{F}_q(t)$ being a constant is new and would be interesting to prove or disprove over a number field $\mathbb{Q}$. It also remains to count the remaining ten cases (classified by the fundamental [Mazur, Theorem 8]) of the torsion subgroups with $|G| > 4$ over $\mathbb{F}_q(t)$ by bounded discriminant height and compare with analogous counting over $\mathbb{Q}$.

Now, let's consider instead elliptic curves with m-marked rational points. To count the number of certain curves of arithmetic genus one over global function fields $\mathbb{F}_q(t)$ with m-markings, we need to again extend the notion of (nonsingular) elliptic curves that admits desired m-markings. Here, we consider the $(m-1)$-stable m-marked curves of arithmetic genus one (defined by Smyth in [Smyth, §1.1] for characteristic $\neq 2, 3$, extended to lower characteristic with mild conditions by [LP, Definition 1.5.3]), see Definition A.9 for a precise definition. Note that if $\mathrm{char}(\mathbb{F}_q) > 3$ and $m = 1$, then 0-stable 1-marked curves are exactly stable elliptic curves as in [DM]. We now consider the following definition:

Definition A.2. Fix an integral reduced K-scheme B, where K is a field. Then a non-isotrivial flat morphism $\pi : X \to B$ is a m-marked $(m-1)$-stable genus one fibration over B if any fiber of π is a $(m-1)$-stable m-marked curves of arithmetic genus one.

Observe that if $\mathrm{char}(K) = 0$ or > 3, then a m-marked $(m-1)$-stable genus one fibration $X \to \mathbb{P}^1_K$ has a discriminant $\Delta \subset \mathbb{P}^1_K$, and if $K = \mathbb{F}_q$, then $0 < ht(\Delta) := q^{\deg \Delta}$.

Now, define $\mathcal{Z}^m_{\mathbb{F}_q(t)}(\mathcal{B})$ as follows:

$$\mathcal{Z}^m_{\mathbb{F}_q(t)}(\mathcal{B}) := |\{m\text{-marked } (m-1)\text{-stable genus one fibrations over } \mathbb{P}^1_{\mathbb{F}_q} \text{ with } 0 < ht(\Delta) \le \mathcal{B}\}|$$

Note that when $m = 1$, $\mathcal{Z}^1_{\mathbb{F}_q(t)}(\mathcal{B})$ counts the stable elliptic fibrations, which is described in [HP, Theorem 3] as $\mathcal{Z}_{\mathbb{F}_q(t)}(\mathcal{B})$ (by identifying stable elliptic fibrations with nonsingular semistable elliptic surfaces, see [HP, Proposition 11]). When $2 \le m \le 5$, we acquire the following sharp estimate of $\mathcal{Z}^m_{\mathbb{F}_q(t)}(\mathcal{B})$:

Theorem A.3 (Sharp estimate of $\mathcal{Z}^m_{\mathbb{F}_q(t)}(\mathcal{B})$). *If $char(\mathbb{F}_q) \neq 2, 3$, then the function $\mathcal{Z}^m_{\mathbb{F}_q(t)}(\mathcal{B})$, which counts the number of m-marked $(m-1)$-stable genus one fibration over*

$\mathbb{P}^1_{\mathbb{F}_q}$ *for* $2 \le m \le 5$ *ordered by* $0 < ht(\Delta) = q^{12n} \le \mathcal{B}$*, satisfies:*

$$\mathcal{Z}^2_{\mathbb{F}_q(t)}(\mathcal{B}) \le \frac{(q^{11} + q^{10} - q^8 - q^7)}{(q^9 - 1)} \cdot (\mathcal{B}^{\frac{3}{4}} - 1) + \frac{(q^7 - q^5)}{(q^6 - 1)} \cdot (\mathcal{B}^{\frac{1}{2}} - 1)$$

$$\mathcal{Z}^3_{\mathbb{F}_q(t)}(\mathcal{B}) \le \frac{(q^{11} + q^{10} + q^9 - q^7 - q^6 - q^5)}{(q^8 - 1)} \cdot (\mathcal{B}^{\frac{2}{3}} - 1) + \frac{(q^5 - q^3)}{(q^4 - 1)} \cdot (\mathcal{B}^{\frac{1}{3}} - 1)$$

$$\mathcal{Z}^4_{\mathbb{F}_q(t)}(\mathcal{B}) \le \frac{(q^{11} + q^{10} + q^9 + q^8 - q^6 - q^5 - q^4 - q^3)}{(q^7 - 1)} \cdot (\mathcal{B}^{\frac{7}{12}} - 1) + \frac{(q^5 - q^3)}{(q^4 - 1)} \cdot (\mathcal{B}^{\frac{1}{3}} - 1)$$

$$\mathcal{Z}^5_{\mathbb{F}_q(t)}(\mathcal{B}) \le \frac{(q^{11} + q^{10} + q^9 + q^8 + q^7 - q^5 - q^4 - q^3 - q^2 - q^1)}{(q^6 - 1)} \cdot (\mathcal{B}^{\frac{1}{2}} - 1)$$

which is an equality when $\mathcal{B} = q^{12n}$ *with* $n \in \mathbb{Z}_{\ge 1}$ *implying that the upper bound is a sharp estimate, i.e., the upper bound is equal to the function at infinitely many values of* $\mathcal{B} \in \mathbb{N}$*.*

Proof. The proof is at the end of §A.2. ∎

A.1 Arithmetic of the moduli of generalized elliptic curves over $\mathbb{P}^1$ with level structures

The essential geometrical idea in acquiring the sharp estimate is to consider the moduli stack of rational curves on a compactified modular curve as in [HP]. The various compactified modular curves $\overline{\mathcal{M}}_{1,1}[\Gamma]$ are isomorphic to the weighted projective stacks $\mathcal{P}(a, b)$.

Proposition A.4. *The moduli stack* $\overline{\mathcal{M}}_{1,1}[\Gamma]$ *of generalized elliptic curves with* $[\Gamma]$*-structures is isomorphic to the following when over a field* K*:*

1. *if* $\mathrm{char}(K) \ne 2$*, the tame Deligne–Mumford moduli stack of generalized elliptic curves with* $[\Gamma_1(2)]$*-structures is isomorphic to*

   $$(\overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])_K \cong [(\mathrm{Spec}\ K[a_2, a_4] - (0,0))/\mathbb{G}_m] = \mathcal{P}_K(2,4),$$

2. *if* $\mathrm{char}(K) \ne 3$*, the tame Deligne–Mumford moduli stack of generalized elliptic curves with* $[\Gamma_1(3)]$*-structures is isomorphic to*

   $$(\overline{\mathcal{M}}_{1,1}[\Gamma_1(3)])_K \cong [(\mathrm{Spec}\ K[a_1, a_3] - (0,0))/\mathbb{G}_m] = \mathcal{P}_K(1,3),$$

3. *if* $\mathrm{char}(K) \ne 2$*, the tame Deligne–Mumford moduli stack of generalized elliptic curves with* $[\Gamma_1(4)]$*-structures is isomorphic to*

   $$(\overline{\mathcal{M}}_{1,1}[\Gamma_1(4)])_K \cong [(\mathrm{Spec}\ K[a_1, a_2] - (0,0))/\mathbb{G}_m] = \mathcal{P}_K(1,2),$$

4. *if* $\mathrm{char}(K) \ne 2$*, the tame Deligne–Mumford moduli stack of generalized elliptic curves with* $[\Gamma(2)]$*-structures is isomorphic to*

   $$(\overline{\mathcal{M}}_{1,1}[\Gamma(2)])_K \cong [(\mathrm{Spec}\ K[a_2, a_2] - (0,0))/\mathbb{G}_m] = \mathcal{P}_K(2,2),$$

where $\lambda \cdot a_i = \lambda^i a_i$ for $\lambda \in \mathbb{G}_m$ and $i = 1, 2, 3, 4$. Thus, the a_i's have degree i respectively. Moreover, the discriminant divisors of $(\overline{\mathcal{M}}_{1,1}[\Gamma])_K \cong \mathcal{P}_K(i, j)$ as above have degree 12.

Proof. The moduli stack $\overline{\mathcal{M}}_{1,1}[\Gamma_1(2)]$ of generalized elliptic curves with $[\Gamma_1(2)]$-level structure has an isomorphism $\overline{\mathcal{M}}_{1,1}[\Gamma_1(2)] \cong \mathcal{P}(2, 4)$ as in [Behrens, §1.3] through the universal equation

$$Y^2Z = X^3 + a_2X^2Z + a_4XZ^2\,,$$

over $\mathrm{Spec}(\mathbb{Z}[1/2])$. And the moduli stack $\overline{\mathcal{M}}_{1,1}[\Gamma_1(3)]$ of generalized elliptic curves with $[\Gamma_1(3)]$-level structure has an isomorphism $\overline{\mathcal{M}}_{1,1}[\Gamma_1(3)] \cong \mathcal{P}(1, 3)$ as in [HMe, Proposition 4.5] through the universal equation

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3\,,$$

over $\mathrm{Spec}(\mathbb{Z}[1/3])$. And the moduli stack $\overline{\mathcal{M}}_{1,1}[\Gamma_1(4)]$ of generalized elliptic curves with $[\Gamma_1(4)]$-level structure has an isomorphism $\overline{\mathcal{M}}_{1,1}[\Gamma_1(4)] \cong \mathcal{P}(1, 2)$ as in [Meier, Examples 2.1] through the universal equation

$$Y^2Z + a_1XYZ + a_1a_2YZ^2 = X^3 + a_2X^2Z\,,$$

over $\mathrm{Spec}(\mathbb{Z}[1/2])$. And the moduli stack $\overline{\mathcal{M}}_{1,1}[\Gamma(2)]$ of generalized elliptic curves with $[\Gamma(2)]$-level structure has an isomorphism $\overline{\mathcal{M}}_{1,1}[\Gamma(2)] \cong \mathcal{P}(2, 2)$ as in [Stojanoska, Proposition 7.1] through the universal equation

$$Y^2Z = X^3 + (\lambda_1 + \lambda_2)X^2Z + \lambda_1\lambda_2XZ^2\,,$$

over $\mathrm{Spec}(\mathbb{Z}[1/2])$ where the degree of each λ_i is 2.

By Proposition 2.2, the weighted projective stacks are tame Deligne–Mumford as well.

For the degree of the discriminant, it suffices to find the weight of the $\mathbb{G}_m$-action. First, the four papers cited above explicitly construct universal families of elliptic curves over the schematic covers $(\mathrm{Spec}\, K[a_i, a_j] - (0, 0)) \to \mathcal{P}_K(i, j)$ of the corresponding moduli stacks. The explicit defining equation of the respective universal family implies that the $\lambda \in \mathbb{G}_m$ also acts on the discriminant of the universal family by multiplying λ^{12}. Therefore, the discriminant has degree 12. ■

We now consider the moduli stack $\mathcal{L}_{1,12n}^{[\Gamma]} := \mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma])$ of generalized elliptic curves over $\mathbb{P}^1$ with $[\Gamma]$-structures.

Proposition A.5. *Assume* $\mathrm{char}(K) = 0$ *or* $\neq 2$ *for* $[\Gamma] = [\Gamma_1(2)], [\Gamma_1(4)], [\Gamma(2)]$, *and* $\mathrm{char}(K) \neq 3$ *for* $[\Gamma] = [\Gamma_1(3)]$. *Then, the moduli stack* $\mathcal{L}_{1,12n}^{[\Gamma]}$ *of generalized elliptic curves over* $\mathbb{P}^1$ *with discriminant degree* $12n > 0$ *and* $[\Gamma]$*-structures is the tame Deligne–Mumford stack* $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma])$ *parameterizing the* K*-morphisms* $f : \mathbb{P}^1 \to \overline{\mathcal{M}}_{1,1}[\Gamma]$ *such that* $f^*\mathcal{O}_{\overline{\mathcal{M}}_{1,1}[\Gamma]}(1) \cong \mathcal{O}_{\mathbb{P}^1}(n)$.

Proof. Without the loss of generality, we prove the $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])$ case over a field K with $\mathrm{char}(K) \neq 2$. The proof for the other cases are analogous. By the definition of the universal family p, any generalized elliptic curves $\pi : Y \to \mathbb{P}^1$ with $[\Gamma_1(2)]$-structures comes from a morphism $f : \mathbb{P}^1 \to \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)]$ and vice versa. As this correspondence also works in families, the moduli stack of generalized elliptic curves over $\mathbb{P}^1$ with $[\Gamma_1(2)]$-structures is isomorphic to $\mathrm{Hom}(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])$.

Since the discriminant degree of f is $12 \deg f^*\mathcal{O}_{\overline{\mathcal{M}}_{1,1}[\Gamma_1(2)]}(1)$ by Proposition A.4, the substack $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])$ parametrizing such f's with $\deg f^*\mathcal{O}_{\overline{\mathcal{M}}_{1,1}[\Gamma_1(2)]}(1) = n$ is the desired moduli stack. Since $\deg f^*\mathcal{O}_{\overline{\mathcal{M}}_{1,1}[\Gamma_1(2)]}(1) = n$ is an open condition, $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])$ is an open substack of $\mathrm{Hom}(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])$, which is tame Deligne–Mumford by [HP2, Proposition 3.6] as $\overline{\mathcal{M}}_{1,1}[\Gamma_1(2)]$ itself is tame Deligne–Mumford by Proposition A.4. This shows that $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])$ satisfies the desired properties as well. ■

We recall the motives & weighted point counts over finite fields of $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a, b))$ worked out in [PS, Corollary 1.2] which is an extension of [HP, Theorem 1].

Corollary A.6 (Corollary 1.2 of [PS]). *If* $\mathrm{char}(K) \nmid a, b$, *then*

$$[\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a, b))] = \mathbb{L}^{(a+b)n+1} - \mathbb{L}^{(a+b)n-1} \in K_0(\mathrm{Stck}_{/K}),$$

and if $\mathrm{char}(\mathbb{F}_q) \nmid a, b$, *then we have the weighted* $\mathbb{F}_q$*-point count*

$$|\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a, b))(\mathbb{F}_q)| := \sum_{x \in \mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))(\mathbb{F}_q)} \frac{1}{|\mathrm{Aut}(x)|} = q^{(a+b)n+1} - q^{(a+b)n-1}.$$

We now acquire the exact number $|\mathcal{L}^{[\Gamma]}_{1,12n}(\mathbb{F}_q)/\sim|$ of $\mathbb{F}_q$–isomorphism classes of $\mathbb{F}_q$–points (i.e., the non–weighted point count) of the moduli stack $\mathcal{L}^{[\Gamma]}_{1,12n} \cong \mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma])$ of generalized elliptic curves over $\mathbb{P}^1$ with discriminant degree $12n > 0$ and $[\Gamma]$-structures.

Proposition A.7. *If* $\mathrm{char}(\mathbb{F}_q) \neq 2$, *then*

$$|\mathcal{L}^{[\Gamma_1(2)]}_{1,12n}(\mathbb{F}_q)/\sim| = 2 \cdot \#_q \left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(2, 4))\right) = 2(q^{6n+1} - q^{6n-1})$$

$$|\mathcal{L}^{[\Gamma_1(4)]}_{1,12n}(\mathbb{F}_q)/\sim| = \#_q \left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(1, 2))\right) = q^{3n+1} - q^{3n-1}$$

$$|\mathcal{L}^{[\Gamma(2)]}_{1,12n}(\mathbb{F}_q)/\sim| = 2 \cdot \#_q \left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(2, 2))\right) = 2(q^{4n+1} - q^{4n-1})$$

If $\mathrm{char}(\mathbb{F}_q) \neq 3$, *then*

$$|\mathcal{L}^{[\Gamma_1(3)]}_{1,12n}(\mathbb{F}_q)/\sim| = \#_q \left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(1, 3))\right) = q^{4n+1} - q^{4n-1}$$

Proof. Fix $n \in \mathbb{Z}_{\geq 1}$. Since any $\varphi_g \in \mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a, b))$ is surjective, the generic stabilizer group $\mu_{\mathrm{gcd}(a,b)}$ of $\mathcal{P}(a, b)$ is the automorphism group of φ_g. Using the identification from Proposition A.5 and the weighted point counts of Hom stacks as in Corollary A.6 gives the desired formula as

$$|\mathcal{L}^{[\Gamma]}_{1,12n}(\mathbb{F}_q)/\sim| = |\mu_{\mathrm{gcd}(a,b)}| \cdot (q^{(a+b)n+1} - q^{(a+b)n-1})$$

where the factor of 2 comes from the hyperelliptic involution when $\mu_{\mathrm{gcd}(a,b)} = \mu_2$. ■

Remark A.8. For weighted projective lines $\mathcal{P}(a,b)$ as in the cases of $\mathcal{L}^{[\Gamma]}_{1,12n}$, the inertia stack of the relevant Hom stack $\{\mathcal{I}\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))\right)\}$ is a sum of $\left\{\mathrm{Hom}_n(\mathbb{P}^1_{\kappa(g)}, \mathcal{P}_{\kappa(g)}(a,b))\right\}$ for each closed point $g \in \mathbb{G}_m$ with $\mathrm{ord}(g) \mid \gcd(a,b)$, as the only possible generic stabilizer of positive dimensional substacks of $\mathcal{P}(a,b)$. On the other hand, the terms with division function $\delta(r, q-1)$ do not occur in $|\mathcal{L}^{[\Gamma]}_{1,12n}(\mathbb{F}_q)/\sim|$ as the characteristic condition required to identify $\overline{\mathcal{M}}_{1,1}[\Gamma]$ as a weighted projective line implies that $\gcd(a,b) \mid q-1$. See [HP2, §4.3] for more details.

We now finally prove the Theorem 1.2 using the above arithmetic invariants as follows:

Proof of Theorem 1.2. Without the loss of the generality, we prove the $[\Gamma_1(2)]$–structures case over $\mathrm{char}(\mathbb{F}_q) \neq 2$. The proof for the other cases are analogous. By Proposition A.5 and Proposition A.7, we know the number of $\mathbb{F}_q$-isomorphism classes of generalized elliptic curves of discriminant degree $12n$ with $[\Gamma_1(2)]$-structures over $\mathbb{P}^1_{\mathbb{F}_q}$ is $|\mathcal{L}^{[\Gamma_1(2)]}_{1,12n}(\mathbb{F}_q)/\sim| = 2\cdot(q^{6n+1} - q^{6n-1})$. Using this, we can explicitly compute the sharp bound on $\mathcal{Z}^{[\Gamma_1(2)]}_{\mathbb{F}_q(t)}(\mathcal{B})$ as the following,

$$\begin{aligned}
\mathcal{Z}^{[\Gamma_1(2)]}_{\mathbb{F}_q(t)}(\mathcal{B}) &= \sum_{n=1}^{\left\lfloor \frac{log_q \mathcal{B}}{12} \right\rfloor} |\mathcal{L}^{[\Gamma_1(2)]}_{1,12n}(\mathbb{F}_q)/\sim| = \sum_{n=1}^{\left\lfloor \frac{log_q \mathcal{B}}{12} \right\rfloor} 2\cdot(q^{6n+1} - q^{6n-1}) \\
&= 2\cdot(q^1 - q^{-1}) \sum_{n=1}^{\left\lfloor \frac{log_q \mathcal{B}}{12} \right\rfloor} q^{6n} \le 2\cdot(q^1 - q^{-1})\left(q^6 + \cdots + q^{6\cdot(\frac{log_q \mathcal{B}}{12})}\right) \\
&= 2\cdot(q^1 - q^{-1})\frac{q^6(\mathcal{B}^{\frac{1}{2}} - 1)}{(q^6-1)} = 2\cdot\frac{(q^7 - q^5)}{(q^6-1)}\cdot(\mathcal{B}^{\frac{1}{2}} - 1)
\end{aligned}$$

On the second line of the equations above, inequality becomes an equality if and only if $n := \frac{log_q \mathcal{B}}{12} \in \mathbb{N}$, i.e., $\mathcal{B} = q^{12n}$ with $n \in \mathbb{Z}_{\geq 1}$. This implies that the acquired upper bound on $\mathcal{Z}^{[\Gamma_1(2)]}_{\mathbb{F}_q(t)}(\mathcal{B})$ is a sharp estimate, i.e., the upper bound is equal to the function at infinitely many values of $\mathcal{B} \in \mathbb{N}$. ■

A.2 Arithmetic of the moduli of m-marked genus one fibrations over $\mathbb{P}^1$

We proceed to estimate the sharp bound on the number of m-marked $(m-1)$-stable genus one fibrations over $\mathbb{P}^1_{\mathbb{F}_q}$ for $2 \le m \le 5$. First, we state the definition of m-marked $(m-1)$-stability from [LP, Definition 1.5.3], which is a modification of the Deligne–Mumford stability [DM]:

Definition A.9. Let K be a field and m be a positive integer. Then, a tuple $(C, p_1, \ldots, p_m)$, of a geometrically connected, geometrically reduced, and proper K-curve C of arithmetic genus one with m distinct K-rational points p_i in the smooth locus of C, is a $(m-1)$-stable m-marked curve of arithmetic genus one if the curve $C_{\overline{K}} := C \times_K \overline{K}$ and the divisor $\Sigma := \{p_1, \ldots, p_m\}$ satisfy the following properties, where $\overline{K}$ is the algebraic closure of K:

1. $C_{\overline{K}}$ has only nodes and elliptic u-fold points as singularities (see below), where $u < m$,

2. $C_{\overline{K}}$ has no disconnecting nodes, and

3. every irreducible component of $C_{\overline{K}}$ contains at least one marked point.

Remark A.10. A singular point of a curve over $\overline{K}$ is an elliptic u-fold singular point if it is Gorenstein and étale locally isomorphic to a union of u general lines in $\mathbb{P}^{u-1}_{\overline{K}}$ passing through a common point.

Note that the name "$(m-1)$-stability" comes from [Smyth, §1.1], which is defined when $\mathrm{char}(K) \neq 2, 3$. By [LP, Proposition 1.5.4], the above definition (by [LP, Definition 1.5.3]) coincides with that of Smyth when $\mathrm{char}(K) \neq 2, 3$, hence we adapt Smyth's naming convention on Lekili and Polishchuk's definition. Regardless, we focus on the case when $\mathrm{char}(K) \neq 2, 3$, so that the moduli stack of such curves behaves reasonably.

By [Smyth, Theorem 3.8], we are able to formulate the moduli stack of $(m-1)$-stable m-marked curves of arithmetic genus one over any field of characteristic $\neq 2, 3$:

Theorem A.11. *There exists a proper irreducible Deligne–Mumford moduli stack* $\overline{\mathcal{M}}_{1,m}(m-1)$ *of* $(m-1)$*-stable* m*-marked curves arithmetic genus one over* $\mathrm{Spec}(\mathbb{Z}[1/6])$

Note that when $m = 1$, $\overline{\mathcal{M}}_{1,1}(0) \cong \overline{\mathcal{M}}_{1,1}$ is the Deligne–Mumford moduli stack of stable elliptic curves.

In fact, the construction of $\overline{\mathcal{M}}_{1,m}(m-1)$ extends to $\mathrm{Spec}\,\mathbb{Z}$ by [LP, Theorem 1.5.7] (called $\overline{\mathcal{M}}^{\infty}_{1,m}$ in loc.cit.) as an algebraic stack, which is proper over $\mathrm{Spec}\,\mathbb{Z}[1/N]$ where N depends on m:

- if $m \geq 3$, then $N = 1$,
- if $m = 2$, then $N = 2$, and
- if $m = 1$, then $N = 6$.

However, even with those assumptions above, $\overline{\mathcal{M}}_{1,m}(m-1)$ is not necessarily Deligne–Mumford. Nevertheless, by [LP, Theorem 1.5.7.], we obtain the explicit descriptions of $\overline{\mathcal{M}}_{1,m}(m-1)$:

Proposition A.12. *The moduli stack* $\overline{\mathcal{M}}_{1,m}(m-1)$ *of* m*-marked* $(m-1)$*-stable curves of arithmetic genus one for* $2 \leq m \leq 5$ *is isomorphic to the following, for a field* K*:*

1. *if* $\mathrm{char}(K) \neq 2, 3$*, the tame Deligne–Mumford moduli stack of 2-marked 1-stable curves of arithmetic genus one is isomorphic to*

$$(\overline{\mathcal{M}}_{1,2}(1))_K \cong [(\mathrm{Spec}\ K[a_2, a_3, a_4] - 0)/\mathbb{G}_m] = \mathcal{P}_K(2, 3, 4),$$

2. *if* $\mathrm{char}(K) \neq 2, 3$*, the tame Deligne–Mumford moduli stack of 3-marked 2-stable curves of arithmetic genus one is isomorphic to*

$$(\overline{\mathcal{M}}_{1,3}(2))_K \cong [(\mathrm{Spec}\ K[a_1, a_2, a_2, a_3] - 0)/\mathbb{G}_m] = \mathcal{P}_K(1, 2, 2, 3),$$

3. *if* $\mathrm{char}(K) \neq 2$*, the tame Deligne–Mumford moduli stack of 4-marked 3-stable curves of arithmetic genus one is isomorphic to*

$$(\overline{\mathcal{M}}_{1,4}(3))_K \cong [(\mathrm{Spec}\ K[a_1, a_1, a_1, a_2, a_2] - 0)/\mathbb{G}_m] = \mathcal{P}_K(1,1,1,2,2),$$

4. *the moduli stack of 5-marked 4-stable curves of arithmetic genus one is isomorphic to a scheme*

$$(\overline{\mathcal{M}}_{1,5}(4))_K \cong [(\mathrm{Spec}\ K[a_1, a_1, a_1, a_1, a_1, a_1] - 0)/\mathbb{G}_m] = \mathbb{P}_K(1,1,1,1,1,1) \cong \mathbb{P}^5_K,$$

where $\lambda \cdot a_i = \lambda^i a_i$ *for* $\lambda \in \mathbb{G}_m$ *and* $i = 1,2,3,4$*. Thus, the* a_i*'s have degree* i *respectively. Furthermore, if* $\mathrm{char}(K) \neq 2,3$*, then the discriminant divisors of such* $\overline{\mathcal{M}}_{1,m}(m-1)$ *have degree 12.*

Proof. Proof of [LP, Theorem 1.5.7.] gives the corresponding isomorphisms $\overline{\mathcal{M}}_{1,m}(m-1) \cong \mathcal{P}(\vec{\lambda})$. By Proposition 2.2, the weighted projective stacks are tame Deligne–Mumford as well, and in fact, smooth.

For the degree of the discriminant when $\mathrm{char}(K) \neq 2,3$, it suffices to describe the discriminant divisor, the locus of singular curves in $\overline{\mathcal{M}}_{1,m}(m-1)$. First, [LP, Theorem 1.5.7.] shows that in the above case, where $\overline{\mathcal{M}}_{1,m}(m-1) \cong \mathcal{P}(\vec{\lambda})$, the line bundle $\mathcal{O}_{\mathcal{P}(\vec{\lambda})}(1)$ of degree one is isomorphic to $\lambda := \pi_*\omega_\pi$, where $\pi : \overline{\mathcal{C}}_{1,m}(m-1) \to \overline{\mathcal{M}}_{1,m}(m-1)$ is the universal family of $(m-1)$-stable m-marked curves of arithmetic genus one. Since $\overline{\mathcal{M}}_{1,m}(m-1)$ is smooth and the Picard rank is one (generated by λ), the discriminant divisor is Cartier. In fact, by [Smyth2, §3.1], it coincides with the locus Δ_{irr} of curves with non-disconnecting nodes or non-nodal singular points. Then [Smyth2, Remark 3.3] (which assumes $\mathrm{char}(K) \neq 2,3$) implies that $\Delta_{irr} \sim 12\lambda$, thus the discriminant divisor has degree 12. ∎

We now consider the moduli stacks of m-marked $(m-1)$-stable genus one fibrations over $\mathbb{P}^1_K$ for any field K of $\mathrm{char}(K) = 0$ or > 3:

Proposition A.13. *Assume* $\mathrm{char}(K) = 0$ *or* > 3*. If* $2 \leq m \leq 5$*, then the moduli stack* $\mathcal{L}^m_{1,12n}$ *of* m*-marked* $(m-1)$*-stable genus one fibrations over* $\mathbb{P}^1_K$ *with discriminant degree* $12n$ *is the tame Deligne–Mumford stack* $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,m}(m-1))$ *parameterizing the* K*-morphisms* $f : \mathbb{P}^1 \to \overline{\mathcal{M}}_{1,m}(m-1)$ *such that* $f^*\mathcal{O}_{\mathcal{P}(\vec{\lambda})}(1) \cong \mathcal{O}_{\mathbb{P}^1}(n)$*.*

Proof. Without the loss of the generality, we prove the 2-marked 1-stable curves of arithmetic genus one case $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,2}(1))$ over $\mathrm{char}(\mathbb{F}_q) \neq 2,3$. The proof for the other cases are analogous. By the definition of the universal family p, any 2-marked 1-stable arithmetic genus one curves $\pi : Y \to \mathbb{P}^1$ with discriminant degree $12n$ comes from a morphism $f : \mathbb{P}^1 \to \overline{\mathcal{M}}_{1,2}(1)$ and vice versa. As this correspondence also works in families, the moduli stack of 2-marked 1-stable curves of arithmetic genus one over $\mathbb{P}^1_K$ is isomorphic to $\mathrm{Hom}(\mathbb{P}^1, \overline{\mathcal{M}}_{1,2}(1))$.

Since the discriminant degree of f is $12 \deg f^*\mathcal{O}_{\overline{\mathcal{M}}_{1,2}(1)}(1)$ by Proposition A.12, the substack $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,2}(1))$ parametrizing such f's with $\deg f^*\mathcal{O}_{\overline{\mathcal{M}}_{1,2}(1)}(1) = n$ is the desired moduli stack. Since $\deg f^*\mathcal{O}_{\overline{\mathcal{M}}_{1,2}(1)}(1) = n$ is an open condition, $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,2}(1))$ is an open substack of $\mathrm{Hom}(\mathbb{P}^1, \overline{\mathcal{M}}_{1,2}(1))$, which is tame Deligne–Mumford by [HP2, Proposition 3.6] as $\overline{\mathcal{M}}_{1,2}(1)$ itself is tame Deligne–Mumford by Proposition A.12. This shows that $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,2}(1))$ satisfies the desired properties as well. ∎

We now acquire the exact number $|\mathcal{L}^m_{1,12n}(\mathbb{F}_q)/\sim|$ of $\mathbb{F}_q$–isomorphism classes of $\mathbb{F}_q$–points of the moduli stack $\mathcal{L}^m_{1,12n} \cong \mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,m}(m-1))$ of m-marked $(m-1)$-stable genus one fibrations over $\mathbb{P}^1$ with discriminant degree $12n > 0$.

Proposition A.14. *If* $\mathrm{char}(\mathbb{F}_q) \neq 2, 3$, *then*

$$
\begin{aligned}
|\mathcal{L}^2_{1,12n}(\mathbb{F}_q)/\sim| &= \#_q\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(2,3,4))\right) + \#_q\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(2,4))\right) \\
&= (q^{9n+2} + q^{9n+1} - q^{9n-1} - q^{9n-2}) + (q^{6n+1} - q^{6n-1})
\end{aligned}
$$

$$
\begin{aligned}
|\mathcal{L}^3_{1,12n}(\mathbb{F}_q)/\sim| &= \#_q\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(1,2,2,3))\right) + \#_q\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(2,2))\right) \\
&= (q^{8n+3} + q^{8n+2} + q^{8n+1} - q^{8n-1} - q^{8n-2} - q^{8n-3}) + (q^{4n+1} - q^{4n-1})
\end{aligned}
$$

$$
\begin{aligned}
|\mathcal{L}^4_{1,12n}(\mathbb{F}_q)/\sim| &= \#_q\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(1,1,1,2,2))\right) + \#_q\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(2,2))\right) \\
&= (q^{7n+4} + q^{7n+3} + q^{7n+2} + q^{7n+1} - q^{7n-1} - q^{7n-2} - q^{7n-3} - q^{7n-4}) \\
&\quad + (q^{4n+1} - q^{4n-1}) \\
|\mathcal{L}^5_{1,12n}(\mathbb{F}_q)/\sim| &= \#_q\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathbb{P}(1,1,1,1,1,1) \cong \mathbb{P}^5)\right) \\
&= q^{6n+5} + q^{6n+4} + q^{6n+3} + q^{6n+2} + q^{6n+1} - q^{6n-1} - q^{6n-2} - q^{6n-3} - q^{6n-4} - q^{6n-5}
\end{aligned}
$$

Proof. Note that $\overline{\mathcal{M}}_{1,2}(1) \cong \mathcal{P}(2,3,4)$ has the substack $\mathcal{P}(2,4)$ with the generic stabilizer of order 2. This implies that the number of isomorphism classes of $\mathbb{F}_q$-points of $\mathcal{L}^2_{1,12n}$ with discriminant degree $12n$ is $|\mathcal{L}^2_{1,12n}(\mathbb{F}_q)/\sim| = (q^{9n+2}+q^{9n+1}-q^{9n-1}-q^{9n-2})+(q^{6n+1}-q^{6n-1})$ by summing the weighted point counts of Hom stacks as in [HP2, Proposition 4.10]. Similarly, $\overline{\mathcal{M}}_{1,3}(2) \cong \mathcal{P}(1,2,2,3)$ and $\overline{\mathcal{M}}_{1,4}(3) \cong \mathcal{P}(1,1,1,2,2)$ has the substack $\mathcal{P}(2,2)$ with the generic stabilizer of order 2. This implies that adding $(q^{4n+1} - q^{4n-1})$ to the corresponding weighted points count gives the desired non–weighted point counts. Finally, $\overline{\mathcal{M}}_{1,5}(4) \cong \mathbb{P}^5$, so that the non-weighted point count coincides with the weighted point count. ∎

We now finally prove the Theorem A.3 using the above arithmetic invariants as follows:

Proof. Without the loss of the generality, we prove the 2-marked 1-stable curves of arithmetic genus one case $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,2}(1) \cong \mathcal{P}(2,3,4))$ over $\mathrm{char}(\mathbb{F}_q) \neq 2, 3$. The proof for the other cases are analogous. Knowing the number of $\mathbb{F}_q$-isomorphism classes of 1-stable arithmetic genus one curves over $\mathbb{P}^1$ with discriminant degree $12n$ and 2-marked Weierstrass sections over $\mathbb{F}_q$ is $|\mathcal{L}^2_{1,12n}(\mathbb{F}_q)/\sim| = (q^{9n+2} + q^{9n+1} - q^{9n-1} - q^{9n-2}) + (q^{6n+1} - q^{6n-1})$ by Proposition A.14, we can explicitly compute the sharp bound on $\mathcal{Z}^2_{\mathbb{F}_q(t)}(\mathcal{B})$ as the following,

$$
\begin{aligned}
\mathcal{Z}^2_{\mathbb{F}_q(t)}(\mathcal{B}) &= \sum_{n=1}^{\left\lfloor \frac{log_q \mathcal{B}}{12} \right\rfloor} |\mathcal{L}^2_{1,12n}(\mathbb{F}_q)/\sim| = \sum_{n=1}^{\left\lfloor \frac{log_q \mathcal{B}}{12} \right\rfloor} (q^{9n+2} + q^{9n+1} - q^{9n-1} - q^{9n-2}) + (q^{6n+1} - q^{6n-1}) \\
&= (q^2 + q^1 - q^{-1} - q^{-2}) \sum_{n=1}^{\left\lfloor \frac{log_q \mathcal{B}}{12} \right\rfloor} q^{9n} + (q^1 - q^{-1}) \sum_{n=1}^{\left\lfloor \frac{log_q \mathcal{B}}{12} \right\rfloor} q^{6n}
\end{aligned}
$$

$$
\begin{aligned}
&\leq (q^2+q^1-q^{-1}-q^{-2})\left(q^9+\cdots+q^{9\cdot(\frac{log_q\mathcal{B}}{12})}\right)+(q^1-q^{-1})\left(q^6+\cdots+q^{6\cdot(\frac{log_q\mathcal{B}}{12})}\right)\\
&=(q^2+q^1-q^{-1}-q^{-2})\cdot\frac{q^9(\mathcal{B}^{\frac{3}{4}}-1)}{(q^9-1)}+(q^1-q^{-1})\frac{q^6(\mathcal{B}^{\frac{1}{2}}-1)}{(q^6-1)}\\
&=\frac{(q^{11}+q^{10}-q^8-q^7)}{(q^9-1)}\cdot(\mathcal{B}^{\frac{3}{4}}-1)+\frac{(q^7-q^5)}{(q^6-1)}\cdot(\mathcal{B}^{\frac{1}{2}}-1)
\end{aligned}
$$

On the third line of the equations above, inequality becomes an equality if and only if $n := \frac{log_q\mathcal{B}}{12} \in \mathbb{N}$, i.e., $\mathcal{B} = q^{12n}$ with $n \in \mathbb{Z}_{\geq 1}$. This implies that the acquired upper bound on $\mathcal{Z}^2_{\mathbb{F}_q(t)}(\mathcal{B})$ is a sharp estimate, i.e., the upper bound is equal to the function at infinitely many values of $\mathcal{B} \in \mathbb{N}$. ■

Acknowledgements

I would like to acknowledge the help and encouragement I have had from Changho Han. His insights were crucial in formulation of the moduli stacks and subsequent enumerations via arithmetic invariants throughout the paper. The author was supported by the Max Planck Institute for Mathematics and thanks the MPIM-Bonn for its hospitality.

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