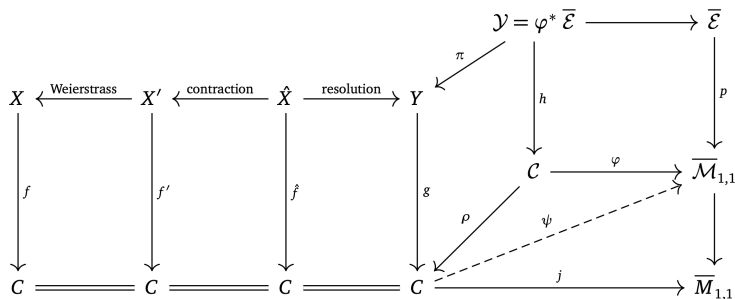


Geometric Interpretation of Tate's Algorithm



Here f is a Weierstrass model, ψ is the associated weighted linear series viewed as a rational map to $\bar{\mathcal{M}}_{1,1}$, φ is a twisted morphism from the universal tuning stack \mathcal{C} which induces a stable stack-like model $h : \mathcal{Y} \rightarrow \mathcal{C}$ where $g : Y \rightarrow C$ is the twisted model via coarse moduli maps, \hat{f} is a resolution of Y , and f' is the relative minimal model obtained by contracting relative (-1) -curves.

Suppose that normalized base multiplicity $m = 3$. This occurs if and only if $(\nu(a_4), \nu(a_6)) = (1, \geq 2)$. Then $r = 12/\gcd(3, 12) = 4$ and $a = 3/\gcd(3, 12) = 1$. Thus the stabilizer of the twisted curve acts on the central fiber of the twisted model via the character $\mu_4 \rightarrow \mu_4, \zeta_4 \mapsto \zeta_4^{-1}$. In particular, the central fiber E of \mathcal{Y} has $j = 1728$. The μ_4 action on E has two fixed points, and there is an orbit of size two with stabilizer $\mu_2 \subset \mu_4$. Let E_0 be the image of E in the twisted model Y . As E appears with multiplicity 4, Y has $\frac{1}{4}(-1, -1)$ quotient singularities at the images of the the fixed points and a $\frac{1}{2}(-1, -1)$ singularity at the image of the orbit of size two. Each of these singularities is resolved by a single blowup to obtain \hat{X} with central fiber $4\tilde{E}_0 + E_1 + E_2 + E_3$ where E_i are the exceptional divisors of the resolution for $i = 1, 2, 3$ and $E_1^2 = E_2^2 = -4$ with $E_3^2 = -2$. Then \tilde{E}_0 is a (-1) -curve so it needs to be contracted. After this contraction E_2 becomes a (-1) curve and must also be contracted. Since E_i for $i = 1, 2, 3$ are incident and pairwise transverse after blowing down \tilde{E}_0 , then the images of E_1 and E_2 must be tangent after blowing down E_3 . Moreover, they are now (-2) -curves and the relatively minimal model for type III.

Tate's Algorithm via Twisted Morphisms

Theorem (Dori Bejleri–JP–Matthew Satriano; April 2024)

If $\text{char}(K) \neq 2, 3$. Then the twisting condition (r, a) and the order of vanishing of j at $j = \infty$ determine the Kodaira fiber type, and (r, a) is in turn determined by $m = \min\{3\nu(a_4), 2\nu(a_6)\}$.

$\gamma : (\nu(a_4), \nu(a_6))$	Reduction type with $j \in \overline{M}_{1,1}$	$\Gamma : (r, a)$
$(\geq 1, 1)$	II with $j = 0$	$(6, 1)$
$(1, \geq 2)$	III with $j = 1728$	$(4, 1)$
$(\geq 2, 2)$	IV with $j = 0$	$(3, 1)$
$(2, 3)$	$I_{k>0}^*$ with $j = \infty$ I_0^* with $j \neq 0, 1728$	$(2, 1)$
$(\geq 3, 3)$	I_0^* with $j = 0$	$(2, 1)$
$(2, \geq 4)$	I_0^* with $j = 1728$	$(2, 1)$
$(\geq 3, 4)$	IV^* with $j = 0$	$(3, 2)$
$(3, \geq 5)$	III^* with $j = 1728$	$(4, 3)$
$(\geq 4, 5)$	II^* with $j = 0$	$(6, 5)$

Geometric Meaning of Height Moduli Framework

1. So one can run the resolution / minimal model. As these are *algebraic surfaces* it can be done over $\text{char}(K) = p > 0$
2. A twisted morphism $\varphi : \mathcal{C} \rightarrow \overline{\mathcal{M}}_{1,1}$ with its twisting data Γ from the universal tuning stack \mathcal{C} induces a stable stack-like model $h : \mathcal{Y} \rightarrow \mathcal{C}$ as a unique pullback of the universal family $p : \overline{\mathcal{E}} \rightarrow \overline{\mathcal{M}}_{1,1}$. All the ensuing birational geometry is natural.
3. True purpose of a **representable classifying morphism** is in the universal principle that φ intrinsically contains all the algebro-geometric data necessary to uniquely determine a fibration with singular fibers. This is the very essence of the inner arithmetic of rational points on moduli stacks over K .

Algebraic Geometry \cap Topology \iff Arithmetic

1. Consider the fact that $\overline{\mathcal{M}}_{1,1}$ could have been any other algebraic stack \mathcal{X} (such as $\overline{\mathcal{M}}_g$ or $\overline{\mathcal{A}}_g$) which is the representing object for certain moduli functor as the fine moduli stack together with the universal family $p : \overline{\mathcal{E}} \rightarrow \mathcal{X}$.
2. Representable classifying morphisms as twisted morphisms $\varphi : \mathcal{C} \rightarrow \mathcal{X}$ uniquely determines certain families of varieties (of algebraic curves or abelian varieties) with non-abelian stabilizers ($g \geq 2$). And they naturally have corresponding “Tate’s algorithm”, counting statements and so on.
3. Geometrizing $\mathcal{X}(K)$ leads to Height moduli space $\mathcal{M}_n(\mathcal{X}, \mathcal{L})$ and once we have a **space** (AG), we compute its **invariants** (AT) naturally having various kinds of **consequences** (NT).

Theorem (Dori Bejleri–JP–Matthew Satriano; April 2024)

$$\left\{ \mathcal{W}_{n=1}^{\min}(\mathcal{P}(\vec{\lambda})) \right\} = \{\mathbb{P}^N\}(\mathbb{L}^{|\vec{\lambda}|} - \mathbb{L}) + \mathbb{L}^{N+1} \{\mathbb{P}^{|\vec{\lambda}|-N-2}\}$$

$$\left\{ \mathcal{W}_{n \geq 2}^{\min}(\mathcal{P}(\vec{\lambda})) \right\} = \mathbb{L}^{(n-2)|\vec{\lambda}|+N+2} (\mathbb{L}^{|\vec{\lambda}|-1} - 1) \{\mathbb{P}^{|\vec{\lambda}|-1}\}$$

Take $|\vec{\lambda}| = 10$ and $N = 1$ as $\overline{\mathcal{M}}_{1,1} \cong \mathcal{P}(4, 6)$ over $\mathbb{Z}[1/6]$.

1. When $n = 1$, X is a **Rational elliptic surface**.

$$\{\mathcal{W}_1^{\min}\} = \mathbb{L}^{11} + \mathbb{L}^{10} + \mathbb{L}^9 + \mathbb{L}^8 + \mathbb{L}^7 + \mathbb{L}^6 + \mathbb{L}^5 + \mathbb{L}^4 + \mathbb{L}^3 - \mathbb{L}$$

2. When $n = 2$, X is algebraic $K3$ surface with elliptic fibration (i.e., **Projective elliptic K3 surface with moduli dim. 18**).

$$\{\mathcal{W}_2^{\min}\} = \mathbb{L}^{21} + \mathbb{L}^{20} + \mathbb{L}^{19} + \mathbb{L}^{18} + \mathbb{L}^{17} + \mathbb{L}^{16} + \mathbb{L}^{15} + \mathbb{L}^{14} + \mathbb{L}^{13} - \mathbb{L}^{11} - \mathbb{L}^{10} - \mathbb{L}^9 - \mathbb{L}^8 - \mathbb{L}^7 - \mathbb{L}^6 - \mathbb{L}^5 - \mathbb{L}^4 - \mathbb{L}^3$$

$$= \mathbb{L}(\mathbb{L}^2 - 1) \left(\mathbb{L}^{18} + \mathbb{L}^{17} + 2\mathbb{L}^{16} + 2\mathbb{L}^{15} + 3\mathbb{L}^{14} + 3\mathbb{L}^{13} + 4\mathbb{L}^{12} + 4\mathbb{L}^{11} + 5\mathbb{L}^{10} + 4\mathbb{L}^9 + 4\mathbb{L}^8 + 3\mathbb{L}^7 + 3\mathbb{L}^6 + 2\mathbb{L}^5 + 2\mathbb{L}^4 + \mathbb{L}^3 + \mathbb{L}^2 \right)$$