Totality of Rational points on Moduli stacks

Counting Families of Varieties : Lecture 1

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KIAS-LFANT Winter School on Number Theory

Rational Points on Projective Varieties over $\mathbb{Q}$

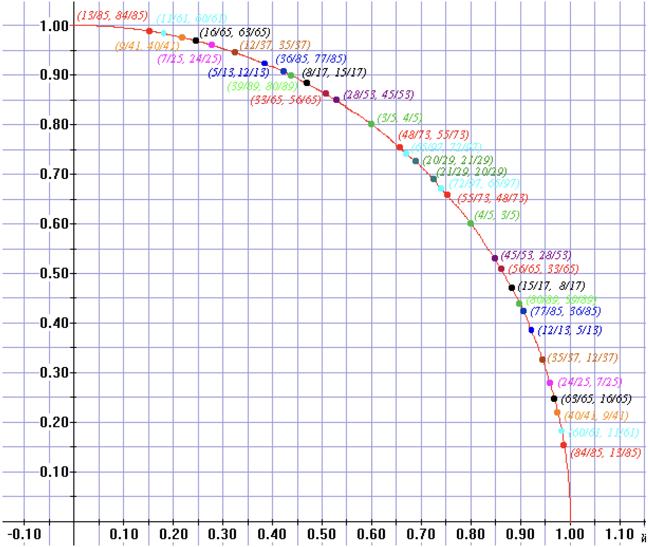


Figure 1: Rational points on $x^2 + y^2 = 1$ over $\mathbb{Q}$ - Pythagorean Triples

Why should we be happy?

1. Height of a rational number a/b with $\gcd(a,b)=1$ is $ht(a/b)=\max(|a|,|b|)$. Therefore, $ht(4/10)=5$. Bigger denominator allows more possibilities for numerator thus more rational points.

2. Geometry is enlightening and the quadratic formula is awesome as we have found / parametrized all rational points $(x,y)=\left(\frac{t^2-1}{t^2+1},\frac{2t}{t^2+1}\right)\in\mathbb{Q}^2$ on the unit circle over $\mathbb{Q}$

3. Integral points $[X:Y:Z]=[a^2-b^2:2ab:a^2+b^2]\in\mathbb{Z}^3$ on $C:=V(X^2+Y^2-Z^2)$ correspond to *"Pythagorean Triples"*

4. On **projective varieties**, the integral and the rational points coincide i.e., $X(\mathbb{Q})=X(\mathbb{Z})$. Bear in mind $\gcd(a,b)=1$.

Why should we be unhappy?

1. If we don't have a rational point to begin the process then we cannot apply quadratic formula. For example, $x^2 + y^2 = 3$ and turns out $X(\mathbb{Q}) = \emptyset$. We need Arithmetic to prove this.

2. Take $x^4 + y^4 = 1$ then we have *"Fermat's Last Theorem"* regarding $x^n + y^n = 1$ with $n = 4$. By Wiles-Taylor, we **know** it has only 4 rational points $X(\mathbb{Q}) = \{(\pm 1, 0), (0, \pm 1)\}$. Recalling Mordell-Faltings, we **know** it had $X(\mathbb{Q}) < \infty$

3. Take $y^2 = x^3 + Ax + B$ this is 1 polynomial in 2 variables of degree 3 (the Weierstrass cubic for an elliptic curve over $\mathbb{Q}$). **What are $E(\mathbb{Q})$?** Shockingly, *we still cannot answer this.*

4. Actually, we know there is at least 1 rational point, the point at $\infty = [0 : 1 : 0]$ for $E : V(Y^2Z - X^3 - AXZ^2 - BZ^3)$

Degree of countable infinity, the Rank

1. By Mordell-Weil, the set $E(\mathbb{Q})$ of rational points on $E/\mathbb{Q}$ has a finitely-generated abelian group structure $E(\mathbb{Q}) = \mathbb{Z}^r \oplus T$ with algebraic rank $r \in \mathbb{Z}_{\geq 0}$ and torsion subgroup T
2. The rank r of $E(\mathbb{Q})$ is **not** well understood.
   2.1 An algorithm that is guaranteed to correctly compute r?
   2.2 Which values of r can occur? *How often do they occur?*
   2.3 Is there an upper limit, or can r be arbitrarily large?
3. When r is small, computational methods exist but when r is large, often the best we can do is a lower bound; we now know, assuming GRH, there is an $E/\mathbb{Q}$ with $r \geq 29$ by Elkies-Klagsbrun (2024).

Demography of Elliptic Curves $E/\mathbb{Q}$

Trying to find / parametrize all the rational points on a given $E/\mathbb{Q}$ is a dead-end. Thus we rotate our entry. We would like to think about *the Question of Distribution and Proportion* over all $E/\mathbb{Q}$

Naive height for $E : y^2 = x^3 + Ax + B$ with no $p^4|A$ and $p^6|B$ (minimal Weierstrass model) is $ht(E) := \max(4|A|^3, 27B^2)$.

Conjecture (Minimality + Parity; Goldfeld and Katz-Sarnak)

Over any number field, 50% of all elliptic curves (when ordered by height) have Mordell-Weil rank $r = 0$ and the other 50% have Mordell-Weil rank $r = 1$. Moreover, higher Mordell-Weil ranks $r \geq 2$ constitute 0% of all elliptic curves, even though there may exist infinitely many such elliptic curves. Therefore, a suitably-defined average rank would be $\frac{1}{2}$.

What does this really mean? To talk about Average, we need the **"Total number of elliptic curves over $\mathbb{Q}$ up to isomorphism"**.

Triangle of Rational Dedekind Domains

Consider not only $E/\mathbb{Q}$ but also $E/\mathbb{F}_q(t)$ as well as $E/\mathbb{C}(z)$

1. The rational number field $\mathbb{Q}$ consisting of ratio of integer numbers in $\mathbb{Z}$ is **the rational global field of char** $= 0$
2. The rational function field $\mathbb{F}_q(t)$ with *coefficients* in $\mathbb{F}_q = \mathbb{F}_{p^r}$ consisting of ratio of polynomial functions in $\mathbb{F}_q[t]$ is **the rational global field of char** $= p > 0 \Leftrightarrow$ Projective line $\mathbb{P}^1_{\mathbb{F}_q}$
3. The meromorphic function field $\mathbb{C}(z)$ with *coefficients* in $\mathbb{C}$ consisting of ratio of holomorphic functions in $\mathbb{C}[z]$ is **NOT** the rational global field of char $= 0 \Leftrightarrow$ Riemann sphere $\mathbb{CP}^1$
4. $K = \mathbb{C}(z)$ is not a global field since the residue field $k = \mathbb{C}$ is not finite (transcendental 2nd order infinity). Similarly, $K = \overline{\mathbb{F}}_q(t)$ is not a global field since the residue field $k = \overline{\mathbb{F}}_q$ is not finite (countable 1st order infinity).

   Let us count ALL elliptic curves over $K = \mathbb{F}_q(t)$ wrt height.

∞ $r = \frac{1}{2}$, $R = 4$

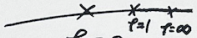
-1 1 $\mathrm{Vol}(\mathbb{CP}^1) = \pi$

0

$b_0 = 1$

$b_2 = 1$

$e = 2$



$t=0$ $t=1$ $t=\infty$

$[u:v]$ $t = u/v$ $|\mathbb{P}^1(\mathbb{F}_q)| = q+1$

$\mathbb{CP}^1$ —— G.B.F —— S^2 —— G.L.T.F —— $\mathbb{P}^1_{\mathbb{F}_q}$

Chow W.C. W.T.

$\mathbb{C}(z)$ - - - $\overline{\mathbb{F}}_q(t)$ —∞— $\mathbb{F}_q(t)$ — Poly/poly

Holo/Holo

G.F.A.

$\mathbb{Q}$

— Int/Int

Ab/$\mathbb{C}$ AT/k Ab/$\mathbb{F}_q$

KT

CV CA

NT

$\mathbb{Z} \sim \mathbb{F}_q[t]$ 'As integers so polynomials'

Let $\mathcal{C}$ be a suitable cat. of sheaves then

$\mathcal{C}(\mathrm{Spec}\,\mathbb{Z} \setminus T) \simeq \mathcal{C}(\mathbb{P}^1_{\mathbb{F}_q} \setminus S)$

Analogy

'Aware of each others'

The Sharp Enumeration over Rational Function Field

Define *height of discriminant* Δ *over* $\mathbb{F}_q(t)$ as $ht(\Delta) := q^{\deg \Delta}$

- Elliptic case: $Deg(\Delta) = 12n \implies ht(\Delta) = q^{12n}$ for $n \in \mathbb{Z}_{\geq 0}$

We consider the counting function $\mathcal{N}(\mathbb{F}_q(t),\ B) :=$

$$\left|\left\{\text{Minimal elliptic curves over } \mathbb{P}^1_{\mathbb{F}_q} \text{ with } 0 < ht(\Delta) \leq B\right\}\right|$$

Theorem (Dori Bejleri–JP–Matthew Satriano; April 2024)

Let $\mathrm{char}(\mathbb{F}_q) > 3$ and $\delta(x) := \begin{cases} 1 & \text{if } x \text{ divides } q-1, \\ 0 & \text{otherwise.} \end{cases}$, then

$$\begin{aligned} \mathcal{N}(\mathbb{F}_q(t),\ B) = 2\left(\frac{q^9-1}{q^8-q^7}\right)B^{5/6} - 2B^{1/6} \\ + \delta(6)\cdot 4\left(\frac{q^5-1}{q^5-q^4}\right)B^{1/2} + \delta(4)\cdot 2\left(\frac{q^3-1}{q^3-q^2}\right)B^{1/3} \\ + \delta(6)\cdot 4 + \delta(4)\cdot 2 \end{aligned}$$

Precise proportions of E/K motivated by NT

Theorem (Generic Torsion Freeness; Phillips)

The set of torsion-free elliptic curves over global function fields has density 1. i.e., 'Most elliptic curves over K are torsion free'.

Theorem (Boundedness; Tate-Shafarevich & Ulmer)

The ranks of non-constant elliptic curves over $\mathbb{F}_q(t)$ are unbounded (in both the **isotrivial** *and* **non-isotrivial** *cases).*

Projective Elliptic K3 Surface of height $n = 2$

$$y^2 = x^3 + a_4(u:v)x + a_6(u:v)$$

Weierstrass data for elliptic fibration on algebraic K3 surface,

$$\begin{cases} a_4(u:v) & = -3u^4v^4, \text{ degree } 8 = 4 \times 2, \\ a_6(u:v) & = u^5v^5(u^2+v^2), \text{ degree } 12 = 6 \times 2. \end{cases}$$

Then we have $\Delta = 4a_4^3 + 27a_6^2$ and $j = 1728 \cdot 4a_4^3/\Delta$

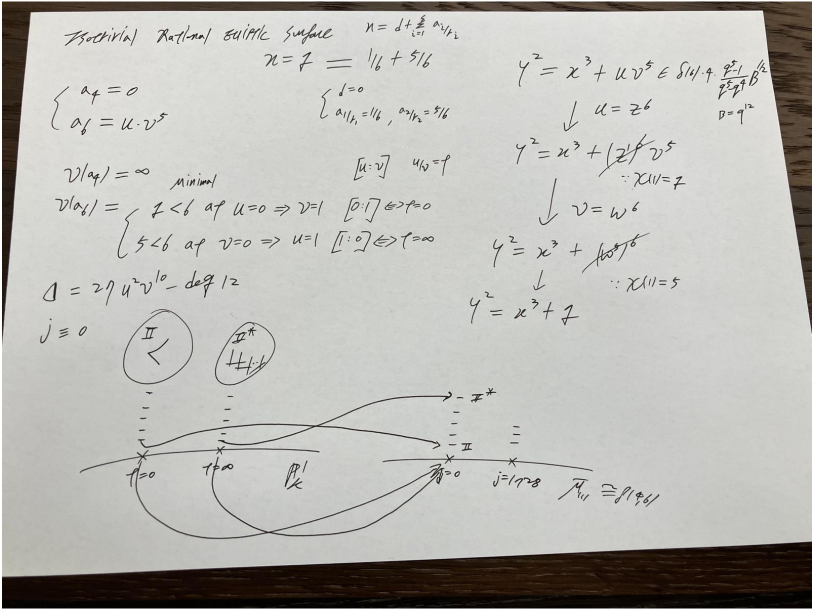
$$\begin{cases} \Delta & = 27u^{10}v^{10}(u-v)^2(u+v)^2, \text{ degree } 24 = 12 \times 2, \\ j & = 1728 \cdot -\frac{4u^2v^2}{(u-v)^2(u+v)^2}, \text{ degree 4! NOT 24.} \end{cases}$$

Wait, where did degree 20 go?

After all, we should have $j : \mathbb{P}^1 \to \overline{M}_{1,1} \cong \mathbb{P}^1$ of degree 24?

Well, it can get whole lot worse.

Isotrivial Rational Elliptic Surface of height $n = 1$



Precise proportions of E/K motivated by NT

We consider the counting function $\mathcal{N}_T^r(\mathbb{F}_q(t),\ B) :=$

$|\{\text{Minimal } E/\mathbb{F}_q(t) \text{ with algebraic rank } r, \text{ torsion } T \text{ and } ht(\Delta) \leq B\}|$

If we combine the above two Theorems and the Rank Distribution Conjecture, we are led to the following conclusion.

Quantitative Rank Distribution Conjecture over $K = \mathbb{F}_q(t)$

$$\mathcal{N}_{T=0}^{r=0}(\mathbb{F}_q(t),\ B) = \left(\frac{q^9-1}{q^8-q^7}\right) B^{5/6} + o(B^{\frac{5}{6}}),$$

$$\mathcal{N}_{T=0}^{r=1}(\mathbb{F}_q(t),\ B) = \left(\frac{q^9-1}{q^8-q^7}\right) B^{5/6} + o(B^{\frac{5}{6}}),$$

$$\mathcal{N}_{T}^{r\geq 2}(\mathbb{F}_q(t),\ B) = o(B^{\frac{5}{6}}),\ \text{where all o are little-o.}$$

† $|E(K)| = 1$ and $E(K) = \mathbb{Z}$ each corresponds to 50% of all elliptic curves over K ordered by discriminant height having *equal* main leading term $B^{5/6}$ with *identical* leading coefficient $\left(\frac{q^9-1}{q^8-q^7}\right)$.