

# Arithmetic of the Moduli of Fibrations

## Arithmetic Moduli of Elliptic Surfaces

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We call an algebraic surface  $X$  to be an **elliptic surface**, if it admits an elliptic fibration  $f : X \rightarrow C$  which is a flat and proper morphism  $f$  from a nonsingular surface  $X$  to  $C$  where  $C$  is a nonsingular curve, such that a generic fiber is a smooth curve of genus one.

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While this is the most general setup, it is natural to work with the case when the base curve is the smooth projective line  $\mathbb{P}^1$  and there exists a section  $S : \mathbb{P}^1 \hookrightarrow X$  coming from the identity points of the elliptic fibres and not passing through the singular points.

## Elliptic surfaces over $\mathbb{P}^1$ with a section

Here we list the properties of an elliptic surface  $X$  with discriminant degree  $12n$ . This also works for any field  $K$  with  $\text{char}(K) \neq 2, 3$ .

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1. When  $n = 1$ ,  $X$  is a **Rational elliptic surface** with the Kodaira dimension  $\kappa = -\infty$  which has 12 nodal singular fibers generically. It is acquired from a pencil of cubic curves in  $\mathbb{P}^2$  by blowing up a base locus of nine points coming from the intersection of two general cubic curves.

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2. When  $n = 2$ ,  $X$  is a  $K3$  surface with an elliptic fibration (i.e., **Elliptic K3 surface**) which has the Kodaira dimension  $\kappa = 0$  that has 24 nodal singular fibers generically. Note that  $X$  is a minimal surface.

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3. When  $n \geq 3$ ,  $X$  is called a **Properly elliptic surface** with Kodaira dimension  $\kappa = 1$  that has  $12n$  nodal singular fibers generically. Note that  $X$  is also a minimal surface.

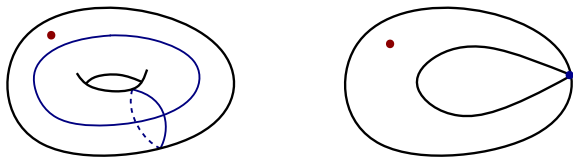


## Deligne–Mumford stack $\overline{\mathcal{M}}_{1,1}$ of stable elliptic curves

Let us recall that  $\overline{\mathcal{M}}_{1,1}$  is a smooth proper Deligne–Mumford stack of stable elliptic curves with a coarse moduli space  $\overline{M}_{1,1} \cong \mathbb{P}^1$ . This  $\mathbb{P}^1$  parametrizes the  $j$ -invariants of elliptic curves.

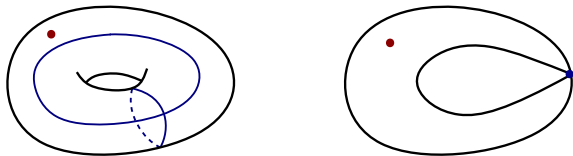
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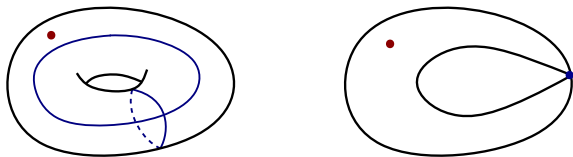
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When the characteristic of the field  $K$  is not equal to 2 or 3,  $(\overline{\mathcal{M}}_{1,1})_K \cong [(Spec K[a_4, a_6] - (0, 0))/\mathbb{G}_m] =: \mathcal{P}_K(4, 6)$  through the short Weierstrass equation:  $Y^2 = X^3 + a_4X + a_6$

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Stabilizers are the orbifold points  $[1 : 0]$  &  $[0 : 1]$  with  $\mu_4$  &  $\mu_6$  respectively and the generic stacky points such as  $[1 : 1]$  with  $\mu_2$

## Moduli stack of stable elliptic surfaces

The fine moduli  $\overline{\mathcal{M}}_{1,1}$  comes with universal family  $p : \overline{\mathcal{C}}_{1,1} \rightarrow \overline{\mathcal{M}}_{1,1}$  of stable elliptic curves. Thus, a stable elliptic surface  $g : Y \rightarrow \mathbb{P}^1$  is induced from a morphism  $\varphi_f : \mathbb{P}^1 \rightarrow \overline{\mathcal{M}}_{1,1}$  and vice versa.

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*The moduli stack of stable elliptic surfaces over  $\mathbb{P}^1$  with  $12n$  nodal singular fibers and a section is the Hom stack  $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$ .*

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Here, we fixed the parameterization of the domain  $\mathbb{P}^1$  which is good for 'Global Fields Analogy' (since  $\mathbb{Q}$  has the *unique* ring of integers called  $\mathbb{Z}$ ) but not natural from Geometric perspective.



## Group actions on stacks for stack quotients of stacks

It is natural to consider the action of  $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2$  on  $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$  by composing the stable elliptic surface  $g : Y \rightarrow \mathbb{P}^1$  with an automorphism of  $\mathbb{P}^1$ .

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It is easy to see that this action is induced by an action on the ambient weighted projective stack  $\mathcal{P}(V)$ .

$$(A, B) \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4n)) \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(6n)) =: V \quad (2)$$

define the so-called *Weierstrass data* of the fibration. Indeed, the action of an element of  $\mathrm{PGL}_2$  on the homogeneous coordinates  $X, Y$  of  $\mathbb{P}^1$  translates to an action on the global sections  $A, B$  of  $\mathcal{O}_{\mathbb{P}^1}(4n), \mathcal{O}_{\mathbb{P}^1}(6n)$  which are the homogeneous coordinates of  $\mathcal{P}(V) = \mathcal{P}(\underbrace{4, \dots, 4}_{4n+1 \text{ times}}, \underbrace{6, \dots, 6}_{6n+1 \text{ times}}) \in \mathbb{Z}^{10n+2}$ .

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Note that since both  $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$  and  $\mathcal{P}(V)$  are themselves stacks, the formal definition of these actions requires one to use the notion of group actions on stacks presented in [Romagny].

# Group actions on stacks for stack quotients of stacks

We have the following commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}) & \hookrightarrow & \mathrm{Rat}_n^\gamma(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}) & \hookrightarrow & \mathcal{P}(V) \\ \downarrow / \mathrm{PGL}_2 & & \downarrow / \mathrm{PGL}_2 & & \downarrow / \mathrm{PGL}_2 \\ \mathcal{W}_{\mathrm{sf},n} & \hookrightarrow & \mathcal{W}_{\mathrm{min},n} & \hookrightarrow & [\mathcal{P}(V) / \mathrm{PGL}_2] \end{array}$$

where the horizontal arrows are open embeddings of moduli stacks.

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## Theorem (Johannes Schmitt, J.-Y. Park)

*Fix a degree  $n \in \mathbb{Z}_{\geq 1}$  and a base field  $K$  with  $\mathrm{char}(K) \neq 2, 3$ . Inside the quotient stack  $[\mathcal{P}(V) / \mathrm{PGL}_2]$ , the open substacks  $\mathcal{W}_{\mathrm{min},n}$  (for  $n \geq 2$ ) of minimal Weierstrass fibrations and  $\mathcal{W}_{\mathrm{sf},n}$  (for  $n \geq 1$ ) of stable Weierstrass fibrations are smooth, irreducible and separated Deligne–Mumford stacks of finite type with affine diagonal for  $\mathrm{char}(K) \nmid n$ , which are tame for  $\mathrm{char}(K) > 12n$ .*

## Group actions on stacks for stack quotients of stacks

The Weierstrass fibration associated to  $[A : B] = [X^{4n} : Y^{6n}]$  is invariant under scaling  $X$  by an element of  $\mu_{4n}$  and by scaling  $Y$  under an element of  $\mu_{6n}$ . Together, these transformations generate a copy of  $\mu_{12n}$  inside  $\mathrm{PGL}_2$  which acts as an automorphism of the fibration, and the quotient is not tame when  $\mathrm{char}(K)$  divides  $12n$ .

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The stack  $\mathcal{W}_{\min,1}$  contains points with positive dimensional stabilizers, thus it is no longer of Deligne–Mumford type. These points are precisely  $\mathrm{PGL}_2$ -orbit of the Weierstrass data  $[A : B]$

$$[A : B] = [0 : XY^5], [XY^3 : 0], [0 : X^2Y^4] \text{ and } [a_0X^2Y^2 : a_1X^3Y^3],$$

where in each case we have a nontrivial action of  $\mathbb{G}_m$  on the coordinates  $X, Y$  fixing the fibrations. They are the four types of rational elliptic surfaces with two singular fibres

$[\mathrm{II}, \mathrm{II}^*], [\mathrm{III}, \mathrm{III}^*], [\mathrm{IV}, \mathrm{IV}^*], [\mathrm{I}_0^*, \mathrm{I}_0^*]$  both of which are additive type in dual pair. One can see that the open substack  $\mathcal{W}'_{\min,1}$  of  $\mathcal{W}_{\min,1}$  obtained by removing these four points is indeed Deligne–Mumford for  $\mathrm{char}(K) \nmid n$  and tame for  $\mathrm{char}(K) > 12$ .

## Grothendieck ring $K_0(\mathrm{Stck}_K)$ of $K$ -stacks

Ekedahl in 2009 introduced the Grothendieck ring  $K_0(\mathrm{Stck}_K)$  of algebraic stacks extending the classical Grothendieck ring  $K_0(\mathrm{Var}_K)$  of varieties first defined by Grothendieck in 1964.



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### Definition

Fix a field  $K$ . Then the *Grothendieck ring  $K_0(\mathrm{Stck}_K)$  of algebraic stacks of finite type over  $K$  all of whose stabilizer group schemes are affine* is an abelian group generated by isomorphism classes of algebraic stacks  $\{\mathcal{X}\}$  modulo relations:

- ▶  $\{\mathcal{X}\} = \{\mathcal{Z}\} + \{\mathcal{X} \setminus \mathcal{Z}\}$  for  $\mathcal{Z} \subset \mathcal{X}$  a closed substack,
- ▶  $\{\mathcal{E}\} = \{\mathcal{X} \times \mathbb{A}^n\}$  for  $\mathcal{E}$  a vector bundle of rank  $n$  on  $\mathcal{X}$ .

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The weighted point count of  $\mathcal{X}$  over  $\mathbb{F}_q$  is defined as a sum:  
 $\#_q(\mathcal{X}) := \sum_{x \in \mathcal{X}(\mathbb{F}_q)/\sim} \frac{1}{|\mathrm{Aut}(x)|}$  where  $\mathcal{X}(\mathbb{F}_q)/\sim$  is the set of  $\mathbb{F}_q$ -isomorphism classes of  $\mathbb{F}_q$ -points of  $\mathcal{X}$ .

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When  $K = \mathbb{F}_q$ , the point counting measure  $\{\mathcal{X}\} \mapsto \#_q(\mathcal{X})$  gives a well-defined ring homomorphism  $\#_q : K_0(\mathrm{Stck}_{\mathbb{F}_q}) \rightarrow \mathbb{Q}$ .

# Motive/Point count of $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$ over finite fields

Theorem (Changho Han, Hunter Spink, Johannes Schmitt, J.-Y. Park)

*The class  $\{\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})\}$  in  $K_0(\mathrm{Stck}_K)$  for  $\mathrm{char}(K) \neq 2, 3$  of the moduli stack for stable elliptic fibrations over the parameterized  $\mathbb{P}^1$  with  $12n$  nodal singular fibers and a section is equivalent to*

$$\{\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})\} = \mathbb{L}^{10n+1} - \mathbb{L}^{10n-1}$$

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$$\{\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})\} = \mathbb{L}^{10n+1} - \mathbb{L}^{10n-1}$$

*and over an unparameterized  $\mathbb{P}^1$  with an odd degree  $n$*

$$\{\mathcal{W}_{\mathrm{sf},n}\} = \mathbb{L}^{10n-2}$$

*where  $\mathbb{L} = \{\mathbb{A}^1\}$  is the Lefschetz motive and  $\{\mathrm{PGL}_2\} = \mathbb{L}(\mathbb{L}^2 - 1)$ .*

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*where  $\mathbb{L} = \{\mathbb{A}^1\}$  is the Lefschetz motive and  $\{\mathrm{PGL}_2\} = \mathbb{L}(\mathbb{L}^2 - 1)$ .*

Then, by using  $\#_q : K_0(\mathrm{Stck}_{\mathbb{F}_q}) \rightarrow \mathbb{Q}$  to count  $\mathbb{F}_q$ -points when  $\mathrm{char}(\mathbb{F}_q) \neq 2, 3$ , we acquire the weighted point counts of the moduli of stable elliptic surfaces over (un)parameterized  $\mathbb{P}^1$ .

# Motive/Point count of $\text{Rat}_n^\gamma(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$ over the finite fields

Theorem (work in progress; Matthew Satriano, J.-Y. Park)

Let  $\text{char}(K) \neq 2, 3$ . Then the motive for the moduli stack of minimal elliptic surfaces over the parameterized  $\mathbb{P}^1$  with a section and discriminant degree  $12n$  having one additive singular fiber is

Fiber type, $j$	$\{\text{Rat}_n^\gamma(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})\}/\{\text{PGL}_2\}$
II, $j = 0$	$\mathbb{L}^{10n-3} - \mathbb{L}^{4n-2}$
III, $j = 1728$	$\mathbb{L}^{10n-4} - \mathbb{L}^{6n-3}$
IV, $j = 0$	$\mathbb{L}^{10n-5} - \mathbb{L}^{4n-3}$
$I_{k \geq 0}^*, j = \infty$ $I_0^*, j \neq 0, 1728$	$\mathbb{L}^{10n-6} - \mathbb{L}^{10n-7} - \mathbb{L}^{6n-4} + \mathbb{L}^{6n-5} - \mathbb{L}^{4n-3} + \mathbb{L}^{4n-4}$
$I_0^*, j = 1728$	$\mathbb{L}^{10n-7} - \mathbb{L}^{4n-3}$
$I_0^*, j = 0$	$\mathbb{L}^{10n-7} - \mathbb{L}^{4n-4}$
$IV^*, j = 0$	$\mathbb{L}^{10n-8} - \mathbb{L}^{4n-4}$
$III^*, j = 1728$	$\mathbb{L}^{10n-9} - \mathbb{L}^{6n-6}$
$II^*, j = 0$	$\mathbb{L}^{10n-10} - \mathbb{L}^{4n-5}$

## Counting elliptic curves over global fields

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## Definition

Define the height of a point  $\mathfrak{p}$  to be  $ht(\mathfrak{p}) := |\mathcal{O}_K/\mathfrak{p}|$  the cardinality of the residue field  $\mathcal{O}_K/\mathfrak{p}$ .

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As the discriminant divisor  $\Delta(X)$  of degree  $12n$  is an invariant of the choice of semistable model  $f : X \rightarrow \mathbb{P}^1$ , we count the number of isomorphism classes of semistable elliptic surfaces on the function field  $\mathbb{F}_q(t)$  by the bounded height of  $\Delta(X)$ .

$$ht(\Delta(X)) = \prod_{i=1}^{\mu} |\mathbb{F}_q|^{k_i} = q^{k_1} \cdots q^{k_i} \cdots q^{k_{\mu}} = q^{k_1 + \cdots + k_{\mu}} = q^{12n}$$

## Counting elliptic curves over global fields

Now consider  $\mathcal{Z}_{\mathbb{F}_q(t)}(\mathcal{B}) =$

$$|\{\text{Semistable elliptic curves over } \mathbb{P}_{\mathbb{F}_q}^1 \text{ with } 0 < ht(\Delta) \leq \mathcal{B}\}|.$$

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Theorem (Changho Han, J.-Y. Park)

*The counting  $\mathcal{Z}_{\mathbb{F}_q(t)}(\mathcal{B})$  by  $ht(\Delta) = q^{12n} \leq \mathcal{B}$  satisfies*

$$\mathcal{Z}_{\mathbb{F}_q(t)}(\mathcal{B}) \leq 2 \cdot \frac{(q^{11} - q^9)}{(q^{10} - 1)} \cdot \left(\mathcal{B}^{\frac{5}{6}} - 1\right)$$

*which is an equality when  $\mathcal{B} = q^{12n}$  for some  $n \in \mathbb{N}$  implying that the acquired upper bound is a sharp enumeration, i.e., the upper bound is equal to the function at infinitely many values of  $\mathcal{B} \in \mathbb{N}$ .*

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Note that we have the lower order term of zeroth order (constant).

# Counting elliptic curves over global fields

Theorem (work in progress; Matthew Satriano, J.-Y. Park)

If  $\text{char}(\mathbb{F}_q) > 3$ , then the function  $\mathcal{Z}_{\mathbb{F}_q(t)}^\gamma(\mathcal{B})$ , which counts the number of minimal elliptic curves with one additive singular fiber of  $\gamma$  type over the parameterized  $\mathbb{P}_{\mathbb{F}_q}^1$  ordered by  $0 < ht(\Delta) = q^{12n} \leq \mathcal{B}$ , satisfies:

$$\mathcal{Z}_{\mathbb{F}_q(t)}^\gamma(\mathcal{B}) \leq a_q \cdot \mathcal{B}^{\frac{5}{6}} + b_q \cdot \mathcal{B}^{\frac{1}{3}} + c_q, \text{ when } \gamma = \text{II}, \text{II}^*, \text{IV}, \text{IV}^* \text{ or } \text{I}_0^*$$

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which is an equality when  $\mathcal{B} = q^{12n}$  with  $n \in \mathbb{Z}_{\geq 1}$  implying that the upper bounds are sharp enumerations.

## Heuristics for counting elliptic curves over number fields

Switching to the number field realm with  $K = \mathbb{Q}$  and  $\mathcal{O}_K = \mathbb{Z}$ , one could choose the minimal integral Weierstrass model with the given discriminant divisor  $\Delta$  which is already a number.

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In order to match the counting with the function field, we define the  $ht(\Delta)$  to be the cardinality of ring of functions on subscheme  $\text{Spec}(\mathbb{Z}/(\Delta))$ . This leads to the following analogue  $\mathcal{Z}_{\mathbb{Q}}(\mathcal{B})$ .



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Conjecture (work in progress; Heuristic on  $\mathcal{Z}_{\mathbb{Q}}(\mathcal{B})$ )

The function  $\mathcal{Z}_{\mathbb{Q}}(\mathcal{B})$ , which counts the number of elliptic curves with one additive bad reduction over  $\mathbb{Z}$  and  $0 < ht(\Delta) \leq \mathcal{B}$ , has the asymptotic behavior:

$$a\mathcal{B}^{\frac{5}{6}} + b\mathcal{B}^{\frac{1}{2}} + c\mathcal{B}^{\frac{1}{3}} + \text{lower order terms}$$

with the main leading term  $\mathcal{O}\left(\mathcal{B}^{\frac{5}{6}}\right)$ , the secondary term  $\mathcal{O}\left(\mathcal{B}^{\frac{1}{2}}\right)$  and the tertiary term  $\mathcal{O}\left(\mathcal{B}^{\frac{1}{3}}\right)$ .

Thank you :)

Thank you to the **organizers & everyone** for listening!