A Counting elliptic surfaces with prescribed torsion or multiple sections

In this appendix, we determine the sharp enumerations on the number of elliptic curves over $\mathbb{P}^1_{\mathbb{F}_q}$ with prescribed level structures or multiple marked points by extending the method as in [HP, Theorem 3] regarding the number of semistable elliptic curves over $\mathbb{P}^1_{\mathbb{F}_q}$.

Specifically, we acquire the sharp enumerations on the number of elliptic curves over global function fields $\mathbb{F}_q(t)$ with level structures $[\Gamma_1(n)]$ for $2 \le n \le 4$ or $[\Gamma(2)]$. Recall that a level structure $[\Gamma_1(n)]$ on an elliptic curve E is a choice of point $P \in E$ of exact order n in the smooth part of E such that over every geometric point of the base scheme every irreducible component of E contains a multiple of P (see [KM, §1.4]). And a level structure $[\Gamma(2)]$ on an elliptic curve E is a choice of isomorphism $\phi : \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to E(2)$ where $E(2)$ is the scheme of 2-torsion Weierstrass points (i.e., kernel of the multiplication-by-2 map $[2] : E \to E$) (see [DR, II.1.18 & IV.2.3]).

Additionally, we consider curves of arithmetic genus one over $\mathbb{F}_q(t)$ with m-marked rational points for $2 \le m \le 5$ by acquiring sharp enumeration on the number of $(m-1)$-stable m-marked curves of arithmetic genus one formulated originally by the works of [Smyth, Smyth2].

To enumerate the number of certain elliptic curves over global function fields $\mathbb{F}_q(t)$ with level structures $[\Gamma(n)]$ or $[\Gamma_1(n)]$, we need to first extend the notion of (nonsingular) elliptic curves (semistable in the case of [HP]) that admits desired level structures. By the work of Deligne and Rapoport [DR] (summarized in [Niles, §2]), we consider the generalized elliptic curves over $\mathbb{P}^1_K$ with $[\Gamma]$–structures (where Γ is $\Gamma(n)$ or $\Gamma_1(n)$) over a field K (focusing on $K = \mathbb{F}_q$). Roughly, a generalized elliptic curve X over $\mathbb{P}^1_K$ can be thought of as a flat family of semistable elliptic curves admitting a group structure, such that a finite group scheme $\mathcal{G} \to \mathbb{P}^1_K$ (determined by Γ) embeds into X and its image meets every irreducible component of every geometric fibers of X. Again, we only consider the *non-isotrivial* generalized elliptic curves. If X is as above, then Δ is the discriminant of a generalized elliptic curve and if $K = \mathbb{F}_q$, then $0 < ht(\Delta) := q^{\deg \Delta}$.

Now, define $\mathcal{Z}^{[\Gamma]}_{\mathbb{F}_q(t)}(\mathcal{B})$ as follows:

$$\mathcal{Z}^{[\Gamma]}_{\mathbb{F}_q(t)}(\mathcal{B}) := |\{\text{Generalized elliptic curves over } \mathbb{P}^1_{\mathbb{F}_q} \text{ with } [\Gamma]\text{–structures and } 0 < ht(\Delta) \le \mathcal{B}\}|$$

Then, we acquire the following descriptions of $\mathcal{Z}^{[\Gamma]}_{\mathbb{F}_q(t)}(\mathcal{B})$:

Theorem A.1 (Sharp enumeration $\mathcal{Z}^{[\Gamma]}_{\mathbb{F}_q(t)}(\mathcal{B})$)**.** *The function $\mathcal{Z}^{[\Gamma]}_{\mathbb{F}_q(t)}(\mathcal{B})$, which counts the number of generalized elliptic curves with $[\Gamma]$-structures over $\mathbb{P}^1_{\mathbb{F}_q}$ with $char(\mathbb{F}_q) \neq 2$ $\big(char(\mathbb{F}_q) \neq 3$ for $\mathcal{Z}^{[\Gamma_1(3)]}_{\mathbb{F}_q(t)}(\mathcal{B})\big)$ ordered by $0 < ht(\Delta) = q^{12n} \le \mathcal{B}$, satisfies:*

$$\mathcal{Z}^{[\Gamma_1(2)]}_{\mathbb{F}_q(t)}(\mathcal{B}) \le 2 \cdot \frac{(q^7 - q^5)}{(q^6 - 1)} \cdot \left(\mathcal{B}^{\frac{1}{2}} - 1\right)$$

$$\mathcal{Z}^{[\Gamma_1(3)]}_{\mathbb{F}_q(t)}(\mathcal{B}) \le \frac{(q^5 - q^3)}{(q^4 - 1)} \cdot \left(\mathcal{B}^{\frac{1}{3}} - 1\right)$$

$$\mathcal{Z}^{[\Gamma_1(4)]}_{\mathbb{F}_q(t)}(\mathcal{B}) \le \frac{(q^4 - q^2)}{(q^3 - 1)} \cdot \left(\mathcal{B}^{\frac{1}{4}} - 1\right)$$

$$\mathcal{Z}^{[\Gamma(2)]}_{\mathbb{F}_q(t)}(\mathcal{B}) \le 2 \cdot \frac{(q^5 - q^3)}{(q^4 - 1)} \cdot \left(\mathcal{B}^{\frac{1}{3}} - 1\right)$$

which is an equality when $\mathcal{B} = q^{12n}$ with $n \in \mathbb{Z}_{\ge 1}$ implying that the upper bound is a sharp enumeration, i.e., the upper bound is equal to the function at infinitely many values of $\mathcal{B} \in \mathbb{N}$.

Proof. The proof is at the end of §A.1. ■

The main leading term of the acquired sharp enumerations over $\mathbb{F}_q(t)$ matches the analogous asymptotic counts ordered by bounded naïve height of underlying elliptic curves over $\mathbb{Q}$ by Harron and Snowden in [HS, Theorem 1.2] (see also [Duke, Grant]). The lower order term over global function fields $\mathbb{F}_q(t)$ being a constant is new and would be interesting to prove or disprove over a number field $\mathbb{Q}$. It also remains to count the remaining ten cases (classified by the fundamental [Mazur, Theorem 8]) of the torsion subgroups with $|G| > 4$ over $\mathbb{F}_q(t)$ by bounded discriminant height and compare with analogous counting over $\mathbb{Q}$.

Now, let's consider instead elliptic curves with m-marked rational points. To count the number of certain curves of arithmetic genus one over global function fields $\mathbb{F}_q(t)$ with m-markings, we need to again extend the notion of (nonsingular) elliptic curves that admits desired m-markings. Here, we consider the $(m-1)$-stable m-marked curves of arithmetic genus one (defined by Smyth in [Smyth, §1.1] for characteristic $\neq 2, 3$, extended to lower characteristic with mild conditions by [LP, Definition 1.5.3]), see Definition A.9 for a precise definition. Note that if $\mathrm{char}(\mathbb{F}_q) > 3$ and $m = 1$, then 0-stable 1-marked curves are exactly stable elliptic curves as in [DM]. We now consider the following definition:

Definition A.2. Fix an integral reduced K-scheme B, where K is a field. Then a non-isotrivial flat morphism $\pi : X \to B$ is a m-marked $(m-1)$-stable genus one fibration over B if any fiber of π is a $(m-1)$-stable m-marked curves of arithmetic genus one.

Observe that if $\mathrm{char}(K) = 0$ or > 3, then a m-marked $(m-1)$-stable genus one fibration $X \to \mathbb{P}^1_K$ has a discriminant $\Delta \subset \mathbb{P}^1_K$, and if $K = \mathbb{F}_q$, then $0 < ht(\Delta) := q^{\deg \Delta}$.

Now, define $\mathcal{Z}^m_{\mathbb{F}_q(t)}(\mathcal{B})$ as follows:

$$\mathcal{Z}^m_{\mathbb{F}_q(t)}(\mathcal{B}) := |\{m\text{-marked } (m-1)\text{-stable genus one fibrations over } \mathbb{P}^1_{\mathbb{F}_q} \text{ with } 0 < ht(\Delta) \le \mathcal{B}\}|$$

Note that when $m = 1$, $\mathcal{Z}^1_{\mathbb{F}_q(t)}(\mathcal{B})$ counts the stable elliptic fibrations, which is described in [HP, Theorem 3] as $\mathcal{Z}_{\mathbb{F}_q(t)}(\mathcal{B})$ (by identifying stable elliptic fibrations with nonsingular semistable elliptic surfaces, see [HP, Proposition 11]). When $2 \le m \le 5$, we acquire the following sharp enumeration of $\mathcal{Z}^m_{\mathbb{F}_q(t)}(\mathcal{B})$:

Theorem A.3 (Sharp enumeration $\mathcal{Z}^m_{\mathbb{F}_q(t)}(\mathcal{B})$)**.** *If char*$(\mathbb{F}_q) \neq 2,3$*, then the function* $\mathcal{Z}^m_{\mathbb{F}_q(t)}(\mathcal{B})$*, which counts the number of m-marked* $(m-1)$*-stable genus one fibration over* $\mathbb{P}^1_{\mathbb{F}_q}$ *for* $2 \leq m \leq 5$ *ordered by* $0 < ht(\Delta) = q^{12n} \leq \mathcal{B}$*, satisfies:*

$$\mathcal{Z}^2_{\mathbb{F}_q(t)}(\mathcal{B}) \leq \frac{(q^{11}+q^{10}-q^8-q^7)}{(q^9-1)} \cdot (\mathcal{B}^{\frac{3}{4}}-1) + \frac{(q^7-q^5)}{(q^6-1)} \cdot (\mathcal{B}^{\frac{1}{2}}-1)$$

$$\mathcal{Z}^3_{\mathbb{F}_q(t)}(\mathcal{B}) \leq \frac{(q^{11}+q^{10}+q^9-q^7-q^6-q^5)}{(q^8-1)} \cdot (\mathcal{B}^{\frac{2}{3}}-1) + \frac{(q^5-q^3)}{(q^4-1)} \cdot (\mathcal{B}^{\frac{1}{3}}-1)$$

$$\mathcal{Z}^4_{\mathbb{F}_q(t)}(\mathcal{B}) \leq \frac{(q^{11}+q^{10}+q^9+q^8-q^6-q^5-q^4-q^3)}{(q^7-1)} \cdot (\mathcal{B}^{\frac{7}{12}}-1) + \frac{(q^5-q^3)}{(q^4-1)} \cdot (\mathcal{B}^{\frac{1}{3}}-1)$$

$$\mathcal{Z}^5_{\mathbb{F}_q(t)}(\mathcal{B}) \leq \frac{(q^{11}+q^{10}+q^9+q^8+q^7-q^5-q^4-q^3-q^2-q^1)}{(q^6-1)} \cdot (\mathcal{B}^{\frac{1}{2}}-1)$$

which is an equality when $\mathcal{B} = q^{12n}$ *with* $n \in \mathbb{Z}_{\geq 1}$ *implying that the upper bound is a sharp enumeration, i.e., the upper bound is equal to the function at infinitely many values of* $\mathcal{B} \in \mathbb{N}$*.*

Proof. The proof is at the end of §A.2. ■

A.1 Arithmetic of the moduli of generalized elliptic curves over $\mathbb{P}^1$ with level structures

The essential geometrical idea in acquiring the sharp enumeration is to consider the moduli stack of rational curves on a compactified modular curve as in [HP]. The various compactified modular curves $\overline{\mathcal{M}}_{1,1}[\Gamma]$ are isomorphic to the weighted projective stacks $\mathcal{P}(a,b)$.

Proposition A.4. *The moduli stack* $\overline{\mathcal{M}}_{1,1}[\Gamma]$ *of generalized elliptic curves with* $[\Gamma]$*-structures is isomorphic to the following when over a field* K*:*

1. *if* char$(K) \neq 2$*, the tame Deligne–Mumford moduli stack of generalized elliptic curves with* $[\Gamma_1(2)]$*-structures is isomorphic to*

$$(\overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])_K \cong [(\text{Spec } K[a_2,a_4]-(0,0))/\mathbb{G}_m] = \mathcal{P}_K(2,4),$$

2. *if* char$(K) \neq 3$*, the tame Deligne–Mumford moduli stack of generalized elliptic curves with* $[\Gamma_1(3)]$*-structures is isomorphic to*

$$(\overline{\mathcal{M}}_{1,1}[\Gamma_1(3)])_K \cong [(\text{Spec } K[a_1,a_3]-(0,0))/\mathbb{G}_m] = \mathcal{P}_K(1,3),$$

3. *if* char$(K) \neq 2$*, the tame Deligne–Mumford moduli stack of generalized elliptic curves with* $[\Gamma_1(4)]$*-structures is isomorphic to*

$$(\overline{\mathcal{M}}_{1,1}[\Gamma_1(4)])_K \cong [(\text{Spec } K[a_1,a_2]-(0,0))/\mathbb{G}_m] = \mathcal{P}_K(1,2),$$

4. *if* $\mathrm{char}(K) \neq 2$, *the tame Deligne–Mumford moduli stack of generalized elliptic curves with* $[\Gamma(2)]$*-structures is isomorphic to*

$$(\overline{\mathcal{M}}_{1,1}[\Gamma(2)])_K \cong [(\mathrm{Spec}\ K[a_2, a_2] - (0,0))/\mathbb{G}_m] = \mathcal{P}_K(2,2),$$

where $\lambda \cdot a_i = \lambda^i a_i$ *for* $\lambda \in \mathbb{G}_m$ *and* $i = 1,2,3,4$. *Thus, the* a_i*'s have degree* i *respectively. Moreover, the discriminant divisors of* $(\overline{\mathcal{M}}_{1,1}[\Gamma])_K \cong \mathcal{P}_K(i,j)$ *as above have degree 12.*

Proof. The moduli stack $\overline{\mathcal{M}}_{1,1}[\Gamma_1(2)]$ of generalized elliptic curves with $[\Gamma_1(2)]$-level structure has an isomorphism $\overline{\mathcal{M}}_{1,1}[\Gamma_1(2)] \cong \mathcal{P}(2,4)$ as in [Behrens, §1.3] through the universal equation

$$Y^2Z = X^3 + a_2X^2Z + a_4XZ^2\,,$$

over $\mathrm{Spec}(\mathbb{Z}[1/2])$. And the moduli stack $\overline{\mathcal{M}}_{1,1}[\Gamma_1(3)]$ of generalized elliptic curves with $[\Gamma_1(3)]$-level structure has an isomorphism $\overline{\mathcal{M}}_{1,1}[\Gamma_1(3)] \cong \mathcal{P}(1,3)$ as in [HMe, Proposition 4.5] through the universal equation

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3\,,$$

over $\mathrm{Spec}(\mathbb{Z}[1/3])$. And the moduli stack $\overline{\mathcal{M}}_{1,1}[\Gamma_1(4)]$ of generalized elliptic curves with $[\Gamma_1(4)]$-level structure has an isomorphism $\overline{\mathcal{M}}_{1,1}[\Gamma_1(4)] \cong \mathcal{P}(1,2)$ as in [Meier, Examples 2.1] through the universal equation

$$Y^2Z + a_1XYZ + a_1a_2YZ^2 = X^3 + a_2X^2Z\,,$$

over $\mathrm{Spec}(\mathbb{Z}[1/2])$. And the moduli stack $\overline{\mathcal{M}}_{1,1}[\Gamma(2)]$ of generalized elliptic curves with $[\Gamma(2)]$-level structure has an isomorphism $\overline{\mathcal{M}}_{1,1}[\Gamma(2)] \cong \mathcal{P}(2,2)$ as in [Stojanoska, Proposition 7.1] through the universal equation

$$Y^2Z = X^3 + (\lambda_1 + \lambda_2)X^2Z + \lambda_1\lambda_2XZ^2\,,$$

over $\mathrm{Spec}(\mathbb{Z}[1/2])$ where the degree of each λ_i is 2.

By Proposition 2.2, the weighted projective stacks are tame Deligne–Mumford as well.

For the degree of the discriminant, it suffices to find the weight of the $\mathbb{G}_m$-action. First, the four papers cited above explicitly construct universal families of elliptic curves over the schematic covers $(\mathrm{Spec}\ K[a_i, a_j] - (0,0)) \to \mathcal{P}_K(i,j)$ of the corresponding moduli stacks. The explicit defining equation of the respective universal family implies that the $\lambda \in \mathbb{G}_m$ also acts on the discriminant of the universal family by multiplying λ^{12}. Therefore, the discriminant has degree 12. ∎

We now consider the moduli stack $\mathcal{L}_{1,12n}^{[\Gamma]} := \mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma])$ of generalized elliptic curves over $\mathbb{P}^1$ with $[\Gamma]$-structures.

Proposition A.5. *Assume* $\mathrm{char}(K) = 0$ *or* $\neq 2$ *for* $[\Gamma] = [\Gamma_1(2)], [\Gamma_1(4)], [\Gamma(2)]$, *and* $\mathrm{char}(K) \neq 3$ *for* $[\Gamma] = [\Gamma_1(3)]$. *Then, the moduli stack* $\mathcal{L}_{1,12n}^{[\Gamma]}$ *of generalized elliptic curves over* $\mathbb{P}^1$ *with discriminant degree* $12n > 0$ *and* $[\Gamma]$*-structures is the tame Deligne–Mumford stack* $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma])$ *parameterizing the* K*-morphisms* $f : \mathbb{P}^1 \to \overline{\mathcal{M}}_{1,1}[\Gamma]$ *such that* $f^*\mathcal{O}_{\overline{\mathcal{M}}_{1,1}[\Gamma]}(1) \cong \mathcal{O}_{\mathbb{P}^1}(n)$.

Proof. Without the loss of generality, we prove the $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])$ case over a field K with $\mathrm{char}(K) \neq 2$. The proof for the other cases are analogous. By the definition of the universal family p, any generalized elliptic curves $\pi : Y \to \mathbb{P}^1$ with $[\Gamma_1(2)]$-structures comes from a morphism $f : \mathbb{P}^1 \to \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)]$ and vice versa. As this correspondence also works in families, the moduli stack of generalized elliptic curves over $\mathbb{P}^1$ with $[\Gamma_1(2)]$-structures is isomorphic to $\mathrm{Hom}(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])$.

Since the discriminant degree of f is $12 \deg f^*\mathcal{O}_{\overline{\mathcal{M}}_{1,1}[\Gamma_1(2)]}(1)$ by Proposition A.4, the substack $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])$ parametrizing such f's with $\deg f^*\mathcal{O}_{\overline{\mathcal{M}}_{1,1}[\Gamma_1(2)]}(1) = n$ is the desired moduli stack. Since $\deg f^*\mathcal{O}_{\overline{\mathcal{M}}_{1,1}[\Gamma_1(2)]}(1) = n$ is an open condition, $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])$ is an open substack of $\mathrm{Hom}(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])$, which is tame Deligne–Mumford by [HP2, Proposition 3.6] as $\overline{\mathcal{M}}_{1,1}[\Gamma_1(2)]$ itself is tame Deligne–Mumford by Proposition A.4. This shows that $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])$ satisfies the desired properties as well. ∎

We recall the motives & weighted point counts over finite fields of $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))$ worked out in [PS, Corollary 1.2] which is an extension of [HP, Theorem 1].

Corollary A.6 (Corollary 1.2 of [PS])**.** *If* $\mathrm{char}(K) \nmid a, b$, *then*

$$[\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))] = \mathbb{L}^{(a+b)n+1} - \mathbb{L}^{(a+b)n-1} \in K_0(\mathrm{Stck}_{/K}),$$

and if $\mathrm{char}(\mathbb{F}_q) \nmid a, b$, *then we have the weighted* $\mathbb{F}_q$*-point count*

$$|\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))(\mathbb{F}_q)| := \sum_{x \in \mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))(\mathbb{F}_q)} \frac{1}{|\mathrm{Aut}(x)|} = q^{(a+b)n+1} - q^{(a+b)n-1}.$$

We now acquire the exact number $|\mathcal{L}_{1,12n}^{[\Gamma]}(\mathbb{F}_q)/\sim|$ of $\mathbb{F}_q$–isomorphism classes of $\mathbb{F}_q$–points (i.e., the non–weighted point count) of the moduli stack $\mathcal{L}_{1,12n}^{[\Gamma]} \cong \mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma])$ of generalized elliptic curves over $\mathbb{P}^1$ with discriminant degree $12n > 0$ and $[\Gamma]$-structures.

Proposition A.7. *If* $\mathrm{char}(\mathbb{F}_q) \neq 2$, *then*

$$|\mathcal{L}_{1,12n}^{[\Gamma_1(2)]}(\mathbb{F}_q)/\sim| = 2 \cdot \#_q \left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(2,4))\right) = 2(q^{6n+1} - q^{6n-1})$$

$$|\mathcal{L}_{1,12n}^{[\Gamma_1(4)]}(\mathbb{F}_q)/\sim| = \#_q \left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(1,2))\right) = q^{3n+1} - q^{3n-1}$$

$$|\mathcal{L}_{1,12n}^{[\Gamma(2)]}(\mathbb{F}_q)/\sim| = 2 \cdot \#_q \left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(2,2))\right) = 2(q^{4n+1} - q^{4n-1})$$

If $\mathrm{char}(\mathbb{F}_q) \neq 3$, *then*

$$|\mathcal{L}_{1,12n}^{[\Gamma_1(3)]}(\mathbb{F}_q)/\sim| = \#_q \left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(1,3))\right) = q^{4n+1} - q^{4n-1}$$

Proof. Fix $n \in \mathbb{Z}_{\geq 1}$. Since any $\varphi_g \in \mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))$ is surjective, the generic stabilizer group $\mu_{\gcd(a,b)}$ of $\mathcal{P}(a,b)$ is the automorphism group of φ_g. Using the identification from Proposition A.5 and the weighted point counts of Hom stacks as in Corollary A.6 gives the desired formula as

$$|\mathcal{L}_{1,12n}^{[\Gamma]}(\mathbb{F}_q)/\sim| = |\mu_{\gcd(a,b)}| \cdot (q^{(a+b)n+1} - q^{(a+b)n-1})$$

where the factor of 2 comes from the hyperelliptic involution when $\mu_{\gcd(a,b)} = \mu_2$. ∎

Remark A.8. For weighted projective lines $\mathcal{P}(a,b)$ as in the cases of $\mathcal{L}_{1,12n}^{[\Gamma]}$, the inertia stack of the relevant Hom stack $\{\mathcal{I}\left(\mathrm{Hom}_n(\mathbb{P}^1,\mathcal{P}(a,b))\right)\}$ is a sum of $\left\{\mathrm{Hom}_n(\mathbb{P}^1_{\kappa(g)},\mathcal{P}_{\kappa(g)}(a,b))\right\}$ for each closed point $g\in\mathbb{G}_m$ with $\mathrm{ord}(g)\mid\gcd(a,b)$, as the only possible generic stabilizer of positive dimensional substacks of $\mathcal{P}(a,b)$. On the other hand, the terms with division function $\delta(r,q-1)$ do not occur in $|\mathcal{L}_{1,12n}^{[\Gamma]}(\mathbb{F}_q)/\sim|$ as the characteristic condition required to identify $\overline{\mathcal{M}}_{1,1}[\Gamma]$ as a weighted projective line implies that $\gcd(a,b)\mid q-1$. See [HP2, §4.3] for more details.

We now finally prove the Theorem A.1 using the above arithmetic invariants as follows:

Proof of Theorem A.1. Without the loss of the generality, we prove the $[\Gamma_1(2)]$–structures case over $\mathrm{char}(\mathbb{F}_q)\neq 2$. The proof for the other cases are analogous. By Proposition A.5 and Proposition A.7, we know the number of $\mathbb{F}_q$-isomorphism classes of generalized elliptic curves of discriminant degree $12n$ with $[\Gamma_1(2)]$-structures over $\mathbb{P}^1_{\mathbb{F}_q}$ is $|\mathcal{L}_{1,12n}^{[\Gamma_1(2)]}(\mathbb{F}_q)/\sim| = 2\cdot(q^{6n+1}-q^{6n-1})$. Using this, we can explicitly compute the sharp bound on $\mathcal{Z}_{\mathbb{F}_q(t)}^{[\Gamma_1(2)]}(\mathcal{B})$ as the following,

$$\begin{aligned}
\mathcal{Z}_{\mathbb{F}_q(t)}^{[\Gamma_1(2)]}(\mathcal{B}) &= \sum_{n=1}^{\left\lfloor\frac{log_q\mathcal{B}}{12}\right\rfloor}|\mathcal{L}_{1,12n}^{[\Gamma_1(2)]}(\mathbb{F}_q)/\sim| = \sum_{n=1}^{\left\lfloor\frac{log_q\mathcal{B}}{12}\right\rfloor} 2\cdot(q^{6n+1}-q^{6n-1}) \\
&= 2\cdot(q^1-q^{-1})\sum_{n=1}^{\left\lfloor\frac{log_q\mathcal{B}}{12}\right\rfloor} q^{6n} \leq 2\cdot(q^1-q^{-1})\left(q^6+\cdots+q^{6\cdot(\frac{log_q\mathcal{B}}{12})}\right) \\
&= 2\cdot(q^1-q^{-1})\frac{q^6(\mathcal{B}^{\frac{1}{2}}-1)}{(q^6-1)} = 2\cdot\frac{(q^7-q^5)}{(q^6-1)}\cdot(\mathcal{B}^{\frac{1}{2}}-1)
\end{aligned}$$

On the second line of the equations above, inequality becomes an equality if and only if $n:=\frac{log_q\mathcal{B}}{12}\in\mathbb{N}$, i.e., $\mathcal{B}=q^{12n}$ with $n\in\mathbb{Z}_{\geq 1}$. This implies that the acquired upper bound on $\mathcal{Z}_{\mathbb{F}_q(t)}^{[\Gamma_1(2)]}(\mathcal{B})$ is a sharp enumeration, i.e., the upper bound is equal to the function at infinitely many values of $\mathcal{B}\in\mathbb{N}$. ■

A.2 Arithmetic of the moduli of m-marked genus one fibrations over $\mathbb{P}^1$

We proceed to determine the sharp enumeration on the number of m-marked $(m-1)$-stable genus one fibrations over $\mathbb{P}^1_{\mathbb{F}_q}$ for $2\leq m\leq 5$. First, we state the definition of m-marked $(m-1)$-stability from [LP, Definition 1.5.3], which is a modification of the Deligne–Mumford stability [DM]:

Definition A.9. Let K be a field and m be a positive integer. Then, a tuple $(C,p_1,\ldots,p_m)$, of a geometrically connected, geometrically reduced, and proper K-curve C of arithmetic genus one with m distinct K-rational points p_i in the smooth locus of C, is a $(m-1)$-stable m-marked curve of arithmetic genus one if the curve $C_{\overline{K}}:=C\times_K\overline{K}$ and the divisor $\Sigma:=\{p_1,\ldots,p_m\}$ satisfy the following properties, where $\overline{K}$ is the algebraic closure of K:

1. $C_{\overline{K}}$ has only nodes and elliptic u-fold points as singularities (see below), where $u < m$,

2. $C_{\overline{K}}$ has no disconnecting nodes, and

3. every irreducible component of $C_{\overline{K}}$ contains at least one marked point.

Remark A.10. A singular point of a curve over $\overline{K}$ is an elliptic u-fold singular point if it is Gorenstein and étale locally isomorphic to a union of u general lines in $\mathbb{P}^{u-1}_{\overline{K}}$ passing through a common point.

Note that the name "$(m-1)$-stability" comes from [Smyth, §1.1], which is defined when $\mathrm{char}(K) \neq 2, 3$. By [LP, Proposition 1.5.4], the above definition (by [LP, Definition 1.5.3]) coincides with that of Smyth when $\mathrm{char}(K) \neq 2, 3$, hence we adapt Smyth's naming convention on Lekili and Polishchuk's definition. Regardless, we focus on the case when $\mathrm{char}(K) \neq 2, 3$, so that the moduli stack of such curves behaves reasonably.

By [Smyth, Theorem 3.8], we are able to formulate the moduli stack of $(m-1)$-stable m-marked curves of arithmetic genus one over any field of characteristic $\neq 2, 3$:

Theorem A.11. *There exists a proper irreducible Deligne–Mumford moduli stack* $\overline{\mathcal{M}}_{1,m}(m-1)$ *of* $(m-1)$*-stable* m*-marked curves arithmetic genus one over* $\mathrm{Spec}(\mathbb{Z}[1/6])$

Note that when $m = 1$, $\overline{\mathcal{M}}_{1,1}(0) \cong \overline{\mathcal{M}}_{1,1}$ is the Deligne–Mumford moduli stack of stable elliptic curves.

In fact, the construction of $\overline{\mathcal{M}}_{1,m}(m-1)$ extends to $\mathrm{Spec}\,\mathbb{Z}$ by [LP, Theorem 1.5.7] (called $\overline{\mathcal{M}}^{\infty}_{1,m}$ in loc.cit.) as an algebraic stack, which is proper over $\mathrm{Spec}\,\mathbb{Z}[1/N]$ where N depends on m:

- if $m \geq 3$, then $N = 1$,
- if $m = 2$, then $N = 2$, and
- if $m = 1$, then $N = 6$.

However, even with those assumptions above, $\overline{\mathcal{M}}_{1,m}(m-1)$ is not necessarily Deligne–Mumford. Nevertheless, by [LP, Theorem 1.5.7.], we obtain the explicit descriptions of $\overline{\mathcal{M}}_{1,m}(m-1)$:

Proposition A.12. *The moduli stack* $\overline{\mathcal{M}}_{1,m}(m-1)$ *of* m*-marked* $(m-1)$*-stable curves of arithmetic genus one for* $2 \leq m \leq 5$ *is isomorphic to the following, for a field* K*:*

1. *if* $\mathrm{char}(K) \neq 2, 3$*, the tame Deligne–Mumford moduli stack of 2-marked 1-stable curves of arithmetic genus one is isomorphic to*

$$(\overline{\mathcal{M}}_{1,2}(1))_K \cong [(\mathrm{Spec}\ K[a_2, a_3, a_4] - 0)/\mathbb{G}_m] = \mathcal{P}_K(2, 3, 4),$$

2. *if* $\mathrm{char}(K) \neq 2, 3$*, the tame Deligne–Mumford moduli stack of 3-marked 2-stable curves of arithmetic genus one is isomorphic to*

$$(\overline{\mathcal{M}}_{1,3}(2))_K \cong [(\mathrm{Spec}\ K[a_1, a_2, a_2, a_3] - 0)/\mathbb{G}_m] = \mathcal{P}_K(1, 2, 2, 3),$$

3. *if* $\mathrm{char}(K) \neq 2$*, the tame Deligne–Mumford moduli stack of 4-marked 3-stable curves of arithmetic genus one is isomorphic to*

$$(\overline{\mathcal{M}}_{1,4}(3))_K \cong [(\mathrm{Spec}\ K[a_1, a_1, a_1, a_2, a_2] - 0)/\mathbb{G}_m] = \mathcal{P}_K(1, 1, 1, 2, 2),$$

4. *the moduli stack of 5-marked 4-stable curves of arithmetic genus one is isomorphic to a scheme*

$$(\overline{\mathcal{M}}_{1,5}(4))_K \cong [(\mathrm{Spec}\ K[a_1, a_1, a_1, a_1, a_1, a_1] - 0)/\mathbb{G}_m] = \mathbb{P}_K(1, 1, 1, 1, 1, 1) \cong \mathbb{P}^5_K,$$

where $\lambda \cdot a_i = \lambda^i a_i$ *for* $\lambda \in \mathbb{G}_m$ *and* $i = 1, 2, 3, 4$*. Thus, the* a_i*'s have degree i respectively. Furthermore, if* $\mathrm{char}(K) \neq 2, 3$*, then the discriminant divisors of such* $\overline{\mathcal{M}}_{1,m}(m-1)$ *have degree 12.*

Proof. Proof of [LP, Theorem 1.5.7.] gives the corresponding isomorphisms $\overline{\mathcal{M}}_{1,m}(m-1) \cong \mathcal{P}(\vec{\lambda})$. By Proposition 2.2, the weighted projective stacks are tame Deligne–Mumford as well, and in fact, smooth.

For the degree of the discriminant when $\mathrm{char}(K) \neq 2, 3$, it suffices to describe the discriminant divisor, the locus of singular curves in $\overline{\mathcal{M}}_{1,m}(m-1)$. First, [LP, Theorem 1.5.7.] shows that in the above case, where $\overline{\mathcal{M}}_{1,m}(m-1) \cong \mathcal{P}(\vec{\lambda})$, the line bundle $\mathcal{O}_{\mathcal{P}(\vec{\lambda})}(1)$ of degree one is isomorphic to $\lambda := \pi_*\omega_\pi$, where $\pi : \overline{\mathcal{C}}_{1,m}(m-1) \to \overline{\mathcal{M}}_{1,m}(m-1)$ is the universal family of $(m-1)$-stable m-marked curves of arithmetic genus one. Since $\overline{\mathcal{M}}_{1,m}(m-1)$ is smooth and the Picard rank is one (generated by λ), the discriminant divisor is Cartier. In fact, by [Smyth2, §3.1], it coincides with the locus Δ_{irr} of curves with non-disconnecting nodes or non-nodal singular points. Then [Smyth2, Remark 3.3] (which assumes $\mathrm{char}(K) \neq 2, 3$) implies that $\Delta_{irr} \sim 12\lambda$, thus the discriminant divisor has degree 12. ∎

We now consider the moduli stacks of m-marked $(m-1)$-stable genus one fibrations over $\mathbb{P}^1_K$ for any field K of $\mathrm{char}(K) = 0$ or > 3:

Proposition A.13. *Assume* $\mathrm{char}(K) = 0$ *or* > 3*. If* $2 \leq m \leq 5$*, then the moduli stack* $\mathcal{L}^m_{1,12n}$ *of m-marked* $(m-1)$*-stable genus one fibrations over* $\mathbb{P}^1_K$ *with discriminant degree* $12n$ *is the tame Deligne–Mumford stack* $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,m}(m-1))$ *parameterizing the K-morphisms* $f : \mathbb{P}^1 \to \overline{\mathcal{M}}_{1,m}(m-1)$ *such that* $f^*\mathcal{O}_{\mathcal{P}(\vec{\lambda})}(1) \cong \mathcal{O}_{\mathbb{P}^1}(n)$*.*

Proof. Without the loss of the generality, we prove the 2-marked 1-stable curves of arithmetic genus one case $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,2}(1))$ over $\mathrm{char}(\mathbb{F}_q) \neq 2, 3$. The proof for the other cases are analogous. By the definition of the universal family p, any 2-marked 1-stable arithmetic genus one curves $\pi : Y \to \mathbb{P}^1$ with discriminant degree $12n$ comes from a morphism $f : \mathbb{P}^1 \to \overline{\mathcal{M}}_{1,2}(1)$ and vice versa. As this correspondence also works in families, the moduli stack of 2-marked 1-stable curves of arithmetic genus one over $\mathbb{P}^1_K$ is isomorphic to $\mathrm{Hom}(\mathbb{P}^1, \overline{\mathcal{M}}_{1,2}(1))$.

Since the discriminant degree of f is $12 \deg f^*\mathcal{O}_{\overline{\mathcal{M}}_{1,2}(1)}(1)$ by Proposition A.12, the substack $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,2}(1))$ parametrizing such f's with $\deg f^*\mathcal{O}_{\overline{\mathcal{M}}_{1,2}(1)}(1) = n$ is the desired moduli stack. Since $\deg f^*\mathcal{O}_{\overline{\mathcal{M}}_{1,2}(1)}(1) = n$ is an open condition, $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,2}(1))$ is an open substack of $\mathrm{Hom}(\mathbb{P}^1, \overline{\mathcal{M}}_{1,2}(1))$, which is tame Deligne–Mumford by [HP2, Proposition 3.6] as $\overline{\mathcal{M}}_{1,2}(1)$ itself is tame Deligne–Mumford by Proposition A.12. This shows that $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,2}(1))$ satisfies the desired properties as well. ∎

We now acquire the exact number $|\mathcal{L}^m_{1,12n}(\mathbb{F}_q)/\sim|$ of $\mathbb{F}_q$–isomorphism classes of $\mathbb{F}_q$–points of the moduli stack $\mathcal{L}^m_{1,12n} \cong \mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,m}(m-1))$ of m-marked $(m-1)$-stable genus one fibrations over $\mathbb{P}^1$ with discriminant degree $12n > 0$.

Proposition A.14. *If* $\mathrm{char}(\mathbb{F}_q) \neq 2, 3$*, then*

$$\begin{aligned}|\mathcal{L}^2_{1,12n}(\mathbb{F}_q)/\sim| &= \#_q\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(2,3,4))\right) + \#_q\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(2,4))\right)\\ &= (q^{9n+2} + q^{9n+1} - q^{9n-1} - q^{9n-2}) + (q^{6n+1} - q^{6n-1})\end{aligned}$$

$$\begin{aligned}|\mathcal{L}^3_{1,12n}(\mathbb{F}_q)/\sim| &= \#_q\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(1,2,2,3))\right) + \#_q\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(2,2))\right)\\ &= (q^{8n+3} + q^{8n+2} + q^{8n+1} - q^{8n-1} - q^{8n-2} - q^{8n-3}) + (q^{4n+1} - q^{4n-1})\end{aligned}$$

$$\begin{aligned}|\mathcal{L}^4_{1,12n}(\mathbb{F}_q)/\sim| &= \#_q\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(1,1,1,2,2))\right) + \#_q\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(2,2))\right)\\ &= (q^{7n+4} + q^{7n+3} + q^{7n+2} + q^{7n+1} - q^{7n-1} - q^{7n-2} - q^{7n-3} - q^{7n-4})\\ &\quad + (q^{4n+1} - q^{4n-1})\\ |\mathcal{L}^5_{1,12n}(\mathbb{F}_q)/\sim| &= \#_q\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathbb{P}(1,1,1,1,1,1) \cong \mathbb{P}^5)\right)\\ &= q^{6n+5} + q^{6n+4} + q^{6n+3} + q^{6n+2} + q^{6n+1} - q^{6n-1} - q^{6n-2} - q^{6n-3} - q^{6n-4} - q^{6n-5}\end{aligned}$$

Proof. Note that $\overline{\mathcal{M}}_{1,2}(1) \cong \mathcal{P}(2,3,4)$ has the substack $\mathcal{P}(2,4)$ with the generic stabilizer of order 2. This implies that the number of isomorphism classes of $\mathbb{F}_q$-points of $\mathcal{L}^2_{1,12n}$ with discriminant degree $12n$ is $|\mathcal{L}^2_{1,12n}(\mathbb{F}_q)/\sim| = (q^{9n+2}+q^{9n+1}-q^{9n-1}-q^{9n-2})+(q^{6n+1}-q^{6n-1})$ by summing the weighted point counts of Hom stacks as in [HP2, Proposition 4.10]. Similarly, $\overline{\mathcal{M}}_{1,3}(2) \cong \mathcal{P}(1,2,2,3)$ and $\overline{\mathcal{M}}_{1,4}(3) \cong \mathcal{P}(1,1,1,2,2)$ has the substack $\mathcal{P}(2,2)$ with the generic stabilizer of order 2. This implies that adding $(q^{4n+1} - q^{4n-1})$ to the corresponding weighted points count gives the desired non–weighted point counts. Finally, $\overline{\mathcal{M}}_{1,5}(4) \cong \mathbb{P}^5$, so that the non-weighted point count coincides with the weighted point count. ■

We now finally prove the Theorem A.3 using the above arithmetic invariants as follows:

Proof. Without the loss of the generality, we prove the 2-marked 1-stable curves of arithmetic genus one case $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,2}(1) \cong \mathcal{P}(2,3,4))$ over $\mathrm{char}(\mathbb{F}_q) \neq 2, 3$. The proof for the other cases are analogous. Knowing the number of $\mathbb{F}_q$-isomorphism classes of 1-stable arithmetic genus one curves over $\mathbb{P}^1$ with discriminant degree $12n$ and 2-marked Weierstrass sections over $\mathbb{F}_q$ is $|\mathcal{L}^2_{1,12n}(\mathbb{F}_q)/\sim| = (q^{9n+2} + q^{9n+1} - q^{9n-1} - q^{9n-2}) + (q^{6n+1} - q^{6n-1})$ by Proposition A.14, we can explicitly compute the sharp bound on $\mathcal{Z}^2_{\mathbb{F}_q(t)}(\mathcal{B})$ as the following,

$$\begin{aligned}\mathcal{Z}^2_{\mathbb{F}_q(t)}(\mathcal{B}) = \sum_{n=1}^{\left\lfloor \frac{log_q \mathcal{B}}{12} \right\rfloor} |\mathcal{L}^2_{1,12n}(\mathbb{F}_q)/\sim| &= \sum_{n=1}^{\left\lfloor \frac{log_q \mathcal{B}}{12} \right\rfloor} (q^{9n+2} + q^{9n+1} - q^{9n-1} - q^{9n-2}) + (q^{6n+1} - q^{6n-1})\\ &= (q^2 + q^1 - q^{-1} - q^{-2}) \sum_{n=1}^{\left\lfloor \frac{log_q \mathcal{B}}{12} \right\rfloor} q^{9n} + (q^1 - q^{-1}) \sum_{n=1}^{\left\lfloor \frac{log_q \mathcal{B}}{12} \right\rfloor} q^{6n}\end{aligned}$$

$$\leq (q^2 + q^1 - q^{-1} - q^{-2}) \left(q^9 + \cdots + q^{9\cdot(\frac{log_q \mathcal{B}}{12})}\right) + (q^1 - q^{-1}) \left(q^6 + \cdots + q^{6\cdot(\frac{log_q \mathcal{B}}{12})}\right)$$
$$= (q^2 + q^1 - q^{-1} - q^{-2}) \cdot \frac{q^9(\mathcal{B}^{\frac{3}{4}} - 1)}{(q^9 - 1)} + (q^1 - q^{-1}) \frac{q^6(\mathcal{B}^{\frac{1}{2}} - 1)}{(q^6 - 1)}$$
$$= \frac{(q^{11} + q^{10} - q^8 - q^7)}{(q^9 - 1)} \cdot (\mathcal{B}^{\frac{3}{4}} - 1) + \frac{(q^7 - q^5)}{(q^6 - 1)} \cdot (\mathcal{B}^{\frac{1}{2}} - 1)$$

On the third line of the equations above, inequality becomes an equality if and only if $n := \frac{log_q \mathcal{B}}{12} \in \mathbb{N}$, i.e., $\mathcal{B} = q^{12n}$ with $n \in \mathbb{Z}_{\geq 1}$. This implies that the acquired upper bound on $\mathcal{Z}^2_{\mathbb{F}_q(t)}(\mathcal{B})$ is a sharp enumeration, i.e., the upper bound is equal to the function at infinitely many values of $\mathcal{B} \in \mathbb{N}$. ■

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