Totality of Rational points on Moduli stacks

Counting Families of Varieties : Lecture 1

June Park

The University of Sydney

KIAS-LFANT Winter School on Number Theory

Rational Points on Projective Varieties over $\mathbb{Q}$

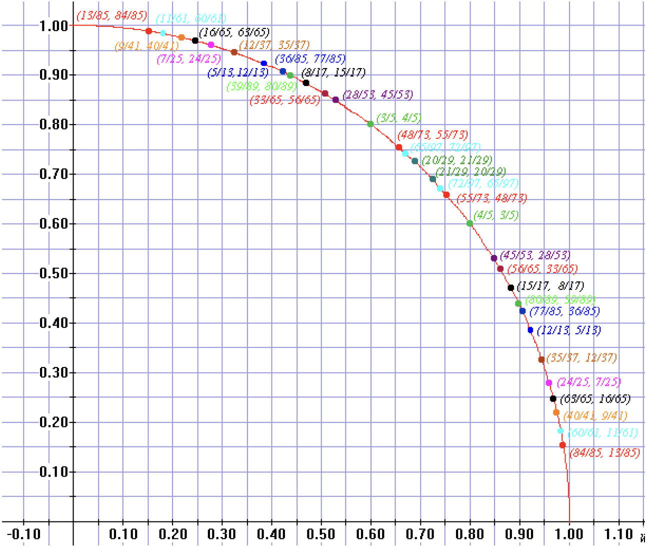


Figure 1: Rational points on $x^2 + y^2 = 1$ over $\mathbb{Q}$ - Pythagorean Triples

Why should we be happy?

1. Height of a rational number a/b with $\gcd(a,b)=1$ is $ht(a/b)=\max(|a|,|b|)$. Therefore, $ht(4/10)=5$. Bigger denominator allows more possibilities for numerator thus more rational points.

2. Geometry is enlightening and the quadratic formula is awesome as we have found / parametrized all rational points $(x,y)=\left(\frac{t^2-1}{t^2+1},\frac{2t}{t^2+1}\right)\in\mathbb{Q}^2$ on the unit circle over $\mathbb{Q}$

3. Integral points $[X:Y:Z]=[a^2-b^2:2ab:a^2+b^2]\in\mathbb{Z}^3$ on $C:=V(X^2+Y^2-Z^2)$ correspond to *"Pythagorean Triples"*

4. On **projective varieties**, the integral and the rational points coincide i.e., $X(\mathbb{Q})=X(\mathbb{Z})$. Bear in mind $\gcd(a,b)=1$.

Why should we be unhappy?

1. If we don't have a rational point to begin the process then we cannot apply quadratic formula. For example, $x^2 + y^2 = 3$ and turns out $X(\mathbb{Q}) = \emptyset$. We need Arithmetic to prove this.

2. Take $x^4 + y^4 = 1$ then we have *"Fermat's Last Theorem"* regarding $x^n + y^n = 1$ with $n = 4$. By Wiles-Taylor, we **know** it has only 4 rational points $X(\mathbb{Q}) = \{(\pm 1, 0), (0, \pm 1)\}$. Recalling Mordell-Faltings, we **know** it had $X(\mathbb{Q}) < \infty$

3. Take $y^2 = x^3 + Ax + B$ this is 1 polynomial in 2 variables of degree 3 (the Weierstrass cubic for an elliptic curve over $\mathbb{Q}$). **What are $E(\mathbb{Q})$?** Shockingly, *we still cannot answer this.*

4. Actually, we know there is at least 1 rational point, the point at $\infty = [0:1:0]$ for $E : V(Y^2Z - X^3 - AXZ^2 - BZ^3)$

Degree of countable infinity, the Rank

1. By Mordell-Weil, the set $E(\mathbb{Q})$ of rational points on $E/\mathbb{Q}$ has a finitely-generated abelian group structure $E(\mathbb{Q}) = \mathbb{Z}^r \oplus T$ with algebraic rank $r \in \mathbb{Z}_{\geq 0}$ and torsion subgroup T

2. The rank r of $E(\mathbb{Q})$ is **not** well understood.

   2.1 An algorithm that is guaranteed to correctly compute r?

   2.2 Which values of r can occur? *How often do they occur?*

   2.3 Is there an upper limit, or can r be arbitrarily large?

3. When r is small, computational methods exist but when r is large, often the best we can do is a lower bound; we now know, assuming GRH, there is an $E/\mathbb{Q}$ with $r \geq 29$ by Elkies-Klagsbrun (2024).

Demography of Elliptic Curves $E/\mathbb{Q}$

Trying to find / parametrize all the rational points on a given $E/\mathbb{Q}$ is a dead-end. Thus we rotate our entry. We would like to think about *the Question of Distribution and Proportion* over all $E/\mathbb{Q}$

Naive height for $E : y^2 = x^3 + Ax + B$ with no $p^4|A$ and $p^6|B$ (minimal Weierstrass model) is $ht(E) := \max(4|A|^3, 27B^2)$.

Conjecture (Minimality + Parity; Goldfeld and Katz-Sarnak)

Over any number field, 50% of all elliptic curves (when ordered by height) have Mordell-Weil rank $r = 0$ and the other 50% have Mordell-Weil rank $r = 1$. Moreover, higher Mordell-Weil ranks $r \geq 2$ constitute 0% of all elliptic curves, even though there may exist infinitely many such elliptic curves. Therefore, a suitably-defined average rank would be $\frac{1}{2}$.

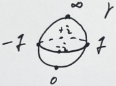
What does this really mean? To talk about Average, we need the **"Total number of elliptic curves over $\mathbb{Q}$ up to isomorphism"**.

Triangle of Rational Dedekind Domains

Consider not only $E/\mathbb{Q}$ but also $E/\mathbb{F}_q(t)$ as well as $E/\mathbb{C}(z)$

1. The rational number field $\mathbb{Q}$ consisting of ratio of integer numbers in $\mathbb{Z}$ is **the rational global field of char** $= 0$
2. The rational function field $\mathbb{F}_q(t)$ with *coefficients* in $\mathbb{F}_q = \mathbb{F}_{p^r}$ consisting of ratio of polynomial functions in $\mathbb{F}_q[t]$ is **the rational global field of char** $= p > 0 \Leftrightarrow$ Projective line $\mathbb{P}^1_{\mathbb{F}_q}$
3. The meromorphic function field $\mathbb{C}(z)$ with *coefficients* in $\mathbb{C}$ consisting of ratio of holomorphic functions in $\mathbb{C}[z]$ is **NOT** the rational global field of char $= 0 \Leftrightarrow$ Riemann sphere $\mathbb{CP}^1$

Let us count ALL elliptic curves over $K = \mathbb{F}_q(t)$ wrt height.



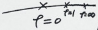
$r=\frac{1}{2}$, $\mathcal{K}=4$

$\mathrm{Vol}(\mathbb{CP}^1)=\pi$

$b_0=1$

$b_2=1$

$e=2$



$[u:v]$ $t=u/v$

$|\mathbb{P}^1(\mathbb{F}_q)|=q+1$

$\mathbb{CP}^1$ — G.B.F — S^2 — G.L.T.F — $\mathbb{P}^1_{\mathbb{F}_q}$

Chow

W.C.

W.T.

$\mathbb{C}(z)$ --- $\mathbb{F}_q(t)$ —∞— $\mathbb{F}_q(t)$ — Poly/Poly

Holo/Holo

G.F.A.

$\mathbb{Q}$ — Int/Int

AG/$\mathbb{C}$ AT/k AG/$\mathbb{F}_q$

KT

CV CA

NT

$\mathbb{Z} \sim \mathbb{F}_q[t]$ 'As integers so polynomials'

Let $\mathcal{E}$ be a suitable cat. of sheaves then

$\mathcal{E}(\mathrm{Spec}\,\mathbb{Z}\setminus T) \simeq \mathcal{E}(\mathbb{P}^1_{\mathbb{F}_q}\setminus S)$

Analogy

'Aware of each others'

The Sharp Enumeration over Rational Function Field

Define *height of discriminant* Δ *over* $\mathbb{F}_q(t)$ as $ht(\Delta) := q^{\deg \Delta}$

- Elliptic case: $Deg(\Delta) = 12n \implies ht(\Delta) = q^{12n}$ for $n \in \mathbb{Z}_{\geq 0}$

We consider the counting function $\mathcal{N}(\mathbb{F}_q(t),\ B) :=$

$$\left|\left\{\text{Minimal elliptic curves over } \mathbb{P}^1_{\mathbb{F}_q} \text{ with } 0 < ht(\Delta) \leq B\right\}\right|$$

Theorem (Dori Bejleri–JP–Matthew Satriano; April 2024)

Let $\mathrm{char}(\mathbb{F}_q) > 3$ and $\delta(x) := \begin{cases} 1 & \text{if } x \text{ divides } q-1, \\ 0 & \text{otherwise.} \end{cases}$, then

$$\begin{aligned}\mathcal{N}(\mathbb{F}_q(t),\ B) = 2\left(\frac{q^9-1}{q^8-q^7}\right)B^{5/6} - 2B^{1/6} \\ + \delta(6)\cdot 4\left(\frac{q^5-1}{q^5-q^4}\right)B^{1/2} + \delta(4)\cdot 2\left(\frac{q^3-1}{q^3-q^2}\right)B^{1/3} \\ + \delta(6)\cdot 4 + \delta(4)\cdot 2\end{aligned}$$

Precise proportions of E/K motivated by NT

Theorem (Generic Torsion Freeness; Phillips)

The set of torsion-free elliptic curves over global function fields has density 1. i.e., 'Most elliptic curves over K are torsion free'.

Theorem (Boundedness; Tate-Shafarevich & Ulmer)

The ranks of <u>non-constant</u> elliptic curves over $\mathbb{F}_q(t)$ are unbounded (in both the **isotrivial** *and* **non-isotrivial** *cases).*

Projective Elliptic K3 Surface of height $n = 2$

$$y^2 = x^3 + a_4(u:v)x + a_6(u:v)$$

Weierstrass data for elliptic fibration on algebraic K3 surface,

$$\begin{cases} a_4(u:v) & = -3u^4v^4, \text{ degree } 8 = 4 \times 2, \\ a_6(u:v) & = u^5v^5(u^2+v^2), \text{ degree } 12 = 6 \times 2. \end{cases}$$

Then we have $\Delta = 4a_4^3 + 27a_6^2$ and $j = 1728 \cdot 4a_4^3/\Delta$

$$\begin{cases} \Delta & = 27u^{10}v^{10}(u-v)^2(u+v)^2, \text{ degree } 24 = 12 \times 2, \\ j & = 1728 \cdot -\frac{4u^2v^2}{(u-v)^2(u+v)^2}, \text{ degree 4! NOT 24.} \end{cases}$$

Wait, where did degree 20 go?

After all, we should have $j : \mathbb{P}^1 \to \overline{M}_{1,1} \cong \mathbb{P}^1$ of degree 24?

Well, it can get whole lot worse.

Isotrivial Rational Elliptic Surface of height $n = 1$

Isotrivial Rational Elliptic Surface $\quad n = d + \sum_{i=1}^{s} a_i/r_i$

$$n = 1 = 1/6 + 5/6$$

$$\begin{cases} a_4 = 0 \\ a_6 = u \cdot v^5 \end{cases} \qquad \begin{cases} d = 0 \\ a_1/r_1 = 1/6, \quad a_2/r_2 = 5/6 \end{cases}$$

$[u:v] \quad u/v = t$

$v(a_4) = \infty$

minimal

$$v(a_6) = \begin{cases} 1 < 6 \text{ at } u = 0 \Rightarrow v = 1 \quad [0:1] \Leftrightarrow t = 0 \\ 5 < 6 \text{ at } v = 0 \Rightarrow u = 1 \quad [1:0] \Leftrightarrow t = \infty \end{cases}$$

$\Delta = 27 u^2 v^{10}$ – deg 12

$j \equiv 0$

$$y^2 = x^3 + u v^5 \in \mathcal{S}(6) \cdot 4 \cdot \frac{q^5 - 1}{q^5 - q^4} B^{1/2} \qquad B = q^{12}$$

$\downarrow \; u = z^6$

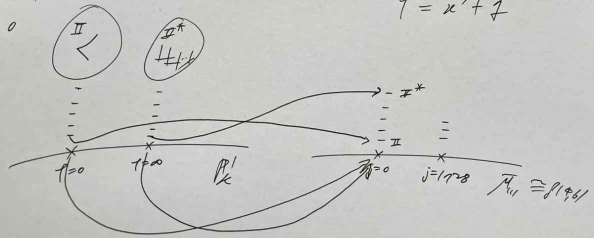
$$y^2 = x^3 + \cancel{(z^1)^6} v^5 \qquad \because \chi(1) = 1$$

$\downarrow \; v = w^6$

$$y^2 = x^3 + \cancel{(w^5)^6} \qquad \because \chi(1) = 5$$

$\downarrow$

$$y^2 = x^3 + 1$$



Precise proportions of E/K motivated by NT

We consider the counting function $\mathcal{N}_T^r(\mathbb{F}_q(t),\ B) :=$

$|\{\text{Minimal } E/\mathbb{F}_q(t) \text{ with algebraic rank } r, \text{ torsion } T \text{ and } ht(\Delta) \leq B\}|$

If we combine the above two Theorems and the Rank Distribution Conjecture, we are led to the following conclusion.

Quantitative Rank Distribution Conjecture over $K = \mathbb{F}_q(t)$

$$\mathcal{N}_{T=0}^{r=0}(\mathbb{F}_q(t),\ B) = \left(\frac{q^9-1}{q^8-q^7}\right) B^{5/6} + o(B^{\frac{5}{6}}),$$

$$\mathcal{N}_{T=0}^{r=1}(\mathbb{F}_q(t),\ B) = \left(\frac{q^9-1}{q^8-q^7}\right) B^{5/6} + o(B^{\frac{5}{6}}),$$

$$\mathcal{N}_{T}^{r\geq 2}(\mathbb{F}_q(t),\ B) = o(B^{\frac{5}{6}}),\ \text{where all o are little-o.}$$

† $|E(K)| = 1$ and $E(K) = \mathbb{Z}$ each corresponds to 50% of all elliptic curves over K ordered by discriminant height having *equal* main leading term $B^{5/6}$ with *identical* leading coefficient $\left(\frac{q^9-1}{q^8-q^7}\right)$.

Totality of Rational points on Moduli stacks

Counting Families of Varieties : Lecture 2

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Grothendieck ring $K_0(\mathrm{Stck}_k)$ of k-algebraic stacks

Ekedahl in 2009 introduced the Grothendieck ring $K_0(\mathrm{Stck}_k)$ of algebraic stacks extending the classical Grothendieck ring $K_0(\mathrm{Var}_k)$ of varieties first defined by Grothendieck in 1964.

Definition

Fix a field k. Then the *Grothendieck ring* $K_0(\mathrm{Stck}_k)$ *of algebraic stacks of finite type over k all of whose stabilizer group schemes are affine* is an abelian group generated by isomorphism classes of algebraic stacks $\{\mathcal{X}\}$ modulo relations:

- $\{\mathcal{X}\} = \{\mathcal{Z}\} + \{\mathcal{X} \setminus \mathcal{Z}\}$ for $\mathcal{Z} \subset \mathcal{X}$ a closed substack,
- $\{\mathcal{E}\} = \{\mathcal{X} \times \mathbb{A}^n\}$ for $\mathcal{E}$ a vector bundle of rank n on $\mathcal{X}$.

Multiplication on $K_0(\mathrm{Stck}_k)$ is induced by $\{\mathcal{X}\}\{\mathcal{Y}\} \coloneqq \{\mathcal{X} \times \mathcal{Y}\}$. A distinguished element $\mathbb{L} \coloneqq \{\mathbb{A}^1\}$ is called the *Lefschetz motive*.

$\{\mathbb{P}^1\} = \{\mathrm{II}\} = \mathbb{L}+1,\ \{\mathbb{P}^N\} = \mathbb{L}^N+\ldots+1,\ \{\mathbb{G}_m\} = \mathbb{L}-1,\ \{E\} = ?$

Universal Property for Additive Invariants

For any ring R and any function $\tilde{\nu} : \mathrm{Stck}_k \to R$ satisfying relations
1) $\tilde{\nu}(\mathcal{X}) = \tilde{\nu}(\mathcal{Y})$ whenever $\mathcal{X} \cong \mathcal{Y}$,
2) $\tilde{\nu}(\mathcal{X}) = \tilde{\nu}(\mathcal{U}) + \tilde{\nu}(\mathcal{X} \setminus \mathcal{U})$ for $\mathcal{U} \hookrightarrow \mathcal{X}$ an open immersion,
2) $\tilde{\nu}(\mathcal{X} \times \mathcal{Y}) = \tilde{\nu}(\mathcal{X}) \cdot \tilde{\nu}(\mathcal{Y})$,
there is a unique ring homomorphism $\nu : K_0(\mathrm{Stck}_k) \to R$

$$\begin{tikzcd} & \mathrm{Stck}_k \arrow[dl, "\{\ \}"'] \arrow[dr, "\tilde{\nu}"] & \\ K_0(\mathrm{Stck}_k) \arrow[rr, "\nu"'] & & R \end{tikzcd}$$

Such homomorphism ν are called **motivic measures**.

$\therefore$ When $k = \mathbb{F}_q$, the point counting measure $\{\mathcal{X}\} \mapsto \#_q(\mathcal{X})$ is a well-defined ring homomorphism $\#_q : K_0(\mathrm{Stck}_{\mathbb{F}_q}) \to \mathbb{Q}$ giving the weighted point count $\#_q(\mathcal{X})$ of $\mathcal{X}$ over $\mathbb{F}_q$.

$$|\mathbb{P}^N(\mathbb{F}_q)| = q^N + \ldots + 1,\ q + 1 - 2\sqrt{q} \leq |E(\mathbb{F}_q)| \leq q + 1 + 2\sqrt{q}$$

V. Arnol'd, J. Milnor, M. Atiyah, G. Segal

1. Hom space $\mathrm{Hom}_n(\mathbb{P}^1_D, \mathbb{P}^1_T)$ is the moduli space of morphisms $f : \mathbb{P}^1_D \to \mathbb{P}^1_T$ of degree n as $f^*\mathcal{O}_{\mathbb{P}^1_T}(1) \cong L_{\mathbb{P}^1_D} \cong \mathcal{O}_{\mathbb{P}^1_D}(n)$.

2. A morphism $f : \mathbb{P}^1_D \to \mathbb{P}^1_T$ consists of global sections (global homogeneous polynomials) $f = (s_0(u:v), s_1(u:v))$ where $\deg(s_0) = \deg(s_1) = n$ and are coprime i.e. $\mathrm{Res}(s_0, s_1) \neq 0$.

3. Consider $f = (-27u^{12}v^{12}, 27u^{14}v^{10} - 54u^{12}v^{12} + 27u^{10}v^{14})$ is a **degree 4** morphism as the common factor is $27u^{10}v^{10}$

4. The rational maps and the morphisms coincide i.e. $f : \mathbb{P}^1_D \dashrightarrow \mathbb{P}^1_T = f : \mathbb{P}^1_D \to \mathbb{P}^1_T$ ($\mathbb{P}^1_D$ smooth $\mathbb{P}^1_T$ projective) **after cancellation of common factors** i.e. $\gcd(s_0, s_1) = 1$

5. $\mathbb{P}^1_T(k(t))_n = \mathbb{P}^1_T(k[t])_n$ for $\mathbb{P}^1_D$ with function field $k(t)$ and ring of integers $\mathcal{O}_{k(t)} = k[t] \sim \mathbb{P}^1_T(\mathbb{Q})_{ht(a/b)} = \mathbb{P}^1_T(\mathbb{Z})_{ht(a/b)}$

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The j-map $j : \mathbb{P}^1 \to \overline{M}_{1,1} \cong \mathbb{P}^1$ is always a morphism but **lost the valuation data crucial for Tate's algorithm** to find out what are (additive) singular fibers at $[0:1]$ for $t = 0$ and $[1:0]$ for $t = \infty$.

Arithmetic of $X_n := \mathrm{Hom}_n(\mathbb{P}^1_D, \mathbb{P}^1_T)$

1. $X_n = \mathbb{P}^{2n+1} - V(\mathrm{Res}(s_0, s_1))$ is the open complement of **Resultant hypersurface** $\mathrm{Res}(s_0, s_1) = 0$ in $\mathbb{P}^{2n+1}$ thus it is an open quasiprojective variety of dimension $2n+1$

2. By Farb-Wolfson's seminal work (2016) $\{X_n\} = \mathbb{L}^{2n+1} - \mathbb{L}^{2n-1} \to |X_n(\mathbb{F}_q)| = q^{2n+1} - q^{2n-1}$

3. Both domain $\mathbb{P}^1_D$ and target $\mathbb{P}^1_T$ are **unparameterized** and the action of an element of PGL_2 on the homogeneous coordinates $[u : v]$ of $\mathbb{P}^1_D$ translates to an action on the global sections s_i of $\mathcal{O}_{\mathbb{P}^1_D}(n)$ for $i = 0, 1$ which are the homogeneous coordinates of $\mathbb{P}(V) = \mathcal{P}(\underbrace{1, \ldots, 1}_{n+1 \text{ times}}, \underbrace{1, \ldots, 1}_{n+1 \text{ times}}) = \mathbb{P}^{2n+1}$

4. $\mathbb{L}^{2n+1} - \mathbb{L}^{2n-1} = \mathbb{L}(\mathbb{L}^2 - 1) \cdot \mathbb{L}^{2n-2}$ as $\{\mathrm{PGL}_2\} = \mathbb{L}(\mathbb{L}^2 - 1)$

Topology of $X_n := \mathrm{Hom}_n(\mathbb{P}^1_D, \mathbb{P}^1_T)$

1. $\mathrm{Hom}^*_n(\mathbb{P}^1_D, \mathbb{P}^1_T) \hookrightarrow \mathrm{Hom}_n(\mathbb{P}^1_D, \mathbb{P}^1_T) \to \mathbb{P}^1_T$ via the evaluation morphism $\mathrm{ev}_\infty : \mathrm{Hom}_n(\mathbb{P}^1_D, \mathbb{P}^1_T) \to \mathbb{P}^1_T$ with $f \mapsto f(\infty) \in \mathbb{P}^1_T$
2. Fiber $\mathrm{Hom}^*_n(\mathbb{P}^1_D, \mathbb{P}^1_T)$ is the based mapping space which is identical to the space of coprime polynomials $\mathrm{Poly}_1^{(n,n)}$

Definition

Fix a field K with algebraic closure $\overline{K}$. Fix $k, l \geq 0$. Define $\mathrm{Poly}_1^{(k,l)}$ to be the set of pairs (u, v) of monic polynomials in $K[z]$ so that:

2.1 $\deg u = k$ and $\deg v = l$.

2.2 u and v have no common root in $\overline{K}$.

3. ev_∞ is a Zariski-locally trivial fibration via the transitive action of $Aut(\mathbb{P}^1_T) = \mathrm{PGL}_2$
4. $\mathbb{L}^{2n+1} - \mathbb{L}^{2n-1} = (\mathbb{L} + 1) \cdot (\mathbb{L}^{2n} - \mathbb{L}^{2n-1})$ as $\{\mathrm{Hom}^*_n(\mathbb{P}^1_D, \mathbb{P}^1_T)\} = \{\mathrm{Poly}_1^{(n,n)}\} = \mathbb{L}^{2n} - \mathbb{L}^{2n-1}$

Arithmetic of Algebraic Stacks over Finite Fields

The weighted point count of $\mathcal{X}$ over $\mathbb{F}_q$ is defined as a sum: $\#_q(\mathcal{X}) := \sum_{x \in \mathcal{X}(\mathbb{F}_q)/\sim} \frac{1}{|\mathrm{Aut}(x)|}$ where $\mathcal{X}(\mathbb{F}_q)/\sim$ is the set of $\mathbb{F}_q$–isomorphism classes of $\mathbb{F}_q$–points of $\mathcal{X}$.

What we really need is the unweighted point count $|\mathcal{X}(\mathbb{F}_q)/\sim|$. But this is immune to the Grothendieck-Lefschetz trace formula.

We clarify the arithmetic role of the *inertia stack* $\mathcal{I}(\mathcal{X})$ of an algebraic stack $\mathcal{X}$ over $\mathbb{F}_q$ which parameterizes pairs $(x, Aut(x))$.

Theorem (Changho Han–JP)

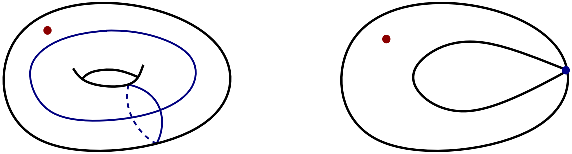
Let $\mathcal{X}$ be an algebraic stack over $\mathbb{F}_q$ of finite type with affine diagonal. Then,

$$|\mathcal{X}(\mathbb{F}_q)/\sim| = \#_q(\mathcal{I}(\mathcal{X}))$$

Thus the weighted point count $\#_q(\mathcal{I}(\mathcal{X}))$ of the inertia stack $\mathcal{I}(\mathcal{X})$ is the unweighted point count $|\mathcal{X}(\mathbb{F}_q)/\sim|$ of $\mathcal{X}$ over $\mathbb{F}_q$.

Deligne–Mumford stack $\overline{\mathcal{M}}_{1,1}$ of stable elliptic curves

Fine moduli stack $\overline{\mathcal{M}}_{1,1}$ parametrizes isomorphism classes $[E]$ of stable elliptic curves with the coarse moduli space $\overline{M}_{1,1} \cong \mathbb{P}^1$ parametrizing the j-invariant $j([E]) = 1728 \cdot 4a_4^3/(4a_4^3 + 27a_6^2)$



When the characteristic of the field k is not equal to 2 or 3, $(\overline{\mathcal{M}}_{1,1})_k \cong [(Spec\ k[a_4, a_6] - (0,0))/\mathbb{G}_m] =: \mathcal{P}_k(4,6)$ through the short Weierstrass equation: $y^2 = x^3 + a_4x + a_6$

Stabilizers are the orbifold points $[1:0]$ & $[0:1]$ with μ_4 & μ_6 respectively and the generic stacky points such as $[1:1]$ with μ_2

The fine moduli stack $\overline{\mathcal{M}}_{1,1}$ comes equipped with the universal family $p : \overline{\mathcal{E}}_{1,1} \to \overline{\mathcal{M}}_{1,1}$ of stable elliptic curves.

Boundary Divisor $\overline{\mathcal{M}}_{1,1} \setminus \mathcal{M}_{1,1} = [\infty]$ for I_1 nodal fiber

1. Take the nodal curve $y^2 = x^3 + x^2$, then *complete the cubic* via $x = x' - \frac{1}{3}$. This is why we require $\mathrm{char}(k) \neq 2, 3$.

2. We get $y^2 = x^3 - \frac{1}{3}x + \frac{2}{27}$. Coefficients should be integral thus we take $\lambda = 3$ to multiply λ^4 to $-\frac{1}{3}$ and λ^6 to $+\frac{2}{27}$. Notice here *weighted homogeneous coordinate* of $\mathcal{P}(4, 6)$.

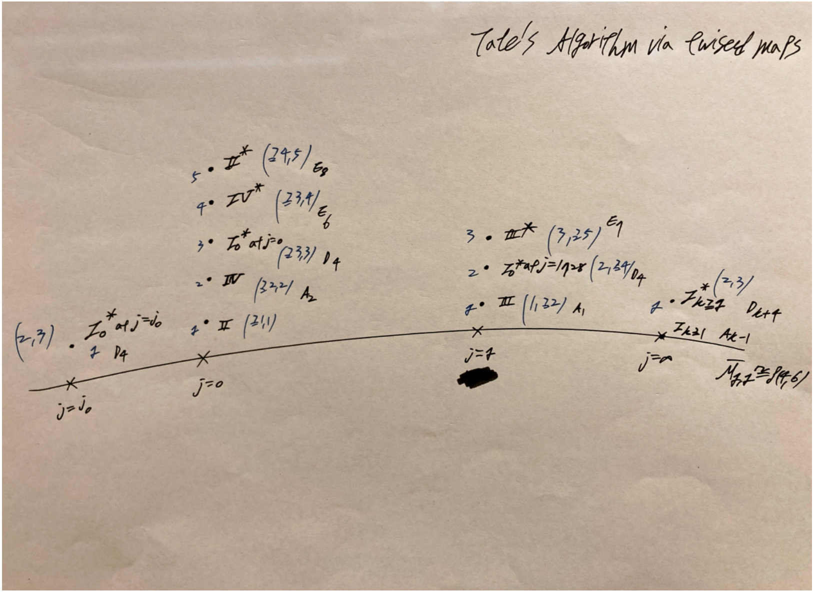
3. We arrive at $y^2 = x^3 - 27x + 54$ thus $[-\frac{1}{3} : \frac{2}{27}] = [-27 : 54]$. Curve is singular $\Delta = 4(-27)^3 + 27(54)^2 = 0$ thus $j = \infty$. Written as I_1 multiplicative reduction in Kodaira notation.

4. Remember the isomorphism, for any $\lambda \in \mathbb{G}_m$

$$\left[y^2 = x^3 + Ax + B\right] \cong \left[y^2 = x^3 + \lambda^4 \cdot Ax + \lambda^6 \cdot B\right]$$

via $x \mapsto \lambda^{-2} \cdot x$ and $y \mapsto \lambda^{-3} \cdot y$.

Geometric Tate's algorithm



Tate's algorithm via Twisted maps; correspondence

Theorem (Dori Bejleri–JP–Matthew Satriano; April 2024)

If $\mathrm{char}(K) \neq 2, 3$. Then the twisting condition (r, a) and the order of vanishing of j at $j = \infty$ determine the Kodaira fiber type, and (r, a) is in turn determined by $m = \min\{3\nu(a_4), 2\nu(a_6)\}$.

$\gamma : (\nu(a_4),\ \nu(a_6))$	*Reduction type with* $j \in \overline{M}_{1,1}$	$\Gamma : (r, a)$
$(\geq 1, 1)$	II *with* $j = 0$	$(6, 1)$
$(1, \geq 2)$	III *with* $j = 1728$	$(4, 1)$
$(\geq 2, 2)$	IV *with* $j = 0$	$(3, 1)$
$(2, 3)$	$\mathrm{I}^*_{k>0}$ *with* $j = \infty$ I^*_0 *with* $j \neq 0, 1728$	$(2, 1)$
$(\geq 3, 3)$	I^*_0 *with* $j = 0$	$(2, 1)$
$(2, \geq 4)$	I^*_0 *with* $j = 1728$	$(2, 1)$
$(\geq 3, 4)$	IV^* *with* $j = 0$	$(3, 2)$
$(3, \geq 5)$	III^* *with* $j = 1728$	$(4, 3)$
$(\geq 4, 5)$	II^* *with* $j = 0$	$(6, 5)$

How many elliptic curves over $k = \mathbb{F}_q$ upto isom?

The inertia stack $\mathcal{I}\overline{\mathcal{M}}_{1,1}$ parametrizes $[E]$ and automorphism groups $([E], \mathrm{Aut}[E])$. To keep track of the primitive roots of unity contained in $\mathbb{F}_q$, define function $\delta(x) := \begin{cases} 1 & \text{if } x \text{ divides } q-1, \\ 0 & \text{otherwise.} \end{cases}$

Grothendieck class in $K_0(\mathrm{Stck}_k)$ with $\mathrm{char}(k) \neq 2, 3$,

$$\{\mathcal{I}\overline{\mathcal{M}}_{1,1}\} = 2 \cdot (\mathbb{L} + 1) + 2 \cdot \delta(4) + 4 \cdot \delta(6)$$

Weighted point count over $\mathbb{F}_q$ with $\mathrm{char}(\mathbb{F}_q) \neq 2, 3$,

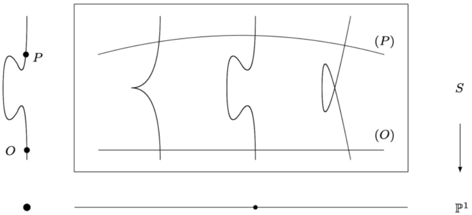
$$\#_q(\mathcal{I}\overline{\mathcal{M}}_{1,1}) = 2 \cdot (q + 1) + 2 \cdot \delta(4) + 4 \cdot \delta(6)$$

Exact number of $\mathbb{F}_q$–isomorphism classes with $\mathrm{char}(\mathbb{F}_q) \neq 2, 3$,

$$|\overline{\mathcal{M}}_{1,1}(\mathbb{F}_q)/\sim| = 2 \cdot (q + 1) + 2 \cdot \delta(4) + 4 \cdot \delta(6)$$

Elliptic surfaces $/k$ = Families of elliptic curves $/K$

The study of **fibrations of algebraic curves** lies at the heart of the Enriques-Kodaira classification of algebraic surfaces.



We call an algebraic surface S to be an **elliptic surface**, if it admits an elliptic fibration $f : S \to C$ which is a flat proper morphism f from a nonsingular surface S to a nonsingular curve C, such that a generic fiber is a smooth curve of genus 1.

While this is the most general setup, it is natural to work with the case when the base curve is the smooth projective line $\mathbb{P}^1$ and there exists a section $O : \mathbb{P}^1 \hookrightarrow S$ coming from the identity points of the elliptic fibres and not passing through the singular points.

Moduli stack of stable elliptic fibrations

Thus, a stable elliptic fibration $g : Y \to \mathbb{P}^1$ is induced by a morphism $\varphi_f : \mathbb{P}^1 \to \overline{\mathcal{M}}_{1,1}$ and vice versa.

$$\begin{tikzcd} X \arrow[r, "\nu"] \arrow[d, "f"] & Y = \varphi_f^*(\overline{\mathcal{E}}_{1,1}) \arrow[r] \arrow[d, "g"] & \overline{\mathcal{E}}_{1,1} \arrow[d, "p"] \\ \mathbb{P}^1 \arrow[r, equal] & \mathbb{P}^1 \arrow[r, "\varphi_f"] & \overline{\mathcal{M}}_{1,1} \end{tikzcd}$$

X is the non-singular semistable elliptic surface; Y is the stable elliptic fibration; $\nu : X \to Y$ is the minimal resolution.

The moduli stack $\mathcal{L}_{12n}$ of stable elliptic fibrations over the $\mathbb{P}^1$ with $12n$ nodal singular fibers and a marked section **is** the Hom stack $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$ where $\varphi_f^*\mathcal{O}_{\overline{\mathcal{M}}_{1,1}}(1) \cong \mathcal{O}_{\mathbb{P}^1}(n)$.

A morphism $\varphi_f : \mathbb{P}^1 \to \overline{\mathcal{M}}_{1,1}$ consists of global sections (homogeneous polynomials in $[u : v]$) $\varphi_f = (a_4(u, v), a_6(u, v))$ where $\deg(a_4) = 4n$ and $\deg(a_6) = 6n$ **(!)** and $\mathrm{Res}(a_4, a_6) \neq 0$.

Motivic Analytic Number Theory Praxis

Moduli of minimal stable $E/\mathbb{F}_q(t)$ is $\mathcal{L}_{12n} = \mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$

Theorem (Changho Han–JP)

Grothendieck class in $K_0(\mathrm{Stck}_k)$ with $\mathrm{char}(k) \neq 2, 3$,

$$\{\mathcal{L}_{12n}\} = \mathbb{L}^{10n+1} - \mathbb{L}^{10n-1}$$

Weighted point count over $\mathbb{F}_q$ with $\mathrm{char}(\mathbb{F}_q) \neq 2, 3$,

$$\#_q(\mathcal{L}_{12n}) = q^{10n+1} - q^{10n-1}$$

Exact number of $\mathbb{F}_q$–isomorphism classes with $\mathrm{char}(\mathbb{F}_q) \neq 2, 3$,

$$|\mathcal{L}_{12n}(\mathbb{F}_q)/\sim| = \#_q(\mathcal{I}\mathcal{L}_{12n}) = 2 \cdot (q^{10n+1} - q^{10n-1})$$

$$\mathcal{Z}_{\mathbb{F}_q(t)}(\mathcal{B}) = \sum_{n=1}^{\left\lfloor \frac{log_q \mathcal{B}}{12} \right\rfloor} |\mathcal{L}_{1,12n}(\mathbb{F}_q)/\sim| = 2 \cdot \frac{(q^{11} - q^9)}{(q^{10} - 1)} \cdot \left(\mathcal{B}^{\frac{5}{6}} - 1\right)$$

Totality of Rational points on Moduli stacks

Counting Families of Varieties : Lecture 3

June Park

The University of Sydney

KIAS-LFANT Winter School on Number Theory

Stacky Heights on Algebraic Stacks

Ellenberg, Zureick-Brown, and Satriano extends the rational point $x \in \mathcal{X}(K)$ to a stacky curve, called a *tuning stack* $(\mathcal{C}, \pi, \overline{x})$ for x.

$$
\begin{tikzcd}
Spec(K) \arrow[r] \arrow[rr, bend left, "x"] \arrow[dr] & \mathcal{C} \arrow[r, "\overline{x}"] \arrow[d, "\pi"] & \mathcal{X} \\
& C &
\end{tikzcd}
$$

$\mathcal{C}$ is a normal, π is a birational coarse space map.

Definition

If $\mathcal{V}$ is a vector bundle on $\mathcal{X}$ and $x \in \mathcal{X}(K)$, the *height of x with respect to $\mathcal{V}$* is defined as

$$\mathrm{ht}_{\mathcal{V}}(x) := -\deg(\pi_* \overline{x}^* \mathcal{V}^\vee)$$

for any choice of tuning stack $(\mathcal{C}, \pi, \overline{x})$.

Height Moduli Space on Cyclotomic Stacks

There is a height moduli stack $\mathcal{M}_n(\mathcal{X}, \mathcal{L})$ parametrizing all rational points on general proper polarized cyclotomic stacks of stacky height n and that the spaces of twisted maps yield a stratification of $\mathcal{M}_n(\mathcal{X}, \mathcal{L})$ corresponding to fixing the local contributions to the stacky height. The fact that $\mathcal{M}_n(\mathcal{X}, \mathcal{L})$ is of finite type is a geometric incarnation of the Northcott property.

Theorem (Dori Bejleri–JP–Matthew Satriano; April 2024)

Let $(\mathcal{X}, \mathcal{L})$ be a proper polarized cyclotomic stack over a perfect field k. Fix a smooth projective curve C/k with function field $K = k(C)$ and $n, d \in \mathbb{Q}_{\geq 0}$.

1. *There exists a separated Deligne–Mumford stack $\mathcal{M}_{n,C}(\mathcal{X}, \mathcal{L})$ of finite type over k with a quasi-projective coarse space and a canonical bijection of k-points*

$$\mathcal{M}_{n,C}(\mathcal{X}, \mathcal{L})(k) = \{P \in \mathcal{X}(K) \mid \mathrm{ht}_{\mathcal{L}}(P) = n\}.$$

1. There is a finite locally closed stratification

$$\bigsqcup_{\Gamma,d} \mathcal{H}^{\Gamma}_{d,C}(\mathcal{X},\mathcal{L})/S_{\Gamma} \to \mathcal{M}_{n,C}(\mathcal{X},\mathcal{L})$$

where $\mathcal{H}^{\Gamma}_{d,C}$ are moduli spaces of twisted maps and the union runs over all possible admissible local conditions

$$\Gamma = (\{r_1, a_1\}, \ldots, \{r_s, a_s\})$$

and degrees d for a twisted map to $(\mathcal{X},\mathcal{L})$ satisfying

$$n = d + \sum_{i=1}^{s} \frac{a_i}{r_i}$$

and S_{Γ} is a subgroup of the symmetric group on s letters that permutes the stacky points of the twisted map.

2. Under the bijection in part (1), each k-point of $\mathcal{H}^{\Gamma}_{d,C}(\mathcal{X},\mathcal{L})/S_{\Gamma}$ corresponds to a K-point P with the stable height and local contributions given by

$$\mathrm{ht}^{st}_{\mathcal{L}}(P) = d \qquad \left\{\delta_i = \frac{a_i}{r_i}\right\}_{i=1}^{s}.$$

Geometric Interpretation of Tate's Algorithm

$$
\begin{tikzcd}
 & & & & \mathcal{Y} = \varphi^* \overline{\mathcal{E}} \arrow[r] \arrow[dl, "\pi"'] \arrow[dd, "h"] & \overline{\mathcal{E}} \arrow[dd, "p"] \\
X \arrow[dd, "f"] & X' \arrow[l, "\text{Weierstrass}"'] \arrow[dd, "f'"] & \hat{X} \arrow[l, "\text{contraction}"'] \arrow[r, "\text{resolution}"] \arrow[dd, "\hat{f}"] & Y \arrow[dd, "g"] & & \\
 & & & & \mathcal{C} \arrow[r, "\varphi"] \arrow[dl, "\rho"'] & \overline{\mathcal{M}}_{1,1} \arrow[d] \\
C \arrow[r, equal] & C \arrow[r, equal] & C \arrow[r, equal] & C \arrow[rr, "j"'] \arrow[urr, dashed, "\psi"] & & \overline{M}_{1,1}
\end{tikzcd}
$$

Here f is a Weierstrass model, ψ is the associated weighted linear series viewed as a rational map to $\overline{\mathcal{M}}_{1,1}$, φ is a twisted morphism from the universal tuning stack $\mathcal{C}$ which induces a stable stack-like model $h : \mathcal{Y} \to \mathcal{C}$ where $g : Y \to C$ is the twisted model via coarse moduli maps, $\hat{f}$ is a resolution of Y, and f' is the relative minimal model obtained by contracting relative (-1)-curves.

Suppose that normalized base multiplicity $m = 3$. This occurs if and only if $(\nu(a_4), \nu(a_6)) = (1, \geq 2)$. Then $r = 12/\gcd(3, 12) = 4$ and $a = 3/\gcd(3, 12) = 1$. Thus the stabilizer of the twisted curve acts on the central fiber of the twisted model via the character $\mu_4 \to \mu_4$, $\zeta_4 \mapsto \zeta_4^{-1}$. In particular, the central fiber E of $\mathcal{Y}$ has $j = 1728$. The μ_4 action on E has two fixed points, and there is an orbit of size two with stabilizer $\mu_2 \subset \mu_4$. Let E_0 be the image of E in the twisted model Y. As E appears with multiplicity 4, Y has $\frac{1}{4}(-1, -1)$ quotient singularities at the images of the the fixed points and a $\frac{1}{2}(-1, -1)$ singularity at the image of the orbit of size two. Each of these singularities is resolved by a single blowup to obtain $\hat{X}$ with central fiber $4\tilde{E}_0 + E_1 + E_2 + E_3$ where E_i are the exceptional divisors of the resolution for $i = 1, 2, 3$ and $E_1^2 = E_2^2 = -4$ with $E_3^2 = -2$. Then $\tilde{E}_0$ is a (-1)-curve so it needs to be contracted. After this contraction E_2 becomes a (-1) curve and must also be contracted. Since E_i for $i = 1, 2, 3$ are incident and pairwise transverse after blowing down $\tilde{E}_0$, then the images of E_1 and E_2 must be tangent after blowing down E_3. Moreover, they are now (-2)-curves and the relatively minimal model for type III.

Tate's Algorithm via Twisted Morphisms

Theorem (Dori Bejleri–JP–Matthew Satriano; April 2024)

If $\mathrm{char}(K) \neq 2, 3$. Then the twisting condition (r, a) and the order of vanishing of j at $j = \infty$ determine the Kodaira fiber type, and (r, a) is in turn determined by $m = \min\{3\nu(a_4), 2\nu(a_6)\}$.

$\gamma : (\nu(a_4),\ \nu(a_6))$	*Reduction type with* $j \in \overline{M}_{1,1}$	$\Gamma : (r, a)$
$(\geq 1, 1)$	II *with* $j = 0$	$(6, 1)$
$(1, \geq 2)$	III *with* $j = 1728$	$(4, 1)$
$(\geq 2, 2)$	IV *with* $j = 0$	$(3, 1)$
$(2, 3)$	$\mathrm{I}^*_{k>0}$ *with* $j = \infty$ I^*_0 *with* $j \neq 0, 1728$	$(2, 1)$
$(\geq 3, 3)$	I^*_0 *with* $j = 0$	$(2, 1)$
$(2, \geq 4)$	I^*_0 *with* $j = 1728$	$(2, 1)$
$(\geq 3, 4)$	IV^* *with* $j = 0$	$(3, 2)$
$(3, \geq 5)$	III^* *with* $j = 1728$	$(4, 3)$
$(\geq 4, 5)$	II^* *with* $j = 0$	$(6, 5)$

Geometric Meaning of Height Moduli Framework

1. So one can run the resolution / minimal model. As these are *algebraic surfaces* it can be done over $\mathrm{char}(K) = p > 0$

2. A twisted morphism $\varphi : \mathcal{C} \to \overline{\mathcal{M}}_{1,1}$ with its twisting data Γ from the universal tuning stack $\mathcal{C}$ induces a stable stack-like model $h : \mathcal{Y} \to \mathcal{C}$ as a unique pullback of the universal family $p : \overline{\mathcal{E}} \to \overline{\mathcal{M}}_{1,1}$. All the ensuing birational geometry is natural.

3. True purpose of a **representable classifying morphism** is in the <u>universal principle</u> that φ intrinsically contains all the algebro-geometric data necessary to uniquely determine a fibration with singular fibers. This is the very essence of the inner arithmetic of rational points on moduli stacks over K.

$\mathcal{A}$lgebraic $\mathcal{G}$eometry $\Cap$ $\mathcal{T}$opology $\Longleftrightarrow$ $\mathbb{A}$rithmetic

1. Consider the fact that $\overline{\mathcal{M}}_{1,1}$ could have been any other algebraic stack $\mathcal{X}$ (such as $\overline{\mathcal{M}}_g$ or $\overline{\mathcal{A}}_g$) which is the representing object for certain moduli functor as the fine moduli stack together with the universal family $p : \overline{\mathcal{E}} \to \mathcal{X}$.

2. Representable classifying morphisms as twisted morphisms $\varphi : \mathcal{C} \to \mathcal{X}$ uniquely determines certain families of varieties (of algebraic curves or abelian varieties) with non-abelian stabilizers ($g \geq 2$). And they naturally have corresponding "Tate's algorithm", counting statements and so on.

3. Geometrizing $\mathcal{X}(K)$ leads to Height moduli space $\mathcal{M}_n(\mathcal{X}, \mathcal{L})$ and once we have a **space** (AG), we compute its **invariants** (AT) naturally having various kinds of **consequences** (NT).

Motivic Height Zeta Function as Generating Series

Definition

A $\vec{\lambda}$-weighted linear series $(L, s_0, \ldots, s_N)$ is *minimal* if for each indeterminacy point $x \in C$, there exists an j such that $\nu_x(s_j) < \lambda_i$.

Definition

The motivic height zeta function of $\mathcal{P}(\lambda_0, \ldots, \lambda_N)$ is the formal power series

$$Z_{\vec{\lambda}}(t) := \sum_{n \geq 0} \{\mathcal{W}_n^{min}\}\, t^n \in K_0(\mathrm{Stck})[\![t]\!]$$

where $\mathcal{W}_n^{min}$ is the space of minimal weighted linear series on $\mathbb{P}^1$ of height n. We also define the variant

$$\mathcal{I}Z_{\vec{\lambda}}(t) := \sum_{n \geq 0} \{\mathcal{I}\mathcal{W}_n^{min}\}\, t^n \in K_0(\mathrm{Stck}_k)[\![t]\!]$$

Stratification on Ambient Projective Stacks

Minimality defect e measures the degree of failure of a weighted linear series to be minimal (not a rational point of height n).

Definition

Let μ be the normalized base profile. We can divide each part μ_i by κ to obtain $\mu_i = \kappa q_i + r_i$. We define $q(\mu)$ and $r(\mu)$ to be the partitions with parts q_i and r_i respectively.
The minimality defect of μ is the size of the quotient $e = |q(\mu)|$.

Corollary (Dori Bejleri–JP–Matthew Satriano; April 2024)

The disjoint union of $\psi_{n,e}$

$$\psi_n : \bigsqcup_{e=0}^{n} \mathcal{W}_{n-e}^{min} \times \mathbb{P}(V_e^1) \to \mathcal{P}\left(\bigoplus_{i=0}^{N} V_n^{\lambda_j}\right)$$

is an isomorphism after stratifying the source and target.

1. We denote the usual motivic zeta function of $\mathbb{P}^1$ by

$$Z(t) = \sum \{\mathrm{Sym}^e \, \mathbb{P}^1\} t^e = \frac{1}{(1-\mathbb{L}t)(1-t)}$$

2. **We stratify by minimality defect** e to obtain an equality

$$\left\{ \mathcal{P}\left(\bigoplus_{i=0}^{N} V_n^{\lambda_i} \right) \right\} = \sum_{e=0}^{n} \{\mathcal{W}_{n-e}^{min}\}\{\mathrm{Sym}^e \, \mathbb{P}^1\}$$

which implies

$$\sum_{n \geq 0} \left\{ \mathcal{P}\left(\bigoplus_{i=0}^{N} V_n^{\lambda_i} \right) \right\} t^n = Z_{\vec{\lambda}}(t) \cdot Z(t) \tag{1}$$

3. *Homogeneous polynomials* live in compact ambient stack!

$$\sum_{n \geq 0} \left\{ \mathcal{P}\left(\bigoplus_{i=0}^{N} V_n^{\lambda_i} \right) \right\} t^n = \frac{\{\mathbb{P}^N\} + \mathbb{L}^{N+1}\{\mathbb{P}^{|\vec{\lambda}|-N-2}\}t}{(1-t)(1-\mathbb{L}^{|\vec{\lambda}|}t)}$$

Rationality of Motivic Height Zeta Function

Fix weights $\vec{\lambda} = (\lambda_0, \ldots, \lambda_N)$ and let $|\vec{\lambda}| \colon = \sum_{i=0}^{N} \lambda_i$. Suppose for simplicity that k contains all $\mathrm{lcm} = \mathrm{lcm}(\lambda_0, \ldots, \lambda_N)$ roots of unity.

Theorem (Dori Bejleri–JP–Matthew Satriano; April 2024)

For $k, \vec{\lambda}$ as above and $C = \mathbb{P}^1_k$, consider $\mathcal{W}_n^{min}$ and its inertia stack $\mathcal{I}\mathcal{W}_n^{min}$. We have the following formulas over $K_0(\mathrm{Stck}_k)$.

$$\sum_{n \geq 0} \{\mathcal{W}_n^{min}\} t^n = \frac{1 - \mathbb{L}t}{1 - \mathbb{L}^{|\vec{\lambda}|} t} \left(\{\mathbb{P}^N\} + \mathbb{L}^{N+1} \{\mathbb{P}^{|\vec{\lambda}| - N - 2}\} t \right)$$

$$\sum_{n \geq 0} \{\mathcal{I}\mathcal{W}_n^{min}\} t^n = \sum_{g \in \mu_{\mathrm{lcm}}(k)} \frac{1 - \mathbb{L}t}{1 - \mathbb{L}^{|\vec{\lambda_g}|} t} \left(\{\mathbb{P}^{N_g}\} + \mathbb{L}^{N_g+1} \{\mathbb{P}^{|\vec{\lambda_g}| - N_g - 2}\} t \right)$$

where g runs over the lcm roots of unity and $\vec{\lambda}_g$ is a subset of $\vec{\lambda}$ of size $N_g + 1$ depending explicitly on the order of g.

Theorem (Dori Bejleri–Changho Han–JP–Matthew Satriano)

Let $\mathrm{char}(k) \neq 2, 3$. *The motives (modulo* $\{\mathrm{PGL}_2\}$*) of moduli stacks* $\mathcal{W}^{\Theta}_{\min,n}$ *of minimal Weierstrass fibrations with a single Kodaira fiber* Θ *and at worst multiplicative reduction elsewhere is*

Reduction type Θ *with* $j \in \overline{M}_{1,1}$	$\lvert\gamma\rvert$	$\{\mathcal{W}^{\Theta}_{\min,n}\} \in K_0(\mathrm{Stck}_K)$
$\mathrm{I}_{k>0}$ *with* $j = \infty$	0	$\mathbb{L}^{10n-2}$
II *with* $j = 0$	2	$\mathbb{L}^{10n-3}$
III *with* $j = 1728$	3	$\mathbb{L}^{10n-4}$
IV *with* $j = 0$	4	$\mathbb{L}^{10n-5}$
$\mathrm{I}^*_{k>0}$ *with* $j = \infty$ I^*_0 *with* $j \neq 0, 1728$	5	$\mathbb{L}^{10n-6} - \mathbb{L}^{10n-7}$
I^*_0 *with* $j = 0, 1728$	6	$\mathbb{L}^{10n-7}$
IV^* *with* $j = 0$	7	$\mathbb{L}^{10n-8}$
III^* *with* $j = 1728$	8	$\mathbb{L}^{10n-9}$
II^* *with* $j = 0$	9	$\mathbb{L}^{10n-10}$

Theorem (Dori Bejleri–JP–Matthew Satriano; April 2024)

$$\left\{\mathcal{W}^{\min}_{n=1}(\mathcal{P}(\vec{\lambda}))\right\} = \{\mathbb{P}^N\}(\mathbb{L}^{|\vec{\lambda}|} - \mathbb{L}) + \mathbb{L}^{N+1}\{\mathbb{P}^{|\vec{\lambda}|-N-2}\}$$

$$\left\{\mathcal{W}^{\min}_{n\geq 2}(\mathcal{P}(\vec{\lambda}))\right\} = \mathbb{L}^{(n-2)|\vec{\lambda}|+N+2}(\mathbb{L}^{|\vec{\lambda}|-1} - 1)\{\mathbb{P}^{|\vec{\lambda}|-1}\}$$

Take $|\vec{\lambda}| = 10$ and $N = 1$ as $\overline{\mathcal{M}}_{1,1} \cong \mathcal{P}(4,6)$ over $\mathbb{Z}[1/6]$.

1. When $n = 1$, X is a **Rational elliptic surface**.

$$\left\{\mathcal{W}^{\min}_1\right\} = \mathbb{L}^{11}+\mathbb{L}^{10}+\mathbb{L}^{9}+\mathbb{L}^{8}+\mathbb{L}^{7}+\mathbb{L}^{6}+\mathbb{L}^{5}+\mathbb{L}^{4}+\mathbb{L}^{3}-\mathbb{L}$$

2. When $n = 2$, X is algebraic $K3$ surface with elliptic fibration (i.e., **Projective elliptic K3 surface with moduli dim. 18**).

$$\left\{\mathcal{W}^{\min}_2\right\} = \mathbb{L}^{21}+\mathbb{L}^{20}+\mathbb{L}^{19}+\mathbb{L}^{18}+\mathbb{L}^{17}+\mathbb{L}^{16}+\mathbb{L}^{15}+\mathbb{L}^{14}+\mathbb{L}^{13}-\mathbb{L}^{11}-\mathbb{L}^{10}-\mathbb{L}^{9}-\mathbb{L}^{8}-\mathbb{L}^{7}-\mathbb{L}^{6}-\mathbb{L}^{5}-\mathbb{L}^{4}-\mathbb{L}^{3}$$

$$= \mathbb{L}(\mathbb{L}^2-1)\left(\mathbb{L}^{18}+\mathbb{L}^{17}+2\mathbb{L}^{16}+2\mathbb{L}^{15}+3\mathbb{L}^{14}+3\mathbb{L}^{13}+4\mathbb{L}^{12}+4\mathbb{L}^{11}+5\mathbb{L}^{10}+4\mathbb{L}^{9}+4\mathbb{L}^{8}+3\mathbb{L}^{7}+3\mathbb{L}^{6}+2\mathbb{L}^{5}+2\mathbb{L}^{4}+\mathbb{L}^{3}+\mathbb{L}^{2}\right)$$