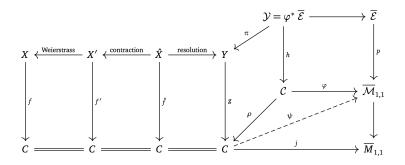
Geometric Interpretation of Tate's Algorithm



Here f is a Weierstrass model, ψ is the associated weighted linear series viewed as a rational map to $\overline{\mathcal{M}}_{1,1}$, φ is a twisted morphism from the universal tuning stack $\mathcal C$ which induces a stable stack-like model $h: \mathcal Y \to \mathcal C$ where $g: Y \to \mathcal C$ is the twisted model via coarse moduli maps, $\hat f$ is a resolution of Y, and f' is the relative minimal model obtained by contracting relative (-1)-curves.

Suppose that normalized base multiplicity m = 3. This occurs if and only if $(\nu(a_4), \nu(a_6)) = (1, \ge 2)$. Then $r = 12/\gcd(3, 12) = 4$ and $a = 3/\gcd(3,12) = 1$. Thus the stabilizer of the twisted curve acts on the central fiber of the twisted model via the character $\mu_4 \to \mu_4$, $\zeta_4 \mapsto \zeta_4^{-1}$. In particular, the central fiber E of Y has j=1728. The μ_4 action on E has two fixed points, and there is an orbit of size two with stabilizer $\mu_2 \subset \mu_4$. Let E_0 be the image of Ein the twisted model Y. As E appears with multiplicity 4, Y has $\frac{1}{4}(-1,-1)$ quotient singularities at the images of the the fixed points and a $\frac{1}{2}(-1,-1)$ singularity at the image of the orbit of size two. Each of these singularities is resolved by a single blowup to obtain \hat{X} with central fiber $4\tilde{E}_0 + E_1 + E_2 + E_3$ where E_i are the exceptional divisors of the resolution for i = 1, 2, 3 and $E_1^2 = E_2^2 = -4$ with $E_3^2 = -2$. Then \tilde{E}_0 is a (-1)-curve so it needs to be contracted. After this contraction E_2 becomes a (-1) curve and must also be contracted. Since E_i for i = 1, 2, 3 are incident and pairwise transverse after blowing down \tilde{E}_0 , then the images of E_1 and E_2 must be tangent after blowing down E_3 . Moreover, they are now (-2)-curves and the relatively minimal model for type III.

Tate's Algorithm via Twisted Morphisms

Theorem (Dori Bejleri-JP-Matthew Satriano; April 2024)

If $\operatorname{char}(K) \neq 2,3$. Then the twisting condition (r,a) and the order of vanishing of j at $j=\infty$ determine the Kodaira fiber type, and (r,a) is in turn determined by $m=\min\{3\nu(a_4),2\nu(a_6)\}$.

| $\gamma:(u(a_4),\ u(a_6))$ | Reduction type with $j \in \overline{M}_{1,1}$ | Γ : (<i>r</i> , <i>a</i>) |
|------------------------------|--|-----------------------------|
| $(\geq 1,1)$ | II with $j = 0$ | (6,1) |
| $(1, \geq 2)$ | III with $j = 1728$ | (4,1) |
| (≥ 2, 2) | IV with $j = 0$ | (3, 1) |
| (2,3) | $I_{k>0}^*$ with $j=\infty$ | (2,1) |
| | I_0^* with $j \neq 0, 1728$ | |
| $(\geq 3, 3)$ | I_0^* with $j=0$ | (2,1) |
| $(2, \geq 4)$ | I_0^* with $j = 1728$ | (2,1) |
| (≥ 3, 4) | IV^* with $j=0$ | (3, 2) |
| $(3,\geq 5)$ | III^* with $j = 1728$ | (4, 3) |
| (≥ 4, 5) | II^* with $j=0$ | (6,5) |

Geometric Meaning of Height Moduli Framework

- 1. So one can run the resolution / minimal model. As these are algebraic surfaces it can be done over char(K) = p > 0
- 2. A twisted morphism $\varphi: \mathcal{C} \to \overline{\mathcal{M}}_{1,1}$ with its twisting data Γ from the universal tuning stack \mathcal{C} induces a stable stack-like model $h: \mathcal{Y} \to \mathcal{C}$ as a unique pullback of the universal family $p: \overline{\mathcal{E}} \to \overline{\mathcal{M}}_{1,1}$. All the ensuing birational geometry is natural.
- 3. True purpose of a **representable classifying morphism** is in the <u>universal principle</u> that φ intrinsically contains all the algebro-geometric data necessary to uniquely determine a fibration with singular fibers. This is the very essence of the inner arithmetic of rational points on moduli stacks over K.

\mathcal{A} lgebraic \mathcal{G} eometry \cap \mathcal{T} opology \iff \mathbb{A} rithmetic

- 1. Consider the fact that $\overline{\mathcal{M}}_{1,1}$ could have been any other algebraic stack \mathcal{X} (such as $\overline{\mathcal{M}}_g$ or $\overline{\mathcal{A}}_g$) which is the representing object for certain moduli functor as the fine moduli stack together with the universal family $p:\overline{\mathcal{E}}\to\mathcal{X}$.
- 2. Representable classifying morphisms as twisted morphisms $\varphi:\mathcal{C}\to\mathcal{X}$ uniquely determines certain families of varieties (of algebraic curves or abelian varieties) with non-abelian stabilizers $(g\geq 2)$. And they naturally have corresponding "Tate's algorithm", counting statements and so on.
- 3. Geometrizing $\mathcal{X}(K)$ leads to Height moduli space $\mathcal{M}_n(\mathcal{X}, \mathcal{V})$. Once we have a **space**, we compute its **invariants**, consider all invariants simultaneously via generating series and show the motivic height zeta function's **rationality**, naturally having various kinds of **consequences**.