Étale cohomological stability of the moduli of stable elliptic fibrations and their arithmetic

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Abstract

We compute the étale cohomology with Frobenius weights of $\mathrm{Hom}_n(C,\mathscr{P}(\vec{\lambda}))$, the moduli stack of degree n morphisms from a smooth projective curve C to the weighted projective stack $\mathscr{P}(\vec{\lambda})$, the latter being a stacky quotient defined by $\mathscr{P}(\vec{\lambda}) := \left[\mathbb{A}^N - \{0\}/\mathbb{G}_m\right]$, where $\mathbb{G}_m$ acts by weights $\vec{\lambda} = (\lambda_0, \cdots, \lambda_N) \in \mathbb{Z}_+^N$. Our key ingredient is étale cohomological descent over the indexing category ΔS, the symmetric (semi)simplicial category. Our stability result has interesting arithmetic consequences. For example, over a finite field $\mathbb{F}_q$, with q coprime to 2 or 3, we obtain an upper bound on the number of $\mathbb{F}_q$–points of the moduli stack of stable elliptic fibrations over C, specializing to an equality when $C = \mathbb{P}^1$. Additionally, we establish the weighted projective bundle formula in the rational Chow ring via intersection theory. In the appendix, we similarly consider the moduli of generalized elliptic fibrations.

1 Introduction

Fix a base field K. Let C be a smooth, projective and geometrically connected curve of genus g over K. The moduli space of morphisms of degree n from C to the projective space $\mathbb{P}^N$, which we denote by $\mathrm{Hom}_n(C,\mathbb{P}^N)$, has been studied extensively for decades, using techniques ranging from scanning maps in topology (see e.g. [Segal, CCMM]), to more algebraic approaches (see e.g. [FW, Banerjee] and the references therein). Motivated by moduli theory, we will consider a generalization of a target space $\mathbb{P}^N$ to a target stack $\mathscr{P}(\vec{\lambda})$. That is, given a vector $\vec{\lambda} = (\lambda_0, \ldots, \lambda_N)$ of positive weights $\lambda_i \in \mathbb{Z}_{\geq 1}$, the N-dimensional weighted projective stack $\mathscr{P}(\vec{\lambda}) := [(\mathbb{A}^{N+1}_{x_0,\ldots,x_N} \setminus 0)/\mathbb{G}_m]$ where $\zeta \in \mathbb{G}_m$ acts by $\zeta \cdot (x_0, \ldots, x_N) = (\zeta^{\lambda_0} x_0, \ldots, \zeta^{\lambda_N} x_N)$. This naturally leads us to consider the following Hom stack of degree $n \in \mathbb{Z}_{\geq 1}$ morphisms

$$
\begin{aligned}
\mathrm{Hom}_n(C,\mathscr{P}(\vec{\lambda})) &:= \left\{ f : C \to \mathscr{P}(\vec{\lambda}) : f \text{ morphism with } f^*\mathscr{O}_{\mathscr{P}(\vec{\lambda})}(1) = \mathscr{O}_C(n) \right\} \\
&= \Big\{ \big(L, [s_0 : \ldots : s_N]\big) : \text{ degree of } L = n, \;\; s_i \in H^0(C, L^{\otimes \lambda_i}), \\
&\qquad\qquad s_0, \ldots, s_N \text{ have no common zeroes} \Big\} / \mathbb{G}_m
\end{aligned}
$$

where $\mathbb{G}_m$ acts on the ith component $H^0(C, L^{\otimes \lambda_i})$ by weight λ_i.

Henceforth we assume K is algebraically closed, that its characteristic is coprime to $l.c.m(\vec{\lambda})$ and that ℓ is a fixed positive integer always coprime to *char* K.

Notations: A bit of notation before we state our theorems. For a Deligne-Mumford stack $\mathscr{X}$ over K, we denote the ℓ-adic cohomology group with rational coefficients by $H^i(\mathscr{X};\mathbb{Q}_\ell)$ (warning: this is not the étale cohomology with $\mathbb{Q}_\ell$ coefficients, see e.g. [GL, Warning 3.2.1.9], or any text on étale cohomology of schemes e.g. [Milne]); by the same token a sheaf of $\mathbb{Q}_\ell$ vector spaces is a $\mathbb{Z}_\ell$ sheaf $\mathscr{F} = (\mathscr{F}_n)$ and

$$H^i(\mathscr{X};\mathscr{F}) := \varprojlim H^i(\mathscr{X};\mathscr{F}_n) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

(similar to the notations set up in [Milne, Section 19].) For brevity and convenience, and noting there is no scope of confusion since we are always working over rational coefficients, we will write $H^i(\mathscr{X})$ to stand for $H^i(\mathscr{X};\mathbb{Q}_\ell)$. Finally, let us denote a vector space spanned by $\{a_1,\ldots,a_k\}$ over $\mathbb{Q}_\ell$ by $\mathbb{Q}_\ell\{a_1,\ldots,a_k\}$.

In our first theorem, we determine the stable ℓ-adic cohomology of $\mathrm{Hom}_n(C,\mathscr{P}(\vec{\lambda}))$ as a $\mathbb{Q}_\ell$-algebra.

Main Theorem 1.1 (Cohomological stability I)**.** *Let C be a smooth projective curve of genus g, and let N and n be fixed positive integers such that $n \geq 2g$, and $N \leq n-g$. Let $n_0 := n-2g$. Then there exists a second quadrant spectral sequence, which converges to $H^*(\mathrm{Hom}_n(C,\mathscr{P}(\vec{\lambda}));\mathbb{Q}_\ell)$ an an algebra, which has the following description. The E_2 term is a bigraded algebra that collapses on $E_2^{-p,q}\Big|_{p\leq n_0}$. Furthermore, $E_2^{-p,q}\Big|_{p\leq n_0}$ is a quotient of the graded commutative $\mathbb{Q}_\ell$-algebra*

$$H^*(J(C);\mathbb{Q}_\ell)[h]/h^N \otimes \wedge\mathbb{Q}_\ell\{t\} \otimes \mathrm{Sym}\,\mathbb{Q}_\ell\{\alpha_1,\ldots,\alpha_{2g}\},$$

where $H^i(J(C);\mathbb{Q}_\ell)$ has degree $(0,i)$, h has degree $(0,2)$, t has degree $(-1,2N+2)$ and α_i has degree $(-1,2N+1)$ for all i, modulo elements of degree $(-i,j)$ with $i > n_0$. Furthermore this is a spectral sequence of mixed Hodge structures, with $\mathbb{Q}_\ell\{\alpha_1,\ldots,\alpha_{2g}\}$ and $\mathbb{Q}_\ell\{t\}$ each carrying a pure Hodge structure of weight $2(N+1)$, and h is of type $(1,1)$.

The special case of $C = \mathbb{P}^1$ deserves a mention in its own right :

Theorem 1.2 (Cohomological stability II)**.** *Let N and n be positive integers satisfying $N \leq n$. Then*

$$H^*(\mathrm{Hom}_n(\mathbb{P}^1,\mathscr{P}(\vec{\lambda}));\mathbb{Q}_\ell) \cong \frac{\mathbb{Q}_\ell[h]}{h^N} \otimes \wedge\mathbb{Q}_\ell\{t\}$$

where h has cohomological degree 2 and Hodge type $(1,1)$, and t has cohomological degree $2N+1$ and Hodge type $(N+1,N+1)$. In particular over a field κ, with algebraic closure $\overline{\kappa}$, we have an isomorphism of $Gal(\overline{\kappa}/\kappa)$-representations:

$$H^i(\mathrm{Hom}_n(\mathbb{P}^1,\mathscr{P}(\vec{\lambda}));\mathbb{Q}_\ell) = \begin{cases} \mathbb{Q}_\ell(-j) & i = 2j, 0 \leq j \leq N-1 \\ \mathbb{Q}_\ell(-(j+1)) & i = 2j+1, N \leq j \leq 2N-1 \\ 0 & \textit{otherwise.} \end{cases}$$

It is interesting to see that the cohomology not only stabilizes with respect to $n \in \mathbb{Z}_{\geq 1}$, but additionally, in the case of $C = \mathbb{P}^1$, the whole cohomology ring is independent of the degree n of the rational curve even as the dimension of the moduli stack $(a+b)n+N$ grows with n for a fixed $\mathscr{P}(\vec{\lambda})$. For $N = 1$ case, the ℓ-adic rational cohomology type of the moduli stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathscr{P}(a,b))$ is a 3-sphere $\mathbb{S}^3$ with an odd-dimensional class at $i = 3$ which is independent of both n and (a,b). In light of the Manin's conjecture on rational points of bounded height over global function fields $\mathbb{F}_q(t)$ which is equivalent to the point counts over $\mathbb{F}_q$ of the moduli spaces of rational curves with bounded degree, we can see this as an instance of a cohomological stability of the moduli stack of rational curves on $\mathscr{P}(\vec{\lambda})$.

The ℓ-adic étale cohomology with Frobenius weights of the Hom stack naturally gives the following weighted point count of $\mathrm{Hom}_n(C, \mathscr{P}(\vec{\lambda}))$ over a finite field $\mathbb{F}_q$ with $\mathrm{char}(\mathbb{F}_q)$ not dividing $\lambda_i \in \mathbb{N}$ for every i.

Theorem 1.3. *Let* $\mathrm{Hom}_n(C, \mathscr{P}(\vec{\lambda}))$ *be the Hom stack of degree* $n \geq 2g$ *morphisms from a genus* g *smooth projective curve* C *to the* N*-dimensional weighted projective stack* $\mathscr{P}(\vec{\lambda}) = \mathscr{P}(\lambda_0, \ldots, \lambda_N)$ *with* $|\vec{\lambda}| := \sum_{i=0}^{N} \lambda_i$. *Then the weighted point count of* $\mathrm{Hom}_n(C, \mathscr{P}(\vec{\lambda}))$ *over* $\mathbb{F}_q$ *is a finite sum given by*

$$\#_q\left(\mathrm{Hom}_n(C, \mathscr{P}(\vec{\lambda}))\right) = q^{|\vec{\lambda}|n+N-2g} + a_{\frac{1}{2}} \cdot q^{|\vec{\lambda}|n+N-2g-\frac{1}{2}} + a_1 \cdots q^{|\vec{\lambda}|n+N-2g-1} + \ldots + a_i \cdot q^{|\vec{\lambda}|n+N-2g-i} + \ldots$$

where $i \in \mathbb{Z}_+[\frac{1}{2}]$, *the coefficients* a_i *for* $i < n-2g$ *are independent of* n, *and for* $i \geq n-2g$ *we have* $a_i \cdot q^{|\vec{\lambda}|n+N-2g-i} \ll q^{|\vec{\lambda}|n+N-2g}$.

As a natural application, we focus on the case $\mathscr{P}(\vec{\lambda}) = \mathscr{P}(a,b)$ as the Deligne–Mumford stack $\overline{\mathscr{M}}_{1,1}$ of stable elliptic curves is isomorphic to $\mathscr{P}(4,6)$ over $\mathrm{Spec}(\mathbb{Z}[1/6])$ (see Example 2.3 for various cases). Thus the Hom stack $\mathrm{Hom}_n(C, \mathscr{P}(4,6))$ is isomorphic to the moduli stack $\mathscr{L}_{12n,g}$ of stable elliptic fibrations with $12n$ nodal singular fibers and a section over the parameterized basecurve C_K of genus g (see [Han-Park, §3] for details on the formulation of the moduli stack as Hom stack). Arithmetically, we have the following enumeration of semistable elliptic curves (the weighted point count of $\mathscr{L}_{12n,g}$ gives the same number as that of the moduli of semistable elliptic surfaces c.f. [Han-Park, Proposition 11]) over global function fields $\mathbb{F}_q(C)$ ordered by bounded height of discriminant $ht(\Delta) = q^{12n}$ for $n \geq 2g$.

Corollary 1.4. *Let* $\mathscr{L}_{12n,g}$ *be the moduli stack of stable elliptic fibrations with section and discriminant degree* $12n$ *over the smooth projective curve* $C_{\mathbb{F}_q}$ *of genus* g. *If* $\mathrm{char}(\mathbb{F}_q) \neq 2,3$ *and* $n \geq 2g$, *then the weighted point count of* $\mathscr{L}_{12n,g}$ *over* $\mathbb{F}_q$ *is the finite sum*

$$\#_q\left(\mathscr{L}_{12n,g}\right) = q^{10n+1-2g} + a_{\frac{1}{2}} \cdot q^{10n+1-2g-\frac{1}{2}} + a_1 \cdot q^{10n+1-2g-1} + \ldots + a_i \cdot q^{10n+1-2g-i} + \cdots$$

where $i \in \mathbb{Z}_+[\frac{1}{2}]$, the coefficients a_i for $i < n-2g$ are independent of n, and for $i \geq n-2g$ we have $a_i \cdot q^{10n+1-2g-i} \ll q^{10n+1-2g}$.

Consequently, we have an estimate of the counting function $\mathcal{N}(\mathbb{F}_q(C), 0 < B \leq q^{12n})$, which counts the number of semistable elliptic surfaces over a genus g projective curve $C_{\mathbb{F}_q}$ ordered by $0 < ht(\Delta) = q^{12n} \leq B$ for $n \geq 2g$

$$\mathcal{N}(\mathbb{F}_q(C), 0 < B \leq q^{12n}) = \sum_{n=1}^{\left\lfloor \frac{log_q B}{12} \right\rfloor} 2 \cdot \#_q\left(\mathcal{L}_{12n,g}\right) = 2 \cdot \frac{(q^{11-2g} - q^{9-2g})}{(q^{10}-1)} \cdot B^{\frac{5}{6}} + o(B^{\frac{5}{6}})$$

where the factor of 2 comes from the hyperelliptic involution.

The attractive feature of our enumeration results is its topological origin and the significance of this result lies in having the enumerations over *general* global function fields $\mathbb{F}_q(C)$ with the main leading term $\mathcal{O}_q\left(B^{\frac{5}{6}}\right)$ and the leading coefficient as the explicit rational function of q and g thus extending [Han-Park, Theorem 3]. Here, we make a remark that [de Jong] established for the first time counting semistable elliptic surfaces over $\mathbb{P}^1_{\mathbb{F}_q}$ via working directly with the generalized Weierstrass equation which works also in characteristic 2 and 3. In [Han-Park] and [Park-Spink] established the enumeration on semistable elliptic surfaces over $\mathbb{P}^1_K$ over $\mathrm{char}(K) \neq 2,3$ via the motives inside $K_0(\mathrm{Stck}_K)$ the *Grothendieck ring of K-stacks* introduced by [Ekedahl] in 2009.

Lastly, we now compute the integral Picard group. It turns out that it is always cyclic, and generated by the line bundle $\mathcal{O}_{\mathrm{Hom}_n(\mathbb{P}^1,\mathcal{P}(\vec{\lambda}))}(1)$. This line bundle is dual to the tautological line bundle $\mathcal{O}_{\mathrm{Hom}_n(\mathbb{P}^1,\mathcal{P}(\vec{\lambda}))}(-1)$, whose fibres are the one-dimensional spaces $K \cdot (s_0, \ldots, s_N)$ inside the direct sum of the $H^0(C, L^{\otimes \lambda_i})$.

Proposition 1.5. *If $\mathrm{char}(K)$ does not divide $\lambda_i \in \mathbb{Z}_{\geq 1}$ for every i, then the Picard group of the Hom stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ is a cyclic group generated by the line bundle $\mathcal{O}_{\mathrm{Hom}_n(\mathbb{P}^1,\mathcal{P}(\vec{\lambda}))}(1)$ and is isomorphic to*

$$\mathrm{Pic}(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))) \cong \begin{cases} \mathbb{Z}/(n(\lambda_0+\lambda_1))\mathbb{Z} & \text{for } N = 1, \\ \mathbb{Z} & \text{for } N > 1. \end{cases}$$

The torsion Picard group $\mathrm{Pic}(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))) \cong \mathbb{Z}/((a+b)n)\mathbb{Z}$ does depend on the degree n and the weights (a,b) which leads one to conjecture that the corresponding fundamental group $\pi_1(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))) \cong \mathbb{Z}/((a+b)n)\mathbb{Z}$ as confirmed by $\pi_1(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(1,1) \cong \mathbb{P}^1)) \cong \mathbb{Z}/2n\mathbb{Z}$ by the work of [EP]. It is natural to recognize that we are dealing with a Lens space from the perspective of homotopy theory. Lens spaces are important in homotopy theory as they are the only 3-manifolds with non-trivial finite cyclic fundamental group.

Methods & Outline

Theorem 1.1 is an immediate consequence of [Banerjee, Theorem 2]. To be able to apply [Banerjee, Theorem 2] we observe that the completion of the Hom-stack

$\widetilde{\mathrm{Hom}}_n(C, \mathscr{P}(\vec{\lambda}))$ by a $\Delta_{inj}S$ which is defined by

$$\widetilde{\mathrm{Hom}}_n(C, \mathscr{P}(\vec{\lambda})) := \{L,\ [s_0 : \ldots : s_N] : L \in \mathrm{Pic}^n(C),\ s_i \in H^0(C, L^{\otimes \lambda_i}) \text{ for all } i\} \quad (1)$$

admits *symmetric semi-simplicial filtration by* C (defined in [Banerjee]). Our proof can be split into two steps:

- We compute $H^*(\widetilde{\mathrm{Hom}}_n(C, \mathscr{P}(\vec{\lambda}); \mathbb{Q}_\ell)$, which is a weighted projective bundle on $\mathrm{Pic}^n C$ (more precisely, the projection to the first factor is a flat proper morphism whose fibres are isomorphic to a weighted projective stack).

  In this regard we study more generally the intersection theory of weighted projective bundles over an arbitrary smooth base S. For this, assume we have a vector bundle $\mathscr{E}/S$ and a splitting

  $$\mathscr{E} = \mathscr{E}_0 \oplus \ldots \oplus \mathscr{E}_N\,,$$

  for vector bundles $\mathscr{E}_i$. Given a vector $\vec{\lambda}$ of positive integers, we can again form the weighted projective bundle

  $$\mathscr{P}_S(\mathscr{E}, \vec{\lambda}) = [(\mathrm{Tot}(\mathscr{E}) \setminus 0)/\mathbb{G}_m] \to S\,, \quad (2)$$

  where the torus $\mathbb{G}_m$ acts on $\mathscr{E}_i$ with weight λ_i.

  To state the result for the Chow group of $\mathscr{P}_S(\mathscr{E}, \vec{\lambda})$, we introduce a notion of *twisted Chern classes*. For this recall that the (standard) Chern polynomial is given by

  $$c_t(\mathscr{E}) = 1 + t \cdot c_1(\mathscr{E}) + t^2 c_2(\mathscr{E}) + \ldots$$

  For a vector bundle $\mathscr{E}$ splitting into $n+1$ summands as above, the Whitney sum formula implies

  $$c_t(\mathscr{E}) = \prod_{i=0}^{N} c_t(\mathscr{E}_i)\,.$$

  Given a vector $\vec{\eta} = (\eta_0, \ldots, \eta_N) \in \mathbb{Z}_{>0}^{N+1}$ of positive integers, we define the $\vec{\eta}$-twisted Chern polynomial of $\mathscr{E}$ as

  $$c_t^{\vec{\eta}}(\mathscr{E}) = \prod_{i=0}^{N} c_{\eta_i t}(\mathscr{E}_i)\,.$$

  Similarly, an individual Chern class $c_j^{\vec{\eta}}(\mathscr{E})$ is defined as the coefficient of t^j in this twisted Chern polynomial. Thinking in terms of the Chern roots of the bundle $\mathscr{E}$, the twisted Chern classes correspond to multiplying the Chern roots of the summand $\mathscr{E}_i$ by η_i.

  Theorem 1.6. *Let S be a smooth Deligne-Mumford stack with a vector bundle $\mathscr{E} = \bigoplus_{i=0}^{N} \mathscr{E}_i$ and let $\vec{\lambda} \in \mathbb{Z}_{\geq 1}^{N+1}$ be a vector of positive integers. Let $L = \mathrm{lcm}(\vec{\lambda})$ and consider the vector $\vec{\eta} = (L/\lambda_0, \ldots, L/\lambda_N)$. Then we have*

  $$A^*(\mathscr{P}_S(\mathscr{E}, \vec{\lambda}), \mathbb{Q}_\ell) = A^*(S, \mathbb{Q}_\ell)[\zeta]/(\zeta^{N+1} + c_1^{\vec{\eta}}(\mathscr{E})\zeta^N + \ldots + c_{N+1}^{\vec{\eta}}(\mathscr{E}))\,, \quad (3)$$

  where $\zeta = L \cdot c_1(\mathscr{O}_{\mathscr{P}_S(\mathscr{E}, \vec{\lambda})}(1))$.

- We show $\widetilde{\mathrm{Hom}}_n(C,\mathscr{P}(\vec{\lambda}))$ admits *symmetric semi-simplicial filtration by C* and use [Banerjee, Theorem 2] which gives us a weighted projective analogue of [Banerjee, Theorem 3] (which makes the same computation in principle, but just for 'non-weighted' projective spaces).

2 Preliminaries

We first recall the definition of a weighted projective stack $\mathscr{P}(\vec{\lambda})$.

Definition 2.1. Let $\vec{\lambda} = (\lambda_0,\dots,\lambda_N) \in \mathbb{Z}_{\geq 1}^{N+1}$ be a vector of $N+1$ positive integers. Consider the affine space $U_{\vec{\lambda}} = \mathbb{A}^{N+1}_{x_0,\dots,x_N}$ endowed with the action of $\mathbb{G}_m$ with weights $\vec{\lambda}$, i.e. an element $\zeta \in \mathbb{G}_m$ acts by

$$\zeta \cdot (x_0,\dots,x_N) = (\zeta^{\lambda_0}x_0,\dots,\zeta^{\lambda_N}x_N). \tag{4}$$

The N-dimensional weighted projective stack $\mathscr{P}(\vec{\lambda})$ is then defined as the quotient stack

$$\mathscr{P}(\vec{\lambda}) = \left[(U_{\vec{\lambda}} \setminus \{0\})/\mathbb{G}_m\right].$$

Remark 2.2. When we wish to emphasize the field K of definition of $\mathscr{P}(\vec{\lambda})$, we use the notation $\mathscr{P}_K(\vec{\lambda})$. All weighted projective stacks are smooth. The stack $\mathscr{P}(\vec{\lambda})$ is Deligne–Mumford if and only if all weights λ_i are prime to the characteristic; in this case, $\mathscr{P}(\vec{\lambda})$ is in fact tame Deligne–Mumford as in [AOV]. Notice that $\mathscr{P}(1,p)$ is not Deligne–Mumford in characteristic p since it has a point with automorphism group μ_p which is not formally unramified. When $\mathscr{P}(\vec{\lambda})$ is Deligne–Mumford, it is an orbifold if and only if $\gcd(\lambda_0,\dots,\lambda_N) = 1$; this is because $\mathscr{P}(\vec{\lambda})$ has generic stabilizer $\mu_{\gcd(\lambda_0,\dots,\lambda_N)}$.

The natural morphism $U_{\vec{\lambda}} \to \mathscr{P}(\vec{\lambda})$ is the total space of the *tautological line bundle* $\mathcal{O}_{\mathscr{P}(\vec{\lambda})}(-1)$ on $\mathscr{P}(\vec{\lambda})$. As in the classical case, we denote by $\mathcal{O}_{\mathscr{P}(\vec{\lambda})}(1)$ the dual of this line bundle.

The fine modular curves are quintessential moduli stacks. Under mild condition on the characteristic of the base field K (i.e., a field K with $\mathrm{char}(K)$ does not divide $\lambda_i \in \mathbb{N}$ for every i), the genus 0 modular curves are isomorphic to the weighted projective stacks $\mathscr{P}(a,b)$.

Example 2.3. An example that will play an important role throughout this paper is the moduli stack of stable elliptic curves. When $\mathrm{char}(K) \neq 2,3$, we have an explicit isomorphism

$$(\overline{\mathscr{M}}_{1,1})_K \cong [(\mathrm{Spec}\, K[a_4,a_6] - (0,0))/\mathbb{G}_m] = \mathscr{P}_K(4,6)$$

given by the short Weierstrass equation $y^2 = x^3 + a_4x + a_6$, where $\zeta \cdot a_i = \zeta^i a_i$ for $\zeta \in \mathbb{G}_m$ and $i = 4,6$. See, e.g., [Hassett, Proposition 3.6].

Similarly, one could consider the stack $\overline{\mathscr{M}}_{1,1}[\Gamma]$ of generalized elliptic curves with $[\Gamma]$-level structure by the work of Deligne and Rapoport [DR] (summarized in [Conrad, §2] and also in [Niles, §2]). (see Proposition A.4).

Also, one could consider the stack $\overline{\mathscr{M}}_{1,m}(m-1)$ of m-marked $(m-1)$-stable curves of arithmetic genus one formulated originally by the works of [Smyth, Smyth2] (see Proposition A.12).

For higher genus curves, we recall the notion of quasi–admissible covers whereby the general member of C is *not* an admissible cover of $\mathbb{P}^1$ and have been studied in depth by [Stankova, §2.4.] as the closest covers to the original families of stable curves. In this regard, [Fedorchuk] introduced the proper Deligne–Mumford stack $\mathscr{H}_{2g}[2g-1]$ of quasi-admissible genus $g \geq 2$ curves. For the case of monic odd–degree hyperelliptic curves with a generalized Weierstrass equation $y^2 = x^{2g+1} + a_4x^{2g-1} + a_6x^{2g-2} + a_8x^{2g-3} + \cdots + a_{4g+2}$ we have

$$\mathscr{H}_{2g}[2g-1] \cong \mathscr{P}(4,6,8,\ldots,4g+2)$$

by [Fedorchuk, Proposition 4.2(1)] over $\mathrm{char}(K)=0$ and by [HP2, Proposition 5.9] over $\mathrm{char}(K) > 2g+1$. The $g=2$ case by $y^2 = x^5 + a_4x^3 + a_6x^2 + a_8x + a_{10}$ with $\mathscr{H}_4[3] \cong \mathscr{P}(4,6,8,10)$ is of special interest as all genus 2 curves are hyperelliptic.

We recall the definition of *weighted point count* of an algebraic stack $\mathscr{X}$ over $\mathbb{F}_q$:

Definition 2.4. The weighted point count of $\mathscr{X}$ over $\mathbb{F}_q$ is defined as a sum:

$$\#_q(\mathscr{X}) := \sum_{x \in \mathscr{X}(\mathbb{F}_q)/\sim} \frac{1}{|\mathrm{Aut}(x)|},$$

where $\mathscr{X}(\mathbb{F}_q)/\sim$ is the set of $\mathbb{F}_q$–isomorphism classes of $\mathbb{F}_q$–points of $\mathscr{X}$ (i.e., the set of non–weighted points of $\mathscr{X}$ over $\mathbb{F}_q$), and we take $\frac{1}{|\mathrm{Aut}(x)|}$ to be 0 when $|\mathrm{Aut}(x)| = \infty$.

A priori, the weighted point count can be ∞, but when $\mathscr{X}$ is of finite type, then the stratification of $\mathscr{X}$ by schemes as in [Behrend, Proof of Lemma 3.2.2] implies that $\mathscr{X}(\mathbb{F}_q)/\sim$ is a finite set, so that $\#_q(\mathscr{X}) < \infty$.

The weighted point count $\#_q(\mathscr{X})$ of an algebraic stack $\mathscr{X}$ over $\mathbb{F}_q$ is *algebro-topological* under the framework of the Weil conjectures as it is equal to the alternating sum of trace of geometric Frobenius. In this regard, we recall the *Grothendieck-Lefschetz trace formula* for Artin stacks by [Behrend, Sun].

Theorem 2.5 (Theorem 1.1. of [Sun])**.** *Let $\mathscr{X}$ be an Artin stack of finite type over $\mathbb{F}_q$. Let Frob_q be the geometric Frobenius on $\mathscr{X}$. Let ℓ be a prime number different from the characteristic of $\mathbb{F}_q$, and let $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$ be an isomorphism of fields. For an integer i, let $H^i_{\acute{e}t,c}(\mathscr{X}_{/\overline{\mathbb{F}}_q}; \overline{\mathbb{Q}}_\ell)$ be the cohomology with compact support of the constant sheaf $\overline{\mathbb{Q}}_\ell$ on $\mathscr{X}$. Then the infinite sum regarded as a complex series via ι*

$$\sum_{i \in \mathbb{Z}} (-1)^i \cdot tr\Big(\mathrm{Frob}_q^* : H^i_c(\mathscr{X}_{/\overline{\mathbb{F}}_q}; \overline{\mathbb{Q}}_\ell) \to H^i_c(\mathscr{X}_{/\overline{\mathbb{F}}_q}; \overline{\mathbb{Q}}_\ell)\Big) \tag{5}$$

is absolutely convergent to the weighted point count $\#_q(\mathscr{X})$ of $\mathscr{X}$ over $\mathbb{F}_q$.

Lastly, we recall that the cohomology in torsion-free coefficient coincide for the fine moduli stack and its coarse moduli space. We show this for the étale cohomology over base field $\overline{\mathbb{F}}_q$ in $\mathbb{Q}_\ell$-coefficient by following the proof of the [Sun, Proposition 7.3.2].

Lemma 2.6. *Let $\mathfrak{X}$ be a smooth separated tame Deligne–Mumford stack of finite type over $\overline{\mathbb{F}}_q$ and the coarse moduli map $c : \mathfrak{X} \to X$ giving the coarse moduli space X. Then for all i, the pullback map*

$$c^* : H^i_{\acute{e}t}(X_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell) \cong H^i_{\acute{e}t}(\mathfrak{X}_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$$

is an isomorphism.

Proof. As $\mathfrak{X}$ is a smooth separated tame Deligne–Mumford stack of finite type over $\overline{\mathbb{F}}_q$, we can cover X by étale charts U such that pull-back of U in $\mathfrak{X}$ is the quotient stack of an algebraic space by a finite group [AOV, Theorem 3.2.]. The lemma follows from the ℓ-adic Leray spectral sequence as in [Behrend, Theorem 1.2.5] once we have shown that the canonical map $\mathbb{Q}_\ell \to \mathrm{R}c_*\mathbb{Q}_\ell$ is an isomorphism. It suffices to show the isomorphism étale locally on X and hence we assume $\mathfrak{X} = [V/G]$ for some algebraic space V under the action of finite group G where $\mathrm{char}(\overline{\mathbb{F}}_q)$ does not divide $|G|$. Let $q : V \to \mathfrak{X}$ be the canonical morphism. Observe that we have $\mathbb{Q}_\ell \simeq (q_*\mathbb{Q}_\ell)^G$. As both q and $c \circ q$ are finite maps and $\mathbb{Q}[G]$ for a finite group G is a semisimple $\mathbb{Q}$-algebra by the Maschke's Theorem, we acquire

$$\mathrm{R}c_*\mathbb{Q}_\ell \simeq \mathrm{R}c_*(q_*\mathbb{Q}_\ell)^G \simeq ((c \circ q)_*\mathbb{Q}_\ell)^G \simeq \mathbb{Q}_\ell$$

■

3 Intersection theory of weighted projective bundles

Many people have studied the cohomology of weighted projective spaces and bundles, starting with [Kawasaki], where the base S is a point. In [Al Amrani] the author computes the integral cohomology of the coarse moduli space of $\mathscr{P}_S(\mathscr{E}, \vec{\lambda})$ as a $\mathbb{Q}$-vector space, and its multiplicative structure in case that $\mathscr{E}$ splits as a sum of line bundles and $\vec{\lambda}$ satisfies a certain divisibility condition (with similar results in étale cohomology). The divisibility condition was later removed in [BFR].

Our proof of the theorem above is independent of the previous work, and proceeds in two steps: first, in the case where $\mathscr{E}$ splits as a sum of line bundles, the weighted projective bundle $\mathscr{P}_S(\mathscr{E}, \vec{\lambda})$ admits a natural finite cover to a *non-weighted* projective bundle. Via the usual projective bundle formula, this suffices to find its Chow groups. Here we explicitly use that we work with $\mathbb{Q}$-coefficients. In the second step, we use a splitting theorem to reduce to the first case.

Let S be a smooth Deligne-Mumford stack and let

$$\mathscr{E} = \mathscr{L}_0 \oplus \cdots \oplus \mathscr{L}_N / S \tag{6}$$

be a vector bundle on S decomposing into a direct sum of $N+1$ line bundles $\mathscr{L}_i$. We denote by $\mathrm{Tot}(\mathscr{E}) \to S$ the total space of this vector bundle, with zero section $0 \subseteq \mathrm{Tot}(\mathscr{E})$. Given a vector $\vec{\lambda} = (\lambda_0, \dots, \lambda_N)$ of positive integers, there exists an action of $\mathbb{G}_m$ on $\mathrm{Tot}(\mathscr{E})$ locally given by $\zeta \cdot (s_0, \dots, s_N) = (\zeta^{\lambda_0} s_0, \dots, \zeta^{\lambda_N} s_N)$. In the following, we want to study the intersection theory of the *weighted projective bundle*

$$\mathscr{P}_S(\mathscr{E}, \vec{\lambda}) = [(\mathrm{Tot}(\mathscr{E}) \setminus 0)/\mathbb{G}_m] \to S. \tag{7}$$

This is a (Zariski) locally trivial bundle with fibre $\mathscr{P}(\vec{\lambda})$ over S.

To present the answer, let $\ell = \mathrm{lcm}(\vec{\lambda})$ be the least common multiple of the λ_i, consider the vector $\vec{\eta} = (\ell/\lambda_0, \dots, \ell/\lambda_N)$ and the modified bundle

$$\mathscr{E}_\eta = \mathscr{L}_0^{\otimes \eta_0} \oplus \cdots \oplus \mathscr{L}_N^{\otimes \eta_N} / S\,. \tag{8}$$

Denote by $\mathbb{P}(\mathscr{E}_\eta) \to S$ the (unweighted) projective bundle associated to this vector bundle. Then there exists a natural map

$$\Phi : \mathscr{P}_S(\mathscr{E}, \vec{\lambda}) \to \mathbb{P}(\mathscr{E}_\eta), [s_0 : \dots : s_N] \mapsto [s_0^{\otimes \eta_0} : \dots : s_N^{\otimes \eta_N}] \tag{9}$$

over S.

Theorem 3.1. *The map Φ is proper, flat and quasi-finite of degree $d = \ell^N / \prod_{i=0}^N \lambda_i$ and the pullback*

$$\Phi^* : A^*(\mathbb{P}(\mathscr{E}_\eta), \mathbb{Q}) \to A^*(\mathscr{P}_S(\mathscr{E}, \vec{\lambda}), \mathbb{Q})$$

induces an isomorphism on the Chow groups with $\mathbb{Q}$-coefficients, whose inverse is given by $1/d \cdot \Phi_*$. In particular, we have

$$A^*(\mathscr{P}_S(\mathscr{E}, \vec{\lambda}), \mathbb{Q}) = A^*(S, \mathbb{Q})[\zeta]/(\zeta^{N+1} + c_1(\mathscr{E}_\eta)\zeta^N + \ldots + c_{N+1}(\mathscr{E}_\eta))\,, \tag{10}$$

where $\zeta = \ell \cdot c_1(\mathscr{O}_{\mathscr{P}_S(\mathscr{E}, \vec{\lambda})}(1))$ and where the Chern classes of $\mathscr{E}_\eta$ can be computed as

$$c_i(\mathscr{E}_\eta) = e_i(\eta_0 c_1(\mathscr{L}_0), \dots, \eta_N c_1(\mathscr{L}_N))\,,$$

with e_i the i-th elementary symmetric polynomial.

For the proof, we need some more preparatory results.

Lemma 3.2. *Let G be an algebraic group acting on varieties X, Y over K such that the action on X is transitive and let $x_0 \in X$ be a K-point with stabilizer G_{x_0}. Then there exists a natural isomorphism*

$$f : [X \times Y/G] \xrightarrow{\sim} [Y/G_{x_0}] \tag{11}$$

whose inverse is induced by the G_{x_0}-equivariant map

$$Y \to \{x_0\} \times Y \subseteq X \times Y\,.$$

Proof. To construct f consider the incidence variety

$$I = \{(g,x,y) : gx = x_0\} \subseteq G \times X \times Y \tag{12}$$

Let $p : I \to X \times Y$ be the projection on the second and third factor. By the assumption on transitivity, the map p is surjective. Moreover, for the action of the group G_{x_0} on I given by left-translation on the factor G (and the *trivial* action on X, Y), we claim that p is a principal G_{x_0}-bundle. An fppf cover of X trivializing this bundle is given by the projection $p : I \to X \times Y$ itself.

Indeed, the fibre product $F = I \times_{X \times Y} I$ parameterizes tuples $(g, g', x, y) \in G \times G \times X \times Y$ such that $gx = x_0$, $g'x = x_0$. Setting $\widetilde{g} = g' \circ g^{-1} \in G_{x_0}$ this is equivalent to the data of $(g, \widetilde{g}, x, y)$ such that $gx = x_0$ and such that $\widetilde{g} \in G_{x_0}$, which defines a trivial G_{x_0}-bundle over I as desired.

Now we note that the map $\widetilde{f} : I \to Y, (g,x,y) \mapsto gy$ is G_{x_0}-equivariant (with respect to G_0-action on the target induced by its given G-action). Using that $p : I \to X \times Y$ is a principal G_{x_0}-bundle together with the definition of the stack $[Y/G_{x_0}]$ it gives rise to a map $\overline{f} : X \times Y \to [Y/G_{x_0}]$. On the other hand, it is not hard to see that this map is invariant under the G-action on $X \times Y$ and thus factors through the quotient stack $[X \times Y/G]$ via the desired map $f : [X \times Y/G] \to [Y/G_{x_0}]$. It is then straightforward to check that the map (12) induces the inverse map of f. ■

Proposition 3.3. *Given a vector $\vec{\lambda} \in \mathbb{Z}_{>0}^{N+1}$ of positive integers, the weighted projective space $\mathcal{P}(\vec{\lambda})$ has Chow group*

$$A^*(\mathcal{P}(\vec{\lambda}), \mathbb{Q}) = \mathbb{Q}[\zeta]/(\zeta^{N+1}),$$

where $\zeta = c_1(\mathcal{O}_{\mathcal{P}(\vec{\lambda})}(1))$ is the first Chern class of the bundle $\mathcal{O}_{\mathcal{P}(\vec{\lambda})}(1)$. Moreover, the space $\mathcal{P}(\vec{\lambda})$ satisfies the Chow-Künneth property*, i.e. for any smooth Deligne-Mumford stack X the natural map*

$$\underbrace{A^*(X,\mathbb{Q}) \otimes_{\mathbb{Q}} A^*(\mathcal{P}(\vec{\lambda}), \mathbb{Q})}_{=A^*(X,\mathbb{Q})[\zeta]/(\zeta^{N+1})} \to A^*(X \times \mathcal{P}(\vec{\lambda}), \mathbb{Q})$$

is an isomorphism.

Note that the intersection theory of $\mathcal{P}(\vec{\lambda})$ was studied before in various settings (see [Kawasaki, Al Amrani, BFR]). We give a self-contained proof in the language of Chow groups with $\mathbb{Q}$-coefficients, which gives us the additional Chow-Künneth property that we need later.

Proof of Proposition 3.3. For our proof we want to use the fact that $\mathcal{P}(\vec{\lambda})$ has a cellular decomposition (similar to the concept in [Fulton]), which in our context means a locally closed stratification in stacks which are isomorphic to finite quotients of affine spaces $\mathbb{A}^n$. The closures of these strata then form a basis of $A^*(\mathcal{P}(\vec{\lambda}), \mathbb{Q})$.

To make this more precise, we first note that for dimension reasons we indeed have $\zeta^{N+1} = 0$, so that there is a well-defined map from $\mathbb{Q}[\zeta]/(\zeta^{N+1})$ to $A^*(\mathcal{P}(\vec{\lambda}), \mathbb{Q})$. Our goal is to show that this map is surjective and injective.

For the surjectivity, consider the open substack

$$\mathscr{U}_0 = \{[x_0 : \ldots : x_N] : x_0 \neq 0\} \subseteq \mathscr{P}(\vec{\lambda})$$

with complement $\mathscr{Z}_0 \cong \mathscr{P}(\lambda_1, \ldots, \lambda_N)$. Then we have the excision sequence

$$A_*(\mathscr{Z}_0, \mathbb{Q}) \xrightarrow{i_*} A_*(\mathscr{P}(\vec{\lambda}), \mathbb{Q}) \to A_*(\mathscr{U}_0, \mathbb{Q}) \to 0\,. \tag{13}$$

By induction (starting with the trivial case $N = 0$) we have $A_*(\mathscr{Z}_0, \mathbb{Q}) \cong \mathbb{Q}[\zeta]/(\zeta^{N+1})$. Moreover, since $\mathscr{Z}_0 = \{x_0 = 0\}$ is the zero set of the section x_0 of $\mathscr{O}_{\mathscr{P}(\vec{\lambda})}(\lambda_0)$, we have that the image of the map i_* equals the image of $(\lambda_0\zeta) \cdot \mathbb{Q}[\zeta]/(\zeta^N) \subseteq \mathbb{Q}[\zeta]/(\zeta^{N+1})$. On the other hand, the open substack $\mathscr{U}_0$ is isomorphic to $[\mathbb{G}_m \times \mathbb{A}^N/\mathbb{G}_m]$. Then observe that the action of $\mathbb{G}_m$ on itself of weight λ_0 is transitive and the stabilizer of 1 equals μ_{λ_0}. Thus using Lemma 3.2 we have

$$\mathscr{U}_0 \cong [\mathbb{A}^N/\mu_{\lambda_0}].$$

This is a vector bundle over $B\mu_{\lambda_0}$ of rank N and thus its Chow group is isomorphic to $A^*(B\mu_{\lambda_0}, \mathbb{Q}) = \mathbb{Q} \cdot [B\mu_{\lambda_0}]$ and hence trivial. Thus the excision sequence (13) has the form

$$\mathbb{Q}[\zeta]/(\zeta^N) \xrightarrow{\cdot \lambda_0 \zeta} A_*(\mathscr{P}(\vec{\lambda}), \mathbb{Q}) \to \mathbb{Q} \to 0\,,$$

which implies that $\mathbb{Q}[\zeta]/(\zeta^{N+1})$ surjects onto the Chow group of $\mathscr{P}(\vec{\lambda})$. For injectivity, note that the kernel of the natural map $\mathbb{Q}[\zeta] \to A_*(\mathscr{P}(\vec{\lambda}), \mathbb{Q})$ must contain the ideal (ζ^{N+1}) and thus be of the form (ζ^m) for some $m \leq N+1$. However, since $\zeta^N \neq 0$ as it is a zero-cycle of positive degree, the kernel is indeed equal to (ζ^{N+1}).

Given the stratification into $\mathscr{U}_0, \mathscr{Z}_0$ as above, by [BaeSchmitt, Proposition 2.10] and an inductive argument, the Chow-Künneth property of $\mathscr{P}(\vec{\lambda})$ follows if we can show the analogous property for $\mathscr{U}_0 \cong [\mathbb{A}^N/\mu_{\lambda_0}]$. This amounts to showing that

$$A^*(X, \mathbb{Q}) \cong A^*(X \times [\mathbb{A}^N/\mu_{\lambda_0}], \mathbb{Q}).$$

But again $X \times [\mathbb{A}^N/\mu_{\lambda_0}]$ is a vector bundle over $X \times B\mu_{\lambda_0}$ and thus the Chow groups of the two spaces agree. On the other hand, the coarse moduli space of $X \times B\mu_{\lambda_0}$ agrees with the coarse moduli space $|X|$ of X and since the Chow group with $\mathbb{Q}$-coefficients can be computed on such a coarse moduli space, we get the desired chain of isomorphisms

$$A^*(X \times [\mathbb{A}^N/\mu_{\lambda_0}], \mathbb{Q}) \cong A^*(X \times B\mu_{\lambda_0}, \mathbb{Q}) \cong A^*(|X|, \mathbb{Q}) \cong A^*(X, \mathbb{Q})\,.$$

■

Proof of Theorem 3.1. Since both $\mathscr{P}(\mathscr{E}, \vec{\lambda})$ and $\mathbb{P}(\mathscr{E}_\eta)$ are Zariski-locally trivial bundles over S, the properties of the map Φ which are local on the target (proper, flat, quasi-finite and the degree) can be verified for the trivial base $S = \mathrm{Spec}(K)$. Thus consider the map $\Phi : \mathscr{P}(\vec{\lambda}) \to \mathbb{P}^N$. The open subsets $U_i = \{x_i \neq 0\} \subseteq \mathbb{P}^N$ form a

Zariski cover and the preimage of U_i under Φ is precisely $\mathscr{U}_i = \{s_i \neq 0\} \subseteq \mathscr{P}(\vec{\lambda})$, which is isomorphic to $[\mathbb{A}^N/\mu_{\lambda_i}]$. Then we have a diagram

$$\begin{tikzcd} \mathbb{A}^N \arrow[d] \arrow[dr] & \\ \mathscr{U}_i \arrow[r] & U_i \cong \mathbb{A}^N \end{tikzcd}$$

and the diagonal map $\mathbb{A}^N \to \mathbb{A}^N$ is the finite, flat map of degree $\ell^N/\prod_{j\neq i}\lambda_j$ given by

$$(s_0, \ldots, \widehat{s_i}, \ldots, s_N) \mapsto (s_0^{\ell/\lambda_0}, \ldots, \widehat{s_i^{\ell/\lambda_i}}, \ldots, s_N^{\ell/\lambda_N}),$$

where the hat indicates that we omit the i-th entry of the vector. Since the map $\mathbb{A}^N \to \mathscr{U}_i$ is of degree λ_i, this shows all local properties of the map Φ.

Returning to the case of a general base S, it remains to show that Φ^* is an isomorphism with inverse $1/d \cdot \Phi_*$. Indeed, once this is established, the presentation (10) is simply the standard formula for the Chow group of the projective bundle $\mathbb{P}(\mathscr{E}_\eta)$, where in addition we use that the natural line bundle $\mathscr{O}_{\mathbb{P}(\mathscr{E}_\eta)}(1)$ pulls back to $\mathscr{O}_{\mathscr{P}_S(\mathscr{E},\vec{\lambda})}(\ell)$ under Φ.

To complete the proof, we note that the composition $(1/d \cdot \Phi_*) \circ \Phi^*$ is clearly equal to the identity by the projection formula (see [Fulton, Example 1.7.4] for the argument in the case of schemes). This shows that Φ^* is injective, so we conclude by proving its surjectivity.

We begin by noting, that since we have a diagram

$$\begin{tikzcd} \mathscr{P}_S(\mathscr{E},\vec{\lambda}) \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & & \mathbb{P}(\mathscr{E}_\eta) \arrow[dl, "\pi'"] \\ & S & \end{tikzcd}$$

it follows that the image of Φ^* contains all classes pulled back from S itself. On the other hand, the natural line bundle $\mathscr{O}_{\mathbb{P}(\mathscr{E}_\eta)}(1)$ pulls back to $\mathscr{O}_{\mathscr{P}_S(\mathscr{E},\vec{\lambda})}(\ell)$, so that the class

$$\widetilde{\xi} = c_1(\mathscr{O}_{\mathscr{P}_S(\mathscr{E},\vec{\lambda})}(1)) \in A^1(\mathscr{P}_S(\mathscr{E},\vec{\lambda}),\mathbb{Q})$$

is likewise contained in the image of Φ^*. We want to show that these two types of classes generate $A^*(\mathscr{P}_S(\mathscr{E},\vec{\lambda}),\mathbb{Q})$ as a $\mathbb{Q}$-algebra, i.e. that the natural map

$$\Psi_S : A^*(S,\mathbb{Q})[\widetilde{\zeta}] \to A^*(\mathscr{P}_S(\mathscr{E},\vec{\lambda}),\mathbb{Q}), \quad \alpha \cdot \widetilde{\zeta}^m \mapsto \pi^*(\alpha) \cdot \widetilde{\zeta}^m$$

is surjective. For this, let $V \subseteq S$ be an open substack on which all summands $\mathscr{L}_i$ of $\mathscr{E}$ are trivial. Then we have $\pi^{-1}(V) \cong V \times \mathscr{P}(\vec{\lambda})$ is a trivial product over V. Denoting $K = S \setminus V$ the complement of V, we have two excision sequences

$$\begin{tikzcd} A_*(\pi^{-1}(K),\mathbb{Q}) \arrow[r] & A_*(\mathscr{P}_S(\mathscr{E},\vec{\lambda}),\mathbb{Q}) \arrow[r] & A_*(V \times \mathscr{P}(\vec{\lambda}),\mathbb{Q}) \arrow[r] & 0 \\ A_*(K,\mathbb{Q})[\widetilde{\zeta}] \arrow[r] \arrow[u, "\Psi_K"] & A_*(S,\mathbb{Q})[\widetilde{\zeta}] \arrow[r] \arrow[u, "\Psi_S"] & A_*(V,\mathbb{Q})[\widetilde{\zeta}] \arrow[r] \arrow[u, "\Psi_V"] & 0 \end{tikzcd} \quad . \qquad (14)$$

Here the bottom row of the diagram is obtained from the excision sequence of $V \subseteq S$ by tensoring with $\mathbb{Q}[\widetilde{\zeta}]$, which is a right-exact operation. Now we see that the map Ψ_V is surjective by Proposition 3.3, whereas the surjectivity of Ψ_K follows by Noetherian induction. By the four lemma, it follows that Ψ_S is surjective, which finishes the proof. ■

3.1 Proof of Theorem 1.6

Based on the theorem above, we can now also compute the Chow group of a weighted projective bundle, where the bundle does not split into line bundles, but rather into sub-vector bundles of possibly higher ranks.

For the proof, we need a variant of the *splitting principle* inspired by an answer of Angelo Vistoli on mathoverflow.

Lemma 3.4. *Given a Deligne-Mumford stack S and a vector bundle $\mathscr{E}$ on S, there exists a flat morphism $f : \widehat{S} \to S$ such that for any morphism $W \to S$ the pullback*

$$f^* : A^*(W, \mathbb{Q}) \to A^*(\widehat{S} \times_S W, \mathbb{Q})$$

is injective and such that $f^*\mathscr{E}$ splits as a direct sum of line bundles.

Proof. Let $P \to S$ be the frame bundle of $\mathscr{E}$, whose fibre over $s \in S$ is the set of all bases of the vector space $\mathscr{E}_s$. It carries a natural action of GL_r. Let $T \subseteq B \subseteq \mathrm{GL}_r$ be the maximal torus and Borel subgroup of GL_r, respectively. Then we claim that the map $f : \widehat{S} = [P/T] \to S$ satisfies the properties of the lemma. Indeed, since $\widehat{S}$ over any point $s \in S$ parameterizes a basis of $\mathscr{E}_s$ up to scaling, the pullback $\mathscr{E}|_{\widehat{S}}$ naturally splits into line bundles. On the other hand, the map f is the composition $[P/T] \to [P/B] \to S = [P/\mathrm{GL}_r]$ of an affine bundle and a projective morphism, both of which have injective pullbacks in Chow (which remains true after base change with $W \to S$). ■

Proof of Theorem 1.6. When the rank of all bundles $\mathscr{E}_i$ is 1, the result is precisely Theorem 3.1. In the more general setting, we can use the fact that the bundles $\mathscr{E}_i$ split into line bundles Zariski locally and repeat the excision sequence arguments from the proof of Theorem 3.1 to conclude that the Chow group of $\mathscr{P}_S(\mathscr{E}, \vec{\lambda})$ is generated as a $\mathbb{Q}$-algebra by classes pulled back from S and by the class ζ. In other words, we have a natural surjection

$$\Psi : A^*(S, \mathbb{Q})[\zeta] \to A^*(\mathscr{P}_S(\mathscr{E}, \vec{\lambda}), \mathbb{Q}).$$

Given $f : \widehat{S} \to S$ as in Lemma 3.4, we can ensure that all $f^*\mathscr{E}_i$ split into line bundles (applying the lemma $N + 1$ times if necessary). Then we can compute the Chow group of $\mathscr{P}_{\widehat{S}}(f^*\mathscr{E}, \vec{\lambda}) = \mathscr{P}_S(\mathscr{E}, \vec{\lambda}) \times_S \widehat{S}$ and have a sequence of maps

$$A^*(S, \mathbb{Q})[\zeta] \xrightarrow{\Psi} A^*(\mathscr{P}_S(\mathscr{E}, \vec{\lambda}), \mathbb{Q}) \xrightarrow{f^*} A^*(\mathscr{P}_S(\mathscr{E}, \vec{\lambda}), \mathbb{Q}) = A^*(\widehat{S}, \mathbb{Q})[\zeta]/(\zeta^{N+1} + \ldots + f^* c_{N+1}^{\vec{\eta}}(\mathscr{E}))$$

with f^* injective. It follows that the element

$$Q = \zeta^{N+1} + \zeta^N c_1^{\vec{\eta}}(\mathscr{E}) \ldots + c_{N+1}^{\vec{\eta}}(\mathscr{E})$$

must be in the kernel of Ψ. To show that it generates the kernel (as an ideal), note that the quotient $A^*(S,\mathbb{Q})[\zeta]/(Q)$ has a direct sum decomposition

$$A^*(S,\mathbb{Q})[\zeta]/(Q) = \bigoplus_{i=0}^{N} A^*(S,\mathbb{Q}) \cdot \zeta^i \tag{15}$$

into $N+1$ copies of $A^*(S,\mathbb{Q})$. Let $\pi : \mathscr{P}_S(\mathscr{E},\vec{\lambda}) \to S$ be the projection to the base, then we have a linear map

$$G : A^*(S,\mathbb{Q})[\zeta]/(Q) \to \bigoplus_{i=0}^{N} A^*(S,\mathbb{Q}), \alpha \mapsto \left(\pi_*(\Psi(\alpha)\cdot\zeta^i)\right)_{i=0,\dots,N} .$$

We claim that with respect to the direct sum decomposition (15), the map G is given by multiplication with a matrix of elements in $A^*(S,\mathbb{Q})$ with vanishing entries 1 on the anti-diagonal, and vanishing entries above this anti-diagonal. Assuming the claim, the map G is injective (up to reordering it is a triangular base change) and since it factors via $\Psi : A^*(S,\mathbb{Q})[\zeta]/(Q) \to A^*(\mathscr{P}_S(\mathscr{E},\vec{\lambda}),\mathbb{Q})$, the map Ψ must also be injective. We have already seen it to be surjective, so it is an isomorphism and thus the theorem is proven.

To show the claim, note that $\pi_*\zeta^N = [S]$, whereas $\pi_*\zeta^i = 0$ for $i < N$. It follows that for an element $\alpha = \sum_{j=0}^{N} \alpha_j \zeta^j$ in the domain of G (with $\alpha_j \in A^*(S,\mathbb{Q})$), we have that the ith component of $G(\alpha)$ is given by

$$G(\alpha)_i = \sum_{j=0}^{N} \pi_*(\alpha_j \cdot \zeta^{j+i}) = \sum_{j=0}^{N} \alpha_j \cdot \underbrace{\pi_*(\zeta^{j+i})}_{=0 \text{ for } j<N-i} .$$

But as stated before, we have $\pi_*(\zeta^{j+i}) = [S]$ for $j = N-i$ and $\pi_*(\zeta^{j+i}) = 0$ for $j < N-i$, which is precisely the shape of the matrix describing G that was claimed. ■

The weighted projective bundle formula in the rational Chow ring holds in the ℓ-adic rational étale cohomology ring as well.

Corollary 3.5. *Let S be a smooth Deligne-Mumford stack with a vector bundle $\mathscr{E} = \bigoplus_{i=0}^{N} \mathscr{E}_i$ and let $\vec{\lambda} \in \mathbb{Z}_{\geq 1}^{N+1}$ be a vector of positive integers. Let $L = \mathrm{lcm}(\vec{\lambda})$ and consider the vector $\vec{\eta} = (L/\lambda_0,\dots,L/\lambda_N)$. Then we have*

$$H^*(\mathscr{P}_S(\mathscr{E},\vec{\lambda}),\mathbb{Q}_\ell) = H^*(S,\mathbb{Q}_\ell)[\zeta]/(\zeta^{N+1} + c_1^{\vec{\eta}}(\mathscr{E})\zeta^N + \ldots + c_{N+1}^{\vec{\eta}}(\mathscr{E})), \tag{16}$$

where $\zeta = L \cdot c_1(\mathscr{O}_{\mathscr{P}_S(\mathscr{E},\vec{\lambda})}(1))$.

Proof. The rational cohomology of the smooth separated tame Deligne–Mumford stack is the same as its coarse space by Lemma 2.6. This allows us to use the weighted projective bundle for the étale cohomology proved in the [Al Amrani, §III] and [BFR] which are proved via the application of Leray-Hirsch Theorem. ■

We conclude this Section by proving Proposition 1.5.

3.2 Proof of Proposition 1.5

Proof. We have that $\mathrm{Hom}_n(\mathbb{P}^1, \mathscr{P}(\vec{\lambda}))$ is naturally an open substack of the weighted projective stack $\mathscr{P}(\vec{\Lambda})$ of dimension $|\vec{\lambda}|n+N$, whose complement $Z \subseteq \mathscr{P}(\vec{\Lambda})$ is the locus of points $[s_0 : \ldots s_N]$ where all s_i *do* vanish simultaneously at some $q \in \mathbb{P}^1$. We have an isomorphism $\mathrm{Pic}(\mathscr{P}(\vec{\Lambda})) = \mathbb{Z}$ by [Noohi, Proposition 6.4.] and for the Picard group of $\mathrm{Hom}_n(\mathbb{P}^1, \mathscr{P}(\vec{\lambda}))$, we claim that the codimension of Z in $\mathscr{P}(\vec{\Lambda})$ is precisely N, implying that for $N \geq 2$ we have

$$\mathrm{Pic}(\mathrm{Hom}_n(\mathbb{P}^1, \mathscr{P}(\vec{\lambda}))) = \mathrm{Pic}(\mathscr{P}(\vec{\Lambda}) \setminus Z) = \mathrm{Pic}(\mathscr{P}(\vec{\Lambda})) = \mathbb{Z}$$

by [Fringuelli, (ii) Theorem 2.1.4.]. To bound the codimension of Z, consider the incidence variety

$$\widehat{Z} = \{(q, [s_0, \ldots, s_N]) \in \mathbb{P}^1 \times \mathscr{P}(\vec{\Lambda}) : s_i(q) = 0 \text{ for } i = 0, \ldots, N\} \subseteq \mathbb{P}^1 \times \mathscr{P}(\vec{\Lambda}).$$

For a fixed $q \in \mathbb{P}^1$, the set of points $(s_0, \ldots, s_N) \in \vec{\Lambda}$ such that all s_i vanish at q is a linear subspace of codimension $N+1$ in $\vec{\Lambda}$, and in fact, the projection $\widehat{Z} \to \mathbb{P}^1$ is a projective bundle of the corresponding rank. In particular, we have that $\widehat{Z} \subset \mathbb{P}^1 \times \mathscr{P}(\vec{\Lambda})$ is irreducible of codimension $N+1$. On the other hand, we have that $Z \subset \mathscr{P}(\vec{\Lambda})$ is the image of $\widehat{Z}$ under the projection $\mathbb{P}^1 \times \mathscr{P}(\vec{\Lambda}) \to \mathscr{P}(\vec{\Lambda})$. Since the fibres of $\widehat{Z} \to Z$ are finite, it follows that the codimension of Z is indeed N.

On the one hand, for $N = 1$ with $\mathscr{P}(\lambda_0, \lambda_1)$ we have that the subvariety Z of $\mathscr{P}(\vec{\Lambda})$ is cut out by the resultant $\mathrm{Res}(s_0, s_1)$. The resultant is an irreducible polynomial in the coefficients of s_0, s_1 ([GKZ, Chapter 8, Proposition-Definition 1.1]), which is homogeneous of degree $n(\lambda_0 + \lambda_1)$.

Then, by [Fringuelli, (iv) Theorem 2.1.4.] we have an exact sequence

$$\mathbb{Z} \to \mathrm{Pic}(\mathscr{P}(\vec{\Lambda})) \to \mathrm{Pic}(\mathrm{Hom}_n(\mathbb{P}^1, \mathscr{P}(\lambda_0, \lambda_1))) \to 0$$

where the first map is defined by $1 \mapsto 1 \cdot V(\mathrm{Res})$. As $\mathrm{Pic}(\mathscr{P}(\vec{\Lambda})) \cong \mathbb{Z}$, we have that

$$\mathrm{Pic}(\mathrm{Hom}_n(\mathbb{P}^1, \mathscr{P}(\lambda_0, \lambda_1))) \cong \mathbb{Z}/\mathrm{Deg}(\mathrm{Res})\mathbb{Z} = \mathbb{Z}/((\lambda_0 + \lambda_1)n)\mathbb{Z}.$$

∎

4 Stable cohomology of the Hom stacks with weights

Recall that for a Deligne-Mumford stack $\mathscr{X}$ over K, we denote the ℓ-adic cohomology group with rational coefficients by $H^i(\mathscr{X}; \mathbb{Q}_\ell)$ (warning: this is not the étale cohomology with $\mathbb{Q}_\ell$ coefficients, see e.g. [GL, Warning 3.2.1.9], or any text on étale cohomology of schemes e.g. [Milne]); by the same token a sheaf of $\mathbb{Q}_\ell$ vector spaces is a $\mathbb{Z}_\ell$ sheaf $\mathscr{F} = (\mathscr{F}_n)$ and

$$H^i(\mathscr{X}; \mathscr{F}) := \varprojlim H^i(\mathscr{X}; \mathscr{F}_n) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

(similar to the notations set up in [Milne, Section 19].) For brevity and convenience, and noting there is no scope of confusion since we are always working over rational coefficients, we will write $H^i(\mathscr{X})$ to stand for $H^i(\mathscr{X};\mathbb{Q}_\ell)$.

For $\vec{\lambda} = (\lambda_0, \cdots, \lambda_N) \in \mathbb{Z}_{>0}^{N+1}$ and positive integers a, b, we denote by $a\vec{\lambda}+b$ the vector $(a\lambda_0 + b, a\lambda_1 + b, \cdots, a\lambda_N + b)$. Furthermore, given $\vec{\lambda} = (\lambda_0, \cdots, \lambda_N)$ with $|\vec{\lambda}| := \sum_{i=0}^{N} \lambda_i$ and a vector space $V := V_0 \oplus \cdots \oplus V_N$ of dimension $|\vec{\lambda}|n + (N+1) - (N+1)g$, we denote by $\mathscr{P}(\oplus V_i, \vec{\lambda})$ the weighted projective stack where $\mathbb{G}_m$ acts on the direct summand V_i of V by weight λ_i. If $\mathscr{E}$ is a vector bundle on a space $\mathscr{X}$ such that $\mathscr{E} = \mathscr{E}_0 \oplus \cdots \oplus \mathscr{E}_N$ then by $\mathscr{P}_{\mathscr{X}}(\oplus_i \mathscr{E}_i, \vec{\lambda})$ we denote the weighted projectivization of $\mathscr{E}$ where $\mathbb{G}_m$ acts on the sub-bundle $\mathscr{E}_i$ by weight λ_i. Note that when we write $\mathscr{P}_{\mathscr{X}}(\oplus_i \mathscr{E}_i, \vec{\lambda})$ we implicitly assume that i runs from 0 to N and that $\vec{\lambda} \in \mathbb{Z}_{>0}^{N+1}$.

4.1 Proof of Main Theorem 1.1

Theorem 1.1 (and in turn Main Theorem 1.2) is a direct consequence of [Banerjee, Theorem 2]. In particular, we construct an object over ΔS (called the *symmetric simplicial category*, see [Banerjee, Definition 2.10]) in the category of (smooth proper) Deligne-Mumford stacks.

Now a degree n morphism $C \to \mathscr{P}(\vec{\lambda})$ is equivalent to the following data:

- a line bundle L of degree n on C,
- an $(N+1)$-tuple $(s_0, \ldots, s_N)$ where $s_i \in H^0(C, L^{\otimes \lambda_i})$
- the sections $s_0, \ldots, s_N$ satisfy the condition that they have no common zeroes (also known as $\{s_0, \ldots, s_N\}$ is *basepoint free*).

Then, $\mathrm{Hom}_n(C, \mathscr{P}(\vec{\lambda}))$ is a Zariski open dense subset of $\widetilde{\mathrm{Hom}}_n(C, \mathscr{P}(\vec{\lambda}))$ defined by

$$\widetilde{\mathrm{Hom}}_n(C, \mathscr{P}(\vec{\lambda})) := \{L,\ [s_0 : \ldots : s_N] : L \in \mathrm{Pic}^n(C),\ s_i \in H^0(C, L^{\otimes \lambda_i}) \text{ for all } i\}.$$

Note that $\widetilde{\mathrm{Hom}}_n(C, \mathscr{P}(\vec{\lambda}))$ is isomorphic to the weighted projectivization of a vector bundle $\mathscr{E}$ on $\mathrm{Pic}^n(C)$ (assuming $n \geq 2g$) whose fibre over L is given by $V_L := \oplus_i H^0(C, L^{\otimes \lambda_i})$, where via the Riemann-Roch theorem we have $H^0(C, L^{\otimes \lambda_i}) \cong \mathbb{A}^{\lambda_i n - g + 1}$ and $\mathbb{G}_m$ acts by weight λ_i on $H^0(C, L^{\otimes \lambda_i})$; fibre over $L \in \mathrm{Pic}^n C$ isomorphic to the weighted projective stack $\mathscr{P}(\oplus_i H^0(\mathbb{P}^1, \mathscr{O}(n)^{\otimes \lambda_i}), \vec{\lambda})$.

More precisely, if

$$\nu : C \times \mathrm{Pic}^n C \to \mathrm{Pic}^n C$$

is the projection to the second factor and $P(n)$ denotes a Poincare bundle of degree n on $C \times \mathrm{Pic}^n C$, then we let $\Gamma_{\lambda_i}(n) := \nu_* P(n)^{\otimes \lambda_i}$ (and when $\lambda_i = 1$ we write $\Gamma(n)$). Then $\widetilde{\mathrm{Hom}}_n(C, \mathscr{P}(\vec{\lambda}))$ is the weighted projectivization of $\oplus_i \Gamma_{\lambda_i}(n)$ where $\mathbb{G}_m$ acts on $\Gamma_{\lambda_i}(n)$ by weight λ_i. By Theorem 1.6 we know the cohomology of $\widetilde{\mathrm{Hom}}_n(C, \mathscr{P}(\vec{\lambda}))$- a key player in the proof of Theorem 1.1 as we shall soon see.

Even though Corollary 1.2 follows from Theorem 1.1 when we take the curve C to be of genus 0, literature shows that not only is the genus 0 case worth proving in it's own right, in this manuscript it also portrays how the basic strategy of proving the higher genus case is essentially the same as that of the genus 0 case; in other words if the reader understands the proof of Corollary 1.2, he knows the proof of Theorem 1.1 as well. Keeping this in mind, we first prove Corollary 1.2.

Proof of Corollary 1.2. First observe that $\widetilde{\mathrm{Hom}}_n(\mathbb{P}^1, \mathscr{P}(\vec{\lambda}))$ is a weighted projective stack isomorphic to $\mathscr{P}(\oplus_i H^0(\mathbb{P}^1, \mathscr{O}(n)^{\lambda_i}, \vec{\lambda})$. Define a symmetric semisimplicial space (see [Banerjee, Defintion 2.10]) as follows. Let

$$\mathscr{T}_0 := \mathbb{P}^1 \times \mathscr{P}(\oplus_i H^0(\mathbb{P}^1, \mathscr{O}(n\lambda_i - 1)), \vec{\lambda}) \tag{17}$$

and let

$$\mathscr{T}_p := (\mathbb{P}^1)^{p+1} \times \mathscr{P}(\oplus_i H^0(\mathbb{P}^1, \mathscr{O}(n\lambda_i - p)), \vec{\lambda}). \tag{18}$$

Letting $\mathscr{T}_{-1}$ denote $\widetilde{\mathrm{Hom}}_n(\mathbb{P}^1, \mathscr{P}(\vec{\lambda}))$, we observe that there are natural *face maps* (which are finite morphisms of smooth proper Deligne-Mumford stacks) $f_i : \mathscr{T}_p \to \mathscr{T}_{p-1}$ for all $p \geq 0$ given by *adding a basepoint* from the i^{th} factor i.e.

$$\begin{gathered} f_i : (\mathbb{P}^1)^{p+1} \times \mathscr{P}(\oplus_i H^0(\mathbb{P}^1, \mathscr{O}(n\lambda_i - (p+1)), \vec{\lambda}) \to (\mathbb{P}^1)^p \times \mathscr{P}(\oplus_i H^0(\mathbb{P}^1, \mathscr{O}(n\lambda_i - p), \vec{\lambda}) \\ \big([a_0 : b_0], \dots, [a_p : b_p]\big),\ [s_0 : \dots : s_N] \mapsto \\ \big([a_0, b_0], \dots, [\widehat{a_i : b_i}], \dots [a_p : b_p]\big),\ [(b_i x - a_i y)s_0 : \dots : (b_i x - a_i y)s_N] \end{gathered} \tag{19}$$

where $[\widehat{a_i : b_i}]$ denotes removing the i^{th} entry $[a_i : b_i]$. In other words, the hypercover under consideration is the following:

$$\begin{gathered} \cdots (\mathbb{P}^1)^3 \times \mathscr{P}(\oplus_i H^0(\mathbb{P}^1, \mathscr{O}(n\lambda_i - 3)), \vec{\lambda}) \substack{\rightarrow\\\rightarrow\\\rightarrow} (\mathbb{P}^1)^2 \times \mathscr{P}(\oplus_i H^0(\mathbb{P}^1, \mathscr{O}(n\lambda_i - 2)), \vec{\lambda}) \\ \rightrightarrows \mathbb{P}^1 \times \mathscr{P}(\oplus_i H^0(\mathbb{P}^1, \mathscr{O}(n\lambda_i - 1)), \vec{\lambda}) \to \mathscr{P}(\oplus_i H^0(\mathbb{P}^1, \mathscr{O}(n\lambda_i)), \vec{\lambda}) \end{gathered}$$

with the unlabelled arrows denoting the face maps f_i.

Let $\mathscr{Z} \subset \widetilde{\mathrm{Hom}}_n(\mathbb{P}^1, \mathscr{P}(\vec{\lambda}))$ be the "discriminant locus" given by

$$\begin{aligned} \mathscr{Z} = \{[s_0 : \cdots : s_N] : s_i \in H^0(\mathbb{P}^1, \mathscr{O}(n)^{\otimes \lambda_i}) \text{ for all } i = 0, \dots, N, \\ s_0, \dots, s_N \text{ have at least one common root}\}. \end{aligned}$$

Now we make the following observation (an almost immediate consequence of [Conrad, Theorem 7.1] or [Deligne])):

Claim 4.1. *The hypercovering $\mathscr{T}_\bullet \to \mathscr{Z}$ is universally of cohomological descent.*

To prove it we show a more general lemma:

Lemma 4.2. *Let $\pi_\bullet : \mathcal{T}_\bullet \to \mathcal{X}$ be a proper hypercovering in the category of Deligne-Mumford stacks, with the étale topology, such that for all p, and a morphism*

$$x : \operatorname{Spec} K \to \mathcal{T}_p$$

the kernel $\operatorname{Isom}_{\mathcal{T}_p}(x,x) \to \operatorname{Isom}_{\mathcal{X}}(x,x)$ *is trivial, then it is universally of cohomological descent.*

Proof of Lemma 4.2. The proof of this lemma follows immediately by noting that the claim is about étale sheaves on the base Deligne-Mumford stack $\mathcal{X}$, so it suffices to check the claim étale locally. Now thanks to [Stacks, Lemma 100.6.2] the face maps are representable by algebraic spaces, so étale locally we can simply apply the result for schemes, which is precisely [Conrad, Theorem 7.1]: that a proper hypercovering in the category of schemes is universally of cohomological descent. ∎

Proof of 4.1. In the special case when $\mathcal{T}_\bullet$ is the specific simplicial space defined above (4.1) and $\mathcal{X}$ is the discriminant locus, the condition in the claim of the kernel being trivial is clearly satisfied, and thus the observation follows. ∎

Having settled that $T_\bullet \to \mathcal{X}$ as a semisimplicial space is indeed of cohomological descent, by virtue of having the additional structure of a $\Delta_{inj}S$ object, it further satisfies the conditions of [Banerjee, Theorem 2] [1] and results in a second quadrant spectral sequence whose E_1 page reads as

$$E_1^{-p,q} = \begin{cases} H^q(\mathcal{P}(\oplus_i H^0(\mathbb{P}^1, \mathcal{O}(n\lambda_i)), \vec{\lambda}))(0) & p=0, \\ H^{q-2N}(\mathbb{P}^1 \times \mathcal{P}(\oplus_i H^0(\mathbb{P}^1, \mathcal{O}(n\lambda_i - 1))(-1) & p=1, \\ H^0(\mathbb{P}^1) \otimes H^2(\mathbb{P}^1) \otimes H^{q-4N-2}(\mathcal{P}(\oplus_i H^0(\mathbb{P}^1, \mathcal{O}(n\lambda_i - 2))(-2) & p=2, \\ 0 & \text{otherwise}, \end{cases}$$

with the differentials given by the alternating sum of the Gysin pushforwards induced by the face maps, which is what we shall compute now.

- *Computing $d_1^{1,q} : E_1^{-1,q} \to E_1^{0,q}$.*

  For simplicity we denote the differential by d_1^1. Let

$$\iota : \mathcal{P}(\oplus_i H^0(\mathbb{P}^1, \mathcal{O}(n\lambda_i - 1)), \vec{\lambda}) \hookrightarrow \mathcal{P}(\oplus_i H^0(\mathbb{P}^1, \mathcal{O}(n\lambda_i), \vec{\lambda})$$

[1] In fact one can make a more general statement at no extra cost, simply by translating cohomological descent in the ∞-category of ℓ-adic sheaves $\mathrm{Sh}_\ell(\mathcal{X})$ (as defined, for example, by Gaitsgory-Lurie in [GL]) over to the indexing category ΔS, essentially giving an ∞-categorical version of [Banerjee, Lemma 2.11], as follows. Given a $S_\bullet$ hypercovering $\pi_\bullet : \mathcal{T}_\bullet \to \mathcal{X}$ satisfying the conditions of Lemma 4.2, there is an equivalence of endofunctors

$$\mathrm{id} \to \left(\mathrm{R}\pi_{\bullet *}\pi_\bullet^* \otimes \mathrm{sgn}_{S_\bullet}\right)^{S_\bullet}$$

in the ∞-category $\mathrm{Fun}(\mathrm{Sh}_\ell(\mathcal{X}), \mathrm{Sh}_\ell(\mathcal{X}))$, where $S_\bullet$ denotes the symmetric simplicial group given by S_n denoting the symmetric group on $(n+1)$ elements. However, we do not need the full power of this statement in for our immediate computation. The interested reader can refer to [Banerjee2].

denote the inclusion given by adding a basepoint.

Choose generators $\mathbf{1} \in H^0(\mathbb{P}^1)$ and $e \in H^2(\mathbb{P}^1)$, and let h denote the hyperplane class in $\mathscr{P}(\oplus_i H^0(\mathbb{P}^1, \mathscr{O}(n\lambda_i), \vec{\lambda})$. Then we claim that:

$$\begin{gathered} d_1^1 = f_{0*} : H^{*-2N}(\mathbb{P}^1 \times \mathscr{P}(\oplus_i H^0(\mathbb{P}^1, \mathscr{O}(n\lambda_i - 1)), \vec{\lambda})) \to \\ H^*(\mathscr{P}(\oplus_i H^0(\mathbb{P}^1, \mathscr{O}(n\lambda_i), \vec{\lambda})) \\ \mathbf{1} \otimes \iota^*\alpha + e \otimes \iota^*\beta \mapsto \alpha h^N + \beta h^{N+1} \end{gathered}$$

is a map of $H^*(\mathscr{P}(\oplus_i H^0(\mathbb{P}^1, \mathscr{O}(n\lambda_i), \vec{\lambda}))$-modules, where

$$\alpha, \beta \in H^*(\mathscr{P}(\oplus_i H^0(\mathbb{P}^1, \mathscr{O}(n\lambda_i), \vec{\lambda}).$$

To see this, first note that

$$\iota^* : H^*(\mathscr{P}(\oplus_i H^0(\mathbb{P}^1, \mathscr{O}(n\lambda_i - 1), \vec{\lambda})) \to H^*(\mathscr{P}(\oplus_i H^0(\mathbb{P}^1, \mathscr{O}(n\lambda_i), \vec{\lambda}))$$

is a surjection; next, the image of the fundamental class

$$[\mathbb{P}^1 \times \mathscr{P}(\oplus_i H^0(\mathbb{P}^1, \mathscr{O}(n\lambda_i - 1), \vec{\lambda})] \in H^0(\mathbb{P}^1 \times \mathscr{P}(\oplus_i H^0(\mathbb{P}^1, \mathscr{O}(n\lambda_i - 1), \vec{\lambda}))$$

is the locus of elements in $\mathscr{P}(\oplus_i H^0(\mathbb{P}^1, \mathscr{O}(n\lambda_i), \vec{\lambda})$ that has a basepoint i.e. $\mathscr{X}$, which is rationally equivalent, and thus cohomologous, to (a multiple of) h^N; and finally, for a fixed point $[a : b] \in \mathbb{P}^1$, the locus given by

$$\{[s_0 : \ldots : s_N] \in \mathscr{P}(\oplus_i H^0(\mathbb{P}^1, \mathscr{O}(n\lambda_i), \vec{\lambda}) : s_i([a : b]) = 0\}$$

is rationally equivalent, and in turn cohomologous, to (a multiple of) h^{N+1}. For the sake of simplicity we won't bother ourselves with the scalar multiples, which is fine because we're working over $\mathbb{Q}$. The Gysin pushforward $d_1^1 = f_{0*}$ surjects onto the ideal generated by h^N in $H^*(\mathscr{P}(\oplus_i H^0(\mathbb{P}^1, \mathscr{O}(n\lambda_i, \vec{\lambda}))$. Indeed, the preimage of h^{N+i} is given by

$$\begin{gathered} d_1^1(\mathbf{1} \otimes \iota^* h^i) = h^{N+i} = d_1^1(e \otimes \iota^* h^{i-1}) \text{ for } i \geq 1 \\ d_1^1([\mathbb{P}^1 \times \mathscr{P}(\oplus_i H^0(\mathbb{P}^1, \mathscr{O}(n\lambda_i - 1), \vec{\lambda})) = h^N, \end{gathered}$$

which shows that the image of d_1^1 is the ideal generated by h^N in $H^*(\mathscr{P}(\oplus_i H^0(\mathbb{P}^1, \mathscr{O}(n\lambda_i), \vec{\lambda}))$. The kernel of d_1^1 is given by elements of the form $\iota^*(\alpha)(h - e)$ for all $\alpha \in H^*(\mathscr{P}(\oplus_i H^0(\mathbb{P}^1, \mathscr{O}(n\lambda_i), \vec{\lambda}))$.

The upshot is that on the E_2 page, for $p = 0$ we have:

$$E_2^{0,q} = \begin{cases} \mathbb{Q}(0) & q = 2j,\ 0 \leq j \leq 2(N-1) \\ 0 & \text{otherwise.} \end{cases} \tag{20}$$

- *Computing* $d_1^{2,q} : E_1^{-2,q} \to E_1^{-1,q}$. For simplicity, we denote the differential by d_1^2. Like before, let $\iota : \mathscr{P}(\oplus_i H^0(\mathbb{P}^1, \mathscr{O}(n\lambda_i - 1), \vec{\lambda}) \hookrightarrow \mathscr{P}(\oplus_i H^0(\mathbb{P}^1, \mathscr{O}(n\lambda_i), \vec{\lambda})$

denote the inclusion given by adding a basepoint, and let h denote the hyperplane class in $\mathscr{P}(\oplus_i H^0(\mathbb{P}^1, \mathscr{O}(n\lambda_i - 1), \vec{\lambda})$. Then the way we computed f_{0*} above works verbatim, and we have

$$\begin{gathered} f_{0*} : H^0(\mathbb{P}^1) \otimes H^2(\mathbb{P}^1) \otimes H^{*-2N-2}(\mathscr{P}(\oplus_i H^0(\mathbb{P}^1, \mathscr{O}(n\lambda_i - 2)), \vec{\lambda})) \\ \to H^*(\mathbb{P}^1 \times \mathscr{P}(\oplus_i H^0(\mathbb{P}^1, \mathscr{O}(n\lambda_i - 1), \vec{\lambda}) \\ \mathbf{1} \otimes e \otimes \alpha \mapsto e \otimes \alpha h^N \end{gathered}$$

and

$$\begin{gathered} f_{1*} : H^0(\mathbb{P}^1) \otimes H^2(\mathbb{P}^1) \otimes H^{*-2N-2}(\mathscr{P}(\oplus_i H^0(\mathbb{P}^1, \mathscr{O}(n\lambda_i - 2), \vec{\lambda})) \\ \to H^*(\mathbb{P}^1 \times \mathscr{P}(\oplus_i H^0(\mathbb{P}^1, \mathscr{O}(n\lambda_i - 1), \vec{\lambda})) \\ \mathbf{1} \otimes e \otimes \alpha \mapsto \mathbf{1} \otimes \alpha h^{N+1}, \end{gathered}$$

and therefore

$$d_1^2(\mathbf{1} \otimes e \otimes \alpha) = \mathbf{1} \otimes \alpha h^{N+1} - e \otimes \alpha h^N.$$

Note that d_1^2 is injective, and the image is generated by h^N in

$$H^*(\mathbb{P}^1 \times \mathscr{P}(\oplus_i H^0(\mathbb{P}^1, \mathscr{O}(n\lambda_i - 1), \vec{\lambda})).$$

Consequently, on the E_2 page we have:

$$E_2^{-1,q} = \begin{cases} \mathbb{Q}_\ell(-N) & p = 1, q = 2j + 2N + 2,\ 0 \le j \le 2(N-1) \\ 0 & \text{otherwise} \end{cases},$$
$$E_2^{-2,q} = 0, \text{ for all } q.$$

In effect on the E_2 page all differentials vanish; the spectral sequence degenerates and we obtain

$$H^*(\mathrm{Hom}_n(\mathbb{P}^1, \mathscr{P}(\vec{\lambda})); \mathbb{Q}) \cong \frac{\mathbb{Q}[h]}{h^N} \otimes \wedge \mathbb{Q}\{t\}$$

where h has cohomological degree 2, and t (which corresponds to $e - h \in Ker(d_1^1)$) has cohomological degree $2N + 1$. Furthermore, over a field κ, with algebraic closure $\overline{\kappa}$, we have an isomorphism of $Gal(\overline{\kappa}/\kappa)$-representations:

$$H^i_{\acute{e}t}(\mathrm{Hom}_n(C, \mathscr{P}(\vec{\lambda})); \mathbb{Q}_\ell) = \begin{cases} \mathbb{Q}_\ell(-j) & i = 2j, 0 \le j \le N-1 \\ \mathbb{Q}_\ell(-(j+1)) & i = 2j+1, N \le j \le 2N-1 \\ 0 & \text{otherwise.} \end{cases}$$

This completes the proof of Corollary 1.2.

∎

Following the same plan, we now proof Main Theorem 1.1.

Proof of Theorem 1.1. As already observed above, the space $\mathrm{Hom}_n(C, \mathscr{P}(\vec{\lambda}))$ is open and dense in $\widetilde{\mathrm{Hom}}_n(C, \mathscr{P}(\vec{\lambda}))$, and we have

$$\widetilde{\mathrm{Hom}}_n(C, \mathscr{P}(\vec{\lambda})) \cong \mathscr{P}_{\mathrm{Pic}^n C}(\oplus_i \nu_* P(n)^{\otimes \lambda_i}, \vec{\lambda}).$$

Let $\mathscr{X}$ denote the discriminant locus i.e. the complement of $\mathrm{Hom}_n(C, \mathscr{P}(\vec{\lambda}))$ in $\widetilde{\mathrm{Hom}}_n(C, \mathscr{P}(\vec{\lambda}))$.

Now we construct a hypercover that is equipped with the additional structure of a ΔS object in the category of Deligne-Mumford stacks, and which admits universal cohomological descent.

To this end, we define spaces $\mathscr{T}_0$ and $\mathscr{I}_0$ as certain fibre products. First, consider the following commutative diagram:

$$\begin{tikzcd}
\mathscr{T}_0 \arrow[rr] \arrow[dr, dashed, "\pi_0"'] \arrow[dd] & & C \times \mathscr{P}_{\Pi_i \mathrm{Pic}^{\lambda_i n-1} C}(\oplus_i \nu_* P(\lambda_i n - 1), \vec{\lambda}) \arrow[dr, "A"] \arrow[dd] & \\
& \mathscr{P}_{\mathrm{Pic}^n C}(\oplus_i \nu_* P(n)^{\otimes \lambda_i}, \vec{\lambda}) \arrow[rr] \arrow[dd] & & \mathscr{P}_{\Pi_i \mathrm{Pic}^{\lambda_i n} C}(\oplus_i \nu_* P(\lambda_i n), \vec{\lambda}) \arrow[dd] \\
\mathscr{I}_0 \arrow[rr] \arrow[dr, dashed, "A"] & & C \times \Pi_i \mathrm{Pic}^{\lambda_i n-1} C \arrow[dr, "A"] & \\
& \mathrm{Pic}^n C \arrow[rr, "\otimes^{\vec{\lambda}}"] & & \Pi_i \mathrm{Pic}^{\lambda_i n} C
\end{tikzcd}$$

where the maps above are defined by:

$$\begin{aligned} \otimes^{\vec{\lambda}} : \mathrm{Pic}^n &\to \Pi_i \mathrm{Pic}^{\lambda_i n} C \\ L &\mapsto (L^{\otimes \lambda_0}, \cdots, L^{\otimes \lambda_N}) \end{aligned} \tag{21}$$

henceforth often denoting $(L^{\otimes \lambda_0}, \cdots, L^{\otimes \lambda_N})$ by $L^{\otimes \vec{\lambda}}$;

$$\begin{aligned} A : C \times \Pi_i \mathrm{Pic}^{\lambda_i n-1} C &\to \Pi_i \mathrm{Pic}^{\lambda_i n} C \\ x, (L_0, \ldots, L_N) &\mapsto L_0 \otimes \mathscr{O}_C(x), \cdots, L_N \otimes \mathscr{O}_C(x) \end{aligned} \tag{22}$$

which is the map of 'adding a point':

$$\mathscr{I}_0 := \mathrm{Pic}^n \times_{\Pi_i \mathrm{Pic}^{\lambda_i n} C} (C \times \Pi_i \mathrm{Pic}^{\lambda_i n-1} C)$$

completes the commutative square at the 'lower face' of the cube and we abuse notation and still denote the resulting 'adding a point' map by $A : \mathscr{I}_0 \to \mathrm{Pic}^n C$; the 'upper face' of the cube consists of spaces which are essentially the space of global sections of suitable Poincare bundles, i.e. the vertical arrows all correspond to taking fibrewise global sections over the moduli of line bundles; and $\mathscr{T}_0$, quite like $\mathscr{I}_0$, is defined to complete the square on the upper face of the cube, and admits natural map to $\mathscr{I}_0$ so that each of the side faces are also naturally commutative squares. Whereas the 'adding a point' map on the lower face of the cube adds points to line bundles, on the upper level they effectively add 'basepoints' to global sections; in other words

$$\pi_0 : \mathscr{T}_0 \to \mathscr{P}_{\mathrm{Pic}^n C}(\oplus_i \nu_* P(n)^{\otimes \lambda_i}, \vec{\lambda})$$

simply captures the notion of adding a basepoint.

We define spaces $\mathscr{T}_p$ for all $p \geq 0$ likewise. Consider the following commutative diagram:

$$\begin{tikzcd}
\mathscr{T}_p \arrow[rr] \arrow[dr, dashed, "\pi_p"] \arrow[dd] & & C^{p+1} \times \mathscr{P}_{\Pi_i \mathrm{Pic}^{\lambda_i n-(p+1)} C}(\oplus_i \nu_* P(\lambda_i n-(p+1), \vec{\lambda}) \arrow[dr, "f_1"] \arrow[dd] & \\
& \mathscr{P}_{\mathrm{Pic}^n C}(\oplus_i \nu_* P(n)^{\otimes \lambda_i}, \vec{\lambda}) \arrow[rr] \arrow[dd] & & \mathscr{P}_{\Pi_i \mathrm{Pic}^{\lambda_i n} C}(\oplus_i \nu_* P(\lambda_i n), \vec{\lambda}) \arrow[dd, "h"] \\
\mathscr{I}_p \arrow[rr] \arrow[dr, dashed, "A"] & & C^{p+1} \times \Pi_i \mathrm{Pic}^{\lambda_i n-(p+1)} C \arrow[dr, "A"] & \\
& \mathrm{Pic}^n C \arrow[rr, "\otimes^{\vec{\lambda}}"] & & \Pi_i \mathrm{Pic}^{\lambda_i n} C
\end{tikzcd}$$

where $\otimes^{\vec{\lambda}}$ is defined like before; A, by abusing notation, now denotes the addition of $(p+1)$ points to a line bundle i.e.

$$\begin{aligned}
A : C^{p+1} \times \Pi_i \mathrm{Pic}^{\lambda_i n-(p+1)} C &\to \Pi_i \mathrm{Pic}^{\lambda_i n} C \\
(x_0, \dots, x_p), (L_0, \dots, L_N) &\mapsto L_0(\sum x_i), \cdots, L_N(\sum x_i);
\end{aligned} \tag{23}$$

and moreover observe that we have commutative cubes of the following form for all p:

$$\begin{tikzcd}
\mathscr{T}_p \arrow[rr] \arrow[dr, "f_i"] \arrow[dd] & & C^{p+1} \times \mathscr{P}_{\Pi_i \mathrm{Pic}^{\lambda_i n-(p+1)} C}(\oplus_i \nu_* P(\lambda_i n-(p+1), \vec{\lambda}) \arrow[dr] \arrow[dd] & \\
& \mathscr{T}_{p-1} \arrow[rr] \arrow[dd] & & C^p \times \mathscr{P}_{\Pi_i \mathrm{Pic}^{\lambda_i n-p} C}(\oplus_i \nu_* P(\lambda_i n-p, \vec{\lambda}) \arrow[dd] \\
\mathscr{I}_p \arrow[rr] \arrow[dr, dashed, "A"] & & C^{p+1} \times \Pi_i \mathrm{Pic}^{\lambda_i n-(p+1)} C \arrow[dr, "A"] & \\
& \mathscr{I}_{p-1} \arrow[rr, "\otimes^{\vec{\lambda}}"] & & C^p \times \Pi_i \mathrm{Pic}^{\lambda_i n-p} C
\end{tikzcd}$$

and the all the down-right arrows are maps corresponding to adding a point from the i^{th} factor of C^{p+1} i.e.

$$\begin{gathered}
A : C^{p+1} \times \Pi_i \mathrm{Pic}^{\lambda_i n-(p+1)} C \to C^p \times \Pi_i \mathrm{Pic}^{\lambda_i n-p} C \\
(x_0, \dots, x_p), (L_0, \dots, L_N) \mapsto \\
(x_0, \dots, \widehat{x_i}, \dots, x_p), (L_0(x_i), \dots, L_N(x_i))
\end{gathered}$$

where $\widehat{x_i}$ implies we delete the i^{th} entry; most importantly, we obtain $f_i : \mathscr{T}_p \to \mathscr{T}_{p-1}$ as the i^{th} face map of the semisimplicial space $\mathscr{T}_\bullet$.

It is easy to see that $\mathscr{I}_p \cong C^{p+1} \times \mathrm{Pic}^n C$ for all $p \geq 0$: for $p = 0$ the map

$$\mathrm{Pic}^n C \times_i \Pi_i \mathrm{Pic}^{\lambda_i n} C.C \times \Pi_i \mathrm{Pic}^{\lambda_i n - 1} C \to C \times \mathrm{Pic}^n C$$
$$L, (x, (L_0, \cdots, L_N)) \mapsto (x, L)$$

has a natural inverse thereby giving an isomorphism; for higher values of p the observation follows likewise. Along those lines, it is not hard to show for $p \leq n - 2g + 1$ that $\mathscr{T}_p \cong C^{p+1} \times \mathscr{P}_{\mathrm{Pic}^n C}(\oplus_i \mathscr{E}_i, \lambda_i)$ (the isomorphism is not canonical) for some sub-vector budnles $\mathscr{E}_i \subset \nu_* P(n)^{\otimes \lambda_i}$ for each i, of rank $\lambda_i n - g + 1 - (p+1)$, in the following way. For any line bundle L on C and any effective divisor D on C there is a natural injection

$$H^0(C, L) \to H^0(C, L(D))$$
$$s \mapsto s(D)$$

coming from the short exact sequence of the corresponding locally free sheaves

$$0 \to L \to L(D) \to \mathscr{O}_D \to 0;$$

by definition we have

$$\mathscr{T}_p := \{[s_0 : \ldots : s_N], ((x_0, \cdots, x_p), [\widetilde{s_0} : \ldots : \widetilde{s_N}]) : s_i \in H^0(C, L^{\lambda_i}) \text{ where } L \in \mathrm{Pic}^n C,$$
$$\widetilde{s_i} \in H^0(C, L_i) \text{ where } L_i \in \mathrm{Pic}^{\lambda_i n - (p+1)} C,$$
$$\widetilde{s_i}(\sum x_j) = t^{\lambda_i} s_i \text{ for some } t \in \mathbb{G}_m \text{ and for all } i\}.$$

Now fix a general unordered $(p+1)$ subset $\{z_0, \cdots, z_p\} \in C$ and define

$$M_p := \{[\widetilde{s_0} : \ldots : \widetilde{s_N}] : \widetilde{s_i} \in H^0(C, L_i) \text{ where } L_i \in \mathrm{Pic}^{\lambda_i n - (p+1)} C, \text{ there exists } t \in \mathbb{G}_m$$
$$\text{such that for each } i, \ \widetilde{s_i}(\sum_j c_j) = t^{\lambda_i} s_i,$$
$$\text{where } s_i \in H^0(C, L^{\lambda_i}), \quad L \in \mathrm{Pic}^n C\}$$

Then one can define a bijective (easy to check) morphism that only depends on the choice of the points $z_0, \ldots, z_p \in C$:

$$C^{p+1} \times M_p \to \mathscr{T}_p$$
$$(x_0, \ldots, x_p), [\widetilde{s_0} : \ldots : \widetilde{s_N}] \mapsto$$
$$[\widetilde{s_0}(\sum_j x_j) : \ldots : \widetilde{s_N}(\sum_j x_j)], ((x_0, \cdots, x_p), [\widetilde{s_0} : \ldots : \widetilde{s_N}]) \tag{24}$$

where the only non-trivial part is to check that the map is well-defined and that fact follows from the observation that degree 0 line bundles always have λ^{th} roots for all positive integers λ. Now M_p is naturally a fibre bundle over $\mathrm{Pic}^n C$, in fact it is a weighted projectivization of the sub-vector bundle of $\oplus \nu_* P(n)^{\otimes \lambda_i}$ whose fibres are given by $(N+1)$-tuples of sections in $\oplus \nu_* P(n)^{\otimes \lambda_i}$ having common zeroes at $z_0, \ldots, z_p$; we denote that sub-vector bundle by $\oplus \nu_* P(n)^{\otimes \lambda_i}_{z_0 + \cdots + z_p}$. In other words

$$M_p \cong \mathscr{P}_{\mathrm{Pic}^n C}(\oplus_i \nu_* P(n)^{\otimes \lambda_i}_{z_0+\cdots+z_p}, \vec{\lambda}),$$

and for all $0 \leq p \leq n-2g+1$ we abuse notation and denote the fibre bundle by ρ i.e.

$$\rho : M_p \to \mathrm{Pic}^n C.$$

Deviating from the norm, we set $\mathscr{T}_{-1} := \widetilde{\mathrm{Hom}}_n(C, \mathscr{P}(\vec{\lambda})) \cong \mathscr{P}_{\mathrm{Pic}^n C}(\oplus_i \nu_* P(n)^{\otimes \lambda_i}, \vec{\lambda}))$ (whereas the norm would be to set $\mathscr{T}_{-1}$ as $\mathscr{X}$- this a minor deviation just to simplify the notations involved in the subsequent computation). From [Banerjee, Definition 2.1], $\mathscr{T}_\bullet$ satisfies the conditions of being a ΔS object. We therefore use [Banerjee, Theorem 1.2] to get a second quadrant spectral sequence that reads as:

$$E_1^{-p,q} = H^{q-2pN}\big(\mathscr{T}_p \otimes sgn_{p+1}\big)^{S_{(p+1)}}(-(p+1)N) \implies H^{q+p}(\mathrm{Hom}_n(C, \mathscr{P}(\vec{\lambda})).$$

where sgn_{p+1} denote the sign action of the symmetric group S_{p+1} on the $(p+1)$ factors of T_p by permutation; and the differentials of this spectral sequence are given by the alternating sum of the Gysin pushforwards induced by the face maps. So we split the rest of the proof into the following parts:

1. *Computing the E_1 terms.* To this end note that a complete understanding of

   $$H^*(\mathscr{P}_{\mathrm{Pic}^n C}(\oplus_i \nu_* P(n)^{\otimes \lambda_i}_{z_0+\cdots+z_p}, \vec{\lambda}))$$

   (at least in a stable range) gives us full knowledge of the E_1 terms (in that range). The Chern classes of $\nu_* P(n)^{\otimes \lambda_i}$ can be computed for example, directly using Grothendieck-Riemann-Roch, or via ad-hoc methods:

   $$c_i(\mathscr{E}_{\lambda_i}) = (-1)^i \frac{\theta^i}{i!} \;\; i = 0, \ldots, g$$

   where θ is the fundamental class of the theta divisor (several proofs are available in [ACGH, Sections 4, 5, Chapter VII and Section 1, Chapter VIII]). Using the Whitney sum formula we can express the twisted Chern classes in terms of θ.

   In turn, let $N_0 := (n-g+1)(N+1)$, which is the dimension of the fibres of $\mathscr{E} \to \mathrm{Pic}^n(C)$, and let h denote the relative hyperplane class i.e. $h = c_1(\mathscr{O}_\rho(1)) \in H^2(\mathscr{P}_{\mathrm{Pic}^n C}(\oplus_i \nu_* P(n)^{\otimes \lambda_i}, \vec{\lambda})$, then $H^*(\mathscr{P}_{\mathrm{Pic}^n C}(\oplus_i \nu_* P(n)^{\otimes \lambda_i}, \vec{\lambda})$, which is an algebra on $H^*(\mathrm{Pic}^n(C)) \cong \wedge(H^1(C))$, is given by (using (16)):

   $$\begin{gathered} H^*(\mathscr{P}_{\mathrm{Pic}^n C}(\oplus_i \nu_* P(n)^{\otimes \lambda_i}, \vec{\lambda}) \\ \cong \\ \frac{H^*(\mathrm{Pic}^n(C))[h]}{h^{N_0} + \rho^* c_1^{\vec{\eta}}(\oplus_i \nu_* P(n)^{\otimes \lambda_i}, \vec{\lambda}) h^{N_0-1} + \ldots + \rho^* c_g^{\vec{\eta}}(\oplus_i \nu_* P(n)^{\otimes \lambda_i}, \vec{\lambda}) h^{N_0-g}}. \end{gathered} \tag{25}$$

Let p be such that $n-p \geq 2g$ and let $N_p := (n-p-g+1)(N+1) = N_0 - p(N+1)$, the dimension of the fibres of $\mathscr{P}_{\mathrm{Pic}^n C}(\oplus_i \nu_* P(n)^{\otimes \lambda_i}_{z_0+\cdots+z_p}, \vec{\lambda}) \to \mathrm{Pic}^n(C)$, then combining (16) and (1) we have a complete description of the E_1 terms of the spectral sequence above. We remark here that since $n-p \geq 2g$, we have that $N_p - g = (n-p-g+1)(N+1) - g \geq N$. This remark will be useful later.

2. *Computing the differentials* $d_1^p : E_1^{-p,*} \to E_1^{-(p-1),*+2N}$.

   Following previously introduced notations, let $h = c_1(\mathscr{O}_{\rho_n}(1))$, and for all p satisfying $n-p \geq 2g$, let

   $$\iota : \mathscr{P}_{\mathrm{Pic}^n C}(\oplus_i \nu_* P(n)^{\otimes \lambda_i}_{z_0+\cdots+z_{p-1}}, \vec{\lambda}) \to \mathscr{P}_{\mathrm{Pic}^n C}(\oplus_i \nu_* P(n)^{\otimes \lambda_i}_{z_0+\cdots+z_p}, \vec{\lambda})$$

   denote the closed embedding induced by adding a basepoint x (an abuse of notation that won't cause any confusion down the way). Note that ι is fibrewise linear embedding, up to translation of $\mathrm{Pic}^n(C)$ by x. Finally, let $e \in H^2(C)$ be the class of a point, $\mathbb{1}$ the fundamental class of C, and let $\gamma_1, \ldots, \gamma_{2g}$ be the standard basis of $H^1(C)$ and because $H^*(\mathrm{Pic}^n(C)) \cong \wedge H^1(C)$, let $\overline{\gamma_1}, \ldots, \overline{\gamma_{2g}}$ be the image of $\gamma_1, \ldots, \gamma_{2g}$ under the aforementioned isomorphism.

   First, we observe that

   $$\begin{aligned} d_1^1 : H^*(C \times \mathscr{P}_{\mathrm{Pic}^n C}(\oplus_i \nu_* P(n)^{\otimes \lambda_i}_{z_0}, \vec{\lambda})) &\to H^*(\mathscr{P}_{\mathrm{Pic}^n C}(\oplus_i \nu_* P(n)^{\otimes \lambda_i}, \vec{\lambda})) \\ [C \times \mathscr{P}_{\mathrm{Pic}^n C}(\oplus_i \nu_* P(n)^{\otimes \lambda_i}_{z_0}, \vec{\lambda})] &\mapsto h^N \\ e &\mapsto h^{N+1} \\ \gamma_i &\mapsto \overline{\gamma_i} h^N, \text{ for all } i. \end{aligned}$$

   is a map of $H^*(\mathscr{P}_{\mathrm{Pic}^n C}(\oplus_i \nu_* P(n)^{\otimes \lambda_i}, \vec{\lambda}))$-modules, and in turn

   $$\iota^* \alpha + e \iota^* \beta + \sum_{i=1}^{2g} \gamma_i \iota^* \gamma_i \overset{d_1^1}{\mapsto} \alpha h^N + \beta h^{N+1} + \sum_{i=1}^{2g} \overline{\gamma_i} \gamma_i h^N,$$

   where $\alpha, \beta, \gamma_1, \ldots, \gamma_{2g} \in H^*(\mathscr{P}_{\mathrm{Pic}^n C}(\oplus_i \nu_* P(n)^{\otimes \lambda_i}, \vec{\lambda}))$. Indeed, the justification for the formula for d_1^1 in the previous case of $C = \mathbb{P}^1$ holds almost verbatim here. We know

   $$\iota^* : H^*(\mathscr{P}_{\mathrm{Pic}^n C}(\oplus_i \nu_* P(n)^{\otimes \lambda_i}, \vec{\lambda})) \to H^*(\mathscr{P}_{\mathrm{Pic}^n C}(\oplus_i \nu_* P(n)^{\otimes \lambda_i}_{z_0}, \vec{\lambda}))$$

   is a surjection; next, for a fixed point $x \in C$, the image $t_x^r(\mathscr{P}_{\mathrm{Pic}^n C}(\oplus_i \nu_* P(n)^{\otimes \lambda_i}_{z_0}, \vec{\lambda}))$ is rationally equivalent, and in turn cohomologous, to (a multiple of) h^{N+1}, and finally, that the image of the fundamental class $[C \times \mathscr{P}_{\mathrm{Pic}^n C}(\oplus_i \nu_* P(n)^{\otimes \lambda_i}_{z_0}, \vec{\lambda})] \in H^0(C \times \mathscr{P}_{\mathrm{Pic}^n C}(\oplus_i \nu_* P(n)^{\otimes \lambda_i}_{z_0}, \vec{\lambda}))$ is rationally equivalent, and thus cohomologous, to (a multiple of) h^N, can be seen as in the following way. Recall that a Poincaré bundle $P(n)$ is ν-relatively very ample for all $n \geq 2g-1$, which in turn induces a relative embedding of $C \times \mathrm{Pic}^n(C) \xrightarrow{i_n} \mathbb{P}(\nu_* P(n))$ over $\mathrm{Pic}^n(C)$ and we have a natural sequence of maps over $\mathrm{Pic}^n C$

$$\begin{tikzcd} C \times \mathrm{Pic}^n C \arrow[r, hook] \arrow[dr] & \mathbb{P}(\nu_* P(n)) \arrow[r] \arrow[d] & \mathscr{P}_{\mathrm{Pic}^n C}(\oplus_i \nu_* P(n)^{\lambda_i}, \vec{\lambda}) \arrow[dl] \\ & \mathrm{Pic}^n C & \end{tikzcd}$$

and we continue to denote the composition mapping $C \times \mathrm{Pic}^n C$ to $\mathscr{P}_{\mathrm{Pic}^n C}(\oplus_i \nu_* P(n)^{\lambda_i}, \vec{\lambda})$ over $\mathrm{Pic}^n C$ by i_n. This makes $i_n(C \times \mathrm{Pic}^n(C))$ in $\mathscr{P}_{\mathrm{Pic}^n C}(\oplus_i \nu_* P(n)^{\otimes \lambda_i}, \vec{\lambda})$ homologous to (a scalar multiple of) the (relative, over the base $\mathrm{Pic}^n C$) Poincaré dual of $h \in H^2(\mathscr{P}_{\mathrm{Pic}^n C}(\oplus_i \nu_* P(n)^{\otimes \lambda_i}, \vec{\lambda}))$. In turn, the image of the $[C \times \mathscr{P}_{\mathrm{Pic}^n C}(\oplus_i \nu_* P(n)^{\otimes \lambda_i}_{z_0}, \vec{\lambda})]$ under the Gysin map f_{0*} is given by

$$\begin{aligned} f_{0*}\big([C \times \mathscr{P}_{\mathrm{Pic}^n C}(\oplus_i \nu_* P(n)^{\otimes \lambda_i}_{z_0}, \vec{\lambda})]\big) &= h^{N+1} \frown i_n(C \times \mathrm{Pic}^n(C)) \\ &= h^N. \end{aligned}$$

Yet again, for the sake of simplicity we won't bother ourselves with the scalar multiples, which is fine because we're working over $\mathbb{Q}$. Noting that

$$\overline{\gamma_i}(e-h) + h(\gamma_i - \overline{\gamma_i}) = \gamma_i h - e\overline{\gamma_i},$$

it is now easy to check that the kernel of d_1^1 is given by:

$$\begin{aligned} & H^*(\mathscr{P}_{\mathrm{Pic}^n C}(\oplus_i \nu_* P(n)^{\otimes \lambda_i}_{z_0}, \vec{\lambda}))(e-h)[2N] \\ \bigoplus & H^*(\mathscr{P}_{\mathrm{Pic}^n C}(\oplus_i \nu_* P(n)^{\otimes \lambda_i}_{z_0}, \vec{\lambda}))(\gamma_i - \overline{\gamma_i})[2N], \qquad (i = 1, \ldots, 2g) \end{aligned}$$

where $[2N]$ denotes a shift in the cohomological degree by $2N$, and which is viewed as a $\iota^* H^*(\mathscr{P}_{\mathrm{Pic}^n C}(\oplus_i \nu_* P(n)^{\otimes \lambda_i}, \vec{\lambda})) \cong H^*(\mathscr{P}_{\mathrm{Pic}^n C}(\oplus_i \nu_* P(n)^{\otimes \lambda_i}_{z_0}, \vec{\lambda}))$-module. The cokernel of d_1^1, which forms $E_2^{0,*}$ is given by

$$\frac{H^*(\mathrm{Pic}^n(C))[h]}{h^N}$$

(where note that as remarked before $r < N_0 - g$, see (1)).

Now we work out the differential for $p = 2$ by computing the Gysin push-fowards by each of the face maps:

$$\begin{gathered} f_{0*}(\mathbb{1} \otimes e) = eh^N, \;\; f_{1*}(\mathbb{1} \otimes e) = h^{N+1} \implies d_1^2(\mathbb{1} \otimes e) = (e-h)h^N, \\ f_{0*}(e \otimes \gamma_i) = \gamma_i h^{N+1}, \;\; f_{1*}(e \otimes \gamma_i) = e\overline{\gamma_i} h^N \implies d_1^2(e \otimes \gamma_i) = (\gamma_i h - e\overline{\gamma_i})h^N, \\ f_{0*}(\mathbb{1} \otimes \gamma_i) = \gamma_i h^N, \;\; f_{1*}(\mathbb{1} \otimes \gamma_i) = \overline{\gamma_i} h^N \implies d_1^2(\mathbb{1} \otimes \gamma_i) = (\gamma_i - \overline{\gamma_i})h^N, \\ d_1^2(\gamma_i \gamma_j) = 0, \end{gathered}$$

where the last equality follows form the fact that on $\mathrm{Sym}^p H^1(C)$ for $p \geq 2$, the alternating sum of face maps is, by definition, 0. Recalling our earlier remark that $N < N_1 - g$, we see that the $E_2^{-1,*}$ terms, as an $H^*(\mathscr{P}_{\mathrm{Pic}^n C}(\oplus_i \nu_* P(n)^{\otimes \lambda_i}_{z_0+z_1}, \vec{\lambda}))$-module, are given by:

$$\frac{H^*(\mathrm{Pic}^n(C);\mathbb{Q}(-N))[h]}{h^N}(e-h)[2N]$$
$$\bigoplus_{1\leq i\leq 2g} \frac{H^*(\mathrm{Pic}^n(C);\mathbb{Q}(-r))[h]}{h^N}(\gamma_i - \overline{\gamma_i})[2N].$$

Whereas the kernel of d_1^2 is generated by exactly what one expects: as a $H^*(\mathscr{P}_{\mathrm{Pic}^n C}(\oplus_i \nu_* P(n)^{\otimes \lambda_i}_{z_0+z_1}, \vec{\lambda}))$-module, we have

$$\mathrm{Ker}(d_1^2) = \bigoplus_{1\leq i\leq 2g} H^*(\mathscr{P}_{\mathrm{Pic}^n C}(\oplus_i \nu_* P(n)^{\otimes \lambda_i}_{z_0+z_1}, \vec{\lambda}))\big(e\otimes\gamma_i - 1\otimes\gamma_i h + \mathbb{1}\otimes e\overline{\gamma_i}\big)[4N]$$
$$\bigoplus_{1\leq i,j\leq 2g} H^*(\mathscr{P}_{\mathrm{Pic}^n C}(\oplus_i \nu_* P(n)^{\otimes \lambda_i}_{z_0+z_1}, \vec{\lambda}))(\gamma_i\gamma_j)[4N].$$

For $p = 3$ we have $d_1^3 : E_1^{-3,*} \to E_1^{-2,*}$ given by:

$$d_1^3(\mathbb{1}\otimes e\otimes\gamma_i) = e\otimes\gamma_i h^N - \mathbb{1}\otimes\gamma_i h^{N+1} + \mathbb{1}\otimes e\overline{\gamma_i} h^N \Longleftarrow \begin{cases} f_{0*}(\mathbb{1}\otimes e\otimes\gamma_i) = e\otimes\gamma_i h^N, \\ f_{1*}(\mathbb{1}\otimes e\otimes\gamma_i) = \mathbb{1}\otimes\gamma_i h^{N+1} \\ f_{2*}(\mathbb{1}\otimes e\otimes\gamma_i) = \mathbb{1}\otimes e\overline{\gamma_i} h^N \end{cases}$$
$$d_1^3(e\otimes\gamma_i\gamma_j) = \gamma_i\gamma_j h^{N+1},$$
$$d_1^3(\mathbb{1}\otimes\gamma_i\gamma_j) = \gamma_i\gamma_j h^N,$$
$$d_1^3(\gamma_i\gamma_j\gamma_k) = 0,$$

where, for the last three equalities, recall again that on $\mathrm{Sym}^p H^1(C)$ for $p \geq 2$, the alternating sum of face maps is, by definition, 0. Therefore the $E_1^{-2,*}$ terms defined by $\mathrm{Ker}(d_1^2)/\mathrm{Coker}(d_1^3)$ is given by:

$$\bigoplus_{1\leq i\leq 2g} \frac{H^*(\mathrm{Pic}^n(C);\mathbb{Q}(-2N))[h]}{h^N}\big(e\otimes\gamma_i - 1\otimes\gamma_i h + \mathbb{1}\otimes e\overline{\gamma_i}\big)[4N]$$
$$\bigoplus_{1\leq i,j\leq 2g} \frac{H^*(\mathrm{Pic}^n(C);\mathbb{Q}(-2N))[h]}{h^N}(\gamma_i\gamma_j)[4N].$$

The formula for the differentials in the case of $p \geq 3$ mimics that of $p = 3$, and we have:

$$\mathbb{1}\otimes e\otimes c_1^{\vec{\eta}}\dots c_{p-2}^{\vec{\eta}} \mapsto \Big((e\otimes c_1^{\vec{\eta}}\dots c_{p-2}^{\vec{\eta}}) - (\mathbb{1}\otimes c_1\dots c_{p-2})h\Big)h^N,$$
$$e\otimes c_1^{\vec{\eta}}\dots c_{p-1}^{\vec{\eta}} \mapsto c_1^{\vec{\eta}}\dots c_{p-1}^{\vec{\eta}} h^{N+1},$$
$$\mathbb{1}\otimes c_1^{\vec{\eta}}\dots c_{p-1}^{\vec{\eta}} \mapsto c_1^{\vec{\eta}}\dots c_{p-1}^{\vec{\eta}} h^r$$
$$c_1^{\vec{\eta}}\dots c_p^{\vec{\eta}} \mapsto 0$$

It is now easy to check that

$$\mathrm{Ker}(d_1^p)/\mathrm{Coker}(d_1^{p+1}) =$$
$$\bigoplus_{1\le i\le 2g} \frac{H^*(\mathrm{Pic}^n(C);\mathbb{Q}(-pN))[h]}{h^N}\big(e\otimes c_1^{\vec{\eta}}\dots c_{p-1}^{\vec{\eta}} - \mathbb{1}\otimes c_1^{\vec{\eta}}\dots c_{p-1}^{\vec{\eta}}\big)[2pN]$$
$$\bigoplus_{1\le i,j\le 2g} \frac{H^*(\mathrm{Pic}^n(C);\overline{\mathbb{Q}}_\ell(-pN))[h]}{h^N}(c_1^{\vec{\eta}}\dots c_p^{\vec{\eta}})[2pN].$$

Now we are left with analysing the resulting E_2 page. That the differentials on the E_2 page vanish for $p\le n-2g$ follow simply from weight considerations- the space $\mathscr{T}_\bullet$ consists of smooth projective varieties and thus their n^{th} cohomology is pure of weight n. Now observe the following: we have an equality

$$\mathrm{R}\Gamma_c(\mathscr{P}_{\mathrm{Pic}^nC}(\oplus_i \nu_*P(n)^{\otimes\lambda_i},\vec{\lambda}), C^\bullet(\underline{\mathbb{Q}_\ell}_{\mathscr{P}_{\mathrm{Pic}^nC}(\oplus_i \nu_*P(n)^{\otimes\lambda_i},\vec{\lambda})})) =$$
$$\mathrm{R}\Gamma_c(\mathscr{P}_{\mathrm{Pic}^nC}(\oplus_i \nu_*P(n)^{\otimes\lambda_i},\vec{\lambda}), j_!\underline{\mathbb{Q}_\ell}_{\mathrm{Hom}_n(C,\mathscr{P}(\vec{\lambda}))})$$

in the derived category of constructible sheaves over $\mathscr{P}_{\mathrm{Pic}^nC}(\oplus_i \nu_*P(n)^{\otimes\lambda_i},\vec{\lambda})$ where $C^\bullet(\mathbb{Q}_{\mathscr{P}_{\mathrm{Pic}^nC}(\oplus_i \nu_*P(n)^{\otimes\lambda_i},\vec{\lambda})})$ denotes the complex

$$\begin{aligned}0\to j_!j^*\underline{\mathbb{Q}_\ell}_{\mathscr{T}_{-1}} \to \underline{\mathbb{Q}_\ell}_{\mathscr{T}_{-1}} \to \pi_{0_*}\pi_0^*\underline{\mathbb{Q}_\ell}_{\mathscr{T}_{-1}} \to (\pi_{1_*}\pi_1^*\underline{\mathbb{Q}_\ell}_{\mathscr{T}_{-1}}\otimes sgn_2)^{S_2}\dots\to\\ \dots\to(\pi_{p_*}\pi_p^*\underline{\mathbb{Q}_\ell}_{\mathscr{T}_{-1}}\otimes sgn_{p+1})^{S_{p+1}}\to\cdots\end{aligned} \tag{26}$$

on the other hand, for any $m\in\mathbb{N}$ we have

$$\mathrm{R}^i\Gamma_c(\mathscr{P}_{\mathrm{Pic}^nC}(\oplus_i \nu_*P(n)^{\otimes\lambda_i},\vec{\lambda}), C^\bullet(\underline{\mathbb{Q}_\ell}_{\mathscr{P}_{\mathrm{Pic}^nC}(\oplus_i \nu_*P(n)^{\otimes\lambda_i},\vec{\lambda})})) \cong$$
$$\mathrm{R}^i\Gamma_c(\mathscr{P}_{\mathrm{Pic}^nC}(\oplus_i \nu_*P(n)^{\otimes\lambda_i},\vec{\lambda}), C^\bullet(\underline{\mathbb{Q}_\ell}_{\mathscr{P}_{\mathrm{Pic}^nC}(\oplus_i \nu_*P(n)^{\otimes\lambda_i},\vec{\lambda})})/\tau_{\ge m}C^\bullet(\underline{\mathbb{Q}_\ell}_{\mathscr{P}_{\mathrm{Pic}^nC}(\oplus_i \nu_*P(n)^{\otimes\lambda_i},\vec{\lambda})})$$

for all $i\ge 2(m+1)-2N$, where $\tau_{\ge m}C^\bullet(\mathbb{Q}_{\mathscr{P}_{\mathrm{Pic}^nC}(\oplus_i \nu_*P(n)^{\otimes\lambda_i},\vec{\lambda})})$ denotes the truncated complex up to the $(N-1)$ term and this is because $\tau_{\ge m}C^\bullet(\mathbb{Q}_{\mathscr{P}_{\mathrm{Pic}^nC}(\oplus_i \nu_*P(n)^{\otimes\lambda_i},\vec{\lambda})})$ is supported on complex codimension m in $\mathscr{P}_{\mathrm{Pic}^nC}(\oplus_i \nu_*P(n)^{\otimes\lambda_i},\vec{\lambda})$. Therefore the cohomology of $\mathrm{Hom}_n(C,\mathscr{P}(\vec{\lambda}))$ up to degree $n-2g$ is solely dictated by the E_2 page.

To this end, let

$$t := (e-h)$$

which has degree $(-1, 2N+2)$ and let

$$\alpha_i := \gamma_i - \overline{\gamma_i},\quad i = 1,\dots,2g$$

which has degree $(-1,2N+1)$. Clearly for $3\le p\le n-2g$, the element $t\alpha_{i_1}\dots\alpha_{i_p}$, which is of degree $(-(p+1), 2N+2+p(2N+1))$, when expanded, gives us

$$
\begin{aligned}
&t\alpha_{i_1}\dots\alpha_{i_p} \\
&\qquad = (e-h)(c_{i_1}-\overline{c_{i_1}})\dots(c_{i_p}-\overline{c_{i_p}}) \\
&\qquad = (e-h)\prod_{j=1}^{p} c_{i_j} + \left\{\text{lower order terms as a polynomial on } c_{i_1},\dots,c_{i_p}\right\} \\
&\qquad = (e-h)\prod_{j=1}^{p} c_{i_j}
\end{aligned}
$$

because the lower order terms are all 0 in

$$\left(H^2(C)\oplus H^0(C)\right)\bigotimes \mathrm{Sym}^p H^1(C)\otimes H^*(\mathrm{Pic}^{n-(p+1)}(C))[h]/h^N,$$

thanks to the alternating action of S_{p+1}. Whereas $\alpha_{i_1}\dots\alpha_{i_{p+1}}$, which is of degree $(-(p+1),(p+1)(2N+1))$, when expanded, gives us

$$
\begin{aligned}
&\alpha_{i_1}\dots\alpha_{i_{p+1}} \\
&\qquad = (c_{i_1}-\overline{c_{i_1}})\dots(c_{i_{p+1}}-\overline{c_{i_{p+1}}}) \\
&\qquad = \prod_{j=1}^{p+1} c_{i_j} + \left\{\text{lower order terms as a polynomial on } c_{i_1},\dots,c_{i_{p+1}}\right\} \\
&\qquad = \prod_{j=1}^{p+1} c_{i_j}
\end{aligned}
$$

because again, the lower order terms are all 0 for the exact same reason cited above.

Now as for $p=2$, we have

$$
\begin{aligned}
t\alpha_i = (e-h)(\gamma_i-\overline{\gamma_i}) &= e\gamma_i - \gamma_i h + e\overline{\gamma_i} + h\overline{\gamma_i} \\
&= e\gamma_i - \gamma_i h + e\overline{\gamma_i}
\end{aligned}
$$

because the alternating action of S_2 kills $H^0(C^2)\otimes H^*(\mathscr{P}_{\mathrm{Pic}^n C}(\oplus_i \nu_* P(n)^{\otimes\lambda_i},\vec{\lambda}))$, and in turn, $h\overline{\gamma_i}$. This give us the algebra structure on the E_2 page for $p\le n-2g$ and thus completes the proof of Theorem 1.1.

■

A Enumerating elliptic fibrations over $\mathbb{F}_q(t)$ with torsion/multiple markings

In this appendix, we determine the sharp enumerations on the number of elliptic curves over $\mathbb{P}^1_{\mathbb{F}_q}$ with prescribed level structures or multiple marked points by extending the method as in [Han-Park, Theorem 3] regarding the number of semistable elliptic curves over $\mathbb{P}^1_{\mathbb{F}_q}$.

Recall that a level structure $[\Gamma_1(n)]$ on an elliptic curve E is a choice of point $P \in E$ of exact order n in the smooth part of E such that over every geometric point of the base scheme every irreducible component of E contains a multiple of P (see [KM, §1.4]). And a level structure $[\Gamma(2)]$ on an elliptic curve E is a choice of isomorphism $\phi : \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to E(2)$ where $E(2)$ is the scheme of 2-torsion Weierstrass points (i.e., kernel of the multiplication-by-2 map $[2] : E \to E$) (see [DR, II.1.18 & IV.2.3]).

Additionally, we consider curves of arithmetic genus one over $\mathbb{F}_q(t)$ with m-marked rational points for $2 \le m \le 5$ by acquiring sharp enumeration on the number of $(m-1)$-stable m-marked curves of arithmetic genus one formulated originally by the works of [Smyth, Smyth2].

To enumerate the number of certain elliptic curves over global function fields $\mathbb{F}_q(t)$ with level structures $[\Gamma(n)]$ or $[\Gamma_1(n)]$, we need to first extend the notion of (nonsingular) elliptic curves (semistable in the case of [Han-Park]) that admits desired level structures. By the work of Deligne and Rapoport [DR] (summarized in [Niles, §2]), we consider the generalized elliptic curves over $\mathbb{P}^1_K$ with $[\Gamma]$–structures (where Γ is $\Gamma(n)$ or $\Gamma_1(n)$) over a field K (focusing on $K = \mathbb{F}_q$). Roughly, a generalized elliptic curve X over $\mathbb{P}^1_K$ can be thought of as a flat family of semistable elliptic curves admitting a group structure, such that a finite group scheme $\mathcal{G} \to \mathbb{P}^1_K$ (determined by Γ) embeds into X and its image meets every irreducible component of every geometric fibers of X. Again, we only consider the *non-isotrivial* generalized elliptic curves. If X is as above, then Δ is the discriminant of a generalized elliptic curve and if $K = \mathbb{F}_q$, then $0 < ht(\Delta) := q^{\deg \Delta}$.

Now, define $\mathcal{N}(\mathbb{F}_q(t), [\Gamma], 0 < B \le q^{12n})$

$:= |\{$Generalized elliptic curves over $\mathbb{P}^1_{\mathbb{F}_q}$ with $[\Gamma]$–structures and $0 < ht(\Delta) \le B\}|$

Then, we acquire the following descriptions of $\mathcal{N}(\mathbb{F}_q(t), [\Gamma], 0 < B \le q^{12n})$:

Theorem A.1. *The function $\mathcal{N}(\mathbb{F}_q(t), [\Gamma], 0 < B \le q^{12n})$, which counts the number of generalized elliptic curves with $[\Gamma]$-structures over $\mathbb{P}^1_{\mathbb{F}_q}$ with char$(\mathbb{F}_q) \neq 2$ $\big($char$(\mathbb{F}_q) \neq 3$ for $\mathcal{N}(\mathbb{F}_q(t), [\Gamma_1(3)], 0 < B \le q^{12n})$ ordered by $0 < ht(\Delta) = q^{12n} \le B$, satisfies:*

$$\mathcal{N}(\mathbb{F}_q(t), [\Gamma_1(2)],\ B \le q^{12n}) = 2 \cdot \frac{(q^7 - q^5)}{(q^6 - 1)} \cdot \left(B^{\frac{1}{2}} - 1\right)$$

$$\mathcal{N}(\mathbb{F}_q(t), [\Gamma_1(3)],\ B \le q^{12n}) = \frac{(q^5 - q^3)}{(q^4 - 1)} \cdot \left(B^{\frac{1}{3}} - 1\right)$$

$$\mathcal{N}(\mathbb{F}_q(t), [\Gamma_1(4)],\ B \le q^{12n}) = \frac{(q^4 - q^2)}{(q^3 - 1)} \cdot \left(B^{\frac{1}{4}} - 1\right)$$

$$\mathcal{N}(\mathbb{F}_q(t),[\Gamma(2)],\ B \le q^{12n}) = 2\cdot\frac{(q^5-q^3)}{(q^4-1)}\cdot\left(B^{\frac{1}{3}}-1\right)$$

$$\mathcal{N}(\mathbb{F}_q(t),[\Gamma_1(m)],\ B \le q^{12n}) = \frac{(q^3-q^1)}{(q^2-1)}\cdot\left(B^{\frac{1}{6}}-1\right)$$

m is for $m = 5,6,7,8,9,10$ or 12.

Proof. They are direct consequence of Proposition A.5 and Proposition A.7 proven at the end of §A (see [Han-Park, §5] for details). ■

The main leading term of the acquired sharp enumerations over $\mathbb{F}_q(t)$ matches the analogous asymptotic counts ordered by bounded naïve height of underlying elliptic curves over $\mathbb{Q}$ by Harron and Snowden in [HS, Theorem 1.2] (see also [Duke, Grant]).

Now, let's consider instead elliptic curves with m-marked rational points. To count the number of certain curves of arithmetic genus one over global function fields $\mathbb{F}_q(t)$ with m-markings, we need to again extend the notion of (nonsingular) elliptic curves that admits desired m-markings. Here, we consider the $(m-1)$-stable m-marked curves of arithmetic genus one (defined by Smyth in [Smyth, §1.1] for characteristic $\neq 2,3$, extended to lower characteristic with mild conditions by [LP, Definition 1.5.3]), see Definition A.9 for a precise definition. Note that if $\mathrm{char}(\mathbb{F}_q) > 3$ and $m = 1$, then 0-stable 1-marked curves are exactly stable elliptic curves as in [DM]. We now consider the following definition:

Definition A.2. Fix an integral reduced K-scheme B, where K is a field. Then a non-isotrivial flat morphism $\pi : X \to B$ is a m-marked $(m-1)$-stable genus one fibration over B if any fiber of π is a $(m-1)$-stable m-marked curves of arithmetic genus one.

Observe that if $\mathrm{char}(K) = 0$ or > 3, then a m-marked $(m-1)$-stable genus one fibration $X \to \mathbb{P}^1_K$ has a discriminant $\Delta \subset \mathbb{P}^1_K$, and if $K = \mathbb{F}_q$, then $0 < ht(\Delta) := q^{\deg \Delta}$.

Now, define $\mathcal{N}(\mathbb{F}_q(t), m, 0 < B \le q^{12n})$

$:= |\{m$-marked $(m-1)$-stable genus one fibrations over $\mathbb{P}^1_{\mathbb{F}_q}$ with $0 < ht(\Delta) \le B\}|$

Note that when $m = 1$, $\mathcal{N}(\mathbb{F}_q(t), 1, 0 < B \le q^{12n})$ counts the stable elliptic fibrations, which is described in [Han-Park, Theorem 3] as $\mathcal{Z}_{\mathbb{F}_q(t)}(B)$ (by identifying stable elliptic fibrations with nonsingular semistable elliptic surfaces, see [Han-Park, Proposition 11]). When $2 \le m \le 5$, we acquire the following sharp enumeration of $\mathcal{N}(\mathbb{F}_q(t), m, 0 < B \le q^{12n})$:

Theorem A.3. *If char*$(\mathbb{F}_q) \neq 2,3$*, then the function* $\mathcal{N}(\mathbb{F}_q(t), m, 0 < B \le q^{12n})$*, which counts the number of m-marked* $(m-1)$*-stable genus one fibration over* $\mathbb{P}^1_{\mathbb{F}_q}$ *for*

$2 \leq m \leq 5$ *ordered by* $0 < ht(\Delta) = q^{12n} \leq B$*, satisfies:*

$$\mathcal{N}(\mathbb{F}_q(t), 2,\ B \leq q^{12n}) = \frac{(q^{11} + q^{10} - q^8 - q^7)}{(q^9 - 1)} \cdot (B^{\frac{3}{4}} - 1) + \frac{(q^7 - q^5)}{(q^6 - 1)} \cdot (B^{\frac{1}{2}} - 1)$$

$$\mathcal{N}(\mathbb{F}_q(t), 3,\ B \leq q^{12n}) = \frac{(q^{11} + q^{10} + q^9 - q^7 - q^6 - q^5)}{(q^8 - 1)} \cdot (B^{\frac{2}{3}} - 1) + \frac{(q^5 - q^3)}{(q^4 - 1)} \cdot (B^{\frac{1}{3}} - 1)$$

$$\begin{aligned}\mathcal{N}(\mathbb{F}_q(t), 4,\ B \leq q^{12n}) &= \frac{(q^{11} + q^{10} + q^9 + q^8 - q^6 - q^5 - q^4 - q^3)}{(q^7 - 1)} \cdot (B^{\frac{7}{12}} - 1) \\ &\quad + \frac{(q^5 - q^3)}{(q^4 - 1)} \cdot (B^{\frac{1}{3}} - 1)\end{aligned}$$

$$\mathcal{N}(\mathbb{F}_q(t), 5,\ B \leq q^{12n}) = \frac{(q^{11} + q^{10} + q^9 + q^8 + q^7 - q^5 - q^4 - q^3 - q^2 - q^1)}{(q^6 - 1)} \cdot (B^{\frac{1}{2}} - 1)$$

Proof. They are direct consequence of Proposition A.13 and Proposition A.14 proven at the end of §A (see [Han-Park, §5] for details). ■

Arithmetic moduli of generalized elliptic fibrations over $\mathbb{P}^1$ with level structures

The essential geometrical idea in acquiring the sharp enumeration is to consider the moduli stack of rational curves on a compactified modular curve as in [Han-Park]. The key fact we recall is that the genus 0 modular curves $\overline{\mathscr{M}}_{1,1}[\Gamma]$ are isomorphic to the weighted projective stacks $\mathscr{P}(a, b)$.

Proposition A.4. *The moduli stack* $\overline{\mathscr{M}}_{1,1}[\Gamma]$ *of generalized elliptic curves with* $[\Gamma]$*-level structure is isomorphic to the following when over a field* K*:*

1. *if* $\mathrm{char}(K) \neq 2$*, the tame Deligne–Mumford moduli stack of generalized elliptic curves with* $[\Gamma_1(2)]$*-structures is isomorphic to*

$$(\overline{\mathscr{M}}_{1,1}[\Gamma_1(2)])_K \cong [(\mathrm{Spec}\ K[a_2, a_4] - (0,0))/\mathbb{G}_m] = \mathscr{P}_K(2,4),$$

2. *if* $\mathrm{char}(K) \neq 3$*, the tame Deligne–Mumford moduli stack of generalized elliptic curves with* $[\Gamma_1(3)]$*-structures is isomorphic to*

$$(\overline{\mathscr{M}}_{1,1}[\Gamma_1(3)])_K \cong [(\mathrm{Spec}\ K[a_1, a_3] - (0,0))/\mathbb{G}_m] = \mathscr{P}_K(1,3),$$

3. *if* $\mathrm{char}(K) \neq 2$*, the tame Deligne–Mumford moduli stack of generalized elliptic curves with* $[\Gamma_1(4)]$*-structures is isomorphic to*

$$(\overline{\mathscr{M}}_{1,1}[\Gamma_1(4)])_K \cong [(\mathrm{Spec}\ K[a_1, a_2] - (0,0))/\mathbb{G}_m] = \mathscr{P}_K(1,2),$$

4. *if* char$(K) \neq m$*, the tame Deligne–Mumford moduli stack of generalized elliptic curves with* $[\Gamma_1(m)]$*-structures for* $m = 5, 6, 7, 8, 9, 10$ or 12 *is isomorphic to*

$$(\overline{\mathscr{M}}_{1,1}[\Gamma_1(m)])_K \cong [(\mathrm{Spec}\, K[a_1, a_1] - (0,0))/\mathbb{G}_m] = \mathscr{P}_K(1,1) \cong \mathbb{P}^1,$$

5. *if* char$(K) \neq 2$*, the tame Deligne–Mumford moduli stack of generalized elliptic curves with* $[\Gamma(2)]$*-structures is isomorphic to*

$$(\overline{\mathscr{M}}_{1,1}[\Gamma(2)])_K \cong [(\mathrm{Spec}\, K[a_2, a_2] - (0,0))/\mathbb{G}_m] = \mathscr{P}_K(2,2),$$

where $\lambda \cdot a_i = \lambda^i a_i$ *for* $\lambda \in \mathbb{G}_m$ *and* $i = 1, 2, 3, 4$*. Thus, the* a_i*'s have degree* i *respectively. Moreover, the discriminant divisors of* $(\overline{\mathscr{M}}_{1,1}[\Gamma])_K \cong \mathscr{P}_K(i,j)$ *as above have degree 12.*

Proof. The moduli stack $\overline{\mathscr{M}}_{1,1}[\Gamma_1(2)]$ of generalized elliptic curves with $[\Gamma_1(2)]$-level structure has an isomorphism $\overline{\mathscr{M}}_{1,1}[\Gamma_1(2)] \cong \mathscr{P}(2,4)$ over Spec$(\mathbb{Z}[1/2])$ as in [Behrens, §1.3] through the universal equation

$$Y^2Z = X^3 + a_2X^2Z + a_4XZ^2.$$

And the moduli stack $\overline{\mathscr{M}}_{1,1}[\Gamma_1(3)]$ of generalized elliptic curves with $[\Gamma_1(3)]$-level structure has an isomorphism $\overline{\mathscr{M}}_{1,1}[\Gamma_1(3)] \cong \mathscr{P}(1,3)$ over Spec$(\mathbb{Z}[1/3])$ as in [HMe, Proposition 4.5] through the universal equation

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3.$$

And the moduli stack $\overline{\mathscr{M}}_{1,1}[\Gamma_1(4)]$ of generalized elliptic curves with $[\Gamma_1(4)]$-level structure has an isomorphism $\overline{\mathscr{M}}_{1,1}[\Gamma_1(4)] \cong \mathscr{P}(1,2)$ over Spec$(\mathbb{Z}[1/2])$ as in [Meier, Examples 2.1] through the universal equation

$$Y^2Z + a_1XYZ + a_1a_2YZ^2 = X^3 + a_2X^2Z.$$

And the moduli stack $\overline{\mathscr{M}}_{1,1}[\Gamma(2)]$ of generalized elliptic curves with $[\Gamma(2)]$-level structure has an isomorphism $\overline{\mathscr{M}}_{1,1}[\Gamma(2)] \cong \mathscr{P}(2,2)$ over Spec$(\mathbb{Z}[1/2])$ as in [Stojanoska, Proposition 7.1] through the universal equation (where the degree of each λ_i is 2)

$$Y^2Z = X^3 + (\lambda_1 + \lambda_2)X^2Z + \lambda_1\lambda_2XZ^2.$$

Finally, the moduli stack $\overline{\mathscr{M}}_{1,1}[\Gamma(m)]$ of generalized elliptic curves with $[\Gamma(m)]$-level structure for $m = 5, 6, 7, 8, 9, 10$ or 12 has an isomorphism $\overline{\mathscr{M}}_{1,1}[\Gamma(m)] \cong \mathbb{P}^1$ over Spec$(\mathbb{Z}[1/m])$ as in [Meier, Example 2.5].

By Remark 2.2, the weighted projective stacks are tame Deligne–Mumford as well.

For the degree of the discriminant, it suffices to find the weight of the $\mathbb{G}_m$-action. First, the four papers cited above explicitly construct universal families of elliptic curves over the schematic covers $(\mathrm{Spec}\, K[a_i, a_j] - (0,0)) \to \mathscr{P}_K(i,j)$ of the corresponding moduli stacks. The explicit defining equation of the respective universal family implies that the $\lambda \in \mathbb{G}_m$ also acts on the discriminant of the universal family by multiplying λ^{12}. Therefore, the discriminant has degree 12. ∎

We now consider the moduli stack $\mathcal{L}_{1,12n}^{[\Gamma]} := \mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma])$ of generalized elliptic curves over $\mathbb{P}^1$ with $[\Gamma]$-structures.

Proposition A.5. *Assume* $\mathrm{char}(K) = 0$ *or* $\neq 2$ *for* $[\Gamma] = [\Gamma_1(2)], [\Gamma_1(4)], [\Gamma(2)]$, $\mathrm{char}(K) \neq 3$ *for* $[\Gamma] = [\Gamma_1(3)]$ *and* $\mathrm{char}(K) \neq m$ *for* $[\Gamma_1(m)]$*-structures for* $m = 5, 6, 7, 8, 9, 10$ or 12. *Then, the moduli stack* $\mathcal{L}_{1,12n}^{[\Gamma]}$ *of generalized elliptic curves over* $\mathbb{P}^1$ *with discriminant degree* $12n > 0$ *and* $[\Gamma]$*-structures is the tame Deligne–Mumford stack* $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma])$ *parameterizing the* K*-morphisms* $f : \mathbb{P}^1 \to \overline{\mathcal{M}}_{1,1}[\Gamma]$ *such that* $f^*\mathcal{O}_{\overline{\mathcal{M}}_{1,1}[\Gamma]}(1) \cong \mathcal{O}_{\mathbb{P}^1}(n)$.

Proof. Without the loss of generality, we prove the $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])$ case over a field K with $\mathrm{char}(K) \neq 2$. The proof for the other cases are analogous. By the definition of the universal family p, any generalized elliptic curves $\pi : Y \to \mathbb{P}^1$ with $[\Gamma_1(2)]$-structures comes from a morphism $f : \mathbb{P}^1 \to \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)]$ and vice versa. As this correspondence also works in families, the moduli stack of generalized elliptic curves over $\mathbb{P}^1$ with $[\Gamma_1(2)]$-structures is isomorphic to $\mathrm{Hom}(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])$.

Since the discriminant degree of f is $12 \deg f^*\mathcal{O}_{\overline{\mathcal{M}}_{1,1}[\Gamma_1(2)]}(1)$ by Proposition A.4, the substack $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])$ parametrizing such f's with $\deg f^*\mathcal{O}_{\overline{\mathcal{M}}_{1,1}[\Gamma_1(2)]}(1) = n$ is the desired moduli stack. Since $\deg f^*\mathcal{O}_{\overline{\mathcal{M}}_{1,1}[\Gamma_1(2)]}(1) = n$ is an open condition, $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])$ is an open substack of $\mathrm{Hom}(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])$, which is tame Deligne–Mumford by [HP2, Proposition 3.6] as $\overline{\mathcal{M}}_{1,1}[\Gamma_1(2)]$ itself is tame Deligne–Mumford by Proposition A.4. This shows that $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])$ satisfies the desired properties as well. ∎

We recall the motives & weighted point counts over finite fields of $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))$ worked out in [Park-Spink, Corollary 1.2] which is an extension of [Han-Park, Theorem 1].

Corollary A.6 (Corollary 1.2 of [Park-Spink]). *If* $\mathrm{char}(K) \nmid a, b$, *then*

$$[\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))] = \mathbb{L}^{(a+b)n+1} - \mathbb{L}^{(a+b)n-1} \in K_0(\mathrm{Stck}_{/K}),$$

and if $\mathrm{char}(\mathbb{F}_q) \nmid a, b$, *then we have the weighted* $\mathbb{F}_q$*-point count*

$$|\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))(\mathbb{F}_q)| := \sum_{x \in \mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))(\mathbb{F}_q)} \frac{1}{|\mathrm{Aut}(x)|} = q^{(a+b)n+1} - q^{(a+b)n-1}.$$

We now acquire the exact number $|\mathcal{L}_{1,12n}^{[\Gamma]}(\mathbb{F}_q)/\sim|$ of $\mathbb{F}_q$–isomorphism classes of $\mathbb{F}_q$–points (i.e., the non–weighted point count) of the moduli stack $\mathcal{L}_{1,12n}^{[\Gamma]} \cong \mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma])$ of generalized elliptic curves over $\mathbb{P}^1$ with discriminant degree $12n > 0$ and $[\Gamma]$-structures.

Proposition A.7. *If* $\mathrm{char}(\mathbb{F}_q) \neq 2$, *then*

$$|\mathcal{L}_{1,12n}^{[\Gamma_1(2)]}(\mathbb{F}_q)/\sim| = 2 \cdot \#_q\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(2,4))\right) = 2(q^{6n+1} - q^{6n-1})$$

$$|\mathscr{L}_{1,12n}^{[\Gamma_1(4)]}(\mathbb{F}_q)/\sim| = \#_q\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathscr{P}(1,2))\right) = q^{3n+1} - q^{3n-1}$$

$$|\mathscr{L}_{1,12n}^{[\Gamma(2)]}(\mathbb{F}_q)/\sim| = 2 \cdot \#_q\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathscr{P}(2,2))\right) = 2(q^{4n+1} - q^{4n-1})$$

If $\mathrm{char}(\mathbb{F}_q) \neq 3$*, then*

$$|\mathscr{L}_{1,12n}^{[\Gamma_1(3)]}(\mathbb{F}_q)/\sim| = \#_q\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathscr{P}(1,3))\right) = q^{4n+1} - q^{4n-1}$$

If $\mathrm{char}(\mathbb{F}_q) \neq m$*, then*

$$|\mathscr{L}_{1,12n}^{[\Gamma_1(m)]}(\mathbb{F}_q)/\sim| = \#_q\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathscr{P}(1,1) \cong \mathbb{P}^1)\right) = q^{2n+1} - q^{2n-1}$$

Proof. Fix $n \in \mathbb{Z}_{\geq 1}$. Since any $\varphi_g \in \mathrm{Hom}_n(\mathbb{P}^1, \mathscr{P}(a,b))$ is surjective, the generic stabilizer group $\mu_{\mathrm{gcd}(a,b)}$ of $\mathscr{P}(a,b)$ is the automorphism group of φ_g. Using the identification from Proposition A.5 and the weighted point counts of Hom stacks as in Corollary A.6 gives the desired formula as

$$|\mathscr{L}_{1,12n}^{[\Gamma]}(\mathbb{F}_q)/\sim| = |\mu_{\mathrm{gcd}(a,b)}| \cdot (q^{(a+b)n+1} - q^{(a+b)n-1})$$

where the factor of 2 comes from the hyperelliptic involution when $\mu_{\mathrm{gcd}(a,b)} = \mu_2$. ■

Remark A.8. For weighted projective lines $\mathscr{P}(a,b)$ as in the cases of $\mathscr{L}_{1,12n}^{[\Gamma]}$, the inertia stack of the relevant Hom stack $\left\{\mathscr{I}\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathscr{P}(a,b))\right)\right\}$ is a sum of

$$\left\{\mathrm{Hom}_n(\mathbb{P}^1_{\kappa(g)}, \mathscr{P}_{\kappa(g)}(a,b))\right\}$$

for each closed point $g \in \mathbb{G}_m$ with $\mathrm{ord}(g) \mid \mathrm{gcd}(a,b)$, as the only possible generic stabilizer of positive dimensional substacks of $\mathscr{P}(a,b)$. On the other hand, the terms with division function $\delta(r, q-1)$ do not occur in $|\mathscr{L}_{1,12n}^{[\Gamma]}(\mathbb{F}_q)/\sim|$ as the characteristic condition required to identify $\overline{\mathscr{M}}_{1,1}[\Gamma]$ as a weighted projective line implies that $\mathrm{gcd}(a,b) \mid q-1$. See [HP2, §4.3] for more details.

Arithmetic moduli of m-marked genus one fibrations over $\mathbb{P}^1$

We proceed to determine the sharp enumeration on the number of m-marked $(m-1)$-stable genus one fibrations over $\mathbb{P}^1_{\mathbb{F}_q}$ for $2 \leq m \leq 5$. First, we state the definition of m-marked $(m-1)$-stability from [LP, Definition 1.5.3], which is a modification of the Deligne–Mumford stability [DM]:

Definition A.9. Let K be a field and m be a positive integer. Then, a tuple $(C, p_1, \ldots, p_m)$, of a geometrically connected, geometrically reduced, and proper K-curve C of arithmetic genus one with m distinct K-rational points p_i in the smooth locus of C, is a $(m-1)$-stable m-marked curve of arithmetic genus one if the curve $C_{\overline{K}} := C \times_K \overline{K}$ and the divisor $\Sigma := \{p_1, \ldots, p_m\}$ satisfy the following properties, where $\overline{K}$ is the algebraic closure of K:

1. $C_{\overline{K}}$ has only nodes and elliptic u-fold points as singularities (see below), where $u < m$,

2. $C_{\overline{K}}$ has no disconnecting nodes, and

3. every irreducible component of $C_{\overline{K}}$ contains at least one marked point.

Remark A.10. A singular point of a curve over $\overline{K}$ is an elliptic u-fold singular point if it is Gorenstein and étale locally isomorphic to a union of u general lines in $\mathbb{P}^{u-1}_{\overline{K}}$ passing through a common point.

Note that the name "$(m-1)$-stability" comes from [Smyth, §1.1], which is defined when $\mathrm{char}(K) \neq 2,3$. By [LP, Proposition 1.5.4], the above definition (by [LP, Definition 1.5.3]) coincides with that of Smyth when $\mathrm{char}(K) \neq 2,3$, hence we adapt Smyth's naming convention on Lekili and Polishchuk's definition. Regardless, we focus on the case when $\mathrm{char}(K) \neq 2,3$, so that the moduli stack of such curves behaves reasonably.

By [Smyth, Theorem 3.8], we are able to formulate the moduli stack of $(m-1)$-stable m-marked curves of arithmetic genus one over any field of characteristic $\neq 2,3$:

Theorem A.11. *There exists a proper irreducible Deligne–Mumford moduli stack* $\overline{\mathscr{M}}_{1,m}(m-1)$ *of* $(m-1)$*-stable m-marked curves arithmetic genus one over* $\mathrm{Spec}(\mathbb{Z}[1/6])$

Note that when $m=1$, $\overline{\mathscr{M}}_{1,1}(0) \cong \overline{\mathscr{M}}_{1,1}$ is the Deligne–Mumford moduli stack of stable elliptic curves.

In fact, the construction of $\overline{\mathscr{M}}_{1,m}(m-1)$ extends to $\mathrm{Spec}\,\mathbb{Z}$ by [LP, Theorem 1.5.7] (called $\overline{\mathscr{M}}^{\infty}_{1,m}$ in loc.cit.) as an algebraic stack, which is proper over $\mathrm{Spec}\,\mathbb{Z}[1/N]$ where N depends on m:

- if $m \geq 3$, then $N = 1$,
- if $m = 2$, then $N = 2$, and
- if $m = 1$, then $N = 6$.

However, even with those assumptions above, $\overline{\mathscr{M}}_{1,m}(m-1)$ is not necessarily Deligne–Mumford. Nevertheless, by [LP, Theorem 1.5.7.], we obtain the explicit descriptions of $\overline{\mathscr{M}}_{1,m}(m-1)$:

Proposition A.12. *The moduli stack* $\overline{\mathscr{M}}_{1,m}(m-1)$ *of m-marked* $(m-1)$*-stable curves of arithmetic genus one for* $2 \leq m \leq 5$ *is isomorphic to the following, for a field K:*

1. *if* $\mathrm{char}(K) \neq 2,3$*, the tame Deligne–Mumford moduli stack of 2-marked 1-stable curves of arithmetic genus one is isomorphic to*

$$(\overline{\mathscr{M}}_{1,2}(1))_K \cong [(\mathrm{Spec}\,K[a_2,a_3,a_4]-0)/\mathbb{G}_m] = \mathscr{P}_K(2,3,4),$$

2. *if* $\mathrm{char}(K) \neq 2, 3$*, the tame Deligne–Mumford moduli stack of 3-marked 2-stable curves of arithmetic genus one is isomorphic to*

$$(\overline{\mathscr{M}}_{1,3}(2))_K \cong [(\mathrm{Spec}\, K[a_1, a_2, a_2, a_3] - 0)/\mathbb{G}_m] = \mathscr{P}_K(1, 2, 2, 3),$$

3. *if* $\mathrm{char}(K) \neq 2$*, the tame Deligne–Mumford moduli stack of 4-marked 3-stable curves of arithmetic genus one is isomorphic to*

$$(\overline{\mathscr{M}}_{1,4}(3))_K \cong [(\mathrm{Spec}\, K[a_1, a_1, a_1, a_2, a_2] - 0)/\mathbb{G}_m] = \mathscr{P}_K(1, 1, 1, 2, 2),$$

4. *the moduli stack of 5-marked 4-stable curves of arithmetic genus one is isomorphic to a scheme*

$$(\overline{\mathscr{M}}_{1,5}(4))_K \cong [(\mathrm{Spec}\, K[a_1, a_1, a_1, a_1, a_1, a_1]–0)/\mathbb{G}_m] = \mathbb{P}_K(1, 1, 1, 1, 1, 1) \cong \mathbb{P}^5_K,$$

where $\lambda \cdot a_i = \lambda^i a_i$ *for* $\lambda \in \mathbb{G}_m$ *and* $i = 1, 2, 3, 4$*. Thus, the* a_i*'s have degree* i *respectively. Furthermore, if* $\mathrm{char}(K) \neq 2, 3$*, then the discriminant divisors of such* $\overline{\mathscr{M}}_{1,m}(m-1)$ *have degree 12.*

Proof. The moduli stack $\overline{\mathscr{M}}_{1,2}(1)$ of 2-marked at ∞ and $(0, 0)$ Smyth's 1-stable curves of arithmetic genus one has an isomorphism $\overline{\mathscr{M}}_{1,2}(1) \cong \mathscr{P}(2, 3, 4)$ over $\mathrm{Spec}(\mathbb{Z}[1/6])$ as in [LP, Theorem 1.5.7.] through the universal equation

$$Y^2Z + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2\,,$$

with discriminant $\Delta = -16a_2^3a_3^2 + 16a_2^2a_4^2 - 64a_4^3 - 27a_3^4 + 56a_2a_4a_3^2$. Similarly, the Proof of [LP, Theorem 1.5.7.] gives the corresponding isomorphisms $\overline{\mathscr{M}}_{1,m}(m-1) \cong \mathscr{P}(\vec{\lambda})$. By Remark 2.2, the weighted projective stacks are tame Deligne–Mumford as well, and in fact, smooth.

For the degree of the discriminant when $\mathrm{char}(K) \neq 2, 3$, it suffices to describe the discriminant divisor, the locus of singular curves in $\overline{\mathscr{M}}_{1,m}(m-1)$. First, [LP, Theorem 1.5.7.] shows that in the above case, where $\overline{\mathscr{M}}_{1,m}(m-1) \cong \mathscr{P}(\vec{\lambda})$, the line bundle $\mathscr{O}_{\mathscr{P}(\vec{\lambda})}(1)$ of degree one is isomorphic to $\lambda := \pi_*\omega_\pi$, where $\pi : \overline{\mathscr{C}}_{1,m}(m-1) \to \overline{\mathscr{M}}_{1,m}(m-1)$ is the universal family of $(m-1)$-stable m-marked curves of arithmetic genus one. Since $\overline{\mathscr{M}}_{1,m}(m-1)$ is smooth and the Picard rank is one (generated by λ), the discriminant divisor is Cartier. In fact, by [Smyth2, §3.1], it coincides with the locus Δ_{irr} of curves with non-disconnecting nodes or non-nodal singular points. Then [Smyth2, Remark 3.3] (which assumes $\mathrm{char}(K) \neq 2, 3$) implies that $\Delta_{irr} \sim 12\lambda$, thus the discriminant divisor has degree 12. ∎

We now consider the moduli stacks of m-marked $(m-1)$-stable genus one fibrations over $\mathbb{P}^1_K$ for any field K of $\mathrm{char}(K) = 0$ or > 3:

Proposition A.13. *Assume* $\mathrm{char}(K) = 0$ *or* > 3*. If* $2 \leq m \leq 5$*, then the moduli stack* $\mathscr{L}^m_{1,12n}$ *of* m*-marked* $(m-1)$*-stable genus one fibrations over* $\mathbb{P}^1_K$ *with discriminant degree* $12n$ *is the tame Deligne–Mumford stack* $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathscr{M}}_{1,m}(m-1))$ *parameterizing the* K*-morphisms* $f : \mathbb{P}^1 \to \overline{\mathscr{M}}_{1,m}(m-1)$ *such that* $f^*\mathscr{O}_{\mathscr{P}(\vec{\lambda})}(1) \cong \mathscr{O}_{\mathbb{P}^1}(n)$*.*

Proof. Without the loss of the generality, we prove the 2-marked 1-stable curves of arithmetic genus one case $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathscr{M}}_{1,2}(1))$ over $\mathrm{char}(\mathbb{F}_q) \neq 2,3$. The proof for the other cases are analogous. By the definition of the universal family p, any 2-marked 1-stable arithmetic genus one curves $\pi : Y \to \mathbb{P}^1$ with discriminant degree $12n$ comes from a morphism $f : \mathbb{P}^1 \to \overline{\mathscr{M}}_{1,2}(1)$ and vice versa. As this correspondence also works in families, the moduli stack of 2-marked 1-stable curves of arithmetic genus one over $\mathbb{P}^1_K$ is isomorphic to $\mathrm{Hom}(\mathbb{P}^1, \overline{\mathscr{M}}_{1,2}(1))$.

Since the discriminant degree of f is $12 \deg f^*\mathscr{O}_{\overline{\mathscr{M}}_{1,2}(1)}(1)$ by Proposition A.12, the substack $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathscr{M}}_{1,2}(1))$ parametrizing such f's with $\deg f^*\mathscr{O}_{\overline{\mathscr{M}}_{1,2}(1)}(1) = n$ is the desired moduli stack. Since $\deg f^*\mathscr{O}_{\overline{\mathscr{M}}_{1,2}(1)}(1) = n$ is an open condition, $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathscr{M}}_{1,2}(1))$ is an open substack of $\mathrm{Hom}(\mathbb{P}^1, \overline{\mathscr{M}}_{1,2}(1))$, which is tame Deligne–Mumford by [HP2, Proposition 3.6] as $\overline{\mathscr{M}}_{1,2}(1)$ itself is tame Deligne–Mumford by Proposition A.12. This shows that $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathscr{M}}_{1,2}(1))$ satisfies the desired properties as well. ■

We now acquire the exact number $|\mathscr{L}^m_{1,12n}(\mathbb{F}_q)/\sim|$ of $\mathbb{F}_q$–isomorphism classes of $\mathbb{F}_q$–points of the moduli stack $\mathscr{L}^m_{1,12n} \cong \mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathscr{M}}_{1,m}(m-1))$ of m-marked $(m-1)$-stable genus one fibrations over $\mathbb{P}^1$ with discriminant degree $12n > 0$.

Proposition A.14. *If* $\mathrm{char}(\mathbb{F}_q) \neq 2,3$*, then*

$$\begin{aligned}
|\mathscr{L}^2_{1,12n}(\mathbb{F}_q)/\sim| &= \#_q\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathscr{P}(2,3,4))\right) + \#_q\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathscr{P}(2,4))\right) \\
&= (q^{9n+2} + q^{9n+1} - q^{9n-1} - q^{9n-2}) + (q^{6n+1} - q^{6n-1})
\end{aligned}$$

$$\begin{aligned}
|\mathscr{L}^3_{1,12n}(\mathbb{F}_q)/\sim| &= \#_q\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathscr{P}(1,2,2,3))\right) + \#_q\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathscr{P}(2,2))\right) \\
&= (q^{8n+3} + q^{8n+2} + q^{8n+1} - q^{8n-1} - q^{8n-2} - q^{8n-3}) + (q^{4n+1} - q^{4n-1})
\end{aligned}$$

$$\begin{aligned}
|\mathscr{L}^4_{1,12n}(\mathbb{F}_q)/\sim| &= \#_q\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathscr{P}(1,1,1,2,2))\right) + \#_q\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathscr{P}(2,2))\right) \\
&= (q^{7n+4} + q^{7n+3} + q^{7n+2} + q^{7n+1} - q^{7n-1} - q^{7n-2} - q^{7n-3} - q^{7n-4}) \\
&\quad + (q^{4n+1} - q^{4n-1}) \\
|\mathscr{L}^5_{1,12n}(\mathbb{F}_q)/\sim| &= \#_q\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathbb{P}(1,1,1,1,1,1) \cong \mathbb{P}^5)\right) \\
&= q^{6n+5} + q^{6n+4} + q^{6n+3} + q^{6n+2} + q^{6n+1} - q^{6n-1} - q^{6n-2} - q^{6n-3} \\
&\quad - q^{6n-4} - q^{6n-5}
\end{aligned}$$

Proof. Note that $\overline{\mathscr{M}}_{1,2}(1) \cong \mathscr{P}(2,3,4)$ has the substack $\mathscr{P}(2,4)$ with the generic stabilizer of order 2. This implies that the number of isomorphism classes of $\mathbb{F}_q$-points of $\mathscr{L}^2_{1,12n}$ with discriminant degree $12n$ is $|\mathscr{L}^2_{1,12n}(\mathbb{F}_q)/\sim| = (q^{9n+2} + q^{9n+1} - q^{9n-1} - q^{9n-2}) + (q^{6n+1} - q^{6n-1})$ by summing the weighted point counts of Hom stacks as in [HP2, Proposition 4.10]. Similarly, $\overline{\mathscr{M}}_{1,3}(2) \cong \mathscr{P}(1,2,2,3)$ and $\overline{\mathscr{M}}_{1,4}(3) \cong \mathscr{P}(1,1,1,2,2)$ has the substack $\mathscr{P}(2,2)$ with the generic stabilizer of order 2. This implies that adding $(q^{4n+1} - q^{4n-1})$ to the corresponding weighted points count gives the desired non–weighted point counts. Finally, $\overline{\mathscr{M}}_{1,5}(4) \cong \mathbb{P}^5$, so that the non-weighted point count coincides with the weighted point count. ■

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