Geometric Interpretation of Tate's Algorithm

$$
\begin{tikzcd}
 & & & & \mathcal{Y} = \varphi^* \overline{\mathcal{E}} \arrow[r] \arrow[dl, "\pi"'] \arrow[dd, "h"] & \overline{\mathcal{E}} \arrow[dd, "p"] \\
X \arrow[ddd, "f"] & X' \arrow[l, "\text{Weierstrass}"'] \arrow[ddd, "f'"] & \hat{X} \arrow[l, "\text{contraction}"'] \arrow[r, "\text{resolution}"] \arrow[ddd, "\hat{f}"] & Y \arrow[ddd, "g"] & & \\
 & & & & \mathcal{C} \arrow[r, "\varphi"] \arrow[dl, "\rho"'] & \overline{\mathcal{M}}_{1,1} \arrow[d] \\
C \arrow[r, equal] & C \arrow[r, equal] & C \arrow[r, equal] & C \arrow[rr, "j"] \arrow[urr, dashed, "\psi"] & & \overline{M}_{1,1}
\end{tikzcd}
$$

Here f is a Weierstrass model, ψ is the associated weighted linear series viewed as a rational map to $\overline{\mathcal{M}}_{1,1}$, φ is a twisted morphism from the universal tuning stack $\mathcal{C}$ which induces a stable stack-like model $h : \mathcal{Y} \to \mathcal{C}$ where $g : Y \to C$ is the twisted model via coarse moduli maps, $\hat{f}$ is a resolution of Y, and f' is the relative minimal model obtained by contracting relative (-1)-curves.

Suppose that normalized base multiplicity $m = 3$. This occurs if and only if $(\nu(a_4), \nu(a_6)) = (1, \geq 2)$. Then $r = 12/\gcd(3,12) = 4$ and $a = 3/\gcd(3,12) = 1$. Thus the stabilizer of the twisted curve acts on the central fiber of the twisted model via the character $\mu_4 \to \mu_4$, $\zeta_4 \mapsto \zeta_4^{-1}$. In particular, the central fiber E of $\mathcal{Y}$ has $j = 1728$. The μ_4 action on E has two fixed points, and there is an orbit of size two with stabilizer $\mu_2 \subset \mu_4$. Let E_0 be the image of E in the twisted model Y. As E appears with multiplicity 4, Y has $\frac{1}{4}(-1,-1)$ quotient singularities at the images of the the fixed points and a $\frac{1}{2}(-1,-1)$ singularity at the image of the orbit of size two. Each of these singularities is resolved by a single blowup to obtain $\hat{X}$ with central fiber $4\tilde{E}_0 + E_1 + E_2 + E_3$ where E_i are the exceptional divisors of the resolution for $i = 1,2,3$ and $E_1^2 = E_2^2 = -4$ with $E_3^2 = -2$. Then $\tilde{E}_0$ is a (-1)-curve so it needs to be contracted. After this contraction E_2 becomes a (-1) curve and must also be contracted. Since E_i for $i = 1,2,3$ are incident and pairwise transverse after blowing down $\tilde{E}_0$, then the images of E_1 and E_2 must be tangent after blowing down E_3. Moreover, they are now (-2)-curves and the relatively minimal model for type III.

Tate's Algorithm via Twisted Morphisms

Theorem (Dori Bejleri–JP–Matthew Satriano; April 2024)

If $\mathrm{char}(K) \neq 2, 3$. Then the twisting condition (r, a) and the order of vanishing of j at $j = \infty$ determine the Kodaira fiber type, and (r, a) is in turn determined by $m = \min\{3\nu(a_4), 2\nu(a_6)\}$.

$\gamma : (\nu(a_4),\ \nu(a_6))$	*Reduction type with* $j \in \overline{M}_{1,1}$	$\Gamma : (r, a)$
$(\geq 1, 1)$	II *with* $j = 0$	$(6, 1)$
$(1, \geq 2)$	III *with* $j = 1728$	$(4, 1)$
$(\geq 2, 2)$	IV *with* $j = 0$	$(3, 1)$
$(2, 3)$	$\mathrm{I}^*_{k>0}$ *with* $j = \infty$ I^*_0 *with* $j \neq 0, 1728$	$(2, 1)$
$(\geq 3, 3)$	I^*_0 *with* $j = 0$	$(2, 1)$
$(2, \geq 4)$	I^*_0 *with* $j = 1728$	$(2, 1)$
$(\geq 3, 4)$	IV^* *with* $j = 0$	$(3, 2)$
$(3, \geq 5)$	III^* *with* $j = 1728$	$(4, 3)$
$(\geq 4, 5)$	II^* *with* $j = 0$	$(6, 5)$

Geometric Meaning of Height Moduli Framework

1. So one can run the resolution / minimal model. As these are *algebraic surfaces* it can be done over $\mathrm{char}(K) = p > 0$

2. A twisted morphism $\varphi : \mathcal{C} \to \overline{\mathcal{M}}_{1,1}$ with its twisting data Γ from the universal tuning stack $\mathcal{C}$ induces a stable stack-like model $h : \mathcal{Y} \to \mathcal{C}$ as a unique pullback of the universal family $p : \overline{\mathcal{E}} \to \overline{\mathcal{M}}_{1,1}$. All the ensuing birational geometry is natural.

3. True purpose of a **representable classifying morphism** is in the <u>universal principle</u> that φ intrinsically contains all the algebro-geometric data necessary to uniquely determine a fibration with singular fibers. This is the very essence of the inner arithmetic of rational points on moduli stacks over K.

$\mathcal{A}$lgebraic $\mathcal{G}$eometry $\Cap$ $\mathcal{T}$opology $\Longleftrightarrow$ $\mathbb{A}$rithmetic

1. Consider the fact that $\overline{\mathcal{M}}_{1,1}$ could have been any other algebraic stack $\mathcal{X}$ (such as $\overline{\mathcal{M}}_g$ or $\overline{\mathcal{A}}_g$) which is the representing object for certain moduli functor as the fine moduli stack together with the universal family $p : \overline{\mathcal{E}} \to \mathcal{X}$.

2. Representable classifying morphisms as twisted morphisms $\varphi : \mathcal{C} \to \mathcal{X}$ uniquely determines certain families of varieties (of algebraic curves or abelian varieties) with non-abelian stabilizers ($g \geq 2$). And they naturally have corresponding "Tate's algorithm", counting statements and so on.

3. Geometrizing $\mathcal{X}(K)$ leads to Height moduli space $\mathcal{M}_n(\mathcal{X}, \mathcal{V})$. Once we have a **space**, we compute its **invariants**, consider all invariants simultaneously via generating series and show the motivic height zeta function's **rationality**, naturally having various kinds of **consequences**.

Theorem (Dori Bejleri–JP–Matthew Satriano; April 2024)

$$\left\{\mathcal{W}^{\min}_{n=1}(\mathcal{P}(\vec{\lambda}))\right\} = \{\mathbb{P}^N\}(\mathbb{L}^{|\vec{\lambda}|} - \mathbb{L}) + \mathbb{L}^{N+1}\{\mathbb{P}^{|\vec{\lambda}|-N-2}\}$$

$$\left\{\mathcal{W}^{\min}_{n\geq 2}(\mathcal{P}(\vec{\lambda}))\right\} = \mathbb{L}^{(n-2)|\vec{\lambda}|+N+2}(\mathbb{L}^{|\vec{\lambda}|-1} - 1)\{\mathbb{P}^{|\vec{\lambda}|-1}\}$$

Take $|\vec{\lambda}| = 10$ and $N = 1$ as $\overline{\mathcal{M}}_{1,1} \cong \mathcal{P}(4,6)$ over $\mathbb{Z}[1/6]$.

1. When $n = 1$, X is a **Rational elliptic surface**.

$$\left\{\mathcal{W}^{\min}_1\right\} = \mathbb{L}^{11}+\mathbb{L}^{10}+\mathbb{L}^{9}+\mathbb{L}^{8}+\mathbb{L}^{7}+\mathbb{L}^{6}+\mathbb{L}^{5}+\mathbb{L}^{4}+\mathbb{L}^{3}-\mathbb{L}$$

2. When $n = 2$, X is algebraic $K3$ surface with elliptic fibration (i.e., **Projective elliptic K3 surface with moduli dim. 18**).

$$\left\{\mathcal{W}^{\min}_2\right\} = \mathbb{L}^{21}+\mathbb{L}^{20}+\mathbb{L}^{19}+\mathbb{L}^{18}+\mathbb{L}^{17}+\mathbb{L}^{16}+\mathbb{L}^{15}+\mathbb{L}^{14}+\mathbb{L}^{13}-\mathbb{L}^{11}-\mathbb{L}^{10}-\mathbb{L}^{9}-\mathbb{L}^{8}-\mathbb{L}^{7}-\mathbb{L}^{6}-\mathbb{L}^{5}-\mathbb{L}^{4}-\mathbb{L}^{3}$$

$$= \mathbb{L}(\mathbb{L}^2-1)\left(\mathbb{L}^{18}+\mathbb{L}^{17}+2\mathbb{L}^{16}+2\mathbb{L}^{15}+3\mathbb{L}^{14}+3\mathbb{L}^{13}+4\mathbb{L}^{12}+4\mathbb{L}^{11}+5\mathbb{L}^{10}+4\mathbb{L}^{9}+4\mathbb{L}^{8}+3\mathbb{L}^{7}+3\mathbb{L}^{6}+2\mathbb{L}^{5}+2\mathbb{L}^{4}+\mathbb{L}^{3}+\mathbb{L}^{2}\right)$$