Arithmetic of the Moduli of Fibrations

Arithmetic Moduli of Elliptic Surfaces

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2021 KMS Annual Meeting – Trends in Arithmetic Geometry

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While this is the most general setup, it is natural to work with the case when the base curve is the smooth projective line $\mathbb{P}^1$ and there exists a section $S : \mathbb{P}^1 \hookrightarrow X$ coming from the identity points of the elliptic fibres and not passing through the singular points.

Elliptic surfaces over $\mathbb{P}^1$ with a section

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1. When $n = 1$, X is a **Rational elliptic surface** with the Kodaira dimension $\kappa = -\infty$ which has 12 nodal singular fibers generically. It is acquired from a pencil of cubic curves in $\mathbb{P}^2$ by blowing up a base locus of nine points coming from the intersection of two general cubic curves.

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2. When $n = 2$, X is a $K3$ surface with an elliptic fibration (i.e., **Elliptic K3 surface**) which has the Kodaira dimension $\kappa = 0$ that has 24 nodal singular fibers generically. Note that X is a minimal surface.

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3. When $n \geq 3$, X is called a **Properly elliptic surface** with Kodaira dimension $\kappa = 1$ that has $12n$ nodal singular fibers generically. Note that X is also a minimal surface.

Deligne–Mumford stack $\overline{\mathcal{M}}_{1,1}$ of stable elliptic curves

Let us recall that $\overline{\mathcal{M}}_{1,1}$ is a smooth proper Deligne-Mumford stack of stable elliptic curves with a coarse moduli space $\overline{M}_{1,1} \cong \mathbb{P}^1$. This $\mathbb{P}^1$ parametrizes the j–invariants of elliptic curves.

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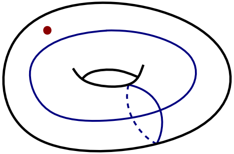

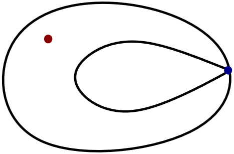

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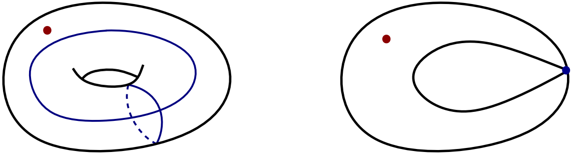

When the characteristic of the field K is not equal to 2 or 3, $(\overline{\mathcal{M}}_{1,1})_K \cong [(Spec\ K[a_4, a_6] - (0,0))/\mathbb{G}_m] =: \mathcal{P}_K(4,6)$ through the short Weierstrass equation: $Y^2 = X^3 + a_4X + a_6$

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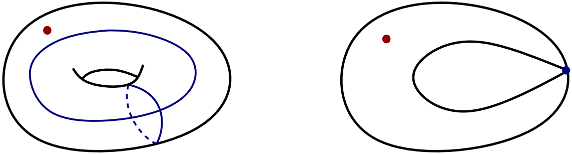

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Stabilizers are the orbifold points $[1:0]$ & $[0:1]$ with μ_4 & μ_6 respectively and the generic stacky points such as $[1:1]$ with μ_2

Moduli stack of stable elliptic surfaces

The fine moduli $\overline{\mathcal{M}}_{1,1}$ comes with universal family $p : \overline{\mathcal{C}}_{1,1} \to \overline{\mathcal{M}}_{1,1}$ of stable elliptic curves. Thus, a stable elliptic surface $g : Y \to \mathbb{P}^1$ is induced from a morphism $\varphi_f : \mathbb{P}^1 \to \overline{\mathcal{M}}_{1,1}$ and vice versa.

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$$\begin{tikzcd} X \arrow[r, "\nu"] \arrow[d, "f"] & Y = \varphi_f^*(\overline{\mathcal{C}}_{1,1}) \arrow[r] \arrow[d, "g"] & \overline{\mathcal{C}}_{1,1} \arrow[d, "p"] \\ \mathbb{P}^1 \arrow[r, equal] & \mathbb{P}^1 \arrow[r, "\varphi_f"] & \overline{\mathcal{M}}_{1,1} \end{tikzcd} \tag{1}$$

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Here, we *fixed the parameterization of the domain* $\mathbb{P}^1$ which is good for 'Global Fields Analogy' (since $\mathbb{Q}$ has the *unique* ring of integers called $\mathbb{Z}$) but not natural from Geometric perspective.

Group actions on stacks for stack quotients of stacks

It is natural to consider the action of $\mathrm{Aut}(\mathbb{P}^1) = \mathsf{PGL}_2$ on $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$ by composing the stable elliptic surface $g : Y \to \mathbb{P}^1$ with an automorphism of $\mathbb{P}^1$.

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It is easy to see that this action is induced by an action on the ambient weighted projective stack $\mathcal{P}(V)$.

$$(A, B) \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4n)) \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(6n)) =: V \qquad (2)$$

define the so-called *Weierstrass data* of the fibration. Indeed, the action of an element of PGL_2 on the homogeneous coordinates X, Y of $\mathbb{P}^1$ translates to an action on the global sections A, B of $\mathcal{O}_{\mathbb{P}^1}(4n), \mathcal{O}_{\mathbb{P}^1}(6n)$ which are the homogeneous coordinates of $\mathcal{P}(V) = \mathcal{P}(\underbrace{4, \dots, 4}_{4n+1 \text{ times}}, \underbrace{6, \dots, 6}_{6n+1 \text{ times}}) \in \mathbb{Z}^{10n+2}$.

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Note that since both $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$ and $\mathcal{P}(V)$ are themselves stacks, the formal definition of these actions requires one to use the notion of group actions on stacks presented in [Romagny].

Group actions on stacks for stack quotients of stacks

We have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}) \arrow[r, hook] \arrow[d, "/\,\mathsf{PGL}_2"] & \mathrm{Rat}_n^\gamma(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}) \arrow[r, hook] \arrow[d, "/\,\mathsf{PGL}_2"] & \mathcal{P}(V) \arrow[d, "/\,\mathsf{PGL}_2"] \\
\mathcal{W}_{\mathrm{sf},n} \arrow[r, hook] & \mathcal{W}_{\min,n} \arrow[r, hook] & {[\mathcal{P}(V)/\,\mathsf{PGL}_2]}
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where the horizontal arrows are open embeddings of moduli stacks.

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Theorem (Johannes Schmitt, J.-Y. Park)

Fix a degree $n \in \mathbb{Z}_{\geq 1}$ and a base field K with $\mathrm{char}(K) \neq 2, 3$. Inside the quotient stack $[\mathcal{P}(V)/\,\mathsf{PGL}_2]$, the open substacks $\mathcal{W}_{\mathrm{min},n}$ (for $n \geq 2$) of minimal Weierstrass fibrations and $\mathcal{W}_{\mathrm{sf},n}$ (for $n \geq 1$) of stable Weierstrass fibrations are smooth, irreducible and separated Deligne–Mumford stacks of finite type with affine diagonal for $\mathrm{char}(K) \nmid n$, which are tame for $\mathrm{char}(K) > 12n$.

Group actions on stacks for stack quotients of stacks

The Weierstrass fibration associated to $[A : B] = [X^{4n} : Y^{6n}]$ is invariant under scaling X by an element of μ_{4n} and by scaling Y under an element of μ_{6n}. Together, these transformations generate a copy of μ_{12n} inside PGL_2 which acts as an automorphism of the fibration, and the quotient is not tame when $\mathrm{char}(K)$ divides $12n$.

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The stack $\mathcal{W}_{\mathrm{min},1}$ contains points with positive dimensional stabilizers, thus it is no longer of Deligne–Mumford type. These points are precisely PGL_2-orbit of the Weierstrass data $[A : B]$

$$[A : B] = [0 : XY^5], [XY^3 : 0], [0 : X^2Y^4] \text{ and } [a_0X^2Y^2 : a_1X^3Y^3],$$

where in each case we have a nontrivial action of $\mathbb{G}_m$ on the coordinates X, Y fixing the fibrations. They are the four types of rational elliptic surfaces with two singular fibres $[\mathrm{II}, \mathrm{II}^*], [\mathrm{III}, \mathrm{III}^*], [\mathrm{IV}, \mathrm{IV}^*], [\mathrm{I}_0^*, \mathrm{I}_0^*]$ both of which are additive type in dual pair. One can see that the open substack $\mathcal{W}'_{\mathrm{min},1}$ of $\mathcal{W}_{\mathrm{min},1}$ obtained by removing these four points is indeed Deligne–Mumford for $\mathrm{char}(K) \nmid n$ and tame for $\mathrm{char}(K) > 12$.

Grothendieck ring $K_0(\mathrm{Stck}_K)$ of K-stacks

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Definition

Fix a field K. Then the *Grothendieck ring $K_0(\mathrm{Stck}_K)$ of algebraic stacks of finite type over K all of whose stabilizer group schemes are affine* is an abelian group generated by isomorphism classes of algebraic stacks $\{\mathcal{X}\}$ modulo relations:

- $\{\mathcal{X}\} = \{\mathcal{Z}\} + \{\mathcal{X} \setminus \mathcal{Z}\}$ for $\mathcal{Z} \subset \mathcal{X}$ a closed substack,
- $\{\mathcal{E}\} = \{\mathcal{X} \times \mathbb{A}^n\}$ for $\mathcal{E}$ a vector bundle of rank n on $\mathcal{X}$.

Multiplication on $K_0(\mathrm{Stck}_K)$ is induced by $\{\mathcal{X}\}\{\mathcal{Y}\} := \{\mathcal{X} \times_K \mathcal{Y}\}$.

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The weighted point count of $\mathcal{X}$ over $\mathbb{F}_q$ is defined as a sum: $\#_q(\mathcal{X}) := \sum_{x \in \mathcal{X}(\mathbb{F}_q)/\sim} \frac{1}{|\mathrm{Aut}(x)|}$ where $\mathcal{X}(\mathbb{F}_q)/\sim$ is the set of $\mathbb{F}_q$–isomorphism classes of $\mathbb{F}_q$–points of $\mathcal{X}$.

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When $K = \mathbb{F}_q$, the point counting measure $\{\mathcal{X}\} \mapsto \#_q(\mathcal{X})$ gives a well-defined ring homomorphism $\#_q : K_0(\mathrm{Stck}_{\mathbb{F}_q}) \to \mathbb{Q}$.

Motive/Point count of $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$ over finite fields

Theorem (Changho Han, Hunter Spink, Johannes Schmitt, J.)

The class $\{\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})\}$ *in* $K_0(\mathrm{Stck}_K)$ *for* $\mathrm{char}(K) \neq 2, 3$ *of the moduli stack for stable elliptic fibrations over the parameterized* $\mathbb{P}^1$ *with* $12n$ *nodal singular fibers and a section is equivalent to*

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$$\{\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})\} = \mathbb{L}^{10n+1} - \mathbb{L}^{10n-1}$$

and over an unparameterized $\mathbb{P}^1$ *with an odd degree* n

$$\{\mathcal{W}_{\mathrm{sf},n}\} = \mathbb{L}^{10n-2}$$

where $\mathbb{L} = \{\mathbb{A}^1\}$ *is the Lefschetz motive and* $\{\mathrm{PGL}_2\} = \mathbb{L}(\mathbb{L}^2 - 1)$.

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Then, by using $\#_q : K_0(\mathrm{Stck}_{\mathbb{F}_q}) \to \mathbb{Q}$ to count $\mathbb{F}_q$–points when $\mathrm{char}(\mathbb{F}_q) \neq 2, 3$, we acquire the weighted point counts of the moduli of stable elliptic surfaces over (un)parameterized $\mathbb{P}^1$.

Motive/Point count of $\mathrm{Rat}_n^{\gamma}(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$ over the finite fields

Theorem (work in progress; Dori Bejleri, Matthew Satriano, J.)

Let $\mathrm{char}(K) \neq 2, 3$. *Then the motive for the moduli stack of minimal elliptic surfaces over the parameterized* $\mathbb{P}^1$ *with a section and discriminant degree* $12n$ *having one additive singular fiber is*

Fiber type, j	$\{\mathrm{Rat}_n^{\gamma}(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})\}/\{\mathrm{PGL}_2\}$
$\mathrm{II},\ j = 0$	$\mathbb{L}^{10n-3} - \mathbb{L}^{4n-3}$
$\mathrm{III},\ j = 1728$	$\mathbb{L}^{10n-4} - \mathbb{L}^{6n-4}$
$\mathrm{IV},\ j = 0$	$\mathbb{L}^{10n-5} - \mathbb{L}^{4n-3}$
$\mathrm{I}^*_{k \geq 0},\ j = \infty$ $\mathrm{I}^*_0,\ j \neq 0, 1728$	$\mathbb{L}^{10n-6} - \mathbb{L}^{10n-7} + \mathbb{L}^{6n-3} - \mathbb{L}^{6n-4} - \mathbb{L}^{4n-3} + \mathbb{L}^{4n-2}$
$\mathrm{I}^*_0,\ j = 1728$	$\mathbb{L}^{10n-7} - \mathbb{L}^{6n-3}$
$\mathrm{I}^*_0,\ j = 0$	$\mathbb{L}^{10n-7} - \mathbb{L}^{4n-4}$
$\mathrm{IV}^*,\ j = 0$	$\mathbb{L}^{10n-8} - \mathbb{L}^{6n-4}$
$\mathrm{III}^*,\ j = 1728$	$\mathbb{L}^{10n-9} - \mathbb{L}^{4n-3}$
$\mathrm{II}^*,\ j = 0$	$\mathbb{L}^{10n-10} - \mathbb{L}^{6n-4}$

Counting elliptic curves over global fields

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Definition

Define the height of a point $\mathfrak{p}$ to be $ht(\mathfrak{p}) := |\mathcal{O}_K/\mathfrak{p}|$ the cardinality of the residue field $\mathcal{O}_K/\mathfrak{p}$.

As the discriminant divisor $\Delta(X)$ of degree $12n$ is an invariant of the choice of semistable model $f : X \to \mathbb{P}^1$, we count the number of isomorphism classes of semistable elliptic surfaces on the function field $\mathbb{F}_q(t)$ by the bounded height of $\Delta(X)$.

$$ht(\Delta(X)) = \prod_{i=1}^{\mu} |\mathbb{F}_q|^{k_i} = q^{k_1} \cdots q^{k_i} \cdots q^{k_\mu} = q^{k_1 + \cdots + k_\mu} = q^{12n}$$

Counting elliptic curves over global fields

Now consider $\mathcal{Z}_{\mathbb{F}_q(t)}(\mathcal{B}) =$

$$|\{\text{Semistable elliptic curves over } \mathbb{P}^1_{\mathbb{F}_q} \text{ with } 0 < ht(\Delta) \leq \mathcal{B}\}|.$$

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Theorem (Changho Han, J.-Y. Park)

The counting $\mathcal{Z}_{\mathbb{F}_q(t)}(\mathcal{B})$ by $ht(\Delta) = q^{12n} \leq \mathcal{B}$ satisfies

$$\mathcal{Z}_{\mathbb{F}_q(t)}(\mathcal{B}) \leq 2 \cdot \frac{(q^{11} - q^9)}{(q^{10} - 1)} \cdot \left(\mathcal{B}^{\frac{5}{6}} - 1\right)$$

which is an equality when $\mathcal{B} = q^{12n}$ for some $n \in \mathbb{N}$ implying that the acquired upper bound is a sharp enumeration, i.e., the upper bound is equal to the function at infinitely many values of $\mathcal{B} \in \mathbb{N}$.

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Note that we have the lower order term of zeroth order (constant).

Counting elliptic curves over global fields

Theorem (work in progress; Dori Bejleri, Matthew Satriano, J.)

If char$(\mathbb{F}_q) > 3$, then the function $\mathcal{Z}^{\gamma}_{\mathbb{F}_q(t)}(\mathcal{B})$, which counts the number of minimal elliptic curves with one additive singular fiber of γ type over the parameterized $\mathbb{P}^1_{\mathbb{F}_q}$ ordered by $0 < ht(\Delta) = q^{12n} \leq \mathcal{B}$, satisfies:

$$\mathcal{Z}^{\gamma}_{\mathbb{F}_q(t)}(\mathcal{B}) \leq a_q \cdot \mathcal{B}^{\frac{5}{6}} + b_q \cdot \mathcal{B}^{\frac{1}{3}} + c_q, \text{ when } \gamma = \mathrm{II}, \mathrm{II}^*, \mathrm{IV}, \mathrm{IV}^* \text{ or } \mathrm{I}_0^*$$

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which is an equality when $\mathcal{B} = q^{12n}$ with $n \in \mathbb{Z}_{\geq 1}$ implying that the upper bounds are sharp enumerations.

Heuristics for counting elliptic curves over number fields

Switching to the number field realm with $K = \mathbb{Q}$ and $\mathcal{O}_K = \mathbb{Z}$, one could choose the minimal integral Weierstrass model with the given discriminant divisor Δ which is already a number.

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In order to match the counting with the function field, we define the $ht(\Delta)$ to be the cardinality of ring of functions on subscheme $Spec(\mathbb{Z}/(\Delta))$. This leads to the following analogue $\mathcal{Z}_{\mathbb{Q}}(\mathcal{B})$.

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Conjecture (work in progress; Heuristic on $\mathcal{Z}_{\mathbb{Q}}(\mathcal{B})$)

The function $\mathcal{Z}_{\mathbb{Q}}(\mathcal{B})$, which counts the number of elliptic curves with at least one additive bad reduction over $\mathbb{Z}$ and $0 < ht(\Delta) \leq \mathcal{B}$, has the asymptotic behavior:

$$a\mathcal{B}^{\frac{5}{6}} + b\mathcal{B}^{\frac{1}{2}} + c\mathcal{B}^{\frac{1}{3}} + o(\mathcal{B}^{\frac{1}{3}})$$

with main leading term $\mathcal{O}\left(\mathcal{B}^{\frac{5}{6}}\right)$, secondary term $\mathcal{O}\left(\mathcal{B}^{\frac{1}{2}}\right)$, tertiary term $\mathcal{O}\left(\mathcal{B}^{\frac{1}{3}}\right)$ and lower order terms.

Thank you :)

Thank you to the **organizers & everyone** for listening!