Geometric Interpretation of Tate's Algorithm

$$
\begin{tikzcd}
 & & & & \mathcal{Y} = \varphi^* \overline{\mathcal{E}} \arrow[dl, "\pi"'] \arrow[dd, "h"] \arrow[r] & \overline{\mathcal{E}} \arrow[dd, "p"] \\
X \arrow[ddd, "f"] & X' \arrow[l, "\text{Weierstrass}"'] \arrow[ddd, "f'"] & \hat{X} \arrow[l, "\text{contraction}"'] \arrow[r, "\text{resolution}"] \arrow[ddd, "\hat{f}"] & Y \arrow[ddd, "g"] & & \\
 & & & & & \\
 & & & & \mathcal{C} \arrow[r, "\varphi"] \arrow[dl, "\rho"'] & \overline{\mathcal{M}}_{1,1} \arrow[d] \\
C \arrow[r, equal] & C \arrow[r, equal] & C \arrow[r, equal] & C \arrow[urr, dashed, "\psi"] \arrow[rr, "j"] & & \overline{M}_{1,1}
\end{tikzcd}
$$

Here f is a Weierstrass model, ψ is the associated weighted linear series viewed as a rational map to $\overline{\mathcal{M}}_{1,1}$, φ is a twisted morphism from the universal tuning stack $\mathcal{C}$ which induces a stable stack-like model $h : \mathcal{Y} \to \mathcal{C}$ where $g : Y \to C$ is the twisted model via coarse moduli maps, $\hat{f}$ is a resolution of Y, and f' is the relative minimal model obtained by contracting relative (-1)-curves.

Suppose that normalized base multiplicity $m = 3$. This occurs if and only if $(\nu(a_4), \nu(a_6)) = (1, \geq 2)$. Then $r = 12/\gcd(3, 12) = 4$ and $a = 3/\gcd(3, 12) = 1$. Thus the stabilizer of the twisted curve acts on the central fiber of the twisted model via the character $\mu_4 \to \mu_4$, $\zeta_4 \mapsto \zeta_4^{-1}$. In particular, the central fiber E of $\mathcal{Y}$ has $j = 1728$. The μ_4 action on E has two fixed points, and there is an orbit of size two with stabilizer $\mu_2 \subset \mu_4$. Let E_0 be the image of E in the twisted model Y. As E appears with multiplicity 4, Y has $\frac{1}{4}(-1, -1)$ quotient singularities at the images of the the fixed points and a $\frac{1}{2}(-1, -1)$ singularity at the image of the orbit of size two. Each of these singularities is resolved by a single blowup to obtain $\hat{X}$ with central fiber $4\tilde{E}_0 + E_1 + E_2 + E_3$ where E_i are the exceptional divisors of the resolution for $i = 1, 2, 3$ and $E_1^2 = E_2^2 = -4$ with $E_3^2 = -2$. Then $\tilde{E}_0$ is a (-1)-curve so it needs to be contracted. After this contraction E_2 becomes a (-1) curve and must also be contracted. Since E_i for $i = 1, 2, 3$ are incident and pairwise transverse after blowing down $\tilde{E}_0$, then the images of E_1 and E_2 must be tangent after blowing down E_3. Moreover, they are now (-2)-curves and the relatively minimal model for type III.

Tate's Algorithm via Twisted Morphisms

Theorem (Dori Bejleri–JP–Matthew Satriano; April 2024)

If $\mathrm{char}(K) \neq 2, 3$. Then the twisting condition (r, a) and the order of vanishing of j at $j = \infty$ determine the Kodaira fiber type, and (r, a) is in turn determined by $m = \min\{3\nu(a_4), 2\nu(a_6)\}$.

$\gamma : (\nu(a_4),\ \nu(a_6))$	*Reduction type with* $j \in \overline{M}_{1,1}$	$\Gamma : (r, a)$
$(\geq 1, 1)$	II *with* $j = 0$	$(6, 1)$
$(1, \geq 2)$	III *with* $j = 1728$	$(4, 1)$
$(\geq 2, 2)$	IV *with* $j = 0$	$(3, 1)$
$(2, 3)$	$\mathrm{I}^*_{k>0}$ *with* $j = \infty$ I^*_0 *with* $j \neq 0, 1728$	$(2, 1)$
$(\geq 3, 3)$	I^*_0 *with* $j = 0$	$(2, 1)$
$(2, \geq 4)$	I^*_0 *with* $j = 1728$	$(2, 1)$
$(\geq 3, 4)$	IV^* *with* $j = 0$	$(3, 2)$
$(3, \geq 5)$	III^* *with* $j = 1728$	$(4, 3)$
$(\geq 4, 5)$	II^* *with* $j = 0$	$(6, 5)$

Geometric Meaning of Height Moduli Framework

1. So one can run the resolution / minimal model. As these are algebraic surfaces it can be done over $\mathrm{char}(K) = p > 0$

2. A twisted morphism $\varphi : \mathcal{C} \to \overline{\mathcal{M}}_{1,1}$ with its twisting data Γ from the universal tuning stack $\mathcal{C}$ induces a stable stack-like model $h : \mathcal{Y} \to \mathcal{C}$ as unique pullback of the universal family $p : \overline{\mathcal{E}} \to \overline{\mathcal{M}}_{1,1}$. All the ensuing birational geometry is natural.

3. True purpose of a **representable classifying morphism** is in the <u>universal principle</u> that φ intrinsically contains all the algebro-geometric data necessary to uniquely determine a fibration with singular fibers. This is the very essence of the inner arithmetic of rational points on moduli stacks over K.

Geometric Meaning of Height Moduli Framework

1. Consider the fact that $\overline{\mathcal{M}}_{1,1}$ could have been any other algebraic stack $\mathcal{X}$ (such as $\overline{\mathcal{M}}_g$ or $\overline{\mathcal{A}}_g$) which is the representing object for certain moduli functor as the fine moduli stack together with the universal family. Then families of varieties (of algebraic curves or abelian varieties) with non-abelian stabilizers ($g \geq 2$) should have corresponding "Tate's algorithm", counting statements and so on.

2. Representable classifying morphisms as twisted morphisms $\varphi : \mathcal{C} \to \mathcal{X}$ uniquely determines families of varieties (of algebraic curves or abelian varieties) with non-abelian stabilizers ($g \geq 2$) which should have corresponding "Tate's algorithm", counting statements and so on.

3. Geometrizing $\mathcal{X}(K)$ is Height moduli space $\mathcal{M}_n(\mathcal{X}, \mathcal{L})$ and once we have a **space** (AG), we compute its **invariants** (AT) naturally having various kinds of **consequences** (NT).