# HEIGHT MODULI ON CYCLOTOMIC STACKS AND COUNTING ELLIPTIC CURVES OVER FUNCTION FIELDS

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*In memory of Yuri Manin, 1937–2023*

ABSTRACT. For proper stacks, unlike schemes, there is a distinction between rational and integral points; we show that this distinction exactly accounts for the main term and lower order terms appearing in counts of elliptic curves over function fields. More generally, using the theory of twisted stable maps and the stacky height functions recently introduced by Ellenberg, Zureick-Brown, and the third author, we construct finite type moduli spaces which parametrize rational points of fixed height on a large class of stacks, so-called cyclotomic stacks. As a by product, we obtain the Northcott property as well as a generalization of Tate's algorithm for cyclotomic stacks, and propose an answer to a question of Venkatesh.

## 1. INTRODUCTION

For many asymptotic counting problems in analytic number theory, it is of interest to determine the main leading term as well as the lower order terms. While the lower order terms are interesting theoretically, having good control of these lower order terms is often also important for numerical computations. A well-known example is the enumeration of cubic fields ordered by height of discriminant. The number of cubic number fields of height $\leq B$ for certain constants $a, b > 0$ is

$$aB - bB^{5/6} + o(B^{\frac{5}{6}})$$

The existence of the precise second main term of order $B^{5/6}$ was conjectured by Roberts in [Rob01] and also implicitly in an earlier paper of Datskovsky–Wright in [DW88]. This was proven by the remarkable works of Bhargava–Shankar–Tsimerman in [BST13] and also by Taniguchi–Thorne in [TT13] along the lines of establishing the Davenport–Heilbronn theorems [DH69, DH71] on cubic fields and 3-torsion in class groups of quadratic fields. In general, however, one has very little understanding of the origin of these lower order main terms (c.f. [VE10, §2.6]). In this regard, Venkatesh in [GGW21, Problem 5] asks the following question:

*What is the topological meaning of secondary terms appearing in asymptotic counts in number theory?*

In the case of elliptic curves over a function field $K = k(C)$ over $k = \mathbb{F}_q$, it is well known that the asymptotic counts of semi-stable elliptic curves, that is, the integral points of the moduli stack $\overline{\mathcal{M}}_{1,1}$, are governed by the geometry of the moduli space of morphisms $C \to \overline{\mathcal{M}}_{1,1}$ (see e.g. [HP19, BPS22]). We generalize this story to show that integral points of $\overline{\mathcal{M}}_{1,1}$ account for the main leading term in the asymptotic count of all elliptic curves over a function field, while rational points of $\overline{\mathcal{M}}_{1,1}$ that do not extend to integral points account for the lower order main terms, thus answering the question of Venkatesh in this case.

Namely, we establish the following sharp enumerations of elliptic curves over a global function field $K = \mathbb{F}_q(t)$ with precise second or third main terms. Recall that the height of the discriminant of an elliptic curve $E$ over $K$ is given by $ht(\Delta) := q^{\deg \Delta} = q^{12n}$ for some integer $n$ (sometimes called the Faltings height of $E$). Recall that the singular fibers of an elliptic fibration, and thus the possible types of bad reductions of $E$, were classified by Kodaira and Néron. Now define the counting function $\mathcal{N}(\mathbb{F}_q(t),\ \Theta,\ B)$ to be the number of elliptic curves over $K$ with $ht(\Delta) \leq B$ that have an additive reduction of Kodaira type $\Theta$ at a single place and at worst multiplicative reductions elsewhere.

**Theorem 1.1.** *Let char$(\mathbb{F}_q) \neq 2, 3$. Then there exist explicit rational functions $a_q, b_q, c_q, d_q \in \mathbb{Q}(q)$ depending only on $\Theta$ (see Theorem 8.9) such that*

$$\mathcal{N}(\mathbb{F}_q(t),\ \Theta,\ B) = a_q B^{\frac{5}{6}} + b_q B^{\frac{1}{3}} + c_q, \text{ if } \Theta = \mathrm{II}, \mathrm{II}^*, \mathrm{IV}, \mathrm{IV}^* \text{ or } \mathrm{I}_0^* \text{ with } j = 0;$$
$$\mathcal{N}(\mathbb{F}_q(t),\ \Theta,\ B) = a_q B^{\frac{5}{6}} + b_q B^{\frac{1}{2}} + c_q, \text{ if } \Theta = \mathrm{III}, \mathrm{III}^* \text{ or } \mathrm{I}_0^* \text{ with } j = 1728; \text{ and}$$
$$\mathcal{N}(\mathbb{F}_q(t),\ \Theta,\ B) = a_q B^{\frac{5}{6}} + b_q B^{\frac{1}{2}} + c_q B^{\frac{1}{3}} + d_q, \text{ if } \Theta = \mathrm{I}_{k>0}^* \text{ or } \mathrm{I}_0^* \text{ with } j \neq 0, 1728.$$

To prove Theorem 1.1, we construct a *height moduli space* $\mathcal{H}^{\Gamma}_{K,n}(\overline{\mathcal{M}}_{1,1})$ whose $\mathbb{F}_q$-points correspond to rational points of $\overline{\mathcal{M}}_{1,1}$ of height $n$ over the function field $K$ with prescribed local conditions $\Gamma$ (in this theorem $\Gamma$ encodes the local condition of having a single fiber of Kodaira type $\Theta$). We count the number of $\mathbb{F}_q$-points, and more generally compute the motive in the Grothendieck ring of stacks $K_0(\mathrm{Stck}_{\mathbb{F}_q})$ of $\mathcal{H}^{\Gamma}_{K,n}(\overline{\mathcal{M}}_{1,1})$. This theorem is an application of a much more general framework that we introduce in this paper which applies to a large class of moduli stacks[1] as well as to higher genus function fields. We conjecture that this geometric framework answers Venkatesh's question in many of these cases.

Fix $k$ a perfect field and $C/k$ a smooth proper geometrically connected curve with function field $K = k(C)$. For $X$ a projective scheme with ample line bundle $L$, the height of a point $P \in X(K)$ is simply the degree, measured with respect to $L$, of the unique extension $C \to X$. Thus for any integer $n$, we have a moduli space

$$\mathrm{Hom}_n(C, X)$$

of degree $n$ maps to $X$ whose points are identified with $K$-points of height $n$. Our goal is to generalize this construction when $X$ is replaced by an algebraic stack $\mathcal{X}$. In this paper we focus on the class of cyclotomic stacks, whose properties best resemble those of projective varieties. Introduced by Abramovich and Hassett in [AH11], these are algebraic stacks whose stabilizers are $\mu_r$ and equipped with a uniformizing line bundle $\mathcal{L}$ (see Definition 2.12) which plays the role of an ample line bundle. This includes familiar examples such as weighted projective stacks, various fine modular curves and in particular, $\overline{\mathcal{M}}_{1,1} \cong \mathcal{P}(4,6)$ over $\mathbb{Z}\left[\frac{1}{6}\right]$.

One of the subtleties for proper algebraic stacks is that not every rational point $C \dashrightarrow \mathcal{X}$ extends to an integral point $C \to \mathcal{X}$. Instead, a rational point always extends to a *twisted map* $\mathcal{C} \to \mathcal{X}$ from an orbifold curve $\mathcal{C}$ with coarse moduli space $C$ (see Section 2.3). This observation was one of the key insights of [ESZB21] for

[1] See Section 6 for moduli stacks of torsors or algebraic curves for which this framework applies.

defining the *stacky height* of a rational point of $P \in \mathcal{X}(K)$ (see Section 2.1). As in the case of schemes, twisted maps form a moduli space $\mathcal{H}_d^{\Gamma}(\mathcal{X}, \mathcal{L})$ where

$$\Gamma = \{(r_1, a_1), \dots, (r_s, a_s)\}$$

encodes the orbifold structure of $\mathcal{C}$ as well as the stabilizer action on the fiber of $\mathcal{L}|_{\mathcal{C}}$ (Definition 2.28) and $d$ is the degree. On the other hand, given $(\mathcal{X}, \mathcal{L})$, the aforementioned stacky height $\mathrm{ht}_{\mathcal{L}}(P)$ decomposes as $\mathrm{ht}^{st}_{\mathcal{L}}(P) + \sum_v \delta_{\mathcal{L},v}(P)$, where $\delta_{\mathcal{L},v}(P)$ are local contributions coming from finitely many places $v$ of $K$ and $\mathrm{ht}^{st}_{\mathcal{L}}(P)$ is the so-called stable height, which is stable under base change. We show (Theorem 3.3) that we can identify the terms in the stacky height with the data $d, \Gamma$ for a twisted map:

$$d = \mathrm{ht}^{st}_{\mathcal{L}}(P), \quad \delta_{\mathcal{L},v_i}(P) = \frac{a_i}{r_i}.$$

Our main theorem is that there is a height moduli space $\mathcal{M}_n(\mathcal{X}, \mathcal{L})$ parametrizing all rational points of stacky height $n$ and that the spaces of twisted maps yield a stratification of $\mathcal{M}_n(\mathcal{X}, \mathcal{L})$ corresponding to fixing the local contributions to the stacky height.

**Theorem 1.2.** *Let $(\mathcal{X}, \mathcal{L})$ be a proper polarized cyclotomic stack over a perfect field $k$. Fix a smooth projective curve $C/k$ with function field $K = k(C)$ and $n, d \in \mathbb{Q}_{\geq 0}$.*

*(1) There exists a separated Deligne–Mumford stack $\mathcal{M}_{n,C}(\mathcal{X}, \mathcal{L})$ of finite type over $k$ with a quasi-projective coarse space and a canonical bijection of $k$-points*

$$\mathcal{M}_{n,C}(\mathcal{X}, \mathcal{L})(k) = \{P \in \mathcal{X}(K) \mid \mathrm{ht}_{\mathcal{L}}(P) = n\}.$$

*(2) There is a finite locally closed stratification*

$$\bigsqcup_{\Gamma, d} \mathcal{H}^{\Gamma}_{d,C}(\mathcal{X}, \mathcal{L})/S_{\Gamma} \to \mathcal{M}_{n,C}(\mathcal{X}, \mathcal{L})$$

*where the union runs over all possible admissible local conditions (see Definition 2.28)*

$$\Gamma = (\{r_1, a_1\}, \dots, \{r_s, a_s\})$$

*and degrees $d$ for a twisted map to $(\mathcal{X}, \mathcal{L})$ such that*

$$n = d + \sum_{i=1}^{s} \frac{a_i}{r_i}$$

*and $S_{\Gamma}$ is a subgroup of the symmetric group on $s$ letters (Definition 2.31).*

*(3) Under the bijection in part (1), each $k$-point of $\mathcal{H}^{\Gamma}_{d,C}(\mathcal{X}, \mathcal{L})/S_{\Gamma}$ corresponds to a $K$-point $P$ with the stable height $\mathrm{ht}^{st}_{\mathcal{L}}(P) = d$ and a set of local contributions $\left\{\frac{a_i}{r_i}\right\}_{i=1}^{s}$.*

We obtain Theorem 1.1 from Theorem 1.2 as follows. When $\mathcal{X} = \overline{\mathcal{M}}_{1,1}$ with $\mathcal{L}$ the Hodge line bundle and $P \in \overline{\mathcal{M}}_{1,1}(K)$ a rational point corresponding to an elliptic curve $E/K$, we show in Theorem 7.12 that Tate's algorithm in [Tat75] extends to a correspondence between the conditions $\Gamma$ of the twisted map and the Kodaira fiber types $\Theta$ of the minimal model of $E$. Thus the asymptotic counts in Theorem 1.1 are the counts of $\mathbb{F}_q$-points of the stratum $\mathcal{H}^{\Gamma}_{d,\mathbb{P}^1}(\overline{\mathcal{M}}_{1,1}, \mathcal{L})$ where $\Gamma$ corresponds to the Kodaira fiber type $\Theta$ under our generalized Tate algorithm. Notably, in the asymptotic count of elliptic curves over $K = \mathbb{F}_q(t)$, the point count of semi-stable elliptic surfaces, i.e. of the space $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$, only contributes to the *leading term* $B^{5/6}$ with *constant lower order term* (c.f. [dJ02, HP19, BPS22]); in contrast, the *non-constant lower order terms* $B^{1/2}$ or $B^{1/3}$ we obtain in Theorem 1.1 arise from the point counts on strata $\mathcal{H}^{\Gamma}_{d,\mathbb{P}^1}$ which parametrize rational points $\overline{\mathcal{M}}_{1,1}(K)$ that do *not* extend to integral points. Thus, in the context of elliptic curves over global function fields, an answer to Venkatesh's question is that the lower order main terms correspond geometrically to the strata $\mathcal{H}^{\Gamma}_{d,\mathbb{P}^1}$; moreover, it is precisely the distinction between integral points and rational points on $\overline{\mathcal{M}}_{1,1}$ that accounts for the difference between the leading term and lower order main terms in the point counts over finite fields.

We conjecture that the motives of the moduli spaces $\mathcal{M}_{n,C}(\mathcal{X}, \mathcal{L})$ and $\mathcal{H}^{\Gamma}_{d,C}(\mathcal{X}, \mathcal{L})$ should satisfy a type of motivic stabilization as in [VW15] and that the asymptotic motives of $\mathcal{H}^{\Gamma}_{d,C}(\mathcal{X}, \mathcal{L})$ determine the lower order terms in the point counts over $\mathbb{F}_q$. We state this precisely in the special case of weighted projective stacks.

**Conjecture 1.3.** Let $(\mathcal{X}, \mathcal{L}) = (\mathcal{P}(\vec{\lambda}), \mathcal{O}_{\mathcal{P}(\vec{\lambda})}(1))$ be a weighted projective stack. Then the generating series for the classes $\{\mathcal{M}_{n,C}(\mathcal{X}, \mathcal{L})\}$ and $\{\mathcal{H}^{\Gamma}_{d,C}(\mathcal{X}, \mathcal{L})\}$ in the Grothendieck ring of stacks $K_0(\mathrm{Stck}_k)$ is a rational function depending explicitly on the motivic zeta function of $C$.

Moreover, as the spaces $\mathcal{H}^{\Gamma}_{d,C}(\mathcal{X}, \mathcal{L})$ are twisted analogues of the mapping stacks $\mathrm{Hom}_n(C, \mathcal{X})$, it is natural to ask if the cohomology of these spaces stabilizes as the degree goes to infinity.

**Conjecture 1.4.** The cohomology of the spaces $\mathcal{H}^{\Gamma}_{d,C}(\mathcal{X}, \mathcal{L})$ equipped with the action of $S_\Gamma$ exhibits representation stability as $d \to \infty$ (see e.g. [CEF14]).

These conjectures should yield an answer to the Venkatesh's question in many other cases beyond $\overline{\mathcal{M}}_{1,1}$.

We prove Theorem 1.2 by reducing to the case where $\mathcal{X}$ is a weighted projective stack $\mathcal{P}(\vec{\lambda}) := \mathcal{P}(\lambda_1, \dots, \lambda_n)$ and $\mathcal{L} = \mathcal{O}(1)$. In this case, we obtain even more precise information in Theorem 4.28: we construct $\mathcal{M}_{n,C}(\mathcal{P}(\vec{\lambda}), \mathcal{O}(1))$ as a moduli space of $\vec{\lambda}$-weighted linear series $(L, s_0, \dots, s_n)$ on the curve $C$. Here $L$ is a line bundle and $s_i \in H^0(C, L^{\otimes \lambda_i})$. We show in Theorem 3.3 that there is a correspondence between the twisting conditions $(r, a)$ at a point $x \in C$ and the order of vanishing of the sections $s_i$ in a weighted linear series. This correspondence induces a bijection between twisted maps and weighted linear series satisfying a *minimality condition* on the orders of vanishing of $s_i$ (Definition 3.2). Moreover, this correspondence is a generalization of Tate's algorithm to weighted projective stacks (Section 7). Finally, to prove Theorem 1.2 for weighted projective stacks, we construct moduli spaces of minimal weighted linear series and of weighted linear series with vanishing conditions (Proposition 4.15 & Theorem 4.28) and identify them with the height moduli space $\mathcal{M}_{n,C}$ and twisted maps strata $\mathcal{H}^{\Gamma}_{d,C}$ respectively (Section 5).

Specializing to the case of $\overline{\mathcal{M}}_{1,1} \cong \mathcal{P}(4,6)$, a weighted linear series on $\mathbb{P}^1$ of height $n$ consists of Weierstrass coefficients $a_4 \in H^0(\mathbb{P}^1, \mathcal{O}(4n))$ and $a_6 \in H^0(\mathbb{P}^1, \mathcal{O}(6n))$

and the orders of vanishing at a point can be encoded in a vector $\gamma = (\nu(a_4), \nu(a_6))$ which corresponds by Tate's algorithm to some twisting data $\Gamma = (r, a)$. The moduli space of such weighted linear series is denoted $\mathcal{W}^{\gamma}_{n,\mathbb{P}^1}$ and is isomorphic to the stratum $\mathcal{H}^{\Gamma}_{d,\mathbb{P}^1}$. Tate's algorithm and the Grothendieck ring computations in this case are summarized in the theorems below.

**Theorem 1.5.** *If* $\mathrm{char}(K) \neq 2, 3$. *Then Tate's algorithm via twisted maps is*

| $\gamma : \big(\nu(a_4),\ \nu(a_6)\big)$ | *Reduction type with* $j \in \overline{M}_{1,1}$ | $\Gamma : (r, a)$ |
|---|---|---|
| $(\geq 1, 1)$ | II *with* $j = 0$ | $(6, 1)$ |
| $(1, \geq 2)$ | III *with* $j = 1728$ | $(4, 1)$ |
| $(\geq 2, 2)$ | IV *with* $j = 0$ | $(6, 2)$ |
| $(2, 3)$ | $\mathrm{I}^*_{k>0}$ *with* $j = \infty$<br>$\mathrm{I}^*_0$ *with* $j \neq 0, 1728$ | $(2, 1)$ |
| $(\geq 3, 3)$ | $\mathrm{I}^*_0$ *with* $j = 0$ | $(6, 3)$ |
| $(2, \geq 4)$ | $\mathrm{I}^*_0$ *with* $j = 1728$ | $(4, 2)$ |
| $(\geq 3, 4)$ | $\mathrm{IV}^*$ *with* $j = 0$ | $(6, 4)$ |
| $(3, \geq 5)$ | $\mathrm{III}^*$ *with* $j = 1728$ | $(4, 3)$ |
| $(\geq 4, 5)$ | $\mathrm{II}^*$ *with* $j = 0$ | $(6, 5)$ |

**Theorem 1.6.** *If* $\mathrm{char}(K) \neq 2, 3$. *Then motives* $\{\mathcal{W}^{\gamma}_{n,\mathbb{P}^1}\} \in K_0(\mathrm{Stck}_K)$ *of moduli stacks of elliptic surfaces with a specified Kodaira fiber is*

| *Reduction type with* $j \in \overline{M}_{1,1}$ | *Motivic class* $\{\mathcal{W}^{\gamma}_{n,\mathbb{P}^1}\} \in K_0(\mathrm{Stck}_K)$ |
|---|---|
| II *with* $j = 0$ | $\mathbb{L}^{10n} - \mathbb{L}^{10n-2} - \mathbb{L}^{4n+1} + \mathbb{L}^{4n-1}$ |
| III *with* $j = 1728$ | $\mathbb{L}^{10n-1} - \mathbb{L}^{10n-3} - \mathbb{L}^{6n} + \mathbb{L}^{6n-2}$ |
| IV *with* $j = 0$ | $\mathbb{L}^{10n-2} - \mathbb{L}^{10n-4} - \mathbb{L}^{4n} + \mathbb{L}^{4n-2}$ |
| $\mathrm{I}^*_{k>0}$ *with* $j = \infty$<br>$\mathrm{I}^*_0$ *with* $j \neq 0, 1728$ | $\mathbb{L}^{10n-3} - \mathbb{L}^{10n-4} - \mathbb{L}^{10n-5} + \mathbb{L}^{10n-6} - \mathbb{L}^{6n-1} + \mathbb{L}^{6n-2}$<br>$+\mathbb{L}^{6n-3} - \mathbb{L}^{6n-4} - \mathbb{L}^{4n} + \mathbb{L}^{4n-1} + \mathbb{L}^{4n-2} - \mathbb{L}^{4n-3}$ |
| $\mathrm{I}^*_0$ *with* $j = 0$ | $\mathbb{L}^{10n-4} - \mathbb{L}^{10n-6} - \mathbb{L}^{4n-1} + \mathbb{L}^{4n-3}$ |
| $\mathrm{I}^*_0$ *with* $j = 1728$ | $\mathbb{L}^{10n-4} - \mathbb{L}^{10n-6} - \mathbb{L}^{6n-2} + \mathbb{L}^{6n-4}$ |
| $\mathrm{IV}^*$ *with* $j = 0$ | $\mathbb{L}^{10n-5} - \mathbb{L}^{10n-7} - \mathbb{L}^{4n-1} + \mathbb{L}^{4n-3}$ |
| $\mathrm{III}^*$ *with* $j = 1728$ | $\mathbb{L}^{10n-6} - \mathbb{L}^{10n-8} - \mathbb{L}^{6n-3} + \mathbb{L}^{6n-5}$ |
| $\mathrm{II}^*$ *with* $j = 0$ | $\mathbb{L}^{10n-7} - \mathbb{L}^{10n-9} - \mathbb{L}^{4n-2} + \mathbb{L}^{4n-4}$ |

In Appendix A, we show the result of applying the same analysis to the moduli space $\overline{\mathcal{M}}_1(2) \cong \mathcal{P}(2,4)$ over $\mathbb{Z}\left[\frac{1}{2}\right]$ leading to similar correspondence & counting results for elliptic curves over $K = \mathbb{F}_q(t)$ with level-2 structure (see §6.3).

1.1. **Relation to other work.** There has been a lot of recent activity in counting points on weighted projective stacks [BGS20, Dar21, SS22], computing asymptotics counts of elliptic curves with additive reduction [CS20, CJ20, Phi22a, Phi22b], and stacky approaches to heights [DY22]. We intend to explore the relationship between our method and theirs in these papers in the future.

1.2. **Outline of the paper.** In Section 2, we discuss heights on cyclotomic stacks from the point of view of twisted maps and construct the moduli space $\mathcal{H}^{\Gamma}_{d}(\mathcal{X}, \mathcal{L})$. In Section 3, we show the bijection between twisted maps and minimal weighted linear series. In Section 4, we construct the moduli spaces of minimal weighted linear series. In Section 5, we prove Theorem 1.2. In Section 6, we give several examples of height moduli on cyclotomic stacks. In Section 7, we state Tate's algorithm via twisting data and interpret $\mathcal{H}^{\Gamma}_{d,C}(\overline{\mathcal{M}}_{1,1})$ as moduli of elliptic surfaces with specified Kodaira fibers. In Section 8, we compute the classes of some height moduli in the Grothendieck ring of stacks and prove Theorems 1.1 and 1.6. In the Appendix A, we include the results of the analogous computations for $\overline{\mathcal{M}}_1(2)$.

## 2. Heights on cyclotomic stacks and twisted maps

We review the definition of heights on stacks from [ESZB21] and specialize it to the class of cyclotomic stacks. We assume that $\mathcal{X}$ is normal throughout.

2.1. **Heights on stacks.** In [ESZB21], the authors introduced height functions on stacks and used them to give point-counting conjectures generalizing the Batyrev–Manin and Malle Conjectures, see [ESZB21, Conjecture 4.14]. When the stack is a scheme, these stacky heights recover the usual Weil heights. On the other hand, when the stack is $\mathcal{B}G$ and the rational point $\mathcal{B}G(K)$ corresponds to a Galois $G$-extension $L/K$, the heights recover the discriminant of the extension.

Unlike the schemes, the Weil height machine fails for stacks. One can see this by considering the universal $n$-torsion line bundle $\mathcal{L}$ on $\mathcal{B}\mu_n$; if the Weil height machine were to hold, then $n$ times the height of $\mathcal{L}$ would be trivial. Also, unlike the case of schemes, one cannot define heights via embeddings into projective space since all stacks that admit such embeddings are necessarily schemes. Instead the definition of heights on stacks is given by extending the rational point to a stacky curve, called a tuning stack.

**Definition 2.1** ([ESZB21, Definition 2.1]). Let $C$ be a smooth proper curve over $K$ and let $p \colon \mathcal{X} \to C$ be a proper map from a normal Artin stack $\mathcal{X}$ with finite diagonal. Let $x \in \mathcal{X}(K)$ be a rational point. A *tuning stack* for $x$ is a diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[rr, bend left, "x"] \arrow[dr] & \mathcal{C} \arrow[r, "\overline{x}"] \arrow[d, "\pi"] & \mathcal{X} \arrow[dl, "p"] \\ & C & \end{tikzcd}$$

where $\mathcal{C}$ is a normal Artin stack with finite diagonal and $\pi$ is a birational coarse space map.

A morphism of tuning stacks $(\mathcal{C}', \pi', \overline{x}') \to (\mathcal{C}, \pi, \overline{x})$ is a map $f : \mathcal{C}' \to \mathcal{C}$ such that $\pi \circ f = \pi'$ and $\overline{x} \circ f = \overline{x}'$. A tuning stack is said to be *universal* if it is terminal among all tuning stacks.

**Remark 2.2.** In [ESZB21, Corollary 2.6], it is shown that a tuning stack $(\mathcal{C}, \pi, \overline{x})$ is universal if and only if $\overline{x}$ is representable.

Heights of a rational point on a stack are then given as follows.

**Definition 2.3** ([ESZB21, Definition 2.11])**.** With hypotheses as in Definition 2.1, if $\mathcal{V}$ is a vector bundle on $\mathcal{X}$ and $x \in \mathcal{X}(K)$, the *height of $x$ with respect to $\mathcal{V}$* is defined as

$$\mathrm{ht}_{\mathcal{V}}(x) := -\deg(\pi_* \overline{x}^* \mathcal{V}^\vee)$$

for any choice of tuning stack $(\mathcal{C}, \pi, \overline{x})$.

Notice that the height $\mathrm{ht}_{\mathcal{V}}(x)$ is defined via pullback then pushforward before taking degree. One may alternatively consider the height function obtained by pullback and then taking degree directly. This is known as the stable height:

**Definition 2.4** ([ESZB21, Definition 2.12])**.** With hypotheses as in Definition 2.1, if $\mathcal{V}$ is a vector bundle on $\mathcal{X}$ and $x \in \mathcal{X}(K)$, the *stable height of $x$ with respect to $\mathcal{V}$* is defined as

$$\mathrm{ht}^{\mathrm{st}}_{\mathcal{V}}(x) := -\deg_{\mathcal{C}} \overline{x}^* \mathcal{V}^\vee$$

for any choice of tuning stack $(\mathcal{C}, \pi, \overline{x})$.

**Remark 2.5.** The height and stable height functions are shown to be independent of the choice of tuning stack in [ESZB21, Proposition 2.13].

When $X$ is a scheme, we can take $\mathcal{C} = C$ and so stable height and height are the same. More generally, height agrees with stable height whenever the vector bundle $\mathcal{V}$ is pulled back from a vector bundle on a scheme.

**Remark 2.6.** The stable height $\mathrm{ht}^{\mathrm{st}}_{\mathcal{V}}(x)$ is stable under base change as opposed to the height $\mathrm{ht}_{\mathcal{V}}(x)$ (c.f. [ESZB21, Proposition 2.14]).

Later, we compute the stable height as the coarse map degree plus the local contributions from the ramified base changes in terms of $r, a$ (see Definition 3.1)

2.2. **Cyclotomic stacks and weighted projective stacks.** Here we review the basic definitions of cyclotomic stacks following [AH11, Section 2]. This is a class of stacks whose properties best resemble those of projective varieties. The central example is that of a *weighted projective stack*.

**Definition 2.7.** Let $\vec{\lambda} = (\lambda_0, \dots, \lambda_N) \in \mathbb{Z}^{N+1}_{\geq 1}$ be a vector of $N+1$ positive integers. Consider the affine space $U_{\vec{\lambda}} = \mathbb{A}^{N+1}_{x_0,\dots,x_N}$ endowed with the action of $\mathbb{G}_m$ with weights $\vec{\lambda}$, i.e. an element $\zeta \in \mathbb{G}_m$ acts by

$$\zeta \cdot (x_0, \dots, x_N) = (\zeta^{\lambda_0} x_0, \dots, \zeta^{\lambda_N} x_N)\,. \tag{1}$$

The $N$-dimensional weighted projective stack $\mathcal{P}(\vec{\lambda})$ is then defined as the quotient stack

$$\mathcal{P}(\vec{\lambda}) = \left[(U_{\vec{\lambda}} \smallsetminus \{0\})/\mathbb{G}_m\right].$$

**Remark 2.8.** When we wish to emphasize a base field $k$ of definition of $\mathcal{P}(\vec{\lambda})$, we use the notation $\mathcal{P}_k(\vec{\lambda})$. The stack $\mathcal{P}(\vec{\lambda})$ is a smooth and proper tame Artin stack. It is Deligne–Mumford if and only if all weights $\lambda_i$ are prime to the characteristic. For example $\mathcal{P}(1,p)$ is not Deligne–Mumford in characteristic $p$ since it has a point with automorphism group $\mu_p$ which is not formally unramified. When $\mathcal{P}(\vec{\lambda})$ is Deligne–Mumford, it is an orbifold if and only if $\gcd(\lambda_0,\ldots,\lambda_N)=1$. More generally, the natural map

$$\mathcal{P}(d\lambda_0,\ldots,d\lambda_N)\to\mathcal{P}(\lambda_0,\ldots,\lambda_N)$$

is a $\mu_d$-gerbe.

The natural morphism $U_{\vec{\lambda}}\smallsetminus 0\to\mathcal{P}(\vec{\lambda})$ is the total space of the *tautological line bundle* $\mathcal{O}_{\mathcal{P}(\vec{\lambda})}(-1)$ on $\mathcal{P}(\vec{\lambda})$. As in the classical case, we denote by $\mathcal{O}_{\mathcal{P}(\vec{\lambda})}(1)$ the dual of this line bundle.

**Definition 2.9.** A $\vec{\lambda}$*–weighted linear series* on a scheme $B$ is the data of a line bundle $L$ and sections

$$s_i:\mathcal{O}_B\to L^{\lambda_i}.$$

The *set theoretic base locus* of $(L,s_0,\ldots,s_N)$ is the reduced closed subscheme $Z\subset B$ of points $b\in B$ where $s_j(b)=0$ for all $j=0,\ldots,N$. A point $b\in Z$ will be called an *indeterminacy* of the weighted linear series.

**Proposition 2.10.** [AH11, Lemma 2.1.3] *The stack $\mathcal{P}(\vec{\lambda})$ with universal line bundle $\mathcal{O}_{\mathcal{P}(\vec{\lambda})}(1)$ is equivalent to the stack of $\vec{\lambda}$–weighted linear series with empty base locus.*

**Remark 2.11.** The open embedding $\mathcal{P}(\vec{\lambda})\subset[U_{\vec{\lambda}}/\mathbb{G}_m]$ corresponds to the inclusion of basepoint-free linear series into the stack of all $\vec{\lambda}$-weighted linear series.

A weighted projective stack is an example of the following.

**Definition 2.12.** A separated algebraic stack $\mathcal{X}$ of finite type over a field $k$ is *cyclotomic* if the stabilizer $\mathrm{Aut}(x)$ of each geometric point $x:\operatorname{Spec}K\to\mathcal{X}$ is isomorphic to $\mu_r$ for some $r$. A *uniformizing line bundle* on $\mathcal{X}$ is a line bundle $\mathcal{L}$ such that for each geometric point $x:\operatorname{Spec}K\to\mathcal{X}$, the natural map

$$\mathrm{Aut}(x)\to\mathrm{Aut}(\mathcal{L}|_x)$$

is injective.

We denote the coarse moduli space of $\mathcal{X}$ by $\pi:\mathcal{X}\to X$. By [AH11, Proposition 2.3.10], the condition that $\mathcal{L}$ is uniformizing if and only if the map $\mathcal{X}\to\mathcal{B}\mathbb{G}_m$ classyfing $\mathcal{L}$ is representable. Moreover, by [AH11, Lemma 2.3.7], there exists an $M$ such that $\mathcal{L}^{\otimes M}\cong\pi^*L$ for some line bundle $L$ on $X$.

**Definition 2.13.** A uniformizing line bundle $\mathcal{L}$ is a *polarizing line bundle* if $\mathcal{L}^{\otimes M}\cong\pi^*L$ where for some $M$ where $L$ is an ample line bundle on $X$. We say that the pair $(\mathcal{X},\mathcal{L})$ is a *polarized cyclotomic stack*.

**Example 2.14.** If $\mathcal{X}\subset\mathcal{P}(\vec{\lambda})$ is a locally closed substack, then the pullback $\mathcal{O}_{\mathcal{P}(\vec{\lambda})}(1)|_{\mathcal{X}}$ is polarizing. More generally, the same is true if $\mathcal{X}\to\mathcal{P}(\vec{\lambda})$ is representable quasi-finite and $\mathcal{X}$ is Noetherian.

**Proposition 2.15.** [AH11, Propositions 2.4.2 & 2.4.3, Corollary 2.4.4] *Let $(\mathcal{X},\mathcal{L})$ be a polarized cyclotomic stack and suppose that $\mathcal{X}$ is proper. Then there exists a weighted projective stack $\mathcal{P}(\vec{\lambda})$ and a closed embedding $\mathcal{X}\subset\mathcal{P}(\vec{\lambda})$ such that*

$$\mathcal{L}\cong\mathcal{O}_{\mathcal{P}(\vec{\lambda})}(1)|_{\mathcal{X}}.$$

A key observation we will exploit in the sequel is that Proposition 2.15 reduces questions about the height $\mathrm{ht}_{\mathcal{L}}$ on a polarized cyclotomic stack $(\mathcal{X},\mathcal{L})$ to the case of $\mathrm{ht}_{\mathcal{O}(1)}$ on $\mathcal{P}(\vec{\lambda})$. Compare this to the fact that the Weil height with respect to a very ample line bundle on a projective variety is determined by the naive height on $\mathbb{P}^N$.

2.3. **Twisted maps to cyclotomic stacks.** In Section 3, we will see that the universal tuning stack of a cyclotomic stack is always a *twisted curve*. In this section we review some background on twisted curves and construct the moduli of twisted curves on cyclotomic stacks.

**Definition 2.16.** A stacky genus $g$ curve $\mathcal{C}$ is a 1-dimensional smooth proper tame Artin stack whose coarse moduli space is isomorphic to an irreducible projective genus-$g$ curve $\pi:\mathcal{C}\to C$. A twisted genus $g$ curve is a stacky genus $g$ curve $\mathcal{C}$ which is generically a scheme and such that for each geometric point $\bar{x}\to C$, there exists a non-negative integer $r$ such that

$$\mathrm{Spec}(\mathcal{O}_{C,\bar{x}}^{sh})\times_C\mathcal{C}\cong\left[\mathrm{Spec}\,k(\bar{x})[t]^{sh}/\mu_r\right]$$

where $\mathcal{O}_{C,\bar{X}}^{sh}$ is the strict Henselization at $\bar{x}$ and $r$ acts by $t\mapsto\zeta t$ for $\zeta\in\mu_r$.

Note that a twisted curve is a root stack (c.f. [Ols16, §10.3]). In particular, $\mathcal{C}$ is uniquely determined up to an isomorphism by the points $q_1,\dots,q_s\in C_g$ and their inertia group orders $r_1,\dots,r_s$. The reduced preimages $\Sigma_i:=\pi^{-1}(q_i)_{red}$ are residual $\mu_{r_i}$-gerbes at the points $p_i$ lying over $q_i$.

**Remark 2.17.** In [AOV11], a more general definition of twisted curve is used where $C$ is allowed to have at worst nodal singularities. Our definition is exactly the ones of *loc. cit.* which are smooth.

**Definition 2.18.** A family of twisted curves of genus $g$ with inertia groups $\mu_{r_1},\dots,\mu_{r_s}$ over $B$ is a tuple $(\mathcal{C}\to B,\Sigma_i)$ where

(1) $\mathcal{C}\to B$ is a flat and proper morphism with $\mathcal{C}_b$ an orbifold curve of genus $g$ for all $b\in B$,
(2) $\Sigma_i\subset\mathcal{C}$ closed substacks such that the composition $\Sigma_i\to B$ is a $\mu_{r_i}$-gerbe, and
(3) $\mathcal{C}\smallsetminus\bigsqcup\Sigma_i\to B$ is representable.

A family of twisted curves is an object as above for some $g$ and tuple $(r_1,\dots,r_s)$.

Now we can define the stack of twisted maps.

**Definition 2.19.** Let $\mathcal{X}$ be a proper and tame Artin stack. A family of twisted maps to $\mathcal{X}$ over $B$ is a tuple $(\mathcal{C}\to B,\Sigma_i,f)$ where $(\mathcal{C},\Sigma_i)\to B$ is a family of twisted curves and $f:\mathcal{C}\to\mathcal{X}$ is a representable morphism.

When $\mathcal{X} = \mathcal{P}(\lambda_j)$, a map $f : \mathcal{C} \to \mathcal{P}(\lambda_j)$ is the same as the data of a line bundle $\mathcal{L} \in \mathrm{Pic}(\mathcal{C})$ and sections $s_j \in \mathcal{L}^{\lambda_j}$ which don't simultaneously vanish. Since $\mathcal{P}(\lambda_j) \to \mathcal{B}\mathbb{G}_m$ is representable, then $f$ is representable if and only if the map $\mathcal{C} \to \mathcal{B}\mathbb{G}_m$ defined by the line bundle $\mathcal{L}$ is representable. In order to describe line bundles on $\mathcal{C}$, we consider the coarse map $\pi : \mathcal{C} \to C$ and denote as above $q_1, \ldots, q_s \in C$ the points where $\pi$ is not an isomorphism with $p_i$ the point of $\mathcal{C}$ lying over $q_i$ and with stabilizer $\mu_{r_i}$.

**Proposition 2.20.** *The pullback map $\pi^*$ induces an injection $\pi^* : \mathrm{Pic}(C) \to \mathrm{Pic}(\mathcal{C})$ and a short exact sequence*

$$0 \to \mathrm{Pic}(C) \to \mathrm{Pic}(\mathcal{C}) \to \bigoplus_i^s \mathbb{Z}/r_i\mathbb{Z} \to 0 \tag{2}$$

*Moreover,* $\mathrm{Pic}(\mathcal{C})$ *is generated by* $\mathrm{Pic}(C)$ *and $\mathcal{O}_{\mathcal{C}}(p_i)$ for $i = 1, \ldots, s$ and $\mathcal{O}_{\mathcal{C}}(p_i)$ map to generators of the cokernel of $\pi^*$.*

*Proof.* Consider the Leray spectral sequence for $R\pi_*\mathbb{G}_{m,\mathcal{C}}$. First since $\pi_*\mathcal{O}_{\mathcal{C}} = \mathcal{O}_C$, we conclude that $\pi_*\mathbb{G}_{m,\mathcal{C}} = \mathbb{G}_{m,C}$.

Next we compute $R^1\pi_*\mathbb{G}_{m,\mathcal{C}}$. The stalk $(R^1\pi_*\mathbb{G}_{m,\mathcal{C}})_x = 0$ for any point $x$ where $\pi$ is an isomorphism. Thus $R^1\pi_*\mathbb{G}_{m,\mathcal{C}}$ is a direct sum of local contributions over stacky points. By flat base change, it suffices to compute it in the special case that $\mathcal{C} = [\mathrm{Spec} A/\mu_r]$ where $A$ is a 1 dimensional regular Henselian local ring. In this case, $R^1\pi_*\mathbb{G}_{m,\mathcal{C}}$ is simply $\mathrm{Pic}(\mathcal{C})$ but $\mathrm{Pic}(\mathrm{Spec} A)$ is trivial so this is the same as the character group of $\mu_r$ which we can identify with $\mathbb{Z}/r\mathbb{Z}$.

Putting this together with the exact sequence coming from low degree terms of the Leray spectral sequence for $R\pi_*\mathbb{G}_{m,\mathcal{C}}$, we get

$$0 \to \mathrm{Pic}(C) \to \mathrm{Pic}(\mathcal{C}) \to \bigoplus_{i=1}^s \mathbb{Z}/r_i\mathbb{Z}$$

where the last map can be identified as the restriction $\mathrm{Pic}(\mathcal{C}) \to \mathrm{Pic}(\mathcal{B}\mu_{r_i})$.

To prove exactness on the right, it suffices to construct a line bundle whose character at the $i^{th}$ marked gerbe $p_i$ generates $\mathbb{Z}/r_i\mathbb{Z}$ and is trivial away from $p_i$. The line bundle $\mathcal{O}_{\mathcal{C}}(p_i)$ does the job and this also proves the last claim. ∎

**Remark 2.21.** Given a line bundle $\mathcal{L}$ on $\mathcal{C}$ and a point $p$ with stabilizer $\mu_{r_i}$, we obtain a character $\chi^{-a} : \zeta \in \mu_{r_i} \mapsto \zeta^{-a} \in \mathbb{G}_m$ via the action of $\mu_{r_i}$ on the fiber $\mathcal{L}|_{p_i}$. This corresponds to the image $\mathcal{L}$ consisting of $a \mod r_i$ in the $i^{th}$ component of the cokernel $\pi^*$. Indeed, as in the proof of the proposition, we can identify the generator of $\mathbb{Z}/r_i\mathbb{Z}$ with the pullback of $\mathcal{O}(p_i)$ to the strict Henselization $[\mathrm{Spec}\, \overline{k(p_i)}[t]^{sh}/\mu_r]$ where $t$ is a uniformizer at $p_i$. The pullback of $\mathcal{O}(p_i)$ is generated by $t^{-1}$ so the fiber of $\mathcal{O}(p_i)$ carries the character $\chi^{-1}$.

**Corollary 2.22.** *There is a presentation of abelian groups*

$$\mathrm{Pic}(\mathcal{C}) = \left(\mathrm{Pic}(C) \oplus \bigoplus_{i=1}^s \mathbb{Z}[\mathcal{O}(p_i)]\right) / \left(\{r_i[\mathcal{O}(p_i)] = [\mathcal{O}(q_i)]\}_{i=1}^s\right)$$

*and of component groups*

$$\pi_0(\mathrm{Pic}(\mathcal{C})) = (\mathbb{Z} \oplus \mathbb{Z}c_1 \oplus \ldots \oplus \mathbb{Z}c_s)/(\{r_i c_i = [k(q_i) : k]\}_{1=1}^s).$$

*Moreover, there is a well defined degree homomorphism*

$$\deg : \operatorname{Pic}(\mathcal{C}) \to \mathbb{Q}$$

*extending the usual degree map* $\deg : \operatorname{Pic}(C) \to \mathbb{Z}$ *such that* $\deg(c_i) = \frac{1}{r_i}[k(q_i) : k]$.

**Corollary 2.23.** *Every line bundle $\mathcal{L}$ on $\mathcal{C}$ can be written uniquely as*

$$\pi^* L\left(\sum_{i=1}^{s} a_i p_i\right)$$

*where $0 \le a_i < r_i$ where $L \in \operatorname{Pic}(C)$. Moreover, there is an isomorphism $\pi_* \mathcal{L} \cong L$.*

*Proof.* The first part follows immediately from the presentation of $\operatorname{Pic}(\mathcal{C})$. For the second part, consider the exact sequence

$$0 \to \pi^* L \to \mathcal{L} \to \mathcal{L} \otimes \mathcal{O}_{\sum a_i p_i} \to 0.$$

Applying $\pi_*$ and using the fact that $\mathcal{C}$ is tame, it suffices to show that $\pi_*(\mathcal{L} \otimes \mathcal{O}_{\sum a_i p_i}) = 0$. By induction on $s$, it suffices to show that $\pi_*(\mathcal{L} \otimes \mathcal{O}_{ap}) = 0$ where $p$ is a point with stabilizer $\mu_r$ and $0 < a < r$. After passing to the strict Henselization, we can compute $\mathcal{L} \otimes \mathcal{O}_{ap}$ explicitly as the $\mu_r$ representation on $k(p)[t]^{sh}/(t^a)t^{-a}$ where $\mu_r$ acts on $t$ as in the definition of twisted curve. In particular, $\pi_*(\mathcal{L} \otimes \mathcal{O}_{ap}) = (k(p)[t]^{sh}/(t^a)t^{-a})^{\mu_r} = 0$ as required. ■

**Corollary 2.24.** *If $\mathcal{C} \to B$ is a family of twisted curves over a connected base, then the quotient $\operatorname{Pic}_{\mathcal{C}/B} / \operatorname{Pic}_{C/B}$ is a finite étale group scheme over $B$ with fibers isomorphic to $\oplus_{i=1}^{s} \mathbb{Z}/r_i\mathbb{Z}$ for some fixed tuple $(r_1, \ldots, r_s)$.*

**Definition 2.25.** If $(\mathcal{X}, \mathcal{L})$ is a proper cyclotomic stack with a polarizing line bundle $\mathcal{L}$, we define the degree of a map $f : \mathcal{C} \to \mathcal{X}$ as $\deg(f^*\mathcal{L})$.

**Remark 2.26.** Suppose that $\mathcal{L}^{\otimes M} = \pi^* L$ where $\pi : \mathcal{X} \to X$ is the coarse map and $L$ is ample. Then the degree of $f$ in our definition agrees with $\frac{1}{M}\deg(C \to X)$ where the degree of the coarse map is measured with respect to $L$.

**Lemma 2.27.** *Let $(\mathcal{X}, \mathcal{L})$ be a proper polarized cyclotomic stack.*

*(1) If $\mathcal{C}$ is a twisted curve and $f : \mathcal{C} \to \mathcal{X}$ any morphism, then $f$ is representable if and only if for each $i = 1, \ldots, s$, the projection of the class $[f^*\mathcal{L}]$ to $\mathbb{Z}/r_i\mathbb{Z}$ under the sequence 2 is a unit.*

*(2) If $f : \mathcal{C}/B \to \mathcal{X}$ is a family of twisted maps to $\mathcal{X}$, then $\deg(f_b)$ and the class of $[f_b^*\mathcal{L}]$ in $\operatorname{Pic}(\mathcal{C}_b)/\operatorname{Pic}(C_b)$ are locally constant functions on $B$.*

**Definition 2.28.** We let $\Gamma = (\{r_1, a_1\}, \ldots, \{r_s, a_s\})$ denote a tuple of pairs of integers where $r_i > 1$ and $0 < a_i < r_i$ is a unit mod $r_i$. We call $\Gamma$ the tuple of local conditions for a twisted map. We say that $\Gamma$ is admissible for $(\mathcal{X}, \mathcal{L})$ if $r_i \mid M$ for all $i$ where $M > 0$ is the smallest positive integer such that $\mathcal{L}^{\otimes M} = \pi^* L$ where $\pi$ is the coarse moduli map.

We are now ready to construct the stack of twisted maps from a fixed curve $C$ to a polarized cyclotomic stack $(\mathcal{X}, \mathcal{L})$.

**Definition 2.29.** Fix $(\mathcal{X}, \mathcal{L})$, $C$ and $\Gamma$ as above and an integer $d$. A family of twisted maps from $C$ to $\mathcal{X}$ of type $\Gamma$ and degree $d$ over a scheme $B$ is a family of twisted maps $(\mathcal{C} \to B, \Sigma_i, f)$ with inertia groups $\mu_{r_1}, \ldots, \mu_{r_s}$ and a map $\pi : \mathcal{C} \to C \times B$ over $B$ such that

(1) $\pi$ is the coarse moduli space of $\mathcal{C}$,
(2) for all $b \in B, \deg(f_b) = d$, and
(3) for all $b \in B$ the class of $f_b^*\mathcal{L}$ in $\operatorname{Pic}(\mathcal{C}_b)/\operatorname{Pic}(C)$ is given by

$$\sum_{i=1}^{s}(a_i \mod r_i).$$

**Theorem 2.30.** *Let $(\mathcal{X}, \mathcal{L})$ be a proper polarized cyclotomic stack over $k$ with coarse moduli space $\pi : \mathcal{X} \to X$ and fix an integer $M$ and ample line bundle $L$ on $X$ such that $\mathcal{L}^{\otimes M} = \pi^* L$. For each smooth projective curve $C$ of genus $g$, local conditions $\Gamma$ and degree $d$, there exists a finite type separated algebraic stack $\mathcal{H}^{\Gamma}_{d,C} = \mathcal{H}^{\Gamma}_{d,C}(\mathcal{X}, \mathcal{L})$ with quasi-projective coarse moduli space parametrizing twisted maps from $C$ to $\mathcal{X}$ of type $\Gamma$ and degree $d$. Moreover, $\mathcal{H}^{\Gamma}_{d,C}$ is quasi-finite over the scheme* $\operatorname{Hom}_{Md}((C, q_1, \ldots, q_s), X)$ *of $s$-pointed degree $Md$ maps $(C, q_1, \ldots, q_s) \to X$.*

*Proof.* We first consider the stack of twisted curves with coarse moduli space $C$ and inertia groups $\mu_{r_1}, \ldots, \mu_{r_s}$. This is the stack of data $(\mathcal{C} \to B, \Sigma_i, \pi)$ where $(\mathcal{C} \to B, \Sigma_i)$ is a twisted curve with stabilizer $\mu_{r_i}$ along $\Sigma_i$ and $\pi : \mathcal{C} \to C \times B$ is a map over $B$ which exhibits $C \times B$ as the coarse moduli space of $\mathcal{C}$. The images of $\Sigma_i$ under $\pi$ define $s$ disjoint sections $\sigma_i : B \to C \times B$. The data of $\sigma_i$ defines a map $B \to \operatorname{Conf}_s(C)$, the configuration space of $s$ distinct points on $C$ and the association

$$(\mathcal{C} \to B, \Sigma_i, \pi) \mapsto (B \to \operatorname{Conf}_s(C))$$

is functorial since coarse moduli space commutes with basechange for tame stacks [AOV08, Corollary 3.3].

On the other hand, given a map $B \to \operatorname{Conf}_s(C)$ induced by a family of $s$ disjoint sections $\sigma_1, \ldots, \sigma_n : B \to B \times C$, the images of $\sigma_i$ are relative effective Cartier divisors. We can then take the $r_i$ root stack along the image of $\sigma_i$ for each $i = 1, \ldots, s$ to obtain coarse moduli map $\pi : \mathcal{C} \to C \times B$. The reduced preimages $\Sigma_i = \pi^*(\sigma_i)_{red}$ are disjoint $\mu_{r_i}$-gerbes over $B$ and $(\mathcal{C} \to B, \Sigma_i, \pi)$ is a family of twisted curves with coarse moduli space $C$ and inertia groups $\mu_{r_i}$. Thus this stack is representable by the configuration space $Z := \operatorname{Conf}_s(C)$.

Let $(\mathcal{C} \to Z, \Sigma_i)$ be the universal family of twisted curves over the configuration space and let

$$\mathcal{H} = \operatorname{Hom}_Z(\mathcal{C}, \mathcal{X} \times Z)$$

be the relative Hom-stack over $Z$. Then $\mathcal{H}$ is an algebraic stack locally of finite type with quasi-compact and separated diagonal by [AOV08, Theorem C.2]. By Lemma 2.27, the degree and the class in $\operatorname{Pic}(\mathcal{C})/\operatorname{Pic}(C)$ are locally constant on $\mathcal{H}$ and so there is an open and closed (and thus algebraic) substack $\mathcal{H}^{\Gamma}_{d,C} \subset \mathcal{H}$ parametrizing those maps of degree $d$ and local twisting conditions $\Gamma$.

To see that $\mathcal{H}^{\Gamma}_{d,C}$ is separated and has finite inertia, we can suppose that the base field is algebraically closed. Composing with the closed embedding $\mathcal{X} \to \mathcal{P}(\vec{\lambda})$ yields a representable monomorphism

$$\operatorname{Hom}_Z(\mathcal{C}, \mathcal{X} \times Z) \to \operatorname{Hom}_Z(\mathcal{C}, \mathcal{P}(\vec{\lambda}) \times Z)$$

which preserves the degree $d$ and twisting condition $\Gamma$. Thus it suffices to prove that $\mathcal{H}^{\Gamma}_{d,C}$ is separated with finite inertia for target weighted projective stack. In this case, the claim follows from the proof of Theorem 1.2 for weighted projective stacks as in §5.1, where it is shown that $\mathcal{H}^{\Gamma}_{d,C}$ is isomorphic to a locally closed substack of a separated stack with finite inertia $\mathcal{R}^{\mu}_{n}$ (c.f. Definition 4.19).

Taking a tuple $(\mathcal{C} \to B, \Sigma_i, f, \pi)$ to its coarse moduli space $(C \times B \to X, \sigma_i)$ produces a morphism

$$\rho : \mathcal{H}^{\Gamma}_{d,C} \to \mathrm{Hom}_{Md}((C, q_1, \ldots, q_s), X).$$

Note that the family of maps $C \times B \to X$ has degree $Md$ by Remark 2.26. The map $\rho$ is quasi-finite by [AOV08, Proposition 4.4] and its proof. Let

$$\psi : H^{\Gamma}_{d,C} \to \mathrm{Hom}_{Md}((C, q_1, \ldots, q_s), X)$$

be the factorization through the coarse moduli space. This map is also quasi-finite so by Zariski's Main Theorem, we can factor it as an open immersion $H^{\Gamma}_{d,C} \subset Y$ followed by a finite morphism $\bar{\psi} : Y \to \mathrm{Hom}_{Md}((C, q_1, \ldots, q_s), X)$. Since the target is a quasi-projective scheme, it follows that $Y$ and thus $H^{\Gamma}_{d,C}$ is quasi-projective. This completes the proof. ■

Later we will need to work with twisted maps where the gerbes are not marked.

**Definition 2.31.** Fix a tuple of local twisting conditions $\Gamma = (\{r_1, a_1\}, \ldots, \{r_s, a_s\})$. Let $S_\Gamma \subset \mathrm{Aut}\{1, \ldots, s\}$ be the subset of permutations that only permute those indices which have the same pair of $\{r_j, a_j\}$.

**Proposition 2.32.** *The group $S_\Gamma$ acts on $\mathcal{H}^{\Gamma}_{d,C}$ and the quotient $\overline{\mathcal{H}}^{\Gamma}_{d,C} := \mathcal{H}^{\Gamma}_{d,C}/S_\Gamma$ is the stack of diagrams $(\mathcal{C} \to B, f : \mathcal{C} \to \mathcal{X}, \pi : \mathcal{C} \to C \times B)$ such that for each geometric point $\bar{t} \in B$, $\mathcal{C}_{\bar{t}}$ is a twisted curve with inertia groups $\mu_{r_1}, \ldots, \mu_{r_s}$ and satisfying* (1), (2) *and* (3) *of Definition 2.29.*

## 3. Correspondence between weighted linear series and twisted maps

Let $K = K(C)$ be the function field of a smooth projective curve $C$. Then rational points $P \in \mathcal{P}(\vec{\lambda})(K)$ correspond to rational maps $C \dashrightarrow \mathcal{P}(\vec{\lambda})$. In this section, we show how the universal tuning stack of $P$ is explicitly constructed from the data of a $\vec{\lambda}$-weighted linear series (Definition 2.9). Furthermore, we show that every rational map $C \dashrightarrow \mathcal{P}(\vec{\lambda})$ is determined by a unique *minimal linear series* defined below. Finally, we compute the height of a point in terms of the minimal linear series.

The indeterminacies of the associated map $C \dashrightarrow \mathcal{P}(\vec{\lambda})$ are the set of points $x \in C$ such that $s_i(x) = 0$ for all $i$, that is, the set theoretic base locus of the weighted linear series. Two $\vec{\lambda}$-weighted linear series are said to be equivalent if they induce the same rational map $f : C \dashrightarrow \mathcal{P}(\vec{\lambda})$, or equivalently if they induce the same rational point $P \in \mathcal{P}(\vec{\lambda})(K)$. We use the notation $\nu_x(s_i)$ to denote the order of vanishing of $s_i$ at $x$.

**Definition 3.1.** Let $(L, s_0, \ldots, s_N)$ be a $\vec{\lambda}$-weighted linear series on a curve $C$. For every $x \in C$, we let

$$r_{\min}(x; L, s_0, \ldots, s_N) := \frac{\lambda_j}{\gcd(\nu_x(s_j), \lambda_j)}$$
$$a_{\min}(x; L, s_0, \ldots, s_N) := \frac{\nu_x(s_j)}{\gcd(\nu_x(s_j), \lambda_j)}$$

where $j$ is a choice of index such that

$$\frac{\nu_x(s_j)}{\lambda_j} = \min_i \left\{ \frac{\nu_x(s_i)}{\lambda_i} \right\}.$$

**Definition 3.2.** A $\vec{\lambda}$-weighted linear series $(L, s_0, \ldots, s_N)$ is *minimal* if for each indeterminacy point $x \in C$, there exists an $j$ such that $\nu_x(s_j) < \lambda_i$.

We now state the main result of this section.

**Theorem 3.3.** *Let $K = K(C)$ be the function field of a smooth projective curve $C$, let $f : C \dashrightarrow \mathcal{P}(\vec{\lambda})$ be a rational map, and let $P \in \mathcal{P}_C(\vec{\lambda})(K)$ denote the corresponding rational point, where $\mathcal{P}_C(\vec{\lambda}) = \mathcal{P}(\vec{\lambda}) \times C \to C$ is the constant family. Let $\{x_j\}$ be the indeterminacy points of $f$. Assume the $\lambda_i$ are prime to the characteristic of the ground field.*

*(1) Let $(L, s_0, \ldots, s_N)$ be any $\vec{\lambda}$-weighted linear series inducing $f$. Then the universal tuning stack $(\mathcal{C}, \pi, \overline{P})$ of $P$ is the root stack of $C$ obtained by taking the $r_j$-th root at $x_j$, where $r_j = r_{\min}(x_j; L, s_0, \ldots, s_N)$. Moreover, the induced morphism on stabilizers over $x_j$ is given by the character $\chi_j^{-a_j}$ where $a_j = a_{\min}(x_j, L, s_0, \ldots, s_N)$.*

*(2) There exists a unique minimal $\vec{\lambda}$-weighted linear series inducing $f$.*

*(3) The stacky height $\mathrm{ht}_{\mathcal{O}(1)}(P)$ is equal to $\deg L$ where $(L, s_0, \ldots, s_N)$ is the unique minimal linear series. Moreover, the stable height is given by $\mathrm{ht}^{st}_{\mathcal{O}(1)}(P) = \deg \overline{P}^* \mathcal{O}(1)$ and the local contribution at $x_j$ is given by $\delta_{x_j}(P) = \frac{a_j}{r_j}[k(x_j) : k]$.*

**Remark 3.4.** Since the quotient map $\mathbb{A}^{N+1} \smallsetminus 0 \to \mathcal{P}(\vec{\lambda})$ can be identified with the total space of $\mathcal{O}(-1)$, the morphism of stabilizers $\mu_r \to \mathbb{G}_m$ induced by the structure map $\mathcal{P}(\vec{\lambda}) \to \mathcal{B}\mathbb{G}_m$ can be identified with the character of the stabilizer $\mu_r$ acting on the fiber of $\mathcal{O}(-1)$. By the theorem, the action of $\mu_{r_j}$ on the fiber of the line bundle $\bar{P}^* \mathcal{O}(1)$ on the universal tuning stack $\mathcal{C}$ is then given by $\chi_j^{a_j}$. Combining this with Proposition 2.20 and Remark 2.21, we conclude that the the class in $\mathrm{Pic}(\mathcal{C})/\mathrm{Pic}(C)$ of $\overline{P}^* \mathcal{O}(1)$ is

$$\sum_j (-a_{\min}(x_j) \mod r_{\min}(x_j)) \in \bigoplus_j \mathbb{Z}/r_{\min}(x_j)\mathbb{Z}$$

where the sum runs over the indeterminacy points of the weighted linear series.

3.1. **Characterizing the universal tuning stack.** In this subsection, we prove Theorem 3.3 (1). The following is the key local computation needed to prove the result.

**Proposition 3.5.** *Let $R$ be a DVR over a field $k$ and let $\Omega$ be the fraction field of $R$. Consider the $\Omega$-point $(s_0,\dots,s_N)$ of $\mathbb{A}^{N+1}\smallsetminus 0$ which yields a map $f\colon \operatorname{Spec}\Omega\to\mathcal{P}(\vec{\lambda})$. Assume $\lambda_0,\dots,\lambda_N$ are prime to the characteristic of $k$. Let $s_j$ have valuation $\nu_j$ and suppose*

$$\frac{\nu_0}{\lambda_0}=\min_j\left\{\frac{\nu_j}{\lambda_j}\right\}. \tag{3}$$

*Let*

$$r=\frac{\lambda_0}{\gcd(\nu_0,\lambda_0)}$$

*and $\mathcal{X}$ be the $r$-root stack of $\operatorname{Spec}R$ at the closed point. Then $f$ extends uniquely (up to unique 2-isomorphism) to a map $g\colon\mathcal{X}\to\mathcal{P}(\vec{\lambda})$. Furthermore, $g$ is representable.*

*Proof.* Upon showing that $f$ extends to $g$, since $\mathcal{C}$ and $\mathcal{P}(\vec{\lambda})$ are separated Deligne–Mumford stacks and $\mathcal{C}$ is normal, [FMN10, Proposition 1.2] shows that $g$ is unique up to unique 2-isomorphism.

We now turn to the construction of $g$. Let $u$ be a uniformizer for $R$. Recall that $\mathcal{X}=[\operatorname{Spec}R'/\mu_r]$, where $R'=R[u']/((u')^r-u)$ and $\mu_r$ acts with weight 1 on $u'$. Let $\Omega'$ be the fraction field of $R'$. Write $s_j=u^{\nu_j}t_j$, where $t_j$ has valuation 0. Our $\Omega$-point $(s_0,\dots,s_N)\in\mathbb{A}^{N+1}\smallsetminus 0$ can be viewed as an $\Omega'$-point; we show that after acting by $\mathbb{G}_m(\Omega')$, we may extend it to an $R'$-point. Notice that $\lambda_0\mid r\nu_0$; acting by $(u')^{-r\nu_0/\lambda_0}\in\mathbb{G}_m(\Omega')$, we obtain

$$s':=(u')^{-\frac{r\nu_0}{\lambda_0}}*(s_0,\dots,s_N)=(t_0,(u')^{r\nu_1-\frac{r\nu_0}{\lambda_0}\lambda_1}t_1,\dots,(u')^{r\nu_N-\frac{r\nu_0}{\lambda_0}\lambda_N}t_N).$$

By (3), we see

$$r\nu_j-\frac{r\nu_0}{\lambda_0}\lambda_j\geq 0$$

and so $s'$ defines an $R'$-point of $\mathbb{A}^{N+1}$. Since the first coordinate of $s'$ has valuation 0, we see $s'$ is an $R'$-point of $\mathbb{A}^{N+1}\smallsetminus 0$. Furthermore, if we let

$$\chi\colon\mu_r\to\mathbb{G}_m,\qquad \chi(\zeta)=\zeta^{-\frac{r\nu_0}{\lambda_0}}$$

then we have a commutative diagram

$$\begin{tikzcd}
\mu_r\times\operatorname{Spec}R' \arrow[r, "\chi\times s'"] \arrow[d, "\sigma'"'] & \mathbb{G}_m\times(\mathbb{A}^{N+1}\smallsetminus 0) \arrow[d, "\sigma"] \\
\operatorname{Spec}R' \arrow[r, "s'"] & \mathbb{A}^{N+1}\smallsetminus 0
\end{tikzcd}$$

where $\sigma$ and $\sigma'$ are the two action maps. As a result, $s'$ induces a map

$$\mathcal{X}=[\operatorname{Spec}R'/\mu_r]\xrightarrow{g}\mathcal{P}(\vec{\lambda})$$

which restricts to $f$ on $[\operatorname{Spec}\Omega'/\mu_r]=\operatorname{Spec}\Omega$.

Lastly, we see $\frac{r\nu_0}{\lambda_0}=\frac{\nu_0}{\gcd(\nu_0,\lambda_0)}$ and $r=\frac{\lambda_0}{\gcd(\nu_0,\lambda_0)}$ are relatively prime. As a result, $\chi$ is injective, and so $g$ is representable. ■

*Proof of Theorem 3.3 (1).* Let $\pi\colon \mathcal{C} \to C$ be the root stack obtained by taking the $r_i$-th root at $x_i$. By construction, $\pi$ is an isomorphism over $C \smallsetminus \{x_i\}$. We show there is a representable map $g\colon \mathcal{C} \to \mathcal{P}(\vec{\lambda})$ which agrees with $f$ on $C \smallsetminus \{x_i\}$. Upon showing this, we see that, by definition, $(\mathcal{C}, \pi, g)$ is a tuning stack; moreover, by Remark 2.2, $(\mathcal{C}, \pi, g)$ is the universal tuning stack.

It remains to prove the existence of a representable map $g\colon \mathcal{C} \to \mathcal{P}(\vec{\lambda})$ extending $f$. We first note that it is enough to show the existence of an étale cover $\{V_\ell \to C\}_\ell$ and representable maps $g_\ell\colon \mathcal{C} \times_C V_\ell \to \mathcal{P}(\vec{\lambda})$ that agree with $f$ over the inverse image of $C \smallsetminus \{x_i\}$ under the map $V_\ell \to C$. Indeed, by [FMN10, Proposition 1.2], any such $g_\ell$ is uniquely determined (up to unique 2-isomorphism) by $f$; hence, the $g_\ell$ descend to yield a map $g\colon \mathcal{C} \to \mathcal{P}(\vec{\lambda})$. Since representability can be checked on geometric points [Con07, Corollary 2.2.7], representability of the $g_\ell$ imply representability of $g$.

To construct our desired étale cover $\{V_\ell \to C\}_\ell$ and representable maps $g_\ell\colon \mathcal{C} \times_C V_\ell \to \mathcal{P}(\vec{\lambda})$, it is enough to do so at the strict Henselization $R_i$ of each point of indeterminacy $x_i$. To see this, we may write $\operatorname{Spec} R_i$ as an inverse limit of affine étale neighborhoods $V_{i,\ell} \to C$ of $x_i$; then by Proposition B.1 and Proposition B.3 of [Ryd15], for any representable map $g_i\colon \mathcal{C} \times_C \operatorname{Spec} R_i \to \mathcal{P}(\vec{\lambda})$ which agrees with $f$, there is some index $\ell$ and an extension $g_{i,\ell}\colon \mathcal{C} \times_C V_{i,\ell} \to \mathcal{P}(\vec{\lambda})$ of $g_i$ where $g_{i,\ell}$ is representable and agrees with $f$.

For the remainder of the proof, we fix a point of indeterminacy $x_i$. Then $R_i$ is a DVR by [Sta18, Tag 0AP3]; let $\Omega_i$ be its fraction field. The rational map $f : C \dashrightarrow \mathcal{P}(\vec{\lambda})$ yields a morphism $f_i\colon \operatorname{Spec} \Omega_i \to \mathcal{P}(\vec{\lambda})$. We continue to denote by $(L, s_0, \ldots, s_N)$ the pullback of the linear series to $\operatorname{Spec} R_i$. Since $L$ is trivial over $\operatorname{Spec} \Omega_i$, we see that $f_i$ lifts to the cover $\operatorname{Spec} \Omega_i \to \mathbb{A}^{N+1} \smallsetminus 0$, where this latter map is defined by the $\Omega_i$-point $(s_0, \ldots, s_N)$. Letting $\nu_j$ denote the order of vanishing of $s_j$ at the closed point of $\operatorname{Spec} R_i$, we may assume without loss of generality that

$$\frac{\nu_0}{\lambda_0} = \min_j \left\{ \frac{\nu_j}{\lambda_j} \right\}.$$

By Proposition 3.5, we see $f_i$ extends to a representable map $g_i\colon \mathcal{C} \times_C \operatorname{Spec} R_i \to \mathcal{P}(\vec{\lambda})$. ■

3.2. **Uniqueness of minimal linear series.** In this subsection, we prove Theorem 2. We begin by associating a minimal linear series on $C$ to any morphism $\mathcal{C} \to \mathcal{P}(\vec{\lambda})$, where $\mathcal{C}$ is a tame root stack over $C$.

**Proposition 3.6.** *Let $C$ be a smooth proper curve over a field $k$, let $x_1, \ldots, x_m \in C$, and let $r_1, \ldots, r_m$ be positive integers prime to the characteristic of $k$. Let $\pi\colon \mathcal{C} \to C$ be the root stack obtained by taking the $r_j$-th root at $x_j$. Let $y_j$ denoted the reduced preimage of $x_j$. By definition, $\mathcal{C}$ carries distinguished line bundles with section $(\mathcal{O}_{\mathcal{C}}(y_j), u_j)$ and isomorphisms $\iota_j\colon \mathcal{O}_{\mathcal{C}}(y_j)^{\otimes r_j} \xrightarrow{\simeq} \pi^*\mathcal{O}_C(x_j)$ with $\iota_j(u_j^{\otimes r_j})$ vanishing to order $1$ at $x_j$.*

*Let $g\colon \mathcal{C} \to \mathcal{P}(\vec{\lambda})$ be a morphism induced by a basepoint-free $\vec{\lambda}$-weighted linear series $(\mathcal{L}, t_0, \ldots, t_N)$. Let $\chi_j\colon \mu_{r_j} \to \mathbb{G}_m$ denote the canonical embedding and suppose that the stabilizer $\mu_{r_j}$ of $y_j$ acts on $\mathcal{L}$ through the character $\chi_j^{a_j}$ with $0 \le a_j < r_j$.*

*Then*

$$\left(\pi_*\mathcal{L}(\sum_j a_j y_j), \pi_*(t_0\prod_j u_j^{\lambda_0 a_j}), \dots, \pi_*(t_N\prod_j u_j^{\lambda_N a_j})\right)$$

*is a minimal weighted linear series on $C$ inducing the rational map $f : C \dashrightarrow \mathcal{P}(\vec{\lambda})$ given by factoring $g$ through the coarse space.*

*Proof.* By construction, the stabilizers of $\mathcal{C}$ act trivially on $\mathcal{L}(\sum_j a_j y_j)$. Thus $\mathcal{L}(\sum_j a_j y_j) = \pi^* L$ for a unique line bundle $L$ on $C$, e.g., by [Alp13, Theorem 10.3]. Since $\pi$ is tame, we have $\pi_*\mathcal{O}_{\mathcal{C}} = \mathcal{O}_C$ and so by the projection formula,

$$\pi_*\mathcal{L}(\sum_j a_j y_j) = \pi_*\pi^* L = L.$$

Now $\pi^*$ is multiplicative so $\mathcal{L}(\sum_j a_j y_j)^k = \pi^* L^k$ and again by the projection formula we have

$$\pi_*(\mathcal{L}(\sum_j a_j y_j)^k) = \pi_*\pi^* L^k = L^k = (\pi_*\mathcal{L}(\sum_j a_j y_j))^k.$$

We conclude that

$$\pi_*(t_i\prod_j u_j^{\lambda_i a_j}) =: s_i$$

is a section of $L^{\lambda_i}$ as required. Moreover, $\pi$ is an isomorphism on $C \smallsetminus \{x_j\}$ so the rational map $f$ induced by the weighted linear series $(L, s_0, \dots, s_N)$ agrees with $g$ on $C \smallsetminus \{x_j\}$.

Finally, we check that $(L, s_0, \dots, s_N)$ is minimal. Since $g$ is a morphism, for each $j$ there exists an $i$ such that $t_i$ does not vanish at $y_j$. Thus the order of vanishing of $t_i\prod_j u_j^{\lambda_i a_j}$ at $y_j$ is $\lambda_i a_j$ and the order of vanishing of $s_i$ at $x_j$ is

$$\frac{\lambda_i a_j}{r_j} < \lambda_i,$$

proving minimality of the weighted linear series. Note that $\frac{\lambda_i a_j}{r_j}$ is integer since $s_i$ does not vanish at $y_j$, showing that $\mathcal{L}^{\lambda_i}$ is trivial when pulled back to the residual gerbe at $y_j$; since the stabilizer $\mu_{r_j}$ acts on $\mathcal{L}^{\lambda_i}$ through the character $\chi_j^{-\lambda_i a_j}$ and since this character must be trivial, we see $r_j \mid \lambda_i a_j$. ■

Next we consider the converse to Proposition 3.6 in the case where $\mathcal{C}$ is the universal tuning stack, characterizing the weighted linear series of $\mathcal{C} \to \mathcal{P}(\vec{\lambda})$ in terms of the weighted linear series of $C \dashrightarrow \mathcal{P}(\vec{\lambda})$.

**Proposition 3.7.** *Let $(L, s_0, \dots, s_N)$ be a weighted linear series on $C$ inducing a rational map $f : C \dashrightarrow \mathcal{P}(\vec{\lambda})$, and assume $\lambda_0, \dots, \lambda_N$ are prime to the characteristic. Let $(\mathcal{C}, \pi, \overline{P})$ be as in Theorem 3.3 (1). Let*

$$a_j = \frac{\nu_{x_j}(s_{i_j})}{\gcd(\nu_{x_j}(s_{i_j}), \lambda_{i_j})}$$

*where*
$$\frac{\nu_{x_j}(s_{i_j})}{\lambda_{i_j}} = \min_i \left\{ \frac{\nu_{x_j}(s_i)}{\lambda_i} \right\}.$$
*Then the morphism $\overline{P}\colon \mathcal{C} \to \mathcal{P}(\vec{\lambda})$ is defined by the weighted linear series $(\mathcal{L}, t_0, \dots, t_N)$ with*
$$\mathcal{L} = \pi^* L(-\sum a_j y_j) \quad \textit{and} \quad t_i = \frac{\pi^* s_i}{\prod_j u_j^{\lambda_i a_j}}.$$
*Furthermore, if $(L, s_0, \dots, s_N)$ is minimal and $(L', s'_0, \dots, s'_N)$ denotes the minimal linear series obtained from $(\mathcal{L}, t_0, \dots, t_N)$ in Proposition 3.6, then there is a canonical isomorphism $\beta\colon L \xrightarrow{\simeq} L'$ such that $\beta(s_i) = s'_i$.*

*Proof.* A priori, $t_i$ is only a rational section of $\mathcal{L}^{\lambda_i}$. We first show it is a regular section. By construction, $\pi\colon \mathcal{C} \to C$ is the $r_j^{th}$ root stack along $x_j$ where
$$r_j = \frac{\lambda_{i_j}}{\gcd(\nu_{x_j}(s_{i_j}), \lambda_{i_j})}.$$
For all $j$, we have
$$\nu_{y_j}(\pi^* s_i) = r_j \nu_{x_j}(s_i) = \frac{\lambda_{i_j} \nu_{x_j}(s_i)}{\gcd(\nu_{x_j}(s_{i_j}), \lambda_{i_j})} \geq \frac{\lambda_i \nu_{x_j}(s_{i_j})}{\gcd(\nu_{x_j}(s_{i_j}), \lambda_{i_j})} = \lambda_i a_j$$
since the index $i_j$ minimizes the ratio $\nu_{x_j}(s_i)/\lambda_i$. Therefore $t_i$ is actually a regular section of $\mathcal{L}^{\lambda_i}$ and $(\mathcal{L}, t_0, \dots, t_N)$ is a well defined $\vec{\lambda}$-weighted linear series on $\mathcal{C}$.

Next, by construction, $\nu_{y_j}(t_{i_j}) = 0$. Therefore $(\mathcal{L}, t_0, \dots, t_N)$ is a basepoint-free weighted linear series and thus induces a morphism $\mathcal{C} \to \mathcal{P}(\vec{\lambda})$. This morphism agrees with $f$ away from the basepoints $x_j$ so by [FMN10, Proposition 1.2], it is uniquely isomorphic to $\overline{P}$ up to unique 2-isomorphism.

Lastly, suppose $(L, s_0, \dots, s_N)$ is minimal. Then, for every $j$, there exists $s_{\ell_j}$ with $\nu_{x_j}(s_{\ell_j}) < \lambda_{\ell_j}$. Since $\frac{\nu_{x_j}(s_{i_j})}{\lambda_{i_j}} \leq \frac{\nu_{x_j}(s_{\ell_j})}{\lambda_{\ell_j}}$, it follows that $\nu_{x_j}(s_{i_j}) < \lambda_{i_j}$, and hence, $0 \leq a_j < r_j$. Let $\chi_j\colon \mu_{r_j} \to \mathbb{G}_m$ denote the canonical embedding; since the stabilizers of $C$ act trivially on $\pi^* L$, we see the stabilizer $\mu_{r_j}$ at $y_j$ acts on $\mathcal{L}$ via the character $\chi_j^{-a_j}$. Having now verified that the hypotheses of Proposition 3.6 hold, we see
$$L' = \pi_* \mathcal{L}(\textstyle\sum_j a_j y_j) = \pi_* \pi^* L \xleftarrow[\simeq]{\beta} L$$
where $\beta$ is the adjuncation map; it is an isomorphism since $\mathcal{C}$ is a tame Deligne–Mumford stack, so we have a canonical isomorphism $\pi_* \mathcal{O}_{\mathcal{C}} \simeq \mathcal{O}_C$ and $\beta$ is the composition of the canonical isomorphisms $\pi_* \pi^* L \simeq L \otimes \pi_* \mathcal{O}_{\mathcal{C}} \simeq L$. Furthermore, by construction
$$\beta(s_i) = \pi_* \pi^*(s_i) = \pi_*(t_i \prod_j u_j^{\lambda_i a_j}) = s'_i. \qquad \blacksquare$$

We are now ready to prove Theorem 3.3 (2), thereby finishing the proof of Theorem 3.3.

*Proof of Theorem 3.3 (2).* Let $f : C \dashrightarrow \mathcal{P}(\vec{\lambda})$ be a rational map. Suppose it is induced by a minimal linear series $(L_1, s_{1,0}, \ldots, s_{1,N})$ and it is also induced by a minimal linear series $(L_2, s_{2,0}, \ldots, s_{2,N})$. We prove there exists a canonical isomorphism $\gamma\colon L_1 \xrightarrow{\simeq} L_2$ with $\gamma(s_{1,i}) = s_{2,i}$.

For $\ell \in \{1,2\}$, let $\pi_\ell\colon \mathcal{C}_\ell \to C$ be the root stack obtained by taking the $r_{\ell,j}$-th root at $x_j$, where $r_{\ell,j} = r_{\min}(x_j; L_\ell, s_{\ell,0}, \ldots, s_{\ell,N})$. Let $g_\ell : \mathcal{C}_\ell \to \mathcal{P}(\vec{\lambda})$ be the induced representable morphism constructed in Theorem 3.3 (1) and let $(\mathcal{L}_\ell, t_{\ell,0}, \ldots, t_{\ell,N})$ be the corresponding basepoint-free weighted linear series. Let $(L'_\ell, s'_{\ell,0}, \ldots, s'_{\ell,N})$ be the minimal linear series on $C$ obtained from $(\mathcal{L}_\ell, t_{\ell,0}, \ldots, t_{\ell,N})$ in Proposition 3.6. By Proposition 3.7, we have canonical isomorphisms $\beta_\ell\colon L_\ell \xrightarrow{\simeq} L'_\ell$ with $\beta_\ell(s_{\ell,i}) = s'_{\ell,i}$.

Theorem 3.3 (1) tells us that $(\mathcal{C}_\ell, \pi_\ell, g_\ell)$ is a universal tuning stack for each $\ell$, so by universality, there is a canonical isomorphism $h\colon \mathcal{C}_2 \to \mathcal{C}_1$ and 2-isomorphism $\alpha\colon g_2 \Rightarrow g_1 \circ h$ with $\pi_2 = \pi_1 \circ h$. This yields an isomorphism $\alpha\colon h^*\mathcal{L}_1 \xrightarrow{\simeq} \mathcal{L}_2$ with $\alpha(h^*(t_{1,i})) = t_{2,i}$. Applying $(\pi_2)_*$, and using the definitions of $(L'_\ell, s'_{\ell,0}, \ldots, s'_{\ell,N})$, we obtain an isomorphism $(\pi_2)_*\alpha\colon L'_1 \xrightarrow{\simeq} L'_2$ sending $s'_{1,i}$ to $s'_{2,i}$. Composing with the isomorphisms $\beta_\ell$ yields our desired isomorphism $\gamma$. ■

3.3. **The height of a minimal linear series.** In this section we finish the proof of Theorem 3.3 by computing the height of a rational point in terms of the minimal linear series.

*Proof of Theorem 3.3 (3).* The stable height $\mathrm{ht}^{st}_{\mathcal{O}(1)}(P) = \deg(\overline{P}^*\mathcal{O}(1)) = \deg \mathcal{L}$ by definition. By Proposition 3.7, we have

$$\mathcal{L} \cong \pi^* L(-\sum a_j y_j)$$

where $(L, s_0, \ldots, s_N)$ is the unique minimal linear series and

$$a_j = a_{min}(x_j; L, s_0, \ldots, s_N).$$

Moreover, by minimality, $0 \leq a_j < r_j$. By Corollary 2.23,

$$\pi_*\mathcal{L}^\vee = \pi_*\left(\pi^* L^\vee\left(\sum a_j y_j\right)\right) = L^\vee$$

and so $\mathrm{ht}_{\mathcal{O}(1)}(P) = -\deg \pi_*\mathcal{L}^\vee = \deg L$. Finally, the local contribution at $x_j$ is given by

$$\delta_{x_j}(P) = \deg\left(\mathrm{coker}(\pi^*\pi_*\mathcal{L}^\vee \to \mathcal{L}^\vee)_{x_j}\right) = \deg \mathcal{O}_{a_j y_j} = \frac{a_j}{r_j}[k(x_j) : k].$$

■

3.4. **The normalized linear series.** In this section, we recast the computation of the universal tuning stack in terms of the *normalized linear series* which will, first, clarify the role of the minimum in the formulas for $r_{min}$ and $a_{min}$ and, second, be easier to work with in families.

Let $\kappa := \mathrm{lcm}\{\lambda_0, \ldots, \lambda_N\}$ and let $\bar{\lambda}_j := \kappa/\lambda_j$ so that $\lambda_j \bar{\lambda}_j = \kappa$. Then there is a natural map

$$\mathcal{P}(\vec{\lambda}) \to \mathcal{P}(\underbrace{\kappa, \ldots, \kappa}_{N+1})$$

induced on $T$-points by taking

$$(L, s_0, \dots, s_N) \mapsto (L, s_0^{\bar{\lambda}_j}, \dots, s_N^{\bar{\lambda}_j})$$

where $s_i^{\bar{\lambda}_j}$ is a section of

$$(L^{\otimes \lambda_j})^{\otimes \bar{\lambda}_j} = L^{\kappa}.$$

**Definition 3.8.** Let $(L, s_0, \dots, s_N)$ be a $\vec{\lambda}$-weighted linear series.

(1) The associated *normalized linear series* is $\mathrm{Span}(s_0^{\bar{\lambda}_0}, \dots, s_N^{\bar{\lambda}_N}) \subset H^0(T, L^{\otimes \kappa})$.
(2) The *normalized base locus* $\overline{Bs}$ is the scheme theoretic base locus of the normalized linear series:

$$\overline{Bs}(L, s_0, \dots, s_N) := \bigcap_{j=1}^{N} \{s_j^{\bar{\lambda}_j} = 0\}$$

**Proposition 3.9.** *Let $(L, s_0, \dots, s_N)$ be a $\vec{\lambda}$-weighted linear series on a curve $C$ and let $x$ be an indeterminacy point such that $\nu_j = \nu_x(s_j)$. Let $m$ be the multiplicity of $\overline{Bs}$ at $x$. Then the stabilizer of the universal tuning stack at $x$ is given by $\mu_r$ with*

$$r = r_{min} = \frac{\kappa}{\gcd(m, \kappa)}$$

*and the character of the $\mu_r$-action on the line bundle $\mathcal{L}$ on the universal tuning stack is given by $\chi^{-a}$ where*

$$a = a_{min} = \frac{m}{\gcd(m, \kappa)}.$$

*Proof.* In Theorem 3.3 (1), we showed $r = r_{min}(x, L, s_0, \dots, s_N)$ and $a = a_{min}(x, L, s_0, \dots, s_N)$. First note that the index which minimizes the ratio $\nu_j/\lambda_j$ is the same as the index which minimizes

$$\kappa \frac{\nu_j}{\lambda_j} = \bar{\lambda}_j \nu_j = \nu_x(s_j^{\bar{\lambda}_j}).$$

On the other hand, the minimum order of vanishing of $\{s_j^{\bar{\lambda}_j}\}$ is exactly the multiplicity of the base locus of the normalized linear series at $x$. Thus, if $j$ is any index minimizing the ratio $\nu_j/\lambda_j$, then we have

$$\kappa \frac{\nu_j}{\lambda_j} = \bar{\lambda}_j \nu_j = m.$$

Now note that for any positive integers $a$ and $b$, $a/\gcd(a, b) = a'$ where $a/b = a'/b'$ is written in lowest terms. In particular, the ratio $a/\gcd(a, b)$ depends only on the fraction $a/b$. Thus, noting that $\kappa/m = \lambda_j/\nu_j$ for $j$ any index minimizing the ratio, we see that

$$r = \frac{\kappa}{\gcd(m, \kappa)}.$$

Finally, we have that for the same index,

$$a = \frac{\nu_j}{\gcd(\nu_j, \lambda_j)} = \frac{\bar{\lambda}_j \nu_j}{\bar{\lambda}_j \gcd(\nu_j, \lambda_j)} = \frac{m}{\gcd(m, \kappa)}.$$

■

**Remark 3.10.** Note that minimality is equivalent to the statement that the for each closed point $x \in C$, the multiplicity $m$ of the normalized base locus satisfies $m < \kappa$.

We are now ready to express the stacky height with respect to the universal bundle $\mathcal{O}_{\mathcal{P}(\vec{\lambda})}(1)$ in terms of the normalized linear series.

**Corollary 3.11.** *Let $K = k(C)$ and $P \in \mathcal{P}(\vec{\lambda})(K)$ a $K$-rational point. Let $(L, s_0, \ldots, s_N)$ be the unique minimal weighted linear series associated to $P$. Then*

$$\mathrm{ht}(P) = \deg L,$$
$$\mathrm{ht}^{st}(P) = \frac{1}{\kappa} \deg(C \to \mathbb{P}^N),$$

*and the local contribution at a closed point $x \in C$ is given by*

$$\delta_x(P) = \frac{a_x}{r_x}[k(x) : k] = \frac{m_x}{\kappa} \deg[k(x) : k]$$

*where $m_x$ is the multiplicity at $x \in C$ of the normalized base locus $\overline{Bs}$ and $C \to \mathbb{P}^N$ is the unique morphism extending the composition $C \dashrightarrow \mathcal{P}(\vec{\lambda}) \to \mathcal{P}(\kappa, \ldots, \kappa) \to \mathbb{P}^N$ with degree measured by $\mathcal{O}_{\mathbb{P}^N}(1)$.*

## 4. Moduli of minimal weighted linear series with vanishing conditions

The goal of this section is to construct the moduli space of minimal weighted linear series of degree $n$ on a curve $C$. By Theorem 3.3, the $k$-points of this moduli space will be canonically in bijection with $k(C)$-points of $\mathcal{P}(\vec{\lambda})$ of height $n$. Our second goal is to construct a natural stratification of this moduli space corresponding to the different universal tuning stacks. In the next section, we will identify these strata with the spaces of twisted maps $\mathcal{H}^{\Gamma}_{d,C}$ for different $d$ and $\Gamma$.

### 4.1. **The ambient stack of weighted linear series.** 
Fix a smooth projective curve $C/k$ with function field $K$.

**Definition 4.1.** Let $B/k$ be a scheme. A family of $\vec{\lambda}$-weighted linear series on $C$ parameterized by $B$ is the data of $(L, s_0, \ldots, s_N)$ where $L$ is a line bundle on $C_B := C \times B$ and $s_i \in H^0(C \times B, L^{\otimes \lambda_i})$. A morphism $(L, s_0, \ldots, s_N)/B \to (L', s'_0, \ldots, s'_N)/B'$ is a morphism $\rho : B \to B'$ and an isomorphism $\psi : \rho^* L' \to L$ such that $\psi^{\lambda_i}(s'_i) = s_i$.

The category of families of weighted linear series $\mathcal{W}_C(\vec{\lambda})$ defines a category fibered in groupoids over $\mathrm{Sch}/k$.

**Definition 4.2.** For $\mathcal{F}$ a coherent sheaf on an algebraic stack $\mathcal{X}$, the *abelian cone* associated to $\mathcal{F}$ is

$$\mathbb{V}(\mathcal{F}) := \mathrm{Spec}_{\mathcal{X}} \mathrm{Sym}^{\bullet} \mathcal{F}.$$

Note that the formation of $\mathbb{V}(\mathcal{F})$ is compatible with flat base change and satisfies fppf descent.

**Proposition 4.3.** *$\mathcal{W}_C(\vec{\lambda})$ is a locally of finite type algebraic stack over $k$. Moreover, the natural morphism $\mathcal{W}_C(\vec{\lambda}) \to \mathcal{P}ic(C)$ to the Picard stack identifies $\mathcal{W}_C(\vec{\lambda})$ with an abelian cone over $\mathcal{P}ic(C)$. In particular, $\mathcal{W}_C(\vec{\lambda})$ is a union*

$$\mathcal{W}_C(\vec{\lambda}) = \bigsqcup_{n \geq 0} \mathcal{W}_{n,C}(\vec{\lambda})$$

*of connected stacks of finite type indexed by $n = \deg(\mathcal{L}_b)$.*

*Proof.* Let $\mathcal{P}ic(C)$ be the Picard stack of $C/k$ and let $\mathcal{L}$ be the universal line bundle on $\pi : C \times \mathcal{P}ic(C) \to \mathcal{P}ic(C)$. The following lemma is a standard corollary of Cohomology and base-change (see [Hal14, Theorem D] for the generalization to algebraic stacks).

**Lemma 4.4.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a proper morphism of algebraic stacks and let $\mathcal{F}$ be a quasicoherent sheaf on $\mathcal{X}$ flat over $\mathcal{Y}$. There exists a quasicoherent sheaf $\mathcal{Q}$ on $\mathcal{Y}$ such that*

$$\mathrm{Hom}_{\mathcal{Y}}(B, \mathbb{V}(\mathcal{Q})) = H^0(\mathcal{X}_B, \mathcal{F}_B).$$

*If $\mathcal{F}$ is finitely presented then so is $\mathcal{Q}$. Moreover, if $f_*\mathcal{F}$ is locally free and its formation commutes with arbitary base change, then $\mathcal{Q} \cong (f_*\mathcal{F})^\vee$*

Applying the lemma to $\pi$ and $\mathcal{L}^{\otimes d}$ we obtain a coherent sheaf $\mathcal{Q}^d$ on $\mathcal{P}ic(C)$ such that for any $B$-point $L \in \mathrm{Pic}(C_B)$,

$$\mathrm{Hom}_{\mathcal{P}ic(C)}(B, \mathbb{V}(\mathcal{Q}^d)) = H^0(C_B, L^{\otimes d}).$$

Denoting $\mathbb{V}(\mathcal{Q}^d) =: \mathbb{V}^d$, we conclude that

$$\mathrm{Hom}_{\mathcal{P}ic(C)}\left(B, \prod_{i=0}^{N} \mathbb{V}^{\lambda_i}\right) = \bigoplus_{i=0}^{N} H^0(C_B, L^{\otimes \lambda_i})$$

and thus

$$\mathrm{Hom}_k\left(B, \prod_{i=0}^{N} \mathbb{V}^{\lambda_i}\right) = \mathcal{W}_C(\vec{\lambda})$$

as required.

Now $\mathcal{P}ic(C) = \bigsqcup_{n\in\mathbb{Z}} \mathcal{P}ic^n(C)$ is a disjoint union of finite connected components parametrizing line bundles of degree $n$. Denoting $\mathbb{V}^d_n := \mathbb{V}^d|_{\mathcal{P}ic^n(C)}$, then $\mathbb{V}^d|_{\mathcal{P}ic^n(C)} = \varnothing$ for $n < 0$ so we conclude that

$$\mathcal{W}_C(\vec{\lambda}) = \bigsqcup_{n \geq 0} \mathcal{W}_{n,C}(\vec{\lambda})$$

where $\mathcal{W}_{n,C}(\vec{\lambda}) = \prod_{i=0}^{N} \mathbb{V}^{\lambda_i}_n$. ∎

**Remark 4.5.** Let $\mu_{d,n} : \mathcal{P}ic^n(C) \to \mathcal{P}ic^{dn}(C)$ be the $d^{th}$ power map induced by $L \mapsto L^{\otimes d}$. Letting $\mathcal{L}$ be the universal line bundle on $C \times \mathcal{P}ic(C)$, we have $\mathcal{L}^{\otimes d}|_{C\times\mathcal{P}ic^n(C)} = \mu^*_{d,n}\mathcal{L}|_{C\times\mathcal{P}ic^{dn}(C)}$. Thus, $\mathbb{V}^d_n = \mu^*_{d,n}\mathbb{V}^1_{nd}$.

**Remark 4.6.** The stack $\mathbb{V}^d_n$ contains an open substack $U^d_n$ given by the complement of the zero section $\mathcal{P}ic^n(C) \to \mathbb{V}^d_n$. This is the open subfunctor parameterizing sections $s \in H^0(C_B, L^{\otimes d})$ where $s_b \not\equiv 0$ for all $b \in B$. In particular, the vanishing locus $D = V(s) \subset C_B$ is flat over $B$ for any $B$-point of $U^d_n$ by [Sta18, Lemma 00ME]. This implies that $D$ is a relative effective Cartier divisor and we get a morphism $U^d_n \to \mathrm{Sym}^{dn}(C)$ which makes the square

$$\begin{tikzcd} U^d_n \arrow[r] \arrow[d] & \mathrm{Sym}^{dn}(C) \arrow[d] \\ \mathcal{P}ic^n(C) \arrow[r, "\mu_{d,n}"] & \mathcal{P}ic^{dn}(C) \end{tikzcd}$$

cartesian.

**Remark 4.7.** When $dn > 2g(C)–2$, $\pi_*\mathcal{L}^{\otimes d}|_{C\times \mathcal{P}ic^n(C)}$ is a vector bundle over $\mathcal{P}ic^n(C)$ whose formation commutes with base change. In this case,

$$\mathbb{V}_n^d = \mathrm{Spec}_{\mathcal{P}ic(C)} \mathrm{Sym}^\bullet(\pi_*\mathcal{L}^d|_{C\times\mathcal{P}ic^n(C)})^\vee.$$

**Remark 4.8.** Let $\pi : \mathcal{X} \to X$ be a $\mathbb{G}_m$-gerbe and $\mathcal{Q}$ a quasicoherent sheaf on $\mathcal{X}$. Then fppf locally (or even étale locally since $\mathbb{G}_m$ is smooth) the gerbe can be trivialized to $\mathcal{B}\mathbb{G}_{m,X}$. In the case where $\mathcal{X}$ is a trivial gerbe, $\mathcal{Q}$ can be identified with a quasicoherent $\mathbb{G}_m$-sheaf on $X$. Then we have a natural identification

$$\mathbb{V}_{\mathcal{X}}(\mathcal{Q}) = [\mathbb{V}_X(\mathcal{Q})/\mathbb{G}_{m,X}].$$

In particular, $\mathbb{V}_{\mathcal{X}}(\mathcal{Q})$ contains as an open substack the weighted projectivization of the abelian cone $\mathbb{V}_X(\mathcal{Q})$. In particular, this applies to the case $\pi : \mathcal{P}ic(C) \to \mathrm{Pic}(C)$, where we can achieve such a trivialization of the gerbe after passing to a field extension $k'/k$ with $C(k') \neq \varnothing$.

**Remark 4.9.** When $C(k) \neq \varnothing$ and $dn > 2g(C)-2$, then $U_n^d$ is isomorphic to a $\mu_d$-gerbe of over the pullback of the projective bundle $\mathrm{Sym}^{dn}(C) \to \mathrm{Pic}^{dn}(C)$. More generally, if $n\lambda_i > 2g(C)-2$ for all $i = 0,\dots,N$, then

$$\prod_{i=0}^N \mathbb{V}_n^{\lambda_i} \smallsetminus 0_{\mathcal{P}ic^n(C)}$$

is isomorphic to a weighted projective bundle over $\mathrm{Pic}^n(C)$ where $\mathbb{V}_n^d$ carries weight $d$. If $C(k) = \varnothing$, it is instead a possibly nontrivial form of this weighted projective bundle.

4.2. **Stratifying by normalized base locus.** As we saw in Section 3.4, the multiplicity of the normalized base locus controls the local behavior of the universal tuning stack. In this section we use this observation to stratify $\mathcal{W}_{n,C}(\vec{\lambda})$ into strata which we will relate to moduli of twisted maps in Section 5.

As $C$ and $\mathcal{P}(\vec{\lambda})$ are fixed, we will denote $\mathcal{W}_{n,C}(\vec{\lambda})$ by $\mathcal{W}_n$ for convenience. Over $\mathcal{W}_n$ we have a universal bundle $\mathcal{L}$ on $C\times\mathcal{W}_n$ with sections $s_i \in H^0(C\times\mathcal{W}_n, \mathcal{L}^{\otimes\lambda_i})$. Let

$$\overline{\mathrm{Bs}} = \overline{\mathrm{Bs}}(\mathcal{L}, s_0,\dots,s_N) = \mathrm{Bs}(\mathcal{L}^{\otimes\kappa}, s_0^{\bar{\lambda}_0},\dots,s_N^{\bar{\lambda}_N})$$

be the normalized base locus of the universal weighted linear series. Now $C\times\mathcal{W}_n \to \mathcal{W}_n$ is a projective morphism so we can apply Grothendieck's functorial flattening stratification to $\overline{\mathrm{Bs}} \to \mathcal{W}_n$ to obtain a stratification of $\mathcal{W}_n$.

**Proposition 4.10.** *The flattening stratification $\{\mathcal{W}_n^m\}$ for $\overline{\mathrm{Bs}} \to \mathcal{W}_n$ is indexed by $m \in \mathbb{N}\cup\infty = \{0,1,\dots,m,\dots,\infty\}$ where*

*(1) $\mathcal{W}_n^0 \cong \mathrm{Hom}_n(C,\mathcal{P}(\vec{\lambda}))$ is the open substack parametrizing basepoint-free weighted linear series, i.e. morphisms $C \to \mathcal{P}(\vec{\lambda})$,*
*(2) $\mathcal{W}_n^m$ for $0 < m < \infty$ is the substack of families of weighted linear series with normalized base locus finite flat of degree $m$,*
*(3) $\mathcal{W}_n^m = \varnothing$ for $n\kappa < m < \infty$, and*
*(4) $\mathcal{W}_n^\infty \cong \mathcal{P}ic^n(C)$ is the closed substack where where $s_i \equiv 0$ for all $i$.*

**Corollary 4.11.** *There is a natural morphism $\mathcal{W}_n^m \to \mathrm{Sym}^m(C)$ for $0 < m < \infty$ sending a $\vec{\lambda}$-weighted linear series to its normalized scheme theoretic base locus $\overline{\mathrm{Bs}}$.*

The symmetric product $\mathrm{Sym}^m(C)$ can be further stratified into strata $\mathrm{Sym}^\mu(C)$ indexed by a partition $\mu \vdash m$. Here a partition $\mu$ is a sequence $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_l > 0$ where

$$\sum_{i=1}^{l} \mu_i = m.$$

We denote $l := l(\mu)$ the length of the partition and $m = |\mu|$ the size of the partition. Then $\mathrm{Sym}^\mu(C)$ is the stratum parametrizing subschemes of $C$ with $l$ irreducible components of multiplicity $\mu_i$. We say that a finite subscheme of length $m$ parametrized by a point in $\mathrm{Sym}^\mu(C)$ has *profile* $\mu$.

**Remark 4.12.** While $\bigsqcup_\mu \mathrm{Sym}^\mu(C)$ is at first glance a set theoretic stratification of $\mathrm{Sym}^m(C)$, in fact these strata have a canonical scheme structure as they can be identified with the *locally trivial* Hilbert schemes parametrizing locally trivial flat families of subschemes in $C$ [GK89, Section 2].

**Definition 4.13.** For any $n \geq 0$ and partition $\mu$, moduli space of *weighted linear series of degree n with normalized base profile* $\mu$ is the pullback

$$\mathcal{W}_n^\mu := \mathcal{W}_n \times_{\mathrm{Sym}^m(C)} \mathrm{Sym}^\mu(C)$$

where $m = |\mu|$.

The $B$-points of $\mathcal{W}_n^\mu$ are families of weighted linear series $(\mathcal{L}, s_0, \ldots, s_N)$ on $C_B$ such that the normalized scheme theoretic base locus $\overline{\mathrm{Bs}}(\mathcal{L}, s_0, \ldots, s_N)$ is finite, flat and locally trivial of profile $\mu$ over $B$.

**Definition 4.14.** We say a normalized base profile $\mu$ is *minimal* if $\mu_i < \kappa$ for all $i$.

**Proposition 4.15.** *There exists an open substack $\mathcal{W}_{n,C}^{min}(\vec{\lambda}) \subset \mathcal{W}_{n,C}(\vec{\lambda})$ parametrizing minimal weighted linear series. Moreover, $\mathcal{W}_{n,C}^{min}(\vec{\lambda})$ has a stratification into locally closed strata $\mathcal{W}_n^\mu$ over minimal base profiles $\mu$.*

*Proof.* A linear series is minimal if and only if the normalized base multiplicity is strictly less than $\kappa$ for all points on $C$. In particular, it is a condition on the partition $\mu$: namely that $\mu_i < \kappa$ for all $i$. Thus, the stratification $\mathcal{W}_n^{min} = \sqcup_{\mu \text{ minimal}} \mathcal{W}_{n,C}^\mu$ is clear, at least set theoretically. To finish the proof, we show that $\mathcal{W}_{n,C}^{min}$ is open in $\mathcal{W}_{n,C}$. This follows from Lemma 4.16 below applied to the universal normalized base locus over $\mathcal{W}_n \smallsetminus \mathcal{W}_n^\infty$.

**Lemma 4.16.** *Let $B$ be a locally Noetherian algebraic stack over $k$ with finite inertia. Let $C$ be a smooth curve and let $Z \subset C \times B$ be a subscheme which is finite over $B$. Then for each non-negative integer $d$, the locus of $b \in B$ such that $Z_b$ has multiplicity $< d$ at each point is open in $B$.*

*Proof.* Without loss of generality, we may suppose that $k$ is algebraically closed. By flattening stratification and further stratifying by the profile $\mu$ we see that the locus of $b \in B$ where $Z_b$ has no point of multiplicity $\geq d$ is constructible. Thus it suffices to show this locus is stable under generalization.

Toward this end, we may suppose $B = \operatorname{Spec} R$ is the spectrum of a DVR $(R, \mathfrak{m})$ with fraction field $K$ and $Z = \operatorname{Spec} A$ for a finite $R$-algebra $A$. Suppose moreover that $\operatorname{Spec} A/\mathfrak{m}A$ has no point of multiplicity at least $d$. We wish to show that $A \otimes_R K$ also satisfies this condition. Let $z_i$ be the points of the special fiber $\operatorname{Spec} A/\mathfrak{m}A$. By assumption,

$$\dim_k (A/\mathfrak{m}A)_{z_i} < d$$

for all $z_i$. Let $z_i'$ be a point of $\operatorname{Spec}(A \otimes_R K)$. Since $Z \to B$ is proper, $z_i'$ specializes to some point, which up reindexing we call $z_i$, lying $\mathfrak{m}$. It suffices to show that $\dim_K (A \otimes_R K)_{z_i'} < d$. Since the question is local, we may replace $A$ by $A_{z_i}$ and suppose without loss of generality that $A$ is local with closed point $z$ and $z'$ is a point of the generic fiber $\operatorname{Spec} A \otimes_R K$. By upper semi-continuity,

$$\dim_K (A \otimes_R K)_{z'} \leq \dim_K (A \otimes_R K) \leq \dim_k (A/\mathfrak{m}A) < d,$$

thus the generic fiber of $\operatorname{Spec} R$ satisfies the required condition. ■

■

**Corollary 4.17.** *For each $K = k(C)$, $n$ and $\vec{\lambda}$, the $k$-points of the finite type Deligne–Mumford stack $\mathcal{W}_{n,C}^{min}(\vec{\lambda})$ are in canonical bijection with $K$-points of $\mathcal{P}(\vec{\lambda})$ of stacky height $n$.*

For some applications it is convenient to parameterize the normalized base locus.

**Definition 4.18.** Let $S_\mu$ be the subgroup of the symmetric group $S_{l(\mu)}$ consisting of permutations $\sigma$ such that $\mu_{\sigma(i)} = \mu_i$ for all $i$.

Let $\operatorname{Conf}_l(C)$ be the configuration space of $l$ ordered distinct points on $C$. Then $S_\mu$ naturally acts on $\operatorname{Conf}_l(C)$ and the quotient $\operatorname{Conf}_l(C)/S_\mu$ is isomorphic to the stratum $\operatorname{Sym}^\mu(C)$.

**Definition 4.19.** The *space of weighted linear series with parametrized normalized base locus of profile* $\mu$ is the fiber product

$$\mathcal{R}_n^\mu := \operatorname{Conf}_{l(\mu)} \times_{\operatorname{Sym}^\mu} \mathcal{W}_n^\mu.$$

A $B$-point of $\mathcal{R}_n^\mu$ consists of a weighted linear series $(L, s_0, \ldots, s_N)$ of degree $n$ on $C_B$ as well as $l$ disjoint sections $\sigma_i : B \to C_B$ such that the $\overline{\operatorname{Bs}}(L, s_0, \ldots, s_N)$ is the relative effective Cartier divisor $\sum \mu_i \sigma_i$.

4.3. **Imposing vanishing conditions.** For certain applications such as counting elliptic surfaces with a fixed fiber type, it is useful to further stratify the moduli space of weighted linear series by specifying vanishing conditions on the $s_i$. We will accomplish this by working over the space of embeddings of infinitesimal discs of the form $\operatorname{Spec} k[t]/(t^n)$ into $C$.

Associated to a partition $\mu$ we have a unique isomorphism type of curvilinear Artinian local scheme with profile $\mu$ and residue field $k$:

$$A_\mu \cong \prod_{i=1}^{l} k[x]/(x^{\mu_i+1}).$$

We denote by $D_\mu := \operatorname{Spec} A_\mu$. In what follows we view $\mu$ as an ordered tuple $(\mu_1, \ldots, \mu_l)$ with $\mu_i \geq 1$, while noting that the isomorphism type of $D_\mu$ and the stratum $\operatorname{Sym}^\mu(C)$ are independent of the choice of ordering.

**Definition 4.20.** Let $G_\mu$ denote the automorphism group $\mathrm{Aut}_k(D_\mu)$.

The following lemma is straightforward from the definitions.

**Lemma 4.21.** *$G_\mu$ is a finite type group scheme over $k$ with component group $S_\mu$ and whose identity component $G_\mu^0 = \prod_{i=1}^{\ell}(\mathbb{G}_m \rtimes U_{\mu_i-1})$, where $U_{\mu_i-1}$ is a split unipotent of dimension $\mu_i - 1$.*

**Definition 4.22.** Let $\mathrm{Emb}(D_\mu, C)$ denote the space of embeddings of $D_\mu$ into $C$.

Note that $\mathrm{Emb}(D_\mu, C)$ is a quasi-projective scheme. Indeed, the Hom scheme $\mathrm{Hom}(D_\mu, C)$ is quasi-projective and $\mathrm{Emb}(D_\mu, C) \subset \mathrm{Hom}(D_\mu, C)$ is open by [GW20, Proposition 12.93]. Note moreover that when $\mu_i = 1$ for all $i = 1, \dots, l$, then $\mathrm{Emb}(D_\mu, C) = \mathrm{Conf}_l(C)$. We define a partial order on ordered $l$-tuples by $\mu' \geq \mu$ if $\mu'_i \geq \mu_i$ for all $i = 1, \dots, l$. Note here that $\mu'$ and $\mu$ are not necessarily partitions of the same number.

**Lemma 4.23.** *We have the following.*

*(i) $G_\mu$ acts freely on $\mathrm{Emb}(D_\mu, C)$ and the quotient is isomorphic to $\mathrm{Sym}^\mu(C)$.*

*(ii) The quotient of $\mathrm{Emb}(D_\mu, C)$ by $G_\mu^0$ is isomorphic to the configuration space of $l = l(\mu)$ ordered distinct points $\mathrm{Conf}_l(C)$ such that the diagram below commutes.*

$$\begin{tikzcd} \mathrm{Emb}(D_\mu, C) \arrow[r, "G_\mu^0"] \arrow[dr, "G_\mu"'] & \mathrm{Conf}_{l(\mu)}(C) \arrow[d, "S_\mu"] \\ & \mathrm{Sym}^\mu(C) \end{tikzcd}$$

*(iii) Let $\mu' \geq \mu$. Then $\mathrm{Emb}(D_{\mu'}, C) \to \mathrm{Emb}(D_\mu, C)$ is a torsor for the subgroup group $G_{\mu'}^{\mu} \subset G_{\mu'}$ of automorphism of $D_{\mu'}$ which are the identity on $D_\mu$.*

*Proof.* We have a map $\mathrm{Emb}(D_\mu, C) \to \mathrm{Sym}^\mu(C)$ given by taking the image of an embedding $D_\mu \hookrightarrow C$ as a subscheme of $C$. Two embeddings have the same image if and only if they differ by a reparametrization of the source, proving $(i)$. For $(ii)$, note that $\mathrm{Conf}_{l(\mu)}(C) = \mathrm{Emb}(\mathrm{Spec} \prod_{i=1}^{l} k, C)$ is the space of embeddings for the partition $1 + \ldots + 1 = l$ with all parts equal to 1. Denoting this partition by $\mu_0$, it is clear that $G_\mu^0 = G_\mu^{\mu_0}$ and so the horizontal map in $(ii)$ is a special case of $(iii)$. The commutative diagram in $(ii)$ then follows by the identification of $S_\mu$ with the component group of $G_\mu$ coming from Lemma 4.21. Finally for $(iii)$, note that the natural map $\mathrm{Emb}(D_{\mu'}, C) \to \mathrm{Emb}(D_\mu, C)$ is induced by composition with the closed embedding $D_\mu \hookrightarrow D_{\mu'}$. Since all embeddings of $D_{\mu'}$ fixing $D_\mu$ have the same image in $\mathrm{Sym}^{\mu'}(C)$ (here we are using that $C$ is a smooth curve) then they differ by a reparametrization by $G_{\mu'}$ but this reparametrization must fix the embedding $D_\mu \hookrightarrow C$ so it must be an element of $G_{\mu'}^{\mu}$. ■

**Remark 4.24.** In fact, the torsor $\mathrm{Emb}(D_{\mu'}, C) \to \mathrm{Emb}(D_\mu, C)$ is a Zariski locally trivial fibration admitting a section and with fiber isomorphic (as a scheme not a group) to

$$\prod_{\mu_i = 1} \mathbb{G}_m \times \mathbb{A}^{\mu_i - 2} \times \prod_{\mu_i \neq 1} \mathbb{A}^{\mu'_i - \mu_i}.$$

This is because $G^{\mu}_{\mu'}$ is an extension of $\mathbb{G}_m$ by a split unipotent group, hence a special group scheme.

**Definition 4.25.** For $\mu' \geq \mu$ and a fixed embedding $f : D_\mu \to C$, we say that an embedding $f' : D_{\mu'} \to C$ extends $(f, D_\mu)$ if $f'|_{D_\mu} = f$.

We are now ready to define moduli spaces of weighted linear series with vanishing conditions. We will need some notation to talk about imposing both equalities and inequalities on the orders of vanishing of the sections. To accomplish this we will use a pair $\{\nu, T\}$ where $\nu = (\nu_1, \dots, \nu_l)$ is an ordered $l$-tuple and $T \subset \{1, \dots, l\}$ to encode the condition

$$\nu_{x_i}(s) \geq \nu_i \text{ with equality if } i \in T.$$

Finally, if $t$ is an integer and $\nu$ is a tuple, denote by $t\nu$ the tuple $(t\nu_1, \dots, t\nu_l)$.

In the following two definitions, $\mu_i$ encodes the multiplicity of the $i$-th component of the normalized base locus $\overline{\mathrm{Bs}}(L, s_0, \dots, s_N)$ and $\mu_i^{(j)}$ encodes the vanishing order of $s_j^{\bar{\lambda}_j}$ along the $i$-th component of $\overline{\mathrm{Bs}}(L, s_0, \dots, s_N)$. As a result, for each $i$, we want $\mu_i^{(j)} \geq \mu_i$ with equality for some $j$.

**Definition 4.26.** For each $j$, fix $\nu^{(j)}$ an $l$-tuple and a subset $T_j \subset \{1, \dots, l\}$ and let $\mu^{(j)} = \bar{\lambda}_j \nu^{(j)}$. Suppose that $\mu^{(j)} \geq \mu$ and that for each $i = 1, \dots, l$, $\mu_i = \min_j \{\mu_i^{(j)}\}$. We call

$$\gamma =: (\{\nu^{(0)}, T_0\}, \dots, \{\nu^{(N)}, T_N\})$$

a *tuple of vanishing orders realizing* $\mu$ and denote by $\gamma_j$ its $j^{th}$ component $\{\nu^{(j)}, T_j\}$.

**Definition 4.27.** A family of *$\vec{\lambda}$-weighted linear series of degree $n$ on $C$ with parametrized normalized base locus and vanishing order $\gamma$* over $B$ is the data of a weighted linear series $(L, s_0, \dots, s_N)$ of relative degree $n$ on $C_B$ and disjoint sections $\{\sigma_i : B \to C_B\}_{i=1}^{l}$ such that

1. $\overline{\mathrm{Bs}}(L, s_0, \dots, s_N)$ is a relative effective Cartier divisor over $B$ which is equal to $\sum \mu_i \sigma_i$,
2. for each $j = 1, \dots, N$ and any family of embeddings $D_{\mu^{(j)}} \times B \to C_B$ with image containing $\sum \mu_i \sigma_i$, the restriction $(L^{\otimes \kappa}, s_j^{\bar{\lambda}_j})|_{D_{\mu^{(j)}}}$ is identically 0, and
3. for each $j = 1, \dots, N$, each $i \in T_j$ and any family of embeddings $D_{\mu_i^{(j)}+1} \times B \to C_B$ with image $\mu_i \sigma_i$, the restriction $(L^{\otimes \kappa}, s_j^{\bar{\lambda}_j})|_{D_{\mu_i^{(j)}+1}}$ is not identically 0.

Note that $S_\mu$ acts on the set of vanishing orders $\gamma_j = \{\nu^{(j)}, T_j\}$ by permuting the $l$ parts of $\nu^{(j)}$ as well as acting on $T_j \subset \{1, \dots, l\}$ via the natural action on subsets of $\{1, \dots, l\}$.

We prove the Main Theorem of this section.

**Theorem 4.28.** *Fix $\vec{\lambda}$, $C$, $n$ and $\mu = (\mu_1, \dots, \mu_l)$ as above and let $\gamma$ be a tuple of vanishing conditions realizing $\mu$ as in Definition 4.26.*

1. *There exists a separated algebraic stack $\mathcal{R}^{\gamma}_{n,C}(\vec{\lambda})$ with a morphism $\mathcal{R}^{\gamma}_{n,C}(\vec{\lambda}) \to \mathrm{Conf}_{l(\mu)}(C)$ whose $B$-points are $\vec{\lambda}$-weighted linear series of degree $n$ with*

*parametrized base locus and vanishing order $\gamma$ over $B$. Moreover, there is a locally closed stratification*

$$\bigsqcup_{\gamma} \mathcal{R}_n^{\gamma} \to \mathcal{R}_n^{\mu}$$

*where the union is over all $\gamma$ realizing $\mu$.*

(2) *Let $\mathcal{W}_{n,C}^{\gamma}(\vec{\lambda}) = \left[\bigsqcup_{\gamma'} \mathcal{R}_{n,C}^{\gamma'}/S_{\mu}\right]$ where the union is taken over all $\gamma'$ in the $S_{\mu}$-orbit of $\gamma$. Then $\mathcal{W}_{n,C}^{\gamma}(\vec{\lambda})$ is a separated algebraic stack and we have an unramified surjective monomorphism*

$$\bigsqcup_{\gamma} \mathcal{W}_n^{\gamma} \to \mathcal{W}_n^{\mu}$$

*where the union is over a set of representatives for the $S_{\mu}$-orbits of the set of $\gamma$ realizing $\mu$.*

*Proof.* For any ordered tuple $\mu' \geq \mu$, we let $\mathcal{R}_n^{\mu,\mu'}$ denote the pullback

$$\mathcal{R}_n^{\mu,\mu'} = \mathrm{Emb}(D_{\mu'}, C) \times_{\mathrm{Conf}_{l(\mu)}} \mathcal{R}_n^{\mu}$$

A point of $\mathcal{R}_n^{\mu,\mu'}$ consists of a weighted linear series $(L, s_0, \ldots, s_N)$ of degree $n$ on $C_B$, $l$ disjoint sections $\sigma_i : T \to C_B$ such that the $\overline{\mathrm{Bs}}(L, s_0, \ldots, s_N)$ is the relative effective Cartier divisor $\sum \mu_i \sigma_i$ as well as an embedding $D_{\mu'} \times B \to C_B$ which contains $\sum \mu_i \sigma_i$ in its image. Note that $\mathcal{R}_n^{\mu,\mu'} \to \mathcal{R}_n^{\mu}$ is an $G_{\mu'}^0$-torsor by Lemma 4.23.

Over $\mathcal{R}_n^{\mu,\mu'}$ we have a universal linear series $(\mathcal{L}_{\mu}, s_0, \ldots, s_N)$ with normalized base profile $\mu$ as well as a universal closed embedding

$$D_{\mu'} \times \mathcal{R}_n^{\mu,\mu'} \hookrightarrow C \times \mathcal{R}_n^{\mu,\mu'}.$$

For each $j$, we can restrict $s_j$ along this closed embedding to obtain a line bundle with section $(\mathcal{L}_{\mu}^{\otimes \lambda_j}, s_j)|_{D_{\mu'}}$. Viewing this data as a morphism

$$D_{\mu'} \times \mathcal{R}_n^{\mu,\mu'} \to [\mathbb{A}^1/\mathbb{G}_m]$$

we obtain for each $j$ a morphism to the Hom-stack

$$\alpha_{\mu',j} : \mathcal{R}_n^{\mu,\mu'} \to \mathrm{Hom}(D_{\mu'}, [\mathbb{A}^1/\mathbb{G}_m]).$$

For each $i$, we can also restrict to $D_{\mu_i'} := \mathrm{Spec}\, k[x]/(x^{\mu_i'+1})$ and obtain a further composition

$$\alpha_{\mu',j,i} : \mathcal{R}_n^{\mu,\mu'} \to \mathrm{Hom}(D_{\mu_i'}, [\mathbb{A}^1/\mathbb{G}_m])$$

given by restricting $(\mathcal{L}_{\mu}^{\otimes \lambda_j}, s_j)$ to the $i^{th}$ jet $D_{\mu_i}$ of $D_{\mu}$. We can also take powers of the line bundle and section for any integer $k$ to obtain morphisms $\alpha_{\mu',j}^k$ and $\alpha_{\mu',j,i}^k$ classifying the restriction of $(\mathcal{L}_{\mu}^{\otimes k\lambda_j}, s_j^k)$.

Let $\tilde{Z}_{\mu',j,i}^k \subset \mathcal{R}_n^{\mu,\mu'}$ be the closed substack defined by the vanishing locus of $\alpha_{\mu',j,i}^k$ and let $\tilde{U}_{\mu',j,i}^k = \mathcal{R}_n^{\mu,\mu'} \setminus \tilde{Z}_{\mu',j,i}^k$. These substacks are invariant under the action of the various groups $G_{\mu'}^{\mu}$ and $G_{\mu}^0$ of reparametrizations of the embedding and so by Lemma 4.23, they descend to closed and open substacks $Z_{\mu',j,i}^k, U_{\mu',j,i}^k \subset \mathcal{R}_n^{\mu}$ respectively. The substack $Z_{\mu',j,i}^k$ (resp. $U_{\mu',j,i}^k$) parametrizes those $\vec{\lambda}$-weighted linear series $(L, s_0, \dots, s_N)$ such that $(L^{\otimes k\lambda_j}, s_j)|_{D_{\mu'_i}}$ vanishes identically (resp. does not vanish identically) for all embeddings $D_{\mu'} \hookrightarrow C$ containing the normalized base locus $\sum \mu_i x_i$ in their image. Then $\mathcal{R}_n^{\gamma}$ is the intersection of $Z^k_{\mu',j,i}$ and $Z^{k'}_{\mu'',j',i'}$ as $\mu', \mu'', j, j', i, i', k, k'$ vary over appropriate values. This implies that $\mathcal{R}_n^{\gamma}$ is locally closed and that they stratify $\mathcal{R}_n^{\mu}$. In particular, $\bigsqcup_{\gamma} \mathcal{R}_n^{\gamma} \to \mathcal{R}_n^{\mu}$ is an unramified surjective monomorphism. Since these properties can be checked étale locally, the same holds for $\bigsqcup_{\gamma} \mathcal{W}_n^{\gamma} \to \mathcal{W}_n^{\mu}$. ■

**Remark 4.29.** The map $\mathcal{W}_n^{\gamma} \to \mathcal{W}_n^{\mu}$ might fail to be a locally closed embedding. The reason is that the $S_\mu$-invariant union $\bigsqcup_{\gamma'} \mathcal{R}_n^{\gamma'}$ over the $S_\mu$-orbit of $\gamma$ may not be locally closed in $\mathcal{R}_n^{\mu}$ even though $\mathcal{R}_n^{\gamma} \to \mathcal{R}_n^{\mu}$ is locally closed embedding. However, $\mathcal{W}_n^{\gamma} \to \mathcal{W}_n^{\mu}$ is locally closed up to a locally closed stratification of the source and target and this is good enough for computing most invariants we will be interested in (e.g. number of points and the class in the Grothendieck ring of stacks).

## 5. Height moduli on cyclotomic stacks

In this section, we prove Theorem 1.2 concerning height moduli on cyclotomic stacks.

**Remark 5.1.** When $(\mathcal{X}, \mathcal{L}) = \big(\mathcal{P}(\vec{\lambda}), \mathcal{O}_{\mathcal{P}(\vec{\lambda})}(1)\big)$, we will see in the next section that the moduli space $\mathcal{M}_{n,C}(\mathcal{X}, \mathcal{L})$ is the space $\mathcal{W}_{n,C}^{min}(\vec{\lambda})$ of minimal weighted linear series from Proposition 4.15 the stratification into spaces of twisted maps $\mathcal{H}_{d,C}^{\Gamma}$ corresponds to the stratification of $\mathcal{W}_{n,C}^{min}$ into $\mathcal{W}_{n,C}^{\mu}$ for minimal partitions $\mu$.

**Remark 5.2.** In fact, the proof of Theorem 1.2 will show that $\mathcal{M}_{n,C}(\mathcal{X}, \mathcal{L})$ is a cyclotomic stack by proving that $\mathcal{M}_{n,C}(\mathcal{X}, \mathcal{L}) \subset \mathcal{M}_{n,C}(\mathcal{P}(\vec{\lambda}), \mathcal{O}_{\mathcal{P}}(\vec{\lambda})(1))$ is a closed substack and that $\mathcal{M}_{n,C}(\mathcal{P}(\vec{\lambda}), \mathcal{O}_{\mathcal{P}}(\vec{\lambda})(1))$ is cyclotomic.

As an immediate corollary we obtain the following Northcott property.

**Corollary 5.3.** *Fix $(\mathcal{X}, \mathcal{L})$ and $C$ as above and suppose $k$ is a finite field. Then for any $B > 0$, the set*

$$\{P \in \mathcal{X}(K) \mid \mathrm{ht}_{\mathcal{L}}(P) \le B\}$$

*is finite.*

**Remark 5.4.** It is not hard to see from the construction of $\mathcal{M}_{n,C}(\mathcal{X}, \mathcal{L})$ that it is compatible with base change in the following sense: if $k'/k$ is a field extension with $k'$ perfect and we denote $(\mathcal{X}', \mathcal{L}') = (\mathcal{X}, \mathcal{L}) \times_k k'$ and $C' = C \times_k k'$, then

$$\mathcal{M}_{n,C}(\mathcal{X}, \mathcal{L}) \times_k k' \cong \mathcal{M}_{n,C'}(\mathcal{X}', \mathcal{L}')$$

as stacks over $k'$.

### 5.1. **The weighted projective case.** We begin by proving Theorem 1.2 for

$$(\mathcal{X}, \mathcal{L}) = \big(\mathcal{P}(\vec{\lambda}), \mathcal{O}_{\mathcal{P}(\vec{\lambda})}(1)\big).$$

The key input is the correspondence in Theorem 3 and our main task is to check that this correspondence holds in families.

*Proof of Theorem 1.2 for weighted projective stacks.* We will check that the space $\mathcal{W}^{min}_{n,C}(\vec{\lambda})$ of minimal linear series on $C$ satisfies the properties of the theorem.

For part (1), note that by definition a $k$-point of $\mathcal{W}^{min}_{n,C}$ is a minimal $\vec{\lambda}$-weighted linear series $(L, s_0, \ldots, s_N)$ on $C$ with $\deg L = n$. This induces a rational map $C \dashrightarrow \mathcal{P}(\vec{\lambda})$ which gives us a rational point $P \in \mathcal{P}(\vec{\lambda})(K)$. By Theorem 3.3 (3), $\mathrm{ht}_{\mathcal{O}(1)}(P) = n$. On the other hand, a rational point $P \in \mathcal{P}(\vec{\lambda})(K)$ of height $n$ spreads out to a unique minimal weighted linear series by Theorem 3.3 (2) and the degree of the minimal weighted linear series is $n$ by Theorem 3.3 (3). This gives us the required canonical bijection

$$\mathcal{W}^{min}_{n,C}(\vec{\lambda})(k) \cong \left\{P \in \mathcal{P}(\vec{\lambda})(K) \mid \mathrm{ht}_{\mathcal{O}(1)}(P) = n\right\}.$$

For part (2), we begin with the following lemma. Let $\kappa := \mathrm{lcm}\{\lambda_0, \ldots, \lambda_N\}$ and $\bar{\lambda}_j := \kappa/\lambda_j$ as usual.

**Lemma 5.5.** *Suppose $\kappa > 1$. Then the map*

$$m \mapsto \left(\frac{\kappa}{\gcd(m,\kappa)}, \frac{m}{\gcd(m,\kappa)}\right)$$

*induces a bijection from the set $\{1, \ldots, \kappa - 1\}$ to the set*

$$\{(r,a) : 1 \leq a < r, r|\kappa, \gcd(r,a) = 1\}$$

*Proof.* By construction, the map lands in the required set so it suffices to construct an inverse. The inverse is simply given by

$$(r,a) \mapsto \frac{\kappa a}{r}$$

which is an integer in $\{1, \ldots, \kappa - 1\}$ by the conditions on $(r,a)$. ■

We have the following immediate corollary.

**Corollary 5.6.** *For each $l \geq 1$, there is a bijection between the set of $l$-tuples of admissible local conditions $\Gamma = (\{r_1, a_1\}, \ldots, \{r_l, a_l\})$ for a representable twisted map to $\mathcal{P}(\vec{\lambda})$ and the set of minimal ordered partitions $\mu$ with $l$ parts.*

*Proof.* A pair $(r,a)$ is admissible if and only if $r|\kappa$ while a partition $\mu$ is minimal if and only if $\mu_i < \kappa$ for each $i$. Thus the required bijection is the $l$-fold product of the bijection in Lemma 5.5. ■

Now by Proposition 4.15, $\mathcal{W}^{min}_n$ has a finite locally closed stratification into $\mathcal{W}^{\mu}_n$ where $\mu$ runs over all minimal partitions with $|\mu| \leq \kappa n$, where the inequality holds by Proposition 4.10 (3). Fix one such partition $\mu$ and suppose $l(\mu) = l$. By the above corollary, there exists a unique $l$-tuple of local conditions $\Gamma = (\{r_1, a_1\}, \ldots, \{r_l, a_l\})$ corresponding to the partition $\mu$. We will show that $\mathcal{H}^{\Gamma}_d/S_{\Gamma} \cong \mathcal{W}^{\mu}_n$ where $n$ and $d$ are related as in the statement of Theorem 1.2. More precisely, we will show the following.

**Proposition 5.7.**

$$\mathcal{H}^{\Gamma}_d \cong \mathcal{R}^{\mu}_n$$

*as stacks over* $\mathrm{Conf}_l(C)$.

Part (2) of the theorem then follows.

*Proof of Proposition 5.7.* In one direction, let $(\mathcal{C} \to B, f : \mathcal{C} \to \mathcal{P}(\vec{\lambda}), \{\Sigma_i\}_{i=1}^l)$ be a family of twisted maps from $C$ to $\mathcal{P}(\vec{\lambda})$ of type $\Gamma$ and degree $d$. Let $\mathcal{L} = f^*\mathcal{O}_{\mathcal{P}(\vec{\lambda})}(1)$ and let $t_j \in H^0(\mathcal{C}, \mathcal{L}^{\otimes \lambda_j})$ be the pullback of the canonical section of $\mathcal{O}_{\mathcal{P}(\vec{\lambda})}(\lambda_j)$. Let $u_i$ be the section of $\mathcal{O}_{\mathcal{C}}(\Sigma_i)$ cutting out the marked $\mu_{r_i}$-gerbe $\Sigma_i$. By construction the line bundle

$$\mathcal{L}\left(\sum_{i=1}^{l} a_i \Sigma_i\right)$$

carries the trivial character at each point of the twisted curve $\mathcal{C}/B$ and thus is the pullback of a line bundle $L$ of degree $n = d + \sum_{i=1}^{l} \frac{a_i}{r_i}$ along the coarse map $\pi : \mathcal{C} \to C_B$. Moreover, as in the proof of Proposition 3.6, $s_j := \pi_*\left(t_j \prod_i u_i^{\lambda_j a_i}\right)$ is a section of $L^{\otimes \lambda_j}$. Thus, $(L, s_0, \ldots, s_N)$ is a family $\vec{\lambda}$-weighted linear series on $C_B \to B$. Moreover, since $\mathcal{C}$ is tame, the coarse moduli map $\pi_*$ is exact and its formation commutes with base-change. In particular, the construction taking the twisted map to the weighted linear series is functorial and by checking on fibers over $b \in B$ and applying Proposition 3.6, we conclude that $(L, s_0, \ldots, s_N)$ is a family of minimal linear series. Next, applying Proposition 3.9 and Corollary 5.6, we conclude that $(L, s_0, \ldots, s_N)$ has base profile $\mu$ along the marked sections $\sigma_i = \pi_*\Sigma_i$. Putting this all together, we see that $(C_B \to B, \sigma_i, L, s_0, \ldots, s_N)$ is a $B$-point of $\mathcal{R}_n^\mu$ as required.

On the other hand, suppose $(C_B \to B, \sigma_i, L, s_0, \ldots, s_N)$ is a $B$-point of $\mathcal{R}_n^\mu$ and let $\Gamma = (\{r_1, a_1\}, \ldots, \{r_l, a_l\})$ be the local conditions corresponding to $\mu$ via Corollary 5.6. Let $\mathcal{C}$ be the iterated root stack of $C_B$ along $\sigma_i$ to order $r_i$ for each $i = 1, \ldots, l$ and let $\Sigma_i$ denote the $\mu_{r_i}$ gerbe corresponding to $\sigma_i$. Let $\mathcal{L}$ on $\mathcal{C}$ be the line bundle

$$\mathcal{L} = \pi^* L\left(-\sum_{i=1}^{l} a_i \Sigma_i\right).$$

Then $\mathcal{L}$ is uniformizing with relative degree $d$ over $B$ and local conditions $\Gamma$ by construction. Moreover,

$$t_j = \frac{\pi^* s_j}{\prod_i u_i^{\lambda_j a_i}} \in H^0(\mathcal{L}^{\otimes \lambda_j})$$

and by checking on fibers over $b \in B$, we conclude that $(L, t_0, \ldots, t_N)$ is a basepoint-free weighted linear series by Proposition 3.7. The formation of root stacks commutes with base change and so this operation is functorial. Moreover these two operations are clearly inverses and so we have an isomorphism $\mathcal{H}_d^\Gamma \cong \mathcal{R}_n^\mu$ as claimed.

Finally, it follows from Corollary 3.11 that $\mathrm{ht}^{st}(P) = d$ and the local contributions to height are $\frac{a_i}{r_i}$ for $P \in \mathcal{P}(\vec{\lambda})(K)$ corresponding to a $k$-point of $\mathcal{H}_d^\Gamma$. ■

This concludes the proof of Theorem 1.2 for the case of weighted projective stacks. ■

5.2. **The general cyclotomic case.** In this section we reduce Theorem 1.2 for general proper polarized cyclotomic stacks $(\mathcal{X}, \mathcal{L})$ to the case of $\mathcal{P}(\vec{\lambda})$.

*Proof of Theorem 1.2 in general.* By Proposition 2.15, there exists a representable closed embedding $\mathcal{X} \subset \mathcal{P}(\vec{\lambda})$ for some $\vec{\lambda}$ such that $\mathcal{L} = \mathcal{O}_{\mathcal{P}(\vec{\lambda})}(1)|_{\mathcal{X}}$. Now $\mathcal{X}(K)$ is a sub-groupoid of $\mathcal{P}(\vec{\lambda})(K)$. For any $P \in \mathcal{X}(K)$, there exsts a unique universal tuning stack with a representable morphism $\mathcal{C} \to \mathcal{X}$ and generic point $P$. The composition $\mathcal{C} \to \mathcal{P}(\vec{\lambda})$ is a representable morphism with generic point $P$ viewed as a point in $\mathcal{P}(\vec{\lambda})(K)$. Thus $\mathcal{C}$ is the universal tuning stack for $P \in \mathcal{P}(\vec{\lambda})(K)$ by uniqueness and $\mathcal{L}|_{\mathcal{C}} = \mathcal{O}_{\mathcal{P}(\vec{\lambda})}(1)|_{\mathcal{C}}$ by functoriality. Thus we have equalities

$$\begin{aligned}&\mathrm{ht}_{\mathcal{L}}(P) = \mathrm{ht}_{\mathcal{O}_{\mathcal{P}(\vec{\lambda})}(1)}(P),\ \mathrm{ht}^{st}_{\mathcal{L}}(P) = \mathrm{ht}^{st}_{\mathcal{O}_{\mathcal{P}(\vec{\lambda})}(1)}(P),\ \text{and}\\ &\delta_x(P,\mathcal{L}) = \delta_x(P,\mathcal{O}_{\mathcal{P}(\vec{\lambda})}(1)) \text{ for all } x \in C.\end{aligned}$$

To finish the proof we show that the condition for a family of $\vec{\lambda}$-weighted linear series to have generic point mapping to $\mathcal{X} \subset \mathcal{P}(\vec{\lambda})$ is closed inside $\mathcal{W}_n^{min}$ and thus cuts out the stack $\mathcal{M}_{n,C}(\mathcal{X},\mathcal{L})$ inside $\mathcal{W}_n^{min}$. To do this, we first consider the twisted map strata $\mathcal{H}^{\Gamma}_{d,C}(\mathcal{P}(\vec{\lambda}),\mathcal{O}(1))/S_{\Gamma}$. Since $\mathcal{X} \subset \mathcal{P}(\vec{\lambda})$ is closed, it is a closed condition in the stack of twisted maps for a $\mathcal{C} \to \mathcal{P}(\vec{\lambda})$ to map to $\mathcal{X}$, and hence the condition for a family of $\vec{\lambda}$-weighted linear series to have generic point mapping to $\mathcal{X} \subset \mathcal{P}(\vec{\lambda})$ is constructible inside $\mathcal{W}_n^{min}$. It therefore remains to prove that this locus is stable under specialization. To this end, let $R$ by a DVR with fraction field $K'$ and residue field $k'$, and let $(L, s_0, \ldots, s_N)$ be a minimal $\vec{\lambda}$-weighted linear series on $C_R$. Let $U \subset C_R$ be the complement of the base locus, which is a non-empty open subset. Then we have a morphism $f : U \to \mathcal{P}(\vec{\lambda})$ and we are assuming that the generic point of $U_{K'}$ maps to $\mathcal{X}$. Since $U \to \operatorname{Spec} R$ is flat, the generic point of $U_{K'}$ specializes to the generic point of $U_{k'}$. Since $\mathcal{X} \subset \mathcal{P}(\vec{\lambda})$ is closed, this implies that the generic point of $U_{k'}$ also maps to $\mathcal{X}$. We have therefore simultaneously shown that $\mathcal{M}_{n,C}$ is a closed substack of $\mathcal{W}_n^{rat}$ and that it comes equipped with the required stratification as in part (2). ■

**Remark 5.8.** Our construction of $\mathcal{M}_{n,C}$ gives us a closed substack of $\mathcal{W}_n^{min}$ with reduced induced structure. This is good enough for applications to point counting, computing classes in the Grothendieck ring, and homological/representation stability. However, it is an interesting question if $\mathcal{M}$ itself carries a natural moduli interpretation, as opposed to just its $k$-points.

**Remark 5.9.** Under the isomorphism $\mathcal{H}^{\Gamma}_d \cong \mathcal{R}^{\mu}_n$ in Proposition (5.7), the stratification of $\mathcal{R}^{\mu}_n$ constructed in Theorem 4.28 yields a stratification of $\mathcal{H}^{\Gamma}_d$. The stratum $\mathcal{R}^{\gamma}_n$ corresponds to stratifying by tangency conditions for the marked gerbes along the toric boundary of $\mathcal{P}(\vec{\lambda})$. This suggests that if one replaces $\mathcal{P}(\vec{\lambda})$ by a smooth Deligne–Mumford toric stack, then one may construct analogues of the strata $\mathcal{R}^{\gamma}_n$ in terms of logarithmic twisted maps.

**Definition 5.10.** We say that a rational point $P \in \mathcal{X}(K)$ is *isotrivial* if the stable height is zero $\mathrm{ht}^{st}(P) = 0$.

**Remark 5.11.** This is equivalent to the usual notion of isotrivial since $\mathrm{ht}^{st}(P) = \deg(\mathcal{C} \to \mathcal{X})$ measured with respect to $\mathcal{L}$. Since $\mathcal{L}^{\otimes M}$ descends to an ample line bundle on the coarse space $X$, the stable height is zero if and only if the morphism on coarse moduli spaces $C \to X$ is constant. Note in particular that the condition of being isotrivial is independent of the choice of polarizing line bundle $\mathcal{L}$.

**Proposition 5.12.** *There is a closed substack $\mathcal{M}^{iso}_{n,C}(\mathcal{X}) \subset \mathcal{M}_{n,C}(\mathcal{X},\mathcal{L})$ parametrizing isotrivial $K$-points.*

*Proof.* By Remark 5.2, $\mathcal{M}_{n,C}(\mathcal{X},\mathcal{L})$ is a closed substack of $\mathcal{M}_{n,C}(\mathcal{P}(\vec{\lambda}),\mathcal{O}_{\mathcal{P}}(\vec{\lambda})(1))$. It is immediate from the definitions that $P \in \mathcal{X}(K)$ is isotrivial if and only if $P \in \mathcal{P}(\vec{\lambda})(K)$ is isotrivial. As a result, $\mathcal{M}^{iso}(\mathcal{X})$ is a closed substack of $\mathcal{M}^{iso}(\mathcal{P}(\vec{\lambda}))$, and hence it suffices to prove the result for $\mathcal{P}(\vec{\lambda})$.

Given $P \in \mathcal{P}(\vec{\lambda})(K)$, let $L, s_0, \ldots, s_N)$ be the corresponding $\vec{\lambda}$-weighted linear series and let $\mathcal{C} \to \mathcal{P}(\vec{\lambda})$ be the universal stack with local conditions

$$\Gamma = (\{r_1, a_1\}, \ldots, \{r_\ell, a_\ell\})$$

Then $P$ is isotrivial if and only if

$$\mathrm{ht}_{\mathcal{O}(1)}(P) = \sum_{i=1}^{\ell} \frac{a_i}{r_i} = \frac{1}{\kappa}\deg(\overline{Bs}(L, s_0, \ldots, s_N)).$$

In other words, $P$ is isotrivial if and only if its normalized base locus has the largest degree possible. The locus of such points $P$ is given by the smallest stratum in the stratification from Proposition 4.10, and hence closed. ■

Finally, we give a simple application to bounding points of bad reduction. A typical situation is that $\mathcal{X}$ is the compactification of a moduli space $\mathcal{U}$ of smooth objects and $\mathcal{X} \setminus \mathcal{U} = \mathcal{D}$ is a Cartier divisor parametrizing singular objects. We call $\mathcal{D}$ the boundary divisor. This is the case for example for the moduli space of elliptic curves $\mathcal{M}_{1,1} \subset \overline{\mathcal{M}}_{1,1}$ with $\mathcal{D} = \infty$.

**Definition 5.13.** Let $(\mathcal{X},\mathcal{L})$ be a polarized cyclotomic stack and let $\mathcal{D} \subset \mathcal{X}$ be a $\mathbb{Q}$-Cartier divisor with open complement $\mathcal{U}$. Let $K = k(C)$ and let $P \in \mathcal{U}(K)$ with universal tuning stack $f : (\mathcal{C}, \Sigma_i) \to \mathcal{X}$. We say $P$ has *bad reduction at $x \in C$* if $x$ is in the image of the locus

$$f^{-1}(\mathcal{D}) \cup \bigsqcup_{i=1}^{l} \Sigma_i.$$

**Proposition 5.14.** *Fix $(\mathcal{X},\mathcal{L},\mathcal{D})$ as in Definition 5.13. Then there is a uniform bound on the number of points of bad reduction depending only on the height of the rational point. More precisely, there is a function $N_{\mathcal{X},\mathcal{L},\mathcal{D}}(n)$ such that for all $K = k(C)$ and all $P \in \mathcal{U}(K) \subset \mathcal{X}(K)$,*

$$\#\{x \in C \mid P \text{ has bad reduction at } x\} \le N_{\mathcal{X},\mathcal{L},\mathcal{D}}(\mathrm{ht}_{\mathcal{L}}(P)).$$

*Proof.* Fix $K = k(C)$ and $P \in \mathcal{U}(K)$. Let $n = \mathrm{ht}_{\mathcal{L}}(P)$ and choose $M$ such that $\mathcal{L}^{\otimes M} = \pi^* L$, where $\pi\colon \mathcal{X} \to X$ is the coarse space map and $L$ is ample on $X$. Then we obtain a map $\bar{f} : C \to X$ and we have $d = \frac{1}{M}\deg(\bar{f})$, where degree is measured with respect to $L$. Letting $\Gamma = (\{r_1, a_1\}, \ldots, \{r_\ell, a_\ell\})$ be the local conditions of the universal tuning stack of $P$, we have $n = d + \sum_{i=1}^{\ell} \frac{a_i}{r_i}$. Since the map $f : \mathcal{C} \to \mathcal{X}$ from the universal tuning stack is representable, we see all $r_i$ are bounded. Since $n$ is also bounded by assumption, we see $d$ and the number of stacky points $\ell$ are bounded. As a result, we have bounded the number of points $P$ of bad reduction which are in the image of $\bigsqcup_{i=1}^{l} \Sigma_i$.

It remains to show that the number of points in the image of $f^{-1}(\mathcal{D})$ are also bounded. Let $D$ be the coarse space of $\mathcal{D}$, so that $D \subset X$ is a divisor. Since $\mathcal{D}$ is Cartier, then $mD$ is Cartier for some fixed $m > 0$. Then since $C$ is not contained in $D$, the number of points in the image of $f^{-1}(\mathcal{D})$ is bounded by the intersection number $C\cdot(mD)$. To bound this quantity, we consider the moduli space of degree $Md$ maps from curves to $X$. This space is quasi-projective so has finitely many irreducible components $Z_1,\dots,Z_m$. For each $i$, there is a non-empty open set $U_i \subset Z_i$ such that for all points $\bar{f}: C' \to X$ of $U_i$, the quantity $C'\cdot(mD)$ is a constant $N_i$. Since $C\cdot(mD)$ is bounded above by $\max_i N_i$, we have therefore bounded the number of points of bad reduction. ■

## 6. Examples of height moduli on cyclotomic stacks

**6.1. Moduli of $\mu_d$-torsors.** The classifying stack $\mathcal{B}\mu_d$ of a finite cyclic group $\mu_d$ is a special case of a weighted projective stack when $N=0$ and the weight is $d$. Let $\mathbb{G}_m$ act on itself via the $d$-th power map; we denote this space by $U_d$. Then

$$\mathcal{B}\mu_d \cong \mathcal{P}(d) = [U_d/\mathbb{G}_m] \subset [\mathbb{A}^1/\mathbb{G}_m]. \tag{4}$$

Thus we can apply the results of the previous sections. Note also that $\mathcal{O}_{\mathcal{P}(d)}(1)$ is the standard representation of $\mu_d$ given by the inclusion $\mu_d \subset \mathbb{G}_m$ so throughout this section height will be with respect to this representation.

For simplicity, let us assume first that $d$ is coprime to the characteristic. Then a $B$-point of $\mathcal{B}\mu_d$ is an étale $\mu_d$-torsor $P \to B$. A $B$-point of $\mathcal{P}(d)$ on the other hand is the data of a line bundle $L$ and a non-vanishing section $s$ of $L^d$. The isomorphism (4) is given by taking $(L,s)$ to the $\mu_d$-torsor

$$\mathrm{Spec}_B\, \mathcal{O}_B[z]/(z^d - s) \to B.$$

By Theorem 3.3 a $k(C)$ point of $B\mu_d$ is equivalent to a minimal $d$-weighted linear series $(L,s)$ on $C$ where $s \in H^0(C, L^{\otimes d})$. Minimality is the condition that $\nu_x(s) < d$ for all $x \in C$. Moreover, an isomorphism $(L,s) \to (L',s')$ is an isomorphism $t: L \to L'$ such that $t^d(s) = s'$. In particular, $\mathbb{G}_m$ acts with weight $d$ on the space of sections of $L$. We gather the basic properties of $(L,s)$ and the associated twisted map below.

**Lemma 6.1.** *Let $K = k(C)$ and $P \in \mathcal{B}\mu_d(K)$ with associated minimal linear series $(L,s)$, where $\deg(L) = n$. Write the vanishing locus $V(s)$ as a divisor $\sum \mu_i x_i$ for $x_i \in C$ closed points where $\mu_i < d$ for all $i$ by minimality. Then $\mathrm{ht}(P) = n$, $\mathrm{ht}^{st}(P) = 0$ and $\delta_{x_i}(P) = \frac{\mu_i}{d}[k(x_i):k]$ and with $\delta_x(P) = 0$ for $x \in C \smallsetminus \{x_1,\dots,x_l\}$. Moreover, the universal tuning stack $\pi: \mathcal{C} \to C$ is the root stack of $C$ along $x_i$ to order*

$$r_i = \frac{d}{\gcd(\mu_i, d)}$$

*and the $i^{th}$ component of the class of $\mathcal{L}$ in $\mathrm{Pic}(\mathcal{C})/\mathrm{Pic}(C)$ is given by*

$$a_i = \frac{\mu_i}{\gcd(\mu_i, d)}.$$

Moreover, by Proposition 3.7, the section $s$ of $L^{\otimes d}$ extends to a non-vanishing section $t$ of $\mathcal{L}^{\otimes d}$ on $\mathcal{C}$ where $\mathcal{L} = \pi^* L\left(-\sum \frac{\mu_i}{d} x_i\right)$. The $\mu_d$ torsor associated to the morphism $\mathcal{C} \to \mathcal{B}\mu_d$ is exactly

$$\mathrm{Spec}_{\mathcal{C}}\, \mathcal{O}_{\mathcal{C}}[z]/(z^d - t) \to \mathcal{C}.$$

The $\mu_d$-cover of $C$ defined by the original $(L,s)$ is branched precisely at the points where $s$ vanishes; introducing stacky structure makes this branched cover étale.

Fixing $\mu$ and denoting by $\Gamma$ the associated local conditions for the twisted map, the moduli space $\mathcal{H}^{\Gamma}_{0,C}(\mathcal{B}\mu_d)$ can be described as the space of tuples $(\mathcal{C}, \Sigma_i, \mathcal{L}, t)$ where $(\mathcal{C}, \Sigma_i)$ is a twisted curve with stabilizers $\mu_{r_i}$, $\mathcal{L}$ is a line bundle on the torsion component of $\mathrm{Pic}(\mathcal{C})$ corresponding to the class $(a_i \mod r_i)$ in $\mathrm{Pic}(\mathcal{C})/\mathrm{Pic}(C)$, and $t$ is a trivialization of $\mathcal{L}^{\otimes d}$. Since the trivialization is unique up to scaling and $\mu_d \subset \mathbb{G}_m$ stabilizes $(\mathcal{C}, \Sigma_i, \mathcal{L}, t)$, we obtain the following.

**Corollary 6.2.** *Let $(\mathcal{C}, \Sigma_i) \to \mathrm{Conf}_l(C) = B$ denote the universal family of twisted curves with stabilizers $\mu_{r_1}, \dots, \mu_{r_l}$ and coarse moduli space $C$. The space $\mathcal{H}^{\Gamma}_{0,C}(\mathcal{B}\mu_d)$ is a $\mu_d$-gerbe over the relative $d$-torsion scheme*

$$\mathrm{Pic}^{a_1,\dots,a_r}_{\tau}(\mathcal{C}/B)[d] \to B$$

*where $\mathrm{Pic}^{a_1,\dots,a_r}_{\tau}$ denotes the unique torsion component of $\mathrm{Pic}(\mathcal{C})$ which maps to $(a_i \mod r_i)$ in the quotient $\mathrm{Pic}(\mathcal{C})/\mathrm{Pic}(C_B)$.*

Next we explicate the height moduli space $\mathcal{M}_{n,C}(\mathcal{B}\mu_d)$. Let $\mathcal{L}$ denote the universal line bundle on $C \times \mathcal{P}ic^n(C)$. Let $\mathbb{V}^d_n$ denote the abelian cone on $\mathcal{P}ic^n(C)$ parametrizing sections of $\mathcal{L}^{\otimes d}$ with $\mathbb{G}_m$ action given by the $d$-power character.

**Corollary 6.3.** *$\mathcal{M}_{n,C}(\mathcal{B}\mu_d) \subset \mathbb{V}^d_n \to \mathcal{P}ic^n(C)$ is the open substack parametrizing sections satisfying the minimality condition.*

When $C$ has a rational point and $nd > 2g-1$ we can be even more explicit using Remarks 4.6, 4.7 and 4.9. In this case, $\mathbb{V}^d_n$ is simply $[V^d_n/\mathbb{G}_m]$ where $V^d_n = \rho_*\mathcal{L}^{\otimes d}$ is a vector bundle on $\mathrm{Pic}^n(C)$ and $\mathbb{G}_m$ acts with weight $d$.

**Proposition 6.4.** *Suppose $C(k) \neq \emptyset$, and let $W^{min}_{nd} \subset \mathrm{Sym}^{nd}(C)$ denote the open subset of the symmetric product of closed subschemes of $C$ with minimal base profile. Then $\mathcal{M}_{n,C}(\mathcal{B}\mu_d)$ is a $\mu_d$ gerbe over the pullback $\mu^*_{d,n}W^{min}_{nd}$ under the $d$-power map $\mu_{d,n} : \mathrm{Pic}^n(C) \to \mathrm{Pic}^{nd}(C)$. If $dn > 2g-1$, then $\mathcal{M}_{n,C}(\mathcal{B}\mu_d)$ is an open substack of a weighted projective bundle*

$$\left[U^d_n/\mathbb{G}_m\right] \subset \left[V^d_n \setminus 0_{\mathrm{Pic}^n(C)}/\mathbb{G}_m\right] \to \mathrm{Pic}^n(C)$$

*where the $\mathbb{G}_m$ action is by weight $d$ and $U^d_n$ is the locus of sections satisfying the minimality condition.*

We can understand the $k$-points of $\mathcal{M}_{n,C}(\mathcal{B}\mu_d)$ using this description as follows. A point of $W^{min}_{nd}$ is the ramification divisor $R$ of the cover $z^d = s$ which satisfies the minimality condition. The pullback along $\mu_{d,n}$ corresponds to choosing a line bundle $L$ such that $L^{\otimes d} \cong \mathcal{O}_C(R)$. The $\mu_d$-gerbe accounts for Galois twists of $z^d = s$.

Finally, if we assume that $k$ contains all $d^{th}$ roots of unity, then by Kummer theory, $\mu_d$-torsors over $K = k(C)$ are the same as cyclic Galois extensions of degree $d$. In this case, we can interpret $\Gamma$ as ramification conditions. Indeed if $\pi : D \to C$ is a cyclic Galois cover associated to a point of $\mathcal{H}^{\Gamma}_{0,C}(\mathcal{B}\mu_d)$ and $x_i \in C$ is a point with twisting conditions $(r_i, a_i)$, then $\pi^{-1}(x_i)$ consists of $d/r_i$ distinct points that are each ramified to order $r_i$. Moreover, étale locally around $q \in \pi^{-1}(x_i)$, the map $\pi$ can be identified with the map

$$t \mapsto t^{a_i}$$

where $t$ is a uniformizer.

**Corollary 6.5.** *Suppose $k$ contains $d^{th}$ roots of unity. Then $\mathcal{M}_{n,C}(\mathcal{B}\mu_d)$ is the moduli space of cyclic Galois extensions $F/k(C)$ of degree $d$ and height $n$. Moreover, $\mathcal{H}^{\Gamma}_{0,C}(\mathcal{B}\mu_d)$ is the moduli space of such cyclic Galois extensions with local ramification conditions encoded by $\Gamma$.*

6.2. **Moduli of $k(t)$-points on weighted projective stacks.** Fix $\vec{\lambda}$, $\kappa$ and $\bar{\lambda}_j$ as above. In this section we explicitly describe the height moduli for $k(t)$-points on $\mathcal{P}(\vec{\lambda})$. By Theorem 3.3, this is equivalent to describing the moduli of weighted linear series on $C = \mathbb{P}^1_k$.

In this case, $\mathcal{P}ic^n(C) \cong \mathcal{B}\mathbb{G}_m$ is a trivial gerbe over $\mathrm{Pic}^n(C) = \operatorname{Spec} k$. The maps $\mu_{d,n} : \mathrm{Pic}^n(C) \to \mathrm{Pic}^{nd}(C)$ are the identity and we can view $\mathcal{W}_n \to \mathcal{B}\mathbb{G}_m$ as the abelian cone associated to the following $\mathbb{G}_m$-representation. Let $V_n^d = H^0(\mathbb{P}^1, \mathcal{O}(dn))$ equipped with a $\mathbb{G}_m$ action of weight $d$ and let

$$\mathcal{Q} = \bigoplus_{j=0}^{N} (V_n^{\lambda_j})^{\vee}$$

viewed as a $\mathbb{G}_m$-equivariant locally free sheaf on $\operatorname{Spec} k$. For a vector space $V$, the abelian cone $\operatorname{Spec}\operatorname{Sym}^{\bullet}(V^{\vee})$ is simply $V$ itself viewed as an affine space. Thus

$$\mathcal{W}_n = [\mathbb{V}_{\operatorname{Spec} k}(\mathcal{Q})/\mathbb{G}_m] \cong \left[\bigoplus_{j=0}^{N} V_n^{\lambda_j}/\mathbb{G}_m\right] \to \mathcal{B}\mathbb{G}_m$$

is simply the stack of tuples $(f_0, \ldots, f_N)$ of binary forms where $\deg f_i = \lambda_i \cdot n$ modulo the action of $\mathbb{G}_m$ given by

$$t \cdot (f_0, \ldots, f_N) = (t^{\lambda_0} f_0, \ldots, t^{\lambda_N} f_N).$$

The stratum $\mathcal{W}_n^{\infty} \subset \mathcal{W}_n$ is the zero section of the projection

$$\left[\bigoplus_{j=0}^{N} V_n^{\lambda_j}/\mathbb{G}_m\right] \to \mathbb{G}_m$$

and thus $\mathcal{W}_n \smallsetminus \mathcal{W}_n^{\infty}$ is identified with the ambient weighted projective stack

$$\mathcal{P}\left(\bigoplus_{j=0}^{N} V_n^{\lambda_j}\right).$$

Fixing coordinates, we can think of this concretely as the weighted projective stack

$$\mathcal{P}(\underbrace{\lambda_0, \ldots, \lambda_0}_{n\lambda_0+1}, \ldots, \underbrace{\lambda_N, \ldots, \lambda_N}_{n\lambda_N+1})$$

parameterizing the coefficients of the homogeneous polynomials $(f_0, \ldots, f_N)$ (see [PS21a, Section 4.1]).

Fix a partition $\mu$ with $l$ parts and $\gamma$ a tuple of vanishing orders realizing $\mu$ as in Definition 4.26. We have natural locally closed subvarieties

$$W_n^{min},\ W_n^{\mu} \subset W_n := \bigoplus_{j=0}^{N} V_n^{\lambda_j} \smallsetminus 0.$$

Here $W_n^{min}$ is the set

$$\left\{(f_0,\ldots,f_N) \mid \min_j\left(\bar{\lambda}_j\nu_x(f_j)\right) \le \kappa \text{ for all } x \in \mathbb{P}^1\right\}$$

which is open in $W_n$ by Lemma 4.16, and $W_n^{\mu}$ is the locus of $(f_0,\ldots,f_N)$ such that $\left(f_0^{\bar{\lambda}_0},\ldots,f_N^{\bar{\lambda}_N}\right)$ simultaneously vanish at exactly $l$ points to order $\mu_i$ for $i=1,\ldots,l$.

**Lemma 6.6.** *The subvarieties $W_n^{min}$ and $W_n^{\mu}$ of $W_n$ are $\mathbb{G}_m$-invariant and we have*

$$\mathcal{W}_n^{min} = \left[W_n^{min}/\mathbb{G}_m\right] \text{ and } \mathcal{W}_n^{\mu} = \left[W_n^{\mu}/\mathbb{G}_m\right].$$

Next we describe the parametrized version $\mathcal{R}_n^{\mu}$. Let $\pi_{n,\mu} : \mathcal{R}_n^{\mu} \to \mathrm{Conf}_{l(\mu)}$ denote the natural projection and let $\pi_{n,\gamma}$ be the restriction of $\pi_{n,\mu}$ to $\mathcal{R}_n^{\gamma} \subset \mathcal{R}_n^{\mu}$. Then $\mathrm{Conf}_{l(\mu)} = (\mathbb{P}^1)^l \smallsetminus \Delta$ where $\Delta$ is the big diagonal. For each $(x_1,\ldots,x_l) \in \mathrm{Conf}_{l(\mu)}(\mathbb{P}^1)$, we define

$$R_n^{\gamma}(x_1,\ldots,x_n) = \left\{(f_0,\ldots,f_N) \mid \nu_{x_i}(f_j) \ge \nu_i^{(j)} \text{ with equality if } i \in T_j\right\}.$$

**Lemma 6.7.** *For each $(x_1,\ldots,x_l) \in \mathrm{Conf}_{l(\mu)}(\mathbb{P}^1)$, the locus $R_n^{\gamma}(x_1,\ldots,x_n) \subset W_n^{\gamma}$ is locally closed and $\mathbb{G}_m$-invariant. Moreover, we have an identification*

$$\pi_{n,\gamma}^{-1}(x_1,\ldots,x_l) = \left[R_n^{\gamma}(x_1,\ldots,x_l)/\mathbb{G}_m\right].$$

We end the section with a discussion of the special case $l=1$ which will feature prominently later in the paper. The weighted linear series has a basepoint at exactly one point $x$ with normalized multiplicity $\mu \in \mathbb{Z}_{>0}$. In this case the vanishing order is just a single number $\nu^{(j)}$ and $T_j$ is either empty, in which case we allow $\nu_x(f_j) \ge \nu^{(j)}$, or $\{1\}$, in which case we require $\nu_x(f_j) = \nu^{(j)}$. For this reason we denote the tuple as a list of numbers, some of which have inequalities.

**Example 6.8.** If $\gamma = (\{1,\varnothing\},\{1,\{1\}\})$, we denote it by the symbol $(\ge 1, 1)$.

**Example 6.9.** If $\gamma = (\{2,\{1\}\},\{3,\{1\}\})$, we denote it by the symbol $(2,3)$.

**Proposition 6.10.** *When $l=1$,*

$$\pi_{n,\gamma} : \mathcal{R}_n^{\gamma} \to \mathbb{P}^1$$

*is a Zariski-locally trivial fibration which can be written as a $\mathbb{G}_m$ quotient of a locally trivial fibration of $\mathbb{P}^1$ with fibers $R_n^{\gamma}(x)$. In particular,*

$$R_n^{\gamma}(x) \cong R_n^{\gamma}(y)$$

*for all $x,y \in \mathbb{P}^1$ and we have an equality*

$$\{\mathcal{R}_n^{\gamma}\} = \{\mathbb{G}_m\}\cdot\{R_n^{\gamma}(0)\}$$

*in the Grothendieck ring of stacks (Section 8.2).*

*Proof.* By Lemma 6.7, $\mathcal{R}_n^{\gamma}$ is isomorphic over $\mathbb{P}^1$ to the $\mathbb{G}_m$ quotient of the locus inside

$$\mathbb{P}^1 \times \bigoplus_{j=0}^{N} V_N^{\lambda_j}$$

of pairs $(x,f_1,\ldots,f_N)$ such that the $f_i$ do not simultaneously vanish on $\mathbb{P}^1 \smallsetminus x$, $\min_j\{\bar{\lambda}_j f_j\} = \mu$, and $\nu_x(f_j) \ge \nu^{(j)}$ with equality if $T_j = \{1\}$; note that this locus agrees with $W_n^\gamma$ since $\ell = 1$. As $\mathbb{G}_m$ is special, it suffices to show that $W_n^\gamma \to \mathbb{P}^1$ is Zariski-locally trivial. Now $W_n^\gamma \to \mathbb{P}^1$ is equivariant for the natural $\mathrm{PGL}_2$ action

$$(x, f_1, \dots, f_N) \mapsto (gx, f_1 \circ g^{-1}, \dots, f_N \circ g^{-1}).$$

Locally in a neighborhood of any given point, say $0 \in \mathbb{P}^1$, there exists a chart $0 \in U \subset \mathbb{P}^1$ and a family of automorphisms $\phi : U \to \mathrm{PGL}_2$ such that $U$ is invariant under $\phi(U)$ and $\phi(u) \cdot 0 = u$. Now let $W_n^\gamma(0)$ denote the fiber of $W_n^\gamma$ over 0 and consider the map

$$U \times W_n^\gamma(0) \to (W_n^\gamma)|_U$$

given by

$$(u, f_1, \dots, f_N) \mapsto (u, f_1 \circ \phi(u)^{-1}, \dots, f_N \circ \phi(u)^{-1})$$

On the other hand, we can act on $(W_n^\gamma)|_U$ by $\phi(u)^{-1}$ to produce a map backwards and it is clear these maps are inverses, proving the required Zariski-local triviality. The computation of the motives then follows from This follows from [Eke09, Proposition 1.1 iii), 1.4]. ■

**Remark 6.11.** Note that when $l = 1$, $\mathrm{Conf}_1 \cong \mathrm{Sym}^{\{\mu\}}(\mathbb{P}^1)$ and $\mathcal{W}_n^\gamma = \mathcal{R}_n^\gamma$ so we conclude that $\mathcal{W}_n^\gamma$ is a locally trivial fibration over $\mathbb{P}^1$ with fiber $\left[R_n^\gamma(0)/\mathbb{G}_m\right]$.

6.3. **Cyclotomic moduli of curves.** Under mild conditions on the characteristic, various moduli stacks of curves are isomorphic to weighted projective stacks, and thus admit height moduli.

An important example that will be our focus for the rest of the paper is the moduli stack of stable elliptic curves. Over $\mathbb{Z}\left[\frac{1}{6}\right]$, we have an explicit isomorphism

$$\overline{\mathcal{M}}_{1,1} \cong \mathcal{P}(4,6)$$

given by the short Weierstrass equation $y^2 = x^3 + a_4x + a_6$, where $\zeta \cdot a_i = \zeta^i a_i$ for $\zeta \in \mathbb{G}_m$ and $i = 4, 6$.

Similarly, one could consider the stack of generalized elliptic curves with level structure, introduced in the work of Deligne and Rapoport [DR73] (summarized in [Con07, §2] and also in [Nil13, §2]). We will consider the case of level-2 structure in this paper.

The moduli space $\overline{\mathcal{M}}_1(2)$ of elliptic curves with level-2 structure parametrizes families $(E, S, P) \to B$ where $(E, S) \to B$ is a semi-stable elliptic curve with section $S$, and $P \in E^{sm}[2](B)$ is a 2-torsion section such that the divisor $P + S$ is relatively ample ([KM85, §1.4]). Over $\mathbb{Z}\left[\frac{1}{2}\right]$ there is an isomorphism

$$\overline{\mathcal{M}}_1(2) \cong \mathcal{P}(2,4),$$

given by the short Weierstrass equation $y^2 = x^3 + a_2x^2 + a_4x$, where $\zeta \cdot a_i = \zeta^i a_i$ for $\zeta \in \mathbb{G}_m$ and $i = 2, 4$ (see e.g. [Beh06, §1.3]). The identity section is the point at infinity while the 2-torsion section is the point $(x, y) = (0, 0)$.

Similarly, recall that full level-2 structure on a semi-stable elliptic curve $(E, S) \to B$ is the choice of 2-torsion sections $P, Q \in E^{sm}[2](B)$ in the smooth locus such that

(a)
$$D = \sum_{(i,j)\in(\mathbb{Z}/2\mathbb{Z})^2} [iP]+[jQ] \subset E^{sm}$$
is a subgroup scheme which is equal to the kernel of the multiplication-by-2 map $E^{sm} \to E^{sm}$.

(b) $D$ is relatively ample as a Cartier divisor.

Here $[nP]$ denotes the image of $P$ under the multiplication by $n$ map. Over $\mathbb{Z}\left[\frac{1}{2}\right]$ the moduli $\overline{\mathcal{M}}(2)$ of elliptic curves with full level-2 structure has an isomorphism
$$\overline{\mathcal{M}}(2) \cong \mathcal{P}(2,2),$$
given by the short Weierstrass equation $y^2 = x(x+\lambda_1)(x+\lambda_2)$, where the degree of each $\lambda_i$ is 2 and $\zeta \cdot \lambda_i = \zeta^i \lambda_i$ for $\zeta \in \mathbb{G}_m$ and $i = 2$ (see e.g. [Sto12, Proposition 7.1]). The full level-2 structure is given by the infinity section and the $x = 0$, $x = -\lambda_1$ and $x = \lambda_2$ sections.

We refer the reader to [BPS22, Proposition 5.1] for a detailed treatment where moduli stacks of generalized elliptic curves with other level structures are shown to be weighted projective stacks.

In [LP19, Theorem A] is shown that the Smyth spaces $\overline{\mathcal{M}}_{1,m}(m-1)$, introduced in [Smy11a, Smy11b], of $m$-marked $(m-1)$-stable curves of arithmetic genus one are cyclotomic stacks (see also [BPS22, Proposition 5.8]).

There are also examples in higher genus. Quasi–admissible curves, studied in [SF00, §2.4.] and [Fed14], are natural degenerations of hyperelliptic genus $g \geq 2$ curves which form a proper Deligne–Mumford stack $\mathcal{H}_{2g}[2g-1]$. For the case of monic odd–degree hyperelliptic curves with a generalized Weierstrass equation $y^2 = x^{2g+1} + a_4x^{2g-1} + a_6x^{2g-2} + a_8x^{2g-3} + \cdots + a_{4g+2}$ we have
$$\mathcal{H}_{2g}[2g-1] \cong \mathcal{P}(4,6,8,\ldots,4g+2)$$
by [Fed14, Proposition 4.2(1)] over $\mathrm{char}(K) = 0$ and by [HP20, Proposition 5.9] over $\mathrm{char}(K) > 2g+1$. Furthermore, when $g = 2$, the case $y^2 = x^5 + a_4x^3 + a_6x^2 + a_8x + a_{10}$ with $\mathcal{H}_4[3] \cong \mathcal{P}(4,6,8,10)$ which is of special interest as all genus 2 curves are hyperelliptic.

## 7. Moduli stacks of elliptic surfaces with specified Kodaira fibers

In this section, we formulate the moduli stacks of elliptic surfaces of stacky height $n$ with fixed singular fibers. Via the isomorphism $\overline{\mathcal{M}}_{1,1} \cong \mathcal{P}(4,6)$ over $\mathbb{Z}\left[\frac{1}{6}\right]$, these are identified with the moduli spaces $\mathcal{H}_d^\Gamma$ and $\mathcal{R}_n^\gamma$ of twisted maps with twisting conditions $\Gamma$ and weighted linear series with vanishing conditions $\gamma$ respectively and the isomorphisms between $\mathcal{H}_d^\Gamma$ and $\mathcal{R}_n^\gamma$ for various $\Gamma, \gamma, d$ and $n$ are interepreted in terms of the birational geometry of elliptic surfaces.

We begin with the basic definitions surrounding elliptic surfaces following [Mir89, SS10] (see also [Sil09, Liu02]).

**Definition 7.1.** An irreducible **elliptic surface** $(f : X \to C, S)$ is an irreducible algebraic surface $X$ together with a surjective flat proper morphism $f : X \to C$ to a smooth curve $C$ and a section $S : C \hookrightarrow X$ such that:

(1) the generic fiber of $f$ is a stable elliptic curve, and

(2) the section is contained in the smooth locus of $f$.

**Definition 7.2.** A **minimal elliptic surface** is an elliptic surface which is relatively-minimal i.e., if none of the fibers contain any $(-1)$-curves.

**Definition 7.3.** A **Weierstrass fibration** is an elliptic surface obtained from an elliptic surface by contracting all fiber components not meeting the section. We call the output of this process a **Weierstrass model**. If starting with a smooth relatively-minimal elliptic surface, we call the result a **minimal Weierstrass model**.

Lastly, the geometry of an elliptic surface is largely influenced by the *fundamental line bundle* $\mathcal{L}$ which is the $L$ in a weighted linear series.

**Definition 7.4.** The **fundamental line bundle** of an elliptic surface is $\mathcal{L} := (f_*\mathcal{N}_{S/X})^\vee$, where $\mathcal{N}_{S/X}$ denotes the normal bundle of $S$ in $X$.

An important observation is that the Weierstrass model over $\mathbb{Z}\left[\frac{1}{6}\right]$ has a global Weierstrass equation of the form

$$\{y^2 = x^3 + Ax + B\} \in \mathbb{P}_C(\mathcal{O} \oplus \mathcal{L}^{\otimes -2} \oplus \mathcal{L}^{\otimes -3}) \tag{5}$$

where $A \in H^0(C, \mathcal{L}^{\otimes 4})$ and $B \in H^0(C, \mathcal{L}^{\otimes 6})$ as in [Mir89, Corollary 2.5] and the minimality condition for the Weierstrass model is exactly the same as the minimality condition as in §3.2 for a weighted linear series.

The singular fibers of elliptic fibrations were classified in the classical works of [Kod63, N$\acute{6}$4]. There are two types of elliptic surfaces depending on what kind of singular fibers (i.e. bad reductions) underlying elliptic fibrations have. When there exists only **multiplicative** bad reductions then we call such a smooth relatively-minimal elliptic surface to be an *semistable elliptic surface*. When there exists at least one **additive** bad reduction then we call such a smooth relatively-minimal elliptic surface to be an *unstable elliptic surface*. By the well-known Tate's algorithm, the classification of singular fibers corresponds to vanishing conditions on the coefficients of the Weierstrass equation (see [Her91, Table 1]).

The following is clear.

**Proposition 7.5.** *The height moduli space $\mathcal{W}^{min}_{n,C}(4,6) = \mathcal{M}_{n,C}(\mathcal{P}(4,6), \mathcal{O}(1))$ of weighted linear series is the moduli stack of minimal Weierstrass models $(f : X \to C, S)$ with the degree of the fundamental line bundle equal to $n$.*

As an immediate corollary we obtain the following.

**Corollary 7.6.** *The stratum $\mathcal{R}^{\gamma}_{n,C}(4,6)$ is the moduli space of Weierstrass elliptic surfaces with unstable fibers specificed by $\gamma$. Moreover, the Faltings height of an elliptic surface given by the degree of the of the fundamental line bundle agrees with the stacky height for a rational point of $\overline{\mathcal{M}}_{1,1} \cong \mathcal{P}(4,6)$ with $\mathcal{O}(1)$.*

On the other hand, the spaces of twisted maps to $\overline{\mathcal{M}}_{1,1}$ also have an interpretation as moduli of elliptic surfaces. This was originally used in [AV02, AB19] to compactify the moduli space of elliptic surfaces. We review the construction here for the benefit of the reader.

**Definition 7.7.** A **stable stack-like** elliptic surface is a tuple $(h : \mathcal{Y} \to \mathcal{C}, \mathcal{S})$ where $h$ is a surjective proper representable morphism from an orbifold surface to a twisted curve and $(\mathcal{Y}, \mathcal{S}) \to \mathcal{C}$ is a flat family of stable elliptic curves with section such that the stabilizers of $\mathcal{C}$ act generically fixed point free on the fibers of $h$. A **twisted model** is the coarse moduli space $(g : Y \to C, S_0)$ of a stable stack-like surface.

A stable stack-like surface induces a representable classifying morphism

$$\varphi : \mathcal{C} \to \overline{\mathcal{M}}_{1,1}$$

where $\mathcal{Y} \cong_{\mathcal{C}} \varphi^* \overline{\mathcal{E}}$ for $\overline{\mathcal{E}} \to \overline{\mathcal{M}}_{1,1}$ the universal family. In particular, the twisted structure on $\mathcal{C}$ and the stabilizer action on $\varphi^*\mathcal{O}(1)$ are encoded by a tuple of local twisting conditions $\Gamma$.

**Proposition 7.8.** *The space $\mathcal{H}^{\Gamma}_{d}(\mathcal{P}(4,6))$ is the moduli of stable stack-like elliptic surfaces with local twisting conditions $\Gamma$ such that $\varphi^*\mathcal{O}(1) \cong \mathcal{L}$ is the fundamental line bundle and $12d = \deg(j)$ where $j : C \to \overline{M}_{1,1}$ is the j-map. In particular, the stable height of an elliptic surface is $\frac{1}{12}\deg(j)$.*

*Proof.* The identification of moduli spaces is clear. The only thing we need to check is that $12d = \deg(j)$ but this follows from the observation that $\pi^*\mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{O}(12)$ where $\pi : \overline{\mathcal{M}}_{1,1} \to \overline{M}_{1,1} \cong \mathbb{P}^1$ is the coarse map. ∎

**Remark 7.9.** In the special case $\Gamma$ is empty, $\mathcal{H}^{\Gamma}_{n}$ is just $\mathrm{Hom}_n(C, \overline{\mathcal{M}}_{1,1})$ studied in [HP19] for $C = \mathbb{P}^1$ case regarding its motive in the Grothendieck ring of stacks and [BPS22] for a general smooth projective curve $C$ of any genus $g$ regarding its $\ell$-adic étale cohomology with Frobenius weights. The identification between $\mathcal{H}^{\emptyset}_{n}$ and $\mathcal{R}^{\emptyset}_{n}$ is simply the fact that a semi-stable elliptic surface can equivalently be described by a Weierstrass equation or by a classifying morphism $C \to \overline{\mathcal{M}}_{1,1}$ and in this case the stacky height is exactly $n = \frac{1}{12}\deg(j)$.

7.1. **Geometric interpretation of Tate's algorithm.** By Tate's algorithm, the base profile $\gamma$ of the Weierstrass equation determines the unstable singular Kodaira fibers of the minimal resolution. On the other hand, by Theorem 1.2 for weighted projective stacks, this is equivalent to local twisting conditions $\Gamma$ which in turn determines the stabilizers of the stable stack-like model. The unstable singular fibers of the twisted model are then determined by the following lemmas.

**Lemma 7.10.** *Let $h : \mathcal{E} \to \mathcal{D}$ be a semistable family of elliptic curves where $\mathcal{D} = \left[\operatorname{Spec} k[t]^{sh}/\mu_r\right]$ and $\varphi : \mathcal{D} \to \overline{\mathcal{M}}_{1,1}$ is a twisted map with local condition $(r, a)$. Let $p \in \mathcal{E}_0$ be a smooth fixed point of the $\mu_r$ action on the central fiber $\mathcal{E}_0$. Then the weights of the $\mu_r$ action on the tangent space $T_{\mathcal{E},p}$ are $(-1, -a)$.*

*Proof.* Since $\mathcal{D}$ is strictly henselian, there is a section of $S$ of $h : \mathcal{E} \to \mathcal{D}$ passing through $p$. Thus the tangent space $T_{\mathcal{E},p}$ splits as a direct sum of fiber and section direction and each of these directions are eigenspaces for the $\mu_r$ action. The section direction is canonically isomorphic to the tangent space $T_{\mathcal{D},0}$ of the base which is isomorphic to $(\mathfrak{m}/\mathfrak{m}^2)^{\vee}$ where $\mathfrak{m}$ is the maximal ideal of 0. Since $t$ is a uniformizer, $\mathfrak{m}/\mathfrak{m}^2$ is rank 1 generated by $t$ which transforms as $t \mapsto \zeta t$, so the dual has weight $(-1)$. On the other hand, $\mathcal{L} = \varphi^*\mathcal{O}(1)$ is the fundamental line bundle of the elliptic fibration and thus isomorphic to $h_*(N_{S/\mathcal{E}})^{\vee}$. Thus the fiber

$$\mathcal{L}^{\vee}|_0 \cong N_{S/\mathcal{E}}|_0 = T_{\mathcal{E}_0,p}$$

and so the weight of $\mu_r$ acting on $T_{\mathcal{E}_0,p}$ is minus the weight on $\mathcal{L}|_0$ which is $-a$. ∎

**Lemma 7.11.** *Let $(h : \mathcal{E} \to \mathcal{D}, \mathcal{S})$ be a stable stack-like elliptic surface over $\mathcal{D} = \left[\operatorname{Spec} k[t]^{sh}/\mu_r\right]$ and let $(g : Y \to D, S_0)$ be the coarse moduli space. Then the central fiber $g^*(0)$ has multiplicity $r$.*

*Proof.* Let $\pi : \mathcal{E} \to Y$ and $\rho : \mathcal{D} \to D$ be the coarse moduli maps. Then $\rho$ is ramified to order $r$ at $0$ so $h^*\rho^*(0) = r\mathcal{E}_0$. By commutativity of $\pi, \rho, h$ and $g$, we have $\pi^*g^*(0) = r\mathcal{E}_0$. On the other hand, $\mu_r$ acts faithfully on $\mathcal{E}_0$. In particular, this action is generically fixed point free so $\pi$ is generically étale along $\mathcal{E}_0$ and of degree 1. Thus $\pi_*\mathcal{E}_0 = Y_0$ so by push-pull, we have $g^*(0) = rY_0$. ∎

Particularly, we remark that this gives a new interpretation of Tate's algorithm as: the vanishing condition $\gamma = (\nu(a_4),\ \nu(a_6))$ of the minimal weighted linear series $\implies$ the local twisting condition $\Gamma = (r, a)$ of the twisted maps $\implies$ the unstable singular fibers of the twisted model $\implies$ specified type of Kodaira fiber by resolving singularities and contracting $(-1)$-curves. This is summarized in the diagram below.

$$\begin{tikzcd}
 & & & & \mathcal{Y} = \varphi^*\,\overline{\mathcal{E}} \arrow[r] \arrow[dl, "\pi"'] \arrow[dd, "h"] & \overline{\mathcal{E}} \arrow[dd, "p"] \\
X \arrow[dd, "f"] & X' \arrow[l, "\text{Weierstrass}"'] \arrow[dd, "f'"] & \hat{X} \arrow[l, "\text{contraction}"'] \arrow[r, "\text{resolution}"] \arrow[dd, "\hat{f}"] & Y \arrow[dd, "g"] & & \\
 & & & & \mathcal{C} \arrow[r, "\varphi"] \arrow[dl, "\rho"'] & \overline{\mathcal{M}}_{1,1} \arrow[d] \\
C \arrow[r, equal] & C \arrow[r, equal] & C \arrow[r, equal] & C \arrow[rr, "j"] \arrow[urr, dashed, "\psi"] & & \overline{M}_{1,1}
\end{tikzcd}$$

Here $f$ is a Weierstrass model, $\psi$ is the associated weighted linear series viewed as a rational map to $\overline{\mathcal{M}}_{1,1}$, $j$ is the $j$-invariant, $\varphi$ is the universal tuning stack which induces a stable stack-like model $h : \mathcal{Y} \to \mathcal{C}$, $g : Y \to C$ is the twisted model, $\hat{f}$ is a resolution of $Y$, and $f'$ is the relative minimal model obtained by contracting relative $(-1)$-curves. Lemmas 7.10 and 7.11 determine singular fibers of $g$ and the singularities of $Y$ which determine $\hat{X}$ and in turn the Kodaira fiber of the minimal model $f'$. By Proposition 3.9, the local twisting conditions for $\varphi$ depend only on the base multiplicity of the normalized linear series. The extra data we need to determine the singular fiber is simply the order of vanishing of the $j$ map at $j = \infty$ as this determines the singularities of $\mathcal{Y}$. Combining these observations, we get.

**Theorem 7.12** (Tate's algorithm via twisted maps)**.** *The local twisting condition $(r, a)$ and the order of vanishing of $j$ at $j = \infty$ determine the Kodaira fiber type of the relative minimal model, and $(r, a)$ is in turn determined by $m = \min\{3\,\nu(a_4), 2\,\nu(a_6)\}$.*

**Remark 7.13.** The fact that Tate's algorithm can be phrased so that the Kodaira fiber depends only on the normalized base multiplicity, that is, the minimum of the normalized orders of vanishing of the Weierstrass equation was first observed in [DD13].

The output of this analysis is summarized in Theorem 1.5. A key point is that Theorem 3.3 and Proposition 3.9 allows us to carry out this analysis for any moduli spaces which are isomorphic to weighted projective stacks and thus gives us a generalization of Tate's algorithm for maps to weighted projective stacks. The output of this analysis for $\overline{\mathcal{M}}_1(2)$ is summarized in Theorem A.1.

**Example 7.14.** Suppose that normalized base multiplicity $m = 3$. This occurs if and only if $(\nu(a_4), \nu(a_6)) = (1, \geq 2)$. Then $r = 12/\gcd(3, 12) = 4$ and $a = 3/\gcd(3, 12) = 1$. Thus the stabilizer of the twisted curve acts on the central fiber of the twisted model via the character $\mu_4 \to \mu_4$, $\zeta_4 \mapsto \zeta_4^{-1}$. In particular, the central fiber $E$ of $\mathcal{Y}$ has $j = 1728$. The $\mu_4$ action on $E$ has two fixed points, and there is an orbit of size two with stabilizer $\mu_2 \subset \mu_4$. Let $E_0$ be the image of $E$ in the twisted model $Y$. By Lemma 7.11, $E$ appears with multiplicity 4 By Lemma 7.10, $Y$ has $\frac{1}{4}(-1,-1)$ quotient singularities at the images of the the fixed points and a $\frac{1}{2}(-1,-1)$ singularity at the image of the orbit of size two. Each of these singularities is resolved by a single blowup to obtain $\hat{X}$ with central fiber $4\tilde{E}_0 + E_1 + E_2 + E_3$ where $E_i$ are the exceptional divisors of the resolution for $i = 1, 2, 3$ and $E_1^2 = E_2^2 = -4$ with $E_3^2 = -2$. Then $\tilde{E}_0$ is a $(-1)$-curve so it needs to be contracted. After this contraction $E_2$ becomes a $(-1)$ curve and must also be contracted. Since $E_i$ for $i = 1, 2, 3$ are incident and pairwise transverse after blowing down $\tilde{E}_0$, then the images of $E_1$ and $E_2$ must be tangent after blowing down $E_3$. Moreover, they are now $(-2)$-curves and so we conclude that this is the relatively minimal model and that it is the type III configuration in Kodaira's classification. See [AB17, Section 4] for more details on these blowup computations.

**Example 7.15.** Suppose $m = 6$ which occurs in the cases $(\nu(a_4), \nu(a_6)) = (2, 3), (\geq 3, 3)$ or $(2, \geq 4)$. Note these three cases are distinguished by their $j$-invariant but are identical from the point of view of twisted maps. Then $(r, a) = (2, 1)$ so we have a $\mu_2$ automorphism on the central fiber $E \subset \mathcal{Y}$. If $j \neq \infty$, $E$ is smooth and the $\mu_2$ action has 4 fixed points, the 2-torsion points. The image $E_0$ of $E$ appears with multiplicity 2 in the central fiber of the twisted model $Y$ and $Y$ has four $\frac{1}{2}(-1,-1)$ singularities. Note these are simply $A_1$ singularities so they are resolved by a single blowup which extracts a $(-2)$-curve. This resolution is already relatively minimal and it is exactly a Kodaira $\mathrm{I}_0^*$ fiber. If $j = \infty$ then $E$ is nodal. There are two smooth fixed points as well as the nodal point. At the nodal point $\mathcal{Y}$ has an $A_{2n-1}$ singularity where $2n$ is the ramification of $\varphi$, or equivalently $n$ is the order of vanishing $j$ at $\infty$. The quotient of the $A_{2n-1}$ singularity by the $\mu_2$ action is an $A_3$ singularity if $n = 1$ and a $D_{n+2}$ singularity if $n \geq 2$. In either case, resolving these singularities yields a $D_{n+4}$ configuration of $(-2)$ curves, that is, an $\mathrm{I}_n^*$ fiber. See [AB17, Lemma 4.2] for more details about this resolution.

7.2. **Identifying the universal families.** In this section we promote the geometric discussion of Tate's algorithm via twisted maps in the previous section (which we now think of as a bijection between Weierstrass elliptic surfaces of height $n$ with vanishing conditions $\mu = (m_1, \ldots, m_s)$ and twisted elliptic surfaces of stable height $d$ with twisting conditions $\Gamma = \{(r_1, a_1), \ldots, (r_s, a_s)\}$) into an identification of respective universal families.

More specifically, by Proposition 5.7, we have a canonical isomorphism

$$\rho : \mathcal{H}_n^{\Gamma} \cong \mathcal{R}_n^{\mu} \tag{6}$$

where $\mu$ and $\Gamma$ are in bijection via Lemma 5.5 and

$$n = d + \sum \frac{a_i}{r_i}.$$

By Corollary 7.6, we have a universal family $(f : X \to C \times \mathcal{R}_n^{\mu}, S, \sigma_1, \ldots, \sigma_s)$ of Weierstrass elliptic surfaces of height $n$ with section $S$ and unstable fibers along the marked points $\sigma_i : \mathcal{R}_n^{\mu} \to C \times \mathcal{R}_n^{\mu}$ with normalized base multiplicity $m_i$. On the other hand, by Proposition 7.8, there is a universal family of stable stack-like elliptic surfaces $(h : \mathcal{Y} \to \mathcal{C} \to \mathcal{H}_d^{\Gamma}, \mathcal{S}, \Sigma_1, \ldots, \Sigma_s)$ where $\Sigma_i$ are a family of marked gerbes with twisting conditions $\Gamma$. By taking coarse space we also have a universal family of twisted surfaces $(g : Y \to C \times \mathcal{H}_d^{\Gamma} \to \mathcal{H}_d^{\Gamma}, S_0, \sigma_i)$.

**Theorem 7.16.** *The isomorphism $\rho$ is induced by a birational transformation of universal families. More precisely, $\rho^*(f : X \to C \times \mathcal{R}_n^{\mu}, S, \sigma_i)$ is the family of Weierstrass models of $(h : \mathcal{Y} \to \mathcal{C} \to \mathcal{H}_d^{\Gamma}, \mathcal{S}, \Sigma_i)$ and conversely $(\rho^{-1})^*(g : Y \to C \times \mathcal{H}_d^{\Gamma} \to \mathcal{H}_d^{\Gamma}, S_0, \sigma_i)$ is the family of twisted models of $(f : X \to C \times \mathcal{R}_n^{\mu} \to \mathcal{R}_n^{\mu}, S, \sigma_i)$.*

*Proof.* The proof follows the same strategy as the general wall-crossing theorems of [AB21, Inc20, ABIP21] for moduli of elliptic surfaces. Namely both twisted models and Weierstrass models are canonical models of the pair $(X', S' + \sum a_i F_i)$ for different coefficients $a_i$. Here $(X' \to C, S')$ is a relatively minimal elliptic surface and $F_i$ are the unstable fibers. Both $\mathcal{H}$ and $\mathcal{R}$ can then be identified as strata in the moduli space of log canonical models and $\rho$ is the restriction of a wall-crossing morphism which relates the moduli space for different values of $a_i$. The content of the theorem then is that the minimal model programs that terminate in the Weierstrass model (resp. the twisted model) can be run on the universal families. In order to avoid the technical details of moduli spaces of canonical models, we sketch a direct proof here.

First note that since our stacks are tame, the formation of the coarse moduli space $g : Y \to C \times \mathcal{H}_d^{\Gamma} \to \mathcal{H}_d^{\Gamma}$ commutes with base change and $Y \to \mathcal{H}_d^{\Gamma}$ is flat. Moreover, for each $i$, we can consider the family $\mathcal{P}_i = \mathcal{S} \cap h^{-1}(\Sigma_i)$. The stabilizer $\mu_{r_i}$ of $\Sigma_i$ acts by an automorphism of the fiber as a pointed elliptic curve and in particular fixes the section $\mathcal{S}$. Thus $\mathcal{P}_i \to \mathcal{H}_d^{\Gamma}$ is an étale $\mu_{r_i}$-gerbe, in fact isomorphic to $\Sigma_i$, and the weight of the $\mu_{r_i}$ action on the tangent space of the fibers of $\mathcal{Y} \to \mathcal{H}_d^{\Gamma}$ is fixed by Lemma 7.10 as we fix the local twisting condition $(r_i, a_i)$. Thus the family of twisted surfaces $Y \to \mathcal{H}_d^{\Gamma}$ is equisingular with singularity $\frac{1}{r_i}(-1, -a_i)$ along the coarse space $P_i = S_0 \cap f^{-1}(\sigma_i) \subset Y$ of $\mathcal{P}_i \subset \mathcal{Y}$. By [Inc20, Proposition 5.9], we can take a partial fiberwise resolution $\mu : \hat{X} \to Y \to \mathcal{H}_d^{\Gamma}$ which resolves the singularities of the family $Y \to \mathcal{H}_d^{\Gamma}$ along $P_i$ for each $i$.

Denote the corresponding family of elliptic fibrations by

$$(\hat{f} : \hat{X} \to C \times \mathcal{H}_d^{\Gamma} \to \mathcal{H}_d^{\Gamma}, \hat{S}, \hat{\sigma}_i).$$

By construction, the section $\hat{S}$ passes through the smooth locus of $\hat{f}$ and the formation of $\mu$ commutes with basechange. Now the Weierstrass model of $\hat{f}$ can be computed as the relative Proj

$$X' = \mathrm{Proj}_{C\times \mathcal{H}_d^\Gamma} \bigoplus_{m\geq 0} \hat{f}_* \mathcal{O}_{\hat{X}}(m\hat{S}) \to C \times \mathcal{H}_d^\Gamma.$$

The cohomology $H^i\left(\hat{f}^{-1}(p), \mathcal{O}_{\hat{X}}(m\hat{S})\big|_{\hat{f}^{-1}(p)}\right) = 0$ for each $p \in C \times \mathcal{H}_d^\Gamma$ and $i > 0$ and so the formation of $X'$ commutes with base change. Thus we have produced a family of Weierstrass models $(f' : X' \to C \times \mathcal{H}_d^\Gamma \to \mathcal{H}_d^\Gamma, S', \sigma_i')$ over $\mathcal{H}_d^\Gamma$. Moreover, for each $\xi \in \mathcal{H}_d^\Gamma$, the fiber $(f_\xi' : X_\xi' \to C, S_\xi', (\sigma_i')_\xi)$ is the Weierstrass model of the fiber $(h : \mathcal{Y}_\xi \to \mathcal{C}_\xi, \mathcal{S}_\xi, (\Sigma_i)_\xi)$. By Tate's algorithm via twisted maps (Theorem 7.12), this family of Weierstrass model has marked unstable fibers with vanishing conditions $\mu$ and thus induces the map $\rho : \mathcal{H}_d^\Gamma \to \mathcal{R}_n^\mu$ such that

$$(f' : X' \to C \times \mathcal{H}_d^\Gamma \to \mathcal{H}_d^\Gamma, S', \sigma_i') = \rho^*(f : X \to C \times \mathcal{R}_n^\mu \to \mathcal{R}_n^\mu, S, \sigma_i)$$

as required. The converse follows since $\rho$ is an isomorphism by Proposition 5.7. ■

The upshot is that under the identification $\rho : \mathcal{H}_d^\Gamma \cong \mathcal{R}_n^\mu$, the height moduli space has two interpretations as the moduli space of elliptic surfaces with specified Kodaira fibers and these two interpretations are equivalent by Tate's algorithm via twisted maps.

## 8. Motives & Point counts of height moduli over finite fields

In this section, we enumerate the number of minimal elliptic surfaces with a specified Kodaira fiber. Following Proposition 6.10, we consider the case of $C = \mathbb{P}_k^1$ (i.e. the basecurve is the projective line) and $l = 1$ (i.e. there is exactly one additive reduction fiber of specified Kodaira type and the rest of the bad reduction fibers are strictly multiplicative).

Let us first review basics on arithmetic of algebraic stacks over finite fields.

### 8.1. **Point counts of algebraic stacks over finite fields.**

The moduli functors we wish to enumerate are often represented by algebraic stacks rather than by schemes (or algebraic spaces) due to the presence of non-trivial automorphisms of the objects we wish to parameterize. To account for these automorphisms, point-counts over $\mathbb{F}_q$ are weighted by the size of their automorphism groups:

**Definition 8.1.** The weighted point count of $\mathcal{X}$ over $\mathbb{F}_q$ is defined as a sum:

$$\#_q(\mathcal{X}) := \sum_{x \in \mathcal{X}(\mathbb{F}_q)/\sim} \frac{1}{|\mathrm{Aut}(x)|},$$

where $\mathcal{X}(\mathbb{F}_q)/\sim$ is the set of $\mathbb{F}_q$–isomorphism classes of $\mathbb{F}_q$–points of $\mathcal{X}$, and we take $\frac{1}{|\mathrm{Aut}(x)|}$ to be 0 when $|\mathrm{Aut}(x)| = \infty$.

A priori, the weighted point count can be $\infty$, but when $\mathcal{X}$ is of finite type, then the stratification of $\mathcal{X}$ by gerbes over schemes as in [Beh93, Proof of Lemma 3.2.2] implies that $\mathcal{X}(\mathbb{F}_q)/\sim$ is a finite set, so that $\#_q(\mathcal{X}) < \infty$. The weighted point count $\#_q(\mathcal{X})$ of an algebraic stack $\mathcal{X}$ over $\mathbb{F}_q$ is *algebro-topological* under the framework of the Weil conjectures as it is equal to the alternating sum of the trace of geometric Frobenius via the *Grothendieck-Lefschetz trace formula* for Artin stack, see [Beh93, Sun12]. Lastly, we note that the weighted point count $\#_q(\mathcal{X})$ is not equal to the exact number $|\mathcal{X}(\mathbb{F}_q)/\sim|$ of $\mathbb{F}_q$–isomorphism classes (i.e., the *non–weighted point count* of $\mathcal{X}$ over $\mathbb{F}_q$) where we do not weight the $\mathbb{F}_q$-rational points by automorphisms of the objects we wish to parameterize. In this regard, when the coarse moduli space $c : \mathcal{X} \to X$ exists, we typically have $|X(\mathbb{F}_q)| \neq |\mathcal{X}(\mathbb{F}_q)/\sim|$, even when $\mathcal{X}$ is Deligne–Mumford. On the other hand, non-weighted point count is equal to *weighted point count of the inertia stack* as $|\mathcal{X}(\mathbb{F}_q)/\sim| = \#_q(\mathcal{I}(\mathcal{X}))$ (c.f. [HP20, Theorem 1.1.]).

8.2. **Grothendieck ring of stacks.** Ekedahl in 2009 introduced the Grothendieck ring $K_0(\mathrm{Stck}_K)$ of algebraic stacks extending the classical Grothendieck ring $K_0(\mathrm{Var}_K)$ of varieties first defined by Grothendieck in 1964 in a letter to Serre. In [Eke09], every algebraic stack $\mathcal{X}$ of finite type over $K$ with affine stabilizers has the motivic class, i.e., the *motive* of $\mathcal{X}$ denoted as $\{\mathcal{X}\} \in K_0(\mathrm{Stck}_K)$.

**Definition 8.2.** [Eke09, §1] Fix a field $K$. Then the *Grothendieck ring $K_0(\mathrm{Stck}_K)$ of algebraic stacks of finite type over $K$ all of whose stabilizer group schemes are affine* is an abelian group generated by isomorphism classes of algebraic stacks $\{\mathcal{X}\}$ modulo relations:

- $\{\mathcal{X}\} = \{\mathcal{Z}\} + \{\mathcal{X} \setminus \mathcal{Z}\}$ for $\mathcal{Z} \subset \mathcal{X}$ a closed substack,
- $\{\mathcal{E}\} = \{\mathcal{X} \times \mathbb{A}^n\}$ for $\mathcal{E}$ a vector bundle of rank $n$ on $\mathcal{X}$.

Multiplication on $K_0(\mathrm{Stck}_K)$ is induced by $\{\mathcal{X}\}\{\mathcal{Y}\} := \{\mathcal{X} \times_K \mathcal{Y}\}$. There is a distinguished element $\mathbb{L} := \{\mathbb{A}^1\} \in K_0(\mathrm{Stck}_K)$, called the *Lefschetz motive*.

As in [Eke09, §2], the Grothendieck ring is universal with respect to additive invariants. When $K = \mathbb{F}_q$, the point counting measure $\{\mathcal{X}\} \mapsto \#_q(\mathcal{X})$ is a well-defined ring homomorphism $\#_q : K_0(\mathrm{Stck}_{\mathbb{F}_q}) \to \mathbb{Q}$ giving the weighted point count $\#_q(\mathcal{X})$ of $\mathcal{X}$ over $\mathbb{F}_q$. This is especially useful when the motive $\{\mathcal{X}\}$ (over finite field $\mathbb{F}_q$) is a mixed Tate motive meaning that $\{\mathcal{X}\}$ is a polynomial in the Lefschetz motive $\mathbb{L}$ as $\#_q(\mathbb{L}) = q$ and hence we can explicitly acquire $\#_q(\mathcal{X})$.

Recall that an algebraic group $G$ is *special* in the sense of [Ser58] and [Gro58], if every $G$-torsor is Zariski-locally trivial; for example $\mathbb{G}_a$, $\mathrm{GL}_d$, $\mathrm{SL}_d$ are special and $\mathrm{PGL}_2$, $\mathrm{PGL}_3$ are non-special. If $\mathcal{X} \to \mathcal{Y}$ is a $G$-torsor and $G$ is special, then we have $\{\mathcal{X}\} = \{G\} \cdot \{\mathcal{Y}\}$ (this is immediate when $\mathcal{Y}$ is a scheme, and it was shown by [Eke09, Proposition 1.1 iii)] when $\mathcal{Y}$ is an algebraic stack).

**Lemma 8.3.** *For any $\mathbb{G}_m$-torsor $\mathcal{X} \to \mathcal{Y}$ of finite type algebraic stacks, we have $\{\mathcal{Y}\} = \{\mathcal{X}\}\{\mathbb{G}_m\}^{-1}$.*

*Proof.* This follows from [Eke09, Proposition 1.1 iii), 1.4] and the definition of $K_0^{\mathrm{Zar}}(\mathrm{Stck}_K)$ in [Eke09, §1]. ∎

8.3. **Motives and point counts of Poly-spaces over finite fields.** The Poly-space is a variety of independent interest which is used throughout the subsequent proofs. We begin with $\mathrm{Poly}_1^{(d_1,\dots,d_m)}$, a slight generalization of [FW16, Definition 1.1].

**Definition 8.4.** Fix $m \in \mathbb{Z}_+$ and $d_1, \dots, d_m \geq 0$. Define $\mathrm{Poly}_1^{(d_1,\dots,d_m)}$ as the set of tuples $(f_1, \dots, f_m)$ of monic polynomials in $K[z]$ so that

(1) $\deg f_i = d_i$ for each $i$, and
(2) $f_1, \dots, f_m$ have no common roots in $\overline{K}$.

Note that $\mathrm{Poly}_1^{(d_1,\dots,d_m)}$ sits inside an affine space parameterizing tuples of monic coprime polynomials. It is the complement of the resultant hypersurface and so can be endowed with the structure of affine variety defined over $\mathbb{Z}$.

The motive of the Poly-space $\mathrm{Poly}_1^{(d_1,d_2)}$ over $K$ is given by

**Proposition 8.5.** *Fix $d_1, d_2 \geq 0$.*

$$\left\{\mathrm{Poly}_1^{(d_1,d_2)}\right\} = \begin{cases} \mathbb{L}^{d_1+d_2} - \mathbb{L}^{d_1+d_2-1} & \text{if } d_1, d_2 > 0\,, \\ \mathbb{L}^{d_1+d_2} & \text{if } d_1 = 0 \text{ or } d_2 = 0\,. \end{cases}$$

*Proof.* We refer the reader to [HP19, Proposition 18] for the details of the proof. The proof in (loc.cit.) is analogous to the proof of [FW16, Theorem 1.2]. Here we point out that the motive formula $\left\{\mathrm{Poly}_1^{(d_1,d_2)}\right\}$ not only holds in $\mathrm{char}(K) = 0$ (c.f. the corrigendum [FW19]) but also holds in any $\mathrm{char}(K) = p$ due to the critical correction from [PS21b, Proposition 3.1] utilizing the Euclidean algorithm. For the more general result on $\left\{\mathrm{Poly}_1^{(d_1,\dots,d_m)}\right\}$ with more tuples of polynomials, we refer the reader to [HP20, Proposition 4.4]. ■

We now consider Poly-spaces of polynomials with a common zero of specified vanishing orders.

**Definition 8.6.** Fix $a, b \in \mathbb{Z}_+$ and $d_1, d_2 \geq 0$. Define $\mathrm{Poly}_{(\geq a,b)}^{(d_1,d_2)}$ as the space parameterizing monic polynomials $(f_1, f_2)$ in $K[z]$ of degrees $(d_1, d_2)$ such that

(1) 0 is the only common root of $f_1$ and $f_2$ over $\overline{K}$,

(2) At 0, $f_1$ vanishes to order at least $a$ and $f_2$ has exact order of vanishing $b$.

We have analogously defined Poly-spaces $\mathrm{Poly}_{(a,\geq b)}^{(d_1,d_2)}$ and $\mathrm{Poly}_{(a,b)}^{(d_1,d_2)}$.

**Proposition 8.7.** *Fix $a, b \in \mathbb{Z}_+$ and $0 \leq d_1 \leq d_2$. Set $\alpha = d_1 - a$ and $\beta = d_2 - b$. The motive of the Poly-space $\mathrm{Poly}_{(\geq a,b)}^{(d_1,d_2)}$ over $K$ is given by*

$$\left\{\mathrm{Poly}_{(\geq a,b)}^{(d_1,d_2)}\right\} = \left\{\mathrm{Poly}_{(b,\geq a)}^{(d_2,d_1)}\right\} = \begin{cases} (\mathbb{L}-1)\cdot \frac{\mathbb{L}^{\alpha+\beta} - \mathbb{L}^{\alpha-\beta}}{\mathbb{L}+1}, & \text{if } 0 < \beta \leq \alpha, \\ (\mathbb{L}-1)\cdot \frac{\mathbb{L}^{\alpha+\beta} + \mathbb{L}^{\beta-\alpha-1}}{\mathbb{L}+1}, & \text{if } 0 \leq \alpha < \beta, \\ \mathbb{L}^{\alpha}, & \text{if } \beta = 0 \end{cases}$$

*The motive of the Poly-space $\mathrm{Poly}_{(a,b)}^{(d_1,d_2)}$ (i.e. the space of monic polynomials $(f_1, f_2)$ in $K[z]$ of degrees $(d_1, d_2)$ having exact order of vanishing $(a, b)$ at 0) over $K$ is given by*

$$\left\{\mathrm{Poly}_{(a,b)}^{(d_1,d_2)}\right\} = \left\{\mathrm{Poly}_1^{(d_1-a,d_2-b)}\right\} - \left\{\mathrm{Poly}_{(\geq(a+1),b)}^{(d_1,d_2)}\right\} - \left\{\mathrm{Poly}_{(a,\geq(b+1))}^{(d_1,d_2)}\right\}$$

*Proof.* We prove the result by induction on $\max(\alpha, \beta)$. If $\alpha = 0$, then $\mathrm{Poly}_{(\geq d_1,b)}^{(d_1,d_2)}$ is the space of polynomials of the form $(z^{d_1}, z^b g(z))$, where $g(z)$ is monic of degree $d_2 - b$ and $g(0) \neq 0$, i.e., the constant term of $g$ is non-zero. Thus, $\mathrm{Poly}_{(\geq d_1,b)}^{(d_1,d_2)} \simeq \mathbb{G}_m \times \mathbb{A}^{\beta-1}$ which has motive $(\mathbb{L}-1)\mathbb{L}^{\beta-1}$. Similarly, if $\beta = 0$, we have $\mathrm{Poly}_{(\geq a,d_2)}^{(d_1,d_2)}$ is the space of polynomials $(z^a g(z), z^{d_2})$ with $g(z)$ monic; this space is isomorphic to $\mathbb{A}^{\alpha}$, so has motive $\mathbb{L}^{\alpha}$.

We may now assume $\alpha$ and $\beta$ are positive. Notice that via the map $(f_1, f_2) \mapsto (z^a f_1, z^b f_2)$, the space $\mathrm{Poly}_1^{(d_1-a,d_2-b)}$ may be identified with the locally closed subscheme $\mathrm{Poly}_{(\geq a,b)}^{(d_1,d_2)} \cup \mathrm{Poly}_{(a,\geq b+1)}^{(d_1,d_2)}$; the union is taking place in the ambient weighted projective stack. Therefore

$$\left\{\mathrm{Poly}_{(\geq a,b)}^{(d_1,d_2)}\right\} = \left\{\mathrm{Poly}_1^{(d_1-a,d_2-b)}\right\} - \left\{\mathrm{Poly}_{(\geq b+1,a)}^{(d_2,d_1)}\right\}.$$

Applying Proposition 8.5, a straightforward induction argument proves the result.

The motive of the Poly-space $\mathrm{Poly}_{(a,b)}^{(d_1,d_2)}$ is immediate from the definitions. ■

8.4. **Derivation of motives for each Kodaira type.** We now compute the exact motive formulas $\mathcal{W}_n^\gamma(\mathbb{P}^1, \mathcal{P}(4,6))$ for each Kodaira type $\gamma : (\nu(a_4),\ \nu(a_6))$ as prescribed in the Theorem 1.5.

**Proposition 8.8.** *The motive of* $\mathcal{W}_n^\gamma := \mathcal{W}_{n,\mathbb{P}^1}^\gamma(4,6)$ *for the vanishing condition* $\gamma : (\nu(a_4),\ \nu(a_6))$ *is*

$$\{\mathcal{W}_n^\gamma\} = \{\mathbb{P}^1\} \cdot \{\mathbb{G}_m\} \cdot \left( \left\{\mathrm{Poly}_\gamma^{(4n,6n)}\right\} + \sum_{k=\nu(a_4)}^{4n-1} \left\{\mathrm{Poly}_\gamma^{(k,6n)}\right\} + \sum_{l=\nu(a_6)}^{6n-1} \left\{\mathrm{Poly}_\gamma^{(4n,l)}\right\} \right)$$

*Proof.* Recall first that by Proposition 6.10 we have $\{\mathcal{W}_n^\gamma\} = \{\mathbb{P}^1\} \cdot \{\mathcal{W}_n^\gamma(0)\}$. So it suffices to compute $\{\mathcal{W}_n^\gamma(0)\}$ which leads us to the stratification into Poly-spaces adopted from the proof of Theorem 1 in [HP19, §4.1].

Consider $T \subset H^0(\mathcal{O}_{\mathbb{P}^1}(4n)) \oplus H^0(\mathcal{O}_{\mathbb{P}^1}(6n)) \setminus \{0\}$ a $\mathbb{G}_m$-equivariant open subset parameterizing pairs $(u, v)$ with one common zero with specified vanishing condition $\gamma : (\nu(a_4),\ \nu(a_6))$, where $\mathbb{G}_m$ acts via $\lambda * (u, v) = (\lambda^4 u, \lambda^6 v)$. Since $\gamma$ consists of a single tuple, in the notation from Definition 4.18, we have $S_\mu$ is the trivial group. Then by Theorem 4.28, we have $\mathcal{W}_{n,C}^\gamma = \mathcal{R}_{n,C}^\gamma$. Using the moduli interpretation for $\mathcal{R}_{n,C}^\gamma$ given after Definition 4.19, we see $\mathcal{W}_n^\gamma(0) = [T/\mathbb{G}_m]$.

Now fix a chart $\mathbb{A}^1 \hookrightarrow \mathbb{P}^1$ with $x \mapsto [1 : x]$, and call $0 = [1 : 0]$ and $\infty = [0 : 1]$. It comes from a homogeneous chart of $\mathbb{P}^1$ by $[Y : X]$ with $x := X/Y$ away from $\infty$. Then for any $(u, v) \in T$, $u$ and $v$ are homogeneous polynomials in $X$ and $Y$ with degrees $4n$ and $6n$ respectively. By plugging in $Y = 1$, we obtain representations of $u$ and $v$ as polynomials in $x$ with degrees at most $4n$ and $6n$, respectively. For instance, $\deg u < 4n$ as a polynomial in $x$ if and only if $u(X, Y)$ is divisible by $Y$, i.e. $u$ vanishes at $\infty$. From now on, $\deg P$ means the degree of $P$ as a polynomial in $x$.

Denoting $\deg u := k$ and $\deg v := l$, then $(u, v) \in T$ is whenever $k = 4n$ or $l = 6n$ (so that they do not simultaneously vanish at $\infty$) and $u, v$ have a unique common root at 0 with specified vanishing condition $\gamma : (\nu(a_4),\ \nu(a_6))$. Since there are many possible degrees for a pair $(u, v) \in T$, consider the locally closed subsets $T_{k,l} := \{(u, v) \in T : \deg u = k,\ \deg v = l\}$. Notice that $T_{k-1,6n} \subset \overline{T}_{k,6n}$ as for any $(u, v) \in T_{k-1,6n}$, $u(X, Y)$ has a description as $Y^{4n-k+1} u'(X, Y)$ which is $u_{[1:0]}(X, Y)$ from a pencil polynomials $u_{[t_0:t_1]}(X, Y) = Y^{4n-k}(t_1 Y - t_0 X) u'(X, Y)$ where $u_{[1:t_1]} \in T_{k,6n}$. Hence, we obtain the following stratification:

$$
\begin{aligned}
T &= T_{4n,6n} \sqcup \left( \bigsqcup_{k=\nu(a_4)}^{4n-1} T_{k,6n} \right) \sqcup \left( \bigsqcup_{l=\nu(a_6)}^{6n-1} T_{4n,l} \right) \\
T &= \overline{T_{4n,6n}} \supsetneq \overline{T_{4n-1,6n}} \supsetneq \cdots \supsetneq \overline{T_{\nu(a_4),6n}} \\
T &= \overline{T_{4n,6n}} \supsetneq \overline{T_{4n,6n-1}} \supsetneq \cdots \supsetneq \overline{T_{4n,\nu(a_6)}} \\
& \overline{T_{4n-k,6n}} \cap \overline{T_{4n,6n-l}} = \varnothing \quad \forall k,l > 0
\end{aligned}
$$

Then,

$$
\{T\} = \{T_{4n,6n}\} + \sum_{k=\nu(a_4)}^{4n-1} \{T_{k,6n}\} + \sum_{l=\nu(a_6)}^{6n-1} \{T_{4n,l}\} \tag{7}
$$

Lastly, taking $\mathbb{G}_m$-quotients and applying Lemma 8.3 yields the result. ∎

*Proof of Theorems 1.6 & A.2.* We now apply Proposition 8.8 to compute the motives for each Kodaira type. Since many of these cases are handled in similar ways, we illustrate the method first with the additive reduction of II type and then with the most difficult of the types, namely the additive reduction of $\mathrm{I}_k^*$ type.

**Additive reduction of II type.** We compute the motive for additive reduction of type II at $j=0$ with the vanishing condition $\gamma = (\geq 1, 1)$. The first equality below follows from Proposition 8.8 and the second equality from Proposition 8.7.

$$
\begin{aligned}
&\{\mathcal{W}_n^{(\geq 1,1)}\} = \{\mathbb{P}^1\} \cdot \{\mathbb{G}_m\} \cdot \left( \left\{ \mathrm{Poly}_{(\geq 1,1)}^{(4n,6n)} \right\} + \sum_{k=1}^{4n-1} \left\{ \mathrm{Poly}_{(\geq 1,1)}^{(k,6n)} \right\} + \sum_{l=1}^{6n-1} \left\{ \mathrm{Poly}_{(\geq 1,1)}^{(4n,l)} \right\} \right) \\
&= (\mathbb{L}^2 - 1) \cdot \left( (\mathbb{L}-1) \left( \frac{\mathbb{L}^{10n-2} + \mathbb{L}^{2n-1}}{(\mathbb{L}+1)} \right) + \sum_{k=1}^{4n-1} (\mathbb{L}-1) \left( \frac{\mathbb{L}^{k+6n-2} + \mathbb{L}^{-k+6n-1}}{(\mathbb{L}+1)} \right) \right) \\
&\quad + (\mathbb{L}^2 - 1) \cdot \left( \sum_{l=1}^{4n} (\mathbb{L}-1) \left( \frac{\mathbb{L}^{l+4n-2} - \mathbb{L}^{-l+4n}}{(\mathbb{L}+1)} \right) + \sum_{l=4n+1}^{6n-1} (\mathbb{L}-1) \left( \frac{\mathbb{L}^{l+4n-2} + \mathbb{L}^{l-4n-1}}{(\mathbb{L}+1)} \right) \right) \\
&= \mathbb{L}(\mathbb{L}^2 - 1) \cdot \left( \mathbb{L}^{10n-3} - \mathbb{L}^{4n-2} \right)
\end{aligned}
$$

**Additive reduction of $\mathrm{I}_k^*$ type.** We next compute the motive for additive reduction of type $\mathrm{I}_{k>0}^*$ at $j=\infty$ or $\mathrm{I}_0^*$ at $j \neq 0, 1728$ with the vanishing condition $\gamma = (2,3)$. The computation of this case is the most subtle as the vanishing constraint is fixed for both tuples. The computation of this motive is the most interesting of the cases as the ultimate expression has terms $\mathbb{L}^\alpha$ for 3 different values of $\alpha$.

By Proposition 8.8, we have

$$
\{\mathcal{W}_n^{(2,3)}\} = \{\mathbb{P}^1\} \cdot \{\mathbb{G}_m\} \cdot \left( \left\{ \mathrm{Poly}_{(2,3)}^{(4n,6n)} \right\} + \sum_{k=2}^{4n-1} \left\{ \mathrm{Poly}_{(2,3)}^{(k,6n)} \right\} + \sum_{l=3}^{6n-1} \left\{ \mathrm{Poly}_{(2,3)}^{(4n,l)} \right\} \right)
$$

Next, Proposition 8.7 tells us

$$
\left\{ \mathrm{Poly}_{(2,3)}^{(d_1,d_2)} \right\} = \left\{ \mathrm{Poly}_1^{(d_1-2,d_2-3)} \right\} - \left\{ \mathrm{Poly}_{(\geq 3,3)}^{(d_1,d_2)} \right\} - \left\{ \mathrm{Poly}_{(2,\geq 4)}^{(d_1,d_2)} \right\}.
$$

This leads to 3 sub-computations corresponding to the 3 summands in the above expression for $\{\mathcal{W}_n^{(2,3)}\}$. First,

$$\left\{\mathrm{Poly}_{(2,3)}^{(4n,6n)}\right\} = \left\{\mathrm{Poly}_{1}^{(4n-2,6n-3)}\right\} - \left\{\mathrm{Poly}_{(\geq 3,3)}^{(4n,6n)}\right\} - \left\{\mathrm{Poly}_{(2,\geq 4)}^{(4n,6n)}\right\}$$
$$= \left(\mathbb{L}^{10n-5} - \mathbb{L}^{10n-6}\right) - (\mathbb{L}-1)\cdot\left(\frac{\mathbb{L}^{10n-6}+\mathbb{L}^{2n-1}}{(\mathbb{L}+1)}\right) - (\mathbb{L}-1)\cdot\left(\frac{\mathbb{L}^{10n-6}-\mathbb{L}^{2n-2}}{(\mathbb{L}+1)}\right)$$

and so

$$\{\mathbb{P}^1\}\cdot\{\mathbb{G}_m\}\cdot\left\{\mathrm{Poly}_{(2,3)}^{(4n,6n)}\right\} = (\mathbb{L}+1)\cdot(\mathbb{L}-1)\cdot\left\{\mathrm{Poly}_{(2,3)}^{(4n,6n)}\right\}$$
$$= (\mathbb{L}-1)\cdot\left(\mathbb{L}^{10n-4} - 2\cdot\mathbb{L}^{10n-5} + \mathbb{L}^{10n-6} - \mathbb{L}^{2n} + 2\cdot\mathbb{L}^{2n-1} - \mathbb{L}^{2n-2}\right).$$

Next,

$$\sum_{k=2}^{4n-1}\left\{\mathrm{Poly}_{(2,3)}^{(k,6n)}\right\} = \sum_{k=2}^{4n-1}\left(\left\{\mathrm{Poly}_{1}^{(k-2,6n-3)}\right\} - \left\{\mathrm{Poly}_{(\geq 3,3)}^{(k,6n)}\right\} - \left\{\mathrm{Poly}_{(2,\geq 4)}^{(k,6n)}\right\}\right)$$

$$= \sum_{k=2}^{4n-1}\left(\left(\mathbb{L}^{k+6n-5} - \mathbb{L}^{k+6n-6}\right) - (\mathbb{L}-1)\cdot\left(\frac{\mathbb{L}^{k+6n-6}+\mathbb{L}^{-k+6n-1}}{(\mathbb{L}+1)}\right) - (\mathbb{L}-1)\cdot\left(\frac{\mathbb{L}^{k+6n-6}-\mathbb{L}^{-k+6n-2}}{(\mathbb{L}+1)}\right)\right)$$

which implies

$$\{\mathbb{P}^1\}\cdot\{\mathbb{G}_m\}\cdot\sum_{k=2}^{4n-1}\left\{\mathrm{Poly}_{(2,3)}^{(k,6n)}\right\} = (\mathbb{L}-1)\cdot\left(\mathbb{L}^{2n} - \mathbb{L}^{2n-1} + \mathbb{L}^{6n-4} - \mathbb{L}^{6n-2} - \mathbb{L}^{10n-6} + \mathbb{L}^{10n-5}\right)$$

Finally,

$$\sum_{l=3}^{6n-1}\left\{\mathrm{Poly}_{(2,3)}^{(4n,l)}\right\} = \sum_{l=3}^{6n-1}\left(\left\{\mathrm{Poly}_{1}^{(4n-2,l-3)}\right\} - \left\{\mathrm{Poly}_{(\geq 3,3)}^{(4n,l)}\right\} - \left\{\mathrm{Poly}_{(2,\geq 4)}^{(4n,l)}\right\}\right)$$

$$= \sum_{l=3}^{6n-1}\left(\mathbb{L}^{l+4n-5} - \mathbb{L}^{l+4n-6}\right)$$
$$- \sum_{l=3}^{4n}(\mathbb{L}-1)\cdot\left(\frac{\mathbb{L}^{l+4n-6}-\mathbb{L}^{4n-l}}{(\mathbb{L}+1)}\right) - \sum_{l=4n+1}^{6n-1}(\mathbb{L}-1)\cdot\left(\frac{\mathbb{L}^{l+4n-6}+\mathbb{L}^{l-4n-1}}{(\mathbb{L}+1)}\right)$$
$$- \sum_{l=3}^{4n+1}(\mathbb{L}-1)\cdot\left(\frac{\mathbb{L}^{l+4n-6}+\mathbb{L}^{4n-l+1}}{(\mathbb{L}+1)}\right) - \sum_{l=4n+2}^{6n-1}(\mathbb{L}-1)\cdot\left(\frac{\mathbb{L}^{l+4n-6}-\mathbb{L}^{l-4n-2}}{(\mathbb{L}+1)}\right)$$

yielding

$$\{\mathbb{P}^1\}\cdot\{\mathbb{G}_m\}\cdot\sum_{l=3}^{6n-1}\left\{\mathrm{Poly}_{(2,3)}^{(4n,l)}\right\} = (\mathbb{L}-1)\cdot\left(-\mathbb{L}^{4n-3} - \mathbb{L}^{4n-2} + \mathbb{L}^{10n-6} + \mathbb{L}^{10n-5}\right)$$
$$-(\mathbb{L}-1)\cdot\left(\mathbb{L}^{8n-5} - \mathbb{L}^{4n-3} - \mathbb{L}^{4n-2} + 1\right) - (\mathbb{L}-1)\cdot\left(\mathbb{L}^{10n-6} + \mathbb{L}^{2n-1} - \mathbb{L}^{8n-5} - 1\right)$$
$$-(\mathbb{L}-1)\cdot\left(\mathbb{L}^{8n-4} - \mathbb{L}^{4n-3} + \mathbb{L}^{4n-1} - 1\right) - (\mathbb{L}-1)\cdot\left(\mathbb{L}^{10n-6} - \mathbb{L}^{2n-2} - \mathbb{L}^{8n-4} + 1\right)$$

Combining these computations, we see

$$\{\mathcal{W}_n^{(2,3)}\} = \{\mathbb{P}^1\}\cdot\{\mathbb{G}_m\}\cdot\left(\left\{\mathrm{Poly}_{(2,3)}^{(4n,6n)}\right\} + \sum_{k=2}^{4n-1}\left\{\mathrm{Poly}_{(2,3)}^{(k,6n)}\right\} + \sum_{l=3}^{6n-1}\left\{\mathrm{Poly}_{(2,3)}^{(4n,l)}\right\}\right)$$
$$= \mathbb{L}(\mathbb{L}^2-1)\cdot(\mathbb{L}-1)\cdot\left(\mathbb{L}^{10n-7}-\mathbb{L}^{6n-5}-\mathbb{L}^{4n-4}\right)$$

The rest of the cases with different $\gamma$ are handled similarly. This completes the Proof of Theorems 1.6 & A.2.

■

We now determine the sharp enumerations ordered by the height of discriminant for the number of minimal elliptic fibrations over $\mathbb{P}^1_{\mathbb{F}_q}$ with a specified additive reduction of Kodaira type $\Theta$ type and the rest of the bad reductions are strictly multiplicative. Let $\Delta$ be the discriminant of a minimal elliptic fibration. Then the height of the discriminant over $\mathbb{F}_q$ is $0 < ht(\Delta) := q^{\deg\Delta} = q^{12n}$ where $n$ is Faltings heignt which by Proposition 7.6 agrees with the stacky height.

**Theorem 8.9.** *Let $n \in \mathbb{Z}_+$ and char$(\mathbb{F}_q) \neq 2,3$. The function $\mathcal{N}(\mathbb{F}_q(t),\ \Theta,\ B)$, which counts the number of elliptic curves over $\mathbb{P}^1_{\mathbb{F}_q}$ with a single specified additive reduction of Kodaira type $\Theta$ and at worst multiplicative reduction otherwise, ordered by the multiplicative height of the discriminant $0 < ht(\Delta) = q^{12n} \leq B$, satisfies:*

$$\mathcal{N}(\mathbb{F}_q(t),\ (\text{II with } j=0),\ B) = 2\cdot\frac{(q^{10}-q^8)}{(q^{10}-1)}\cdot\left(B^{\frac{5}{6}}-1\right) - 2\cdot\frac{(q^5-q^3)}{(q^4-1)}\cdot\left(B^{\frac{1}{3}}-1\right)$$

$$\mathcal{N}(\mathbb{F}_q(t),\ (\text{IV with } j=0),\ B) = 2\cdot\frac{(q^8-q^6)}{(q^{10}-1)}\cdot\left(B^{\frac{5}{6}}-1\right) - 2\cdot\frac{(q^4-q^2)}{(q^4-1)}\cdot\left(B^{\frac{1}{3}}-1\right)$$

$$\mathcal{N}(\mathbb{F}_q(t),\ (\text{I}_0^* \text{ with } j=0),\ B) = 2\cdot\frac{(q^6-q^4)}{(q^{10}-1)}\cdot\left(B^{\frac{5}{6}}-1\right) - 2\cdot\frac{(q^3-q)}{(q^4-1)}\cdot\left(B^{\frac{1}{3}}-1\right)$$

$$\mathcal{N}(\mathbb{F}_q(t),\ (\text{IV}^* \text{ with } j=0),\ B) = 2\cdot\frac{(q^5-q^3)}{(q^{10}-1)}\cdot\left(B^{\frac{5}{6}}-1\right) - 2\cdot\frac{(q^3-q)}{(q^4-1)}\cdot\left(B^{\frac{1}{3}}-1\right)$$

$$\mathcal{N}(\mathbb{F}_q(t),\ (\text{II}^* \text{ with } j=0),\ B) = 2\cdot\frac{(q^3-q)}{(q^{10}-1)}\cdot\left(B^{\frac{5}{6}}-1\right) - 2\cdot\frac{(q^2-1)}{(q^4-1)}\cdot\left(B^{\frac{1}{3}}-1\right)$$

$$\mathcal{N}(\mathbb{F}_q(t),\ (\text{III with } j=1728),\ B) = 2\cdot\frac{(q^9-q^7)}{(q^{10}-1)}\cdot\left(B^{\frac{5}{6}}-1\right) - 2\cdot\frac{(q^6-q^4)}{(q^6-1)}\cdot\left(B^{\frac{1}{2}}-1\right)$$

$$\mathcal{N}(\mathbb{F}_q(t),\ (\text{I}_0^* \text{ with } j=1728),\ B) = 2\cdot\frac{(q^6-q^4)}{(q^{10}-1)}\cdot\left(B^{\frac{5}{6}}-1\right) - 2\cdot\frac{(q^4-q^2)}{(q^6-1)}\cdot\left(B^{\frac{1}{2}}-1\right)$$

$$\mathcal{N}(\mathbb{F}_q(t),\ (\text{III}^* \text{ with } j=1728),\ B) = 2\cdot\frac{(q^4-q^2)}{(q^{10}-1)}\cdot\left(B^{\frac{5}{6}}-1\right) - 2\cdot\frac{(q^3-q)}{(q^6-1)}\cdot\left(B^{\frac{1}{2}}-1\right)$$

$$\mathcal{N}(\mathbb{F}_q(t),\ (\text{I}_{k>0}^* \text{ or } \text{I}_0^*),\ B) = 2(q-1)^2(q+1)\left(\frac{q^4(B^{\frac{5}{6}}-1)}{(q^{10}-1)} - \frac{q^2(B^{\frac{1}{2}}-1)}{(q^6-1)} - \frac{q(B^{\frac{1}{3}}-1)}{(q^4-1)}\right)$$

*Proof of Theorems 1.1 & 8.9.* We acquire the exact weighted point counts $\#_q\left(\mathcal{W}_n^\gamma\right)$ over $\mathbb{F}_q$ from the motive formulas $\{\mathcal{W}_n^\gamma\}$ by $\#_q : \mathbb{L} \mapsto q$. The exact number $|\mathcal{W}_n^\gamma(\mathbb{F}_q)/\sim|$ of $\mathbb{F}_q$–isomorphism classes of $\mathbb{F}_q$–points of the moduli stack $\mathcal{W}_n^\gamma$ over $\mathbb{F}_q$ with char$(\mathbb{F}_q) \neq 2,3$ is $|\mathcal{W}_n^{\gamma}(\mathbb{F}_q)/\sim| = 2\cdot \#_q\left(\mathcal{W}_n^{\gamma}\right)$ where the factor of 2 comes from the hyperelliptic involution i.e., the generic $\mu_{\text{gcd}(4,6)}$ stabilizer. By Theorem 7.16 and Proposition 5.7, this is the count of elliptic surfaces of height $n$ over $\mathbb{P}^1_{\mathbb{F}_q}$ with the additive reduction controlled by vanishing conditions $\gamma$ corresponding to the given Kodaira fiber type $\Theta$.

$$\mathcal{N}(\mathbb{F}_q(t),\ \Theta,\ B) = \sum_{n=1}^{\left\lfloor \frac{log_q B}{12} \right\rfloor} |\mathcal{W}_n^{\Theta}(\mathbb{F}_q)/\sim| = \sum_{n=1}^{\left\lfloor \frac{log_q B}{12} \right\rfloor} 2\cdot \#_q\left(\mathcal{W}_n^{\Theta}\right)$$

In the case of $\Theta = (\mathrm{I}_0^*$ with $j \neq 0, 1728$ or $\mathrm{I}_{k>0}^*$ with $j = \infty)$ we have

$$\begin{aligned}
\mathcal{N}(\mathbb{F}_q(t),\ \Theta,\ B) &= \sum_{n=1}^{\left\lfloor \frac{log_q B}{12} \right\rfloor} |\mathcal{W}_n^{(2,3)}(\mathbb{P}^1, \mathcal{P}(4,6))(\mathbb{F}_q)/\sim| \\
&= \sum_{n=1}^{\left\lfloor \frac{log_q B}{12} \right\rfloor} 2\cdot(q-1)^2\cdot(q+1)\cdot(q^{10n-6}-q^{6n-4}-q^{4n-3}) \\
&= 2\cdot(q-1)^2\cdot(q+1)\cdot\left(\frac{q^4(B^{\frac{5}{6}}-1)}{(q^{10}-1)} - \frac{q^2(B^{\frac{1}{2}}-1)}{(q^6-1)} - \frac{q(B^{\frac{1}{3}}-1)}{(q^4-1)}\right)
\end{aligned}$$

The rest of the cases with different $\Theta$ are handled similarly. ■

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## Appendix A. Applications to stack $\overline{\mathcal{M}}_1(2) \cong \mathcal{P}(2,4)$

We show the result of applying the same analysis to the moduli space $\overline{\mathcal{M}}_1(2) \cong \mathcal{P}(2,4)$ over $\mathbb{Z}\left[\frac{1}{2}\right]$ leading to similar correspondence & counting results for elliptic curves over $K = \mathbb{F}_q(t)$ with level-2 structure (see §6.3).

**Theorem A.1.** *If* char$(K) \neq 2$. *Then correspondence for* $\overline{\mathcal{M}}_1(2) \cong \mathcal{P}(2,4)$ *is*

| $\gamma : \big(\nu(a_2),\ \nu(a_4)\big)$ | *Reduction type with* $j \in \overline{M}_1(2)$ | $\Gamma : (r,a)$ |
|---|---|---|
| $(\geq 1, 1)$ | III *with* $j = 0$ | $(4,1)$ |
| $(1,2)$ | $\mathrm{I}^*_{k>0}$ *with* $j = \infty$<br>$\mathrm{I}^*_0$ *with* $j \neq 0, 1728$ | $(2,1)$ |
| $(\geq 2, 2)$ | $\mathrm{I}^*_0$ *with* $j = 0$ | $(4,2)$ |
| $(1, \geq 3)$ | $\mathrm{I}^*_0$ *with* $j = 1728$ | $(2,1)$ |
| $(\geq 2, 3)$ | $\mathrm{III}^*$ *with* $j = 0$ | $(4,3)$ |

**Theorem A.2.** *If* char$(K) \neq 2$. *Then motive of height moduli on* $\overline{\mathcal{M}}_1(2) \cong \mathcal{P}(2,4)$

| *Reduction type with* $j \in \overline{M}_1(2)$ | *Motivic class* $\{\mathcal{W}^{\gamma}_{\mu_2,n,\mathbb{P}^1}\} \in K_0(\mathrm{Stck}_K)$ |
|---|---|
| III *with* $j = 0$ | $\mathbb{L}^{6n} - \mathbb{L}^{6n-2} - \mathbb{L}^{2n+1} + \mathbb{L}^{2n-1}$ |
| $\mathrm{I}^*_{k>0}$ *with* $j = \infty$<br>$\mathrm{I}^*_0$ *with* $j \neq 0, 1728$ | $\mathbb{L}^{6n-1} - \mathbb{L}^{6n-2} - \mathbb{L}^{6n-3} + \mathbb{L}^{6n-4} - \mathbb{L}^{4n} + \mathbb{L}^{4n-1}$<br>$+\mathbb{L}^{4n-2} - \mathbb{L}^{4n-3} - \mathbb{L}^{2n+1} + \mathbb{L}^{2n} + \mathbb{L}^{2n-1} - \mathbb{L}^{2n-2}$ |
| $\mathrm{I}^*_0$ *with* $j = 0$ | $\mathbb{L}^{6n-2} - \mathbb{L}^{6n-4} - \mathbb{L}^{2n} + \mathbb{L}^{2n-2}$ |
| $\mathrm{I}^*_0$ *with* $j = 1728$ | $\mathbb{L}^{6n-2} - \mathbb{L}^{6n-4} - \mathbb{L}^{4n-1} + \mathbb{L}^{4n-3}$ |
| $\mathrm{III}^*$ *with* $j = 0$ | $\mathbb{L}^{6n-3} - \mathbb{L}^{6n-5} - \mathbb{L}^{2n} + \mathbb{L}^{2n-2}$ |

**Theorem A.3.** *Let char*$(\mathbb{F}_q) \neq 2$. *The function* $\mathcal{N}_{\mu_2}(\mathbb{F}_q(t),\ \Theta,\ B)$*, which counts the number of elliptic curves with level-2 structure over* $\mathbb{P}^1_{\mathbb{F}_q}$ *with the additive reduction of Kodaira type* $\Theta$ *at a single place and at worst multiplicative reductions elsewhere ordered by height of discriminant* $0 < ht(\Delta) = q^{12n} \leq B$ *for* $n \in \mathbb{Z}_+$*, satisfies:*

$$\mathcal{N}_{\mu_2}(\mathbb{F}_q(t),\ \Theta,\ B) = a_q B^{\frac{1}{2}} + b_q B^{\frac{1}{6}} + c_q, \text{ if } \Theta = \mathrm{III}, \mathrm{III}^* \text{ or } \mathrm{I}^*_0 \text{ with } j = 0;$$

$$\mathcal{N}_{\mu_2}(\mathbb{F}_q(t),\ \Theta,\ B) = a_q B^{\frac{1}{2}} + b_q B^{\frac{1}{3}} + c_q, \text{ if } \Theta = \mathrm{I}^*_0 \text{ with } j = 1728; \text{ and}$$

$$\mathcal{N}_{\mu_2}(\mathbb{F}_q(t),\ \Theta,\ B) = a_q B^{\frac{1}{2}} + b_q B^{\frac{1}{3}} + c_q B^{\frac{1}{6}} + d_q, \text{ if } \Theta = \mathrm{I}^*_{k>0} \text{ or } \mathrm{I}^*_0 \text{ with } j \neq 0, 1728.$$

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