ENUMERATING STABLE HYPERELLIPTIC CURVES AND PRINCIPALLY POLARIZED ABELIAN SURFACES OVER GLOBAL FIELDS

CHANGHO HAN AND JUN–YONG PARK

Abstract. We use geometric methods to establish an upper bound for counting stable hyperelliptic curves with a marked Weierstrass section ordered by height of discriminant at most $\mathcal{B}$ over $\mathbb{P}^1_{\mathbb{F}_q}$ with characteristic $p > 2g+1$; the acquired estimate is of order $\mathcal{O}_q\left(\mathcal{B}^{\frac{2g+3}{4g+2}}\right)$. We sharpen the estimate for each genus $g \geq 2$; specifically when $g = 2$, this renders an estimate on the number of principally polarized Abelian surfaces over $\mathbb{F}_q(t)$. Through the global fields analogy, we formulate analogous new heuristics for counting stable hyperelliptic curves with a marked rational Weierstrass point or principally polarized Abelian surfaces over $\mathbb{Q}$. In Appendix, we determine the sharp estimate for counting elliptic curves with prescribed level structure or multiple marked points over $\mathbb{P}^1_{\mathbb{F}_q}$.

1. Introduction

The enumeration of arithmetic curves over an algebraic number field is a central problem in number theory constituting a crucial part in the Gerd Faltings' celebrated proof of the Mordell's conjecture in 1983 [Mordell, Faltings]. In this regard, the finiteness of the number of *non-isotrivial* smooth families of algebraic curves over $\mathbb{Q}$ with bounded bad reduction was a problem first proposed by Igor R. Shafarevich in his 1962 address at the International Congress in Stockholm.

Problem 1.1 (Shafarevich's problem for arithmetic curves)**.** Let S be a finite set of places of a number field K. How many distinct K-isomorphism classes of curves X/K are there, of genus $g \geq 2$ and possessing good reduction at all primes $P \notin S$?

'Shafarevich's conjecture' in [Shafarevich] is the assertion that there is only a finite number for any given (g, K, S) (i.e., *Finiteness principle for curves*). In this paper, we consider a problem of analogous nature where the number field $\mathbb{Q}$ is replaced by the global function field $\mathbb{F}_q(t)$ through the *global fields analogy* (see §5 for further discussion). We effectively prove the geometric Shafarevich's conjecture for stable odd degree hyperelliptic curves over $\mathbb{P}^1_{\mathbb{F}_q}$ with a marked Weierstrass section, which admit squarefree discriminants. Recall that an *odd degree* hyperelliptic curve has a marked rational Weierstrass point at ∞. Specifically, given a stable odd hyperelliptic genus $g \geq 2$ curve X over $\mathbb{P}^1_{\mathbb{F}_q}$ with $\mathrm{char}(\mathbb{F}_q) > 2g+1$, define the *height* of the hyperelliptic discriminant $\Delta_g(X)$ (see Definition 4.15) to be $ht(\Delta_g(X)) := q^{\deg \Delta_g(X)}$ (see Definition 5.5). Then, for a positive real number $\mathcal{B}$, we define the counting function $\mathcal{Z}_{g,\mathbb{F}_q(t)}(\mathcal{B})$ as follows:

$$\mathcal{Z}_{g,\mathbb{F}_q(t)}(\mathcal{B}) := |\{\text{Stable odd hyperelliptic genus } g \geq 2 \text{ curves over } \mathbb{P}^1_{\mathbb{F}_q} \text{ with } 0 < ht(\Delta_g(X)) \leq \mathcal{B}\}|$$

We prove the following upper bound for $\mathcal{Z}_{g,\mathbb{F}_q(t)}(\mathcal{B})$:

Theorem 1.2 (Estimate on $\mathcal{Z}_{g,\mathbb{F}_q(t)}(\mathcal{B})$)**.** *If $char(\mathbb{F}_q) > 2g+1$, then the function $\mathcal{Z}_{g,\mathbb{F}_q(t)}(\mathcal{B})$, which counts the number of stable hyperelliptic genus $g \geq 2$ curves X with a marked Weierstrass section over $\mathbb{P}^1_{\mathbb{F}_q}$ ordered by $0 < ht(\Delta_g(X)) = q^{4g(2g+1)n} \leq \mathcal{B}$, satisfies:*

$$\mathcal{Z}_{g,\mathbb{F}_q(t)}(\mathcal{B}) \leq 2g \cdot \frac{q^{4g(g+1)+1} \cdot (q^{2g-1}-1)(q^{2g}-1)}{(q-1)(q^{2g(2g+3)}-1)} \cdot \left(\mathcal{B}^{\frac{2g+3}{4g+2}} - 1\right)$$

As a consequence of Theorem 1.2, there are only finitely many non-isotrivial smooth families of odd degree hyperelliptic curves with a marked Weierstrass section over an open subset of $\mathbb{P}^1_{\mathbb{F}_q}$

that extend to families of stable odd hyperelliptic curves over $\mathbb{P}^1_{\mathbb{F}_q}$ with squarefree discriminants. Additionally, for each genus $g \geq 2$, we sharpen the above estimate in Theorem 1.15, by finding a sharper upper bound on $\mathcal{Z}_{g,\mathbb{F}_q(t)}(\mathcal{B})$ which has the leading term of order $\mathcal{O}_q\left(\mathcal{B}^{\frac{2g+3}{4g+2}}\right)$ where $\mathcal{O}_q$-constant is an explicit rational function of q with the corresponding lower order terms.

We now consider the number field side where we formulate an analogous heuristic on counting stable odd degree hyperelliptic curves over $\mathbb{Q}$ by passing the above upper bound through the global fields analogy via the bounded *height* of the hyperelliptic discriminant Δ_g (i.e., the norm of the discriminant), where $ht(\Delta_g)$ is the cardinality of the ring of functions on $\mathrm{Spec}(\mathbb{Z}/(\Delta_g))$:

Conjecture 1.3 (Heuristic on $\mathcal{Z}_{g,\mathbb{Q}}(\mathcal{B})$)**.** The function $\mathcal{Z}_{g,\mathbb{Q}}(\mathcal{B})$, which counts the number of stable hyperelliptic genus $g \geq 2$ curves with a marked rational Weierstrass point over $\mathbb{Z}$ with $0 < ht(\Delta_g) \leq \mathcal{B}$, has the same order of magnitude to

$$\mathcal{Z}_{g,\mathbb{Q}}(\mathcal{B}) \asymp \mathcal{B}^{\frac{2g+3}{4g+2}}$$

Since the qualitative finiteness shown by the classical works of [Parshin, Oort] (see [Känel] for partly explicit upper bound), we currently do not have an explicit estimate for the number of stable genus 2 curves over $\mathbb{Q}$ ordered by the bounded height of Δ_2. Similarly, we also lack explicit estimate for the number of stable hyperelliptic genus $g \geq 3$ curves over $\mathbb{Q}$ ordered by the bounded height of Δ_g. We note that counting curves over a global field K by height of discriminant $ht(\Delta_g)$ is more difficult than counting by *naïve height* (see [Brumer, p. 446]). In the case of counting elliptic curves ordered by height of discriminant $ht(\Delta_1)$, the region of lattice points with bounded discriminant has *cusps*, meaning that there are *unbounded points* with large $a_4,\ a_6 \in \mathcal{O}_K$ (i.e., elliptic curves with large naïve height) and small discriminant Δ_1. As explained in [Hortsch], controlling these cusps is difficult, even if the ABC conjecture is assumed.

Remark 1.4. The classical [Ogg, Saito] formula for elliptic curves gives the *Artin conductor = (order of the) discriminant* equality which implies that counting the elliptic curves by discriminant Δ_1 as in [HP, Theorem 3] is the same as counting the elliptic curves by conductor $\mathcal{N}_1$. For genus 2 curves, the work of [Liu3] proves $\mathcal{N}_2 \leq \nu(\Delta_2)$ where he also shows that equality can fail to hold. For hyperelliptic curves of genus g, we have $\mathcal{N}_g \leq \nu(\Delta_g)$ proven recently by the work of [Srinivasan, OSr] implying that our counting $\mathcal{Z}_{g,\mathbb{F}_q(t)}(\mathcal{B})$ by the hyperelliptic discriminant $\Delta_g(X)$ bounds above the parallel counting by the conductor $\mathcal{N}_g$.

To conclude, we consider a closely related problem of enumerating Abelian varieties over $\mathbb{Q}$.

Problem 1.5 (Shafarevich's problem for Abelian varieties)**.** Let S be a finite set of places of a number field K. How many K-isomorphism classes of Abelian varieties of dimension g are there, defined over K, with good reduction at all primes $P \notin S$?

We focus on Abelian varieties of dimension 2, i.e., *Abelian surfaces* over global fields. By the local (i.e., infinitesimal) Torelli theorem in [OS, Theorem 2.6 and 2.7] and [Milne, Theorem 12.1], the Torelli map $\tau_2 : \mathcal{M}_2 \hookrightarrow \mathcal{A}_2$, which sends a smooth projective genus 2 curve X defined over a field K to its principally polarized Jacobian $(\mathrm{Jac}(X), \lambda_\theta)/K$ (where λ_θ is the theta divisor of $\mathrm{Jac}(X)$), is an open immersion. Furthermore, it is shown in [OU, 4. Theorem] (see also [Weil, Satz 2]) that given a principally polarized Abelian surface (A, λ) over a field K, after a finite extension of scalars, is isomorphic to the canonically polarized (generalized) Jacobian variety $(\mathrm{Jac}(X), \lambda_\theta)$ of a stable genus 2 curve X. Recall that if a curve X has good reduction at a place $v \in S$ then so does its Jacobian $\mathrm{Jac}(X)$. With regard to the effective geometric Shafarevich's conjecture for counting the number of *non-isotrivial* smooth families of principally polarized Abelian surfaces over open subsets of $\mathbb{P}^1_{\mathbb{F}_q}$ with bounded squarefree bad reduction, we prove the following Theorem:

Theorem 1.6 (Estimate on $\mathcal{N}_{2,\mathbb{F}_q(t)}(\mathcal{B})$)**.** *If char$(\mathbb{F}_q) \neq 2,3,5$, then the function $\mathcal{N}_{2,\mathbb{F}_q(t)}(\mathcal{B})$, which counts the number of principally polarized Abelian surfaces $A = \mathrm{Jac}(X)$ where X is a stable genus 2 curve with a marked Weierstrass section over $\mathbb{P}^1_{\mathbb{F}_q}$ ordered by $0 < ht(\Delta_2(X)) = q^{40n} \leq \mathcal{B}$, satisfies:*

$$\mathcal{N}_{2,\mathbb{F}_q(t)}(\mathcal{B}) \leq 2 \cdot \frac{(q^{31} + q^{30} + q^{29} - q^{27} - q^{26} - q^{25})}{(q^{28} - 1)} \cdot \left(\mathcal{B}^{\frac{7}{10}} - 1\right) + 2 \cdot \frac{(q^{13} - q^{11})}{(q^{12} - 1)} \cdot \left(\mathcal{B}^{\frac{3}{10}} - 1\right)$$

Proof. Theorem 1.15 provides an explicit upper bound on the number of stable genus 2 curves with a marked Weierstrass section over $\mathbb{P}^1_{\mathbb{F}_q}$ with char$(\mathbb{F}_q) \neq 2,3,5$. The upper bound follows from the properties of the Torelli map τ_2 discussed above (c.f. [OS, Theorem 2.6 and 2.7] & [OU, 4. Theorem]). ■

For higher genus $g \geq 3$, we count the Jacobians of hyperelliptic curves.

Theorem 1.7 (Estimate on $\mathcal{N}_{g,\mathbb{F}_q(t)}(\mathcal{B})$)**.** *If char$(\mathbb{F}_q) > 2g+1$, then the function $\mathcal{N}_{g,\mathbb{F}_q(t)}(\mathcal{B})$, which counts the number of principally polarized hyperelliptic Jacobians $A = \mathrm{Jac}(X)$ where X is a stable hyperelliptic genus $g \geq 3$ curves X with a marked Weierstrass section over $\mathbb{P}^1_{\mathbb{F}_q}$ ordered by $0 < ht(\Delta_g(X)) = q^{4g(2g+1)n} \leq \mathcal{B}$, satisfies:*

$$\mathcal{N}_{g,\mathbb{F}_q(t)}(\mathcal{B}) \leq 2g \cdot \frac{q^{4g(g+1)+1} \cdot (q^{2g-1} - 1)(q^{2g} - 1)}{(q-1)(q^{2g(2g+3)} - 1)} \cdot \left(\mathcal{B}^{\frac{2g+3}{4g+2}} - 1\right)$$

Proof. Theorem 1.2 provides an explicit upper bound on the number of stable hyperelliptic genus $g \geq 3$ curves with a marked Weierstrass section over $\mathbb{P}^1_{\mathbb{F}_q}$ with char$(\mathbb{F}_q) > 2g+1$. The upper bound follows from the open immersion property of the Torelli map τ_g restricted to the hyperelliptic locus (c.f. [Landesman, Theorem 1.2]). Note that one could sharpen the upper bound through the Theorem 1.15. ■

Naturally, we formulate the following conjectures over $\mathbb{Q}$ by passing the above upper bound through the global fields analogy via the bounded *height* of the discriminant Δ_2:

Conjecture 1.8 (Heuristic on $\mathcal{N}_{2,\mathbb{Q}}(\mathcal{B})$)**.** The function $\mathcal{N}_{2,\mathbb{Q}}(\mathcal{B})$, which counts the number of principally polarized Abelian surfaces $A = \mathrm{Jac}(X)$ where X is a stable genus 2 curve over $\mathbb{Z}$ with $0 < ht(\Delta_2(X)) \leq \mathcal{B}$, has the same order of magnitude to

$$\mathcal{N}_{2,\mathbb{Q}}(\mathcal{B}) \asymp \mathcal{B}^{\frac{7}{10}}$$

And similarly for genus $g \geq 3$ via the bounded *height* of the hyperelliptic discriminant Δ_g:

Conjecture 1.9 (Heuristic on $\mathcal{N}_{g,\mathbb{Q}}(\mathcal{B})$)**.** The function $\mathcal{N}_{g,\mathbb{Q}}(\mathcal{B})$, which counts the number of principally polarized hyperelliptic Jacobians $A = \mathrm{Jac}(X)$ where X is a stable hyperelliptic genus $g \geq 3$ curve over $\mathbb{Z}$ with $0 < ht(\Delta_g(X)) \leq \mathcal{B}$, has the same order of magnitude to

$$\mathcal{N}_{g,\mathbb{Q}}(\mathcal{B}) \asymp \mathcal{B}^{\frac{2g+3}{4g+2}}$$

It would be consequential to prove the above effective Shafarevich's conjectures for Abelian varieties over number fields in its original formulation or its translated version in terms of the classical *Faltings height* as in [Faltings, §3]. Lastly, we note that the effective Shafarevich's conjecture for Abelian surfaces over $\mathbb{Q}$ which can be made algorithmic would render the effective Siegel's theorem [Siegel] for genus 2 curves over $\mathbb{Q}$ in the sense of [Levin, Theorem 1.2.].

Methods. The central idea behind the proof of Theorem 1.2 (and related Theorems 1.6 and 1.15) is to count rational points (with bounded height) on the fine moduli stack parameterizing the stable hyperelliptic curves over $\mathbb{P}^1_{\mathbb{F}_q}$ with a marked Weierstrass section. In this regard, efforts to acquire the arithmetic invariants of $\mathrm{Hom}(\mathbb{P}^1, \overline{\mathcal{M}}_g)$ comes up against hard obstructions due to the global geometry of the Deligne–Mumford moduli stack $\overline{\mathcal{M}}_g$ of stable genus g curves formulated in [DM]. For example, the coarse moduli space $\overline{M}_g$ is of general type for $g \geq 24$ by the fundamental works of Harris, Mumford and Eisenbud in [HM, EH] which in turn makes the study of (rational) curves on $\overline{\mathcal{M}}_{g\geq 24}$ ineffective for counting stable curves of sufficiently high genus over $\mathbb{P}^1_{\mathbb{F}_q}$.

On the other hand, all smooth genus 2 curves are hyperelliptic, so $\mathcal{M}_2 \cong \mathcal{H}_2$. Therefore, we are naturally led to count hyperelliptic genus $g \geq 2$ curves. In this paper, we will concentrate on the moduli substack $\mathcal{H}_{g,\underline{1}} \subset \mathcal{M}_{g,1}$ of hyperelliptic genus $g \geq 2$ curves with 1 marked Weierstrass point (which has the same dimension as $\mathcal{H}_g$) as we focus on counting odd degree hyperelliptic genus $g \geq 2$ curves. Since $\mathcal{H}_{g,\underline{1}}$ is not proper, we consider the proper moduli stack $\overline{\mathcal{H}}_{g,\underline{1}} := \overline{\mathcal{H}_{g,\underline{1}}} \subset \overline{\mathcal{M}}_{g,1}$ (meaning the reduced closure) of stable odd hyperelliptic curves. Extracting the exact arithmetic invariants of $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{H}}_{g,\underline{1}})$ is challenging as it is difficult to write down a general member of K–points of $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{H}}_{g,\underline{1}})$ explicitly as an equation $y^2 = f(x,t)$ where t is a parameter of $\mathbb{A}^1_K \subset \mathbb{P}^1_K$ and $f \in K(t)[x]$ of x–degree $2g+1$. Therefore, we are led to impose conditions on characteristic of $\mathbb{F}_q$, then consider a different extension of smooth odd degree hyperelliptic curves, originally introduced as a special case of [Fedorchuk, Definition 2.5]:

Definition 1.10. Fix an integral reduced K-scheme B, where $\mathrm{char}(K) \neq 2$. A flat family $u : C \to B$ of genus $g \geq 2$ curves is *quasi–admissible* if every geometric fiber has at worst A_{2g-1}-singularity (i.e., étale locally defined by $x^2 + y^m$ for some $0 < m \leq 2g$), and factors through a separable morphism $\phi : C \to H$ of degree 2 where H is a $\mathbb{P}^1$-bundle over B with a distinguished section (often called ∞) which is a connected component of the branch locus of u.

For example, if $\mathrm{char}(K) > 2g+1$ or 0, then a quasi–admissible curve over any K-scheme B can be written as

$$y^2 = f(x) = x^{2g+1} + a_4 x^{2g-1} + a_6 x^{2g-2} + a_8 x^{2g-3} + \cdots + a_{4g+2}, \tag{1}$$

where a_i's are appropriate sections of suitable line bundles on B where not all of them simultaneously vanish at anywhere on B. This equation should be thought of as a generalized Weierstrass equation. Here, we identify the section at ∞ as the locus missed by the above affine equation. This identification is a consequence of Theorem 4.9, where we show that the Deligne–Mumford moduli stack $\mathcal{H}_{2g}[2g-1]$ of quasi–admissible curves of genus g is isomorphic to the weighted projective stack $\mathcal{P}(4, 6, 8, \ldots, 4g+2)$ if $\mathrm{char}(K) = 0$ or $> 2g+1$. Throughout the paper, we consider $\mathcal{H}_{2g}[2g-1]$ as defined over some base field K (parameterizing the quasi–admissible K-curves). While we focus on a finite field $K = \mathbb{F}_q$, we frequently also consider a base field K with $\mathrm{char}(K) = 0$ or $> 2g+1$, whenever the relevant results apply to a broader context.

The novel idea here is to relate the arithmetic invariants of $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{H}}_{g,\underline{1}})$ to the arithmetic invariants of $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{H}_{2g}[2g-1])$; through the birational geometry between the family of stable odd degree hyperelliptic curves with generically smooth fiber and the family of quasi–admissible curves. Specifically, as a consequence of Theorem 4.11 over a field K with $\mathrm{char}(K) = 0$ or $> 2g+1$, we show that any given stable odd hyperelliptic model over $\mathbb{P}^1_K$ admits a birational transformation into a unique quasi–admissible curve over $\mathbb{P}^1_K$, and no two non-isomorphic stable odd hyperelliptic models correspond to the isomorphic quasi–admissible curve. *Therefore, counting quasi–admissible genus $g \geq 2$ curves over $\mathbb{P}^1_{\mathbb{F}_q}$ naturally renders an upper bound for counting stable odd hyperelliptic genus $g \geq 2$ curves over $\mathbb{P}^1_{\mathbb{F}_q}$ with generically smooth fibers as in Theorems 1.2 and 1.15.*

To effectively count $\mathbb{F}_q$-points of the Hom stack $\mathrm{Hom}(\mathbb{P}^1, \mathcal{H}_{2g}[2g-1])$ parameterizing the quasi–admissible curves over $\mathbb{P}^1_{\mathbb{F}_q}$, we need to impose a notion of *bounded height* on those $\mathbb{F}_q$–points. Thanks to the works of Lockhart and Liu, we have a natural definition (see Definition 4.15) of a hyperelliptic discriminant Δ_g of quasi–admissible curves as in [Lockhart, Liu2]. In fact, it is a homogeneous polynomial of degree $4g(2g+1)$ on variables a_i's, where each a_i has degree i (a_i's are as in equation (1) where $B = \mathbb{P}^1_{\mathbb{F}_q}$ in this case). Moreover, since $\mathcal{P}(4, 6, 8, \ldots, 4g+2)$ is a weighted projective stack (Definition 2.1) carrying a primitive ample line bundle $\mathcal{O}_{\mathcal{P}(4,6,8,\ldots,4g+2)}(1)$, the degree of the discriminant Δ_g of a given quasi-admissible fibration $f : \mathbb{P}^1 \to \mathcal{H}_{2g}[2g-1] \cong \mathcal{P}(4, 6, 8, \ldots, 4g+2)$ is equal to $4g(2g+1)n$ where $f^*\mathcal{O}_{\mathcal{P}(4,6,8,\ldots,4g+2)}(1) \cong \mathcal{O}_{\mathbb{P}^1}(n)$. Therefore, the Hom stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(4, 6, 8, \ldots, 4g+2))$ parameterizing such morphisms is the moduli stack of quasi-admissible genus $g \geq 2$ fibrations of a fixed discriminant degree $|\Delta_g| \cdot n = 4g(2g+1)n$ and those fibrations have the height of discriminant $ht(\Delta_g(X))$ equal to $q^{4g(2g+1)n}$ (c.f. §5).

All of the above frame our arithmetic counting problem into summing over the $\mathbb{F}_q$-point counts of $\mathcal{L}_{g,|\Delta_g|\cdot n} := \mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(4, 6, 8, \ldots, 4g+2))$ for $n \in \mathbb{N}$ with $0 < ht(\Delta_g(X)) = q^{4g(2g+1)n} \leq \mathcal{B}$.

To acquire this count, we generalize the problem into considering the Hom stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ parameterizing the K-morphisms $f : \mathbb{P}^1 \to \mathcal{P}(\vec{\lambda})$ with $f^*\mathcal{O}_{\mathcal{P}(\vec{\lambda})}(1) \cong \mathcal{O}_{\mathbb{P}^1}(n)$ with $n \in \mathbb{N}$, where the target is a weighted projective stack $\mathcal{P}(\vec{\lambda})$ (see Proposition 2.5). The original motivation of this generalization lies in the classical *geometric Batyrev-Manin conjecture* regarding the study of K-morphisms $\mathbb{P}^1 \to X$ [FMT, BM]. To do so, we first analyze the motive of the moduli stack by considering the class in the Grothendieck ring of K-stacks (see Definition 3.1):

Theorem 1.11 (Motive of the Hom stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ over K)**.** *Fix the weight $\vec{\lambda} = (\lambda_0, \ldots, \lambda_N)$ with $|\vec{\lambda}| := \sum_{i=0}^{N} \lambda_i$. If $\mathrm{char}(K)$ does not divide $\lambda_i \in \mathbb{N}$ for every i, then the motive of the Hom stack $[\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))]$ in the Grothendieck ring of K–stacks $K_0(\mathrm{Stck}_K)$ is equivalent to*

$$\left[\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))\right] = \left(\sum_{i=0}^{N} \mathbb{L}^i\right) \cdot \left(\mathbb{L}^{|\vec{\lambda}|n} - \mathbb{L}^{|\vec{\lambda}|n-N}\right) .$$

where $\mathbb{L}^1 := [\mathbb{A}^1_K]$ is the Lefschetz motive.

When $K = \mathbb{F}_q$, notice that any finite type algebraic $\mathbb{F}_q$-stack $\mathcal{X}$ admits a smooth cover $Y \to \mathcal{X}$ by a $\mathbb{F}_q$-scheme of finite type. This allows us to countably enumerate $\mathbb{F}_q$–points of $\mathcal{X}$ by enumerating $\mathbb{F}_{q^n}$–points of Y that maps to $\mathbb{F}_q$–points of $\mathcal{X}$ for all $n \in \mathbb{N}_{>0}$. Hence, we can define:

Definition 1.12. The weighted point count of $\mathcal{X}$ over $\mathbb{F}_q$ is defined as a sum:

$$\#_q(\mathcal{X}) := \sum_{x \in \mathcal{X}(\mathbb{F}_q)/\sim} \frac{1}{|\mathrm{Aut}(x)|},$$

where $\mathcal{X}(\mathbb{F}_q)/\sim$ is the set of $\mathbb{F}_q$–isomorphism classes of $\mathbb{F}_q$–points of $\mathcal{X}$, and we take $\frac{1}{|\mathrm{Aut}(x)|}$ to be 0 when $|\mathrm{Aut}(x)| = \infty$.

As the Grothendieck ring $K_0(\mathrm{Stck}_K)$ is the universal object for *additive invariants*, it is easy to see that when $K = \mathbb{F}_q$, the assignment $[X] \mapsto \#_q(X)$ gives a well-defined ring homomorphism $\#_q : K_0(\mathrm{Stck}_{\mathbb{F}_q}) \to \mathbb{Q}$ (c.f. [Ekedahl, §2]) rendering the weighted point count of a stack $\mathcal{X}$ (by [Behrend, Lemma 3.2.2], $\#_q(\mathcal{X}) < \infty$ when $\mathcal{X}$ is of finite type).

Corollary 1.13 (Point count of the Hom stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ over $\mathbb{F}_q$)**.** *Fix the weight $\vec{\lambda} = (\lambda_0, \ldots, \lambda_N)$ with $|\vec{\lambda}| := \sum_{i=0}^{N} \lambda_i$. Suppose that $\mathrm{char}(\mathbb{F}_q)$ does not divide $\lambda_i \in \mathbb{N}$ for every i, then the weighted point count of the Hom stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ over $\mathbb{F}_q$ is*

$$\#_q \left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))\right) = \left(\sum_{i=0}^{N} q^i\right) \cdot \left(q^{|\vec{\lambda}|n} - q^{|\vec{\lambda}|n-N}\right) .$$

Denote $\delta := \gcd(\lambda_0, \ldots, \lambda_N)$ and $\omega := \max \gcd(\lambda_i, \lambda_j)$ for $0 \le i, j \le N$. Then the number $|\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))(\mathbb{F}_q)/\sim|$ of $\mathbb{F}_q$–isomorphism classes of $\mathbb{F}_q$–points (i.e., the non–weighted point count over $\mathbb{F}_q$) of $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ satisfies

$$\delta \cdot \#_q \left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))\right) \le \left|\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))(\mathbb{F}_q)/\sim\right| \le \omega \cdot \#_q \left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))\right) .$$

Let us denote $\vec{\lambda_g} := (4, 6, 8, \ldots, 4g + 2)$ such that $\mathcal{P}(\vec{\lambda_g}) \cong \mathcal{H}_{2g}[2g - 1]$ is of dimension $2g - 1$ with $|\vec{\lambda_g}| := \sum_{i=1}^{2g} 2i + 2 = 2g(2g + 3)$. We now present the arithmetic invariants of the moduli stack $\mathcal{L}_{g,|\Delta_g|\cdot n} \cong \mathrm{Hom}_n(\mathbb{P}^1, \mathcal{H}_{2g}[2g - 1])$ of quasi-admissible hyperelliptic genus $g \ge 2$ fibrations over $\mathbb{P}^1$ with $|\Delta_g| \cdot n = 4g(2g + 1)n$ for $n \in \mathbb{N}$.

Corollary 1.14 (Motive and weighted point count of $\mathcal{L}_{g,|\Delta_g|\cdot n}$ over $\mathbb{F}_q$)**.** *If $\mathrm{char}(K) = 0$ or $> 2g+1$, then the motive of $\mathcal{L}_{g,|\Delta_g|\cdot n}$ in the Grothendieck ring of K–stacks $K_0(\mathrm{Stck}_K)$ is equivalent to*

$$\begin{aligned}
[\mathcal{L}_{g,|\Delta_g|\cdot n}] &= \left(\sum_{i=0}^{2g-1} \mathbb{L}^i\right) \cdot \left(\mathbb{L}^{|\vec{\lambda_g}|n} - \mathbb{L}^{|\vec{\lambda_g}|n-2g+1}\right) \\
&= \mathbb{L}^{2g(2g+3)n} \cdot (\mathbb{L}^{2g-1} + \mathbb{L}^{2g-2} + \cdots + \mathbb{L}^2 + \mathbb{L}^1 - \mathbb{L}^{-1} - \mathbb{L}^{-2} - \cdots - \mathbb{L}^{-2g+2} - \mathbb{L}^{-2g+1}).
\end{aligned}$$

If $K = \mathbb{F}_q$ with $\mathrm{char}(\mathbb{F}_q) > 2g + 1$, then

$$\#_q \left(\mathcal{L}_{g,|\Delta_g|\cdot n}\right) = q^{2g(2g+3)n} \cdot (q^{2g-1} + q^{2g-2} + \cdots + q^2 + q^1 - q^{-1} - q^{-2} - \cdots - q^{-2g+2} - q^{-2g+1}) .$$

Theorem 1.2 is deduced from Corollary 1.13 and 1.14. Moreover, by identifying the loci of $\mathcal{P}(\vec{\lambda})$ with the same stabilizer groups, we obtain the non-weighted point count of $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ by using Proposition 3.8, strengthening the second half of Corollary 1.13. Applications of this formula gives Proposition 4.17, which in turn implies the following sharper upper bound reinforcing Theorem 1.2:

Theorem 1.15 (Sharper estimate on $\mathcal{Z}_{g,\mathbb{F}_q(t)}(\mathcal{B})$)**.** *If $char(\mathbb{F}_q) > 2g+1$, then the function $\mathcal{Z}_{g,\mathbb{F}_q(t)}(\mathcal{B})$, which counts the number of stable hyperelliptic genus $g \ge 2$ curves X with a marked Weierstrass section over $\mathbb{P}^1_{\mathbb{F}_q}$ ordered by $0 < ht(\Delta_g(X)) = q^{4g(2g+1)n} \le \mathcal{B}$, satisfies:*

$$\mathcal{Z}_{2,\mathbb{F}_q(t)}(\mathcal{B}) \le 2 \cdot \frac{(q^{31} + q^{30} + q^{29} - q^{27} - q^{26} - q^{25})}{(q^{28} - 1)} \cdot \left(\mathcal{B}^{\frac{7}{10}} - 1\right) + 2 \cdot \frac{(q^{13} - q^{11})}{(q^{12} - 1)} \cdot \left(\mathcal{B}^{\frac{3}{10}} - 1\right)$$

$$\begin{aligned}
\mathcal{Z}_{3,\mathbb{F}_q(t)}(\mathcal{B}) \le{} & 2 \cdot \frac{(q^{59} + q^{58} + \cdots + q^{55} - q^{53} - q^{52} - \cdots - q^{49})}{(q^{54} - 1)} \cdot \left(\mathcal{B}^{\frac{9}{14}} - 1\right) \\
& + 2 \cdot \frac{(q^{26} + q^{25} - q^{23} - q^{22})}{(q^{24} - 1)} \cdot \left(\mathcal{B}^{\frac{2}{7}} - 1\right) + 4 \cdot \frac{(q^{19} - q^{17})}{(q^{18} - 1)} \cdot \left(\mathcal{B}^{\frac{3}{14}} - 1\right)
\end{aligned}$$

$$\begin{aligned}
\mathcal{Z}_{4,\mathbb{F}_q(t)}(\mathcal{B}) \le{} & 2 \cdot \frac{(q^{95} + q^{94} + \cdots + q^{89} - q^{87} - q^{52} - \cdots - q^{81})}{(q^{88} - 1)} \cdot \left(\mathcal{B}^{\frac{11}{18}} - 1\right) \\
& + 2 \cdot \frac{(q^{43} + q^{42} + q^{41} - q^{39} - q^{38} - q^{37})}{(q^{40} - 1)} \cdot \left(\mathcal{B}^{\frac{5}{18}} - 1\right)
\end{aligned}$$

$$+4 \cdot \frac{(q^{38}+q^{37}-q^{35}-q^{34})}{(q^{36}-1)} \cdot \left(\mathcal{B}^{\frac{1}{4}}-1\right) + 4 \cdot \frac{(q^{25}-q^{23})}{(q^{24}-1)} \cdot \left(\mathcal{B}^{\frac{1}{6}}-1\right)$$

For higher genus $g \geq 5$, the corresponding sharper upper bound on $\mathcal{Z}_{g,\mathbb{F}_q(t)}(\mathcal{B})$ rendering a closed-form formula as above can be similarly worked out through Proposition 4.17.

Outline. The outline of the paper is as follows. In §2, we formulate the Hom stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ of rational curves on weighted projective stack $\mathcal{P}(\vec{\lambda})$. In §3, we use the Grothendieck ring of K–stacks $K_0(\mathrm{Stck}_K)$ to acquire the motive $[\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))]$ (Theorem 1.11) which provides the weighted point count $\#_q(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda})))$. We also discuss the related non–weighted point count $|\,\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))(\mathbb{F}_q)/\sim|$ there, proving Corollary 1.13. Afterwards in §4, we formulate the moduli stack $\mathcal{L}_{g,|\Delta_g|\cdot n}$ of quasi-admissible hyperelliptic genus g fibrations over $\mathbb{P}^1$ with the hyperelliptic discriminant Δ_g via the birational geometry of surfaces. We use birational geometry to prove Theorem 4.11. Then we prove the sharp estimate on $\left|\mathcal{L}_{g,|\Delta_g|\cdot n}(\mathbb{F}_q)/\sim\right|$. In §5, we finally prove the (sharper) estimate Theorem 1.2 and Theorem 1.15 which provide the heuristic evidences for Conjecture 1.3 and Conjecture 1.8 through the global fields analogy. In Appendix, we determine the sharp estimate for counting elliptic curves with prescribed level structure or multiple marked points over $\mathbb{P}^1_{\mathbb{F}_q}$.

Conventions. In the present paper, schemes are assumed to be defined over a field K of characteristic not equal to 2, if K is not mentioned explicitly or if such scheme is obviously not defined over any field (e.g., Spec $\mathbb{Z}$). We identify the Weil divisors and the associated divisorial sheaves implicitly (e.g., if X is a Cohen-Macaulay scheme, then the canonical divisor K_X corresponds to the dualizing sheaf $\omega_X \cong \mathcal{O}(K_X)$ of X). Given a finite morphism $f : X \to Y$ of reduced equidimensional schemes, a branch divisor of f on Y means the pushforward of the ramification divisor of f on X. Given a function $f : \mathbb{N} \to \mathbb{R}$ with coefficients depending on an integer q, $f \sim \mathcal{O}_q(x^\alpha)$ means that $f \sim c(q)x^\alpha$ for some function $c(q)$ of q (independent of the input variable x). Also, $f \asymp g$ means that $|f|$ and $|g|$ are bounded by positive constant multiples of each other (as the input variable x tends to infinity) and thus have the same order of magnitude. Finally, a sharp estimate of a function f (with coefficients depending on q) means an upper bound g (with same properties) that coincide with f on infinitely many values of $x \in \mathbb{N}$.

2. Moduli stack of rational curves on weighted projective stack $\mathcal{P}(\vec{\lambda})$

In this section, we formulate the moduli stack of K-morphisms $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ where $\mathcal{P}(\vec{\lambda})$ is a fixed N-dimensional weighted projective stack with the weight $\vec{\lambda} = (\lambda_0, \dots, \lambda_N)$ over a basefield K with $\mathrm{char}(K) \nmid \lambda_i \in \mathbb{N}$ for every i .

We first recall the definition of an N-dimensional weighted projective stack $\mathcal{P}(\vec{\lambda})$.

Definition 2.1. Fix a tuple of nondecreasing positive integers $\vec{\lambda} = (\lambda_0, \dots, \lambda_N)$. The N-dimensional weighted projective stack $\mathcal{P}(\vec{\lambda}) = \mathcal{P}(\lambda_0, \dots, \lambda_N)$ with the weight $\vec{\lambda}$ is defined as a quotient stack

$$\mathcal{P}(\vec{\lambda}) := \left[(\mathbb{A}^{N+1}_{x_0,\dots,x_N} \setminus 0)/\mathbb{G}_m\right]$$

where $\zeta \in \mathbb{G}_m$ acts by $\zeta \cdot (x_0, \dots, x_N) = (\zeta^{\lambda_0}x_0, \dots, \zeta^{\lambda_N}x_N)$. In this case, the degree of x_i's are λ_i's respectively. A line bundle $\mathcal{O}_{\mathcal{P}(\vec{\lambda})}(m)$ is defined to be the line bundle associated to the sheaf of degree m homogeneous rational functions without poles on $\mathbb{A}^{N+1}_{x_0,\dots,x_N} \setminus 0$.

Note that $\mathcal{P}(\vec{\lambda})$ is not an (effective) orbifold when $\gcd(\lambda_0, \dots, \lambda_N) \neq 1$. In this case, the cyclic isotropy group scheme $\mu_{\gcd(\lambda_0,\dots,\lambda_N)}$ is the generic stabilizer of $\mathcal{P}(\vec{\lambda})$. Nevertheless, the following proposition shows that it behaves well in most characteristics as a tame Deligne–Mumford stack:

Proposition 2.2. *The N-dimensional weighted projective stack $\mathcal{P}(\vec{\lambda}) = \mathcal{P}(\lambda_0, \dots, \lambda_N)$ is a tame Deligne–Mumford stack over a basefield K if $\mathrm{char}(K)$ does not divide $\lambda_i \in \mathbb{N}$ for every i.*

Proof. For any algebraically closed field extension $\overline{K}$ of K, any point $y \in \mathcal{P}(\vec{\lambda})(\overline{K})$ is represented by the coordinates $(y_0, \dots, y_N) \in \mathbb{A}^{N+1}_{\overline{K}}$ with its stabilizer group as the subgroup of $\mathbb{G}_m$ fixing $(y_0, \dots, y_N)$. Hence, any stabilizer group of such $\overline{K}$-points is $\mathbb{Z}/u\mathbb{Z}$ where u divides λ_i for some i. Since the characteristic of K does not divide the orders of $\mathbb{Z}/\lambda_i\mathbb{Z}$ for any i, the stabilizer group of y is $\overline{K}$-linearly reductive. Hence, $\mathcal{P}(\vec{\lambda})$ is tame by [AOV, Theorem 3.2]. Note that the stabilizer groups constitute fibers of the diagonal $\Delta : \mathcal{P}(\vec{\lambda}) \to \mathcal{P}(\vec{\lambda}) \times_K \mathcal{P}(\vec{\lambda})$. Since $\mathcal{P}(\vec{\lambda})$ is of finite type and $\mathbb{Z}/u\mathbb{Z}$'s are unramified over K whenever u does not divide λ_i for some i, Δ is unramified as well. Therefore, $\mathcal{P}(\vec{\lambda})$ is also Deligne–Mumford by [Olsson2, Theorem 8.3.3]. ■

The tameness is analogous to flatness for stacks in positive/mixed characteristic as it is preserved under base change by [AOV, Corollary 3.4]. Moreover, if a stack $\mathcal{X}$ is tame and Deligne–Mumford, then the formation of the coarse moduli space $c : \mathcal{X} \to X$ commutes with base change as well by [AOV, Corollary 3.3].

Example 2.3. When the characteristic of the field K is not equal to 2 or 3, [Hassett2, Proposition 3.6] shows that the proper Deligne–Mumford stack of stable elliptic curves $(\overline{\mathcal{M}}_{1,1})_K \cong [(\mathrm{Spec}\, K[a_4, a_6] - (0,0))/\mathbb{G}_m] = \mathcal{P}_K(4,6)$ by using the short Weierstrass equation $y^2 = x^3 + a_4 x + a_6 x$, where $\zeta \cdot a_i = \zeta^i \cdot a_i$ for $\zeta \in \mathbb{G}_m$ and $i = 4, 6$. Thus, a_i's have degree i's respectively. Note that this is no longer true if characteristic of K is 2 or 3, as the Weierstrass equations are more complicated.

One can consider an embedding $\mathcal{X} \subset \mathcal{P}(\vec{\lambda})$ of a given cyclotomic stack $\mathcal{X}$ into an ambient $\mathcal{P}(\vec{\lambda})$.

Definition 2.4. A stack $\mathcal{X}$ is cyclotomic if for every point $p \in \mathcal{X}$, its stabilizer group is a finite cyclic group. Any closed substack of a cyclotomic stack is cyclotomic.

We now generalize the Hom stack formulation to $\mathcal{P}(\vec{\lambda})$ as follows:

Proposition 2.5. *The Hom stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ with the weight $\vec{\lambda} = (\lambda_0, \dots, \lambda_N)$, which parameterize the K-morphisms $f : \mathbb{P}^1 \to \mathcal{P}(\vec{\lambda})$ with $f^*\mathcal{O}_{\mathcal{P}(\vec{\lambda})}(1) \cong \mathcal{O}_{\mathbb{P}^1}(n)$ with $n \in \mathbb{N}$ over a basefield K with $\mathrm{char}(K)$ not dividing $\lambda_i \in \mathbb{N}$ for every i, is a smooth separated tame Deligne–Mumford stack of finite type with $\dim_{\mathrm{K}} \left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda})) \right) = |\vec{\lambda}| n + N$ where $|\vec{\lambda}| := \sum_{i=0}^{N} \lambda_i$ and $\dim_{\mathrm{K}} \left(\mathcal{P}(\vec{\lambda}) \right) = N$.*

Proof. $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ is a smooth Deligne–Mumford stack by [Olsson, Theorem 1.1]. It is isomorphic to the quotient stack $[T/\mathbb{G}_m]$, admitting a smooth schematic cover $T \subset \left(\bigoplus_{i=0}^{N} H^0(\mathcal{O}_{\mathbb{P}^1}(\lambda_i \cdot n)) \right) \setminus 0$, parameterizing the set of tuples $(u_0, \dots, u_N)$ of sections with no common zero. The $\mathbb{G}_m$ action on T is given by $\zeta \cdot (u_0, \dots, u_N) = (\zeta^{\lambda_0} u_0, \dots, \zeta^{\lambda_N} u_N)$. Note that

$$\dim T = \sum_{i=0}^{n} h^0(\mathcal{O}_{\mathbb{P}^1}(\lambda_i \cdot n)) = \sum_{i=0}^{n} (\lambda_i + 1) = |\vec{\lambda}| + N + 1,$$

implying that $\dim \mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda})) = |\vec{\lambda}| + N$ since $\dim \mathbb{G}_m = 1$.

As $\mathbb{G}_m$ acts on T properly with positive weights $\lambda_i > 0$ for every i, the quotient stack $[T/\mathbb{G}_m]$ is separated. It is tame as in [AOV, Theorem 3.2] since $\mathrm{char}(K)$ does not divide λ_i for every i . ■

3. Motive/Point count of $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$

In this section, we show that the Grothendieck class of $[\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))] \in K_0(\mathrm{Stck}_{\mathrm{K}})$ in the Grothendieck ring of K–stacks is expressed as a polynomial in the Lefschetz motive $\mathbb{L} := [\mathbb{A}^1]$.

To perform a weighted point count of $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ over $\mathbb{F}_q$, we use the idea of cut-and-paste by Grothendieck:

Definition 3.1. [Ekedahl, §1] Fix a field K. Then the *Grothendieck ring* $K_0(\mathrm{Stck}_K)$ *of algebraic stacks of finite type over* K *all of whose stabilizer group schemes are affine*, is an abelian group generated by isomorphism classes of K-stacks $[\mathcal{X}]$ of finite type, modulo relations:

- $[\mathcal{X}] = [\mathcal{Z}] + [\mathcal{X} \setminus \mathcal{Z}]$ for $\mathcal{Z} \subset \mathcal{X}$ a closed substack,
- $[\mathcal{E}] = [\mathcal{X} \times \mathbb{A}^n]$ for $\mathcal{E}$ a vector bundle of rank n on $\mathcal{X}$.

Multiplication on $K_0(\mathrm{Stck}_K)$ is induced by $[\mathcal{X}][\mathcal{Y}] := [\mathcal{X} \times_K \mathcal{Y}]$. There is a distinguished element $\mathbb{L} := [\mathbb{A}^1] \in K_0(\mathrm{Stck}_K)$, called the *Lefschetz motive.*

Given an algebraic K-stack $\mathcal{X}$ of finite type with affine diagonal, the *motive* of $\mathcal{X}$ refers to $[\mathcal{X}] \in K_0(\mathrm{Stck}_K)$.

Since many algebraic stacks can be written locally as a quotient of a scheme by an algebraic group $\mathbb{G}_m$, the following lemma (a special case of [Ekedahl, §1]) is very useful:

Lemma 3.2. [HP, Lemma 15] *For any* $\mathbb{G}_m$*-torsor* $\mathcal{X} \to \mathcal{Y}$ *of finite type algebraic stacks, we have* $[\mathcal{Y}] = [\mathcal{X}][\mathbb{G}_m]^{-1}$.

The proof of Theorem 1.11 involves the following variety of its own interest (a slight generalization of [FW, Definition 1.1]) :

Definition 3.3. Fix $m \in \mathbb{Z}_{>0}$ and $d_1, \ldots, d_m \geq 0$. Define $\mathrm{Poly}_1^{(d_1,\ldots,d_m)}$ as the set of tuples $(f_1, \ldots, f_m)$ of monic polynomials in $K[z]$ so that

(1) $\deg f_i = d_i$ for each i, and
(2) $f_1, \ldots, f_m$ have no common roots in $\overline{K}$.

Since the set $\mathrm{Poly}_1^{(d_1,\ldots,d_m)}$ is open inside the affine space (complement of the resultant hypersurface) parameterizing the tuples of monic coprime polynomials of degrees $(d_1, \ldots, d_m)$, we can endow $\mathrm{Poly}_1^{(d_1,\ldots,d_m)}$ with a structure of affine variety defined over $\mathbb{Z}$.

Generalizing the proof of [FW, Theorem 1.2] with the correction from [PS, Proposition 3.1.], we find the motive of $\mathrm{Poly}_1^{(d_1,\ldots,d_m)}$:

Proposition 3.4. *Fix* $0 \leq d_1 \leq d_2 \leq \cdots \leq d_m$. *Then,*

$$\left[\mathrm{Poly}_1^{(d_1,\cdots,d_m)}\right] = \begin{cases} \mathbb{L}^{d_1+\cdots+d_m} - \mathbb{L}^{d_1+\cdots+d_m-m+1}, & \text{if } d_1 \neq 0 \\ \mathbb{L}^{d_1+\cdots+d_m}, & \text{if } d_1 = 0 \end{cases}$$

Proof. The proof is analogous to [FW, Theorem 1.2 (1)], with the correction from [PS, Proposition 3.1.], and is a direct generalization of [HP, Proposition 18]. Here, we recall the differences to the work in [FW, HP, PS].

Step 1: The space of $(f_1, \ldots, f_m)$ monic polynomials of degree $d_1, \ldots, d_m$ is instead the quotient $\mathbb{A}^{d_1} \times \cdots \times \mathbb{A}^{d_m}/(S_{d_1} \times \cdots \times S_{d_m}) \cong \mathbb{A}^{d_1+\cdots+d_m}$. We have the same filtration of $\mathbb{A}^{\sum d_i}$ by $R_{1,k}^{(d_1,\ldots,d_m)}$: the space of monic polynomials $(f_1, \ldots, f_m)$ of degree $d_1, \ldots, d_m$ respectively for which there exists a monic $h \in K[z]$ with $\deg(h) \geq k$ and monic polynomials $g_i \in K[z]$ so that $f_i = g_i h$ for any i. The rest of the arguments follow analogously, keeping in mind that the group action is via $S_{d_1} \times \cdots \times S_{d_m}$.

Step 2: Here, we prove that $R_{1,k}^{(d_1,\ldots,d_m)} - R_{1,k+1}^{(d_1,\ldots,d_m)} \cong \mathrm{Poly}_1^{(d_1-k,\ldots,d_m-k)} \times \mathbb{A}^k$. Just as in [FW], the base case of $k = 0$ follows from the definition. For $k \geq 1$, the rest of the arguments follow analogously just as in Step 2 of loc. cit. where the isomorphism over $\mathbb{Z}$ of the morphism

$$\Psi : \mathrm{Poly}_1^{(d_1-k,\dots,d_m-k)} \times \mathbb{A}^k \to R_{1,k}^{(d_1,\dots,d_m)} \setminus R_{1,k+1}^{(d_1,\dots,d_m)}$$

is provided by the proof of [PS, Proposition 3.1.].

Step 3: By combining Step 1 and 2 as in [FW], we obtain

$$\left[\mathrm{Poly}_1^{(d_1,\dots,d_m)}\right] = \mathbb{L}^{d_1+\dots+d_m} - \sum_{k\geq 1}\left[\mathrm{Poly}_1^{(d_1-k,\dots,d_m-k)}\right]\mathbb{L}^k$$

For the induction on the class $\left[\mathrm{Poly}_1^{(d_1,\dots,d_m)}\right]$, we use lexicographic induction on the pair $(d_1,\dots,d_m)$. For the base case, consider when $d_1 = 0$. Here the monic polynomial of degree 0 is nowhere vanishing, so that any tuple of polynomials of degree d_i for $i > 1$ constitutes a member of $\mathrm{Poly}_1^{(0,d_2,\dots,d_m)}$, so that $\mathrm{Poly}_1^{(0,d_2,\dots,d_m)} \cong \mathbb{A}^{d_2+\dots+d_m}$.

Now assume that $d_1 > 0$. Then, we obtain

$$\begin{aligned}
&\left[\mathrm{Poly}_1^{(d_1,\dots,d_m)}\right] \\
&= \mathbb{L}^{d_1+\dots+d_m} - \sum_{k\geq 1}\left[\mathrm{Poly}_1^{(d_1-k,\dots,d_m-k)}\right]\mathbb{L}^k \\
&= \mathbb{L}^{d_1+\dots+d_m} - \left(\sum_{k=1}^{d_1-1}(\mathbb{L}^{(d_1-k)+\dots+(d_m-k)} - \mathbb{L}^{(d_1-k)+\dots+(d_m-k)-m+1})\mathbb{L}^k + \mathbb{L}^{(d_2-d_1)+\dots+(d_m-d_1)}\mathbb{L}^{d_1}\right) \\
&= \mathbb{L}^{d_1+\dots+d_m} - \left(\sum_{k=1}^{d_1-1}(\mathbb{L}^{d_1+\dots+d_m-(m-1)k} - \mathbb{L}^{d_1+\dots+d_m-(m-1)(k+1)}) + \mathbb{L}^{d_1+\dots+d_m-(m-1)d_1}\right) \\
&= \mathbb{L}^{d_1+\dots+d_m} - \mathbb{L}^{d_1+\dots+d_m-m+1}
\end{aligned}$$

■

Now we are ready to prove Theorem 1.11.

3.1. **Proof of Theorem 1.11.** Let $\vec{\lambda} = (\lambda_0,\dots,\lambda_N)$ and $\lambda_i \in \mathbb{N}$ for every i with $|\vec{\lambda}| := \sum_{i=0}^{N}\lambda_i$. Then the Hom stack $\mathrm{Hom}_n(\mathbb{P}^1,\mathcal{P}(\vec{\lambda})) \cong [T/\mathbb{G}_m]$ is the quotient stack by the proof of Proposition 2.5. By Lemma 3.2, we have $[\mathrm{Hom}_n(\mathbb{P}^1,\mathcal{P}(\vec{\lambda}))] = (\mathbb{L}-1)^{-1}[T]$. Henceforth, it suffices to find the motive $[T]$. To do so, we need to reinterpret T as follows.

Fix a chart $\mathbb{A}^1 \hookrightarrow \mathbb{P}^1$ with $x \mapsto [1 : x]$, and call $0 = [1 : 0]$ and $\infty = [0 : 1]$. It comes from a homogeneous chart of $\mathbb{P}^1$ by $[Y : X]$ with $x := X/Y$ away from ∞. Then for any $u \in H^0(\mathcal{O}_{\mathbb{P}^1}(d))$ with $d \geq 0$, u is a homogeneous polynomial of degree d in X and Y. By substituting in $Y = 1$, we obtain a representation of u as a polynomial in x with degree at most d. For instance, $\deg u < d$ as a polynomial in x if and only if $u(X,Y)$ is divisible by Y (i.e., u vanishes at ∞). From now on, $\deg u$ means the degree of u as a polynomial in x. Conventionally, set $\deg 0 := -\infty$.

Therefore, T parameterizes a N-tuple $(f_0,\dots,f_N)$ of polynomials in $K[x]$ with no common roots in $\overline{K}$, where $\deg f_i \leq n\lambda_i$ for each i with equality for some i. We use this interpretation to construct $\Phi : T \to \mathbb{A}^{N+1}\setminus 0$, $\Phi(f_0,\dots,f_N) = (a_0,\dots,a_N)$, where a_i is the coefficient of degree $n\lambda_i$ term of f_i.

Now, we stratify T by taking preimages under Φ of a stratification of $\mathbb{A}^{N+1} \setminus 0$ by $\sqcup E_J$, where J is any proper subset of $\{0, \ldots, N\}$ and

$$E_J = \{(a_0, \ldots, a_N) \mid a_j = 0 \ \ \forall j \in J\} \cong \mathbb{G}_m^{N+1-|J|}$$

Observe that E_J has the natural free $\mathbb{G}_m^{N+1-|J|}$-action, which lifts to $\Phi^{-1}(E_J)$ via multiplication on $\mathbb{G}_m$-scalars on f_i for $i \notin J$. The action is free on $\Phi^{-1}(E_J)$ as well, so that $\Phi|_{\Phi^{-1}(E_J)}$ is a Zariski-locally trivial fibration with base E_J. Each fiber is isomorphic to $F_J(n\vec{\lambda})$ defined below:

Definition 3.5. Fix $m \in \mathbb{N}$ and $\vec{d} := (d_0, \ldots, d_N) \in \mathbb{Z}_{\geq 0}^{N+1}$. Given $J \subsetneq \{0, \ldots, N\}$, $F_J(\vec{d})$ is defined as a variety consisting of tuples $(f_0, \ldots, f_N)$ of K-polynomials without common roots such that

- for any $j \notin J$, then f_j is monic of degree $n\lambda_j$, and
- for any $j \in J$, then $\deg f_j < n\lambda_j$ (f_j is not necessarily monic).

If instead $J = \{0, \ldots, N\}$, then we define $F_J(\vec{d}) := \emptyset$

This implies that $[\Phi^{-1}(E_J)] = [E_J][F_J(n\vec{\lambda})] = (\mathbb{L} - 1)^{N+1-|J|}[F_J(n\vec{\lambda})]$. Since

$$[T] = \sum_{J \subsetneq \{0,\ldots,N\}} [\Phi^{-1}(E_J)] = \sum_{J \subsetneq \{0,\ldots,N\}} [E_J][F_J(n\vec{\lambda})] \ , \tag{2}$$

it suffices to find $[F_J(n\vec{\lambda})]$ as a polynomial of $\mathbb{L}$.

Proposition 3.6. *$[F_J(n\vec{\lambda})] = [\mathrm{Poly}_1^{(n\lambda_0, \cdots, n\lambda_N)}] = \left(\mathbb{L}^{|\vec{\lambda}| \cdot n} - \mathbb{L}^{|\vec{\lambda}| \cdot n - N}\right)$, where $|\vec{\lambda}| := \sum_i \lambda_i$. In other words, $[F_J(n\vec{\lambda})]$ only depends on $n\vec{\lambda}$.*

Proof. Set $d_i := n\lambda_i > 0$ for the notational convention. Upto S_{N+1}-action on $\{0, \ldots, N\}$ (forgetting that $\lambda_0 \leq \cdots \leq \lambda_N$), consider instead $F_{\langle m \rangle}(\vec{d})$ with $\langle m \rangle = \{0, \cdots, m-1\}$ and $\vec{d} = (d_0, \cdots, d_N)$ with $|\vec{d}| := \sum_{i=0}^{N} d_i$. We now want to show that

$$[F_{\langle m \rangle}(\vec{d})] = [\mathrm{Poly}_1^{(d_0, \cdots, d_N)}] = \left(\mathbb{L}^{|\vec{d}|} - \mathbb{L}^{|\vec{d}| - N}\right). \tag{3}$$

In order to prove this, we first check that if we set $d_i = 0$ for some $i \geq m$, then

$$[F_{\langle m \rangle}(\vec{d})] = [\mathrm{Poly}_1^{(d_0, \cdots, d_N)}] = \mathbb{L}^{|\vec{d}|}.$$

To see this, note that $i \notin \langle m \rangle$, so that f_i is monic of degree $d_i = 0$ for any $(f_0, \ldots, f_N) \in F_{\langle m \rangle}(\vec{d})$; so $f_i = 1$. Therefore, the common root condition from Definition 3.5 is vacuous, so that $[F_{\langle m \rangle}(\vec{d})] = \mathbb{L}^{|\vec{d}|}$ (as the space of monic polynomials of degree d is isomorphic to $\mathbb{A}^d$ and so is the space of polynomials of degree $< d$).

We prove equation (3) by lexicographical induction on the ordered pairs (N, m) such that $N > 0$ and $0 \leq m < N + 1$. There are two base cases to consider:

1. If $m = 0$, then $\langle 0 \rangle = \emptyset$, so that $F_\emptyset(\vec{d}) \cong \mathrm{Poly}_1^{(d_0, \ldots, d_N)} =: \mathrm{Poly}_1^{\vec{d}}$ by Definition 3.3.
2. If $N = 1$, then m is 0 or 1. $m = 0$ follows from above. Now assume $m = 1$. Then $(f_0, f_1) \in F_{\langle 1 \rangle}(\vec{d})$ if and only if $\deg f_0 < d_0$ and $\deg f_1 = d_1 > 0$ with f_1 monic. Observe that f_0 cannot be 0, otherwise f_1 has no roots while having positive degree, which is a contradiction. Since f_0 can be written as $a_0 g_0$ for g_0 monic of degree $\deg f_0$ and $a_0 \in \mathbb{G}_m$, $F_{\langle 1 \rangle}(\vec{d})$ decomposes into the following locally closed subsets:

$$F_{\langle 1 \rangle}(\vec{d}) = \bigsqcup_{l=0}^{d_0 - 1} \mathbb{G}_m \times F_\emptyset(l, d_1) = \mathbb{G}_m \times \bigsqcup_{l=0}^{d_0 - 1} \mathrm{Poly}_1^{(l, d_1)}.$$

Therefore,

$$\begin{aligned}
[F_{\langle 1\rangle}(\vec{d})] &= [\mathbb{G}_m]\sum_{l=0}^{d_0-1}\left[\mathrm{Poly}_1^{(l,d_1)}\right] = (\mathbb{L}-1)\left(\mathbb{L}^{d_1}+\sum_{l=1}^{d_0-1}(\mathbb{L}^{l+d_1}-\mathbb{L}^{l+d_1-1})\right)\\
&= (\mathbb{L}-1)(\mathbb{L}^{d_1}+\mathbb{L}^{d_0+d_1-1}-\mathbb{L}^{d_1}) = (\mathbb{L}-1)\mathbb{L}^{d_0+d_1-1}\\
&= \mathbb{L}^{d_0+d_1}-\mathbb{L}^{d_0+d_1-1}
\end{aligned}$$

In general, assume that the statement is true for any (N', m') whenever $N' < N$ or $N' = N$ and $m' \le m$. If $m+1 < N+1$, then we want to prove the assertion for $(N, m+1)$. We can take the similar decomposition as the base case $(1,1)$, except that we vary the degree of f_m, which is the $(m+1)$-st term of $(f_0,\ldots,f_N) \in F_{\langle m+1\rangle}(\vec{d})$, and f_m can be 0. If $f_m = 0$, then $(f_0,\ldots,\widehat{f_m},\ldots,f_N)$ have no common roots, so that $(f_0,\ldots,\widehat{f_m},\ldots,f_N) \in F_{\langle m\rangle}(d_0,\ldots,\widehat{d_m},\ldots,d_N)$ (and vice versa). Henceforth, as a set,

$$\begin{aligned}
F_{\langle m+1\rangle}(\vec{d}) = F_{\langle m\rangle}(d_0,\ldots,\widehat{d_m},\ldots,d_N) \bigsqcup (\mathbb{G}_m \times F_{\langle m\rangle}(d_0,\ldots,0,\ldots,d_N))\\
\bigsqcup\left(\mathbb{G}_m \times \bigsqcup_{\ell=1}^{d_m-1} F_{\langle m\rangle}(d_0,\ldots,\ell,\ldots,d_N)\right).
\end{aligned}$$

By induction,

$$\begin{aligned}
\left[F_{\langle m+1\rangle}(\vec{d})\right] &= \left[F_{\langle m\rangle}(d_0,\cdots,\widehat{d_m},\cdots,d_N)\right] + (\mathbb{L}-1)\left[F_{\langle m\rangle}(d_0,\cdots,0,\cdots,d_N)\right]\\
&\quad + (\mathbb{L}-1)\sum_{\ell=0}^{d_m-1}\left[F_{\langle m\rangle}(d_0,\cdots,\ell,\cdots,d_N)\right]\\
&= \mathbb{L}^{|\vec{d}|-d_m} - \mathbb{L}^{\vec{d}-d_m-N+1} + (\mathbb{L}-1)\cdot\mathbb{L}^{|\vec{d}|-d_m}\\
&\quad + (\mathbb{L}-1)\sum_{\ell=1}^{d_m-1}\left(\mathbb{L}^{|\vec{d}|-d_m+\ell} - \mathbb{L}^{|\vec{d}|-d_m+\ell-N}\right)\\
&= \mathbb{L}^{|\vec{d}|-d_m} - \mathbb{L}^{|\vec{d}|-d_m-N+1} + \mathbb{L}^{|\vec{d}|-d_m+1} - \mathbb{L}^{|\vec{d}|-d_m}\\
&\quad + (\mathbb{L}-1)\mathbb{L}(\mathbb{L}^{|\vec{d}|-d_m} - \mathbb{L}^{|\vec{d}|-d_m-N})(1+\mathbb{L}+\cdots+\mathbb{L}^{d_m-2})\\
&= \mathbb{L}^{|\vec{d}|-d_m+1} - \mathbb{L}^{|\vec{d}|-d_m-N+1} + \mathbb{L}(\mathbb{L}^{|\vec{d}|-d_m} - \mathbb{L}^{|\vec{d}|-d_m-N})(\mathbb{L}^{d_m-1}-1)\\
&= \mathbb{L}^{|\vec{d}|-d_m+1} - \mathbb{L}^{|\vec{d}|-d_m-N+1} + \mathbb{L}^{|\vec{d}|} - \mathbb{L}^{|\vec{d}|-d_m+1} - \mathbb{L}^{|\vec{d}|-N} + \mathbb{L}^{|\vec{d}|-d_m-N+1}\\
&= \mathbb{L}^{|\vec{d}|} - \mathbb{L}^{|\vec{d}|-N}
\end{aligned}$$

■

Combining (2) and Proposition 3.6 with $\sum_{J\subsetneq\{0,\ldots,N\}} E_J = (\mathbb{A}^{N+1}\setminus 0)$, we finally acquire

$$\begin{aligned}
[\mathrm{Hom}_n(\mathbb{P}^1,\mathcal{P}(\vec{\lambda}))] &= [\mathbb{G}_m]^{-1}[T] = (\mathbb{L}-1)^{-1}\sum_{J\subsetneq\{0,\ldots,N\}}[E_J][\mathrm{Poly}_1^{(n\vec{\lambda})}]\\
&= (\mathbb{L}-1)^{-1}(\mathbb{L}^{N+1}-1)[\mathrm{Poly}_1^{(n\vec{\lambda})}] = \left(\sum_{i=0}^{N}\mathbb{L}^i\right)\cdot\left(\mathbb{L}^{|\vec{\lambda}|\cdot n} - \mathbb{L}^{|\vec{\lambda}|\cdot n-N}\right)
\end{aligned}$$

This finishes the proof of Theorem 1.11.

3.2. **(Non-)Weighted point count of** $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ **over finite fields.** Fix the basefield $K = \mathbb{F}_q$ a finite field of order $q = p^d$. Here, we exhibit facts about the weighted point count $\#_q(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda})))$ and the non–weighted point count $|\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))(\mathbb{F}_q)/\sim|$ (i.e., the number of $\mathbb{F}_q$–isomorphism classes of $\mathbb{F}_q$–points) over $\mathbb{F}_q$ as consequences of Theorem 1.11.

First, we prove the Corollary 1.13 below:

Proof of Corollary 1.13. The first part of the Corollary follows as $\#_q : K_0(\mathrm{Stck}_{\mathbb{F}_q}) \to \mathbb{Q}$ is a ring homomorphism (see §1) with $\#_q(\mathbb{L}) = q$ as $\mathbb{L} = [\mathbb{A}^1_{\mathbb{F}_q}]$. For the second part, notice that for each $\varphi \in \mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))(\mathbb{F}_q)/\sim$, it contributes 1 towards $|\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))(\mathbb{F}_q)/\sim|$ instead of $\frac{1}{|\mathrm{Aut}(\varphi)|}$ for $\#_q(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda})))$. Thus, we need to check that for any $\varphi \in \mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))(\mathbb{F}_q)$ with $\delta := \gcd(\lambda_0, \dots, \lambda_N)$ and $\omega := \max \gcd(\lambda_i, \lambda_j)$ for $0 \le i, j \le N$, the automorphism group satisfies the following :

$$\delta \le |\mathrm{Aut}(\varphi)| \le \omega \ .$$

By Proposition 2.5, we can represent φ as a tuple $(f_0, \dots, f_N)$ of sections $f_i \in H^0(\mathcal{O}_{\mathbb{P}^1_{\mathbb{F}_q}}(n\lambda_i))$, with equivalence relation given by a $\mathbb{G}_m$–action. Since the automorphism group of φ is identified with the subgroup of $\mathbb{G}_m$ fixing $(f_0, \dots, f_N)$, $\mathrm{Aut}(\varphi)$ consists of $u \in \mathbb{G}_m(\mathbb{F}_q)$ such that $u^{\lambda_i} f_i = f_i$ for any i . Since f_i's have no common root and the degree of the morphism φ is $n \in \mathbb{N}$, at least two of those are nonzero; call I to be the set of i's with $f_i \neq 0$. Then, $u^{\lambda_i} = 1$ for any $i \in I$, so that u is a $\gcd(\lambda_i \ : \ i \in I)^{\mathrm{th}}$ root of unity. This shows that $\mathrm{Aut}(\varphi)$ is a finite cyclic group of order $\gcd(\lambda_i \ : \ i \in I)$, proving the second part of the Corollary. ∎

Above proof shows that computing automorphism groups of $\mathbb{F}_q$–points of $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ is the key ingredient for comparing between weighted and non–weighted point counts over $\mathbb{F}_q$. Since the automorphism group of an equivalence class $(f_0, \dots, f_N)$ depends on which f_i is 0, we can characterize such regions as follows :

Definition 3.7. Fix J to be a subset of indices $\{0, 1, \dots, N\}$, where $\vec{\lambda} = (\lambda_0, \dots, \lambda_N)$. Then, $U(J)$ is defined to be a locally closed substack of $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$, consisting of equivalence classes of elements $(f_0, \dots, f_N) \in T$ with $f_j \neq 0$ for any $j \in J$.

Above definition combined with the proof of Corollary 1.13 gives an algorithm for computing $|\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))(\mathbb{F}_q)/\sim|$:

Proposition 3.8.

$$|\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))(\mathbb{F}_q)/\sim| = \sum_{\substack{I \subset \{0,\dots,N\} \\ |I| \ge 2}} \#_q(U(I)) \cdot \gcd(\lambda_i \ : \ i \in I)$$

Note that writing a closed-form formula for $|\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))(\mathbb{F}_q)/\sim|$ is difficult in general, as modular arithmetic is used and computing $\#_q(U(I))$ from $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ involves a fairly involved inclusion-exclusion formula of terms $\#_q(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}_J)))$, where $\vec{\lambda}_J$ is a subtuple of $\vec{\lambda}$ only involving λ_j for $j \in J \subset \{0, \dots, N\}$. Nevertheless, it is possible to obtain a closed-form formula by hand for special cases (Proposition 4.17 is a good example).

4. Moduli stack $\mathcal{L}_{g,|\Delta_g|\cdot n}$ of quasi-admissible hyperelliptic genus g fibrations over $\mathbb{P}^1$

In this section, we first define a rational fibration with a marked section, which allows us to define a hyperelliptic genus g fibration with a marked Weierstrass section as a double cover fibration. Subsequently, we focus on a quasi-admissible hyperelliptic genus g fibration over $\mathbb{P}^1$ with a marked Weierstrass section which extends a family of odd degree hyperelliptic genus g curves over $\mathbb{F}_q(t)$ with a marked Weierstrass point. For detailed references on hyperelliptic fibrations or fibered algebraic/arithmetic surfaces (over an algebraically closed field), we refer the reader to [Liu, Liedtke].

Recall that a hyperelliptic curve C is a separable morphism $\phi : C \to \mathbb{P}^1$ of degree 2. In order to extend the notion of hyperelliptic curve C into family, we first generalize the notion of rational curve $\mathbb{P}^1$ into family.

Definition 4.1. A *rational fibration with a marked section* is given by a flat proper morphism $h : H \to \mathbb{P}^1$ of pure relative dimension 1 with a marked section $s' : \mathbb{P}^1 \to H$ such that

(1) any geometric fiber $h^{-1}(c)$ is a connected rational curve (so that arithmetic genus is 0),
(2) $s'(\mathbb{P}^1)$ is away from the non-reduced locus of any geometric fiber, and
(3) $s'(\mathbb{P}^1)$ is away from the singular locus of H.

If the geometric generic fiber of h is a smooth rational curve, then we call (H, h, s') a $\mathbb{P}^1$*-fibration*.

We will occationally call (H, h, s') a *rational fibration* when there is no ambiguity on the marked section s'. Note that we allow a rational fibration H to be reducible (when generic fiber is a nodal chain), and the total space of a $\mathbb{P}^1$-fibration can be singular. Certain double cover of the rational fibration gives us the hyperelliptic genus g fibration with a marked Weierstrass section.

Definition 4.2. A *hyperelliptic genus g fibration with a marked Weierstrass section* consists of a tuple (X, H, h, f, s, s') of a rational fibration $h : H \to \mathbb{P}^1$, a flat proper morphism $f : X \to H$ of degree 2 with X connected and reduced, and sections $s : \mathbb{P}^1 \to X$ and $s' : \mathbb{P}^1 \to H$ such that

(1) Each geometric fiber $(h \circ f)^{-1}(c)$ is a connected 1–dimensional scheme of arithmetic genus g,
(2) $s(\mathbb{P}^1)$ is contained in the smooth locus of $h \circ f$ and is away from the non-reduced locus of any geometric fiber,
(3) $s' = f \circ s$ and $s(\mathbb{P}^1)$ is a connected component of the ramification locus of f (i.e., $s'(\mathbb{P}^1)$ is a connected component of the branch locus of f), and
(4) if p is a node of a geometric fiber $h^{-1}(c)$, then any $q \in f^{-1}(p)$ is a node of the fiber $(h \circ f)^{-1}(c)$,
(5) if the branch divisor of f contains a node e of a fiber $h^{-1}(t)$ with t a closed geometric point of $\mathbb{P}^1$, then the branch divisor contains either an irreducible component of $h^{-1}(t)$ containing e or an irreducible component of the singular locus of H containing e.

The *underlying genus g fibration* is a tuple $(\pi := h \circ f, s)$ with $\pi : X \to \mathbb{P}^1$ a flat proper morphism with geometric fibers of arithmetic genus g with a marked Weierstrass point given by s.

Note 4.3. An isomorphism between hyperelliptic genus g fibrations $(X_1, H_1, h_1, f_1, s_1, s'_1)$ and $(X_2, H_2, h_2, f_2, s_2, s'_2)$ is given by a pair of isomorphisms $\alpha : X_1 \to X_2$ and $\beta : H_1 \to H_2$ such that

(1) $h_2 \circ \beta = h_1$ and $f_2 \circ \alpha = \beta \circ f_1$ ($\mathbb{P}^1$–isomorphism criteria), and
(2) $\beta \circ s = s'$ (compatibility with sections).

From now on, we only consider non-isotrivial hyperelliptic fibrations, i.e., the underlying genus g fibrations must be non-isotrivial. Thus, non-isotrivialness will be assumed on every statement and discussions below.

Recall that a fibration with a section is said to be *stable* if all of its fibers are stable pointed curves. This leads to the following definition in the hyperelliptic case:

Definition 4.4. A *stable hyperelliptic genus g fibration with a marked Weierstrass section* is a hyperelliptic genus g fibration (X, H, h, f, s, s') with $K_X + s(\mathbb{P}^1)$ is π-ample. We assume that X is not isotrivial, i.e., the trivial hyperelliptic fiber bundle over $\mathbb{P}^1$ with no singular fibers.

Moreover, if the geometric generic fiber is smooth, then (X, H, h, f, s, s') is called a *stable odd hyperelliptic genus g model over* $\mathbb{P}^1$.

Conditions in the above definition implies that $(X, s(\mathbb{P}^1))/\mathbb{P}^1$ is log canonical. In classical language, this means that there are no smooth rational curves of self-intersection -1 and -2 in a fiber without meeting $s(\mathbb{P}^1)$.

Example 4.5. Suppose that (X, H, h, f, s, s') is a stable odd hyperelliptic genus g model with a marked Weierstrass section. Then, it is possible that $f : X \to H$ in a étale local neighborhood of $p \in H$ is the map $\mathbb{A}^2_{x,y} \to \mathbb{A}^2_{x,y}/\mu_2$, where μ_2 acts on $\mathbb{A}^2_{x,y}$ by $(x, y) \mapsto (-x, -y)$. In this case, π can be given by $\mathbb{A}^2_{x,y} \to \mathbb{A}^1_z$ by $z = xy$. Note that H admits an A_1–singularity at p, $f^{-1}(p)$ is a node of a fiber of π, but X is nonsingular. In general, X and H admit at worst A_l–singularities for some l (because geometric fibers of X are nodal curves), where A_u–singularities of surfaces are étale locally given by $w^2 + x^2 + y^{u+1} = 0$. This follows from the fact that 1–parameter deformation of nodes create such singularities. Note that on the neighborhood of such an isolated singular point of H, the branch locus of f is concentrated at the point if it contains the point, which only appears possibly at singular points of the fibers of $h : H \to \mathbb{P}^1$.

Example 4.6. Suppose that (X, H, h, f, s, s') is a stable odd hyperelliptic genus g model with a marked Weierstrass section over a field K. The goal is to classify singularities of the branch divisor of f. By the definition, the branch divisor decomposes into $B \sqcup s'(\mathbb{P}^1_K)$, which is contained in the smooth locus of H by the definition. First, consider a geometric point c in the intersection $B \cap H_t$, where t is a geometric point of $\mathbb{P}^1_K$ and H_t is the fiber $h^{-1}(t)$. Since the corresponding double cover X_t (which is a fiber over t of $h \circ f$) only admits nodes as singularities, the multiplicity m of $B \cap H_t$ at c is at most 2, as $f_t : X_t \to Y_t$ étale locally near c is given by the equation

$$\mathrm{Spec}(\overline{K}[y, z]/(z^2 - y^m)) \to \mathrm{Spec}(\overline{K}[y]), \text{ where } y \text{ is the uniformizer of } c \in H_t.$$

Since B does not contain any irreducible component of geometric fibers of h (as any geometric fiber of $h \circ f$ is reduced), above implies that the multiplicity of B at any geometric point is at most 2. Thus, the support of B possibly admits plane double point curve singularities, étale locally of the form $y^2 - x^m = 0$ with $m \in \mathbb{N}_{\geq 2}$, on the geometrically reduced locus of B, and is smooth elsewhere. Those singularities are in fact A_{m-1} (curve) singularities.

Example 4.5 and 4.6 illustrate that a general stable odd hyperelliptic genus g model often gives a mildly singular $\mathbb{P}^1$-fibration and mildly singular branch divisor on it. On the other hand, we could instead consider the hyperelliptic fibrations with smooth $\mathbb{P}^1$-bundle H, but with X and the branch divisor having worse singularities. Then, each fiber of X is irreducible and is a double cover of $\mathbb{P}^1$ branched over $2g + 2$ number of points, where many of these points could collide. For instance, if l branch points collide, then the preimage has A_{l-1}-singularity on the fiber, given étale locally by an equation $y^2 - x^l = 0$. Such a curve is called the quasi-admissible hyperelliptic curve, defined in Definition 1.10. Quasi-admissible hyperelliptic curves over $\mathbb{P}^1_K$ (which are non-isotrivial) are equivalent to the following:

Definition 4.7. A hyperelliptic fibration (X, H, h, f, s, s') is *quasi-admissible* if for every geometric point $c \in C$, f restricted to the fibers of X and H is quasi-admissible. We assume that X is not isotrivial over $\mathbb{P}^1$, i.e., all geometric fibers are isomorphic.

Remark 4.8. Observe that the Definitions 4.1, 4.2, 4.4, and 4.7 should be interpreted as rational / hyperelliptic / stable / quasi-admissible curves over $\mathbb{P}^1_K$, instead of a point Spec K (just as in Definition 1.10). Thus, these definitions can be extended to corresponding curves over a general scheme T, assuming that any geometric point t of T has the property that the characteristic of the residue field is 0 or larger than $2g + 1$ (when instead $g = 1$, the standard definition of semistable over T is more delicate whenever the characteristic of geometric point is 2 or 3, and is not analogous to the definitions proposed in this paper).

In particular, a quasi-admissible hyperelliptic fibration (X, H, h, f, s, s') has the property that H is a $\mathbb{P}^1$-bundle, and on each geometric fiber of H, each point of the branch divisor away from s' has the multiplicity at most $2g$. Moreover, X is the double cover of H branched along the branch divisor (which coincides with the branch locus).

To parameterize such fibrations, we first consider the moduli stack $\mathcal{H}_{2g}[2g-1]$ of quasi-admissible hyperelliptic genus g curves characterized by [Fedorchuk, Proposition 4.2(1)] :

Theorem 4.9. *If $p := \mathrm{char}(K)$ is 0 or $> 2g+1$, then the moduli stack $\mathcal{H}_{2g}[2g-1]$ of quasi-admissible hyperelliptic genus g curves is a tame Deligne–Mumford stack isomorphic to $\mathcal{P}(4, 6, 8, \ldots, 4g + 2)$, where a point $(a_4, a_6, a_8, \ldots, a_{4g+2})$ of $\mathcal{P}(4, 6, 8, \ldots, 4g + 2)$ corresponds to the quasi-admissible hyperelliptic genus g curve with the Weierstrass equation*

$$y^2 = x^{2g+1} + a_4x^{2g-1} + a_6x^{2g-2} + a_8x^{2g-3} + \cdots + a_{4g+2} \tag{4}$$

Proof. Proof of [Fedorchuk, Proposition 4.2(1)] is originally done when $p = 0$, so it suffices to show that the proof in loc.cit. extends to the case when $p > 2g + 1$.

When $p = 0$, the proof of loc.cit. shows that the quasi-admissible hyperelliptic curves are characterized by the base $\mathbb{P}^1$ with the branch locus of degree $2g + 1$ on $\mathbb{A}^1 = \mathbb{P}^1 \setminus \infty$, of the form

$$x^{2g+1} + a_2x^{2g} + a_4x^{2g-1} + a_6x^{2g-2} + a_8x^{2g-3} + \cdots + a_{4g+2} = 0$$

where $a_2 = 0$ and not all of the rest of a_i's vanish. When $p > 2g + 1$, any monic polynomial of degree $2g + 1$ with not all roots being identical can be written in the same way (via same method) by replacing x by $x - \frac{a_2}{(2g+1)}$ (this is allowed as $2g + 1 < p$ is invertible). Hence, the moduli stack is indeed isomorphic to $\mathcal{P}(4, 6, 8, \ldots, 4g + 2)$, with a_{2i}'s referring to the standard coordinates of $\mathcal{P}(4, 6, 8, \ldots, 4g + 2)$ of degree $2i$.

Since $p > 2g + 1$. $\mathcal{P}(4, 6, 8, \ldots, 4g + 2)$ is tame Deligne–Mumford by Proposition 2.2 . ■

Assigning $\mathcal{H}_{2g}[2g - 1]$ as the target stack, we can now formulate the moduli stack $\mathcal{L}_g$ of quasi-admissible hyperelliptic genus g fibrations with a marked Weierstrass section as the following :

Proposition 4.10. *Assume $\mathrm{char}(K) = 0$ or $> 2g+1$. Then, the moduli stack $\mathcal{L}_g$ of quasi-admissible hyperelliptic genus g fibrations over $\mathbb{P}^1$ with a marked Weierstrass section is the tame Deligne–Mumford stack $\mathrm{Hom}_{>0}(\mathbb{P}^1, \mathcal{H}_{2g}[2g-1])$ parameterizing the K-morphisms $f : \mathbb{P}^1 \to \mathcal{H}_{2g}[2g-1]$ with $\deg f^*\mathcal{O}_{\mathcal{H}_{2g}[2g-1]}(1) > 0$.*

Proof. By the definition of the universal family p, any quasi-admissible hyperelliptic genus g fibration $f : Y \to \mathbb{P}^1$ comes from a morphism $\varphi_f : \mathbb{P}^1 \to \mathcal{H}_{2g}[2g - 1]$ and vice versa. As this correspondence also works in families, the moduli stack $\mathcal{L}_g$ is a substack of $\mathrm{Hom}(\mathbb{P}^1, \mathcal{H}_{2g}[2g - 1])$. As $\mathcal{H}_{2g}[2g - 1]$ is tame Deligne–Mumford by Theorem 4.9, the Hom stack $\mathrm{Hom}(\mathbb{P}^1, \mathcal{H}_{2g}[2g - 1])$ is Deligne–Mumford by [Olsson]. Tameness follows from [AOV], as $\mathcal{H}_{2g}[2g - 1]$ itself is tame. Thus, $\mathcal{L}_g$ is tame Deligne–Mumford as well.

Since any quasi-admissible hyperelliptic genus g fibration f is not isotrivial, φ_f must be a non-trivial morphism, i.e., the image of f in $\mathcal{H}_{2g}[2g - 1]$) is 1-dimensional. Since non-trivialness of a morphism is an clopen condition, the corresponding clopen locus (consisting of the union of connected components) $\mathrm{Hom}_{>0}(\mathbb{P}^1, \mathcal{H}_{2g}[2g - 1])$ is indeed isomorphic to $\mathcal{L}_g$. ■

By working out the birational geometry of surfaces over $\mathrm{char}(K) = 0$ as well as $\mathrm{char}(K) > 2g+1$, we construct a geometric transformation from $\mathcal{S}_g(K)$ the K–points of the moduli functor $\mathcal{S}_g$ of the stable odd hyperelliptic genus $g \geq 2$ models (see Definition 4.4) over $\mathbb{P}^1$ with a marked Weierstrass section to $\mathcal{L}_g(K)$ the K–points of the moduli functor $\mathcal{L}_g$. In fact, this transformation is injective:

Theorem 4.11. *If* $\mathrm{char}(K) = 0$ *or* $\mathrm{char}(K) > 2g+1$*, then there is a canonical fully faithful functor of groupoids* $\mathcal{F} : \mathcal{S}_g(K) \to \mathcal{L}_g(K)$*.*

Proof. The key idea of proof is to construct $\mathcal{F}$ by using relative canonical model, a particular birational transformation from the subject of relative minimal model program. We prove this in a few steps, beginning with a preliminary step. We construct and verify properties of $\mathcal{F}$ in the other steps:

Step 1. Log canonical singularities and log canonical models. The main reference here is [Fujino] when $\mathrm{char}(K) = 0$, and [Tanaka, §5–6] when $\mathrm{char}(K) \neq 0$, noting that both references deal with algebraically closed fields instead.

First, we need the following definition for types of singularities of a pair (S, D) of a normal $\overline{K}$–surface S and an effective $\mathbb{R}$–divisor D on S:

Definition 4.12. ([Fujino, §2.4], [Tanaka, Definition 5.1]) A pair (S, D) is log canonical if

(1) the log canonical divisor $K_S + D$ is $\mathbb{R}$–Cartier,
(2) for any proper birational morphism $\pi : W \to S$ and the divisor D_W defined by

$$K_W + D_W = \pi^*(K_S + D),$$

then $D_W \leq 1$, i.e. when writing $D_W = \sum_i a_i E_i$ as a sum of distinct irreducible divisors E_i, $a_i \leq 1$ for every i.

Moreover, if a pair (S, D) is defined over a non-algebraically closed field K, then it is called log canonical if its base-change to $\overline{K}$ is.

For instance, if S is smooth and D is a reduced simple normal crossing divisor, then (S, D) is log canonical. Similarly, if $w \in \mathbb{R} \cap [0, 1]$, then (S, wD) is log canonical under the same assumptions. Note that we cannot consider $w > 1$ under the same assumptions, as the weight on each irreducible component of D must be at most 1.

For example, consider a stable odd hyperelliptic genus g model (X, H, h, f, s, s') over K, consider the pair $(H_{\overline{K}}, wB_{\overline{K}} + (s'(\mathbb{P}^1_K))_{\overline{K}})$ defined over $\overline{K}$ where the branch divisor of h decomposes as $B \sqcup s'(\mathbb{P}^1_K)$ and $w \in \mathbb{R} \cap (0, 1/2]$ is a weight (since B can have components of multiplicity 2 by Example 4.6, we consider weights at most 1/2). To claim that this pair is log canonical under additional condition on w, it suffices to consider neighborhoods of singular points of $H_{\overline{K}}$ and support of $B_{\overline{K}}$ by the above observation.

First, recall that the isolated singularities of $H_{\overline{K}}$ away from the support of $wB_{\overline{K}} + (s'(\mathbb{P}^1_K))_{\overline{K}}$ is of type $A_{l'}$ for some l' by Example 4.5. Hence, the pair is log canonical at neighborhoods of such points (in fact, those points are called canonical singular points of $H_{\overline{K}}$). Also, at a singular point c of the support of $B_{\overline{K}}$, $H_{\overline{K}}$ is smooth and $B_{\overline{K}}$ is reduced at c but $B_{\overline{K}}$ admits A_l–singularities by Example 4.6. Therefore, the pair has log canonical singularities whenever $w \leq \frac{1}{2} + \frac{1}{l+1}$ by [Järvilehto] (summariezd in [GHM, Introduction], where the log canonical threshold is the supremum of values w that makes the pair log canonical).

To construct a log canonical model, consider a pair (S, D) as the beginning of this step with projective $\overline{K}$–morphism $f : S \to C$ into a $\overline{K}$–variety C, and assume that D is $\mathbb{Q}$–divisor and S is $\mathbb{Q}$–factorial. If (S, D) is log canonical with $K_S + D$ not f–antinef, then [HP, Pages 1750–1751] uses key birational geometry results from [Fujino, Tanaka] to construct the f–log canonical model, defined below. In fact, analogous arguments from [HP, Proof of Proposition 11] implies that the same procedure can be applied to $f : (S, D) \to C$ over a field K, leading to the following definition:

Definition 4.13. Suppose that (S, D) is a log canonical pair over a field K where S is a normal projective $\mathbb{Q}$–factorial surface and D is a $\mathbb{Q}$–divisor. Assume that $f : S \to C$ is a projective morphism into a K–variety C with $K_S + D$ not f–antinef. If K is algebraically closed, then the f–log canonical model is a pair (S', D') with a projective morphism $f' : S' \to C$, where

$$S' := \underline{Proj} \bigoplus_{m \in \mathbb{N}} f_* \mathcal{O}_S(m(K_S + D))$$

and $D' := \phi_* D$ where $\phi : S \to S'$ is the induced birational morphism.

If K is not algebraically closed, then the f–log canonical model is the $\mathrm{Gal}(\overline{K}/K)$–fixed locus of the $f_{\overline{K}}$–log canonical model of $(S_{\overline{K}}, D_{\overline{K}})$.

Step 2. Construction of faithful $\mathcal{F} : \mathcal{S}_g(K) \to \mathcal{L}_g(K)$. Fix any member of $\mathcal{S}_g(K)$, i.e., a stable odd hyperelliptic genus g model (X, H, h, f, s, s') . Denote $B \sqcup s'(\mathbb{P}^1_K)$ to be the divisorial part of the branch locus of $f : X \to H$ (B is also called branch divisor in literature). Notice that h restricted to B has degree $2g+1$. By Step 1 above, $(H, \frac{1}{2g}B + s'(C))$ is log canonical. Take h–log canonical model of $(H, \frac{1}{2g}B + s'(C))/\mathbb{P}^1_K$ to obtain a birational $\mathbb{P}^1_K$–morphism $\varphi : (H, \frac{1}{2g}B + s'(\mathbb{P}^1_K)) \to (H', D')$ where H' is a rational fibration over K and D' is a $\mathbb{R}$–divisor of H' defined over K (c.f. Definition 4.13). Since the only canonical rational curve, defined over an algebraically closed field with $\frac{1}{2g}$ weights on $(2g+1)$ points and weight 1 on another point, is a smooth rational curve where the point of weight 1 is distinct from the other points (of weight $\frac{1}{2g}$), H' is a $\mathbb{P}^1$–bundle (given by $h' : H' \to \mathbb{P}^1_K$). This description shows that D' decomposes into $\frac{1}{2g}A' + T'$ where A' is a divisor of H' and T' consists of weight 1 points on each geometric fiber of $H'/\mathbb{P}^1_K$. Thus, T' comes from a section t' of h' . We will show that H' is the $\mathbb{P}^1$–fibration associated to the desired quasi–admissible hyperelliptic genus g fibration.

To finish the construction of the quasi–admissible fibration, take Stein factorization on $\varphi \circ f$. This gives a finite morphism $f' : X' \to H'$ and a morphism $\psi : X \to X'$ with geometrically connected fibers such that $\varphi \circ f = f' \circ \psi$. Since f is finite of degree 2 and φ is birational, f' is finite of degree 2 and ψ is birational. Moreover, $B' := A' + T'$ is the branch locus of f' . By calling t to be the unique lift of t' on $h' \circ f'$, (X', H', h', f', t, t') is the desired quasi–admissible hyperelliptic fibration. Define $\mathcal{F}(X, H, h, f, s, s') := (X', H', h', f', t, t')$.

To see that $\mathcal{F}$ is faithful, suppose that there are two isomorphisms

$$(\alpha_i, \beta_i) : (X_1, H_1, h_1, f_1, s_1, s'_1) \to (X_2, H_2, h_2, f_2, s_2, s'_2)$$

between stable odd hyperelliptic genus g models that induce the same isomorphism under $\mathcal{F}$:

$$(\alpha', \beta') : \mathcal{F}(X_1, H_1, h_1, f_1, s_1, s'_1) \to \mathcal{F}(X_2, H_2, h_2, f_2, s_2, s'_2)$$

Denote $(X'_j, H'_j, h'_j, f'_j, t_j, t'_j) = \mathcal{F}(X_j, H_j, h_j, f_j, s_j, s'_j)$ for $j = 1, 2$. From the construction of $\mathcal{F}$ shown above, induced morphisms $X_j \to X'_j$ and $H_j \to H'_j$ are birational for each j. Since they are separated varieties over K, (α_1, β_1) must be equal to (α_2, β_2) , hence $\mathcal{F}$ is faithful.

Step 3. Fullness of $\mathcal{F}$. Given any isomorphism ψ between $(X'_i, H'_i, h'_i, f'_i, t_i, t'_i)$'s in $\mathcal{L}_g(K)$ as images of $(X_i, H_i, h_i, f_i, s_i, s'_i) \in \mathcal{S}_g(K)$ under $\mathcal{F}$, notice that h'_i's and $h'_i \circ f'_i$'s have smooth geometric generic fibers for $i = 1, 2$ and ψ comes in pairs of isomorphisms $\psi_1 : X'_1 \to X'_2$ and $\psi_2 : H'_1 \to H'_2$ (so denote $\psi = (\psi_1, \psi_2)$). Then, ψ lifts to a pair of birational maps $\overline{\psi} = (\overline{\psi}_1, \overline{\psi}_2)$ between X_i's and H_i's which induce isomorphisms on geometric generic fibers and irreducible components of any geometric fiber meeting the sections s_i's or s'_i's. To claim that those extend to isomorphisms that induce ψ_i's, it suffices to understand geometric properties of related moduli stacks, as we claim that ψ_i's can be interpreted as an element of $Isom$ spaces of such stacks.

Observe first that for each $i = 1, 2$, H_i is a $\mathbb{Z}/2\mathbb{Z}$–quotient of X_i, and $K_{X_i} + s_i(\mathbb{P}^1_K)$ is ample over $\mathbb{P}^1_K$ by the defintion. Since the branch divisor of f_i is $B_i \sqcup s'_i(\mathbb{P}^1_K)$, the log canonical divisor $K_{H_i} + \frac{1}{2}B_i + s'_i(\mathbb{P}^1_K)$ is also ample over $\mathbb{P}^1_K$ as $f_i^*(K_{H_i} + \frac{1}{2}B_i + s'_i(\mathbb{P}^1_K)) = K_{X_i} + s_i(\mathbb{P}^1_K)$. Since X_i admits nodes as the only singularities of geometric fibers, B_i on each fiber has multiplicity at most 2 at any $\overline{K}$–points in the support. Therefore, fibers of the pair $(H_i, \frac{1}{2}B_i + s_i'(\mathbb{P}^1_K))$ are $((\frac{1}{2}, 2g+1), (1,1))$–stable curves of genus 0 in the sense of [Hassett, §2.1.3], meaning that H_i for each i is a family of such curves over $\mathbb{P}^1_K$. Note that the moduli stack $\overline{\mathcal{M}}_{0,((\frac{1}{2},2g+1),(1,1))}$ of $((\frac{1}{2}, 2g+1), (1,1))$–stable curves of genus 0 is a proper (so separated) Deligne–Mumford stack (it easily follows from loc.cit. and [Hassett, Theorem 2.1]), and H_i is realized as $\alpha_i : \mathbb{P}^1_K \to \overline{\mathcal{M}}_{0,((\frac{1}{2},2g+1),(1,1))}$. Since there is a nonempty open subset $U \subset \mathbb{P}^1_K$ such that ψ_2 induces an isomorphism between $h_i^{-1}(U)$'s, $\overline{\psi}_2$ is an element of $Isom_{\overline{\mathcal{M}}_{0,((\frac{1}{2},2g+1),(1,1))}}(\alpha_1, \alpha_2)(U)$. Then, separatedness of $\overline{\mathcal{M}}_{0,((\frac{1}{2},2g+1),(1,1))}$ implies that $\overline{\psi_2}$ extends to an isomorphism between H_i's.

Similar argument shows that ψ_1 also extends to an isomorphism between X_i's (as $\overline{H}_{g,\underline{1}} \subset \overline{\mathcal{M}}_{2,\overline{1}}$ is a separated Deligne–Mumford stack by [Knudsen]), so it suffices to show that $\overline{\psi}_i$'s commute with f_i's and induce ψ. The commutativity of $\overline{\psi}_i$'s follows because H_i's are $\mathbb{Z}/2\mathbb{Z}$–quotients of X_i's and any isomorphism between families of stable hyperelliptic curves with marked Weierstrass points commute with $\mathbb{Z}/2\mathbb{Z}$–actions. Because the birational morphisms $X_i \to X_i'$ and $H_i \to H_i'$ for any i contract all but irreducible components of fibers over $\mathbb{P}^1_K$ meeting marked sections, $\overline{\psi} := (\overline{\psi}_1, \overline{\psi}_2)$ induce ψ as well. Henceforth, $\overline{\psi}$ maps to ψ under $\mathcal{F}$, proving that $\mathcal{F}$ is full. ■

Remark 4.14. Due to log abundance being a conjecture for higher dimensions, which is a key ingredient of the existence of log canonical models (c.f. [HP, Remark 13]), it is unclear whether $\mathcal{F}$ in the proof above extends to a functor from the moduli of stable odd hyperelliptic genus g models to $\mathcal{L}_g$. If it extends, we expect the functor to be still fully faithful, as opposed to [HP, Remark 13] for birational transformations between semistable elliptic surfaces and stable elliptic curves over $\mathbb{P}^1$. The key obstruction on [HP, Remark 13], assuming that the functor discussed in loc.cit. (which is an equivalence) extends, is that the essential surjectivity may not hold on the extension, whereas the functor from Theorem 4.11 is not even essentially surjective to begin with.

4.1. **Hyperelliptic discriminant Δ_g of quasi-admissible hyperelliptic genus g fibration.** As we consider the algebraic surfaces X as fibrations in genus g curves over $\mathbb{P}^1$, the discriminant $\Delta_g(X)$ is the basic invariant of X. For the quasi-admissible hyperelliptic genus g fibrations over $\mathbb{P}^1$, we have the work of [Lockhart, Liu2] which describes the hyperelliptic discriminant $\Delta_g(X)$.

Definition 4.15. [Lockhart, Definition 1.6, Proposition 1.10] The hyperelliptic discriminant Δ_g of the monic odd degree Weierstrass equation $y^2 = x^{2g+1} + a_4x^{2g-1} + a_6x^{2g-2} + a_8x^{2g-3} + \cdots + a_{4g+2}$ over a basefield K with $\mathrm{char}(K) \neq 2$ is

$$\Delta_g = 2^{4g} \cdot \mathrm{Disc}(x^{2g+1} + a_4x^{2g-1} + a_6x^{2g-2} + \cdots + a_{4g+2})$$

which has $\deg(\Delta_g) := |\Delta_g| = 4g(2g+1)$ formally when we associate each variable a_i with degree i.

Note that when $g = 1$, the discriminant Δ_1 of the short Weierstrass equation $y^2 = x^3 + a_4x + a_6$ coincides with the usual discriminant $-16(4a_4^3 - 27a_6^2)$ of an elliptic curve. We can now formulate the moduli stack $\mathcal{L}_{g,|\Delta_g|\cdot n}$ of quasi-admissible fibration over $\mathbb{P}^1$ with a fixed discriminant degree $|\Delta_g| \cdot n = 4g(2g+1)n$ and a marked Weierstrass section :

Proposition 4.16. *Assume* $\mathrm{char}(K) = 0$ *or* $> 2g+1$. *Then, the moduli stack* $\mathcal{L}_{g,|\Delta_g|\cdot n}$ *of quasi-admissible hyperelliptic genus g fibrations over* $\mathbb{P}^1_K$ *with a marked Weierstrass section and a hyperelliptic discriminant of degree* $|\Delta_g| \cdot n = 4g(2g+1)n$ *over a basefield* K *is the tame Deligne–Mumford Hom stack* $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{H}_{2g}[2g-1])$ *parameterizing the K-morphisms* $f : \mathbb{P}^1 \to \mathcal{H}_{2g}[2g-1]$ *with* $\mathcal{H}_{2g}[2g-1] \cong \mathcal{P}(\vec{\lambda_g}) = \mathcal{P}(4, 6, 8, \ldots, 4g+2)$ *such that* $f^*\mathcal{O}_{\mathcal{P}(\vec{\lambda_g})}(1) \cong \mathcal{O}_{\mathbb{P}^1}(n)$ *with* $n \in \mathbb{N}$.

Proof. Since $\deg f^*\mathcal{O}_{\mathcal{P}(\vec{\lambda_g})}(1) = n$ is an open condition, $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{H}_{2g}[2g-1])$ is an open substack of $\mathrm{Hom}(\mathbb{P}^1, \mathcal{H}_{2g}[2g-1])$. Now, it suffices to show that $\deg f = n$ (i.e., $\deg f^*\mathcal{O}_{\mathcal{P}(\vec{\lambda_g})}(1) = n$) if and only if the discriminant degree of the corresponding quasi-admissible fibration is $4g(2g+1)n$. Note that $\deg f = n$ if and only if the quasi-admissible fibration is given by the Weierstrass equation

$$y^2 = x^{2g+1} + a_4x^{2g-1} + a_6x^{2g-2} + \cdots + a_{4g+2}$$

where a_i's are sections of $\mathcal{O}(in)$, since a_i's represent the coordinates of $\mathcal{P}(4, 6, \ldots, 4g+2)$. Then by Definition 4.15, it is straightforward to check that Δ_g has the discriminant degree $4g(2g+1)n$. ■

Observe that the above Proposition combined with the Corollary 1.13 proves the Corollary 1.14.

4.2. **Exact count $\left|\mathcal{L}_{g,|\Delta_g|\cdot n}(\mathbb{F}_q)/\sim\right|$ of the $\mathbb{F}_q$–isomorphism classes over finite fields.** Fix $n \in \mathbb{Z}^+$, note that a given morphism $\varphi_g \in \mathrm{Hom}_n(\mathbb{P}^1, \mathcal{H}_{2g}[2g-1] \cong \mathcal{P}(\vec{\lambda_g}))$ can map into a special substack of $\mathcal{P}(\vec{\lambda_g})$. By using Corollary 1.13, Proposition 3.8, and Proposition 4.16, we count the exact number of $\mathbb{F}_q$–isomorphism classes $\left|\mathcal{L}_{g,|\Delta_g|\cdot n}(\mathbb{F}_q)/\sim\right|$ of quasi-admissible hyperelliptic fibrations:

Proposition 4.17. *If $\mathrm{char}(\mathbb{F}_q) > 2g+1$, the number $\left|\mathcal{L}_{g,|\Delta_g|\cdot n}(\mathbb{F}_q)/\sim\right|$ of $\mathbb{F}_q$–isomorphism classes of quasi-admissible hyperelliptic genus $g = 2, 3, 4$ fibrations over $\mathbb{P}^1$ with a marked Weierstrass section and a hyperelliptic discriminant of degree $|\Delta_g| \cdot n = 4g(2g+1)n$ is equal to*

$$\begin{aligned}
|\mathcal{L}_{2,40n}(\mathbb{F}_q)/\sim| &= 2 \cdot q^{28n} \cdot (q^3 + q^2 + q^1 - q^{-1} - q^{-2} - q^{-3}) + 2 \cdot q^{12n} \cdot (q^1 - q^{-1}) \\
|\mathcal{L}_{3,84n}(\mathbb{F}_q)/\sim| &= 2 \cdot q^{54n} \cdot (q^5 + \cdots + q^1 - q^{-1} - \cdots - q^{-5}) + 2 \cdot q^{24n} \cdot (q^2 + q^1 - q^{-1} - q^{-2}) \\
&\quad + 4 \cdot q^{18n} \cdot (q^1 - q^{-1}) \\
|\mathcal{L}_{4,144n}(\mathbb{F}_q)/\sim| &= 2 \cdot q^{88n} \cdot (q^7 + \cdots + q^1 - q^{-1} - \cdots - q^{-7}) \\
&\quad + 2 \cdot q^{40n} \cdot (q^3 + q^2 + q^1 - q^{-1} - q^{-2} - q^{-3}) \\
&\quad + 4 \cdot q^{36n} \cdot (q^2 + q^1 - q^{-1} - q^{-2}) + 4 \cdot q^{24n} \cdot (q^1 - q^{-1})
\end{aligned}$$

And for genus $g \geq 5$, the corresponding exact count $\left|\mathcal{L}_{g,|\Delta_g|\cdot n}(\mathbb{F}_q)/\sim\right|$ can be similarly worked out.

Proof. Recall that $\mathcal{P}(\vec{\lambda_g})$ in this case has the property that $\lambda_i = 4 + 2i$ for $i = 0, \ldots, 2g-1$.

For genus 2 case, notice that $\gcd(\lambda_i \,:\, i \in I)$ in Proposition 3.8 is 2 except when $I = \{0, 2\}$ (corresponding to $\mathcal{P}(4, 8) \subsetneq \mathcal{P}(4, 6, 8, 10)$), taking the value 4. Therefore, Proposition 3.8 gives

$$\begin{aligned}
|\mathcal{L}_{2,40n}(\mathbb{F}_q)/\sim| &= \sum_{\substack{I \subset \{0,\ldots,2g+1\} \\ |I| \geq 2}} [U(I)] \cdot \gcd(\lambda_i \,:\, i \in I) = \sum_{\substack{I \subset \{0,\ldots,2g+1\} \\ |I| \geq 2 \\ I \neq \{0,2\}}} 2[U(I)] + 4[U(0,2)] \\
&= 2 \left(\sum_{\substack{I \subset \{0,\ldots,2g+1\} \\ |I| \geq 2}} [U(I)] \right) + 2[U(0,2)] \\
&= 2 \cdot \#_q \left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(4,6,8,10))(\mathbb{F}_q) \right) + 2 \cdot \#_q \left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(4,8))(\mathbb{F}_q) \right) \\
&= 2 \cdot q^{28n} \cdot (q^3 + q^2 + q^1 - q^{-1} - q^{-2} - q^{-3}) + 2 \cdot q^{12n} \cdot (q^1 - q^{-1})
\end{aligned}$$

By following the Proposition 3.8, the explicit counting shown here can straightforwardly generalized to higher genus cases by identifying the special substack of $\mathcal{P}(4, 6, \ldots, 4g+2)$ with the order of the generic stabilizer group bigger than 2.

■

5. Counting stable odd degree hyperelliptic curves over global fields by Δ_g

In this last section, we effectively prove the geometric Shafarevich's conjecture for stable odd degree hyperelliptic curves over $\mathbb{P}^1_{\mathbb{F}_q}$ with a marked Weierstrass section, which admit squarefree discriminants. As the elliptic $g = 1$ case has been previously worked out by both authors in [HP, §5], we assume from now on that $g \geq 2$. Through the arithmetic invariant $|\mathcal{L}_{g,|\Delta_g|\cdot n}(\mathbb{F}_q)/\sim|$ over $\mathbb{F}_q$ with $\mathrm{char}(\mathbb{F}_q) > 2g+1$, we find the sharp estimate on the number $\mathcal{Z}'_{g,\mathbb{F}_q(t)}(\mathcal{B})$ of quasi-admissible hyperelliptic genus g fibrations (Definition 4.7) with the height $ht(\Delta_g(X)) := q^{\deg \Delta_g(X)}$ of the hyperelliptic discriminant $\Delta_g(X)$ bounded by a positive real number $\mathcal{B}$. By Theorem 4.11 which renders the Equation (5), this sharp estimate induces an upper bound on the number $\mathcal{Z}_{g,\mathbb{F}_q(t)}(\mathcal{B})$ of stable hyperelliptic genus g curves over $\mathbb{P}^1_K$ with the height $ht(\Delta_g(X)) := q^{\deg \Delta_g(X)}$ of the hyperelliptic discriminant $\Delta_g(X)$ bounded by a positive real number $\mathcal{B}$. As a result of this observation, we prove both Theorem 1.2 and Theorem 1.15 (which implies Theorem 1.6) and provide the heuristic evidences for Conjecture 1.3 through the global fields analogy.

We first recall the definition of a global field. Let S be the set of *places* (i.e., the set of equivalence classes of nontrivial absolute values) of a field K.

Definition 5.1. A field K is a global field if all completions K_v of K at each place $v \in S$ is a local field, and K satisfies the product formula $\prod_v |\alpha|_v = 1$ for all $\alpha \in K^*$ and over all places $v \in S$ where $|\cdot|_v$ denotes the normalized absolute value for each place $v \in S$.

And we have the fundamental Artin-Whaples Theorem proven in 1945 [AW, AW2] which emphasized the close analogy between the theory of algebraic number fields and the theory of function fields of algebraic curves over finite fields. The axiomatic method used in these papers unified the two global fields from a valuation theoretic perspective by clarifying the role of the product formula.

Theorem 5.2 (Artin-Whaples Theorem)**.** *Every global field is a finite extension of* $\mathbb{Q}$ *or* $\mathbb{F}_q(t)$*.*

Focusing upon the global fields $\mathbb{F}_q(t)$ or $\mathbb{Q}$, in order to draw the analogy, we need to fix an affine chart $\mathbb{A}^1_{\mathbb{F}_q} \subset \mathbb{P}^1_{\mathbb{F}_q}$ and its corresponding ring of functions $\mathbb{F}_q[t]$ interpreted as the ring of integers of the field of fractions $\mathbb{F}_q(t)$ of $\mathbb{P}^1_{\mathbb{F}_q}$. This is necessary since $\mathbb{F}_q[t]$ could come from any affine chart of $\mathbb{P}^1_{\mathbb{F}_q}$, whereas the ring of integers $\mathcal{O}_K$ for the number field K is canonically determined. We denote $\infty \in \mathbb{P}^1_{\mathbb{F}_q}$ to be the unique point not in the chosen affine chart.

Note that for a maximal ideal $\mathfrak{p}$ in $\mathcal{O}_K$, the residue field $\mathcal{O}_K/\mathfrak{p}$ is finite for both of our global fields. One could think of $\mathfrak{p}$ as a point in Spec $\mathcal{O}_K$ and define the *height* of a point $\mathfrak{p}$.

Definition 5.3. Define the height of a point $\mathfrak{p}$ to be $ht(\mathfrak{p}) := |\mathcal{O}_K/\mathfrak{p}|$ the cardinality of the residue field $\mathcal{O}_K/\mathfrak{p}$.

We recall the notion of *bad reduction* & *good reduction*:

Definition 5.4. Let C be an odd degree hyperelliptic genus g curve over K given by the odd degree Weierstrass equation

$$y^2 = x^{2g+1} + a_4x^{2g-1} + a_6x^{2g-2} + \cdots + a_{4g+2},$$

with $a_{2i+2} \in \mathcal{O}_K$ for every $1 \leq i \leq 2g$. Then C has bad reduction at $\mathfrak{p}$ if the fiber $C_\mathfrak{p}$ over $\mathfrak{p}$ is a singular curve of degree $2g+1$. The prime $\mathfrak{p}$ is said to be of good reduction if $C_\mathfrak{p}$ is a smooth hyperelliptic genus g curve.

Consider the case when $K = \mathbb{F}_q(t)$, and a quasi-admissible model $f : X \to \mathbb{P}^1_{\mathbb{F}_q}$ (a quasi–admissible fibration with smooth geometric generic fiber). For simplicity, assume that X does not have a singular fiber over $\infty \in \mathbb{P}^1_{\mathbb{F}_q}$. Note that the primes $\mathfrak{p}_i$ of bad reductions of f are precisely points of the discriminant divisor $\Delta_g(X) = \sum k_i \cdot \mathfrak{p}_i$, as the fiber $X_{\mathfrak{p}_i}$ is singular over $\Delta_g(X)$. When $K = \mathbb{F}_q(t)$ the global function field, we have $\Delta_g(X) \in H^0(\mathbb{P}^1, \mathcal{O}(4g(2g+1)n))$ by the proof of Proposition 4.16. We can define the height of $\Delta_g(X)$ as follows:

Definition 5.5. The height $ht(\Delta_g(X))$ of a discriminant divisor $\Delta_g(X)$ in $\mathbb{P}^1_{\mathbb{F}_q}$ is $q^{\deg \Delta_g(X)}$. As a convention, if a divisor $\Delta_g(X)$ is given as a zero section of any line bundle, then set $ht(\Delta_g(X)) = \infty$.

In general, the height of a hyperelliptic discriminant $\Delta_g(X)$ of any X (without nonsingular fiber assumption over ∞) is defined as $q^{|\Delta_g(X)|}$ where $\deg(\Delta_g(X)) := |\Delta_g(X)|$ is equal to $4g(2g+1)n$.

As the hyperelliptic discriminant divisor $\Delta_g(X)$ is an invariant of a quasi-admissible model $f : X \to \mathbb{P}^1$, we count the number of $\mathbb{F}_q$–isomorphism classes of quasi-admissible hyperelliptic genus g fibrations on the function field $\mathbb{F}_q(t)$ by the bounded height of $\Delta_g(X)$:

$$\mathcal{Z}'_{g,\mathbb{F}_q(t)}(\mathcal{B}) := |\{\text{Quasi-admissible hyperelliptic genus } g \text{ curves over } \mathbb{P}^1_{\mathbb{F}_q} \text{ with } 0 < ht(\Delta_g(X)) \le \mathcal{B}\}|$$

A very important consequence of Theorem 4.11 is that the number $\mathcal{Z}'_{g,\mathbb{F}_q(t)}(\mathcal{B})$ of quasi–admissible genus $g \ge 2$ curves over $\mathbb{P}^1_{\mathbb{F}_q}$ ordered by height of hyperelliptic discriminant $ht(\Delta_g(X))$ gives an upper bound for the number $\mathcal{Z}_{g,\mathbb{F}_q(t)}(\mathcal{B})$ of stable odd hyperelliptic genus $g \ge 2$ models over $\mathbb{P}^1_{\mathbb{F}_q}$ ordered by height of hyperelliptic discriminant $ht(\Delta_g(X))$:

$$\mathcal{Z}_{g,\mathbb{F}_q(t)}(\mathcal{B}) \le \mathcal{Z}'_{g,\mathbb{F}_q(t)}(\mathcal{B}) \tag{5}$$

We now prove the sharp estimates on $\mathcal{Z}'_{g,\mathbb{F}_q(t)}(\mathcal{B})$ which induces the upper bounds on $\mathcal{Z}_{g,\mathbb{F}_q(t)}(\mathcal{B})$

Proof of Theorem 1.15 for $g = 2$. Knowing the number of $\mathbb{F}_q$-isomorphism classes of quasi-admissible genus 2 fibrations of discriminant degree $40n$ over $\mathbb{F}_q$ is

$$|\mathcal{L}_{2,40n}(\mathbb{F}_q)/\sim| = 2 \cdot q^{28n} \cdot (q^3 + q^2 + q^1 - q^{-1} - q^{-2} - q^{-3}) + 2 \cdot q^{12n} \cdot (q^1 - q^{-1})$$

by Proposition 4.17, we explicitly compute the upper bound for $\mathcal{Z}'_{2,\mathbb{F}_q(t)}(\mathcal{B})$ as the following,

$$\begin{aligned}
\mathcal{Z}'_{2,\mathbb{F}_q(t)}(\mathcal{B}) &= \sum_{n=1}^{\left\lfloor \frac{log_q \mathcal{B}}{40} \right\rfloor} |\mathcal{L}_{2,40n}(\mathbb{F}_q)/\sim| \\
&= \sum_{n=1}^{\left\lfloor \frac{log_q \mathcal{B}}{40} \right\rfloor} 2 \cdot q^{28n} \cdot (q^3 + q^2 + q^1 - q^{-1} - q^{-2} - q^{-3}) + 2 \cdot q^{12n} \cdot (q^1 - q^{-1}) \\
&= 2 \cdot (q^3 + q^2 + q^1 - q^{-1} - q^{-2} - q^{-3}) \sum_{n=1}^{\left\lfloor \frac{log_q \mathcal{B}}{40} \right\rfloor} q^{28n} + 2 \cdot (q^1 - q^{-1}) \sum_{n=1}^{\left\lfloor \frac{log_q \mathcal{B}}{40} \right\rfloor} q^{12n} \\
&\le 2 \cdot (q^3 + q^2 + q^1 - q^{-1} - q^{-2} - q^{-3}) \left(q^{28} + \cdots + q^{28 \cdot (\frac{log_q \mathcal{B}}{40})}\right) \\
&\quad + 2 \cdot (q^1 - q^{-1}) \left(q^{12} + \cdots + q^{12 \cdot (\frac{log_q \mathcal{B}}{40})}\right) \\
&= 2 \cdot (q^3 + q^2 + q^1 - q^{-1} - q^{-2} - q^{-3}) \left(\frac{q^{28} \cdot (\mathcal{B}^{\frac{7}{10}} - 1)}{(q^{28} - 1)}\right) \\
&\quad + 2 \cdot (q^1 - q^{-1}) \left(\frac{q^{12} \cdot (\mathcal{B}^{\frac{3}{10}} - 1)}{(q^{12} - 1)}\right)
\end{aligned}$$

$$= 2 \cdot \frac{(q^{31}+q^{30}+q^{29}-q^{27}-q^{26}-q^{25})}{(q^{28}-1)} \cdot \left(\mathcal{B}^{\frac{7}{10}}-1\right) + 2 \cdot \frac{(q^{13}-q^{11})}{(q^{12}-1)} \cdot \left(\mathcal{B}^{\frac{3}{10}}-1\right)$$

On the fourth line of the equations above, inequality becomes an equality if and only if $n := \frac{log_q \mathcal{B}}{40} \in \mathbb{N}$, i.e., $\mathcal{B} = q^{40n}$ with $n \in \mathbb{N}$. This implies that the acquired upper bound on $\mathcal{Z}'_{2,\mathbb{F}_q(t)}(\mathcal{B})$ is a sharp estimate of order $\mathcal{O}_q\left(\mathcal{B}^{\frac{7}{10}}\right)$ with the lower order terms of orders $\mathcal{O}_q\left(\mathcal{B}^{\frac{3}{10}}\right)$ and $\mathcal{O}_q(1)$ where $\mathcal{O}_q$-constant is an explicit rational function of q. ■

As there are non-hyperelliptic curves for higher genus $g \geq 3$ curves, $\mathcal{Z}'_{g\geq 3,\mathbb{F}_q(t)}(\mathcal{B})$ counts the quasi-admissible hyperelliptic genus $g \geq 3$ curves over $\mathbb{P}^1_{\mathbb{F}_q}$ only. We determine $\mathcal{Z}'_{3,\mathbb{F}_q(t)}(\mathcal{B})$ explicitly thereby counting the quasi-admissible hyperelliptic genus 3 curves over $\mathbb{P}^1_{\mathbb{F}_q}$.

Proof of Theorem 1.15 for $g = 3$. Knowing the number of $\mathbb{F}_q$-isomorphism classes of quasi-admissible hyperelliptic genus 3 fibrations of discriminant degree $84n$ over $\mathbb{F}_q$ is $|\mathcal{L}_{3,84n}(\mathbb{F}_q)/\sim| = 2 \cdot q^{54n} \cdot (q^5 + \cdots + q^1 - q^{-1} - \cdots - q^{-5}) + 2 \cdot q^{24n} \cdot (q^2 + q^1 - q^{-1} - q^{-2}) + 4 \cdot q^{18n} \cdot (q^1 - q^{-1})$ by Proposition 4.17, we explicitly compute the upper bound for $\mathcal{Z}'_{3,\mathbb{F}_q(t)}(\mathcal{B})$ similarly as genus 2 case. ■

We conclude with $\mathcal{Z}'_{4,\mathbb{F}_q(t)}(\mathcal{B})$ counting the quasi-admissible hyperelliptic genus 4 curves over $\mathbb{P}^1_{\mathbb{F}_q}$.

Proof of Theorem 1.15 for $g = 4$. Knowing the number of $\mathbb{F}_q$-isomorphism classes of quasi-admissible hyperelliptic genus 4 fibrations of discriminant degree $144n$ over $\mathbb{F}_q$ is $|\mathcal{L}_{4,144n}(\mathbb{F}_q)/\sim| = 2 \cdot q^{88n} \cdot (q^7 + \cdots + q^1 - q^{-1} - \cdots - q^{-7}) + 2 \cdot q^{40n} \cdot (q^3 + q^2 + q^1 - q^{-1} - q^{-2} - q^{-3}) + 4 \cdot q^{36n} \cdot (q^2 + q^1 - q^{-1} - q^{-2}) + 4 \cdot q^{24n} \cdot (q^1 - q^{-1})$ by Proposition 4.17, we explicitly compute the upper bound for $\mathcal{Z}'_{4,\mathbb{F}_q(t)}(\mathcal{B})$ similarly as genus 2 case. ■

The computation of upper bounds for the higher genus cases $\mathcal{Z}'_{g,\mathbb{F}_q(t)}(\mathcal{B})$ can be done similarly after working out $\left|\mathcal{L}_{g,|\Delta_g|\cdot n}(\mathbb{F}_q)/\sim\right|$ by the Proposition 4.17. While the lower order terms vary, the order of the leading term can be found by the following. This computation of upper bound effectively answers the geometric Shafarevich's problem on the stable odd degree hyperelliptic curves over $\mathbb{P}^1_{\mathbb{F}_q}$ with $\mathrm{char}(\mathbb{F}_q) > 2g+1$ and a marked Weierstrass section, which admit squarefree discriminants. Below, we prove Theorem 1.2, which gives a closed-form upper bound formula:

Proof of Theorem 1.2. Note that the automorphism group of minimum order of φ_g is the generic stabilizer group $\mu_\delta = \mu_2$ of $\mathcal{P}(\vec{\lambda_g})$ and the automorphism group of maximum order of φ_g is $\mu_\omega = \mu_{2g}$ as $2g$ is the maximum value of GCD for all possible pairs among $\vec{\lambda_g} = (4, 6, 8, \ldots, 4g+2)$. By Corollary 1.13 we know that the number of $\mathbb{F}_q$-isomorphism classes of quasi-admissible hyperelliptic genus g fibrations of hyperelliptic discriminant degree $|\Delta_g| \cdot n = 4g(2g+1)n$ over $\mathbb{F}_q$ is $2 \cdot \#_q\left(\mathcal{L}_{g,|\Delta_g|\cdot n}\right) \leq \left|\mathcal{L}_{g,|\Delta_g|\cdot n}(\mathbb{F}_q)/\sim\right| \leq 2g \cdot \#_q\left(\mathcal{L}_{g,|\Delta_g|\cdot n}\right)$, we can explicitly compute the upper bound for $\mathcal{Z}'_{g,\mathbb{F}_q(t)}(\mathcal{B})$ as the following,

$$\begin{aligned}
\mathcal{Z}'_{g,\mathbb{F}_q(t)}(\mathcal{B}) &= \sum_{n=1}^{\left\lfloor \frac{log_q \mathcal{B}}{4g(2g+1)} \right\rfloor} |\mathcal{L}_{g,|\Delta_g|\cdot n}(\mathbb{F}_q)/\sim| \\
&\leq \sum_{n=1}^{\left\lfloor \frac{log_q \mathcal{B}}{4g(2g+1)} \right\rfloor} 2g \cdot q^{2g(2g+3)n} \cdot \frac{(q^{2g-1}-1)(q-q^{-2g+1})}{q-1}
\end{aligned}$$

$$
\begin{aligned}
&= 2g \cdot \frac{(q^{2g-1}-1)(q-q^{-2g+1})}{q-1} \sum_{n=1}^{\left\lfloor \frac{log_q \mathcal{B}}{4g(2g+1)} \right\rfloor} q^{2g(2g+3)n} \\
&= 2g \cdot \frac{(q^{2g-1}-1)(q-q^{-2g+1})}{q-1} \left(q^{2g(2g+3)} + \cdots + q^{2g(2g+3)\cdot(\frac{log_q \mathcal{B}}{4g(2g+1)})} \right) \\
&= 2g \cdot \frac{(q^{2g-1}-1)(q-q^{-2g+1})}{q-1} \cdot \left(\frac{q^{2g(2g+3)} \cdot (\mathcal{B}^{\frac{2g+3}{4g+2}} - 1)}{q^{2g(2g+3)} - 1} \right) \\
&= 2g \cdot \frac{q^{4g(g+1)+1} \cdot (q^{2g-1}-1)(q^{2g}-1)}{(q-1)(q^{2g(2g+3)}-1)} \cdot \left(\mathcal{B}^{\frac{2g+3}{4g+2}} - 1 \right)
\end{aligned}
$$

This implies that the sharp estimate (which can be found explicitly from $\left|\mathcal{L}_{g,|\Delta_g|\cdot n}(\mathbb{F}_q)/\sim\right|$ by Proposition 4.17) has the leading term of order $\mathcal{O}_q\left(\mathcal{B}^{\frac{2g+3}{4g+2}}\right)$ where $\mathcal{O}_q$-constant is an explicit rational function of q with the corresponding lower order terms for each genus $g \geq 2$. ■

Switching to the number field with $K = \mathbb{Q}$ and $\mathcal{O}_K = \mathbb{Z}$, one could choose the minimal integral Weierstrass model of a stable odd degree hyperelliptic curve with the given hyperelliptic discriminant divisor Δ_g which is already a number. This renders the Conjecture 1.3, stated in the Introduction.

On a related note, the number of discriminants Δ_1 of an elliptic curve over $\mathbb{Z}$ with smooth generic fiber such that $\Delta_1 \leq \mathcal{B}$ is estimated to be asymptotic to $\mathcal{O}\left(\mathcal{B}^{\frac{5}{6}}\right)$ by [BMc]. The lower order term of order $\mathcal{O}\left(\mathcal{B}^{(7-\frac{5}{27}+\epsilon)/12}\right)$ for counting the stable elliptic curves over $\mathbb{Q}$ by the bounded height of squarefree Δ_1 was suggested by the work of [Baier] improving upon their previous error term in [BB] (see [Hortsch] for asymptotic with bounded Faltings height). In fact, Baier proved his asymptotic under the assumption of the generalized Riemann hypothesis with the twelveth root of the naïve height function on elliptic curves; this gives arise to the prediction above. For global function fields, by considering the moduli of semistable elliptic surfaces and finding its motive/point count, we acquire [HP, Theorem 3] the sharp estimate on $\mathcal{Z}_{1,\mathbb{F}_q(t)}$ for counting the semistable elliptic curves by the bounded height of $\Delta_1(X)$ over $\mathbb{P}^1_{\mathbb{F}_q}$ with $\mathrm{char}(\mathbb{F}_q) \neq 2, 3$ giving the leading term of order $\mathcal{O}_q\left(\mathcal{B}^{\frac{5}{6}}\right)$ and the lower order term of order $\mathcal{O}_q(1)$. The arithmetic invariant which leads to the above counting also has been established in the past via different method by the seminal work of [de Jong]. It would be fascinating if one could indeed show $\mathcal{Z}_{g,\mathbb{Q}}(\mathcal{B})$ has the same order of magnitude $\mathcal{Z}_{g,\mathbb{Q}}(\mathcal{B}) \asymp \mathcal{B}^{\frac{2g+3}{4g+2}}$ for counting the number of stable hyperelliptic genus $g \geq 2$ curves with a marked rational Weierstrass point and bounded height of hyperelliptic discriminant Δ_g over a number field $\mathbb{Q}$ as shown here by the explicit estimate of $\mathcal{Z}_{g,\mathbb{F}_q(t)}$ over $\mathbb{P}^1_{\mathbb{F}_q}$ with $\mathrm{char}(\mathbb{F}_q) > 2g+1$.

6. Appendix - Arithmetic of the moduli of elliptic curves with prescribed level structure or multiple marked points

In this appendix, we extend the sharp estimate on the number of semistable elliptic curves over $\mathbb{P}^1_{\mathbb{F}_q}$ from [HP, Theorem 3] by using the above methods (i.e., Theorem 1.11 and Corollary 1.13).

Specifically, we explicitly estimate the sharp bound on the number of elliptic curves over global function fields $\mathbb{F}_q(t)$ with level structures $[\Gamma_1(n)]$ for $2 \leq n \leq 4$ or $[\Gamma(2)]$. Recall that a level structure $[\Gamma_1(n)]$ on an elliptic curve E is a choice of point $P \in E$ of exact order n in the smooth part of E such that over every geometric point of the base scheme every irreducible component of

E contains a multiple of P (see [KM, §1.4]). And a level structure $[\Gamma(2)]$ on an elliptic curve E is a choice of isomorphism $\phi : \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to E(2)$ where $E(2)$ is the scheme of 2-torsion Weierstrass points (i.e., kernel of the multiplication-by-2 map $[2] : E \to E$) (see [DR, II.1.18 & IV.2.3]).

Additionally, we consider curves of arithmetic genus one over $\mathbb{F}_q(t)$ with m-marked rational points for $2 \leq m \leq 5$ by instead acquiring sharp estimate on the number of $(m-1)$-stable m-marked curves of arithmetic genus one formulated originally by the works of [Smyth, Smyth2].

To estimate the number of certain elliptic curves over global function fields $\mathbb{F}_q(t)$ with level structures $[\Gamma(n)]$ or $[\Gamma_1(n)]$ as in Theorem 1.2, we need to extend the notion of (nonsingular) elliptic curves (semistable in the case of [HP]) that admits desired level structures. By the work of Deligne and Rapoport [DR] (summarized in [Niles, §2]), we consider the generalized elliptic curves over $\mathbb{P}^1_K$ with $[\Gamma]$–structures (where Γ is $\Gamma(n)$ or $\Gamma_1(n)$) over a field K (focusing on $K = \mathbb{F}_q$). Roughly, a generalized elliptic curve X over $\mathbb{P}^1_K$ can be thought of as a flat family of semistable elliptic curves admitting a group structure, such that a finite group scheme $\mathcal{G} \to \mathbb{P}^1_K$ (determined by Γ) embeds into X and its image meets every irreducible component of every geometric fibers of X. Again, we only consider the *non-isotrivial* generalized elliptic curves. If X is as above, then $\Delta(X)$ is the discriminant of a generalized elliptic curve and if $K = \mathbb{F}_q$, then $0 < ht(\Delta(X)) := q^{\deg \Delta(X)}$.

Now, define $\mathcal{Z}^{[\Gamma]}_{1,\mathbb{F}_q(t)}(\mathcal{B})$ as follows:

$$\mathcal{Z}^{[\Gamma]}_{1,\mathbb{F}_q(t)}(\mathcal{B}) := |\{\text{Generalized elliptic curves over } \mathbb{P}^1_{\mathbb{F}_q} \text{ with } [\Gamma]\text{–structures and } 0 < ht(\Delta(X)) \leq \mathcal{B}\}|$$

Then, we acquire the following descriptions of $\mathcal{Z}^{[\Gamma]}_{1,\mathbb{F}_q(t)}(\mathcal{B})$:

Theorem 6.1 (Sharp estimate of $\mathcal{Z}^{[\Gamma]}_{1,\mathbb{F}_q(t)}(\mathcal{B})$)**.** *If $char(\mathbb{F}_q) \neq 2$ ($\neq 3$ for $\mathcal{Z}^{[\Gamma_1(3)]}_{1,\mathbb{F}_q(t)}(\mathcal{B})$), then the function $\mathcal{Z}^{[\Gamma]}_{1,\mathbb{F}_q(t)}(\mathcal{B})$, which counts the number of generalized elliptic curves with $[\Gamma]$-structures over $\mathbb{P}^1_{\mathbb{F}_q}$ ordered by $0 < ht(\Delta(X)) = q^{12n} \leq \mathcal{B}$, satisfies:*

$$\begin{aligned}
\mathcal{Z}^{[\Gamma_1(2)]}_{1,\mathbb{F}_q(t)}(\mathcal{B}) &\leq 2 \cdot \frac{(q^7 - q^5)}{(q^6 - 1)} \cdot (\mathcal{B}^{\frac{1}{2}} - 1) \\
\mathcal{Z}^{[\Gamma_1(3)]}_{1,\mathbb{F}_q(t)}(\mathcal{B}) &\leq \frac{(q^5 - q^3)}{(q^4 - 1)} \cdot (\mathcal{B}^{\frac{1}{3}} - 1) \\
\mathcal{Z}^{[\Gamma_1(4)]}_{1,\mathbb{F}_q(t)}(\mathcal{B}) &\leq \frac{(q^4 - q^2)}{(q^3 - 1)} \cdot (\mathcal{B}^{\frac{1}{4}} - 1) \\
\mathcal{Z}^{[\Gamma(2)]}_{1,\mathbb{F}_q(t)}(\mathcal{B}) &\leq 2 \cdot \frac{(q^5 - q^3)}{(q^4 - 1)} \cdot (\mathcal{B}^{\frac{1}{3}} - 1)
\end{aligned}$$

which is an equality when $\mathcal{B} = q^{12n}$ with $n \in \mathbb{N}$ implying that the acquired upper bound is a sharp estimate with the following terms:

- $[\Gamma_1(2)]$: *leading term is $\mathcal{O}_q\left(\mathcal{B}^{\frac{1}{2}}\right)$, lower order term is $\mathcal{O}_q(1)$.*
- $[\Gamma_1(3)]$: *leading term is $\mathcal{O}_q\left(\mathcal{B}^{\frac{1}{3}}\right)$, lower order term is $\mathcal{O}_q(1)$.*
- $[\Gamma_1(4)]$: *leading term is $\mathcal{O}_q\left(\mathcal{B}^{\frac{1}{4}}\right)$, lower order term is $\mathcal{O}_q(1)$.*
- $[\Gamma(2)]$: *leading term is $\mathcal{O}_q\left(\mathcal{B}^{\frac{1}{3}}\right)$, lower order term is $\mathcal{O}_q(1)$.*

Proof. The proof is at the end of §6.1. ■

Through the global fields analogy, we formulate analogous heuristics over a number field $\mathbb{Q}$ focusing on the lower order term as we have sharp estimates over global function fields $\mathbb{F}_q(t)$.

Conjecture 6.2 (Heuristic on $\mathcal{Z}^{[\Gamma]}_{1,\mathbb{Q}}(\mathcal{B})$)**.** The function $\mathcal{Z}^{[\Gamma]}_{1,\mathbb{Q}}(\mathcal{B})$, which counts the number of generalized elliptic curves with $[\Gamma]$-structures over $\mathbb{Z}$ ordered by $0 < ht(\Delta) \leq \mathcal{B}$, has the leading term of order $\mathcal{O}\left(\mathcal{B}^{\frac{1}{N}}\right)$ for $[\Gamma_1(N)]$ with $N = 2, 3, 4$; $\mathcal{O}\left(\mathcal{B}^{\frac{1}{3}}\right)$ for $[\Gamma(2)]$ and the lower order term of zeroth order $\mathcal{O}(1)$.

The order of the leading terms of the acquired sharp estimate over global function fields $\mathbb{F}_q(t)$ matches the asymptotic order of the analogous counting by Harron and Snowden in [HS, Theorem 1.2] over a number field $\mathbb{Q}$, but uses instead naïve height function of underlying elliptic curves (see also [Duke, Grant]). It remains an intriguing problem to count the remaining ten possibilities (classified by the fundamental [Mazur, Theorem 8]) of the torsion subgroups with $|G| > 4$ over $\mathbb{F}_q(t)$ by height of discriminant and compare with analogous counting over $\mathbb{Q}$.

Now, let's consider instead elliptic curves with m-marked rational points. To count the number of certain curves of arithmetic genus one over global function fields $\mathbb{F}_q(t)$ with m-markings as in Theorem 1.2, we need to again extend the notion of (nonsingular) elliptic curves that admits desired m-markings. Here, we consider the $(m-1)$-stable m-marked curves of arithmetic genus one (defined by Smyth in [Smyth, §1.1] for characteristic $\neq 2, 3$, extended to lower characteristic with mild conditions by [LP, Definition 1.5.3]), see Definition 6.12 for a precise definition. Note that if $\mathrm{char}(\mathbb{F}_q) > 3$ and $m = 1$, then 0-stable 1-marked curves are exactly stable elliptic curves as in [DM]. We now consider the following definition:

Definition 6.3. Fix an integral reduced K-scheme B, where K is a field. Then a non-isotrivial flat morphism $\pi : X \to B$ is a m-marked $(m-1)$-stable genus one fibration over B if any fiber of π is a $(m-1)$-stable m-marked curves of arithmetic genus one.

Observe that if $\mathrm{char}(K) = 0$ or > 3, then a m-marked $(m-1)$-stable genus one fibration $X \to \mathbb{P}^1_K$ has a discriminant $\Delta(X) \subset \mathbb{P}^1_K$, and if $K = \mathbb{F}_q$, then $0 < ht(\Delta(X)) := q^{\deg \Delta(X)}$.

Now, define $\mathcal{Z}_{1,m,\mathbb{F}_q(t)}(\mathcal{B})$ as follows:

$$\mathcal{Z}_{1,m,\mathbb{F}_q(t)}(\mathcal{B}) := |\{m\text{-marked } (m-1)\text{-stable genus one fibrations over } \mathbb{P}^1_{\mathbb{F}_q} \text{ with } 0 < ht(\Delta(X)) \leq \mathcal{B}\}|$$

Note that when $m = 1$, $\mathcal{Z}_{1,m,\mathbb{F}_q(t)}(\mathcal{B})$ counts stable elliptic fibrations, which is described in [HP, Theorem 3] (by identifying stable elliptic fibrations with nonsingular semistable elliptic surfaces, see [HP, Proposition 11]). When $2 \leq m \leq 5$, we acquire the following sharp estimate of $\mathcal{Z}_{1,m,\mathbb{F}_q(t)}(\mathcal{B})$:

Theorem 6.4 (Sharp estimate of $\mathcal{Z}_{1,m,\mathbb{F}_q(t)}(\mathcal{B})$)**.** *If $\mathrm{char}(\mathbb{F}_q) \neq 2, 3$, then the function $\mathcal{Z}_{1,m,\mathbb{F}_q(t)}(\mathcal{B})$, which counts the number of m-marked $(m-1)$-stable genus one fibration over $\mathbb{P}^1_{\mathbb{F}_q}$ for $2 \leq m \leq 5$ ordered by $0 < ht(\Delta(X)) = q^{12n} \leq \mathcal{B}$, satisfies:*

$$\mathcal{Z}_{1,2,\mathbb{F}_q(t)}(\mathcal{B}) \leq \frac{(q^{11} + q^{10} - q^8 - q^7)}{(q^9 - 1)} \cdot (\mathcal{B}^{\frac{3}{4}} - 1) + \frac{(q^7 - q^5)}{(q^6 - 1)} \cdot (\mathcal{B}^{\frac{1}{2}} - 1)$$

$$\mathcal{Z}_{1,3,\mathbb{F}_q(t)}(\mathcal{B}) \leq \frac{(q^{11} + q^{10} + q^9 - q^7 - q^6 - q^5)}{(q^8 - 1)} \cdot (\mathcal{B}^{\frac{2}{3}} - 1) + \frac{(q^5 - q^3)}{(q^4 - 1)} \cdot (\mathcal{B}^{\frac{1}{3}} - 1)$$

$$\mathcal{Z}_{1,4,\mathbb{F}_q(t)}(\mathcal{B}) \leq \frac{(q^{11} + q^{10} + q^9 + q^8 - q^6 - q^5 - q^4 - q^3)}{(q^7 - 1)} \cdot (\mathcal{B}^{\frac{7}{12}} - 1) + \frac{(q^5 - q^3)}{(q^4 - 1)} \cdot (\mathcal{B}^{\frac{1}{3}} - 1)$$

$$\mathcal{Z}_{1,5,\mathbb{F}_q(t)}(\mathcal{B}) \leq \frac{(q^{11} + q^{10} + q^9 + q^8 + q^7 - q^5 - q^4 - q^3 - q^2 - q^1)}{(q^6 - 1)} \cdot (\mathcal{B}^{\frac{1}{2}} - 1)$$

which is an equality when $\mathcal{B} = q^{12n}$ with $n \in \mathbb{N}$ implying that the acquired upper bound is a sharp estimate with the following terms:

- *$m = 2$: leading term is $\mathcal{O}_q\left(\mathcal{B}^{\frac{3}{4}}\right)$, lower order terms are $\mathcal{O}_q\left(\mathcal{B}^{\frac{1}{2}}\right)$ and $\mathcal{O}_q(1)$.*

- $m = 3$: *leading term is* $\mathcal{O}_q\left(\mathcal{B}^{\frac{2}{3}}\right)$*, lower order terms are* $\mathcal{O}_q\left(\mathcal{B}^{\frac{1}{3}}\right)$ *and* $\mathcal{O}_q(1)$*.*
- $m = 4$: *leading term is* $\mathcal{O}_q\left(\mathcal{B}^{\frac{7}{12}}\right)$*, lower order terms are* $\mathcal{O}_q\left(\mathcal{B}^{\frac{1}{3}}\right)$ *and* $\mathcal{O}_q(1)$*.*
- $m = 5$: *leading term is* $\mathcal{O}_q\left(\mathcal{B}^{\frac{1}{2}}\right)$*, lower order term is* $\mathcal{O}_q(1)$*.*

Proof. The proof is at the end of §6.2. ■

Through the global fields analogy, we formulate analogous heuristics over a number field $\mathbb{Q}$ focusing on the lower order term as we have sharp estimates over global function fields $\mathbb{F}_q(t)$.

Conjecture 6.5 (Heuristic on $\mathcal{Z}_{1,m,\mathbb{Q}}(\mathcal{B})$)**.** The function $\mathcal{Z}_{1,m,\mathbb{Q}}(\mathcal{B})$, which counts the number of m-marked $(m-1)$-stable genus one curves for $2 \le m \le 5$ over $\mathbb{Z}$ ordered by $0 < ht(\Delta) \le \mathcal{B}$, has the leading term of order $\mathcal{O}\left(\mathcal{B}^{\frac{3}{4}}\right)$ and the lower order term of orders $\mathcal{O}\left(\mathcal{B}^{\frac{1}{2}}\right)$ and $\mathcal{O}(1)$ for $m = 2$; the leading term of order $\mathcal{O}\left(\mathcal{B}^{\frac{2}{3}}\right)$ and the lower order term of orders $\mathcal{O}\left(\mathcal{B}^{\frac{1}{3}}\right)$ and $\mathcal{O}(1)$ for $m = 3$; the leading term of order $\mathcal{O}\left(\mathcal{B}^{\frac{7}{12}}\right)$ and the lower order term of orders $\mathcal{O}\left(\mathcal{B}^{\frac{1}{3}}\right)$ and $\mathcal{O}(1)$ for $m = 4$; the leading term of order $\mathcal{O}\left(\mathcal{B}^{\frac{1}{2}}\right)$ and the lower order term of order $\mathcal{O}(1)$ for $m = 5$.

6.1. Arithmetic of the moduli of generalized elliptic curves over $\mathbb{P}^1$ with level structures.

The essential geometrical idea in acquiring the sharp estimate is to consider the moduli stack of rational curves on a compactified modular curve as in [HP]. The various compactified modular curves $\overline{\mathcal{M}}_{1,1}[\Gamma]$ are isomorphic to the 1-dimensional weighted projective stacks $\mathcal{P}(a,b)$.

Proposition 6.6. *The moduli stack* $\overline{\mathcal{M}}_{1,1}[\Gamma]$ *of generalized elliptic curves with* $[\Gamma]$*-structures is isomorphic to the following when over a field* K*:*

(1) if $\mathrm{char}(K) \neq 2$*, the tame Deligne–Mumford moduli stack of generalized elliptic curves with* $[\Gamma_1(2)]$*-structures is isomorphic to*

$$(\overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])_K \cong [(\text{Spec } K[a_2,a_4] - (0,0))/\mathbb{G}_m] = \mathcal{P}_K(2,4),$$

(2) if $\mathrm{char}(K) \neq 3$*, the tame Deligne–Mumford moduli stack of generalized elliptic curves with* $[\Gamma_1(3)]$*-structures is isomorphic to*

$$(\overline{\mathcal{M}}_{1,1}[\Gamma_1(3)])_K \cong [(\text{Spec } K[a_1,a_3] - (0,0))/\mathbb{G}_m] = \mathcal{P}_K(1,3),$$

(3) if $\mathrm{char}(K) \neq 2$*, the tame Deligne–Mumford moduli stack of generalized elliptic curves with* $[\Gamma_1(4)]$*-structures is isomorphic to*

$$(\overline{\mathcal{M}}_{1,1}[\Gamma_1(4)])_K \cong [(\text{Spec } K[a_1,a_2] - (0,0))/\mathbb{G}_m] = \mathcal{P}_K(1,2),$$

(4) if $\mathrm{char}(K) \neq 2$*, the tame Deligne–Mumford moduli stack of generalized elliptic curves with* $[\Gamma(2)]$*-structures is isomorphic to*

$$(\overline{\mathcal{M}}_{1,1}[\Gamma(2)])_K \cong [(\text{Spec } K[a_2,a_2] - (0,0))/\mathbb{G}_m] = \mathcal{P}_K(2,2),$$

where $\lambda \cdot a_i = \lambda^i a_i$ *for* $\lambda \in \mathbb{G}_m$ *and* $i = 1,2,3,4$*. Thus, the* a_i*'s have degree* i *respectively. Moreover, the discriminant divisors of* $(\overline{\mathcal{M}}_{1,1}[\Gamma])_K \cong \mathcal{P}_K(i,j)$ *as above have degree 12.*

Proof. Proof of the first, second, third and fourth equivalence can be found in [Behrens, §1.3], [HMe, Proposition 4.5], [Meier, Examples 2.1] and [Stojanoska, Proposition 7.1] respectively. By Proposition 2.2, the weighted projective stacks are tame Deligne–Mumford as well.

For the degree of the discriminant, it suffices to find the weight of the $\mathbb{G}_m$-action. First, the four papers cited above explicitly construct universal families of elliptic curves over the schematic covers $(\text{Spec } K[a_i,a_j] - (0,0)) \to \mathcal{P}_K(i,j)$ of the corresponding moduli stacks. The explicit defining equation of the respective universal family implies that the $\lambda \in \mathbb{G}_m$ also acts on the discriminant of the universal family by multiplying λ^{12}. Therefore, the discriminant has degree 12. ■

We now consider the moduli stack $\mathcal{L}_{1,12n}^{[\Gamma]} := \mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma])$ of generalized elliptic curves over $\mathbb{P}^1$ with $[\Gamma]$-structures.

Proposition 6.7. *Assume* $\mathrm{char}(K) = 0, \neq 2$ *for* $[\Gamma] = [\Gamma_1(2)], [\Gamma_1(4)], [\Gamma(2)], \neq 3$ *for* $[\Gamma] = [\Gamma_1(3)]$. *Then, the moduli stack* $\mathcal{L}_{1,12n}^{[\Gamma]}$ *of generalized elliptic curves over* $\mathbb{P}^1$ *with discriminant degree* $12n > 0$ *and* $[\Gamma]$*-structures is the tame Deligne–Mumford stack* $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma])$ *parameterizing the* K*-morphisms* $f : \mathbb{P}^1 \to \overline{\mathcal{M}}_{1,1}[\Gamma]$ *such that* $f^*\mathcal{O}_{\overline{\mathcal{M}}_{1,1}[\Gamma]}(1) \cong \mathcal{O}_{\mathbb{P}^1}(n)$.

Proof. Without the loss of generality, we prove the $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])$ case over a field K with $\mathrm{char}(K) \neq 2$. The proof for the other cases are analogous. By the definition of the universal family p, any generalized elliptic curves $\pi : Y \to \mathbb{P}^1$ with $[\Gamma_1(2)]$-structures comes from a morphism $f : \mathbb{P}^1 \to \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)]$ and vice versa. As this correspondence also works in families, the moduli stack of generalized elliptic curves over $\mathbb{P}^1$ with $[\Gamma_1(2)]$-structures is isomorphic to $\mathrm{Hom}(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])$.

Since the discriminant degree of f is $12 \deg f^*\mathcal{O}_{\overline{\mathcal{M}}_{1,1}[\Gamma_1(2)]}(1)$ by Proposition 6.6, the substack $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])$ parametrizing such f's with $\deg f^*\mathcal{O}_{\overline{\mathcal{M}}_{1,1}[\Gamma_1(2)]}(1) = n$ is the desired moduli stack. Since $\deg f^*\mathcal{O}_{\overline{\mathcal{M}}_{1,1}[\Gamma_1(2)]}(1) = n$ is an open condition, $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])$ is an open substack of $\mathrm{Hom}(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])$, which is tame Deligne–Mumford by Proposition 2.5 as $\overline{\mathcal{M}}_{1,1}[\Gamma_1(2)]$ itself is tame Deligne–Mumford by Proposition 6.6. This shows that $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])$ satisfies the desired properties as well. ■

We now apply the Theorem 1.11 to the moduli stacks $\mathcal{L}_{1,12n}^{[\Gamma]} \cong \mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma])$ over a field K and acquire the following motives in the Grothendieck ring of K–stacks $K_0(\mathrm{Stck}_K)$.

Corollary 6.8. *If* $\mathrm{char}(K) \neq 2$, *then*

$$\begin{aligned} [\mathcal{L}_{1,12n}^{[\Gamma_1(2)]}] &= \mathbb{L}^{6n+1} - \mathbb{L}^{6n-1} , \\ [\mathcal{L}_{1,12n}^{[\Gamma_1(4)]}] &= \mathbb{L}^{3n+1} - \mathbb{L}^{3n-1} , \\ [\mathcal{L}_{1,12n}^{[\Gamma(2)]}] &= \mathbb{L}^{4n+1} - \mathbb{L}^{4n-1} . \end{aligned}$$

If $\mathrm{char}(K) \neq 3$, *then*

$$[\mathcal{L}_{1,12n}^{[\Gamma_1(3)]}] = \mathbb{L}^{4n+1} - \mathbb{L}^{4n-1} .$$

It is easy to see that when $K = \mathbb{F}_q$, the assignment $[\mathcal{L}_{1,12n}^{[\Gamma]}] \mapsto \#_q(\mathcal{L}_{1,12n}^{[\Gamma]})$ gives the weighted point count of the moduli stack $\mathcal{L}_{1,12n}^{[\Gamma]}$ over $\mathbb{F}_q$.

Corollary 6.9. *If* $\mathrm{char}(\mathbb{F}_q) \neq 2$, *then*

$$\begin{aligned} \#_q(\mathcal{L}_{1,12n}^{[\Gamma_1(2)]}) &= q^{6n+1} - q^{6n-1} , \\ \#_q(\mathcal{L}_{1,12n}^{[\Gamma_1(4)]}) &= q^{3n+1} - q^{3n-1} , \\ \#_q(\mathcal{L}_{1,12n}^{[\Gamma(2)]}) &= q^{4n+1} - q^{4n-1} . \end{aligned}$$

If $\mathrm{char}(\mathbb{F}_q) \neq 3$, *then*

$$\#_q(\mathcal{L}_{1,12n}^{[\Gamma_1(3)]}) = q^{4n+1} - q^{4n-1} .$$

Furthermore, we obtain the number $|\mathcal{L}_{1,12n}^{[\Gamma]}(\mathbb{F}_q)/\sim|$ of $\mathbb{F}_q$–isomorphism classes of $\mathbb{F}_q$–points (i.e., the non–weighted point count).

Proposition 6.10. *If* $\mathrm{char}(\mathbb{F}_q) \neq 2$*, then*

$$
\begin{aligned}
|\mathcal{L}_{1,12n}^{[\Gamma_1(2)]}(\mathbb{F}_q)/\sim| &= 2(q^{6n+1} - q^{6n-1}) \,, \\
|\mathcal{L}_{1,12n}^{[\Gamma_1(4)]}(\mathbb{F}_q)/\sim| &= q^{3n+1} - q^{3n-1} \,, \\
|\mathcal{L}_{1,12n}^{[\Gamma(2)]}(\mathbb{F}_q)/\sim| &= 2(q^{4n+1} - q^{4n-1}) \,.
\end{aligned}
$$

If $\mathrm{char}(\mathbb{F}_q) \neq 3$*, then*

$$|\mathcal{L}_{1,12n}^{[\Gamma_1(3)]}(\mathbb{F}_q)/\sim| = q^{4n+1} - q^{4n-1} \,.$$

Proof. Fix $n > 0$. Since any $\varphi_g \in \mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))$ is surjective, the generic stabilizer group $\mu_{\gcd(a,b)}$ of $\mathcal{P}(a,b)$ is the automorphism group of φ_g. Then the Definition 1.12 and Proposition 3.8 implies that the number $|\mathcal{L}_{1,12n}^{[\Gamma]}(\mathbb{F}_q)/\sim|$ of $\mathbb{F}_q$–isomorphism classes of generalized elliptic curves with discriminant degree $12n$ and $[\Gamma]$-structures is

$$|\mathcal{L}_{1,12n}^{[\Gamma]}(\mathbb{F}_q)/\sim| = (\gcd(a,b)) \cdot (\#_q(\mathcal{L}_{1,12n}^{[\Gamma]}))$$

where the factor $\gcd(a,b)$ comes from the hyperelliptic involution (or lack thereof) depending on the corresponding modular weights (a,b) of $\overline{\mathcal{M}}_{1,1}[\Gamma] \cong \mathcal{P}(a,b)$. ■

Remark 6.11. It is striking to note that while the weighted point counts $\#_q(\mathcal{L}_{1,12n}^{[\Gamma_1(3)]}) = \#_q(\mathcal{L}_{1,12n}^{[\Gamma(2)]})$ are the same due to the equal sum of the modular weights of the compactified modular curves $\overline{\mathcal{M}}_{1,1}[\Gamma_1(3)] \cong \mathcal{P}(1,3)$ and $\overline{\mathcal{M}}_{1,1}[\Gamma(2)] \cong \mathcal{P}(2,2)$, the non–weighted point counts $|\mathcal{L}_{1,12n}^{[\Gamma_1(3)]}(\mathbb{F}_q)/\sim| \neq |\mathcal{L}_{1,12n}^{[\Gamma(2)]}(\mathbb{F}_q)/\sim|$ are different over $\mathrm{char}(\mathbb{F}_q) \neq 2, 3$. This is due to the differences in the generic stabilizer group of the corresponding moduli stacks which is isomorphic to a trivial group for $\mathcal{P}(1,3)$ as $\gcd(1,3) = 1$ whereas for $\mathcal{P}(2,2)$ the generic stabilizer group is isomorphic to μ_2 as $\gcd(2,2) = 2$.

We now finally prove the Theorem 6.1 using the above arithmetic invariants as follows:

Proof of Theorem 6.1. Without the loss of the generality, we prove the $[\Gamma_1(2)]$–structures case over $\mathrm{char}(\mathbb{F}_q) \neq 2$. The proof for the other cases are analogous. By Proposition 6.10, we know the number of $\mathbb{F}_q$-isomorphism classes of generalized elliptic curves of discriminant degree $12n$ with $[\Gamma_1(2)]$-structures over $\mathbb{P}^1_{\mathbb{F}_q}$ is $|\mathcal{L}_{1,12n}^{[\Gamma_1(2)]}(\mathbb{F}_q)| = 2 \cdot (q^{6n+1} - q^{6n-1})$. Using this, we can explicitly estimate the sharp bound on $\mathcal{Z}_{1,\mathbb{F}_q(t)}^{[\Gamma_1(2)]}(\mathcal{B})$ as the following,

$$
\begin{aligned}
\mathcal{Z}_{1,\mathbb{F}_q(t)}^{[\Gamma_1(2)]}(\mathcal{B}) &= \sum_{n=1}^{\left\lfloor \frac{log_q \mathcal{B}}{12} \right\rfloor} |\mathcal{L}_{1,12n}^{[\Gamma_1(2)]}(\mathbb{F}_q)| = \sum_{n=1}^{\left\lfloor \frac{log_q \mathcal{B}}{12} \right\rfloor} 2 \cdot (q^{6n+1} - q^{6n-1}) \\
&= 2 \cdot (q^1 - q^{-1}) \sum_{n=1}^{\left\lfloor \frac{log_q \mathcal{B}}{12} \right\rfloor} q^{6n} \le 2 \cdot (q^1 - q^{-1}) \left(q^6 + \cdots + q^{6 \cdot (\frac{log_q \mathcal{B}}{12})} \right) \\
&= 2 \cdot (q^1 - q^{-1}) \frac{q^6(\mathcal{B}^{\frac{1}{2}} - 1)}{(q^6 - 1)} = 2 \cdot \frac{(q^7 - q^5)}{(q^6 - 1)} \cdot (\mathcal{B}^{\frac{1}{2}} - 1)
\end{aligned}
$$

On the second line of the equations above, inequality becomes an equality if and only if $n := \frac{log_q \mathcal{B}}{12} \in \mathbb{N}$, i.e., $\mathcal{B} = q^{12n}$ with $n \in \mathbb{N}$. This implies that the acquired upper bound on $\mathcal{Z}_{1,\mathbb{F}_q(t)}^{[\Gamma_1(2)]}(\mathcal{B})$ is a sharp estimate with the leading term of order $\mathcal{O}_q\left(\mathcal{B}^{\frac{1}{2}}\right)$ where $\mathcal{O}_q$-constant is an explicit rational function of q with the lower order term of order $\mathcal{O}_q(1)$. ■

6.2. **Arithmetic of the moduli of m-marked genus one fibrations over $\mathbb{P}^1$.** We proceed to explicitly estimate the sharp bound on the number of m-marked $(m-1)$-stable genus one fibrations over $\mathbb{P}^1_{\mathbb{F}_q}$ for $2 \le m \le 5$. First, we state the definition of m-marked $(m-1)$-stability from [LP, Definition 1.5.3], which is a modification of the Deligne–Mumford stability [DM]:

Definition 6.12. Let K be a field and m be a positive integer. Then, a tuple $(C, p_1, \dots, p_m)$, of a geometrically connected, geometrically reduced, and proper K-curve C of arithmetic genus one with m distinct K-rational points p_i in the smooth locus of C, is a $(m-1)$-stable m-marked curve of arithmetic genus one if the curve $C_{\overline{K}} := C \times_K \overline{K}$ and the divisor $\Sigma := \{p_1, \dots, p_m\}$ satisfy the following properties, where $\overline{K}$ is the algebraic closure of K:

1. $C_{\overline{K}}$ has only nodes and elliptic u-fold points as singularities (see below), where $u < m$,
2. $C_{\overline{K}}$ has no disconnecting nodes, and
3. every irreducible component of $C_{\overline{K}}$ contains at least one marked point.

Remark 6.13. A singular point of a curve over $\overline{K}$ is an elliptic u-fold singular point if it is Gorenstein and étale locally isomorphic to a union of u general lines in $\mathbb{P}^{u-1}_{\overline{K}}$ passing through a common point.

Note that the name "$(m-1)$-stability" comes from [Smyth, §1.1], which is defined when $\mathrm{char}(K) \neq 2, 3$. By [LP, Proposition 1.5.4], the above definition (by [LP, Definition 1.5.3]) coincides with that of Smyth when $\mathrm{char}(K) \neq 2, 3$, hence we adapt Smyth's naming convention on Lekili and Polishchuk's definition. Regardless, we focus on the case when $\mathrm{char}(K) \neq 2, 3$, so that the moduli stack of such curves behaves reasonably.

By the work of Smyth [Smyth, Theorem 3.8], we are able to formulate the moduli stack of $(m-1)$-stable m-marked curves of arithmetic genus one over any field of characteristic $\neq 2, 3$:

Theorem 6.14. *There exists a proper irreducible Deligne–Mumford moduli stack $\overline{\mathcal{M}}_{1,m}(m-1)$ of $(m-1)$-stable m-marked curves arithmetic genus one over* $\mathrm{Spec}(\mathbb{Z}[1/6])$

Note that when $m = 1$, $\overline{\mathcal{M}}_{1,1}(0) \cong \overline{\mathcal{M}}_{1,1}$ is the Deligne–Mumford moduli stack of stable elliptic curves.

In fact, the construction of $\overline{\mathcal{M}}_{1,m}(m-1)$ extends to $\mathrm{Spec}\,\mathbb{Z}$ by [LP, Theorem 1.5.7] (called $\overline{\mathcal{M}}^{\infty}_{1,m}$ in loc.cit.) as an algebraic stack, which is proper over $\mathrm{Spec}\,\mathbb{Z}[1/N]$, where N depends on m:

- if $m \geq 3$, then $N = 1$,
- if $m = 2$, then $N = 2$, and
- if $m = 1$, then $N = 6$.

However, even with those assumptions above, $\overline{\mathcal{M}}_{1,m}(m-1)$ is not necessarily Deligne–Mumford. Nevertheless, by [LP, Theorem 1.5.7.], we obtain the explicit descriptions of $\overline{\mathcal{M}}_{1,m}(m-1)$:

Proposition 6.15. *The moduli stack $\overline{\mathcal{M}}_{1,m}(m-1)$ of m-marked $(m-1)$-stable curves of arithmetic genus one for $2 \le m \le 5$ is isomorphic to the following, for a field K:*

(1) if $\mathrm{char}(K) \neq 2, 3$, the tame Deligne–Mumford moduli stack of 2-marked 1-stable curves of arithmetic genus one is isomorphic to

$$(\overline{\mathcal{M}}_{1,2}(1))_K \cong [(\mathrm{Spec}\ K[a_2, a_3, a_4] - 0)/\mathbb{G}_m] = \mathcal{P}_K(2, 3, 4),$$

(2) if $\mathrm{char}(K) \neq 2,3$*, the tame Deligne–Mumford moduli stack of 3-marked 2-stable curves of arithmetic genus one is isomorphic to*

$$(\overline{\mathcal{M}}_{1,3}(2))_K \cong [(\mathrm{Spec}\ K[a_1, a_2, a_2, a_3] - 0)/\mathbb{G}_m] = \mathcal{P}_K(1, 2, 2, 3),$$

(3) if $\mathrm{char}(K) \neq 2$*, the tame Deligne–Mumford moduli stack of 4-marked 3-stable curves of arithmetic genus one is isomorphic to*

$$(\overline{\mathcal{M}}_{1,4}(3))_K \cong [(\mathrm{Spec}\ K[a_1, a_1, a_1, a_2, a_2] - 0)/\mathbb{G}_m] = \mathcal{P}_K(1, 1, 1, 2, 2),$$

(4) the moduli stack of 5-marked 4-stable curves of arithmetic genus one is isomorphic to a scheme

$$(\overline{\mathcal{M}}_{1,5}(4))_K \cong [(\mathrm{Spec}\ K[a_1, a_1, a_1, a_1, a_1, a_1] - 0)/\mathbb{G}_m] = \mathbb{P}_K(1, 1, 1, 1, 1, 1) \cong \mathbb{P}^5_K,$$

where $\lambda \cdot a_i = \lambda^i a_i$ *for* $\lambda \in \mathbb{G}_m$ *and* $i = 1, 2, 3, 4$*. Thus, the* a_i*'s have degree* i *respectively. Furthermore, if* $\mathrm{char}(K) \neq 2, 3$*, then the discriminant divisors of such* $\overline{\mathcal{M}}_{1,m}(m-1)$ *have degree 12.*

Proof. Proof of [LP, Theorem 1.5.7.] gives the corresponding isomorphisms $\overline{\mathcal{M}}_{1,m}(m-1) \cong \mathcal{P}(\vec{\lambda})$. By Proposition 2.2, the weighted projective stacks are tame Deligne–Mumford as well, and in fact, smooth.

For the degree of the discriminant when $\mathrm{char}(K) \neq 2, 3$, it suffices to describe the discriminant divisor, the locus of singular curves in $\overline{\mathcal{M}}_{1,m}(m-1)$. First, [LP, Theorem 1.5.7.] shows that in the above case, where $\overline{\mathcal{M}}_{1,m}(m-1) \cong \mathcal{P}(\vec{\lambda})$, the line bundle $\mathcal{O}_{\mathcal{P}(\vec{\lambda})}(1)$ of degree one is isomorphic to $\lambda := \pi_* \omega_\pi$, where $\pi : \overline{\mathcal{C}}_{1,m}(m-1) \to \overline{\mathcal{M}}_{1,m}(m-1)$ is the universal family of $(m-1)$-stable m-marked curves of arithmetic genus one. Since $\overline{\mathcal{M}}_{1,m}(m-1)$ is smooth and the Picard rank is one (generated by λ), the discriminant divisor is Cartier. In fact, by [Smyth2, §3.1], it coincides with the locus Δ_{irr} of curves with non-disconnecting nodes or non-nodal singular points. Then [Smyth2, Remark 3.3] (which assumes $\mathrm{char}(K) \neq 2, 3$) implies that $\Delta_{irr} \sim 12\lambda$, thus the discriminant divisor has degree 12. ∎

We now consider the moduli stacks of m-marked $(m-1)$-stable genus one fibrations over $\mathbb{P}^1_K$ for any field K of $\mathrm{char}(K) = 0$ or > 3:

Proposition 6.16. *Assume* $\mathrm{char}(K) = 0$ *or* > 3*. If* $2 \leq m \leq 5$*, then the moduli stack* $\mathcal{L}_{1,m,12n}$ *of* m*-marked* $(m-1)$*-stable genus one fibrations over* $\mathbb{P}^1_K$ *with discriminant degree* $12n$ *is the tame Deligne–Mumford stack* $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,m}(m-1))$ *parameterizing the* K*-morphisms* $f : \mathbb{P}^1 \to \overline{\mathcal{M}}_{1,m}(m-1)$ *such that* $f^*\mathcal{O}_{\mathcal{P}(\vec{\lambda})}(1) \cong \mathcal{O}_{\mathbb{P}^1}(n)$*.*

Proof. Without the loss of the generality, we prove the 2-marked 1-stable curves of arithmetic genus one case $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,2}(1))$ over $\mathrm{char}(\mathbb{F}_q) \neq 2, 3$. The proof for the other cases are analogous. By the definition of the universal family p, any 2-marked 1-stable arithmetic genus one curves $\pi : Y \to \mathbb{P}^1$ with discriminant degree $12n$ comes from a morphism $f : \mathbb{P}^1 \to \overline{\mathcal{M}}_{1,2}(1)$ and vice versa. As this correspondence also works in families, the moduli stack of 2-marked 1-stable curves of arithmetic genus one over $\mathbb{P}^1_K$ is isomorphic to $\mathrm{Hom}(\mathbb{P}^1, \overline{\mathcal{M}}_{1,2}(1))$.

Since the discriminant degree of f is $12 \deg f^*\mathcal{O}_{\overline{\mathcal{M}}_{1,2}(1)}(1)$ by Proposition 6.15, the substack $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,2}(1))$ parametrizing such f's with $\deg f^*\mathcal{O}_{\overline{\mathcal{M}}_{1,2}(1)}(1) = n$ is the desired moduli stack. Since $\deg f^*\mathcal{O}_{\overline{\mathcal{M}}_{1,2}(1)}(1) = n$ is an open condition, $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,2}(1))$ is an open substack of $\mathrm{Hom}(\mathbb{P}^1, \overline{\mathcal{M}}_{1,2}(1))$, which is tame Deligne–Mumford by Proposition 2.5 as $\overline{\mathcal{M}}_{1,2}(1)$ itself is tame Deligne–Mumford by Proposition 6.15. This shows that $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,2}(1))$ satisfies the desired properties as well. ∎

We now apply the Theorem 1.11 to the moduli stacks $\mathcal{L}_{1,m,12n} \cong \mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,m}(m-1))$ over a field K and acquire the following motives in the Grothendieck ring $K_0(\mathrm{Stck}_K)$ of K–stacks.

Corollary 6.17. *If K is a field with $\mathrm{char}(K) \neq 2, 3$, then*

$$[\mathcal{L}_{1,2,12n}] = \mathbb{L}^{9n+2} + \mathbb{L}^{9n+1} - \mathbb{L}^{9n-1} - \mathbb{L}^{9n-2} .$$
$$[\mathcal{L}_{1,3,12n}] = \mathbb{L}^{8n+3} + \mathbb{L}^{8n+2} + \mathbb{L}^{8n+1} - \mathbb{L}^{8n-1} - \mathbb{L}^{8n-2} - \mathbb{L}^{8n-3} .$$
$$[\mathcal{L}_{1,4,12n}] = \mathbb{L}^{7n+4} + \mathbb{L}^{7n+3} + \mathbb{L}^{7n+2} + \mathbb{L}^{7n+1} - \mathbb{L}^{7n-1} - \mathbb{L}^{7n-2} - \mathbb{L}^{7n-3} - \mathbb{L}^{7n-4} .$$
$$[\mathcal{L}_{1,5,12n}] = \mathbb{L}^{6n+5} + \mathbb{L}^{6n+4} + \mathbb{L}^{6n+3} + \mathbb{L}^{6n+2} + \mathbb{L}^{6n+1} - \mathbb{L}^{6n-1} - \mathbb{L}^{6n-2} - \mathbb{L}^{6n-3} - \mathbb{L}^{6n-4} - \mathbb{L}^{6n-5} .$$

It is easy to see that when $K = \mathbb{F}_q$, the assignment $[\mathcal{L}_{1,m,12n}] \mapsto \#_q(\mathcal{L}_{1,m,12n})$ gives the weighted point count of the moduli stack $\mathcal{L}_{1,m,12n}$ over $\mathbb{F}_q$.

Corollary 6.18. *If $\mathrm{char}(\mathbb{F}_q) \neq 2, 3$, then*

$$\#_q(\mathcal{L}_{1,2,12n}) = q^{9n+2} + q^{9n+1} - q^{9n-1} - q^{9n-2} .$$
$$\#_q(\mathcal{L}_{1,3,12n}) = q^{8n+3} + q^{8n+2} + q^{8n+1} - q^{8n-1} - q^{8n-2} - q^{8n-3} .$$
$$\#_q(\mathcal{L}_{1,4,12n}) = q^{7n+4} + q^{7n+3} + q^{7n+2} + q^{7n+1} - q^{7n-1} - q^{7n-2} - q^{7n-3} - q^{7n-4} .$$
$$\#_q(\mathcal{L}_{1,5,12n}) = q^{6n+5} + q^{6n+4} + q^{6n+3} + q^{6n+2} + q^{6n+1} - q^{6n-1} - q^{6n-2} - q^{6n-3} - q^{6n-4} - q^{6n-5} .$$

Furthermore, we obtain the number $|\mathcal{L}_{1,m,12n}(\mathbb{F}_q)/\sim|$ of $\mathbb{F}_q$–isomorphism classes of $\mathbb{F}_q$–points (i.e., the non–weighted point count).

Proposition 6.19. *Over $\mathrm{char}(\mathbb{F}_q) \neq 2, 3$,*

$$|\mathcal{L}_{1,2,12n}(\mathbb{F}_q)/\sim| = (q^{9n+2} + q^{9n+1} - q^{9n-1} - q^{9n-2}) + (q^{6n+1} - q^{6n-1}) .$$
$$|\mathcal{L}_{1,3,12n}(\mathbb{F}_q)/\sim| = (q^{8n+3} + q^{8n+2} + q^{8n+1} - q^{8n-1} - q^{8n-2} - q^{8n-3}) + (q^{4n+1} - q^{4n-1}) .$$
$$\begin{aligned}|\mathcal{L}_{1,4,12n}(\mathbb{F}_q)/\sim| &= (q^{7n+4} + q^{7n+3} + q^{7n+2} + q^{7n+1} - q^{7n-1} - q^{7n-2} - q^{7n-3} - q^{7n-4}) \\ &\quad + (q^{4n+1} - q^{4n-1}) .\end{aligned}$$
$$\begin{aligned}|\mathcal{L}_{1,5,12n}(\mathbb{F}_q)/\sim| &= q^{6n+5} + q^{6n+4} + q^{6n+3} + q^{6n+2} + q^{6n+1} \\ &\quad - q^{6n-1} - q^{6n-2} - q^{6n-3} - q^{6n-4} - q^{6n-5} .\end{aligned}$$

Proof. Note that $\overline{\mathcal{M}}_{1,2}(1) \cong \mathcal{P}(2,3,4)$ has the substack $\mathcal{P}(2,4)$ with the generic stabilizer of order 2. This implies that the number of isomorphism classes of $\mathbb{F}_q$-points of $\mathcal{L}_{1,2,12n}$ with discriminant degree $12n$ is $|\mathcal{L}_{1,2,12n}(\mathbb{F}_q)/\sim| = (q^{9n+2} + q^{9n+1} - q^{9n-1} - q^{9n-2}) + (q^{6n+1} - q^{6n-1})$ by Corollary 6.17 and Proposition 3.8. Similarly, $\overline{\mathcal{M}}_{1,3}(2) \cong \mathcal{P}(1,2,2,3)$ and $\overline{\mathcal{M}}_{1,4}(3) \cong \mathcal{P}(1,1,1,2,2)$ has the substack $\mathcal{P}(2,2)$ with the generic stabilizer of order 2. This implies that adding $(q^{4n+1} - q^{4n-1})$ to the corresponding weighted points count gives the desired non–weighted point counts. Finally, $\overline{\mathcal{M}}_{1,5}(4) \cong \mathbb{P}^5$, so that the non-weighted point count coincides with the weighted point count from Corollary 6.18 ■

We now finally prove the Theorem 6.4 using the above arithmetic invariants as follows:

Proof. Without the loss of the generality, we prove the 2-marked 1-stable curves of arithmetic genus one case $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,2}(1) \cong \mathcal{P}(2,3,4))$ over $\mathrm{char}(\mathbb{F}_q) \neq 2, 3$. The proof for the other cases are analogous. Knowing the number of $\mathbb{F}_q$-isomorphism classes of 1-stable arithmetic genus one curves over $\mathbb{P}^1$ with discriminant degree $12n$ and 2-marked Weierstrass sections over $\mathbb{F}_q$ is $|\mathcal{L}_{1,2,12n}(\mathbb{F}_q)| = (q^{9n+2} + q^{9n+1} - q^{9n-1} - q^{9n-2}) + (q^{6n+1} - q^{6n-1})$ by Proposition 6.19, we can explicitly estimate the sharp bound on $\mathcal{Z}_{1,2,\mathbb{F}_q(t)}(\mathcal{B})$ as the following,

$$
\begin{aligned}
\mathcal{Z}_{1,2,\mathbb{F}_q(t)}(\mathcal{B}) = \sum_{n=1}^{\left\lfloor \frac{log_q \mathcal{B}}{12} \right\rfloor} |\mathcal{L}_{1,2,12n}(\mathbb{F}_q)| &= \sum_{n=1}^{\left\lfloor \frac{log_q \mathcal{B}}{12} \right\rfloor} (q^{9n+2} + q^{9n+1} - q^{9n-1} - q^{9n-2}) + (q^{6n+1} - q^{6n-1}) \\
&= (q^2 + q^1 - q^{-1} - q^{-2}) \sum_{n=1}^{\left\lfloor \frac{log_q \mathcal{B}}{12} \right\rfloor} q^{9n} + (q^1 - q^{-1}) \sum_{n=1}^{\left\lfloor \frac{log_q \mathcal{B}}{12} \right\rfloor} q^{6n} \\
&\le (q^2 + q^1 - q^{-1} - q^{-2}) \left(q^9 + \cdots + q^{9 \cdot (\frac{log_q \mathcal{B}}{12})} \right) + (q^1 - q^{-1}) \left(q^6 + \cdots + q^{6 \cdot (\frac{log_q \mathcal{B}}{12})} \right) \\
&= (q^2 + q^1 - q^{-1} - q^{-2}) \cdot \frac{q^9 (\mathcal{B}^{\frac{3}{4}} - 1)}{(q^9 - 1)} + (q^1 - q^{-1}) \frac{q^6 (\mathcal{B}^{\frac{1}{2}} - 1)}{(q^6 - 1)} \\
&= \frac{(q^{11} + q^{10} - q^8 - q^7)}{(q^9 - 1)} \cdot (\mathcal{B}^{\frac{3}{4}} - 1) + \frac{(q^7 - q^5)}{(q^6 - 1)} \cdot (\mathcal{B}^{\frac{1}{2}} - 1)
\end{aligned}
$$

On the third line of the equations above, inequality becomes an equality if and only if $n := \frac{log_q \mathcal{B}}{12} \in \mathbb{N}$, i.e., $\mathcal{B} = q^{12n}$ with $n \in \mathbb{N}$. This implies that the acquired upper bound on $\mathcal{Z}_{1,2,\mathbb{F}_q(t)}(\mathcal{B})$ is a sharp estimate with the leading term of order $\mathcal{O}_q\left(\mathcal{B}^{\frac{3}{4}}\right)$ where $\mathcal{O}_q$-constant is an explicit rational function of q with the lower order terms of orders $\mathcal{O}_q\left(\mathcal{B}^{\frac{1}{2}}\right)$ and $\mathcal{O}_q(1)$. ■

Acknowledgments. We would like to thank Kenneth Ascher, Dori Bejleri, Benson Farb, Joe Harris, Ariyan Javanpeykar, Hunter Spink, Craig Westerland and Jesse Wolfson for helpful conversations and inspirations. Thanks also to Brian Conrad, Minhyong Kim and Qing Liu for a careful reading and valuable comments which greatly improved the exposition of this paper. Jun-Yong Park was supported by the Institute for Basic Science in Korea (IBS-R003-D1). Changho Han acknowledge the partial support of the Natural Sciences and Engineering Research Council of Canada (NSERC), [PGSD3-487436-2016], and of the Research and Training Group in Algebraic Geometry, Algebra and Number Theory, at the University of Georgia.

References

[AOV] D. Abramovich, M. Olsson and A. Vistoli, *Tame stacks in positive characteristic*, Annales de l'Institut Fourier, **58**, No. 4, (2008): 1057–1091.

[AW] E. Artin and G. Whaples, *Axiomatic characterization of fields by the product formula for valuations*, Bulletin of the American Mathematical Society, **51**, No. 7, (1945): 469–492.

[AW2] E. Artin and G. Whaples, *A note on axiomatic characterization of fields*, Bulletin of the American Mathematical Society, **52**, No. 4, (1946): 245–247.

[Baier] S. Baier, *Elliptic curves with square-free* Δ, International Journal of Number Theory, **12**, No. 3, (2016): 737–764.

[BB] S. Baier and T. D. Browning, *Inhomogeneous cubic congruences and rational points on Del Pezzo surfaces*, Journal für die reine und angewandte Mathematik, **680**, (2013): 69–151.

[Behrend] K. A. Behrend, *The Lefschetz trace formula for algebraic stacks*, Inventiones mathematicae, **112**, No. 1, (1993): 127–149.

[Behrens] M. Behrens, *A modular description of the* $K(2)$*-local sphere at the prime 3*, Topology, **45**, No. 2, (2006): 343–402.

[BM] V. V. Batyrev and Y. I. Manin, *Sur le nombre des points rationnels de hauteur borné des variétés algébriques*, Mathematische Annalen, **286**, No. 1-3, (1990): 27–43.

[BMc] A. Brumer and O. McGuinness, *The behavior of the Mordell-Weil group of elliptic curves*, Bulletin of the American Mathematical Society, **23**, No. 2, (1990): 375–382.

[Brumer] A. Brumer, *The average rank of elliptic curves I*, Inventiones mathematicae, **109**, No. 1, (1992): 445–472.

[de Jong] A. J. de Jong, *Counting elliptic surfaces over finite fields*, Moscow Mathematical Journal, **2**, No. 2, (2002): 281–311.

[DM] P. Deligne and D. Mumford, *The irreducibility of the space of curves of given genus*, Publications Mathématiques de l'I.H.É.S., **36**, (1969): 75–109.

[DR] P. Deligne and M. Rapoport, *Les Schémas de Modules de Courbes Elliptiques*, Modular Functions of One Variable II. Lecture Notes in Mathematics, **349**, (1973): 143–316.

[Duke] W. Duke, *Elliptic curves with no exceptional primes*, Comptes Rendus de l'Académie des Sciences - Series I - Mathematics, **325**, No. 8, (1997): 813–818.

[EH] D. Eisenbud and J. Harris, *The Kodaira dimension of the moduli space of curves of genus* $\geq$ *23*, Inventiones mathematicae, **90**, No. 2, (1987): 359–387.

[Ekedahl] T. Ekedahl, *The Grothendieck group of algebraic stacks*, arXiv:0903.3143, (2009).

[Faltings] G. Faltings, *Endlichkeitssätze für abelsche Varietäten über Zahlkörpern*, Inventiones mathematicae, **73**, No. 3, (1983): 349–366.

[Fedorchuk] M. Fedorchuk, *Moduli spaces of hyperelliptic curves with A and D singularities*, Mathematische Zeitschrift, **276**, No. 1-2, (2014): 299–328.

[FMT] J. Franke, Y. I. Manin and Y. Tschinkel, *Rational points of bounded height on Fano varieties*, Inventiones mathematicae, **95**, No. 2, (1989): 421–435.

[Fujino] O. Fujino, *Minimal Model Theory for Log Surfaces*, Publications of the Research Institute for Mathematical Sciences, **48**, No. 2, (2012): 339–371.

[FW] B. Farb and J. Wolfson, *Topology and arithmetic of resultants, I*, New York Journal of Mathematics, **22**, (2016): 801–821.

[GHM] C. Galindo, F. Hernando and F. Monserrat, *The log-canonical threshold of a plane curve*, Mathematical Proceedings of the Cambridge Philosophical Society, **160**, No. 3, (2016): 513–535.

[Grant] D. Grant, *A Formula for the Number of Elliptic Curves with Exceptional Primes*, Compositio Mathematica, **122**, No. 2, (2000): 151–164.

[Hassett] B. Hassett, *Moduli spaces of weighted pointed stable curves*, Advances in Mathematics, **173**, No. 2, (2003): 316–352.

[Hassett2] B. Hassett, *Classical and minimal models of the moduli space of curves of genus two*, Geometric Methods in Algebra and Number Theory, Progress in Mathematics, **235**, Birkhäuser Boston, (2005): 169—192.

[HM] J. Harris and D. Mumford, *On the Kodaira dimension of the moduli space of curves*, Inventiones mathematicae, **67**, No. 1, (1982): 23–86.

[HMe] M. Hill and L. Meier, *The* C_2*-spectrum* $Tmf_1(3)$ *and its invertible modules*, Algebraic & Geometric Topology, **17**, No. 4, (2017): 1953–2011.

[Hortsch] R. Hortsch, *Counting elliptic curves of bounded Faltings height*, Acta Arithmetica, **173**, No. 3, (2016): 239–253.

[HP] C. Han and J. Park, *Arithmetic of the moduli of semistable elliptic surfaces*, Mathematische Annalen, **375**, No. 3–4, (2019): 1745–1760.

[HS] R. Harron and A. Snowden, *Counting elliptic curves with prescribed torsion*, Journal für die reine und angewandte Mathematik, **729**, (2017): 151–170.

[Järvilehto] T. Järvilehto, *Jumping numbers of a simple complete ideal in a two-dimensional regular local ring*, Memoirs of the American Mathematical Society, **214**, No. 1009, (2011).

[Känel] R. V. Känel, *An effective proof of the hyperelliptic Shafarevich conjecture*, Journal de Théorie des Nombres de Bordeaux, **26**, No. 2, (2014): 507–530.

[KM] N. Katz and B. Mazur, *Arithmetic Moduli of Elliptic Curves*, Annals of Mathematics Studies, **108**, Princeton University Press (1985).

[Knudsen] F. F. Knudsen, *The projectivity of the moduli space of stable curves II, III*, Mathematica Scandinavica, **52**, No. 2, (1983): 161–199, 200–212.

[Landesman] A. Landesman, *The Torelli map restricted to the hyperelliptic locus*, arXiv:1911.02084, (2019).

[Levin] A. Levin, *Siegel's theorem and the Shafarevich conjecture*, Journal de Théorie des Nombres de Bordeaux, **24**, No. 3, (2012): 705–727.

[Liedtke] C. Liedtke, *Algebraic Surfaces in Positive Characteristic*, Birational Geometry, Rational Curves, and Arithmetic, Simons Symposia, Springer–Verlag New York (2013): 229–292.

[Liu] Q. Liu, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics, **6**, Oxford University Press (2002).

[Liu2] Q. Liu, *Modèles entiers des courbes hyperelliptiques sur un corps de valuation discrète*, Transactions of the American Mathematical Society, **348**, No. 11, (1996): 4577–4610.

[Liu3] Q. Liu, *Conducteur et discriminant minimal de courbes de genre 2*, Compositio Mathematica, **94**, No. 1, (1994): 51–79.

[Lockhart] P. Lockhart, *On the discriminant of a hyperelliptic curve*, Transactions of the American Mathematical Society, **342**, No. 2, (1994): 729–752.
[LP] Y. Lekili and A. Polishchuk, *A modular compactification of $\mathcal{M}_{1,n}$ from $\mathcal{A}_\infty$-structures*, Journal für die reine und angewandte Mathematik, **755**, (2019): 151–189.
[Mazur] B. Mazur, *Modular curves and the Eisenstein ideal*, Publications Mathématiques de l'I.H.É.S., **47**, (1977): 33–186.
[Meier] L. Meier, *Additive decompositions for rings of modular forms*, arXiv:1710.03461, (2017).
[Milne] J. S. Milne, *Jacobian Varieties*, Arithmetic Geometry, (Storrs, Conn., 1984), Springer, New York, NY, (1986): 167–212.
[Mordell] L. J. Mordell, *On the rational solutions of the indeterminate equation of the third and fourth degrees*, Proceedings of the Cambridge Philosophical Society, **21**, (1922): 179–192.
[Niles] A. Niles, *Moduli of elliptic curves via twisted stable maps*, Algebra and Number Theory, **7**, No. 9, (2013): 2141–2202.
[Olsson] M. Olsson, *Hom-stacks and restriction of scalars*, Duke Mathematical Journal, **134**, No. 1, (2006): 139–164.
[Olsson2] M. Olsson, *Algebraic Spaces and Stacks*, Colloquium Publications, **62**, American Mathematical Society (2016).
[Oort] F. Oort, *Hyperelliptic curves over number fields*, Classification of Algebraic Varieties and Compact Complex Manifolds, Lecture Notes in Mathematics, **412**, Springer–Verlag Berlin Heidelberg (1974): 211–218.
[OS] F. Oort and J. Steenbrink, *The local Torelli problem for algebraic curves*, Journées de Géometrie Algébrique d'Angers, Juillet 1979 / Algebraic Geometry, Angers, Sijthoff & Noordhoff. Alphen aan den Rijn-Germantown Md. 1980, (1979): 157–204.
[OU] F. Oort and K. Ueno, *Principally polarized abelian varieties of dimension two or three are Jacobian varieties*, Journal of the Faculty of Science, University of Tokyo, Section IA Mathematics, **20**, (1973): 377–381.
[Ogg] A. P. Ogg, *Elliptic curves and wild ramification*, American Journal of Mathematics, **89**, No. 1, (1967): 1–21.
[OSr] A. Obus and P. Srinivasan, *Conductor-discriminant inequality for hyperelliptic curves in odd residue characteristic*, arXiv:1910.02589, (2019).
[Parshin] A. N. Parshin, *Minimal models of curves of genus 2 and homomorphisms of abelian varieties defined over a field of finite characteristic*, Izv. Akad. Nauk SSSR Ser. Mat., **36**, No. 1, (1972): 67–109. (Russian). Mathematics of the USSR-Izvestiya, **6**, No. 1, (1972): 65–108. (English).
[PS] J. Park and H. Spink, *Motive of the moduli of rational curves on a weighted projective stack*, arXiv:1909.01030, To appear in PIMS Workshop on Arithmetic Topology conference proceedings, a special issue of Research in the Mathematical Sciences.
[Saito] T. Saito, *Conductor, discriminant, and the Noether formula of arithmetic surfaces*, Duke Mathematical Journal, **57**, No. 1, (1988): 151–173.
[Shafarevich] I. R. Shafarevich, *Algebraic number fields*, Proceedings of the International Congress of Mathematicians in Stockholm, 1962, Institut Mittag-Leffler, Djursholm, (1963): 163–176. (Russian). American Mathematical Society Translations: Series 2, **31**, American Mathematical Society (1963): 25–39. (English).
[Siegel] C. L. Siegel, *Über einige Anwendungen diophantischer Approximationen*, Abhandlungen der Preußischen Akademie der Wissenschaften. Physikalisch-mathematische Klasse, **1**, (1929): 41–69.
[Smyth] D. I. Smyth, *Modular compactifications of the space of pointed elliptic curves I*, Compositio Mathematica, **147**, No. 3, (2011): 877–913.
[Smyth2] D. I. Smyth, *Modular compactifications of the space of pointed elliptic curves II*, Compositio Mathematica, **147**, No. 6, (2011): 1843–1884.
[Stojanoska] V. Stojanoska, *Duality for topological modular forms*, Documenta Mathematica, **17**, (2012): 271–311.
[Srinivasan] P. Srinivasan, *Conductors and minimal discriminants of hyperelliptic curves with rational Weierstrass points*, arXiv:1508.05172, (2015).
[Tanaka] H. Tanaka, *Minimal Models and Abundance for Positive Characteristic Log Surfaces*, Nagoya Mathematical Journal, **216**, (2014): 1–70.
[Weil] A. Weil, *Zum Beweis des Torellischen Satzes*, Nachrichten der Akademie der Wissenschaften in Göttingen. II. Mathematisch-Physikalische Klasse, (1957): 33–53.

Department of Mathematics, University of Georgia, Athens, GA 30602
Email address: `Changho.Han@uga.edu`

Center for Geometry and Physics, Institute for Basic Science (IBS), Pohang 37673, Korea
Email address: `junepark@ibs.re.kr`