ARITHMETIC TOPOLOGY OF THE MODULI OF STABLE ELLIPTIC FIBRATIONS OVER $\mathbb{P}^1$

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Abstract. We determine the Picard group and the ℓ-adic étale cohomology with Frobenius weights of the Hom stack of degree n rational curves on a weighted projective stack which renders those of the moduli stack of stable elliptic fibrations over $\mathbb{P}^1_{\mathbb{F}_q}$ with a marked Weierstrass section or the moduli stack of generalized elliptic curves over $\mathbb{P}^1_{\mathbb{F}_q}$ with prescribed level structure.

1. Introduction

Fix a base field K, and define for the weight $\vec{\lambda} = (\lambda_0, \ldots, \lambda_N)$ with positive weights $\lambda_i \in \mathbb{N}$, the N-dimensional weighted projective stack $\mathcal{P}(\vec{\lambda}) := [(\mathbb{A}^{N+1}_{x_0,\ldots,x_N} \setminus 0)/\mathbb{G}_m]$ where $\zeta \in \mathbb{G}_m$ acts by $\zeta \cdot (x_0, \ldots, x_N) = (\zeta^{\lambda_0} x_0, \ldots, \zeta^{\lambda_N} x_N)$. Now for $n \in \mathbb{N}$, consider the Hom stack

$$\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda})) = \left\{ f : \mathbb{P}^1 \to \mathcal{P}(\vec{\lambda}) : f \text{ morphism with } f^*\mathcal{O}_{\mathcal{P}(\vec{\lambda})}(1) = \mathcal{O}_{\mathbb{P}^1}(n) \right\}$$

of degree n rational curves from $\mathbb{P}^1$ to $\mathcal{P}(\vec{\lambda})$. In this paper, we first compute the Picard group.

Theorem 1.1. *If $\mathrm{char}(K)$ does not divide $\lambda_i \in \mathbb{N}$ for every i, then the Picard group of the Hom stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ is a cyclic group generated by $\mathcal{O}_{\mathrm{Hom}_n(\mathbb{P}^1,\mathcal{P}(\vec{\lambda}))}(1)$ and is isomorphic to*

$$\mathrm{Pic}(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))) \cong \begin{cases} \mathbb{Z}/(n(\lambda_0 + \lambda_1))\mathbb{Z} & \text{for } N = 1, \\ \mathbb{Z} & \text{for } N > 1. \end{cases}$$

For the moduli stack $\mathcal{M}_{1,1}$ of smooth elliptic curves, $\mathrm{Pic}(\mathcal{M}_{1,1}) \cong \mathbb{Z}/12\mathbb{Z}$ by [Mumford, FO]. For the moduli stack $\mathcal{L}_{1,12n} := \mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$ of stable elliptic fibrations over $\mathbb{P}^1$ with discriminant degree $12n$ and a marked Weierstrass section, $\mathrm{Pic}(\mathcal{L}_{1,12n}) \cong \mathbb{Z}/10n\mathbb{Z}$ as $\overline{\mathcal{M}}_{1,1} \cong \mathcal{P}(4,6)$ over $Spec(\mathbb{Z}[1/6])$. This applies similarly for $\overline{\mathcal{M}}_{1,1}[\Gamma] \cong \mathcal{P}(a,b)$ (see Example 2.3). We remark that both moduli $\mathcal{M}_{1,1}$ & $\mathcal{L}_{1,12n}$ are smooth Deligne–Mumford stacks over $Spec(\mathbb{Z}[1/6])$ and not proper.

Next, we compute the ℓ-adic étale cohomology with Frobenius weights of $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))$.

Theorem 1.2. *Hom stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))$ over $\overline{\mathbb{F}}_q$ with $\mathrm{char}(\overline{\mathbb{F}}_q)$ not dividing a or b for $a, b \in \mathbb{N}$ has the following compactly supported ℓ-adic étale cohomology with geometric Frobenius weights*

$$H^i_{\acute{e}t,c}(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell) \cong \begin{cases} \mathbb{Q}_\ell(-((a+b)n+1)) & i = 2(a+b)n+2, \\ \mathbb{Q}_\ell(-((a+b)n-1)) & i = 2(a+b)n-1, \\ 0 & \text{else.} \end{cases}$$

By Poincaré duality, there are isomorphisms of Galois representations of mixed Tate type

$$H^i_{\acute{e}t}(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell) \cong \begin{cases} \mathbb{Q}_\ell(0) & i = 0, \\ \mathbb{Q}_\ell(-2) & i = 3, \\ 0 & \text{else.} \end{cases}$$

Our approach is to first perform an analysis of the evaluation morphism $\mathrm{ev}_\infty : \mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b)) \to \mathcal{P}(a,b)$ showing the cohomology of the fibers are fixed as in Proposition 2.12. Since the base $\mathcal{P}(a,b)$ is simply-connected, we show the associated ℓ-adic Leray spectral sequence is actually computable as Serre spectral sequence leading to ℓ-adic étale Betti numbers as in Proposition 2.13. Consequently, we have the Theorem through the Grothendieck-Lefschetz trace formula (see Theorem 2.9) applied to the motive/weighted $\mathbb{F}_q$-point count of the Hom stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))$ as in Theorem 2.5 acquired from [HP, PS, HP2].

Lastly, we note that the moduli stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ is an algebraic stack generalization of the famous moduli space $\mathrm{Hom}_n(\mathbb{P}^1, \mathbb{P}^N)$ of degree $n \in \mathbb{N}$ rational curves to a projective space $\mathbb{P}^N$ studied in depth by [Segal, CCMM, FP] and many others in various perspectives.

2. Picard group & ℓ-adic étale cohomology with eigenvalues of Frobenius

In this section, we will introduce some definitions, background materials on schemes, algebraic stacks and discuss basic properties of Picard group & ℓ-adic étale cohomology. For further details on the material presented here, the reader is referred to [Liu, Olsson2, DIS, Fringuelli, Behrend].

We first recall the definition of a weighted projective stack $\mathcal{P}(\vec{\lambda})$.

Definition 2.1. Fix a tuple of nondecreasing positive integers $\vec{\lambda} = (\lambda_0, \ldots, \lambda_N)$. The N-dimensional weighted projective stack $\mathcal{P}(\vec{\lambda}) = \mathcal{P}(\lambda_0, \ldots, \lambda_N)$ with the weight $\vec{\lambda}$ is defined as a quotient stack

$$\mathcal{P}(\vec{\lambda}) := \left[(\mathbb{A}^{N+1}_{x_0,\ldots,x_N} \setminus 0)/\mathbb{G}_m\right]$$

where $\zeta \in \mathbb{G}_m$ acts by $\zeta \cdot (x_0, \ldots, x_N) = (\zeta^{\lambda_0} x_0, \ldots, \zeta^{\lambda_N} x_N)$. In this case, the degree of x_i's are λ_i's respectively. A line bundle $\mathcal{O}_{\mathcal{P}(\vec{\lambda})}(m)$ is defined to be the line bundle associated to the sheaf of degree m homogeneous rational functions without poles on $\mathbb{A}^{N+1}_{x_0,\ldots,x_N} \setminus 0$.

Note that $\mathcal{P}(\vec{\lambda})$ is not an (effective) orbifold when $\gcd(\lambda_0, \ldots, \lambda_N) \neq 1$. In this case, the cyclic isotropy group scheme $\mu_{\gcd(\lambda_0,\ldots,\lambda_N)}$ is the generic stabilizer of $\mathcal{P}(\vec{\lambda})$. Nevertheless, the following proposition shows that it behaves well in most characteristics as a tame Deligne–Mumford stack:

Proposition 2.2. *The N-dimensional weighted projective stack $\mathcal{P}(\vec{\lambda}) = \mathcal{P}(\lambda_0, \ldots, \lambda_N)$ is a tame Deligne–Mumford stack over $Spec(\mathbb{Z}[1/\prod \lambda_i])$ with $\lambda_i \in \mathbb{N}$ for every i.*

Proof. For any algebraically closed field extension $\overline{K}$ of K, any point $y \in \mathcal{P}(\vec{\lambda})(\overline{K})$ is represented by the coordinates $(y_0, \ldots, y_N) \in \mathbb{A}^{N+1}_{\overline{K}}$ with its stabilizer group as the subgroup of $\mathbb{G}_m$ fixing $(y_0, \ldots, y_N)$. Hence, any stabilizer group of such $\overline{K}$-points is $\mathbb{Z}/u\mathbb{Z}$ where u divides λ_i for some i. Since the characteristic of K does not divide the orders of $\mathbb{Z}/\lambda_i\mathbb{Z}$ for any i, the stabilizer group of y is $\overline{K}$-linearly reductive. Hence, $\mathcal{P}(\vec{\lambda})$ is tame by [AOV, Theorem 3.2]. Note that the stabilizer groups constitute fibers of the diagonal $\Delta : \mathcal{P}(\vec{\lambda}) \to \mathcal{P}(\vec{\lambda}) \times_K \mathcal{P}(\vec{\lambda})$. Since $\mathcal{P}(\vec{\lambda})$ is of finite type and $\mathbb{Z}/u\mathbb{Z}$'s are unramified over K whenever u does not divide λ_i for some i, Δ is unramified as well. Therefore, $\mathcal{P}(\vec{\lambda})$ is also Deligne–Mumford by [Olsson2, Theorem 8.3.3]. ∎

The tameness is analogous to flatness for stacks in positive/mixed characteristic as it is preserved under base change by [AOV, Corollary 3.4]. Moreover, if a stack $\mathcal{X}$ is tame and Deligne–Mumford, then the formation of the coarse moduli space $c : \mathcal{X} \to X$ commutes with base change as well by [AOV, Corollary 3.3].

Example 2.3. There is a whole array of moduli stacks of curves that are isomorphic to $\mathcal{P}(\vec{\lambda})$ for various weights $\vec{\lambda} = (\lambda_0, \ldots, \lambda_N)$ over a field K with $\mathrm{char}(K)$ does not divide $\lambda_i \in \mathbb{N}$ for every i.

When $\mathrm{char}(K) \neq 2, 3$, [Hassett, Proposition 3.6] shows that one example is given by the proper Deligne–Mumford stack of stable elliptic curves $(\overline{\mathcal{M}}_{1,1})_K \cong [(\mathrm{Spec}\ K[a_4, a_6] - (0,0))/\mathbb{G}_m] = \mathcal{P}_K(4,6)$ by using the short Weierstrass equation $y^2 = x^3 + a_4x + a_6x$, where $\zeta \cdot a_i = \zeta^i a_i$ for $\zeta \in \mathbb{G}_m$ and $i = 4, 6$. This is no longer true when $\mathrm{char}(K) \in \{2, 3\}$, as the Weierstrass equations are more complicated.

Similarly, one could consider $\overline{\mathcal{M}}_{1,1}[\Gamma]$ of generalized elliptic curves with $[\Gamma]$-level structure by the work of Deligne and Rapoport [DR] (summarized in [Conrad, §2] and also in [Niles, §2]) such as $\overline{\mathcal{M}}_{1,1}[\Gamma_1(2)] \cong \mathcal{P}(2,4)$ [Behrens, §1.3], $\overline{\mathcal{M}}_{1,1}[\Gamma_1(3)] \cong \mathcal{P}(1,3)$ [HMe, Proposition 4.5], $\overline{\mathcal{M}}_{1,1}[\Gamma_1(4)] \cong \mathcal{P}(1,2)$ [Meier, Examples 2.1] and $\overline{\mathcal{M}}_{1,1}[\Gamma(2)] \cong \mathcal{P}(2,2)$ [Stojanoska, Proposition 7.1].

Also, one could consider $\overline{\mathcal{M}}_{1,m}(m-1)$ of m-marked $(m-1)$-stable curves of arithmetic genus one formulated originally by the works of [Smyth, Smyth2] such as $\overline{\mathcal{M}}_{1,2}(1) \cong \mathcal{P}(2,3,4)$, $\overline{\mathcal{M}}_{1,3}(2) \cong \mathcal{P}(1,2,2,3)$, $\overline{\mathcal{M}}_{1,4}(3) \cong \mathcal{P}(1,1,1,2,2)$ and $\overline{\mathcal{M}}_{1,5}(4) \cong \mathbb{P}(1,1,1,1,1,1) \cong \mathbb{P}^5$ as shown by [LP, Theorem 1.5.7.].

For higher genus $g \geq 2$, the moduli stack $\mathcal{H}_{2g}[2g-1]$ of quasi-admissible hyperelliptic genus g curves originally introduced by [Fedorchuk] is a Deligne–Mumford stack isomorphic to $\mathcal{P}(4, 6, 8, \ldots, 4g+2)$ if $\mathrm{char}(K) = 0$ [Fedorchuk, Proposition 4.2(1)] or $> 2g+1$ by [HP2, Proposition 5.9].

We refer interested readers to [HP, PS, HP2] for *arithmetic & birational geometric* considerations.

We now generalize the Hom stack formulation to $\mathcal{P}(\vec{\lambda})$ as follows:

Proposition 2.4. *Hom stack* $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ *with weight* $\vec{\lambda} = (\lambda_0, \ldots, \lambda_N)$, *which parameterize degree* $n \in \mathbb{N}$ *morphisms* $f : \mathbb{P}^1 \to \mathcal{P}(\vec{\lambda})$ *with* $f^*\mathcal{O}_{\mathcal{P}(\vec{\lambda})}(1) \cong \mathcal{O}_{\mathbb{P}^1}(n)$ *over Spec*$(\mathbb{Z}[1/\prod \lambda_i])$ *with* $\lambda_i \in \mathbb{N}$ *for every* i *is a smooth separated tame Deligne–Mumford stack of finite type that is an open substack of* $\mathcal{P}(\vec{\Lambda})$ *and* $\dim_K \left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))\right) = |\vec{\lambda}|n + N$ *where* $|\vec{\lambda}| := \sum_{i=0}^{N} \lambda_i$.

Proof. $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ is a smooth Deligne–Mumford stack by [Olsson, Theorem 1.1]. It is isomorphic to the quotient stack $[T/\mathbb{G}_m]$, admitting a smooth schematic cover $T \subset \left(\bigoplus_{i=0}^{N} H^0(\mathcal{O}_{\mathbb{P}^1}(\lambda_i \cdot n))\right) \setminus 0$, parameterizing the set of tuples $(u_0, \ldots, u_N)$ of sections with no common zero (here, we interpret $H^0(\mathcal{O}_{\mathbb{P}^1}(\lambda_i \cdot n))$ as an affine space over K of appropriate dimension, induced by its K-vector space structure). The $\mathbb{G}_m$ action on T is given by $\zeta \cdot (u_0, \ldots, u_N) = (\zeta^{\lambda_0} u_0, \ldots, \zeta^{\lambda_N} u_N)$. Note that

$$\dim T = \sum_{i=0}^{N} h^0(\mathcal{O}_{\mathbb{P}^1}(\lambda_i \cdot n)) = \sum_{i=0}^{N} (\lambda_i \cdot n + 1) = |\vec{\lambda}|n + N + 1,$$

implying that $\dim \mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda})) = |\vec{\lambda}|n + N$ since $\dim \mathbb{G}_m = 1$. Since $h^0(\mathcal{O}_{\mathbb{P}^1}(\lambda_i \cdot n)) = n\lambda_i + 1$, $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ is an open substack of $\mathcal{P}(\vec{\Lambda})$ of dimension $|\vec{\lambda}|n + N$ where

$$\vec{\Lambda} := (\underbrace{\lambda_0, \ldots, \lambda_0}_{n\lambda_0+1 \text{ times}}, \ldots, \underbrace{\lambda_N, \ldots, \lambda_N}_{n\lambda_N+1 \text{ times}}) .$$

As $\mathbb{G}_m$ acts on T properly with positive weights $\lambda_i \in \mathbb{N}$ for every i, the quotient stack $[T/\mathbb{G}_m]$ is separated. It is tame as in [AOV, Theorem 3.2] since $\mathrm{char}(K)$ does not divide λ_i for every i . ■

As a polynomial in $H^0(\mathcal{O}_{\mathbb{P}^1}(\lambda_i \cdot n)) \setminus 0$ is determined by its restriction away from $\infty = [0 : 1]$ (i.e., by evaluating $t = 1$), we may consider T as the set of not-necessarily-monic tuples of nonzero coprime polynomials $(u_0, \ldots, u_N)$ such that for a fixed $j \in \{0, \ldots, N\}$, $\deg u_j = \lambda_j \cdot n$ and the rests have $0 \leq \deg u_i \leq \lambda_i \cdot n$ for every $i \neq j$. Here the degree conditions are precisely ensuring that $\widetilde{u}_0, \ldots, \widetilde{u}_N$ do not share a common zero at ∞.

As $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))$ is a smooth Deligne–Mumford stack, the homomorphism of the Picard group to the first Chow group $c_1 : \mathrm{Pic}(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))) \to A^1(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b)))$ defined by the first Chern class is an isomorphism by [Fringuelli, (iii) Theorem 2.1.4.].

We are now ready to prove Theorem 1.1.

2.1. **Proof of Theorem 1.1.**

Proof. As explained by Proposition 2.4, we have that $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ is naturally an open substack of the weighted projective stack $\mathcal{P}(\vec{\Lambda})$ of dimension $|\vec{\lambda}|n + N$, whose complement $Z \subsetneq \mathcal{P}(\vec{\Lambda})$ is the locus of points $[s_0 : \dots s_N]$ where all s_i *do* vanish simultaneously at some $q \in \mathbb{P}^1$. We have an isomorphism $\mathrm{Pic}(\mathcal{P}(\vec{\Lambda})) = \mathbb{Z}$ by [Noohi2, Proposition 6.4.] and for the Picard group of $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$, we claim that the codimension of Z in $\mathcal{P}(\vec{\Lambda})$ is precisely N, implying that for $N \geq 2$ we have

$$\mathrm{Pic}(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))) = \mathrm{Pic}(\mathcal{P}(\vec{\Lambda}) \setminus Z) = \mathrm{Pic}(\mathcal{P}(\vec{\Lambda})) = \mathbb{Z}$$

by [Fringuelli, (ii) Theorem 2.1.4.]. To bound the codimension of Z, consider the incidence variety

$$\widehat{Z} = \{(q, [s_0, \dots, s_N]) \in \mathbb{P}^1 \times \mathcal{P}(\vec{\Lambda}) : s_i(q) = 0 \text{ for } i = 0, \dots, N\} \subseteq \mathbb{P}^1 \times \mathcal{P}(\vec{\Lambda}).$$

For a fixed $q \in \mathbb{P}^1$, the set of points $(s_0, \dots, s_N) \in \vec{\Lambda}$ such that all s_i vanish at q is a linear subspace of codimension $N+1$ in $\vec{\Lambda}$, and in fact, the projection $\widehat{Z} \to \mathbb{P}^1$ is a projective bundle of the corresponding rank. In particular, we have that $\widehat{Z} \subset \mathbb{P}^1 \times \mathcal{P}(\vec{\Lambda})$ is irreducible of codimension $N+1$. On the other hand, we have that $Z \subset \mathcal{P}(\vec{\Lambda})$ is the image of $\widehat{Z}$ under the projection $\mathbb{P}^1 \times \mathcal{P}(\vec{\Lambda}) \to \mathcal{P}(\vec{\Lambda})$. Since the fibres of $\widehat{Z} \to Z$ are finite, it follows that the codimension of Z is indeed N.

On the one hand, for $N = 1$ we have that the subvariety Z of $\mathcal{P}(\vec{\Lambda})$ is cut out by the resultant $\mathrm{Res}(s_0, s_1)$. The resultant is an irreducible polynomial in the coefficients of s_0, s_1 ([GKZ, Chapter 8, Proposition-Definition 1.1]), which is homogeneous of degree $n(\lambda_0 + \lambda_1)$.

Then, by [Fringuelli, (iv) Theorem 2.1.4.] we have an exact sequence

$$\mathbb{Z} \to \mathrm{Pic}(\mathcal{P}(\vec{\Lambda})) \to \mathrm{Pic}(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))) \to 0$$

where the first map is defined by $1 \mapsto 1 \cdot V(\mathrm{Res})$. As $\mathrm{Pic}(\mathcal{P}(\vec{\Lambda})) \cong \mathbb{Z}$, we have that

$$\mathrm{Pic}(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))) \cong \mathbb{Z}/\mathrm{Deg}(\mathrm{Res})\mathbb{Z} = \mathbb{Z}/((\lambda_0 + \lambda_1)n)\mathbb{Z}.$$

■

We recall the *arithmetic aspect* (i.e., motive/weighted $\mathbb{F}_q$-point count) of $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$.

Theorem 2.5 (Proposition 4.5 & Corollary 4.8 of [HP2]). *Fix the weight $\vec{\lambda} = (\lambda_0, \dots, \lambda_N)$ with $|\vec{\lambda}| := \sum_{i=0}^{N} \lambda_i$. Then the weighted point count of the Hom stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ over $\mathbb{F}_q$ is*

$$\begin{aligned}
\#_q\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))\right) &= \left(\sum_{i=0}^{N} q^i\right) \cdot \left(q^{|\vec{\lambda}|n} - q^{|\vec{\lambda}|n-N}\right) \\
&= q^{|\vec{\lambda}|n} \cdot \left(q^N + q^{N-1} + \cdots + q^2 + q^1 - q^{-1} - q^{-2} - \cdots - q^{-(N-1)} - q^{-N}\right) \\
&= q^{|\vec{\lambda}|n+N} + \cdots + q^{|\vec{\lambda}|n+1} - q^{|\vec{\lambda}|n-1} - \cdots - q^{|\vec{\lambda}|n-N}
\end{aligned}$$

$\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ has the stratification by $\mathrm{Poly}_1^{(d_1,\dots,d_m)}$ as in [HP, §4.1] and [HP2, §4.1].

Definition 2.6. Fix $m \in \mathbb{Z}_{>0}$ and $d_1, \ldots, d_m \in \mathbb{N}$, define

$$\mathrm{Poly}_1^{(d_1,\ldots,d_m)} := \{(f_1(z), \ldots, f_m(z)) : f_i(z) \text{ are monic and coprime of degrees } d_i\}.$$

Equivalently, $\mathrm{Poly}_1^{(d_1,\ldots,d_m)}$ is the complement of the resultant hypersurface $\mathcal{R}^{(d_1,\ldots,d_m)} \subset \prod_{i=1}^{m} \mathrm{Sym}^{d_i} \mathbb{A}^1 = \prod_{i=1}^{m} \mathbb{A}^{d_i}$ of tuples of monic polynomials $(f_1(z), \ldots, f_m(z))$ of degrees d_i which share a factor.

Arithmetic of $\mathrm{Poly}_1^{(d_1,\ldots,d_m)}$ was studied in [HP, PS, HP2] which was inspired by the work of [Segal, FW]. As the resultant hypersurface $\mathcal{R}^{(d_1,\ldots,d_m)}$ is defined over $\mathbb{Z}$, it makes sense to define $\mathrm{Poly}_1^{d_1,\ldots,d_m} = (\prod_{i=1}^{m} \mathbb{A}^{d_i}) \setminus \mathcal{R}^{(d_1,\ldots,d_m)}$ as a variety over $\mathbb{Z}$.

Generalizing the proof of [FW, Theorem 1.2] and [HP, Proposition 18] with the correction from [PS, Proposition 3.1.], we find the exact $\mathbb{F}_q$-point count of $\mathrm{Poly}_1^{(d_1,\ldots,d_m)}$.

Proposition 2.7 (Proposition 4.4. of [HP2])**.** *Fix $0 \le d_1 \le d_2 \le \ldots \le d_m$. Then for any prime power q :*

$$|\mathrm{Poly}_1^{(d_1,\ldots,d_m)}(\mathbb{F}_q)| = \begin{cases} q^{d_1+\ldots+d_m} - q^{d_1+\ldots+d_m-m+1}, & \text{if } d_1 \neq 0 \\ q^{d_1+\ldots+d_m}, & \text{if } d_1 = 0 \end{cases}$$

Before proving Theorem 1.2, we recall the definition of a weighted point count of an algebraic stack $\mathcal{X}$ over $\mathbb{F}_q$:

Definition 2.8. The weighted point count of $\mathcal{X}$ over $\mathbb{F}_q$ is defined as a sum:

$$\#_q(\mathcal{X}) := \sum_{x \in \mathcal{X}(\mathbb{F}_q)/\sim} \frac{1}{|\mathrm{Aut}(x)|},$$

where $\mathcal{X}(\mathbb{F}_q)/\sim$ is the set of $\mathbb{F}_q$–isomorphism classes of $\mathbb{F}_q$–points of $\mathcal{X}$ (i.e., the set of non–weighted points of $\mathcal{X}$ over $\mathbb{F}_q$), and we take $\frac{1}{|\mathrm{Aut}(x)|}$ to be 0 when $|\mathrm{Aut}(x)| = \infty$.

A priori, the weighted point count can be ∞, but when $\mathcal{X}$ is of finite type, then the stratification of $\mathcal{X}$ by schemes as in [Behrend, Proof of Lemma 3.2.2] implies that $\mathcal{X}(\mathbb{F}_q)/\sim$ is a finite set, so that $\#_q(\mathcal{X}) < \infty$.

As the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ acting on the ℓ-adic étale cohomology is a procyclic group that is topologically generated by the geometric Frobenius, the task of finding the eigenvalues of Frobenius map can be achieved through the trace formula where the cardinality of the fixed set of $\mathrm{Frob}_q : \mathfrak{X}(\overline{\mathbb{F}}_q) \to \mathfrak{X}(\overline{\mathbb{F}}_q)$ coincides with $\#_q(\mathfrak{X})$ the weighted $\mathbb{F}_q$–point count. We recall the *Grothendieck-Lefschetz trace formula* [Sun, Behrend] for Artin stacks of finite type over finite fields.

Theorem 2.9 (Theorem 1.1. of [Sun])**.** *Let $\mathcal{X}$ be an Artin stack of finite type over $\mathbb{F}_q$. Let Frob_q be the geometric Frobenius on $\mathcal{X}$. Let ℓ be a prime number different from the characteristic of $\mathbb{F}_q$, and let $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$ be an isomorphism of fields. For an integer i, let $H^i_{\acute{e}t,c}(\mathcal{X}_{/\overline{\mathbb{F}}_q}; \overline{\mathbb{Q}}_\ell)$ be the cohomology with compact support of the constant sheaf $\overline{\mathbb{Q}}_\ell$ on $\mathcal{X}$ as in* [LO]*. Then the infinite sum regarded as a complex series via ι*

$$\sum_{i \in \mathbb{Z}} (-1)^i \cdot tr\big(\,\mathrm{Frob}_q^* : H^i_c(\mathcal{X}_{/\overline{\mathbb{F}}_q}; \overline{\mathbb{Q}}_\ell) \to H^i_c(\mathcal{X}_{/\overline{\mathbb{F}}_q}; \overline{\mathbb{Q}}_\ell)\big) \tag{1}$$

is absolutely convergent to the weighted point count $\#_q(\mathcal{X})$ of $\mathcal{X}$ over $\mathbb{F}_q$.

As $\mathfrak{X} = \mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ is a separated Deligne–Mumford stack of finite type over $\mathbb{F}_q$ (see Proposition 2.4), the corresponding (compactly supported) ℓ-adic étale cohomology for prime number ℓ invertible in $\mathbb{F}_q$ is finite dimensional as a $\mathbb{Q}_\ell$-algebra, making the trace formula hold in $\mathbb{Q}_\ell$-coefficients. Also by the smoothness, the Poincaré duality $H^i_{\acute{e}t,c}(\mathfrak{X}_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell) \cong H^{2dim(\mathfrak{X})-i}_{\acute{e}t}(\mathfrak{X}_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell(-dim(\mathfrak{X})))^\vee$ hold for the dual of ordinary étale cohomology as in [LO, 7.3.1. Theorem]. From now on, we identify the dual spaces with the ordinary étale cohomology as finite dimensional $\mathbb{Q}_\ell$-vector spaces. We denote by $\mathbb{Q}_\ell(-i)$ the $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-representation of rank 1 on which the *geometric Frobenius* acts by q^{-i} and the *arithmetic Frobenius* acts by q^i. Also, recall that the étale cohomology of a stack $\mathfrak{X}$ is *pure* if the absolute values of the eigenvalues of Frob_q on $H^i_{\acute{e}t}$ are all $q^{i/2}$ and if otherwise it is *mixed*. The group $H^i_{\acute{e}t}(\mathfrak{X}_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$ is of *Tate type* if the eigenvalues of Frob_q are all equal to integer powers of q.

We are now ready to prove Theorem 1.2.

2.2. **Proof of Theorem 1.2.**

Proof. We first recall the ℓ-adic étale cohomology of the smooth proper base $\mathcal{P}(\vec{\lambda})$.

Proposition 2.10. *There are isomorphisms of Galois representations*

$$H^i_{\acute{e}t}\left(\mathcal{P}(\vec{\lambda})_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell\right) \cong \begin{cases} \mathbb{Q}_\ell\left(-\frac{i}{2}\right) & i \in \{0, 2, \dots, 2N\} \\ 0 & else \end{cases}$$

Proof. This follows from [Kawasaki, Theorem 1.] where we use the well-known fact that the cohomology of a smooth Deligne–Mumford stack $\mathfrak{X}$ with projective coarse moduli space X is pure. ■

Next, we need the ℓ-adic étale cohomology of $\mathrm{Poly}_1^{(\lambda_0 \cdot n, \dots, \lambda_N \cdot n)}$ which follows from [FW, HP, HP2].

Proposition 2.11. *There are isomorphisms of Galois representations*

$$H^i_{\acute{e}t}\left(\mathrm{Poly}_1^{(\lambda_0 \cdot n, \dots, \lambda_N \cdot n)}{}_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell\right) \cong \begin{cases} \mathbb{Q}_\ell\left(0\right) & i = 0 \\ \mathbb{Q}_\ell\left(-N\right) & i = 2N-1 \\ 0 & else \end{cases}$$

Proof. We can express $\mathrm{Poly}_1^{(\lambda_0 \cdot n, \dots, \lambda_N \cdot n)}$ as the quotient of a finite group $S_{\lambda_0 \cdot n} \times \cdots \times S_{\lambda_N \cdot n}$ acting on the complement of a hyperplane arrangement in $\mathbb{A}^{|\vec{\lambda}|n}$. It is well known by [Lehrer, Shapiro, Kim] that the geometric Frobenius Frob_q^* acts by q^i on $H^i_{\acute{e}t}$ of the complement of a hyperplane arrangement. By transfer and Poincaré duality, we see that the geometric Frobenius acts by $q^{-(|\vec{\lambda}|n)+i}$ on $H^i_{\acute{e}t,c}(\mathrm{Poly}_1^{(\lambda_0 \cdot n, \dots, \lambda_N \cdot n)}{}_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$. Applying the *Grothendieck-Lefschetz trace formula* for compactly supported étale cohomology (as $\mathrm{Poly}_1^{(\lambda_0 \cdot n, \dots, \lambda_N \cdot n)}$ is smooth but not projective variety), we see that

$$\left|\mathrm{Poly}_1^{(\lambda_0 \cdot n, \dots, \lambda_N \cdot n)}(\mathbb{F}_q)\right| = \sum_{i=0}^{2|\vec{\lambda}|n} (-1)^i \cdot q^{-(|\vec{\lambda}|n)+i} \cdot \dim_{\mathbb{Q}_\ell}\left(H^i_{\acute{e}t,c}(\mathrm{Poly}_1^{(\lambda_0 \cdot n, \dots, \lambda_N \cdot n)}{}_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)\right)$$

and we use the $\mathbb{F}_q$-point count of Proposition 2.7 then Poincaré duality to conclude the above. ■

We now consider the morphism $\mathrm{ev}_\infty : \mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a, b)) \to \mathcal{P}(a, b)$, set degree $n \in \mathbb{Z}_{\geq 1}$.

Proposition 2.12. *Evaluation morphism* $\mathrm{ev}_\infty : \mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b)) \to \mathcal{P}(a,b)$ *mapping an unbased morphism* $f : \mathbb{P}^1 \to \mathcal{P}(a,b)$ *of degree* n *with the chosen basepoint* $\infty \in \mathbb{P}^1$ *to the basepoint of the target* $f(\infty) \in \mathcal{P}(a,b)$ *is a smooth surjective morphism. The* ℓ*-adic étale cohomology of the fiber* $H^j_{\acute{e}t}(\mathrm{ev}_\infty^{-1}(p)_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$ *at each geometric point* $p \in \mathcal{P}(a,b)$ *is isomorphic to* $H^j_{\acute{e}t}(\mathrm{Poly}_1^{(an,bn)}{}_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$ *the* ℓ*-adic étale cohomology of the space* $\mathrm{Poly}_1^{(an,bn)} \cong \mathrm{Hom}^*_n(\mathbb{P}^1, \mathcal{P}(a,b))$ *of degree* n *based morphisms.*

Proof. For $\mathcal{P}(a,b) = \mathcal{P}(\lambda_0, \lambda_1)$, recall that T is an open dense $\mathbb{G}_m$-invariant subscheme of an affine space $\oplus H^0(\mathcal{O}_{\mathbb{P}^1}(\lambda_i \cdot n))$ not containing zero, where for each i, $\mathbb{G}_m$ acts on $H^0(\mathcal{O}_{\mathbb{P}^1}(\lambda_i \cdot n))$ with weight λ_i and $(u_0, u_1) \in \oplus H^0(\mathcal{O}_{\mathbb{P}^1}(\lambda_i \cdot n))$ of polynomials in $K[x]$ lies in T with no common roots in $\overline{K}$, where $\deg u_i \le n\lambda_i$ for each i with equality for some i. As an open subset of an affine space, T is smooth. The pre-quotient evaluation morphism $\widetilde{\mathrm{ev}}_\infty : T \to \mathbb{A}^2 \setminus 0$ which is the restriction of a linear map from $\oplus H^0(\mathcal{O}_{\mathbb{P}^1}(\lambda_i \cdot n)) \to \mathbb{A}^2$ remembering the coefficients of $x^{\lambda_i \cdot n}$ in u_i is a surjective linear map and thus $\widetilde{\mathrm{ev}}_\infty$ is a smooth surjective morphism. This implies that a $\mathbb{G}_m$-equivariant evaluation morphism $\mathrm{ev}_\infty : \mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b)) = [T/\mathbb{G}_m] \to \mathcal{P}(a,b) = [(\mathbb{A}^2 \setminus 0)/\mathbb{G}_m]$ is a smooth surjective morphism as well by [Stacks, Tag 02VL]. Note that for such maps and a closed subscheme $(x,y) \in \mathbb{A}^2 \setminus 0$ on which $\mathbb{G}_m$ acts, $\mathrm{ev}_\infty^{-1}([x:y]) \cong \widetilde{\mathrm{ev}}_\infty^{-1}((x,y))/\mathbb{G}_m$. This implies in particular that the special fiber at $[1:0]$ or $[0:1]$ is smooth and also complement of the hyperplane arrangement by the same principle that the generic fiber such as over at $[1:1]$ is complement of the hyperplane arrangement. The fiber at the generic stacky points $[x:y] \in \mathcal{P}(a,b) \setminus \{[0:1],[1:0]\}$ with $\mu_{gcd(a,b)}$ stabilizer can be accounted for by looking at $\mathrm{ev}_\infty^{-1}([1:1])$ where we have $\mathrm{ev}_\infty^{-1}([1:1]) \cong \mathrm{Poly}_1^{(an,bn)}$. This is because a pair of monic coprime polynomials (u,v) with $\deg(u,v) = (an, bn)$ determines, and is determined by, a morphism $f : \mathbb{P}^1 \to \mathcal{P}(a,b)$ such that $f^*\mathcal{O}_{\mathcal{P}(a,b)}(1) \simeq \mathcal{O}_{\mathbb{P}^1}(n)$ and $f(\infty) = [1:1]$ via $f(z) := [u(z) : v(z)]$ under the weighted homogeneous coordinate of the target stack $\mathcal{P}(a,b)$. By the $\mathbb{G}_m$ action, this implies that the cohomology $H^j_{\acute{e}t}(\mathrm{ev}_\infty^{-1}(p)_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$ away from the special orbifold points $[0:1]$ or $[1:0]$ is isomorphic to $H^j_{\acute{e}t}(\mathrm{Poly}_1^{(an,bn)}{}_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$. For the cohomology of the special fibers at $[0:1]$ or $[1:0]$, we have the fiber at $[0:1]$ with μ_b stabilizer that has the stratification $\mathrm{ev}_\infty^{-1}([0:1]) = \left(\coprod_{k=1}^{an} \mathbb{G}_m \times \mathrm{Poly}_1^{(an-k,bn)}\right)$ as it must map to $[0:1]$ by ev_∞. By applying the Grothendieck relation [Ekedahl, §1] on the stratification as in [HP], we have $[\mathbb{G}_m \times \mathrm{Poly}_1^{(an-1,bn)}] + [\mathbb{G}_m \times \mathrm{Poly}_1^{(an-2,bn)}] + \cdots + [\mathbb{G}_m \times \mathrm{Poly}_1^{(0,bn)}] = \mathbb{L}^{an+bn} - \mathbb{L}^{an+bn-1} = [\mathrm{Poly}_1^{(an,bn)}]$. Similarly for $[1:0]$ with μ_a stabilizer and the stratification of the fiber $\mathrm{ev}_\infty^{-1}([1:0]) = \left(\coprod_{k=1}^{bn} \mathbb{G}_m \times \mathrm{Poly}_1^{(an,bn-k)}\right)$ leading to $[\mathbb{G}_m \times \mathrm{Poly}_1^{(an,bn-1)}] + [\mathbb{G}_m \times \mathrm{Poly}_1^{(an,bn-2)}] + \cdots + [\mathbb{G}_m \times \mathrm{Poly}_1^{(an,0)}] = \mathbb{L}^{an+bn} - \mathbb{L}^{an+bn-1} = [\mathrm{Poly}_1^{(an,bn)}]$ which implies that both of the special fibers which are smooth and complement of the hyperplane arrangement have the same exact $\mathbb{F}_q$-point count as $\mathrm{Poly}_1^{(an,bn)}$ by [PS, Proposition 3.1.]. We apply the Proposition 2.11 to conclude that the ℓ-adic étale cohomology of the special fibers at $[1:0]$ or $[0:1]$ are isomorphic to $H^j_{\acute{e}t}(\mathrm{ev}_\infty^{-1}(p)_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell) \cong H^j_{\acute{e}t}(\mathrm{Poly}_1^{(an,bn)}{}_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$ as well. ∎

We note that the smoothness of the surjective evaluation morphism $\mathrm{ev}_\infty : \mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b)) \to \mathcal{P}(a,b)$ together with finite cohomology of the fibers at every geometric points (in fact, isomorphic as expected from the isomorphisms of the specialization maps $(\mathrm{R}\,\mathrm{ev}_{\infty *}\,\mathbb{Q}_\ell)_y \to (\mathrm{R}\,\mathrm{ev}_{\infty *}\,\mathbb{Q}_\ell)_x$) implies that the pushforwards $\mathrm{R}^j\,\mathrm{ev}_{\infty *}\,\mathbb{Q}_\ell$ is locally constant sheaves on $\mathcal{P}(a,b)$ (constructible?). As the base $\mathcal{P}(a,b)$ is simply-connected (c.f. [Noohi, Example 9.2]), we can identify $\mathrm{R}^j\,\mathrm{ev}_{\infty *}\,\mathbb{Q}_\ell$ as a local system that is isomorphic to the cohomology of the fiber $\mathrm{R}^j\,\mathrm{ev}_{\infty *}\,\mathbb{Q}_\ell = H^j_{\acute{e}t}(\mathrm{ev}_\infty^{-1}{}_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell) = H^j_{\acute{e}t}(\mathrm{Poly}_1^{(an,bn)}{}_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$ as in Proposition 2.12 which implies that the ℓ-adic Leray spectral sequence (c.f. [Behrend, Theorem 1.2.5]) has the following E_2 page.

$$E_2^{i,j} = \begin{cases} H^i_{\acute{e}t}(\mathcal{P}(a,b)_{/\overline{\mathbb{F}}_q}; H^j_{\acute{e}t}(\mathrm{Poly}_1^{(an,bn)}{}_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)) & \text{if } i,j \geq 0 \\ 0 & \text{else} \end{cases}$$

and the spectral sequence converges to $H^{i+j}_{\acute{e}t}(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$.

Proposition 2.13. *Hom stack* $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ *has the* ℓ*-adic étale cohomology* $H^i_{\acute{e}t}(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$ *that is one-dimensional in degree* $i \in \{0, 3\}$ *and vanishes for all other* i *showing the cohomology is independent of degree* $n \in \mathbb{N}$*. By the Poincaré duality, we have* $H^i_{\acute{e}t,c}(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$ *one-dimensional for* $i = 2(a+b)n+2$*,* $i = 2(a+b)n-1$ *and vanishes for all other* i*.*

Proof. It suffices to show that the transgression differential is an isomorphism as we have only one possible non-zero differential $d_2^{(2,0)} : H^2_{\acute{e}t}\left(\mathcal{P}(a,b)_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell\right) \to H^1_{\acute{e}t}\left(\mathrm{Poly}_1^{(an,bn)}{}_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell\right)$ on the E_2 page. Assume for the purpose of contradiction that the transgression differential is trivial (i.e., $d_2^{(2,0)} = 0$), then we would have the Picard number of $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))$ is ≥ 1 as $\mathrm{Pic}(\mathcal{P}(a,b)) = \mathbb{Z}$ by [Noohi2, Proposition 6.4.]. This is not the case, however, as the Picard group computation in Theorem 1.1 shows $\mathrm{Pic}(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))) \cong \mathbb{Z}/(n(a+b))\mathbb{Z}$ having the Picard number zero. The contradiction implies that we have the transgression differential is an isomorphism (i.e., $d_2^{(2,0)} = \pm\, m \neq 0$) for $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))$ showing that the ℓ-adic étale Betti numbers $\dim_{\mathbb{Q}_\ell}\left(H^i_{\acute{e}t}(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)\right)$ are equal to one for $i = 0, 3$ and vanishes for other i. ∎

As $H^i_{\acute{e}t,c}(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$ are all one dimensional, the trace of Frob_q on each of these $\mathbb{Q}_\ell$-vector spaces is just the corresponding eigenvalue λ_i of Frob_q.

When $i = 2(an+bn)+2$, the connectedness of $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))$ together with the Poincaré duality implies that $\lambda_{2(an+bn)+2} = q^{(a+b)n+1}$

Plugging all of the above into the *Grothendieck-Lefschetz trace formula* (2.9) :

$$\begin{aligned}\#_q(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))) = q^{(a+b)n+1} - q^{(a+b)n-1} &= \lambda_{2(an+bn)+2} - \lambda_{2(an+bn)-1} \\ &= q^{(a+b)n+1} - \lambda_{2(an+bn)-1}\end{aligned}$$

which implies that $\lambda_{2(an+bn)-1} = q^{(a+b)n-1}$ as claimed.

This finishes the proof of Theorem 1.2.

∎

Acknowledgments. It is a pleasure to thank Dori Bejleri, David Hansen, Changho Han, Johannes Schmitt, Hunter Spink, Craig Westerland, Jesse Wolfson and Mingjia Zhang for many helpful conversations. The author was supported by the Institute for Basic Science in Korea (IBS-R003-D1) and the Max Planck Institute for Mathematics.

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