ENUMERATING ALGEBRAIC CURVES AND ABELIAN VARIETIES OVER GLOBAL FUNCTION FIELDS WITH LOWER ORDER TERMS

CHANGHO HAN AND JUN–YONG PARK

Abstract. Given asymptotic counts in number theory, a question of Venkatesh asks what is the topological nature of lower order terms. We consider the arithmetic aspect of the inertia stack of an algebraic stack over finite fields to partially answer this question. Subsequently, we enumerate algebraic curves and abelian varieties with precise lower order terms ordered by bounded discriminant height over $\mathbb{F}_q(t)$ which renders new heuristics over $\mathbb{Q}$ through the global fields analogy.

1. Introduction

In [GGW, Problem 5], Akshay Venkatesh asks the following question:

What is the topological meaning of secondary terms appearing in asymptotic counts in number theory?

In many interesting number theory problems (e.g., counting number fields, arithmetic curves or abelian varieties over a number field) one has not only a main term in the asymptotic count, but a secondary term or more. For example, the number of cubic fields of discriminant up to $\mathcal{B}$ is

$$a\mathcal{B} + b\mathcal{B}^{5/6} + \text{lower order terms}$$

We have very little understanding of these lower order terms. They are not just of theoretical interest: when one tries to verify the conjectures numerically, one finds that the secondary terms are dominant in the computational range. Following the framework of the Weil conjectures, they do not correspond to stable cohomology classes (these are the main terms), but to some kind of slightly weaker structure, which is still much better behaved than cohomology near middle degree.

Note that the moduli functors we wish to enumerate are often represented by algebraic stacks rather than by schemes (or algebraic spaces) due to the presence of non-trivial automorphisms of the objects we wish to parameterize. If we consider a finite field analogue, the traditional approaches to count the number of rational points on the moduli spaces do not render the lower order terms. This is because the Grothendieck-Lefschetz trace formula (relating point counts and ℓ-adic cohomologies) for algebraic stacks as in [Behrend] counts the rational points with weights (given a rational point x, its weight is $\frac{1}{\mathrm{Aut}(x)}$). Instead, we must acquire the number $|\mathcal{X}(\mathbb{F}_q)/\sim|$ of $\mathbb{F}_q$–isomorphism classes of $\mathbb{F}_q$–points of the algebraic stack $\mathcal{X}$, i.e., the non–weighted point count of $\mathcal{X}$ over $\mathbb{F}_q$. In this regard, the coarse moduli space $c : \mathcal{X} \to X$ is insufficient as $|X(\mathbb{F}_q)| \neq |\mathcal{X}(\mathbb{F}_q)/\sim|$.

This discrepancy naturally raises the following question:

What arithmetic invariant of a specific geometric object $\mathcal{Y}$ is equal to the non–weighted point count $|\mathcal{X}(\mathbb{F}_q)/\sim|$ of the algebraic stack $\mathcal{X}$ over $\mathbb{F}_q$?

Given an algebraic stack of interest $\mathcal{X}$ over $\mathbb{F}_q$, we clarify the arithmetic role of its *inertia stack* $\mathcal{I}(\mathcal{X})$ which parameterizes pairs (x, ξ), where $x \in \mathcal{X}$ and ξ is the conjugate class of $g \in \mathrm{Aut}(x)$.

Theorem 1.1. *Let $\mathcal{X}$ be an algebraic stack over $\mathbb{F}_q$ of finite type with quasi-separated finite type diagonal and let $\mathcal{I}(\mathcal{X})$ be the inertia stack of $\mathcal{X}$. Then,*

$$|\mathcal{X}(\mathbb{F}_q)/\sim| = \#_q(\mathcal{I}(\mathcal{X}))$$

where $\#_q(\mathcal{I}(\mathcal{X}))$ is the weighted point count of the inertia stack $\mathcal{I}(\mathcal{X})$ over $\mathbb{F}_q$.

Before drawing the connection of this Theorem to Venkatesh's question, let us first consider a simpler problem instead where we want to find the non-weighted point count $|\mathcal{X}(\mathbb{F}_q)/\sim|$ of a Deligne-Mumford moduli stack $\mathcal{X}/\mathbb{F}_q$ of finite type with affine diagonal (here, bounded height $\mathcal{B}$ doesn't play any role). On one hand, Theorem 1.1 gives us the exact answer as a single number $\#_q(\mathcal{I}(\mathcal{X}))$, so we can't immediately distinguish the leading and lower order terms. On the other hand, the Grothendieck-Lefschetz trace formula for algebraic stacks by [Behrend, LO, Sun] (see Theorem 2.2) equates this number as the alternating sum of trace of geometric Frobenius:

$$\#_q(\mathcal{I}(\mathcal{X})) = \sum_{i=0}^{2\dim \mathcal{I}(\mathcal{X})} (-1)^i \cdot tr\big(\mathrm{Frob}_q^* : H^i_{\acute{e}t,c}(\mathcal{I}(\mathcal{X})_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell) \to H^i_{\acute{e}t,c}(\mathcal{I}(\mathcal{X})_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)\big)$$

It is standard to consider the natural grading determined by degree i of compactly-supported cohomologies; then, the top degree cohomology (when $i = 2\dim\mathcal{I}(\mathcal{X})$) can be interpreted as the main leading term and rest of the lower degree cohomologies corresponds to the lower order terms.

However, it is very difficult in general to compute the ℓ-adic cohomologies with Frobenius weights of $\mathcal{I}(\mathcal{X})$ for an arbitrary algebraic stack $\mathcal{X}$ with desired conditions, since the geometry of $\mathcal{I}(\mathcal{X})$ can be quite complicated. For example, even if $\mathcal{X}$ is irreducible, $\mathcal{I}(\mathcal{X})$ can be disconnected, with many irreducible components of different dimensions. Also, $\mathcal{I}(\mathcal{X})$ may have intersecting irreducible components which are possibly singular. Nonetheless, we note that the lower degree, compactly-supported cohomologies of $\mathcal{I}(\mathcal{X})$ are affected by the compactly-supported cohomologies of lower-dimensional irreducible components of $\mathcal{I}(\mathcal{X})$ corresponding to different automorphisms.

In a given counting problem of number theory (such as in the cubic field example), one must be aware of the discriminant involved. In some of these cases, the moduli stack $\mathcal{X}$ is not quasi-compact (so cannot be of finite type), but is rather a disjoint union of connected or irreducible components $\mathcal{X}_{\mathcal{B}}$ of finite type, indexed by the ranges of values $0 < ht(\Delta) \le \mathcal{B}$ of the height of discriminant up to $\mathcal{B}$. For example, the function field analogue of the cubic field example above translates as counting the number of degree 3 (also called trigonal in some literature) covers of $\mathbb{P}^1_{\mathbb{F}_q}$ defined over $\mathbb{F}_q$ with bounded discriminant height up to $\mathcal{B}$; in this case, the moduli stack is a disjoint union of Hurwitz stacks (which are irreducible) of degree 3 covers indexed by the degree of discriminant (or equivalently, by the height of discriminant which is $ht(\Delta) = q^{\deg \mathrm{Br(f)}}$ where Br(f) is the branch divisor of degree 3 cover $f : C \to \mathbb{P}^1_{\mathbb{F}_q}$). One could expect to obtain an analogous formula for degree d covers of $\mathbb{P}^1_{\mathbb{F}_q}$ as in the case of cubic fields, but this remains open as the arithmetic geometry of the Hurwitz stacks is not cyclotomic (i.e., the objects we wish to parameterize & enumerate have various non-abelian automorphism groups).

Putting all of the above considerations together, Venkatesh's question over finite fields is then equivalent to understanding the algebro-topological meaning of the lower order terms of the counting function $N(\mathcal{B})$ as a function of the bounded height $\mathcal{B}$ of discriminant:

$$N(\mathcal{B}) := |\hat{\mathcal{X}}_{\mathcal{B}}(\mathbb{F}_q)/\sim| = \#_q(\mathcal{I}(\hat{\mathcal{X}}_{\mathcal{B}})) = \sum_{\mathcal{B}' \le \mathcal{B}} \#_q(\mathcal{I}(\mathcal{X}_{\mathcal{B}'})), \quad \hat{\mathcal{X}}_{\mathcal{B}} := \bigsqcup_{\mathcal{B}' \le \mathcal{B}} \mathcal{X}_{\mathcal{B}'}$$

Then, the lower order terms of $N(\mathcal{B})$ are determined by the growth pattern of the weighted point count $\#_q(\mathcal{I}(\hat{\mathcal{X}}_{\mathcal{B}}))$. If all the automorphism groups of $\mathcal{I}(\hat{\mathcal{X}}_{\mathcal{B}})$ are affine then applying the Grothendieck-Lefschetz trace formula together with the generalized Deligne's upper bound of weights (see [Sun, Theorem 1.4.]), we conclude that the lower order terms of $N(\mathcal{B})$ are detected by looking at how the lower degree, compactly-supported, ℓ-adic cohomologies with geometric Frobenius weights of $\mathcal{I}(\hat{\mathcal{X}}_{\mathcal{B}})$ change with respect to $\mathcal{B}$. A priori, however, the general mechanism that precisely determines which connected component(s) of $\mathcal{I}(\hat{\mathcal{X}}_{\mathcal{B}})$ contribute(s) to a given lower order term of a specific order is unclear. In essence, this analysis, which is an extension of the framework of the Weil conjectures to the rational points of inertia stacks of arithmetic moduli stacks, provides a partial answer to the question of Venkatesh.

Due to the inherent complexity of inertia stacks in general, we instead focus on irreducible algebraic stacks $\mathcal{X}$ of finite type (with conditions on the diagonal). Furthermore, we restrict to the case when $\mathcal{X} \cong [U/G]$ is a quotient stack, which is a testing ground for the strategy above. Then, the inertia stack $\mathcal{I}(\mathcal{X})$ turns out to be a quotient stack as well, of the form $[R_\Delta/G]$ (see Corollary 2.5). If $\mathcal{X}$ is furthermore Deligne-Mumford with affine diagonal, then $\mathcal{I}(\mathcal{X})$ decomposes into a disjoint union of $\mathcal{X}$ and other components, which are fixed loci of nontrivial elements of G (see (5) and Definition 2.6 for more details). By using the idea of cut-and-paste by Grothendieck in $K_0(\mathrm{Stck}_K)$ (this is a natural generalization of the Grothendieck ring of varieties, see Definition 4.1), we acquire the motive $\{\mathcal{I}(\mathcal{X})\}$ which renders $\#_q(\mathcal{I}(\mathcal{X}))$ to be a polynomial in q through the decomposition of $\mathcal{I}(\mathcal{X})$ whenever every piece is reasonably simple (For instance, when $G = \mathbb{G}_m$ then the motive $\{\mathcal{I}(\mathcal{X})\} = \{R_\Delta\}/\{G\}$ by Lemma 4.2).

As an application of this strategy, we consider the Hom stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ parameterizing the degree $n \in \mathbb{N}$ morphisms $f : \mathbb{P}^1 \to \mathcal{P}(\vec{\lambda})$ of rational curves on a weighted projective stack $\mathcal{P}(\vec{\lambda})$ (see Definition 3.1) with $f^*\mathcal{O}_{\mathcal{P}(\vec{\lambda})}(1) \simeq \mathcal{O}_{\mathbb{P}^1}(n)$. Since the Hom stacks are quotient stacks by Remark 3.7, the strategy works out nicely and we prove that the exact weighted point count $\#_q\left(\mathcal{I}\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))\right)\right)$ of the inertia stack over $\mathbb{F}_q$ can be acquired to be a polynomial in q which in turn provides the exact non–weighted point count $\left|\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))(\mathbb{F}_q)/\sim\right|$ of the Hom stack over $\mathbb{F}_q$ by Theorem 1.1. Hom stacks are important classes of Deligne-Mumford stacks as numerous arithmetic moduli problems can naturally be approximated (if not identified) to weighted projective stacks or Hom stacks under mild condition on the characteristic of the base field K. For example, both authors in [HP] showed that $\mathcal{L}_{1,12n} := \mathrm{Hom}_n(\mathbb{P}^1_{\mathbb{F}_q}, \mathcal{P}_{\mathbb{F}_q}(4,6))$ represents the moduli stack of stable elliptic fibrations over $\mathbb{P}^1_{\mathbb{F}_q}$ with discriminant degree $12n$ (as $(\overline{\mathcal{M}}_{1,1})_{\mathbb{F}_q} \cong \mathcal{P}_{\mathbb{F}_q}(4,6)$ is the moduli stack of stable elliptic curves when $2, 3 \nmid q$), and computed the exact non-weighted point count $|\mathcal{L}_{1,12n}(\mathbb{F}_q)/\sim|$ over $\mathbb{F}_q$ through the motive $\{\mathcal{L}_{1,12n}\} \in K_0(\mathrm{Stck}_K)$ (see also [PS]).

For the moduli stack of genus $g \geq 2$ fibrations over $\mathbb{P}^1_{\mathbb{F}_q}$, however, it is difficult to acquire the arithmetic invariants of $\mathcal{I}\left(\mathrm{Hom}(\mathbb{P}^1, \overline{\mathcal{M}}_g)\right)$ due to the global geometry of the Deligne–Mumford moduli stack $\overline{\mathcal{M}}_g$ of stable genus g curves formulated in [DM]. For example, the coarse moduli space $\overline{M}_g$ is of general type for $g \geq 24$ by the fundamental works of Harris, Mumford and Eisenbud in [HM, EH] which in turn makes the study of (rational) curves on $\overline{\mathcal{M}}_{g\geq 24}$ ineffective for counting stable curves of sufficiently high genus over $\mathbb{P}^1_{\mathbb{F}_q}$. Instead, we consider the following strategy:

Could we approximate $\overline{\mathcal{M}}_g$ by $\mathcal{P}(\vec{\lambda_g})$ and show that the non-weighted point count of the Hom stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda_g}))$ is an upper bound for the non-weighted point count of the Moduli stack $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_g)$ of stable genus $g \geq 2$ fibrations over $\mathbb{P}^1_{\mathbb{F}_q}$?

Remarkably, the strategy can be executed successfully if we restrict to hyperelliptic genus $g \geq 2$ curves. Firstly, all smooth genus 2 curves are hyperelliptic, thus $\mathcal{M}_2 \cong \mathcal{H}_2$. In general, recall that an *odd degree* hyperelliptic curve has a marked rational Weierstrass point at ∞. In this paper, we will concentrate on the moduli substack $\mathcal{H}_{g,\underline{1}} \subset \mathcal{M}_{g,1}$ of hyperelliptic genus $g \geq 2$ curves with 1 marked rational Weierstrass point (which has the same dimension as $\mathcal{H}_g$) as we focus on counting odd degree hyperelliptic genus $g \geq 2$ curves. Since $\mathcal{H}_{g,\underline{1}}$ is not proper, we consider the proper moduli stack $\overline{\mathcal{H}}_{g,\underline{1}} := \overline{\mathcal{H}_{g,\underline{1}}} \subset \overline{\mathcal{M}}_{g,1}$ (meaning the reduced closure) of stable odd degree hyperelliptic curves. Similar to $\overline{\mathcal{M}}_g$, extracting the exact arithmetic invariants of $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{H}}_{g,\underline{1}})$ is challenging, so we consider (upto some conditions on characteristic of $\mathbb{F}_q$) a different extension of smooth odd degree hyperelliptic curves such that the compactified moduli stack is a weighted projective stack, originally introduced as a special case of [Fedorchuk, Definition 2.5]:

Definition 1.2. Fix an integral reduced K-scheme B, where $\mathrm{char}(K) \neq 2$. A flat family $u : C \to B$ of genus $g \geq 2$ curves is *quasi–admissible* if every geometric fiber has at worst A_{2g-1}-singularity (i.e., étale locally defined by $x^2 + y^m$ for some $0 < m \leq 2g$), and factors through a separable morphism $\phi : C \to H$ of degree 2 where H is a $\mathbb{P}^1$-bundle over B with a distinguished section (often called ∞) which is a connected component of the branch locus of u.

For example, if $\mathrm{char}(K) > 2g+1$ or 0, then a quasi–admissible curve over any K-scheme B can be written as

$$y^2 = f(x) = x^{2g+1} + a_4 x^{2g-1} + a_6 x^{2g-2} + a_8 x^{2g-3} + \cdots + a_{4g+2}, \tag{1}$$

where a_i's are appropriate sections of suitable line bundles on B where not all of them simultaneously vanish at anywhere on B. This equation should be thought of as a generalized Weierstrass equation. Here, we identify the section at ∞ as the locus missed by the above affine equation. This identification is a consequence of Proposition 5.9, where we show that the Deligne–Mumford moduli stack $\mathcal{H}_{2g}[2g-1]$ of quasi–admissible curves of genus g is isomorphic to the weighted projective stack $\mathcal{P}(\vec{\lambda_g})$ for $\vec{\lambda_g} := (4, 6, 8, \ldots, 4g+2)$ over base field K with $\mathrm{char}(K) = 0$ or $> 2g+1$. This is the desired approximation of $\overline{\mathcal{H}}_{g,\underline{1}}$ as the set of isomorphism classes of quasi-admissible curves of genus g over $\mathbb{P}^1_K$ contains the set of isomorphism classes of stable odd degree hyperelliptic genus g curves over $\mathbb{P}^1_K$ (with smooth generic fiber) by Theorem 5.12 (explicitly via the birational transformation from one family of curves to another).

To effectively count the non–weighted $\mathbb{F}_q$–points of the Hom stack $\mathrm{Hom}(\mathbb{P}^1, \mathcal{H}_{2g}[2g-1] \cong \mathcal{P}(4, 6, 8, \ldots, 4g+2))$, we need to impose a notion of *bounded height* on those $\mathbb{F}_q$–points. Thanks to the works of Lockhart and Liu, we have a natural definition (see Definition 5.16) of a hyperelliptic discriminant Δ_g of quasi–admissible curves as in [Lockhart, Liu2]. In fact, it is a homogeneous polynomial of degree $4g(2g+1)$ on variables a_i's, where each a_i has degree i (a_i's are as in equation (1) where $B = \mathbb{P}^1_{\mathbb{F}_q}$ in this case). Moreover, since $\mathcal{P}(4, 6, 8, \ldots, 4g+2)$ carries a primitive ample line bundle $\mathcal{O}_{\mathcal{P}(4,6,8,\ldots,4g+2)}(1)$, the degree of the discriminant Δ_g of a given quasi-admissible fibration $f : \mathbb{P}^1 \to \mathcal{H}_{2g}[2g-1] \cong \mathcal{P}(4, 6, 8, \ldots, 4g+2)$ is equal to $4g(2g+1)n$ where $f^*\mathcal{O}_{\mathcal{P}(4,6,8,\ldots,4g+2)}(1) \cong \mathcal{O}_{\mathbb{P}^1}(n)$. Therefore, the Hom stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(4, 6, 8, \ldots, 4g+2))$ parameterizing such morphisms is the moduli stack of quasi-admissible genus $g \geq 2$ fibrations of a fixed discriminant degree $|\Delta_g| \cdot n = 4g(2g+1)n$. Consequently, we acquire the exact weighted point count $\mathcal{I}\left(\mathcal{L}_{g,|\Delta_g|\cdot n}\right)$ of the inertia stack over $\mathbb{F}_q$ which is equal to the exact non–weighted point count $\left|\mathcal{L}_{g,|\Delta_g|\cdot n}(\mathbb{F}_q)/\sim\right|$ of the moduli stack over $\mathbb{F}_q$ by Theorem 1.1.

Theorem 1.3. *If* $\mathrm{char}(\mathbb{F}_q) > 2g+1$*, the number* $\left|\mathcal{L}_{g,|\Delta_g|\cdot n}(\mathbb{F}_q)/\sim\right|$ *of* $\mathbb{F}_q$*–isomorphism classes of quasi-admissible hyperelliptic genus* $g = 2, 3, 4$ *fibrations over* $\mathbb{P}^1_{\mathbb{F}_q}$ *with a marked Weierstrass point and a hyperelliptic discriminant of degree* $|\Delta_g| \cdot n = 4g(2g+1)n$ *is equal to*

$$|\mathcal{L}_{2,40n}(\mathbb{F}_q)/\sim| = 2 \cdot q^{28n} \cdot (q^3 + q^2 + q^1 - q^{-1} - q^{-2} - q^{-3}) + \delta(4, q-1) \cdot 2 \cdot q^{12n} \cdot (q^1 - q^{-1})$$

$$\begin{aligned}|\mathcal{L}_{3,84n}(\mathbb{F}_q)/\sim| &= 2 \cdot q^{54n} \cdot (q^5 + \cdots + q^1 - q^{-1} - \cdots - q^{-5}) \\ &\quad + \delta(4, q-1) \cdot 2 \cdot q^{24n} \cdot (q^2 + q^1 - q^{-1} - q^{-2}) + \delta(6, q-1) \cdot 4 \cdot q^{18n} \cdot (q^1 - q^{-1})\end{aligned}$$

$$\begin{aligned}|\mathcal{L}_{4,144n}(\mathbb{F}_q)/\sim| &= 2 \cdot q^{88n} \cdot (q^7 + \cdots + q^1 - q^{-1} - \cdots - q^{-7}) \\ &\quad + \delta(4, q-1) \cdot 2 \cdot q^{40n} \cdot (q^3 + q^2 + q^1 - q^{-1} - q^{-2} - q^{-3})\end{aligned}$$

$$+ \delta(6, q-1) \cdot 4 \cdot q^{36n} \cdot (q^2 + q^1 - q^{-1} - q^{-2}) + \delta(8, q-1) \cdot 4 \cdot q^{24n} \cdot (q^1 - q^{-1})$$

where

$$\delta(a,b) := \begin{cases} 1 & \text{if } a \text{ divides } b, \\ 0 & \text{otherwise.} \end{cases}$$

For genus $g \geq 5$, the corresponding exact non–weighted point count $\left|\mathcal{L}_{g,|\Delta_g|\cdot n}(\mathbb{F}_q)/\sim\right|$ of the moduli stack $\mathcal{L}_{g,|\Delta_g|\cdot n}$ over $\mathbb{F}_q$ can be similarly worked out.

Given a stable odd hyperelliptic genus $g \geq 2$ curve X over $\mathbb{P}^1_{\mathbb{F}_q}$ with $\mathrm{char}(\mathbb{F}_q) > 2g+1$, define the *height* of the hyperelliptic discriminant $\Delta_g(X)$ to be $ht(\Delta_g(X)) := q^{\deg \Delta_g(X)} = q^{4g(2g+1)n}$ (see Definition 6.5). Then, for a positive real number $\mathcal{B}$, we define the counting function $\mathcal{Z}_{g,\mathbb{F}_q(t)}(\mathcal{B})$.

$$\mathcal{Z}_{g,\mathbb{F}_q(t)}(\mathcal{B}) := |\{\text{Stable odd hyperelliptic genus } g \geq 2 \text{ curves over } \mathbb{P}^1_{\mathbb{F}_q} \text{ with } 0 < ht(\Delta_g(X)) \leq \mathcal{B}\}|$$

As we have seen (by Theorem 5.12) above, whenever $\mathrm{char}(\mathbb{F}_q) > 2g+1$, counting the number $\mathcal{Z}'_{g,\mathbb{F}_q(t)}(\mathcal{B})$ of quasi–admissible genus $g \geq 2$ curves over $\mathbb{P}^1_{\mathbb{F}_q}$ renders an upper bound for counting the number $\mathcal{Z}_{g,\mathbb{F}_q(t)}(\mathcal{B})$ of stable odd hyperelliptic genus $g \geq 2$ curves over $\mathbb{P}^1_{\mathbb{F}_q}$ with generically smooth fibers. Using Theorem 1.3, we can compute $\mathcal{Z}'_{g,\mathbb{F}_q(t)}(\mathcal{B})$, leading to (see §6):

Theorem 1.4 (Estimate on $\mathcal{Z}_{g,\mathbb{F}_q(t)}(\mathcal{B})$)**.** *If $\mathrm{char}(\mathbb{F}_q) > 2g+1$, then the function $\mathcal{Z}_{g,\mathbb{F}_q(t)}(\mathcal{B})$, which counts the number of stable hyperelliptic genus $g \geq 2$ curves X with a marked Weierstrass point over $\mathbb{P}^1_{\mathbb{F}_q}$ ordered by $0 < ht(\Delta_g(X)) = q^{4g(2g+1)n} \leq \mathcal{B}$, satisfies:*

$$\begin{aligned}
\mathcal{Z}_{2,\mathbb{F}_q(t)}(\mathcal{B}) &\leq a_{2,2} \cdot \mathcal{B}^{\frac{7}{10}} + a_{2,4} \cdot \mathcal{B}^{\frac{3}{10}} + b_{2,4} \\
\mathcal{Z}_{3,\mathbb{F}_q(t)}(\mathcal{B}) &\leq a_{3,2} \cdot \mathcal{B}^{\frac{9}{14}} + a_{3,4} \cdot \mathcal{B}^{\frac{2}{7}} + a_{3,6} \cdot \mathcal{B}^{\frac{3}{14}} + b_{3,6} \\
\mathcal{Z}_{4,\mathbb{F}_q(t)}(\mathcal{B}) &\leq a_{4,2} \cdot \mathcal{B}^{\frac{11}{18}} + a_{4,4} \cdot \mathcal{B}^{\frac{5}{18}} + a_{4,6} \cdot \mathcal{B}^{\frac{1}{4}} + a_{4,8} \cdot \mathcal{B}^{\frac{1}{6}} + b_{4,8}
\end{aligned}$$

where $\delta(a,b)$ is as in Theorem 1.3, and for each $g, m \in \mathbb{N}_{\geq 2}$, $a_{g,2}(q), a_{g,2m}(q, \delta(2m, q-1))$, and $b_{g,2m}(q, \delta(4, q-1), \delta(6, q-1), \ldots, \delta(2m, q-1))$ are explicit rational functions as in Theorem 6.6.

For higher genus $g \geq 5$, the corresponding upper bound on $\mathcal{Z}_{g,\mathbb{F}_q(t)}(\mathcal{B})$ rendering a closed-form formula with precise lower order terms can be similarly worked out through Theorem 6.6.

Note 1.5. When $g = 2$ and $\delta(4, q-1) = 1$, the secondary term $a_{2,4} \cdot \mathcal{B}^{\frac{3}{10}}$ of $\mathcal{Z}'_{2,\mathbb{F}_q(t)}(\mathcal{B})$ comes exactly from the lower dimensional irreducible component $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(4,8)) \subsetneq \mathcal{I}\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(4,6,8,10))\right)$ corresponding to μ_4 generic stabilizer (see Proposition 4.10 and Theorem 6.6). We also note that the lower order term $b_{2,4}$ of zeroth order is a sum of contributions from both the higher dimensional irreducible component $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(4,6,8,10)) \subsetneq \mathcal{I}\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(4,6,8,10))\right)$ and the lower dimensional irreducible component $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(4,8)) \subsetneq \mathcal{I}\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(4,6,8,10))\right)$. The emergence of lower order terms illustrated here continues similarly for higher genus $g \geq 3$ case $\mathcal{Z}'_{g,\mathbb{F}_q(t)}(\mathcal{B})$ with $\delta(r, q-1)$ for each corresponding component with μ_r generic stabilizer. This gives the partial answer to function field analog of Venkatesh's question for counting stable curves of genus $g \geq 2$.

Using this, we obtain another application regarding the enumeration of abelian varieties of dimension 2, i.e., *abelian surfaces* over global function fields. By the local (i.e., infinitesimal) Torelli theorem in [OS, Theorem 2.6 and 2.7] and [Milne, Theorem 12.1], the Torelli map $\tau_2 : \mathcal{M}_2 \hookrightarrow \mathcal{A}_2$, which sends a smooth projective genus 2 curve X defined over a field K to its principally polarized Jacobian $(\mathrm{Jac}(X), \lambda_\theta)/K$ (where λ_θ is the theta divisor of $\mathrm{Jac}(X)$), is an open immersion. Furthermore, it is shown in [OU, 4. Theorem] (see also [Weil, Satz 2]) that given a principally polarized abelian surface (A, λ) over a field K, after a finite extension of scalars, is isomorphic to the canonically principally polarized (generalized) Jacobian variety $(\mathrm{Jac}(X), \lambda_\theta)$ of a stable genus

2 curve X. Recall that if a curve X has good reduction at a place $v \in S$ then so does its Jacobian $\mathrm{Jac}(X)$. We have the following enumeration (see Corollary 6.8 for higher genus Jacobians).

Corollary 1.6 (Estimate on $\mathcal{N}_{2,\mathbb{F}_q(t)}(\mathcal{B})$)**.** *If char*$(\mathbb{F}_q) \neq 2,3,5$*, then the function* $\mathcal{N}_{2,\mathbb{F}_q(t)}(\mathcal{B})$*, which counts the number of principally polarized abelian surfaces* $A = \mathrm{Jac}(X)$ *where* X *is a stable genus 2 curve with a marked Weierstrass point over* $\mathbb{P}^1_{\mathbb{F}_q}$ *ordered by* $0 < ht(\Delta_2(X)) = q^{40n} \leq \mathcal{B}$*, satisfies:*

$$\mathcal{N}_{2,\mathbb{F}_q(t)}(\mathcal{B}) \leq a_{2,2} \cdot \mathcal{B}^{\frac{7}{10}} + a_{2,4} \cdot \mathcal{B}^{\frac{3}{10}} + b_{2,4}$$

$$a_{2,2}(q) = 2 \cdot \frac{(q^{31}+q^{30}+q^{29}-q^{27}-q^{26}-q^{25})}{(q^{28}-1)}, \quad a_{2,4}(q) = \delta(4,q-1) \cdot 2 \cdot \frac{(q^{13}-q^{11})}{(q^{12}-1)}$$

$$b_{2,4}(q) = -2 \cdot \frac{(q^{31}+q^{30}+q^{29}-q^{27}-q^{26}-q^{25})}{(q^{28}-1)} - \delta(4,q-1) \cdot 2 \cdot \frac{(q^{13}-q^{11})}{(q^{12}-1)}$$

where

$$\delta(4,q-1) := \begin{cases} 1 & \textit{if } 4 \textit{ divides } q-1, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Theorem 1.4 provides an explicit upper bound with precise lower order terms on the number of stable genus 2 curves with a marked Weierstrass point over $\mathbb{P}^1_{\mathbb{F}_q}$ with $\mathrm{char}(\mathbb{F}_q) \neq 2,3,5$. The upper bound follows from the properties of the Torelli map τ_2 as all principally polarized abelian surfaces are isomorphic to Jacobians of genus 2 curves of compact type (i.e., genus 2 curve with dual graph equal to tree) (c.f., [OS, Theorem 2.6 and 2.7] & [OU, 4. Theorem]). ∎

Theorem 1.4 and Corollary 1.6 suggest that the counting fuctions of interest behave similar to the sum of power functions of the discriminant height $\mathcal{B}$. However, the linear sum of power function approximations do not differ by functions of q (which are constants with respect to $\mathcal{B}$), as the counting functions are staircase functions and the sequences of the lengths of individual steps are increasing geometric series. This is exactly because in the function field case (counting varieties over $\mathbb{P}^1_{\mathbb{F}_q}$), the possible values that the height of the discriminant can take are very sparse in $\mathbb{N}$. Translating this to number field case by global fields analogy (where we count varieties over Spec $\mathbb{Z}$ instead), we expect the sparsity issue to go away; so it seems reasonable to use the sum of power functions as the guess for estimating the behavior of the counting functions over $\mathbb{Z}$.

To formulate our conjectures precisely, we define the notion of *"Asymptotic tightness"*.

Definition 1.7. Two functions $f, g : \mathbb{N} \to \mathbb{R}$ are asymptotically tight (by constants), denoted by $f \sim_{\mathcal{O}(1)} g$, if $f - g \in \mathcal{O}(1)$. Moreover, f is asymptotically tight to a sum of power functions if there are constants $c_1, \ldots, c_m \in \mathbb{R}$ such that $f \sim_{\mathcal{O}(1)} \sum c_i \cdot x^{a_i}$

Asymptotic tightness allows us to extract both the leading term and lower order terms of a function f, as long as f grows faster than any constant functions. Using this, we formulate the following conjectures through the global fields analogy (§ 6) based on Theorem 1.4 and Corollary 1.6:

Conjecture 1.8 (Heuristic on $\mathcal{N}_{2,\mathbb{Q}}(\mathcal{B})$)**.** The function $\mathcal{N}_{2,\mathbb{Q}}(\mathcal{B})$, which counts the number of principally polarized abelian surfaces $A = \mathrm{Jac}(X)$ where X is a stable genus 2 curve over $\mathbb{Z}$ with $0 < ht(\Delta_2(X)) \leq \mathcal{B}$, satisfies:

$$\mathcal{N}_{2,\mathbb{Q}}(\mathcal{B}) \sim_{\mathcal{O}(1)} a \cdot \mathcal{B}^{\frac{7}{10}} + b \cdot \mathcal{B}^{\frac{3}{10}} + c$$

having the leading term of degree $\mathcal{O}\left(\mathcal{B}^{\frac{7}{10}}\right)$ with lower order terms of degrees $\mathcal{O}\left(\mathcal{B}^{\frac{3}{10}}\right)$ and $\mathcal{O}(1)$.

And similarly for the higher genus $g \geq 3$ hyperelliptic curves we formulate the following:

Conjecture 1.9 (Heuristic on $\mathcal{Z}_{g,\mathbb{Q}}(\mathcal{B})$)**.** The function $\mathcal{Z}_{g,\mathbb{Q}}(\mathcal{B})$, which counts the number of stable hyperelliptic genus $g \geq 3$ curves with a marked rational Weierstrass point over $\mathbb{Z}$ with $0 < ht(\Delta_g) \leq \mathcal{B}$, satisfies:

$$\mathcal{Z}_{3,\mathbb{Q}}(\mathcal{B}) \sim_{\mathcal{O}(1)} a \cdot \mathcal{B}^{\frac{9}{14}} + b \cdot \mathcal{B}^{\frac{2}{7}} + c \cdot \mathcal{B}^{\frac{3}{14}} + d$$
$$\mathcal{Z}_{4,\mathbb{Q}}(\mathcal{B}) \sim_{\mathcal{O}(1)} a \cdot \mathcal{B}^{\frac{11}{18}} + b \cdot \mathcal{B}^{\frac{5}{18}} + c \cdot \mathcal{B}^{\frac{1}{4}} + d \cdot \mathcal{B}^{\frac{1}{6}} + e$$

For higher genus $g \geq 5$, the corresponding degrees of the leading term and the lower order terms can be similarly worked out through Theorem 1.4.

In Appendix, we determine the sharp estimate for counting elliptic curves with prescribed level structure or multiple marked points over $\mathbb{P}^1_{\mathbb{F}_q}$ and formulate the following Heuristics.

Conjecture 1.10 (Heuristic on $\mathcal{Z}^{[\Gamma]}_{1,\mathbb{Q}}(\mathcal{B})$)**.** The function $\mathcal{Z}^{[\Gamma]}_{1,\mathbb{Q}}(\mathcal{B})$, which counts the number of generalized elliptic curves with $[\Gamma]$-structures over $\mathbb{Z}$ ordered by $0 < ht(\Delta_1) \leq \mathcal{B}$, satisfies:

$$\mathcal{Z}^{[\Gamma_1(2)]}_{1,\mathbb{Q}}(\mathcal{B}) \sim_{\mathcal{O}(1)} a \cdot \mathcal{B}^{\frac{1}{2}} + b$$
$$\mathcal{Z}^{[\Gamma_1(3)]}_{1,\mathbb{Q}}(\mathcal{B}) \sim_{\mathcal{O}(1)} a \cdot \mathcal{B}^{\frac{1}{3}} + b$$
$$\mathcal{Z}^{[\Gamma_1(4)]}_{1,\mathbb{Q}}(\mathcal{B}) \sim_{\mathcal{O}(1)} a \cdot \mathcal{B}^{\frac{1}{4}} + b$$
$$\mathcal{Z}^{[\Gamma(2)]}_{1,\mathbb{Q}}(\mathcal{B}) \sim_{\mathcal{O}(1)} a \cdot \mathcal{B}^{\frac{1}{3}} + b$$

Conjecture 1.11 (Heuristic on $\mathcal{Z}_{1,m,\mathbb{Q}}(\mathcal{B})$)**.** The function $\mathcal{Z}_{1,m,\mathbb{Q}}(\mathcal{B})$, which counts the number of m-marked $(m-1)$-stable genus one curves for $2 \leq m \leq 5$ over $\mathbb{Z}$ ordered by $0 < ht(\Delta_1) \leq \mathcal{B}$, satisfies:

$$\mathcal{Z}_{1,2,\mathbb{Q}}(\mathcal{B}) \sim_{\mathcal{O}(1)} a \cdot \mathcal{B}^{\frac{3}{4}} + b \cdot \mathcal{B}^{\frac{1}{2}} + c$$
$$\mathcal{Z}_{1,3,\mathbb{Q}}(\mathcal{B}) \sim_{\mathcal{O}(1)} a \cdot \mathcal{B}^{\frac{2}{3}} + b \cdot \mathcal{B}^{\frac{1}{3}} + c$$
$$\mathcal{Z}_{1,4,\mathbb{Q}}(\mathcal{B}) \sim_{\mathcal{O}(1)} a \cdot \mathcal{B}^{\frac{7}{12}} + b \cdot \mathcal{B}^{\frac{1}{3}} + c$$
$$\mathcal{Z}_{1,5,\mathbb{Q}}(\mathcal{B}) \sim_{\mathcal{O}(1)} a \cdot \mathcal{B}^{\frac{1}{2}} + b$$

Connection to other Heights. On a related note, the classical [Ogg, Saito] formula gives the *Artin conductor = (order of the) discriminant* equality for elliptic curves; then counting the elliptic curves by discriminant Δ_1 is the same as counting the elliptic curves by conductor $\mathcal{N}_1$. We recall that the number of discriminants Δ_1 of an elliptic curve over $\mathbb{Z}$ with smooth generic fiber such that $\Delta_1 \leq \mathcal{B}$ is estimated to be asymptotic to $\mathcal{O}\left(\mathcal{B}^{\frac{5}{6}}\right)$ by [BMc]. The lower order term of order $\mathcal{O}\left(\mathcal{B}^{(7-\frac{5}{27}+\epsilon)/12}\right)$ for counting the stable elliptic curves over $\mathbb{Q}$ by the bounded height of squarefree Δ_1 was suggested by the work of [Baier] improving upon their previous error term in [BB] (see [Hortsch] for asymptotic with bounded Faltings height). In fact, Baier proved his asymptotic under the assumption of the generalized Riemann hypothesis with the twelveth root of the naïve height function on elliptic curves which gives the prediction above. For global function fields $\mathbb{F}_q(t)$, by considering the moduli of semistable elliptic surfaces and finding its motive/point count, we acquire the sharp estimate [HP, Theorem 3] on $\mathcal{Z}_{1,\mathbb{F}_q(t)}$ for counting the semistable elliptic curves by the bounded height of $\Delta_1(X)$ over $\mathbb{P}^1_{\mathbb{F}_q}$ with $\mathrm{char}(\mathbb{F}_q) \neq 2, 3$ giving the leading term of order $\mathcal{O}_q\left(\mathcal{B}^{\frac{5}{6}}\right)$ and the lower order term of zeroth order $\mathcal{O}_q(1)$ (this is exactly the asymptotic tightness in

Definition 1.7, where $\mathcal{O}_q$ refers that the constants used are functions of q). The arithmetic invariant which leads to the above counting also has been established in the past via different method by the seminal work of [de Jong], which works also in characteristic 2 and 3.

For genus 2 curves, the work of [Liu3] proves $\mathcal{N}_2 \le \nu(\Delta_2)$ where he also shows that equality can fail to hold. And for hyperelliptic genus $g \ge 3$ curves, we have $\mathcal{N}_g \le \nu(\Delta_g)$ proven recently by the work of [Srinivasan, OSr]. Consequently, our enumeration $\mathcal{Z}_{g,\mathbb{F}_q(t)}(\mathcal{B})$ of stable hyperelliptic genus $g \ge 2$ curves with a marked Weierstrass point over $\mathbb{P}^1_{\mathbb{F}_q}$ with $\mathrm{char}(\mathbb{F}_q) > 2g+1$ by the hyperelliptic discriminant $\Delta_g(X)$ bounds above the parallel counting by the conductor $\mathcal{N}_g$. Since the qualitative finiteness shown by the classical works of [Parshin, Oort] (see [BG] for counting by naïve height and [Känel] for partly explicit upper bound), we currently do not have an explicit estimate with precise lower order terms for the number of stable genus 2 curves over $\mathbb{Q}$ ordered by the bounded height of Δ_2 (and similarly for the number of principally polarized abelian surfaces).

We note that counting curves over a global field K by height of discriminant $ht(\Delta_g)$ is more difficult than counting by *naïve height* (see [Brumer, p. 446]). In the case of counting elliptic curves ordered by height of discriminant $ht(\Delta_1)$, the region of lattice points with bounded discriminant has *cusps*, meaning that there are *unbounded points* with large $a_4,\ a_6 \in \mathcal{O}_K$ (i.e., elliptic curves with large naïve height) and small discriminant Δ_1. As explained in [Hortsch], controlling these cusps is difficult, even if the ABC conjecture is assumed.

Organization. In §2, we establish the arithmetic geometric properties of the inertia stack $\mathcal{I}(\mathcal{X})$ of an algebraic stack $\mathcal{X}$ thereby proving the Theorem 1.1 and describing various decompositions of the inertia stacks of quotient stacks. In §3, we formulate the Hom stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ of rational curves on a weighted projective stack $\mathcal{P}(\vec{\lambda})$ and provide a clear decomposition of the inertia stack $\mathcal{I}(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda})))$ (i.e., each summand is the Hom stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}_{I_g})))$. In §4, we use the Grothendieck ring of K–stacks $K_0(\mathrm{Stck}_K)$ to acquire the motive $\left\{\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))\right\}$ (Proposition 4.5) which provides the class $\left\{\mathcal{I}\left(\mathrm{Hom}_n(\mathbb{P}^1_K, \mathcal{P}_K(\vec{\lambda}))\right)\right\}$. We also give an algorithm for computing $|\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))(\mathbb{F}_q)/\sim|$. Afterwards in §5, we formulate the moduli stack $\mathcal{L}_{g,|\Delta_g|\cdot n}$ of quasi-admissible hyperelliptic genus g fibrations over $\mathbb{P}^1$ with the hyperelliptic discriminant Δ_g via the birational geometry of surfaces. We use birational geometry to prove Theorem 5.12. Then we compute the related non–weighted point count of the moduli stack $\mathcal{L}_{g,|\Delta_g|\cdot n}$ over $\mathbb{F}_q$ there, proving Theorem 1.3. In §6, we finally establish the upper bounds with precise lower order terms thereby proving Theorem 1.4; this provides the heuristic evidences for Conjecture 1.8 and Conjecture 1.9 through the global fields analogy. In Appendix, we determine the sharp estimate for counting elliptic curves with prescribed level structure or multiple marked points over $\mathbb{P}^1_{\mathbb{F}_q}$.

Notation and conventions. In the present paper, schemes/stacks are assumed to be defined over a field K, if K is not mentioned explicitly or if such scheme is obviously not defined over any field (e.g., Spec $\mathbb{Z}$). Given a point x of a scheme/stack, $\kappa(g)$ means the field of definition of x (i.e., the residue field). Given a group scheme G defined over a field K, then $Cl(G)$ is the set of conjugate classes of closed points g of G (here, $\kappa(g)$ is not necessarily K); this in general is a strictly larger set than the conjugacy class $Cl(G(K))$ of the group of K-rational points of G.

Here, we use the convention in [Olsson2, §8] that the diagonal of an algebraic stack is representable (by algebraic spaces). For any T-point x of a stack $\mathcal{X}$, $\mathrm{Aut}(x)$ is the group of automorphisms of $x \in \mathcal{X}(T)$ (defined over T). We denote $\underline{\mathrm{Aut}}_x$ to be the automorphism space (as an algebraic space) of $x \in \mathcal{X}$ (see (3) in §2).

We identify the Weil divisors and the associated divisorial sheaves implicitly (e.g., if X is a Cohen-Macaulay scheme, then the canonical divisor K_X corresponds to the dualizing sheaf $\omega_X \cong \mathcal{O}(K_X)$ of X). Given a finite morphism $f : X \to Y$ of reduced equidimensional schemes, a branch divisor of f on Y means the pushforward of the ramification divisor of f on X. Given a morphism $f : X \to Y$ of schemes with an isolated subset $Z \subset Y$ (i.e., Y as a topological space is $Z \sqcup (Y \setminus Z)$ under the Zariski topology), the preimage of Z in X refers to the components of X with their image supported on Z.

2. Arithmetic geometry of the inertia stack $\mathcal{I}(\mathcal{X})$ of an algebraic stack $\mathcal{X}$

In this section, we describe the geometry of inertia stacks associated to algebraic stacks; particularly, we first recall various key properties of inertia stacks. By using these properties, we prove Theorem 1.1. Then we describe the groupoid structure of inertia stacks, in particular, inertia stacks of quotient stacks. For general reference on algebraic stacks, we refer the reader to [Olsson2, Stacks].

Given an algebraic stack $\mathcal{X}$ defined over a field K, its inertia stack $\mathcal{I}(\mathcal{X})$ is defined as:

(1) objects: (x, α) where $x \in \mathcal{X}(T)$ for some scheme T (i.e., $x : T \to \mathcal{X}$) and $\alpha \in \mathrm{Aut}(x)$
(2) morphisms: $\psi : (x, \alpha) \to (y, \beta)$ is given by $\phi : x \to y$ in $\mathrm{Mor}(\mathcal{X})(T)$ such that $\phi \circ \alpha = \beta \circ \phi$, i.e., $\beta = \phi \circ \alpha \circ \phi^{-1}$

Also, $\mathcal{I}(\mathcal{X})$ is characterized by the following Cartesian diagram (by [Olsson2, Definition 8.1.17]):

$$\begin{tikzcd} \mathcal{I}(\mathcal{X}) \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\Delta"] \\ \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times \mathcal{X} \end{tikzcd} \tag{2}$$

Note that if the representable morphism Δ satisfies a property (such as finite type, quasi-separated, etc.,), then this property is also satisfied for the representable morphism $\mathcal{I}(\mathcal{X}) \to \mathcal{X}$. In particular, $\mathcal{I}(\mathcal{X})$ is a $\mathcal{X}$-algebraic space, i.e., $\mathcal{I}(\mathcal{X}) \times_{\mathcal{X}} T$ is an algebraic space for any morphism $T \to \mathcal{X}$ from a scheme T.

To understand $\mathcal{I}(\mathcal{X}) \to \mathcal{X}$, we first pay attention to Δ. Given an object $x : T \to \mathcal{X}$ of $\mathcal{X}$ from a scheme T, recall that the automorphism space $\underline{\mathrm{Aut}}_x$ of x is defined to be the fiber product $\mathcal{X}\ {}_{\Delta}\!\times_{x \times x} T$. This means that S-points of $\underline{\mathrm{Aut}}_x$ are characterized by pairs (s, α) of maps $s : S \to T$ and automorphisms $\alpha : s^*x \to s^*x$ in the groupoid $\mathcal{X}(T)$. Since $x \times x$ factors through Δ, $\underline{\mathrm{Aut}}_x$ fits into the following Cartesian diagram:

$$\begin{tikzcd} \underline{\mathrm{Aut}}_x \arrow[r] \arrow[d] & \mathcal{I}(\mathcal{X}) \arrow[d] \\ T \arrow[r, "x"] & \mathcal{X} \end{tikzcd} \tag{3}$$

As before, representability of Δ implies that $\underline{\mathrm{Aut}}_x \to T$ is a morphism of algebraic spaces, and the group algebraic space structure on $\underline{\mathrm{Aut}}_x$ lift, realizing $\mathcal{I}(\mathcal{X})$ as a group algebraic space over $\mathcal{X}$.

Before proving Theorem 1.1, we recall the definition of a weighted point count of an algebraic stack $\mathcal{X}$ over $\mathbb{F}_q$:

Definition 2.1. The weighted point count of $\mathcal{X}$ over $\mathbb{F}_q$ is defined as a sum:

$$\#_q(\mathcal{X}) := \sum_{x \in \mathcal{X}(\mathbb{F}_q)/\sim} \frac{1}{|\mathrm{Aut}(x)|},$$

where $\mathcal{X}(\mathbb{F}_q)/\sim$ is the set of $\mathbb{F}_q$–isomorphism classes of $\mathbb{F}_q$–points of $\mathcal{X}$ (i.e., the set of non–weighted points of $\mathcal{X}$ over $\mathbb{F}_q$), and we take $\frac{1}{|\mathrm{Aut}(x)|}$ to be 0 when $|\mathrm{Aut}(x)| = \infty$.

A priori, the weighted point count can be ∞, but when $\mathcal{X}$ is of finite type, then the stratification of $\mathcal{X}$ by schemes as in [Behrend, Proof of Lemma 3.2.2] implies that $\mathcal{X}(\mathbb{F}_q)/\sim$ is a finite set, so that $\#_q(\mathcal{X}) < \infty$.

We also recall the *Grothendieck-Lefschetz trace formula* for Artin stacks by [Behrend, LO, Sun].

Theorem 2.2 (Theorem 1.1. of [Sun])**.** *Let $\mathcal{X}$ be an Artin stack of finite type over $\mathbb{F}_q$. Let Frob_q be the geometric Frobenius on $\mathcal{X}$. Let ℓ be a prime number different from the characteristic of $\mathbb{F}_q$, and let $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$ be an isomorphism of fields. For an integer i, let $H^i_{\acute{e}t,c}(\mathcal{X}_{/\overline{\mathbb{F}}_q}; \overline{\mathbb{Q}}_\ell)$ be the cohomology with compact support of the constant sheaf $\overline{\mathbb{Q}}_\ell$ on $\mathcal{X}$ as in* [LO]*. Then the infinite sum regarded as a complex series via ι*

$$\sum_{i\in\mathbb{Z}} (-1)^i \cdot tr\big(\mathrm{Frob}_q^* : H^i_c(\mathcal{I}(\mathcal{X})_{/\overline{\mathbb{F}}_q}; \overline{\mathbb{Q}}_\ell) \to H^i_c(\mathcal{I}(\mathcal{X})_{/\overline{\mathbb{F}}_q}; \overline{\mathbb{Q}}_\ell)\big) \tag{4}$$

is absolutely convergent to the weighted point count $\#_q(\mathcal{X})$ of $\mathcal{X}$ over $\mathbb{F}_q$.

When the stack $\mathcal{X}$ is a Deligne–Mumford stack of finite type over $\mathbb{F}_q$ with affine diagonal, then the corresponding compactly-supported, ℓ-adic étale cohomology for prime number ℓ invertible in $\mathbb{F}_q$ is finite dimensional as a $\mathbb{Q}_\ell$-algebra, making the above trace formula to hold in $\mathbb{Q}_\ell$-coefficients.

We are now ready to prove Theorem 1.1:

Proof of Theorem 1.1. Choose any $x \in \mathcal{X}(\mathbb{F}_q)/\sim$. Then the morphism $x : \mathbb{F}_q \to \mathcal{X}$ factors through a representable morphism $\overline{x} : [\mathrm{Spec}(\mathbb{F}_q)/\underline{\mathrm{Aut}}_x] \to \mathcal{X}$. Note that for any $\mathbb{F}_q$-scheme T and any $y, z \in \mathcal{X}(T)$ such that $y \sim x_T$ and $z \sim x_T$ in $\mathcal{X}(T)$, then y, z factors through x and $\underline{\mathrm{Isom}}_{\mathcal{X}}(y, z) \cong \underline{\mathrm{Isom}}_{[\mathrm{Spec}(\mathbb{F}_q)/\underline{\mathrm{Aut}}_x]}(y', z')$, where $y = \overline{x} \circ y'$ and $z = \overline{x} \circ z'$. Thus, $[\mathrm{Spec}(\mathbb{F}_q)/\underline{\mathrm{Aut}}_x]$ is a substack of $\mathcal{X}$ via $\overline{x}$.

Now consider $\mathcal{I}(\mathcal{X})_{\overline{x}}$ defined by the following Cartesian square:

$$\begin{tikzcd}
\mathcal{I}(\mathcal{X})_{\overline{x}} \arrow[r] \arrow[d] & \mathcal{I}(\mathcal{X}) \arrow[d] \\
{[\mathrm{Spec}(\mathbb{F}_q)/\underline{\mathrm{Aut}}_x]} \arrow[r, "\overline{x}"] & \mathcal{X}
\end{tikzcd}$$

This is a substack of $\mathcal{I}(\mathcal{X})$, and $(y, \beta) \in (\mathcal{I}(\mathcal{X})_{\overline{x}})(\mathbb{F}_q)$ iff $y \sim x$ in $\mathcal{X}(\mathbb{F}_q)$. Since x contributes 1 on the unweighted point count $|\mathcal{X}(\mathbb{F}_q)/\sim|$, it suffices to show that $\#_q(\mathcal{I}(\mathcal{X})_{\overline{x}}) = 1$.

Observe that two points (x, α) and (x, β) in $\mathcal{I}(\mathcal{X})_{\overline{x}}$ are equivalent iff $\beta = \phi \circ \alpha \circ \phi^{-1}$ for some $\phi \in \mathrm{Aut}(x)$. This holds in general if we replace x by $y : U \to \mathcal{X}$ that factors thru x. Thus, $\mathcal{I}(\mathcal{X})_{\overline{x}} \cong [\underline{\mathrm{Aut}}_x/\underline{\mathrm{Aut}}_x]$, where the group space action is the conjugation. Since the diagonal of $\mathcal{X}$ is quasi-separated and of finite type, $\underline{\mathrm{Aut}}_x$ is a quasi-separated group algebraic space of finite type over $\mathbb{F}_q$ by Diagram (3); henceforth, $\mathrm{Aut}(x) = \underline{\mathrm{Aut}}_x(\mathbb{F}_q)$ is a finite group since $\underline{\mathrm{Aut}}_x$ admits a finite stratification by schemes of finite type by [Knutson, II.6.6]. Moreover, $\mathrm{Aut}(x, \alpha)$ is the finite centralizer subgroup $C_{\mathrm{Aut}(x)}(\alpha) \subset \mathrm{Aut}(x)$, and the set $(\mathcal{I}(\mathcal{X})_{\overline{x}})(\mathbb{F}_q)/\sim$ is exactly the set $Cl(\mathrm{Aut}(x))$ of orbits of $\mathrm{Aut}(x)$ under the conjugation. Then, the Orbit-Stabilizer Theorem implies that as a set,

$$\mathrm{Aut}(x) \cong \bigsqcup_{\alpha \in Cl(\mathrm{Aut}(x))} \mathrm{Aut}(x)/C_{\mathrm{Aut}(x)}(\alpha).$$

Finally, we can divide the cardinality of both sides by the finite number $|\mathrm{Aut}(x)|$; then right hand side becomes $\#_q(\mathcal{I}(\mathcal{X})_{\overline{x}})$, proving the statement. ■

The following Lemma shows that certain nice property of $\mathcal{X}$ carries over to $\mathcal{I}(\mathcal{X})$ as well.

Lemma 2.3. *If $\mathcal{X}$ is an algebraic stack of finite type with affine finite type diagonal, then so is $\mathcal{I}(\mathcal{X})$.*

Proof. Since $\mathcal{X}$ is of finite type with finite type diagonal, $\mathcal{I}(\mathcal{X})$ must be of finite type as well by Diagram (2). It remains to show that $\mathcal{I}(\mathcal{X})$ has an affine diagonal. This is equivalent to showing that for any scheme T and any pairs $(x,\alpha),(y,\beta)\in\mathcal{I}(\mathcal{X})(T)$, the Isom space $\underline{\mathrm{Isom}}_{\mathcal{I}(\mathcal{X})}((x,\alpha),(y,\beta))$ is an affine T-scheme by the following Cartesian diagram:

$$\begin{tikzcd}
\underline{\mathrm{Isom}}_{\mathcal{I}(\mathcal{X})}((x,\alpha),(y,\beta)) \arrow[r] \arrow[d] & T \arrow[d, "(x,\alpha)\times(y,\beta)"] \\
\mathcal{I}(\mathcal{X}) \arrow[r, "\Delta"] & \mathcal{I}(\mathcal{X})\times\mathcal{I}(\mathcal{X})
\end{tikzcd}$$

To see the structure of $\underline{\mathrm{Isom}}_{\mathcal{I}(\mathcal{X})}((x,\alpha),(y,\beta))\to T$, observe that $\underline{\mathrm{Isom}}_{\mathcal{X}}(x,y)\to T$ and $\underline{\mathrm{Aut}}_x\to T$ are affine morphisms of finite type by the conditions on the diagonal of $\mathcal{X}$. Then $\underline{\mathrm{Isom}}_{\mathcal{I}(\mathcal{X})}((x,\alpha),(y,\beta))$ is the preimage under the closed subscheme $1_x\in\underline{\mathrm{Aut}}_x$ of a morphism between affine T-schemes:

$$\underline{\mathrm{Isom}}_{\mathcal{X}}(x,y)\to\underline{\mathrm{Aut}}_x$$
$$\phi\mapsto\phi\circ\alpha\circ\phi^{-1}\circ\beta^{-1}$$

Therefore, $\underline{\mathrm{Isom}}_{\mathcal{I}(\mathcal{X})}((x,\alpha),(y,\beta))$ is an affine T-scheme as well. ■

In practice, an algebraic stack $\mathcal{X}$ can be characterized by its smooth cover $U\to\mathcal{X}$ by an algebraic space U (most of the time, U is assumed to be a scheme) with the space of equivalence relations R, i.e., R is defined via the following Cartesian diagram

$$\begin{tikzcd}
R \arrow[r, "t"] \arrow[d, "s"] & U \arrow[d] \\
U \arrow[r] & \mathcal{X}
\end{tikzcd}$$

where $s(r)=x$ and $t(r)=y$ for any directed equivalence relation $r:x\to y$ of $x,y\in U$ via $r\in R$. In this case, $\mathcal{X}\cong[U/R]$ (technically, $[U/R]$ is the sheafification of the presheaf U/R of groupoids over Sch). Note that the following Cartesian square

$$\begin{tikzcd}
R \arrow[r, "s\times t"] \arrow[d] & U\times U \arrow[d] \\
\mathcal{X} \arrow[r, "\Delta"] & \mathcal{X}\times\mathcal{X}
\end{tikzcd}$$

implies that R is an algebraic space as well, since $\Delta:\mathcal{X}\to\mathcal{X}\times\mathcal{X}$ is representable. Given this presentation, we obtain the following presentation of $\mathcal{I}(\mathcal{X})$:

Proposition 2.4. $\mathcal{I}(\mathcal{X})\cong[R_\Delta/(R_\Delta\times_s R)]$, *where R_Δ is defined by the following Cartesian square:*

$$\begin{tikzcd}
R_\Delta \arrow[r] \arrow[d] & R \arrow[d] \\
U \arrow[r, "\Delta"] & U\times U
\end{tikzcd}$$

Proof. By [Stacks, Tag 06PR], R_Δ (denoted G in loc.cit.) is a smooth cover of $\mathcal{I}(\mathcal{X})$. To see that $R_\Delta\times_{\mathcal{I}(\mathcal{X})}R_\Delta$ is isomorphic to $R_\Delta\,{}_t\times_s R$, it suffices to compare their T-points for any scheme T. Recall by the Cartesian diagram above that any T-point of R_Δ is characterized by a pair $(u,r)\in U\times R$ where $r:u\to u$. Then, given any $((u_1,r_1),(u_2,r_2))\in(R_\Delta\times_{\mathcal{I}(\mathcal{X})}R_\Delta)(T)$, then there is $\tau : u_1 \to u_2$ in $R(T)$ such that $\tau \circ r_1 = r_2 \circ \tau$. This gives an element $((u_1, r_1), \tau) \in (R_\Delta \, {}_t\!\times_s R)(T)$. The converse can be recovered, as u_2 is the target of r_1 and $r_2 = \tau \circ r_1 \circ \tau^{-1}$. This establishes the bijection between T-points of $R_\Delta \times_{\mathcal{I}(\mathcal{X})} R_\Delta$ and $R_\Delta \, {}_t\!\times_s R$. ∎

Sometimes, we will denote $R(\mathcal{X})$ (resp, $R_\Delta(\mathcal{X})$) instead of R (resp, R_Δ) when we need to emphasize the algebraic stack $\mathcal{X}$ in question.

Recall that a quotient stack, denoted $[U/G]$, corresponds to U a scheme with the action of a group scheme G. In this case, $R = U \times G$ with s being the first projection and t being the G-action map $t : (u, g) \mapsto g \cdot u$. Then $R_\Delta \subset U \times G$ consists of (u, g) with $t(u, g) = g \cdot u = u$.

Corollary 2.5. *If $\mathcal{X} \cong [U/G]$ is a quotient stack, then $\mathcal{I}(\mathcal{X})$ is also a quotient stack $[R_\Delta/G]$, where $R \cong U \times G$ and G acts on $R_\Delta \subset R$ by $g \cdot (u, h) = (g \cdot u, ghg^{-1})$.*

Proof. By the proof of Proposition 2.4, it suffices to show that $R_\Delta \, {}_t\!\times_s R \cong R_\Delta \times G$ and that the action map

$$\begin{aligned} R_\Delta \, {}_t\!\times_s R &\to R_\Delta \\ ((u, r), \tau) &\mapsto (t(r), \tau \circ r \circ \tau^{-1}), \end{aligned}$$

which coincides with the second projection of $R_\Delta \times_{\mathcal{I}(\mathcal{X})} R_\Delta$, coincides with the conjugate G-action described above. The isomorphism $R_\Delta \, {}_t\!\times_s R \cong R_\Delta \times G$ is given by:

$$\begin{aligned} R_\Delta \times G &\to R_\Delta \, {}_t\!\times_s R \\ ((u, g), h) &\mapsto ((u, g), (g \cdot u = u, h)) \end{aligned}$$

By the description of the action map above, G acts on R_Δ by the conjugation. ∎

Now assume that a quotient stack $\mathcal{X} \cong [U/G]$ of finite type has the affine diagonal. Then, R_Δ is not irreducible in general; in fact, not even connected. Since the image of the second projection $\pi_2 : U \times G \supset R_\Delta \to G$ can have many irreducible components G_i, we have the decomposition $R_\Delta = \cup \pi_2^{-1}(G \cdot G_i)$ (where G acts on itself by conjugation). Note that when $\pi_2(R_\Delta)$ is disconnected, so is R_Δ.

Thus, assume furthermore that $\mathcal{X}$ is a Deligne-Mumford (DM) stack. Since the diagonal of $\mathcal{X}$ is affine (by the previous assumption) and formally unramified (by DM), the diagonal must be finite; this implies that $\pi_2(R_\Delta)$ lies in torsion subset of G. Instead of stratifying $\pi_2(R_\Delta)$ by G-orbits of its irreducible components as above, Abramovich-Graber-Vistoli in [AGV, Definition 3.1.5] stratify $\mathcal{I}(\mathcal{X})$ by looking at orders of automorphism elements: in our language, this induces a coarser stratification of R_Δ:

$$\begin{aligned} R_\Delta(\mathcal{X}) &= \bigsqcup_{r \in \mathbb{Z}_{>0}} R_{\Delta, \mu_r}(\mathcal{X}) \\ \mathcal{I}(\mathcal{X}) &= \bigsqcup_{r \in \mathbb{Z}_{>0}} \mathcal{I}_{\mu_r}(\mathcal{X}) = \bigsqcup_{r \in \mathbb{Z}_{>0}} [R_{\Delta, \mu_r}(\mathcal{X})/G] \end{aligned} \tag{5}$$

where $R_{\Delta, \mu_r}(\mathcal{X})$ is the preimage under π_2 of the subscheme of order r elements of G. However, $R_{\Delta, \mu_r}(\mathcal{X})$ can still be disconnected with many components of different dimensions.

Instead, assume that we have chosen a nice presentation of $\mathcal{X}$ into a quotient stack $[U/G]$ such that the support of $\pi_2(R_\Delta)$ consists of finitely many closed points of G. In this case, $\pi_2(R_\Delta)$ is, as a set, a disjoint union of conjugate classes of some closed points in $\pi_2(R_\Delta)$. Let's use our initial decomposition of R_Δ as above by G-orbits of connected components of $\pi_2(R_\Delta)$. This induces the following stratification:

Definition 2.6. Let $\mathcal{X} \cong [U/G]$ be a Deligne-Mumford quotient stack of finite type with affine diagonal and let R_Δ be as in Corollary 2.5 such that the support of the second projection $\pi_2(R_\Delta)$ in G consists of finitely many closed points of G. Then the decomposition of the inertia stack $\mathcal{I}(\mathcal{X})$ via the conjugacy classes is as follows:

$$R_\Delta(\mathcal{X}) = \bigsqcup_{\alpha \in Cl(G)} R_{\Delta,\alpha}(\mathcal{X}),$$
$$\mathcal{I}(\mathcal{X}) = \bigsqcup_{\alpha \in Cl(G)} \mathcal{I}_\alpha(\mathcal{X}) = \bigsqcup_{\alpha \in Cl(G)} [R_{\Delta,\alpha}(\mathcal{X})/G],$$

where $R_{\Delta,\alpha}(\mathcal{X})$ is the preimage under π_2 of a conjugate class $\alpha \in Cl(G)$, as a finite subset of G.

Note that $R_{\Delta,\alpha} = \sqcup_{g\in\alpha} R_{\Delta,g}$ where $R_{\Delta,g}$ is the preimage under π_2 of $g \in G$; it is the base change by $\kappa(g)/K$ of the fixed locus in U of $g \in G$ (i.e., every point is fixed under the action of g). Observe that $R_{\Delta,hgh^{-1}} = h \cdot R_{\Delta,g}$, which is itself $R_{\Delta,g}$ (then $h \in C_G(g)$) or is disjoint from $R_{\Delta,g}$ by the finiteness of $\pi_2(R_\Delta)$. Therefore, $\mathcal{I}_\alpha(\mathcal{X}) = [R_{\Delta,g}/C_G(g)]$ for any generator $g \in \alpha$, i.e. $\alpha = G \cdot g$.

As a summary, the decomposition in Definition 2.6 is finer than (5) when it exists, but assumes *the finiteness of* $\pi_2(R_\Delta) \subset G$ *as a subset*. We will see that weighted projective stacks (and Hom stacks) defined in §3 satisfy this condition.

Remark 2.7. When $\mathcal{X} \cong [U/G]$, U, G in Definition 2.6 are defined over a perfect field K, the condition, that the support of $\pi_2(R_\Delta)$ in G consists of finitely many closed points of G, is equivalent to the finiteness of the following set:

$$\{g \text{ is a geometric point of } G \mid g \cdot u = u \text{ for some geometric point } u \text{ of } U\}.$$

When $\mathcal{X}$ is Deligne-Mumford and G is an abelian group (such as $\mathbb{G}_m$), this is easy to check. However, when G is a non-abelian group (examples are GIT constructions of moduli of smooth/stable curves), this condition puts restriction on what kind of g can fix an element of U, even when $\mathcal{X}$ is a Deligne-Mumford stack. If $g \cdot u = u$, then $hgh^{-1} \cdot hu = hu$, so that this set above is a union of conjugacy classes as sets. Whenever the centralizer subgroup scheme $C_G(g)$ has lower dimension than G, the conjugacy class (i.e., the orbit of g under conjugation) forms a positive dimensional subscheme, contained in the set above. Since K is perfect, the algebraic closure $\overline{K}$ is infinite, implying that such positive dimensional subschems have infinitely many geometric points by Bertini's Theorem.

3. Hom stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ of rational curves on a weighted projective stack

In this section, we formulate the Hom stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ over a base field K. First, we recall the definition of a weighted projective stack $\mathcal{P}(\vec{\lambda})$ with the weight $\vec{\lambda}$ over K.

Definition 3.1. Fix a tuple of nondecreasing positive integers $\vec{\lambda} = (\lambda_0, \dots, \lambda_N)$. The N-dimensional weighted projective stack $\mathcal{P}(\vec{\lambda}) = \mathcal{P}(\lambda_0, \dots, \lambda_N)$ with the weight $\vec{\lambda}$ is defined as a quotient stack

$$\mathcal{P}(\vec{\lambda}) := \left[(\mathbb{A}^{N+1}_{x_0,\dots,x_N} \setminus 0)/\mathbb{G}_m\right]$$

where $\zeta \in \mathbb{G}_m$ acts by $\zeta \cdot (x_0, \dots, x_N) = (\zeta^{\lambda_0} x_0, \dots, \zeta^{\lambda_N} x_N)$. In this case, the degree of x_i's are λ_i's respectively. A line bundle $\mathcal{O}_{\mathcal{P}(\vec{\lambda})}(m)$ is defined to be the line bundle associated with the sheaf of degree m homogeneous rational functions without poles on $\mathbb{A}^{N+1}_{x_0,\dots,x_N} \setminus 0$.

Note that $\mathcal{P}(\vec{\lambda})$ is not an (effective) orbifold when $\gcd(\lambda_0, \dots, \lambda_N) \neq 1$. In this case, the finite cyclic group scheme $\mu_{\gcd(\lambda_0,\dots,\lambda_N)}$ is the generic stabilizer of $\mathcal{P}(\vec{\lambda})$. When we need to emphasize the field K of definition of $\mathcal{P}(\vec{\lambda})$, we instead use the notation $\mathcal{P}_K(\vec{\lambda})$.

Lemma 3.2. *The N-dimensional weighted projective stack $\mathcal{P}(\vec{\lambda}) = \mathcal{P}(\lambda_0, \dots, \lambda_N)$ over any field K is of finite type with finite type affine diagonal.*

Proof. Since the smooth schematic cover $\mathbb{A}^{N+1}_{x_0,\dots,x_N} \setminus 0$ of $\mathcal{P}(\vec{\lambda})$ is of finite type over K, $\mathcal{P}(\vec{\lambda})$ is of finite type over K as well. It remains to prove the properties of the diagonal of $\mathcal{P}(\vec{\lambda})$. Choose any T-point $x = (x_0, \dots, x_N)$ of $U := \mathbb{A}^{N+1}_K \setminus 0$. The fiber over x of $R_\Delta \to U$ as in Corollary 2.5 is a proper subgroup scheme of $\mathbb{G}_m$ (over T), which is always affine of finite type over T. Henceforth, the diagonal of $\mathcal{P}(\vec{\lambda})$ satisfies the desired properties. ∎

However, when $K = \mathbb{F}_p$ for some prime p, $\mathcal{P}(1,p)$ is not Deligne-Mumford, as $\underline{\mathrm{Aut}}_{[0:1]} \cong \mu_p$, which is not formally unramified over $\mathbb{F}_p$. Nevertheless, the following proposition shows that any $\mathcal{P}(\vec{\lambda})$ behaves well in most characteristics as a tame Deligne–Mumford stack:

Proposition 3.3. *The weighted projective stack $\mathcal{P}(\vec{\lambda}) = \mathcal{P}(\lambda_0, \dots, \lambda_N)$ is a tame Deligne–Mumford stack over K if $\mathrm{char}(K)$ does not divide $\lambda_i \in \mathbb{N}$ for every i.*

Proof. For any algebraically closed field extension $\overline{K}$ of K, any point $y \in \mathcal{P}(\vec{\lambda})(\overline{K})$ is represented by the coordinates $(y_0, \dots, y_N) \in \mathbb{A}^{N+1}_{\overline{K}}$ with its stabilizer group as the subgroup of $\mathbb{G}_m$ fixing $(y_0, \dots, y_N)$. Hence, any stabilizer group of such $\overline{K}$-points is $\mathbb{Z}/u\mathbb{Z}$ where u divides λ_i for some i. Since the characteristic of K does not divide the orders of $\mathbb{Z}/\lambda_i\mathbb{Z}$ for any i, the stabilizer group of y is $\overline{K}$-linearly reductive. Hence, $\mathcal{P}(\vec{\lambda})$ is tame by [AOV, Theorem 3.2]. Note that the stabilizer groups constitute fibers of the diagonal $\Delta : \mathcal{P}(\vec{\lambda}) \to \mathcal{P}(\vec{\lambda}) \times_K \mathcal{P}(\vec{\lambda})$. Since $\mathcal{P}(\vec{\lambda})$ is of finite type and $\mathbb{Z}/u\mathbb{Z}$'s are unramified over K whenever u does not divide λ_i for some i, Δ is unramified as well. Therefore, $\mathcal{P}(\vec{\lambda})$ is also Deligne–Mumford by [Olsson2, Theorem 8.3.3]. ∎

The tameness is analogous to flatness for stacks in positive/mixed characteristic as it is preserved under base change by [AOV, Corollary 3.4]. Moreover, if a stack $\mathcal{X}$ is tame and Deligne–Mumford, then the formation of the coarse moduli space $c : \mathcal{X} \to X$ commutes with base change as well by [AOV, Corollary 3.3].

Example 3.4. When the characteristic of the field K is not equal to 2 or 3, [Hassett2, Proposition 3.6] shows that one example is given by the proper Deligne–Mumford stack of stable elliptic curves $(\overline{\mathcal{M}}_{1,1})_K \cong [(\mathrm{Spec}\ K[a_4, a_6] - (0,0))/\mathbb{G}_m] = \mathcal{P}_K(4,6)$ by using the short Weierstrass equation $y^2 = x^3 + a_4 x + a_6 x$, where $\zeta \cdot a_i = \zeta^i \cdot a_i$ for $\zeta \in \mathbb{G}_m$ and $i = 4, 6$. Thus, a_i's have degree i's respectively. Note that this is no longer true if characteristic of K is 2 or 3, as the Weierstrass equations are more complicated.

In the proof of Lemma 3.2, we have shown that $R_\Delta \to U$ is proper, implying that $\pi_2(R_\Delta) \subset \mathbb{G}_m$ is a proper subgroup scheme, i.e., supported on finitely many closed points. Thus, we can apply the decomposition in Definition 2.6 to the inertia stack $\mathcal{I}(\mathcal{P}(\vec{\lambda}))$:

Proposition 3.5. *For any N-dimensional weighted projective stack $\mathcal{P}_K(\vec{\lambda})$, Definition 2.6 describes connected components of $\mathcal{I}(\mathcal{P}_K(\vec{\lambda}))$:*

$$\mathcal{I}(\mathcal{P}_K(\vec{\lambda})) \cong \bigsqcup_{g \in |(\mathbb{G}_m)_K|} \mathcal{P}_{\kappa(g)}(\vec{\lambda}_{I_g})$$

where $|(\mathbb{G}_m)_K|$ is set of closed points of $(\mathbb{G}_m)_K$, I_g is the largest subset of $\{0, \dots, N\}$ such that $\mathrm{ord}(g)$ divides $\gcd_{i \in I_g}(\lambda_i)$, and $\vec{\lambda}_{I_g}$ is the subtuple of $\vec{\lambda}$ indexed by $I_g \subset \{0, \dots, N\}$.

Note that $I_g = I_{g'}$ when $\mathrm{ord}(g) = \mathrm{ord}(g')$, as any subgroup of $\mathbb{G}_m$ is cyclic. Also, when $|I_g| = 0$, then $\mathcal{P}(\vec{\lambda}_{I_g}) = \emptyset$ vacuously.

Proof of Proposition 3.5. It suffices to show that $R_{\Delta,g}$ is the subspace

$$\{(x_0,\dots,x_N,g)\in(\mathbb{A}^{N+1}\setminus 0)\times\mathbb{G}_m \mid x_i=0 \text{ if } \mathrm{ord}(g) \text{ does not divide } \lambda_i\}$$

as commutativity of $\mathbb{G}_m$ implies that $C_G(g)=\mathbb{G}_m$ for any $g\in\mathbb{G}_m$ (here, g as a closed point of $\mathbb{G}_m$ in above coordinates is equivalent to taking a Galois orbit of a representative of g as a $\kappa(g)$-point of $\mathbb{G}_m$). Note that this space is a $\kappa(g)$-variety as its projection onto $\mathbb{G}_m$ maps to $g\in\mathbb{G}_m$. For any $g\in|(\mathbb{G}_m)_K|$, $x=(x_0,\dots,x_N)\in\mathbb{A}^{N+1}$ is fixed by g iff $g^{\lambda_i}x_i=x_i$ for all i. Whenever $g^{\lambda_i}\neq 1$, x_i must be zero. Thus, x lies in the closed subscheme $\{x_i=0\ :\ \forall i,\ g^{\lambda_i}\neq 1\}$, which is exactly the desired subspace. ■

We now generalize the Hom stack formulation to $\mathcal{P}(\vec\lambda)$ as follows:

Proposition 3.6. *Hom stack* $\mathrm{Hom}_n(\mathbb{P}^1,\mathcal{P}(\vec\lambda))$ *with weight* $\vec\lambda=(\lambda_0,\dots,\lambda_N)$*, which parameterize degree* $n\in\mathbb{N}$ *K-morphisms* $f:\mathbb{P}^1\to\mathcal{P}(\vec\lambda)$ *with* $f^*\mathcal{O}_{\mathcal{P}(\vec\lambda)}(1)\cong\mathcal{O}_{\mathbb{P}^1}(n)$ *over a base field K with* $\mathrm{char}(K)$ *not dividing* $\lambda_i\in\mathbb{N}$ *for every i, is a smooth separated tame Deligne–Mumford stack of finite type with* $\dim_{\mathrm{K}}\left(\mathrm{Hom}_n(\mathbb{P}^1,\mathcal{P}(\vec\lambda))\right)=|\vec\lambda|n+N$ *where* $|\vec\lambda|:=\sum_{i=0}^{N}\lambda_i$.

Proof. $\mathrm{Hom}_n(\mathbb{P}^1,\mathcal{P}(\vec\lambda))$ is a smooth Deligne–Mumford stack by [Olsson, Theorem 1.1]. It is isomorphic to the quotient stack $[T/\mathbb{G}_m]$, admitting a smooth schematic cover $T\subset\left(\bigoplus_{i=0}^{N}H^0(\mathcal{O}_{\mathbb{P}^1}(\lambda_i\cdot n))\right)\setminus 0$, parameterizing the set of tuples $(u_0,\dots,u_N)$ of sections with no common zero (here, we interpret $H^0(\mathcal{O}_{\mathbb{P}^1}(\lambda_i\cdot n))$ as an affine space over K of appropriate dimension, induced by its K-vector space structure). The $\mathbb{G}_m$ action on T is given by $\zeta\cdot(u_0,\dots,u_N)=(\zeta^{\lambda_0}u_0,\dots,\zeta^{\lambda_N}u_N)$. Note that

$$\dim T=\sum_{i=0}^{n}h^0(\mathcal{O}_{\mathbb{P}^1}(\lambda_i\cdot n))=\sum_{i=0}^{n}(\lambda_i+1)=|\vec\lambda|+N+1,$$

implying that $\dim\mathrm{Hom}_n(\mathbb{P}^1,\mathcal{P}(\vec\lambda))=|\vec\lambda|+N$ since $\dim\mathbb{G}_m=1$.

As $\mathbb{G}_m$ acts on T properly with positive weights $\lambda_i\in\mathbb{N}$ for every i, the quotient stack $[T/\mathbb{G}_m]$ is separated. It is tame as in [AOV, Theorem 3.2] since $\mathrm{char}(K)$ does not divide λ_i for every i . ■

Remark 3.7. In the proof of Proposition 3.6, we showed that $\mathrm{Hom}_n(\mathbb{P}^1,\mathcal{P}(\vec\lambda))\cong[T/\mathbb{G}_m]$ where T is an open dense $\mathbb{G}_m$-invariant subscheme of $\oplus H^0(\mathcal{O}_{\mathbb{P}^1}(\lambda_i\cdot n))$ not containing zero, where for each i, $\mathbb{G}_m$ acts on $H^0(\mathcal{O}_{\mathbb{P}^1}(\lambda_i\cdot n))$ with weight λ_i. In fact, this remains true even when the characteristic assumption fails, as the arguments still follow. Since $h^0(\mathcal{O}_{\mathbb{P}^1}(\lambda_i\cdot n))=n\lambda_i+1$, $\mathrm{Hom}_n(\mathbb{P}^1,\mathcal{P}(\vec\lambda))$ is an open substack of $\mathcal{P}(\vec\Lambda)$ where

$$\vec\Lambda:=(\underbrace{\lambda_0,\dots,\lambda_0}_{n\lambda_0+1\text{ times}},\dots,\underbrace{\lambda_N,\dots,\lambda_N}_{n\lambda_N+1\text{ times}}).$$

Furthermore, $(u_0,\dots,u_N)\in\oplus H^0(\mathcal{O}_{\mathbb{P}^1}(\lambda_i\cdot n))$ lies in T iff u_i's have no common zero on $\mathbb{P}^1$. By Lemma 3.2, $\mathrm{Hom}_n(\mathbb{P}^1,\mathcal{P}(\vec\lambda))$ is of finite type with finite type affine diagonal (without any condition on the base field K) ■

Similar to $\mathcal{P}(\vec\lambda)$, the inertia stack $\mathcal{I}(\mathrm{Hom}_n(\mathbb{P}^1,\mathcal{P}(\vec\lambda)))$ also admits a clear decomposition (i.e., each summand is the Hom stack $\mathrm{Hom}_n(\mathbb{P}^1,\mathcal{P}(\vec\lambda_{I_g})))$ that will play a crucial role.

Proposition 3.8. *The inertia stack of the Hom stack* $\mathrm{Hom}_n(\mathbb{P}^1,\mathcal{P}(\vec\lambda))$ *admits the following decomposition into connected components as in Definition 2.6, where I_g and $\vec\lambda_{I_g}$ are the same as in*

Proposition 3.5:

$$\mathcal{I}(\mathrm{Hom}_n(\mathbb{P}^1_K, \mathcal{P}_K(\vec{\lambda}))) \cong \bigsqcup_{g \in |(\mathbb{G}_m)_K|} \mathrm{Hom}_n(\mathbb{P}^1_{\kappa(g)}, \mathcal{P}_{\kappa(g)}(\vec{\lambda}_{I_g}))$$

Note that $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}_{I_g})) = \emptyset$ whenever $|I_g| \leq 1$, as there are no maps from $\mathbb{P}^1$ to $\mathcal{P}(\vec{\lambda}_{I_g})$ where the pullback of $\mathcal{O}(1)$ to $\mathbb{P}^1$ has degree n.

Proof of Proposition 3.8. By Remark 3.7, $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ is an open substack of $\mathcal{P}(\vec{\Lambda})$. Restricting the decomposition of $\mathcal{I}(\mathcal{P}(\vec{\Lambda}))$ as in Proposition 3.5 to $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$,

$$\mathcal{I}(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))) \cong \bigsqcup_{g \in |(\mathbb{G}_m)_K|} \Big[T \cap \{ \vec{u} \in \oplus H^0(\mathcal{O}_{\mathbb{P}^1}(\lambda_i \cdot n)) \ : \ u_i = 0 \text{ if } g^{\lambda_i} \neq 1 \} / \mathbb{G}_m \Big]_{\kappa(g)}$$

Since $\vec{u} \in T$ iff u_i's have no common zeroes, each summand is isomorphic to $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}_{I_g}))$. ■

4. Motive/Point count of Hom and inertia stacks

In this section, we use the idea of cut-and-paste by Grothendieck and acquire the motive $\{\mathcal{X}\} \in K_0(\mathrm{Stck_K})$ of moduli stack $\mathcal{X}$ in the Grothendieck ring of K–stacks; in fact, we show that $\{\mathcal{X}\}$ is a polynomial in the Lefschetz motive $\mathbb{L} := \{\mathbb{A}^1_K\}$. Particularly, we acquire the motive $\{\mathrm{Poly}_1^{(d_1,\ldots,d_m)}\}$ of the space of monic coprime polynomials through filtration which in turn provides the motive $\left\{\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))\right\}$ of the Hom stack through stratification. In the end, we acquire the weighted point count of the inertia stack $\left\{\mathcal{I}\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))\right)\right\}$ of the Hom stack over $\mathbb{F}_q$ through the decomposition of Proposition 3.8.

First, we recall the definition of the Grothendieck ring of algebraic stacks following [Ekedahl].

Definition 4.1. [Ekedahl, §1] Fix a field K. Then the *Grothendieck ring* $K_0(\mathrm{Stck}_K)$ *of algebraic stacks of finite type over* K *all of whose stabilizer group schemes are affine* is an abelian group generated by isomorphism classes of K-stacks $\{\mathcal{X}\}$ of finite type, modulo relations:

- $\{\mathcal{X}\} = \{\mathcal{Z}\} + \{\mathcal{X} \setminus \mathcal{Z}\}$ for $\mathcal{Z} \subset \mathcal{X}$ a closed substack,
- $\{\mathcal{E}\} = \{\mathcal{X} \times \mathbb{A}^n\}$ for $\mathcal{E}$ a vector bundle of rank n on $\mathcal{X}$.

Multiplication on $K_0(\mathrm{Stck}_K)$ is induced by $\{\mathcal{X}\}\{\mathcal{Y}\} := \{\mathcal{X} \times_K \mathcal{Y}\}$. There is a distinguished element $\mathbb{L} := \{\mathbb{A}^1\} \in K_0(\mathrm{Stck}_K)$, called the *Lefschetz motive.*

Given an algebraic K-stack $\mathcal{X}$ of finite type with affine diagonal, the *motive* of $\mathcal{X}$ refers to $\{\mathcal{X}\} \in K_0(\mathrm{Stck}_K)$.

As the Grothendieck ring $K_0(\mathrm{Stck}_K)$ is the universal object for *additive invariants*, it is easy to see that when $K = \mathbb{F}_q$, the assignment $\{X\} \mapsto \#_q(X)$ gives a well-defined ring homomorphism $\#_q : K_0(\mathrm{Stck}_{\mathbb{F}_q}) \to \mathbb{Q}$ (c.f. [Ekedahl, §2]) rendering the weighted point count of a stack $\mathcal{X}$ over $\mathbb{F}_q$. Note that $\#_q(\mathcal{X}) < \infty$ when $\mathcal{X}$ is of finite type (see discussion right below Definition 2.1).

Since many algebraic stacks can be written locally as a quotient of a scheme by an algebraic group $\mathbb{G}_m$, the following lemma (a special case of [Ekedahl, §1]) is very useful:

Lemma 4.2. [HP, Lemma 15] *For any* $\mathbb{G}_m$*-torsor* $\mathcal{X} \to \mathcal{Y}$ *of finite type algebraic stacks, we have* $\{\mathcal{Y}\} = \{\mathcal{X}\}\{\mathbb{G}_m\}^{-1}$.

The subsequent proofs involves the following variety of its own interest (a slight generalization of [FW, Definition 1.1]) :

Definition 4.3. Fix $m \in \mathbb{Z}_{>0}$ and $d_1, \ldots, d_m \geq 0$. Define $\mathrm{Poly}_1^{(d_1,\ldots,d_m)}$ as the set of tuples $(f_1, \ldots, f_m)$ of monic polynomials in $K[z]$ so that

(1) $\deg f_i = d_i$ for each i, and
(2) $f_1, \ldots, f_m$ have no common roots in $\overline{K}$.

Since the set $\mathrm{Poly}_1^{(d_1,\ldots,d_m)}$ is open inside the affine space (complement of the resultant hypersurface) parameterizing the tuples of monic coprime polynomials of degrees $(d_1, \ldots, d_m)$, we can endow $\mathrm{Poly}_1^{(d_1,\ldots,d_m)}$ with a structure of affine variety defined over $\mathbb{Z}$.

Generalizing the proof of [FW, Theorem 1.2] with the correction from [PS, Proposition 3.1.], we find the motive of $\mathrm{Poly}_1^{(d_1,\ldots,d_m)}$:

Proposition 4.4 (Motive of the Poly space $\mathrm{Poly}_1^{(d_1,\cdots,d_m)}$ over K)**.** *Fix* $0 \leq d_1 \leq d_2 \leq \cdots \leq d_m$. *Then,*

$$\left\{\mathrm{Poly}_1^{(d_1,\cdots,d_m)}\right\} = \begin{cases} \mathbb{L}^{d_1+\cdots+d_m} - \mathbb{L}^{d_1+\cdots+d_m-m+1} & \textit{if } d_1 \neq 0 \\ \mathbb{L}^{d_1+\cdots+d_m} & \textit{if } d_1 = 0 \end{cases}$$

Proof. The proof is analogous to [FW, Theorem 1.2 (1)], with the correction from [PS, Proposition 3.1.], and is a direct generalization of [HP, Proposition 18]. Here, we recall the differences to the work in [FW, HP, PS].

Step 1: The space of $(f_1, \ldots, f_m)$ monic polynomials of degree $d_1, \ldots, d_m$ is instead the quotient $\mathbb{A}^{d_1} \times \cdots \times \mathbb{A}^{d_m}/(S_{d_1} \times \cdots \times S_{d_m}) \cong \mathbb{A}^{d_1+\cdots+d_m}$. We have the same filtration of $\mathbb{A}^{\sum d_i}$ by $R_{1,k}^{(d_1,\ldots,d_m)}$: the space of monic polynomials $(f_1, \ldots, f_m)$ of degree $d_1, \ldots, d_m$ respectively for which there exists a monic $h \in K[z]$ with $\deg(h) \geq k$ and monic polynomials $g_i \in K[z]$ so that $f_i = g_i h$ for any i. The rest of the arguments follow analogously, keeping in mind that the group action is via $S_{d_1} \times \cdots \times S_{d_m}$.

Step 2: Here, we prove that $\{R_{1,k}^{(d_1,\ldots,d_m)} - R_{1,k+1}^{(d_1,\ldots,d_m)}\} = \{\mathrm{Poly}_1^{(d_1-k,\ldots,d_m-k)} \times \mathbb{A}^k\}$. Just as in [FW], the base case of $k = 0$ follows from the definition (in fact, loc.cit. shows that the two schemes are indeed isomorphic). For $k \geq 1$, [PS, Proposition 3.1] proves that the map

$$\Psi : \mathrm{Poly}_1^{(d_1-k,\ldots,d_m-k)} \times \mathbb{A}^k \to R_{1,k}^{(d_1,\ldots,d_m)} \setminus R_{1,k+1}^{(d_1,\ldots,d_m)}$$

induces a piecewise isomorphism (where each piece is a locally closed subset, see [PS, Proposition 3.1] for more details); this immediately implies the claim by the definition of the Grothendieck Ring.

Step 3: By combining Step 1 and 2 as in [FW], we obtain

$$\left\{\mathrm{Poly}_1^{(d_1,\ldots,d_m)}\right\} = \mathbb{L}^{d_1+\cdots+d_m} - \sum_{k\geq 1} \left\{\mathrm{Poly}_1^{(d_1-k,\ldots,d_m-k)}\right\} \mathbb{L}^k$$

For the induction on the class $\left\{\mathrm{Poly}_1^{(d_1,\ldots,d_m)}\right\}$, we use lexicographic induction on the pair $(d_1, \ldots, d_m)$. For the base case, consider when $d_1 = 0$. Here the monic polynomial of degree 0 is nowhere vanishing, so that any tuple of polynomials of degree d_i for $i > 1$ constitutes a member of $\mathrm{Poly}_1^{(0,d_2,\ldots,d_m)}$, so that $\mathrm{Poly}_1^{(0,d_2,\ldots,d_m)} \cong \mathbb{A}^{d_2+\cdots+d_m}$.

Now assume that $d_1 > 0$. Then, we obtain

$$\left\{\mathrm{Poly}_1^{(d_1,\ldots,d_m)}\right\}$$

$$
\begin{aligned}
&= \mathbb{L}^{d_1+\cdots+d_m} - \sum_{k\geq 1}\left\{\mathrm{Poly}_1^{(d_1-k,\ldots,d_m-k)}\right\}\mathbb{L}^k \\
&= \mathbb{L}^{d_1+\cdots+d_m} - \left(\sum_{k=1}^{d_1-1}(\mathbb{L}^{(d_1-k)+\cdots+(d_m-k)} - \mathbb{L}^{(d_1-k)+\cdots+(d_m-k)-m+1})\mathbb{L}^k + \mathbb{L}^{(d_2-d_1)+\cdots+(d_m-d_1)}\mathbb{L}^{d_1}\right) \\
&= \mathbb{L}^{d_1+\cdots+d_m} - \left(\sum_{k=1}^{d_1-1}(\mathbb{L}^{d_1+\cdots+d_m-(m-1)k} - \mathbb{L}^{d_1+\cdots+d_m-(m-1)(k+1)}) + \mathbb{L}^{d_1+\cdots+d_m-(m-1)d_1}\right) \\
&= \mathbb{L}^{d_1+\cdots+d_m} - \mathbb{L}^{d_1+\cdots+d_m-m+1}
\end{aligned}
$$

■

4.1. **Motive of Hom stack.** Now we are ready to find the class in Grothendieck ring of the Hom stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$:

Proposition 4.5. *Fix the weight $\vec{\lambda} = (\lambda_0, \ldots, \lambda_N)$ with $|\vec{\lambda}| := \sum_{i=0}^{N} \lambda_i$. Then the motive of the Hom stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ in the Grothendieck ring of K–stacks $K_0(\mathrm{Stck}_K)$ is equivalent to*

$$
\begin{aligned}
\left\{\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))\right\} &= \left(\sum_{i=0}^{N} \mathbb{L}^i\right) \cdot \left(\mathbb{L}^{|\vec{\lambda}|n} - \mathbb{L}^{|\vec{\lambda}|n-N}\right) \\
&= \mathbb{L}^{|\vec{\lambda}|n-N} \cdot \left(\mathbb{L}^{2N} + \cdots + \mathbb{L}^{N+1} - \mathbb{L}^{N-1} - \cdots - 1\right)
\end{aligned}
$$

where $\mathbb{L}^1 := \{\mathbb{A}^1_K\}$ is the Lefschetz motive.

Proof. Let $\vec{\lambda} = (\lambda_0, \ldots, \lambda_N)$ and $\lambda_i \in \mathbb{N}$ for every i with $|\vec{\lambda}| := \sum_{i=0}^{N} \lambda_i$. Then the Hom stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda})) \cong [T/\mathbb{G}_m]$ is the quotient stack by the proof of Proposition 3.6. By Lemma 4.2, we have $\{\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))\} = (\mathbb{L}-1)^{-1}\{T\}$. Henceforth, it suffices to find the motive $\{T\}$, and not worry about the original $\mathbb{G}_m$-action on T. To do so, we need to reinterpret T as follows.

Fix a chart $\mathbb{A}^1 \hookrightarrow \mathbb{P}^1$ with $x \mapsto [1 : x]$, and call $0 = [1 : 0]$ and $\infty = [0 : 1]$. It comes from a homogeneous chart of $\mathbb{P}^1$ by $[Y : X]$ with $x := X/Y$ away from ∞. Then for any $u \in H^0(\mathcal{O}_{\mathbb{P}^1}(d))$ with $d \geq 0$, u is a homogeneous polynomial of degree d in X and Y. By substituting in $Y = 1$, we obtain a representation of u as a polynomial in x with degree at most d. For instance, $\deg u < d$ as a polynomial in x if and only if $u(X, Y)$ is divisible by Y (i.e., u vanishes at ∞). From now on, $\deg u$ means the degree of u as a polynomial in x. Conventionally, set $\deg 0 := -\infty$.

Therefore, T parameterizes a N-tuple $(f_0, \ldots, f_N)$ of polynomials in $K[x]$ with no common roots in $\overline{K}$, where $\deg f_i \leq n\lambda_i$ for each i with equality for some i. We use this interpretation to construct $\Phi : T \to \mathbb{A}^{N+1} \setminus 0$, $\Phi(f_0, \ldots, f_N) = (a_0, \ldots, a_N)$, where a_i is the coefficient of degree $n\lambda_i$ term of f_i.

Now, we stratify T by taking preimages under Φ of a stratification of $\mathbb{A}^{N+1} \setminus 0$ by $\sqcup E_J$, where J is any proper subset of $\{0, \ldots, N\}$ and

$$
E_J = \{(a_0, \ldots, a_N) \mid a_j = 0 \ \ \forall j \in J\} \cong \mathbb{G}_m^{N+1-|J|}
$$

Observe that E_J has the natural free $\mathbb{G}_m^{N+1-|J|}$-action, which lifts to $\Phi^{-1}(E_J)$ via multiplication on $\mathbb{G}_m$-scalars on f_i for $i \notin J$. The action is free on $\Phi^{-1}(E_J)$ as well, so that $\Phi|_{\Phi^{-1}(E_J)}$ is a Zariski-locally trivial fibration with base E_J. Each fiber is isomorphic to $F_J(n\vec{\lambda})$ defined below:

Definition 4.6. Fix $m \in \mathbb{N}$ and $\vec{d} := (d_0, \ldots, d_N) \in \mathbb{Z}_{\geq 0}^{N+1}$. Given $J \subsetneq \{0, \ldots, N\}$, $F_J(\vec{d})$ is defined as a variety consisting of tuples $(f_0, \ldots, f_N)$ of K-polynomials without common roots such that

- for any $j \notin J$, then f_j is monic of degree $n\lambda_j$, and
- for any $j \in J$, then $\deg f_j < n\lambda_j$ (f_j is not necessarily monic).

If instead $J = \{0, \ldots, N\}$, then we define $F_J(\vec{d}) := \emptyset$

This implies that $\{\Phi^{-1}(E_J)\} = \{E_J\}\{F_J(n\vec{\lambda})\} = (\mathbb{L}-1)^{N+1-|J|}\{F_J(n\vec{\lambda})\}$. Since

$$\{T\} = \sum_{J \subsetneq \{0,\ldots,N\}} \{\Phi^{-1}(E_J)\} = \sum_{J \subsetneq \{0,\ldots,N\}} \{E_J\}\{F_J(n\vec{\lambda})\} \; , \tag{6}$$

it suffices to find $\{F_J(n\vec{\lambda})\}$ as a polynomial of $\mathbb{L}$.

Proposition 4.7. *$\{F_J(n\vec{\lambda})\} = \{\mathrm{Poly}_1^{(n\lambda_0, \cdots, n\lambda_N)}\} = \left(\mathbb{L}^{|\vec{\lambda}| \cdot n} - \mathbb{L}^{|\vec{\lambda}| \cdot n - N}\right)$, where $|\vec{\lambda}| := \sum_i \lambda_i$. In other words, $\{F_J(n\vec{\lambda})\}$ only depends on $n\vec{\lambda}$.*

Proof. Set $d_i := n\lambda_i > 0$ for the notational convention. Upto S_{N+1}-action on $\{0, \ldots, N\}$ (forgetting that $\lambda_0 \leq \cdots \leq \lambda_N$), consider instead $F_{\langle m \rangle}(\vec{d})$ with $\langle m \rangle = \{0, \cdots, m-1\}$ and $\vec{d} = (d_0, \cdots, d_N)$ with $|\vec{d}| := \sum_{i=0}^{N} d_i$. We now want to show that

$$\{F_{\langle m \rangle}(\vec{d})\} = \{\mathrm{Poly}_1^{(d_0, \cdots, d_N)}\} = \left(\mathbb{L}^{|\vec{d}|} - \mathbb{L}^{|\vec{d}| - N}\right). \tag{7}$$

In order to prove this, we first check that if we set $d_i = 0$ for some $i \geq m$, then

$$\{F_{\langle m \rangle}(\vec{d})\} = \{\mathrm{Poly}_1^{(d_0, \cdots, d_N)}\} = \mathbb{L}^{|\vec{d}|}.$$

To see this, note that $i \notin \langle m \rangle$, so that f_i is monic of degree $d_i = 0$ for any $(f_0, \ldots, f_N) \in F_{\langle m \rangle}(\vec{d})$; so $f_i = 1$. Therefore, the common root condition from Definition 4.6 is vacuous, so that $\{F_{\langle m \rangle}(\vec{d})\} = \mathbb{L}^{|\vec{d}|}$ (as the space of monic polynomials of degree d is isomorphic to $\mathbb{A}^d$ and so is the space of polynomials of degree $< d$).

We prove equation (7) by lexicographical induction on the ordered pairs (N, m) such that $N > 0$ and $0 \leq m < N+1$. There are two base cases to consider:

1. If $m = 0$, then $\langle 0 \rangle = \emptyset$, so that $F_\emptyset(\vec{d}) \cong \mathrm{Poly}_1^{(d_0, \ldots, d_N)} =: \mathrm{Poly}_1^{\vec{d}}$ by Definition 4.3.
2. If $N = 1$, then m is 0 or 1. $m = 0$ follows from above. Now assume $m = 1$. Then $(f_0, f_1) \in F_{\langle 1 \rangle}(\vec{d})$ if and only if $\deg f_0 < d_0$ and $\deg f_1 = d_1 > 0$ with f_1 monic. Observe that f_0 cannot be 0, otherwise f_1 has no roots while having positive degree, which is a contradiction. Since f_0 can be written as $a_0 g_0$ for g_0 monic of degree $\deg f_0$ and $a_0 \in \mathbb{G}_m$, $F_{\langle 1 \rangle}(\vec{d})$ decomposes into the following locally closed subsets:

$$F_{\langle 1 \rangle}(\vec{d}) = \bigsqcup_{l=0}^{d_0 - 1} \mathbb{G}_m \times F_\emptyset(l, d_1) = \mathbb{G}_m \times \bigsqcup_{l=0}^{d_0 - 1} \mathrm{Poly}_1^{(l, d_1)}.$$

Therefore,

$$\begin{aligned}
\{F_{\langle 1 \rangle}(\vec{d})\} = \{\mathbb{G}_m\} \sum_{l=0}^{d_0 - 1} \left\{\mathrm{Poly}_1^{(l, d_1)}\right\} &= (\mathbb{L} - 1)\left(\mathbb{L}^{d_1} + \sum_{l=1}^{d_0 - 1} (\mathbb{L}^{l + d_1} - \mathbb{L}^{l + d_1 - 1})\right) \\
&= (\mathbb{L} - 1)(\mathbb{L}^{d_1} + \mathbb{L}^{d_0 + d_1 - 1} - \mathbb{L}^{d_1}) = (\mathbb{L} - 1)\mathbb{L}^{d_0 + d_1 - 1} \\
&= \mathbb{L}^{d_0 + d_1} - \mathbb{L}^{d_0 + d_1 - 1}
\end{aligned}$$

In general, assume that the statement is true for any (N', m') whenever $N' < N$ or $N' = N$ and $m' \leq m$. If $m+1 < N+1$, then we want to prove the assertion for $(N, m+1)$. We can take the similar decomposition as the base case $(1,1)$, except that we vary the degree of f_m, which is the $(m+1)$-st term of $(f_0, \dots, f_N) \in F_{\langle m+1 \rangle}(\vec{d})$, and f_m can be 0. If $f_m = 0$, then $(f_0, \dots, \widehat{f_m}, \dots, f_N)$ have no common roots, so that $(f_0, \dots, \widehat{f_m}, \dots, f_N) \in F_{\langle m \rangle}(d_0, \dots, \widehat{d_m}, \dots, d_N)$ (and vice versa). Henceforth, as a set,

$$F_{\langle m+1 \rangle}(\vec{d}) = F_{\langle m \rangle}(d_0, \dots, \widehat{d_m}, \dots, d_N) \bigsqcup (\mathbb{G}_m \times F_{\langle m \rangle}(d_0, \dots, 0, \dots, d_N))$$
$$\bigsqcup \left(\mathbb{G}_m \times \bigsqcup_{\ell=1}^{d_m - 1} F_{\langle m \rangle}(d_0, \dots, \ell, \dots, d_N) \right).$$

By induction,

$$\begin{aligned}
\left\{F_{\langle m+1 \rangle}(\vec{d})\right\} &= \left\{F_{\langle m \rangle}(d_0, \cdots, \widehat{d_m}, \cdots, d_N)\right\} + (\mathbb{L} - 1) \left\{F_{\langle m \rangle}(d_0, \cdots, 0, \cdots, d_N)\right\} \\
&\quad + (\mathbb{L} - 1) \sum_{\ell=0}^{d_m - 1} \left\{F_{\langle m \rangle}(d_0, \cdots, \ell, \cdots, d_N)\right\} \\
&= \mathbb{L}^{|\vec{d}| - d_m} - \mathbb{L}^{\vec{d} - d_m - N + 1} + (\mathbb{L} - 1) \cdot \mathbb{L}^{|\vec{d}| - d_m} \\
&\quad + (\mathbb{L} - 1) \sum_{\ell=1}^{d_m - 1} \left(\mathbb{L}^{|\vec{d}| - d_m + \ell} - \mathbb{L}^{|\vec{d}| - d_m + \ell - N} \right) \\
&= \mathbb{L}^{|\vec{d}| - d_m} - \mathbb{L}^{|\vec{d}| - d_m - N + 1} + \mathbb{L}^{|\vec{d}| - d_m + 1} - \mathbb{L}^{|\vec{d}| - d_m} \\
&\quad + (\mathbb{L} - 1)\mathbb{L}(\mathbb{L}^{|\vec{d}| - d_m} - \mathbb{L}^{|\vec{d}| - d_m - N})(1 + \mathbb{L} + \cdots + \mathbb{L}^{d_m - 2}) \\
&= \mathbb{L}^{|\vec{d}| - d_m + 1} - \mathbb{L}^{|\vec{d}| - d_m - N + 1} + \mathbb{L}(\mathbb{L}^{|\vec{d}| - d_m} - \mathbb{L}^{|\vec{d}| - d_m - N})(\mathbb{L}^{d_m - 1} - 1) \\
&= \mathbb{L}^{|\vec{d}| - d_m + 1} - \mathbb{L}^{|\vec{d}| - d_m - N + 1} + \mathbb{L}^{|\vec{d}|} - \mathbb{L}^{|\vec{d}| - d_m + 1} - \mathbb{L}^{|\vec{d}| - N} + \mathbb{L}^{|\vec{d}| - d_m - N + 1} \\
&= \mathbb{L}^{|\vec{d}|} - \mathbb{L}^{|\vec{d}| - N}
\end{aligned}$$

■

Combining (6) and Proposition 4.7 with $\sum_{J \subsetneq \{0, \dots, N\}} E_J = (\mathbb{A}^{N+1} \setminus 0)$, we finally acquire

$$\begin{aligned}
\{\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))\} &= \{\mathbb{G}_m\}^{-1}\{T\} = (\mathbb{L} - 1)^{-1} \sum_{J \subsetneq \{0, \dots, N\}} \{E_J\}\{\mathrm{Poly}_1^{(n\vec{\lambda})}\} \\
&= (\mathbb{L} - 1)^{-1}(\mathbb{L}^{N+1} - 1)\{\mathrm{Poly}_1^{(n\vec{\lambda})}\} = \left(\sum_{i=0}^{N} \mathbb{L}^i \right) \cdot \left(\mathbb{L}^{|\vec{\lambda}| \cdot n} - \mathbb{L}^{|\vec{\lambda}| \cdot n - N} \right)
\end{aligned}$$

This finishes the proof of Proposition 4.5.

■

4.2. **Point count of Hom stack.** Using Proposition 4.5, we immediately obtain the weighted point count $\#_q \left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda})) \right)$:

Corollary 4.8. *Fix the weight $\vec{\lambda} = (\lambda_0, \dots, \lambda_N)$ with $|\vec{\lambda}| := \sum_{i=0}^{N} \lambda_i$. Then the weighted point count of the Hom stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ over $\mathbb{F}_q$ is*

$$\#_q\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))\right) = \left(\sum_{i=0}^{N} q^i\right) \cdot \left(q^{|\vec{\lambda}|n} - q^{|\vec{\lambda}|n-N}\right)$$

$$= q^{|\vec{\lambda}|n-N} \cdot \left(q^{2N} + \cdots + q^{N+1} - q^{N-1} - \cdots - 1\right)$$

Denote $\delta := \gcd(\lambda_0, \dots, \lambda_N)$ and $\omega := \max\gcd(\lambda_i, \lambda_j)$ for $0 \le i, j \le N$. Then the number $|\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))(\mathbb{F}_q)/\sim|$ of $\mathbb{F}_q$–isomorphism classes of $\mathbb{F}_q$–points (i.e., the non–weighted point count over $\mathbb{F}_q$) of $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ satisfies

$$\delta \cdot \#_q\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))\right) \le \left|\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))(\mathbb{F}_q)/\sim\right| \le \omega \cdot \#_q\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))\right) .$$

Proof. The first part of the Corollary follows as $\#_q : K_0(\mathrm{Stck}_{\mathbb{F}_q}) \to \mathbb{Q}$ is a ring homomorphism with $\#_q(\mathbb{L}) = q$ as $\mathbb{L} = \{\mathbb{A}^1_{\mathbb{F}_q}\}$. For the second part, notice that for each $\varphi \in \mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))(\mathbb{F}_q)/\sim$, it contributes 1 towards $|\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))(\mathbb{F}_q)/\sim|$ instead of $\frac{1}{|\mathrm{Aut}(\varphi)|}$ for $\#_q(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda})))$. Thus, we need to check that for any $\varphi \in \mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))(\mathbb{F}_q)$ with $\delta := \gcd(\lambda_0, \dots, \lambda_N)$ and $\omega := \max\gcd(\lambda_i, \lambda_j)$ for $0 \le i, j \le N$, the automorphism group satisfies the following :

$$\delta \le |\mathrm{Aut}(\varphi)| \le \omega .$$

By Proposition 3.6, we can represent φ as a tuple $(f_0, \dots, f_N)$ of sections $f_i \in H^0(\mathcal{O}_{\mathbb{P}^1_{\mathbb{F}_q}}(n\lambda_i))$, with equivalence relation given by a $\mathbb{G}_m$–action. Since the automorphism group of φ is identified with the subgroup of $\mathbb{G}_m$ fixing $(f_0, \dots, f_N)$, $\mathrm{Aut}(\varphi)$ consists of $u \in \mathbb{G}_m(\mathbb{F}_q)$ such that $u^{\lambda_i} f_i = f_i$ for any i . Since f_i's have no common root and the degree of the morphism φ is $n \in \mathbb{N}$, at least two of those are nonzero; call I to be the set of i's with $f_i \neq 0$. Then, $u^{\lambda_i} = 1$ for any $i \in I$, so that u is a $\gcd(\lambda_i \ : \ i \in I)^{\text{th}}$ root of unity. This shows that $\mathrm{Aut}(\varphi)$ is a finite cyclic group of order $\gcd(\lambda_i \ : \ i \in I)$, proving the second part of the Corollary. ■

Above proof shows that computing automorphism groups of $\mathbb{F}_q$–points of $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ is the key ingredient for comparing between weighted and non–weighted point counts over $\mathbb{F}_q$. Instead, we bypass this issue by using properties of the inertia stack $\mathcal{I}\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))\right)$, particularly by using Theorem 1.1 and Proposition 3.8.

4.3. **Point count of Inertia of Hom stack.** We compute the class $\left\{\mathcal{I}\left(\mathrm{Hom}_n(\mathbb{P}^1_K, \mathcal{P}_K(\vec{\lambda}))\right)\right\}$, which renders the non–weighted point count of the moduli stack $\mathcal{L}_{g,|\Delta_g|\cdot n}$ over $\mathbb{F}_q$.

Proposition 4.9. *Take the same notation as in Proposition 3.5. Then,*

$$\left\{\mathcal{I}\left(\mathrm{Hom}_n(\mathbb{P}^1_K, \mathcal{P}_K(\vec{\lambda}))\right)\right\} = \sum_{g \in |(\mathbb{G}_m)_K|} \left\{\mathrm{Hom}_n(\mathbb{P}^1_{\kappa(g)}, \mathcal{P}_{\kappa(g)}(\vec{\lambda}_{\kappa(g)}))\right\}$$

Proof. This directly follows from Proposition 3.8 and the definition of the Grothendieck Ring. ■

Above definition combined with the proof of Corollary 4.8 gives an algorithm for computing $|\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))(\mathbb{F}_q)/\sim|$:

Proposition 4.10. *Take the same notation as in Proposition 3.8 for $K = \mathbb{F}_q$, and let R be the set of positive integers r (including $r = 1$) that divides $q-1$. Then,*

$$|\operatorname{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))(\mathbb{F}_q)/\sim| = \sum_{r \in R} \varphi(r) \cdot \#_q(\operatorname{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}_r)))$$

where φ is the Euler φ-function and $\vec{\lambda}_r := \vec{\lambda}_{I_{\zeta_r}}$ for some primitive r^{th} root of unity ζ_r in $\mathbb{G}_m$.

Proof. Recall that the multiplicative group $\mathbb{F}_q^*$ of a finite field $\mathbb{F}_q$ is a cyclic group of order $q-1$. By the primitive root condition, we see that $\zeta_r \in \mathbb{G}_m(\mathbb{F}_q)$ iff $r|(q-1)$. This implies that the substack $\operatorname{Hom}_n(\mathbb{P}^1_{\kappa(\zeta_r)}, \mathcal{P}_{\kappa(\zeta_r)}(\vec{\lambda}_r))$ contributes $\mathbb{F}_q$–rational points iff r divides $q-1$, hence the definition of the set R. As $\vec{\lambda}_r$ is independent of the choice of a primitive r^{th} root of unity and there are $\varphi(r)$ number of them, simplifying $\#_q \left\{\mathcal{I}\left(\operatorname{Hom}_n(\mathbb{P}^1_K, \mathcal{P}_K(\vec{\lambda}))\right)\right\}$ gives the desired formula by Theorem 1.1. ∎

Remark 4.11. Note that writing a closed-form formula for $|\operatorname{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))(\mathbb{F}_q)/\sim|$ is difficult in general, as Euler φ-function is used, the sum is over all possible positive factors of $q-1$, and the length of $\vec{\lambda}_r$ can vary. Nevertheless, it is possible to obtain a closed-form formula by hand for special cases with mild assumptions on q (Theorem 1.3 is a good example).

5. Moduli stack $\mathcal{L}_{g,|\Delta_g|\cdot n}$ of quasi-admissible hyperelliptic genus g fibrations over $\mathbb{P}^1$

In this section, we first define a rational fibration with a marked section, which allows us to define a hyperelliptic genus g fibration with a marked Weierstrass section as a double cover fibration. Subsequently, we focus on a quasi-admissible hyperelliptic genus g fibration over $\mathbb{P}^1$ with a marked Weierstrass section which extends a family of odd degree hyperelliptic genus g curves over $\mathbb{F}_q(t)$ with a marked Weierstrass point. For detailed references on hyperelliptic fibrations or fibered algebraic surfaces (over an algebraically closed field), we refer the reader to [Liu, Liedtke].

Recall that a hyperelliptic curve C is a separable morphism $\phi : C \to \mathbb{P}^1$ of degree 2. In order to extend the notion of hyperelliptic curve C into family, we first generalize the notion of rational curve $\mathbb{P}^1$ into family.

Definition 5.1. A *rational fibration with a marked section* is given by a flat proper morphism $h : H \to \mathbb{P}^1$ of pure relative dimension 1 with a marked section $s' : \mathbb{P}^1 \to H$ such that

(1) any geometric fiber $h^{-1}(c)$ is a connected rational curve (so that arithmetic genus is 0),
(2) $s'(\mathbb{P}^1)$ is away from the non-reduced locus of any geometric fiber, and
(3) $s'(\mathbb{P}^1)$ is away from the singular locus of H.

If the geometric generic fiber of h is a smooth rational curve, then we call (H, h, s') a $\mathbb{P}^1$*-fibration.*

We will occationally call (H, h, s') a *rational fibration* when there is no ambiguity on the marked section s'. Note that we allow a rational fibration H to be reducible (when generic fiber is a nodal chain), and the total space of a $\mathbb{P}^1$-fibration can be singular. Certain double cover of the rational fibration gives us the hyperelliptic genus g fibration with a marked Weierstrass section.

Definition 5.2. A *hyperelliptic genus g fibration with a marked Weierstrass section* consists of a tuple (X, H, h, f, s, s') of a rational fibration $h : H \to \mathbb{P}^1$, a flat proper morphism $f : X \to H$ of degree 2 with X connected and reduced, and sections $s : \mathbb{P}^1 \to X$ and $s' : \mathbb{P}^1 \to H$ such that

(1) Each geometric fiber $(h \circ f)^{-1}(c)$ is a connected 1–dimensional scheme of arithmetic genus g,
(2) $s(\mathbb{P}^1)$ is contained in the smooth locus of $h \circ f$ and is away from the non-reduced locus of any geometric fiber,

(3) $s' = f \circ s$ and $s(\mathbb{P}^1)$ is a connected component of the ramification locus of f (i.e., $s'(\mathbb{P}^1)$ is a connected component of the branch locus of f), and
(4) if p is a node of a geometric fiber $h^{-1}(c)$, then any $q \in f^{-1}(p)$ is a node of the fiber $(h \circ f)^{-1}(c)$,
(5) if the branch divisor of f contains a node e of a fiber $h^{-1}(t)$ with t a closed geometric point of $\mathbb{P}^1$, then the branch divisor contains either an irreducible component of $h^{-1}(t)$ containing e or an irreducible component of the singular locus of H containing e.

The *underlying genus g fibration* is a tuple $(\pi := h \circ f, s)$ with $\pi : X \to \mathbb{P}^1$ a flat proper morphism with geometric fibers of arithmetic genus g with a marked Weierstrass point given by s.

Note 5.3. An isomorphism between hyperelliptic genus g fibrations $(X_1, H_1, h_1, f_1, s_1, s_1')$ and $(X_2, H_2, h_2, f_2, s_2, s_2')$ is given by a pair of isomorphisms $\alpha : X_1 \to X_2$ and $\beta : H_1 \to H_2$ such that
(1) $h_2 \circ \beta = h_1$ and $f_2 \circ \alpha = \beta \circ f_1$ ($\mathbb{P}^1$–isomorphism criteria), and
(2) $\beta \circ s = s'$ (compatibility with sections).

From now on, we only consider non-isotrivial hyperelliptic fibrations, i.e., the underlying genus g fibrations must be non-isotrivial. Thus, non-isotrivialness will be assumed on every statement and discussions below.

Recall that a fibration with a section is said to be *stable* if all of its fibers are stable pointed curves. This leads to the following definition in the hyperelliptic case:

Definition 5.4. A *stable hyperelliptic genus g fibration with a marked Weierstrass section* is a hyperelliptic genus g fibration (X, H, h, f, s, s') with $K_X + s(\mathbb{P}^1)$ is π-ample. We assume that X is not isotrivial, i.e., the trivial hyperelliptic fiber bundle over $\mathbb{P}^1$ with no singular fibers.

Moreover, if the geometric generic fiber is smooth, then (X, H, h, f, s, s') is called a *stable odd hyperelliptic genus g model over* $\mathbb{P}^1$.

Conditions in the above definition implies that $(X, s(\mathbb{P}^1))/\mathbb{P}^1$ is log canonical. In classical language, this means that there are no smooth rational curves of self-intersection -1 and -2 in a fiber without meeting $s(\mathbb{P}^1)$.

Example 5.5. Suppose that (X, H, h, f, s, s') is a stable odd hyperelliptic genus g model with a marked Weierstrass section. Then, it is possible that $f : X \to H$ in a étale local neighborhood of $p \in H$ is the map $\mathbb{A}^2_{x,y} \to \mathbb{A}^2_{x,y}/\mu_2$, where μ_2 acts on $\mathbb{A}^2_{x,y}$ by $(x, y) \mapsto (-x, -y)$. In this case, π can be given by $\mathbb{A}^2_{x,y} \to \mathbb{A}^1_z$ by $z = xy$. Note that H admits an A_1–singularity at p, $f^{-1}(p)$ is a node of a fiber of π, but X is nonsingular. In general, X and H admit at worst A_l–singularities for some l (because geometric fibers of X are nodal curves), where A_u–singularities of surfaces are étale locally given by $w^2 + x^2 + y^{u+1} = 0$. This follows from the fact that 1–parameter deformation of nodes create such singularities. Note that on the neighborhood of such an isolated singular point of H, the branch locus of f is concentrated at the point if it contains the point, which only appears possibly at singular points of the fibers of $h : H \to \mathbb{P}^1$.

Example 5.6. Suppose that (X, H, h, f, s, s') is a stable odd hyperelliptic genus g model with a marked Weierstrass section over a field K. The goal is to classify singularities of the branch divisor of f. By the definition, the branch divisor decomposes into $B \sqcup s'(\mathbb{P}^1_K)$, which is contained in the smooth locus of H by the definition. First, consider a geometric point c in the intersection $B \cap H_t$, where t is a geometric point of $\mathbb{P}^1_K$ and H_t is the fiber $h^{-1}(t)$. Since the corresponding double cover X_t (which is a fiber over t of $h \circ f$) only admits nodes as singularities, the multiplicity m of $B \cap H_t$ at c is at most 2, as $f_t : X_t \to Y_t$ étale locally near c is given by the equation

$$\mathrm{Spec}(\overline{K}[y, z]/(z^2 - y^m)) \to \mathrm{Spec}(\overline{K}[y]), \text{ where } y \text{ is the uniformizer of } c \in H_t.$$

Since B does not contain any irreducible component of geometric fibers of h (as any geometric fiber of $h \circ f$ is reduced), above implies that the multiplicity of B at any geometric point is at most 2. Thus, the support of B possibly admits plane double point curve singularities, étale locally of the form $y^2 - x^m = 0$ with $m \in \mathbb{N}_{\geq 2}$, on the geometrically reduced locus of B, and is smooth elsewhere. Those singularities are in fact A_{m-1} (curve) singularities.

Example 5.5 and 5.6 illustrate that a general stable odd hyperelliptic genus g model often gives a mildly singular $\mathbb{P}^1$-fibration and mildly singular branch divisor on it. On the other hand, we could instead consider the hyperelliptic fibrations with smooth $\mathbb{P}^1$-bundle H, but with X and the branch divisor having worse singularities. Then, each fiber of X is irreducible and is a double cover of $\mathbb{P}^1$ branched over $2g+2$ number of points, where many of these points could collide. For instance, if l branch points collide, then the preimage has A_{l-1}-singularity on the fiber, given étale locally by an equation $y^2 - x^l = 0$. Such a curve is called the quasi-admissible hyperelliptic curve, defined in Definition 1.2. Quasi-admissible hyperelliptic curves over $\mathbb{P}^1_K$ (which are non-isotrivial) are equivalent to the following:

Definition 5.7. A hyperelliptic fibration (X, H, h, f, s, s') is *quasi-admissible* if for every geometric point $c \in C$, f restricted to the fibers of X and H is quasi-admissible. We assume that X is not isotrivial over $\mathbb{P}^1$, i.e., all geometric fibers are isomorphic.

Remark 5.8. Observe that the Definitions 5.1, 5.2, 5.4, and 5.7 should be interpreted as rational / hyperelliptic / stable / quasi-admissible curves over $\mathbb{P}^1_K$, instead of a point Spec K (just as in Definition 1.2). Thus, these definitions can be extended to corresponding curves over a general scheme T, assuming that any geometric point t of T has the property that the characteristic of the residue field is 0 or larger than $2g+1$ (when instead $g = 1$, the standard definition of semistable over T is more delicate whenever the characteristic of geometric point is 2 or 3, and is not analogous to the definitions proposed in this paper).

In particular, a quasi-admissible hyperelliptic fibration (X, H, h, f, s, s') has the property that H is a $\mathbb{P}^1$-bundle, and on each geometric fiber of H, each point of the branch divisor away from s' has the multiplicity at most $2g$. Moreover, X is the double cover of H branched along the branch divisor (which coincides with the branch locus).

To parameterize such fibrations, we first consider the moduli stack $\mathcal{H}_{2g}[2g-1]$ of quasi-admissible hyperelliptic genus g curves characterized by [Fedorchuk, Proposition 4.2(1)] :

Proposition 5.9. *If $p := \mathrm{char}(K)$ is 0 or $> 2g+1$, then the moduli stack $\mathcal{H}_{2g}[2g-1]$ of quasi-admissible hyperelliptic genus g curves is a tame Deligne–Mumford stack isomorphic to $\mathcal{P}(4, 6, 8, \ldots, 4g+2)$, where a point $(a_4, a_6, a_8, \ldots, a_{4g+2})$ of $\mathcal{P}(4, 6, 8, \ldots, 4g+2)$ corresponds to the quasi-admissible hyperelliptic genus g curve with the Weierstrass equation*

$$y^2 = x^{2g+1} + a_4 x^{2g-1} + a_6 x^{2g-2} + a_8 x^{2g-3} + \cdots + a_{4g+2} \tag{8}$$

Proof. Proof of [Fedorchuk, Proposition 4.2(1)] is originally done when $p = 0$, so it suffices to show that the proof in loc.cit. extends to the case when $p > 2g+1$.

When $p = 0$, the proof of loc.cit. shows that the quasi-admissible hyperelliptic curves are characterized by the base $\mathbb{P}^1$ with the branch locus of degree $2g+1$ on $\mathbb{A}^1 = \mathbb{P}^1 \setminus \infty$, of the form

$$x^{2g+1} + a_2 x^{2g} + a_4 x^{2g-1} + a_6 x^{2g-2} + a_8 x^{2g-3} + \cdots + a_{4g+2} = 0$$

where $a_2 = 0$ and not all of the rest of a_i's vanish. When $p > 2g+1$, any monic polynomial of degree $2g+1$ with not all roots being identical can be written in the same way (via same method) by replacing x by $x - \frac{a_2}{(2g+1)}$ (this is allowed as $2g+1 < p$ is invertible). Hence, the moduli stack is indeed isomorphic to $\mathcal{P}(4, 6, 8, \ldots, 4g+2)$, with a_{2i}'s referring to the standard coordinates of $\mathcal{P}(4, 6, 8, \ldots, 4g+2)$ of degree $2i$.

Since $p > 2g+1$. $\mathcal{P}(4,6,8,\dots,4g+2)$ is tame Deligne–Mumford by Proposition 3.3 . ■

Assigning $\mathcal{H}_{2g}[2g-1]$ as the target stack, we can now formulate the moduli stack $\mathcal{L}_g$ of quasi-admissible hyperelliptic genus g fibrations with a marked Weierstrass section as the following :

Proposition 5.10. *Assume* $\mathrm{char}(K) = 0$ *or* $> 2g+1$*. Then, the moduli stack* $\mathcal{L}_g$ *of quasi-admissible hyperelliptic genus* g *fibrations over* $\mathbb{P}^1$ *with a marked Weierstrass section is the tame Deligne–Mumford stack* $\mathrm{Hom}_{>0}(\mathbb{P}^1, \mathcal{H}_{2g}[2g-1])$ *parameterizing the* K*-morphisms* $f : \mathbb{P}^1 \to \mathcal{H}_{2g}[2g-1]$ *with* $\deg f^*\mathcal{O}_{\mathcal{H}_{2g}[2g-1]}(1) > 0$*.*

Proof. By the definition of the universal family p, any quasi-admissible hyperelliptic genus g fibration $f : Y \to \mathbb{P}^1$ comes from a morphism $\varphi_f : \mathbb{P}^1 \to \mathcal{H}_{2g}[2g-1]$ and vice versa. As this correspondence also works in families, the moduli stack $\mathcal{L}_g$ is a substack of $\mathrm{Hom}(\mathbb{P}^1, \mathcal{H}_{2g}[2g-1])$. As $\mathcal{H}_{2g}[2g-1]$ is tame Deligne–Mumford by Proposition 5.9, the Hom stack $\mathrm{Hom}(\mathbb{P}^1, \mathcal{H}_{2g}[2g-1])$ is Deligne–Mumford by [Olsson]. Tameness follows from [AOV], as $\mathcal{H}_{2g}[2g-1]$ itself is tame. Thus, $\mathcal{L}_g$ is tame Deligne–Mumford as well.

Since any quasi-admissible hyperelliptic genus g fibration f is not isotrivial, φ_f must be a non-trivial morphism, i.e., the image of f in $\mathcal{H}_{2g}[2g-1])$ is 1-dimensional. Since non-trivialness of a morphism is an clopen condition, the corresponding clopen locus (consisting of the union of connected components) $\mathrm{Hom}_{>0}(\mathbb{P}^1, \mathcal{H}_{2g}[2g-1])$ is indeed isomorphic to $\mathcal{L}_g$. ■

We now have the following arithmetic invariant of the moduli stack $\mathcal{L}_{g,|\Delta_g|\cdot n}$ over $\mathbb{F}_q$.

Corollary 5.11 (Motive and weighted point count of $\mathcal{L}_{g,|\Delta_g|\cdot n}$ over $\mathbb{F}_q$)**.** *If* $\mathrm{char}(K) = 0$ *or* $> 2g+1$*, then the motive of* $\mathcal{L}_{g,|\Delta_g|\cdot n}$ *in the Grothendieck ring of* K*–stacks* $K_0(\mathrm{Stck}_K)$ *is equivalent to*

$$\begin{aligned}\left[\mathcal{L}_{g,|\Delta_g|\cdot n}\right] &= \left(\sum_{i=0}^{2g-1} \mathbb{L}^i\right) \cdot \left(\mathbb{L}^{|\vec{\lambda_g}|n} - \mathbb{L}^{|\vec{\lambda_g}|n-2g+1}\right) \\ &= \mathbb{L}^{2g(2g+3)n} \cdot (\mathbb{L}^{2g-1} + \mathbb{L}^{2g-2} + \dots + \mathbb{L}^2 + \mathbb{L}^1 - \mathbb{L}^{-1} - \mathbb{L}^{-2} - \dots - \mathbb{L}^{-2g+2} - \mathbb{L}^{-2g+1}).\end{aligned}$$

If $K = \mathbb{F}_q$ *with* $\mathrm{char}(\mathbb{F}_q) > 2g+1$*, then*

$$\#_q\left(\mathcal{L}_{g,|\Delta_g|\cdot n}\right) = q^{2g(2g+3)n} \cdot \left(q^{2g-1} + q^{2g-2} + \dots + q^2 + q^1 - q^{-1} - q^{-2} - \dots - q^{-2g+2} - q^{-2g+1}\right).$$

Proof. This follows from combining Proposition 5.9 and Proposition 5.10 with Corollary 4.8. ■

By working out the birational geometry of surfaces over $\mathrm{char}(K) = 0$ as well as $\mathrm{char}(K) > 2g+1$, we construct a geometric transformation from $\mathcal{S}_g(K)$ the K–points of the moduli functor $\mathcal{S}_g$ of the stable odd hyperelliptic genus $g \geq 2$ models (see Definition 5.4) over $\mathbb{P}^1$ with a marked Weierstrass point to $\mathcal{L}_g(K)$ the K–points of the moduli functor $\mathcal{L}_g$. In fact, this transformation is injective:

Theorem 5.12. *If* $\mathrm{char}(K) = 0$ *or* $\mathrm{char}(K) > 2g+1$*, then there is a canonical fully faithful functor of groupoids* $\mathcal{F} : \mathcal{S}_g(K) \to \mathcal{L}_g(K)$*.*

Proof. The key idea of proof is to construct $\mathcal{F}$ by using relative canonical model, a particular birational transformation from the subject of relative minimal model program. We prove this in a few steps, beginning with a preliminary step. We construct and verify properties of $\mathcal{F}$ in the other steps:

Step 1. Log canonical singularities and log canonical models. The main reference here is [Fujino] when $\mathrm{char}(K) = 0$, and [Tanaka, §5–6] when $\mathrm{char}(K) \neq 0$, noting that both references deal with algebraically closed fields instead.

First, we need the following definition for types of singularities of a pair (S, D) of a normal $\overline{K}$–surface S and an effective $\mathbb{R}$–divisor D on S:

Definition 5.13. ([Fujino, §2.4], [Tanaka, Definition 5.1]) A pair (S, D) is log canonical if

(1) the log canonical divisor $K_S + D$ is $\mathbb{R}$–Cartier,
(2) for any proper birational morphism $\pi : W \to S$ and the divisor D_W defined by

$$K_W + D_W = \pi^*(K_S + D),$$

then $D_W \leq 1$, i.e., when writing $D_W = \sum_i a_i E_i$ as a sum of distinct irreducible divisors E_i, $a_i \leq 1$ for every i.

Moreover, if a pair (S, D) is defined over a non-algebraically closed field K, then it is called log canonical if its base-change to $\overline{K}$ is.

For instance, if S is smooth and D is a reduced simple normal crossing divisor, then (S, D) is log canonical. Similarly, if $w \in \mathbb{R} \cap [0, 1]$, then (S, wD) is log canonical under the same assumptions. Note that we cannot consider $w > 1$ under the same assumptions, as the weight on each irreducible component of D must be at most 1.

For example, consider a stable odd hyperelliptic genus g model (X, H, h, f, s, s') over K, consider the pair $(H_{\overline{K}}, wB_{\overline{K}} + (s'(\mathbb{P}^1_K))_{\overline{K}})$ defined over $\overline{K}$ where the branch divisor of h decomposes as $B \sqcup s'(\mathbb{P}^1_K)$ and $w \in \mathbb{R} \cap (0, 1/2]$ is a weight (since B can have components of multiplicity 2 by Example 5.6, we consider weights at most 1/2). To claim that this pair is log canonical under additional condition on w, it suffices to consider neighborhoods of singular points of $H_{\overline{K}}$ and support of $B_{\overline{K}}$ by the above observation.

First, recall that the isolated singularities of $H_{\overline{K}}$ away from the support of $wB_{\overline{K}} + (s'(\mathbb{P}^1_K))_{\overline{K}}$ is of type $A_{l'}$ for some l' by Example 5.5. Hence, the pair is log canonical at neighborhoods of such points (in fact, those points are called canonical singular points of $H_{\overline{K}}$). Also, at a singular point c of the support of $B_{\overline{K}}$, $H_{\overline{K}}$ is smooth and $B_{\overline{K}}$ is reduced at c but $B_{\overline{K}}$ admits A_l–singularities by Example 5.6. Therefore, the pair has log canonical singularities whenever $w \leq \frac{1}{2} + \frac{1}{l+1}$ by [Järvilehto] (summariezd in [GHM, Introduction], where the log canonical threshold is the supremum of values w that makes the pair log canonical).

To construct a log canonical model, consider a pair (S, D) as the beginning of this step with projective $\overline{K}$–morphism $f : S \to C$ into a $\overline{K}$–variety C, and assume that D is $\mathbb{Q}$–divisor and S is $\mathbb{Q}$–factorial. If (S, D) is log canonical with $K_S + D$ not f–antinef, then [HP, Pages 1750–1751] uses key birational geometry results from [Fujino, Tanaka] to construct the f–log canonical model, defined below. In fact, analogous arguments from [HP, Proof of Proposition 11] implies that the same procedure can be applied to $f : (S, D) \to C$ over a field K, leading to the following definition:

Definition 5.14. Suppose that (S, D) is a log canonical pair over a field K where S is a normal projective $\mathbb{Q}$–factorial surface and D is a $\mathbb{Q}$–divisor. Assume that $f : S \to C$ is a projective morphism into a K–variety C with $K_S + D$ not f–antinef. If K is algebraically closed, then the f–log canonical model is a pair (S', D') with a projective morphism $f' : S' \to C$, where

$$S' := \underline{Proj} \bigoplus_{m \in \mathbb{N}} f_* \mathcal{O}_S(m(K_S + D))$$

and $D' := \phi_* D$ where $\phi : S \to S'$ is the induced birational morphism.

If K is not algebraically closed, then the f–log canonical model is the $\mathrm{Gal}(\overline{K}/K)$–fixed locus of the $f_{\overline{K}}$–log canonical model of $(S_{\overline{K}}, D_{\overline{K}})$.

Step 2. Construction of faithful $\mathcal{F} : \mathcal{S}_g(K) \to \mathcal{L}_g(K)$. Fix any member of $\mathcal{S}_g(K)$, i.e., a stable odd hyperelliptic genus g model (X, H, h, f, s, s') . Denote $B \sqcup s'(\mathbb{P}^1_K)$ to be the divisorial part of the branch locus of $f : X \to H$ (B is also called branch divisor in literature). Notice that h restricted to B has degree $2g+1$. By Step 1 above, $(H, \frac{1}{2g} B + s'(C))$ is log canonical. Take h–log canonical model of $(H, \frac{1}{2g}B + s'(C))/\mathbb{P}^1_K$ to obtain a birational $\mathbb{P}^1_K$–morphism $\varphi : (H, \frac{1}{2g}B + s'(\mathbb{P}^1_K)) \to (H', D')$ where H' is a rational fibration over K and D' is a $\mathbb{R}$–divisor of H' defined over K (c.f. Definition 5.14). Since the only canonical rational curve, defined over an algebraically closed field with $\frac{1}{2g}$ weights on $(2g+1)$ points and weight 1 on another point, is a smooth rational curve where the point of weight 1 is distinct from the other points (of weight $\frac{1}{2g}$), H' is a $\mathbb{P}^1$–bundle (given by $h' : H' \to \mathbb{P}^1_K$). This description shows that D' decomposes into $\frac{1}{2g}A' + T'$ where A' is a divisor of H' and T' consists of weight 1 points on each geometric fiber of $H'/\mathbb{P}^1_K$. Thus, T' comes from a section t' of h' . We will show that H' is the $\mathbb{P}^1$–fibration associated to the desired quasi–admissible hyperelliptic genus g fibration.

To finish the construction of the quasi–admissible fibration, take Stein factorization on $\varphi \circ f$. This gives a finite morphism $f' : X' \to H'$ and a morphism $\psi : X \to X'$ with geometrically connected fibers such that $\varphi \circ f = f' \circ \psi$. Since f is finite of degree 2 and φ is birational, f' is finite of degree 2 and ψ is birational. Moreover, $B' := A' + T'$ is the branch locus of f' . By calling t to be the unique lift of t' on $h' \circ f'$, (X', H', h', f', t, t') is the desired quasi–admissible hyperelliptic fibration. Define $\mathcal{F}(X, H, h, f, s, s') := (X', H', h', f', t, t')$.

To see that $\mathcal{F}$ is faithful, suppose that there are two isomorphisms

$$(\alpha_i, \beta_i) : (X_1, H_1, h_1, f_1, s_1, s'_1) \to (X_2, H_2, h_2, f_2, s_2, s'_2)$$

between stable odd hyperelliptic genus g models that induce the same isomorphism under $\mathcal{F}$:

$$(\alpha', \beta') : \mathcal{F}(X_1, H_1, h_1, f_1, s_1, s'_1) \to \mathcal{F}(X_2, H_2, h_2, f_2, s_2, s'_2)$$

Denote $(X'_j, H'_j, h'_j, f'_j, t_j, t'_j) = \mathcal{F}(X_j, H_j, h_j, f_j, s_j, s'_j)$ for $j = 1, 2$. From the construction of $\mathcal{F}$ shown above, induced morphisms $X_j \to X'_j$ and $H_j \to H'_j$ are birational for each j. Since they are separated varieties over K, (α_1, β_1) must be equal to (α_2, β_2) , hence $\mathcal{F}$ is faithful.

Step 3. Fullness of $\mathcal{F}$. Given any isomorphism ψ between $(X'_i, H'_i, h'_i, f'_i, t_i, t'_i)$'s in $\mathcal{L}_g(K)$ as images of $(X_i, H_i, h_i, f_i, s_i, s'_i) \in \mathcal{S}_g(K)$ under $\mathcal{F}$, notice that h'_i's and $h'_i \circ f'_i$'s have smooth geometric generic fibers for $i = 1, 2$ and ψ comes in pairs of isomorphisms $\psi_1 : X'_1 \to X'_2$ and $\psi_2 : H'_1 \to H'_2$ (so denote $\psi = (\psi_1, \psi_2)$). Then, ψ lifts to a pair of birational maps $\overline{\psi} = (\overline{\psi}_1, \overline{\psi}_2)$ between X_i's and H_i's which induce isomorphisms on geometric generic fibers and irreducible components of any geometric fiber meeting the sections s_i's or s'_i's. To claim that those extend to isomorphisms that induce ψ_i's, it suffices to understand geometric properties of related moduli stacks, as we claim that ψ_i's can be interpreted as an element of $Isom$ spaces of such stacks.

Observe first that for each $i = 1, 2$, H_i is a $\mathbb{Z}/2\mathbb{Z}$–quotient of X_i, and $K_{X_i} + s_i(\mathbb{P}^1_K)$ is ample over $\mathbb{P}^1_K$ by the defintion. Since the branch divisor of f_i is $B_i \sqcup s'_i(\mathbb{P}^1_K)$, the log canonical divisor $K_{H_i} + \frac{1}{2}B_i + s'_i(\mathbb{P}^1_K)$ is also ample over $\mathbb{P}^1_K$ as $f_i^*(K_{H_i} + \frac{1}{2}B_i + s'_i(\mathbb{P}^1_K)) = K_{X_i} + s_i(\mathbb{P}^1_K)$. Since X_i admits nodes as the only singularities of geometric fibers, B_i on each fiber has multiplicity at most 2 at any $\overline{K}$–points in the support. Therefore, fibers of the pair $(H_i, \frac{1}{2}B_i + s'_i(\mathbb{P}^1_K))$ are $((\frac{1}{2}, 2g+1), (1, 1))$–stable curves of genus 0 in the sense of [Hassett, §2.1.3], meaning that H_i for each i is a family of such curves over $\mathbb{P}^1_K$. Note that the moduli stack $\overline{\mathcal{M}}_{0,((\frac{1}{2},2g+1),(1,1))}$ of $((\frac{1}{2}, 2g+1), (1, 1))$–stable curves of genus 0 is a proper (so separated) Deligne–Mumford stack (it easily follows from loc.cit. and [Hassett, Theorem 2.1]), and H_i is realized as $\alpha_i : \mathbb{P}^1_K \to \overline{\mathcal{M}}_{0,((\frac{1}{2},2g+1),(1,1))}$. Since there is a nonempty open subset $U \subset \mathbb{P}^1_K$ such that ψ_2 induces an isomorphism between $h_i^{-1}(U)$'s, $\overline{\psi}_2$ is an element of $Isom_{\overline{\mathcal{M}}_{0,((\frac{1}{2},2g+1),(1,1))}}(\alpha_1, \alpha_2)(U)$. Then, separatedness of $\overline{\mathcal{M}}_{0,((\frac{1}{2},2g+1),(1,1))}$ implies that $\overline{\psi_2}$ extends to an isomorphism between H_i's.

Similar argument shows that ψ_1 also extends to an isomorphism between X_i's (as $\overline{H}_{g,\underline{1}} \subset \overline{\mathcal{M}}_{2,\overline{1}}$ is a separated Deligne–Mumford stack by [Knudsen]), so it suffices to show that $\overline{\psi}_i$'s commute with

f_i's and induce ψ. The commutativity of $\overline{\psi}_i$'s follows because H_i's are $\mathbb{Z}/2\mathbb{Z}$–quotients of X_i's and any isomorphism between families of stable hyperelliptic curves with marked Weierstrass points commute with $\mathbb{Z}/2\mathbb{Z}$–actions. Because the birational morphisms $X_i \to X'_i$ and $H_i \to H'_i$ for any i contract all but irreducible components of fibers over $\mathbb{P}^1_K$ meeting marked sections, $\overline{\psi} := (\overline{\psi}_1, \overline{\psi}_2)$ induce ψ as well. Henceforth, $\overline{\psi}$ maps to ψ under $\mathcal{F}$, proving that $\mathcal{F}$ is full. ∎

Remark 5.15. Due to log abundance being a conjecture for higher dimensions, which is a key ingredient of the existence of log canonical models (c.f. [HP, Remark 13]), it is unclear whether $\mathcal{F}$ in the proof above extends to a functor from the moduli of stable odd hyperelliptic genus g models to $\mathcal{L}_g$. If it extends, we expect the functor to be still fully faithful, as opposed to [HP, Remark 13] for birational transformations between semistable elliptic surfaces and stable elliptic curves over $\mathbb{P}^1$. The key obstruction on [HP, Remark 13], assuming that the functor discussed in loc.cit. (which is an equivalence) extends, is that the essential surjectivity may not hold on the extension, whereas the functor from Theorem 5.12 is not even essentially surjective to begin with.

5.1. **Hyperelliptic discriminant Δ_g of quasi-admissible hyperelliptic genus g fibration.** As we consider the algebraic surfaces X as fibrations in genus g curves over $\mathbb{P}^1$, the discriminant $\Delta_g(X)$ is the basic invariant of X. For the quasi-admissible hyperelliptic genus g fibrations over $\mathbb{P}^1$, we have the work of [Lockhart, Liu2] which describes the hyperelliptic discriminant $\Delta_g(X)$.

Definition 5.16. [Lockhart, Definition 1.6, Proposition 1.10] The hyperelliptic discriminant Δ_g of the monic odd degree Weierstrass equation $y^2 = x^{2g+1} + a_4x^{2g-1} + a_6x^{2g-2} + a_8x^{2g-3} + \cdots + a_{4g+2}$ over a base field K with $\mathrm{char}(K) \neq 2$ is

$$\Delta_g = 2^{4g} \cdot \mathrm{Disc}(x^{2g+1} + a_4x^{2g-1} + a_6x^{2g-2} + \cdots + a_{4g+2})$$

which has $\deg(\Delta_g) := |\Delta_g| = 4g(2g+1)$ formally when we associate each variable a_i with degree i.

Note that when $g = 1$, the discriminant Δ_1 of the short Weierstrass equation $y^2 = x^3 + a_4x + a_6$ coincides with the usual discriminant $-16(4a_4^3 - 27a_6^2)$ of an elliptic curve. We can now formulate the moduli stack $\mathcal{L}_{g,|\Delta_g|\cdot n}$ of quasi-admissible fibration over $\mathbb{P}^1$ with a fixed discriminant degree $|\Delta_g| \cdot n = 4g(2g+1)n$ and a marked Weierstrass point :

Proposition 5.17. *Assume* $\mathrm{char}(K) = 0$ *or* $> 2g+1$. *Then, the moduli stack* $\mathcal{L}_{g,|\Delta_g|\cdot n}$ *of quasi-admissible hyperelliptic genus g fibrations over* $\mathbb{P}^1_K$ *with a marked Weierstrass point and a hyperelliptic discriminant of degree* $|\Delta_g| \cdot n = 4g(2g+1)n$ *over a base field K is the tame Deligne–Mumford Hom stack* $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{H}_{2g}[2g-1])$ *parameterizing the K-morphisms* $f : \mathbb{P}^1 \to \mathcal{H}_{2g}[2g-1]$ *with* $\mathcal{H}_{2g}[2g-1] \cong \mathcal{P}(\vec{\lambda}_g) = \mathcal{P}(4,6,8,\ldots,4g+2)$ *such that* $f^*\mathcal{O}_{\mathcal{P}(\vec{\lambda}_g)}(1) \cong \mathcal{O}_{\mathbb{P}^1}(n)$ *with* $n \in \mathbb{N}$.

Proof. Since $\deg f^*\mathcal{O}_{\mathcal{P}(\vec{\lambda}_g)}(1) = n$ is an open condition, $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{H}_{2g}[2g-1])$ is an open substack of $\mathrm{Hom}(\mathbb{P}^1, \mathcal{H}_{2g}[2g-1])$. Now, it suffices to show that $\deg f = n$ (i.e., $\deg f^*\mathcal{O}_{\mathcal{P}(\vec{\lambda}_g)}(1) = n$) if and only if the discriminant degree of the corresponding quasi-admissible fibration is $4g(2g+1)n$. Note that $\deg f = n$ if and only if the quasi-admissible fibration is given by the Weierstrass equation

$$y^2 = x^{2g+1} + a_4x^{2g-1} + a_6x^{2g-2} + \cdots + a_{4g+2}$$

where a_i's are sections of $\mathcal{O}(in)$, since a_i's represent the coordinates of $\mathcal{P}(4,6,\ldots,4g+2)$. Then by Definition 5.16, it is straightforward to check that Δ_g has the discriminant degree $4g(2g+1)n$. ∎

Now we are ready to count the number $|\mathcal{L}_{g,|\Delta_g|\cdot n}(\mathbb{F}_q)/\sim|$ of $\mathbb{F}_q$-isomorphism classes of quasi-admissible genus g fibrations over $\mathbb{P}^1_{\mathbb{F}_q}$:

Proof of Theorem 1.3. By Proposition 4.10, for a fixed g, it suffices to understand when a connected component $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}((\vec{\lambda}_g)_r))$ (indexed by r) of $\mathcal{I}(\mathcal{L}_{g,|\Delta_g|\cdot n})$ is nonempty for $\vec{\lambda}_g = (4,6,\ldots,4g+2)$;

r	$\varphi(r)$	$g=2$	$g=3$	$g=4$
$1,2$	1	$(4,6,8,10)$	$(4,6,8,10,12,14)$	$(4,6,8,10,12,14,16,18)$
3	2	–	$(6,12)$	$(6,12,18)$
4	2	$(4,8)$	$(4,8,12)$	$(4,8,12,16)$
6	2	–	$(6,12)$	$(6,12,18)$
8	4	–	–	$(8,16)$

TABLE 1. Table of all tuples $(\vec{\lambda}_g)_r$ of length at least two for low genus $g = 2, 3, 4$. Entry has – when $(\vec{\lambda}_g)_r$ has length zero or one.

this is equivalent to finding r such that the subtuple $(\vec{\lambda}_g)_r$ has length at least two. Table 1 describes all such possible r's for given low values of $g = 2, 3, 4$:

Summing the weighted point counts of Hom stacks from Proposition 4.5 into Proposition 4.10 gives the desired formula, where we use the division function $\delta(r, q-1)$ (defined in Theorem 1.3) to indicate that we take the contribution of $\#_q(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}_r)))$ whenever $r \in R$ (i.e., r divides $q-1$).

The same method directly applies when $g \geq 5$. ■

6. Counting stable odd degree hyperelliptic curves over global fields by Δ_g

We first recall the definition of a *global field*. Let S be the set of places of a field K and $|\cdot|_v$ be the normalized absolute value for each place $v \in S$.

Definition 6.1. A field K is a global field if all completions K_v of K at each place $v \in S$ is a local field, and K satisfies the product formula $\prod\limits_v |\alpha|_v = 1$ over all places $v \in S$ for all $\alpha \in K^*$.

And we have the fundamental Artin-Whaples Theorem proven in 1945 [AW, AW2] which emphasized the close analogy between the theory of algebraic number fields and the theory of function fields of algebraic curves over finite fields. The axiomatic method used in these papers unified the two global fields from a valuation theoretic perspective by clarifying the role of the product formula.

Theorem 6.2 (Artin-Whaples Theorem)**.** *Every global field is a finite extension of* $\mathbb{Q}$ *or* $\mathbb{F}_q(t)$.

Focusing upon the global fields $\mathbb{F}_q(t)$ or $\mathbb{Q}$, in order to draw the analogy, we need to fix an affine chart $\mathbb{A}^1_{\mathbb{F}_q} \subset \mathbb{P}^1_{\mathbb{F}_q}$ and its corresponding ring of functions $\mathbb{F}_q[t]$ interpreted as the ring of integers of the field of fractions $\mathbb{F}_q(t)$ of $\mathbb{P}^1_{\mathbb{F}_q}$. This is necessary since $\mathbb{F}_q[t]$ could come from any affine chart of $\mathbb{P}^1_{\mathbb{F}_q}$, whereas the ring of integers $\mathcal{O}_K$ for the number field K is canonically determined. We denote $\infty \in \mathbb{P}^1_{\mathbb{F}_q}$ to be the unique point not in the chosen affine chart.

Note that for a maximal ideal $\mathfrak{p}$ in $\mathcal{O}_K$, the residue field $\mathcal{O}_K/\mathfrak{p}$ is finite for both of our global fields. One could think of $\mathfrak{p}$ as a point in Spec $\mathcal{O}_K$ and define the *height* of a point $\mathfrak{p}$.

Definition 6.3. Define the height of a point $\mathfrak{p}$ to be $ht(\mathfrak{p}) := |\mathcal{O}_K/\mathfrak{p}|$ the cardinality of the residue field $\mathcal{O}_K/\mathfrak{p}$.

We recall the notion of *bad reduction* & *good reduction*:

Definition 6.4. Let C be an odd degree hyperelliptic genus g curve over K given by the odd degree Weierstrass equation

$$y^2 = x^{2g+1} + a_4 x^{2g-1} + a_6 x^{2g-2} + \cdots + a_{4g+2},$$

with $a_{2i+2} \in \mathcal{O}_K$ for every $1 \leq i \leq 2g$. Then C has bad reduction at $\mathfrak{p}$ if the fiber $C_{\mathfrak{p}}$ over $\mathfrak{p}$ is a singular curve of degree $2g+1$. The prime $\mathfrak{p}$ is said to be of good reduction if $C_{\mathfrak{p}}$ is a smooth hyperelliptic genus g curve.

Consider the case when $K = \mathbb{F}_q(t)$, and a quasi-admissible model $f : X \to \mathbb{P}^1_{\mathbb{F}_q}$ (a quasi–admissible fibration with smooth geometric generic fiber). For simplicity, assume that X does not have a singular fiber over $\infty \in \mathbb{P}^1_{\mathbb{F}_q}$. Note that the primes $\mathfrak{p}_i$ of bad reductions of f are precisely points of the discriminant divisor $\Delta_g(X) = \sum k_i \cdot \mathfrak{p}_i$, as the fiber $X_{\mathfrak{p}_i}$ is singular over $\Delta_g(X)$. When $K = \mathbb{F}_q(t)$ the global function field, we have $\Delta_g(X) \in H^0(\mathbb{P}^1, \mathcal{O}(4g(2g+1)n))$ by the proof of Proposition 5.17. We can define the height of $\Delta_g(X)$ as follows:

Definition 6.5. The height $ht(\Delta_g(X))$ of a discriminant divisor $\Delta_g(X)$ in $\mathbb{P}^1_{\mathbb{F}_q}$ is $q^{\deg \Delta_g(X)}$. As a convention, if a divisor $\Delta_g(X)$ is given as a zero section of any line bundle, then set $ht(\Delta_g(X)) = \infty$.

In general, the height of a hyperelliptic discriminant $\Delta_g(X)$ of any X (without nonsingular fiber assumption over ∞) is defined as $q^{|\Delta_g(X)|}$ where $\deg(\Delta_g(X)) := |\Delta_g(X)|$ is equal to $4g(2g+1)n$.

As the hyperelliptic discriminant divisor $\Delta_g(X)$ is an invariant of a quasi-admissible model $f : X \to \mathbb{P}^1$, we count the number of $\mathbb{F}_q$–isomorphism classes of quasi-admissible hyperelliptic genus g fibrations on the function field $\mathbb{F}_q(t)$ by the bounded height of $\Delta_g(X)$:

$$\mathcal{Z}'_{g,\mathbb{F}_q(t)}(\mathcal{B}) := |\{\text{Quasi-admissible hyperelliptic genus } g \text{ curves over } \mathbb{P}^1_{\mathbb{F}_q} \text{ with } 0 < ht(\Delta_g(X)) \leq \mathcal{B}\}|$$

A very important consequence of Theorem 5.12 is that the number $\mathcal{Z}'_{g,\mathbb{F}_q(t)}(\mathcal{B})$ of quasi–admissible genus $g \geq 2$ curves over $\mathbb{P}^1_{\mathbb{F}_q}$ ordered by height of hyperelliptic discriminant $ht(\Delta_g(X))$ gives an upper bound for the number $\mathcal{Z}_{g,\mathbb{F}_q(t)}(\mathcal{B})$ of stable odd hyperelliptic genus $g \geq 2$ models over $\mathbb{P}^1_{\mathbb{F}_q}$ ordered by height of hyperelliptic discriminant $ht(\Delta_g(X))$:

$$\mathcal{Z}_{g,\mathbb{F}_q(t)}(\mathcal{B}) \leq \mathcal{Z}'_{g,\mathbb{F}_q(t)}(\mathcal{B}) \tag{9}$$

We now prove the sharp estimates on $\mathcal{Z}'_{g,\mathbb{F}_q(t)}(\mathcal{B})$ which induces the upper bounds on $\mathcal{Z}_{g,\mathbb{F}_q(t)}(\mathcal{B})$:

Theorem 6.6 (Estimate on $\mathcal{Z}_{g,\mathbb{F}_q(t)}(\mathcal{B})$)**.** *If $char(\mathbb{F}_q) > 2g+1$, then the function $\mathcal{Z}_{g,\mathbb{F}_q(t)}(\mathcal{B})$, which counts the number of stable hyperelliptic genus $g \geq 2$ curves X with a marked Weierstrass point over $\mathbb{P}^1_{\mathbb{F}_q}$ ordered by $0 < ht(\Delta_g(X)) = q^{4g(2g+1)n} \leq \mathcal{B}$, satisfies:*

$$\mathcal{Z}_{2,\mathbb{F}_q(t)}(\mathcal{B}) \leq 2\frac{(q^{31}+q^{30}+q^{29}-q^{27}-q^{26}-q^{25})}{(q^{28}-1)} \cdot \left(\mathcal{B}^{\frac{7}{10}}-1\right) + 2\delta(4,q-1)\frac{(q^{13}-q^{11})}{(q^{12}-1)} \cdot \left(\mathcal{B}^{\frac{3}{10}}-1\right)$$

$$\begin{aligned}\mathcal{Z}_{3,\mathbb{F}_q(t)}(\mathcal{B}) \leq\ & 2\frac{(q^{59}+q^{58}+\cdots+q^{55}-q^{53}-q^{52}-\cdots-q^{49})}{(q^{54}-1)} \cdot \left(\mathcal{B}^{\frac{9}{14}}-1\right) \\ & + 2\delta(4,q-1)\frac{(q^{26}+q^{25}-q^{23}-q^{22})}{(q^{24}-1)} \cdot \left(\mathcal{B}^{\frac{2}{7}}-1\right) + 4\delta(6,q-1)\frac{(q^{19}-q^{17})}{(q^{18}-1)} \cdot \left(\mathcal{B}^{\frac{3}{14}}-1\right)\end{aligned}$$

$$
\begin{aligned}
\mathcal{Z}_{4,\mathbb{F}_q(t)}(\mathcal{B}) \leq\ & 2\frac{(q^{95}+q^{94}+\cdots+q^{89}-q^{87}-q^{52}-\cdots-q^{81})}{(q^{88}-1)}\cdot\left(\mathcal{B}^{\frac{11}{18}}-1\right) \\
&+ 2\delta(4,q-1)\frac{(q^{43}+q^{42}+q^{41}-q^{39}-q^{38}-q^{37})}{(q^{40}-1)}\cdot\left(\mathcal{B}^{\frac{5}{18}}-1\right) \\
&+ 4\delta(6,q-1)\frac{(q^{38}+q^{37}-q^{35}-q^{34})}{(q^{36}-1)}\cdot\left(\mathcal{B}^{\frac{1}{4}}-1\right)+4\delta(8,q-1)\frac{(q^{25}-q^{23})}{(q^{24}-1)}\cdot\left(\mathcal{B}^{\frac{1}{6}}-1\right)
\end{aligned}
$$

Proof of Theorem 1.4 for $g=2$. Knowing the number of $\mathbb{F}_q$-isomorphism classes of quasi-admissible genus 2 fibrations of discriminant degree $40n$ over $\mathbb{F}_q$ is

$$|\mathcal{L}_{2,40n}(\mathbb{F}_q)/\sim| = 2q^{28n}\cdot(q^3+q^2+q^1-q^{-1}-q^{-2}-q^{-3})+2\delta(4,q-1)q^{12n}\cdot(q^1-q^{-1})$$

by Theorem 1.3, we explicitly compute the upper bound for $\mathcal{Z}'_{2,\mathbb{F}_q(t)}(\mathcal{B})$ as the following,

$$
\begin{aligned}
\mathcal{Z}'_{2,\mathbb{F}_q(t)}(\mathcal{B}) =\ & \sum_{n=1}^{\left\lfloor\frac{log_q\mathcal{B}}{40}\right\rfloor} |\mathcal{L}_{2,40n}(\mathbb{F}_q)/\sim| \\
=\ & \sum_{n=1}^{\left\lfloor\frac{log_q\mathcal{B}}{40}\right\rfloor} 2\cdot q^{28n}\cdot(q^3+q^2+q^1-q^{-1}-q^{-2}-q^{-3})+2\delta(4,q-1)q^{12n}\cdot(q^1-q^{-1}) \\
=\ & 2\cdot(q^3+q^2+q^1-q^{-1}-q^{-2}-q^{-3})\sum_{n=1}^{\left\lfloor\frac{log_q\mathcal{B}}{40}\right\rfloor} q^{28n}+2\delta(4,q-1)(q^1-q^{-1})\sum_{n=1}^{\left\lfloor\frac{log_q\mathcal{B}}{40}\right\rfloor} q^{12n} \\
\leq\ & 2\cdot(q^3+q^2+q^1-q^{-1}-q^{-2}-q^{-3})\left(q^{28}+\cdots+q^{28\cdot(\frac{log_q\mathcal{B}}{40})}\right) \\
&+2\delta(4,q-1)(q^1-q^{-1})\left(q^{12}+\cdots+q^{12\cdot(\frac{log_q\mathcal{B}}{40})}\right) \\
=\ & 2\cdot(q^3+q^2+q^1-q^{-1}-q^{-2}-q^{-3})\left(\frac{q^{28}\cdot(\mathcal{B}^{\frac{7}{10}}-1)}{(q^{28}-1)}\right) \\
&+2\delta(4,q-1)(q^1-q^{-1})\left(\frac{q^{12}\cdot(\mathcal{B}^{\frac{3}{10}}-1)}{(q^{12}-1)}\right) \\
=\ & 2\frac{(q^{31}+q^{30}+q^{29}-q^{27}-q^{26}-q^{25})}{(q^{28}-1)}\cdot\left(\mathcal{B}^{\frac{7}{10}}-1\right)+2\delta(4,q-1)\frac{(q^{13}-q^{11})}{(q^{12}-1)}\cdot\left(\mathcal{B}^{\frac{3}{10}}-1\right)
\end{aligned}
$$

On the fourth line of the equations above, inequality becomes an equality if and only if $n:=\frac{log_q\mathcal{B}}{40}\in\mathbb{N}$, i.e., $\mathcal{B}=q^{40n}$ with $n\in\mathbb{N}$. This implies that the acquired upper bound on $\mathcal{Z}'_{2,\mathbb{F}_q(t)}(\mathcal{B})$ is a sum of power functions, where each monomial is in $\mathcal{O}_q\left(\mathcal{B}^{\frac{7}{10}}\right),\mathcal{O}_q\left(\mathcal{B}^{\frac{3}{10}}\right)$ and $\mathcal{O}_q(1)$ such that $\mathcal{O}_q$-constant with respect to $\mathcal{B}$ is an explicit rational function of q. ■

As there are non-hyperelliptic curves for higher genus $g\geq 3$ curves, $\mathcal{Z}'_{g\geq3,\mathbb{F}_q(t)}(\mathcal{B})$ counts the quasi-admissible hyperelliptic genus $g\geq3$ curves over $\mathbb{P}^1_{\mathbb{F}_q}$ only. We determine $\mathcal{Z}'_{3,\mathbb{F}_q(t)}(\mathcal{B})$ explicitly thereby counting the quasi-admissible hyperelliptic genus 3 curves over $\mathbb{P}^1_{\mathbb{F}_q}$.

Proof of Theorem 1.4 for $g=3$. Knowing the number of $\mathbb{F}_q$-isomorphism classes of quasi-admissible hyperelliptic genus 3 fibrations of discriminant degree $84n$ over $\mathbb{F}_q$ is $|\mathcal{L}_{3,84n}(\mathbb{F}_q)/\sim| = 2\cdot q^{54n}\cdot(q^5+\cdots+q^1-q^{-1}-\cdots-q^{-5})+2\delta(4,q-1)q^{24n}\cdot(q^2+q^1-q^{-1}-q^{-2})+4\delta(6,q-1)q^{18n}\cdot(q^1-q^{-1})$ by Theorem 1.3, we explicitly compute the upper bound for $\mathcal{Z}'_{3,\mathbb{F}_q(t)}(\mathcal{B})$ similarly as genus 2 case. ■

We conclude with $\mathcal{Z}'_{4,\mathbb{F}_q(t)}(\mathcal{B})$ counting the quasi-admissible hyperelliptic genus 4 curves over $\mathbb{P}^1_{\mathbb{F}_q}$.

Proof of Theorem 1.4 for $g = 4$. Knowing the number of $\mathbb{F}_q$-isomorphism classes of quasi-admissible hyperelliptic genus 4 fibrations of discriminant degree $144n$ over $\mathbb{F}_q$ is $|\mathcal{L}_{4,144n}(\mathbb{F}_q)/\sim| = 2 \cdot q^{88n} \cdot (q^7 + \cdots + q^1 - q^{-1} - \cdots - q^{-7}) + 2\delta(4, q-1)q^{40n} \cdot (q^3 + q^2 + q^1 - q^{-1} - q^{-2} - q^{-3}) + 4\delta(6, q-1)q^{36n} \cdot (q^2 + q^1 - q^{-1} - q^{-2}) + 4\delta(8, q-1)q^{24n} \cdot (q^1 - q^{-1})$ by Theorem 1.3, we explicitly compute the upper bound for $\mathcal{Z}'_{4,\mathbb{F}_q(t)}(\mathcal{B})$ similarly as genus 2 case. ∎

The computation of explicit upper bounds with precise lower order terms for the higher genus cases $\mathcal{Z}'_{g,\mathbb{F}_q(t)}(\mathcal{B})$ can be done similarly after working out $\left|\mathcal{L}_{g,|\Delta_g|\cdot n}(\mathbb{F}_q)/\sim\right|$ by the Theorem 1.3. While the lower order terms vary, the order of the leading term can be found by the following. Below, we prove the following closed-form upper bound for $\mathcal{Z}_{g,\mathbb{F}_q(t)}(\mathcal{B})$:

Proposition 6.7 (Estimate on $\mathcal{Z}_{g,\mathbb{F}_q(t)}(\mathcal{B})$)**.** *If $char(\mathbb{F}_q) > 2g+1$, then the function $\mathcal{Z}_{g,\mathbb{F}_q(t)}(\mathcal{B})$, which counts the number of stable hyperelliptic genus $g \geq 2$ curves X with a marked Weierstrass point over $\mathbb{P}^1_{\mathbb{F}_q}$ ordered by $0 < ht(\Delta_g(X)) = q^{4g(2g+1)n} \leq \mathcal{B}$, satisfies:*

$$\mathcal{Z}_{g,\mathbb{F}_q(t)}(\mathcal{B}) \leq 2g \cdot \frac{q^{4g(g+1)+1} \cdot (q^{2g-1} - 1)(q^{2g} - 1)}{(q-1)(q^{2g(2g+3)} - 1)} \cdot \left(\mathcal{B}^{\frac{2g+3}{4g+2}} - 1\right)$$

Proof. Note that the automorphism group of minimum order of φ_g is the generic stabilizer group $\mu_\delta = \mu_2$ of $\mathcal{P}(\vec{\lambda_g})$ and the automorphism group of maximum order of φ_g is $\mu_\omega = \mu_{2g}$ as $2g$ is the maximum value of GCD for all possible pairs among $\vec{\lambda_g} = (4, 6, 8, \ldots, 4g+2)$. By Corollary 4.8 we know that the number of $\mathbb{F}_q$-isomorphism classes of quasi-admissible hyperelliptic genus g fibrations of hyperelliptic discriminant degree $|\Delta_g| \cdot n = 4g(2g+1)n$ over $\mathbb{F}_q$ is $2 \cdot \#_q\left(\mathcal{L}_{g,|\Delta_g|\cdot n}\right) \leq \left|\mathcal{L}_{g,|\Delta_g|\cdot n}(\mathbb{F}_q)/\sim\right| \leq 2g \cdot \#_q\left(\mathcal{L}_{g,|\Delta_g|\cdot n}\right)$, we can explicitly compute the upper bound for $\mathcal{Z}'_{g,\mathbb{F}_q(t)}(\mathcal{B})$ as the following,

$$\begin{aligned}
\mathcal{Z}'_{g,\mathbb{F}_q(t)}(\mathcal{B}) &= \sum_{n=1}^{\left\lfloor \frac{log_q \mathcal{B}}{4g(2g+1)} \right\rfloor} |\mathcal{L}_{g,|\Delta_g|\cdot n}(\mathbb{F}_q)/\sim| \\
&\leq \sum_{n=1}^{\left\lfloor \frac{log_q \mathcal{B}}{4g(2g+1)} \right\rfloor} 2g \cdot q^{2g(2g+3)n} \cdot \frac{(q^{2g-1} - 1)(q - q^{-2g+1})}{q-1} \\
&= 2g \cdot \frac{(q^{2g-1} - 1)(q - q^{-2g+1})}{q-1} \sum_{n=1}^{\left\lfloor \frac{log_q \mathcal{B}}{4g(2g+1)} \right\rfloor} q^{2g(2g+3)n} \\
&= 2g \cdot \frac{(q^{2g-1} - 1)(q - q^{-2g+1})}{q-1} \left(q^{2g(2g+3)} + \cdots + q^{2g(2g+3)\cdot(\frac{log_q \mathcal{B}}{4g(2g+1)})}\right) \\
&= 2g \cdot \frac{(q^{2g-1} - 1)(q - q^{-2g+1})}{q-1} \cdot \left(\frac{q^{2g(2g+3)} \cdot (\mathcal{B}^{\frac{2g+3}{4g+2}} - 1)}{q^{2g(2g+3)} - 1}\right) \\
&= 2g \cdot \frac{q^{4g(g+1)+1} \cdot (q^{2g-1} - 1)(q^{2g} - 1)}{(q-1)(q^{2g(2g+3)} - 1)} \cdot \left(\mathcal{B}^{\frac{2g+3}{4g+2}} - 1\right)
\end{aligned}$$

This implies that this upper bound has the leading term of order $\mathcal{O}_q\left(\mathcal{B}^{\frac{2g+3}{4g+2}}\right)$ where $\mathcal{O}_q$-constant is an explicit rational function of q with the corresponding lower order terms for each genus $g \geq 2$. ∎

For higher genus $g \geq 3$, we count the Jacobians of hyperelliptic curves.

Corollary 6.8 (Estimate on $\mathcal{N}_{g,\mathbb{F}_q(t)}(\mathcal{B})$)**.** *If char*$(\mathbb{F}_q) > 2g+1$*, then the function* $\mathcal{N}_{g,\mathbb{F}_q(t)}(\mathcal{B})$*, which counts the number of principally polarized hyperelliptic Jacobians* $A = \mathrm{Jac}(X)$ *where* X *is a stable hyperelliptic genus* $g \geq 3$ *curves* X *with a marked Weierstrass point over* $\mathbb{P}^1_{\mathbb{F}_q}$ *ordered by* $0 < ht(\Delta_g(X)) = q^{4g(2g+1)n} \leq \mathcal{B}$*, satisfies:*

$$\mathcal{N}_{g,\mathbb{F}_q(t)}(\mathcal{B}) \leq 2g \cdot \frac{q^{4g(g+1)+1} \cdot (q^{2g-1}-1)(q^{2g}-1)}{(q-1)(q^{2g(2g+3)}-1)} \cdot \left(\mathcal{B}^{\frac{2g+3}{4g+2}} - 1\right)$$

Proof. Proposition 6.7 provides an upper bound on the number of stable hyperelliptic genus $g \geq 3$ curves with a marked Weierstrass point over $\mathbb{P}^1_{\mathbb{F}_q}$ with $\mathrm{char}(\mathbb{F}_q) > 2g+1$. The upper bound follows from the open immersion property of the Torelli map τ_g restricted to the hyperelliptic locus (c.f. [Landesman, Theorem 1.2]). Note that one could sharpen the upper bound through the Theorem 1.3 to acquire the precise lower order terms for each genus $g \geq 5$. ■

Switching to the number field with $K = \mathbb{Q}$ and $\mathcal{O}_K = \mathbb{Z}$, one could choose the minimal integral Weierstrass model of a stable odd degree hyperelliptic curve with the given hyperelliptic discriminant divisor Δ_g which is already a number. This renders the following Conjecture 1.8 and Conjecture 1.9, stated in the Introduction.

7. Appendix - Arithmetic of the moduli of elliptic curves with a prescribed level structure or multiple marked points

In this appendix, we extend the sharp estimate on the number of semistable elliptic curves over $\mathbb{P}^1_{\mathbb{F}_q}$ from [HP, Theorem 3] by using the above methods (i.e., Proposition 4.5 into Proposition 4.10).

Specifically, we explicitly estimate the sharp bound on the number of elliptic curves over global function fields $\mathbb{F}_q(t)$ with level structures $[\Gamma_1(n)]$ for $2 \leq n \leq 4$ or $[\Gamma(2)]$. Recall that a level structure $[\Gamma_1(n)]$ on an elliptic curve E is a choice of point $P \in E$ of exact order n in the smooth part of E such that over every geometric point of the base scheme every irreducible component of E contains a multiple of P (see [KM, §1.4]). And a level structure $[\Gamma(2)]$ on an elliptic curve E is a choice of isomorphism $\phi : \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to E(2)$ where $E(2)$ is the scheme of 2-torsion Weierstrass points (i.e., kernel of the multiplication-by-2 map $[2] : E \to E$) (see [DR, II.1.18 & IV.2.3]).

Additionally, we consider curves of arithmetic genus one over $\mathbb{F}_q(t)$ with m-marked rational points for $2 \leq m \leq 5$ by instead acquiring sharp estimate on the number of $(m-1)$-stable m-marked curves of arithmetic genus one formulated originally by the works of [Smyth, Smyth2].

To estimate the number of certain elliptic curves over global function fields $\mathbb{F}_q(t)$ with level structures $[\Gamma(n)]$ or $[\Gamma_1(n)]$ as in Theorem 1.4, we need to extend the notion of (nonsingular) elliptic curves (semistable in the case of [HP]) that admits desired level structures. By the work of Deligne and Rapoport [DR] (summarized in [Niles, §2]), we consider the generalized elliptic curves over $\mathbb{P}^1_K$ with $[\Gamma]$–structures (where Γ is $\Gamma(n)$ or $\Gamma_1(n)$) over a field K (focusing on $K = \mathbb{F}_q$). Roughly, a generalized elliptic curve X over $\mathbb{P}^1_K$ can be thought of as a flat family of semistable elliptic curves admitting a group structure, such that a finite group scheme $\mathcal{G} \to \mathbb{P}^1_K$ (determined by Γ) embeds into X and its image meets every irreducible component of every geometric fibers of X. Again, we only consider the *non-isotrivial* generalized elliptic curves. If X is as above, then $\Delta(X)$ is the discriminant of a generalized elliptic curve and if $K = \mathbb{F}_q$, then $0 < ht(\Delta(X)) := q^{\deg \Delta(X)}$.

Now, define $\mathcal{Z}^{[\Gamma]}_{1,\mathbb{F}_q(t)}(\mathcal{B})$ as follows:

$$\mathcal{Z}^{[\Gamma]}_{1,\mathbb{F}_q(t)}(\mathcal{B}) := |\{\text{Generalized elliptic curves over } \mathbb{P}^1_{\mathbb{F}_q} \text{ with } [\Gamma]\text{–structures and } 0 < ht(\Delta(X)) \leq \mathcal{B}\}|$$

Then, we acquire the following descriptions of $\mathcal{Z}^{[\Gamma]}_{1,\mathbb{F}_q(t)}(\mathcal{B})$:

Theorem 7.1 (Sharp estimate of $\mathcal{Z}^{[\Gamma]}_{1,\mathbb{F}_q(t)}(\mathcal{B})$)**.** *If* $char(\mathbb{F}_q) \neq 2$*, then the function* $\mathcal{Z}^{[\Gamma]}_{1,\mathbb{F}_q(t)}(\mathcal{B})$ $\big(char(\mathbb{F}_q) \neq 3$ *for* $\mathcal{Z}^{[\Gamma_1(3)]}_{1,\mathbb{F}_q(t)}(\mathcal{B})\big)$*, which counts the number of generalized elliptic curves with* $[\Gamma]$*-structures over* $\mathbb{P}^1_{\mathbb{F}_q}$ *ordered by* $0 < ht(\Delta(X)) = q^{12n} \leq \mathcal{B}$*, satisfies:*

$$\mathcal{Z}^{[\Gamma_1(2)]}_{1,\mathbb{F}_q(t)}(\mathcal{B}) \leq 2 \cdot \frac{(q^7 - q^5)}{(q^6 - 1)} \cdot \mathcal{B}^{\frac{1}{2}} - 2 \cdot \frac{(q^7 - q^5)}{(q^6 - 1)}$$

$$\mathcal{Z}^{[\Gamma_1(3)]}_{1,\mathbb{F}_q(t)}(\mathcal{B}) \leq \frac{(q^5 - q^3)}{(q^4 - 1)} \cdot \mathcal{B}^{\frac{1}{3}} - \frac{(q^5 - q^3)}{(q^4 - 1)}$$

$$\mathcal{Z}^{[\Gamma_1(4)]}_{1,\mathbb{F}_q(t)}(\mathcal{B}) \leq \frac{(q^4 - q^2)}{(q^3 - 1)} \cdot \mathcal{B}^{\frac{1}{4}} - \frac{(q^4 - q^2)}{(q^3 - 1)}$$

$$\mathcal{Z}^{[\Gamma(2)]}_{1,\mathbb{F}_q(t)}(\mathcal{B}) \leq 2 \cdot \frac{(q^5 - q^3)}{(q^4 - 1)} \cdot \mathcal{B}^{\frac{1}{3}} - 2 \cdot \frac{(q^5 - q^3)}{(q^4 - 1)}$$

which is an equality when $\mathcal{B} = q^{12n}$ *with* $n \in \mathbb{N}$ *implying that the upper bound is a sharp estimate, i.e., the upper bound is equal to the function at infinitely many values of* $\mathcal{B} \in \mathbb{N}$*.*

Proof. The proof is at the end of §7.1. ■

Through the global fields analogy, we formulate a heuristic for the analogous asymptotic counts over $\mathbb{Q}$ by passing the above sharp estimates where the bounded height $ht(\Delta)$ of the discriminant on the number field side is the cardinality of the ring of functions on $\mathrm{Spec}(\mathbb{Z}/(\Delta))$.

Conjecture 7.2 (Heuristic on $\mathcal{Z}^{[\Gamma]}_{1,\mathbb{Q}}(\mathcal{B})$)**.** The function $\mathcal{Z}^{[\Gamma]}_{1,\mathbb{Q}}(\mathcal{B})$, which counts the number of generalized elliptic curves with $[\Gamma]$-structures over $\mathbb{Z}$ ordered by $0 < ht(\Delta_1) \leq \mathcal{B}$, satisfies:

$$\mathcal{Z}^{[\Gamma_1(2)]}_{1,\mathbb{Q}}(\mathcal{B}) \sim_{\mathcal{O}(1)} a \cdot \mathcal{B}^{\frac{1}{2}} + b$$
$$\mathcal{Z}^{[\Gamma_1(3)]}_{1,\mathbb{Q}}(\mathcal{B}) \sim_{\mathcal{O}(1)} a \cdot \mathcal{B}^{\frac{1}{3}} + b$$
$$\mathcal{Z}^{[\Gamma_1(4)]}_{1,\mathbb{Q}}(\mathcal{B}) \sim_{\mathcal{O}(1)} a \cdot \mathcal{B}^{\frac{1}{4}} + b$$
$$\mathcal{Z}^{[\Gamma(2)]}_{1,\mathbb{Q}}(\mathcal{B}) \sim_{\mathcal{O}(1)} a \cdot \mathcal{B}^{\frac{1}{3}} + b$$

The leading order term of the acquired sharp estimates over $\mathbb{F}_q(t)$ matches the analogous asymptotic counts by Harron and Snowden in [HS, Theorem 1.2] over $\mathbb{Q}$, but uses instead naïve height function of underlying elliptic curves (see also [Duke, Grant]). The lower order term over global function fields $\mathbb{F}_q(t)$ being a constant is new and would be interesting to prove or disprove over a number field $\mathbb{Q}$. It also remains a worthwhile problem to count the remaining ten possibilities (classified by the fundamental [Mazur, Theorem 8]) of the torsion subgroups with $|G| > 4$ over $\mathbb{F}_q(t)$ by height of discriminant and compare with analogous counting over $\mathbb{Q}$.

Now, let's consider instead elliptic curves with m-marked rational points. To count the number of certain curves of arithmetic genus one over global function fields $\mathbb{F}_q(t)$ with m-markings as in Theorem 1.4, we need to again extend the notion of (nonsingular) elliptic curves that admits desired m-markings. Here, we consider the $(m-1)$-stable m-marked curves of arithmetic genus one (defined by Smyth in [Smyth, §1.1] for characteristic $\neq 2, 3$, extended to lower characteristic with mild conditions by [LP, Definition 1.5.3]), see Definition 7.10 for a precise definition. Note that if $\mathrm{char}(\mathbb{F}_q) > 3$ and $m = 1$, then 0-stable 1-marked curves are exactly stable elliptic curves as in [DM]. We now consider the following definition:

Definition 7.3. Fix an integral reduced K-scheme B, where K is a field. Then a non-isotrivial flat morphism $\pi : X \to B$ is a m-marked $(m-1)$-stable genus one fibration over B if any fiber of π is a $(m-1)$-stable m-marked curves of arithmetic genus one.

Observe that if $\mathrm{char}(K) = 0$ or > 3, then a m-marked $(m-1)$-stable genus one fibration $X \to \mathbb{P}^1_K$ has a discriminant $\Delta(X) \subset \mathbb{P}^1_K$, and if $K = \mathbb{F}_q$, then $0 < ht(\Delta(X)) := q^{\deg \Delta(X)}$.

Now, define $\mathcal{Z}_{1,m,\mathbb{F}_q(t)}(\mathcal{B})$ as follows:

$\mathcal{Z}_{1,m,\mathbb{F}_q(t)}(\mathcal{B}) := |\{m\text{-marked } (m-1)\text{-stable genus one fibrations over } \mathbb{P}^1_{\mathbb{F}_q} \text{ with } 0 < ht(\Delta(X)) \leq \mathcal{B}\}|$

Note that when $m = 1$, $\mathcal{Z}_{1,m,\mathbb{F}_q(t)}(\mathcal{B})$ counts stable elliptic fibrations, which is described in [HP, Theorem 3] (by identifying stable elliptic fibrations with nonsingular semistable elliptic surfaces, see [HP, Proposition 11]). When $2 \leq m \leq 5$, we acquire the following sharp estimate of $\mathcal{Z}_{1,m,\mathbb{F}_q(t)}(\mathcal{B})$:

Theorem 7.4 (Sharp estimate of $\mathcal{Z}_{1,m,\mathbb{F}_q(t)}(\mathcal{B})$)**.** *If $\mathrm{char}(\mathbb{F}_q) \neq 2, 3$, then the function $\mathcal{Z}_{1,m,\mathbb{F}_q(t)}(\mathcal{B})$, which counts the number of m-marked $(m-1)$-stable genus one fibration over $\mathbb{P}^1_{\mathbb{F}_q}$ for $2 \leq m \leq 5$ ordered by $0 < ht(\Delta(X)) = q^{12n} \leq \mathcal{B}$, satisfies:*

$$\mathcal{Z}_{1,2,\mathbb{F}_q(t)}(\mathcal{B}) \leq \frac{(q^{11} + q^{10} - q^8 - q^7)}{(q^9 - 1)} \cdot (\mathcal{B}^{\frac{3}{4}} - 1) + \frac{(q^7 - q^5)}{(q^6 - 1)} \cdot (\mathcal{B}^{\frac{1}{2}} - 1)$$

$$\mathcal{Z}_{1,3,\mathbb{F}_q(t)}(\mathcal{B}) \leq \frac{(q^{11} + q^{10} + q^9 - q^7 - q^6 - q^5)}{(q^8 - 1)} \cdot (\mathcal{B}^{\frac{2}{3}} - 1) + \frac{(q^5 - q^3)}{(q^4 - 1)} \cdot (\mathcal{B}^{\frac{1}{3}} - 1)$$

$$\mathcal{Z}_{1,4,\mathbb{F}_q(t)}(\mathcal{B}) \leq \frac{(q^{11} + q^{10} + q^9 + q^8 - q^6 - q^5 - q^4 - q^3)}{(q^7 - 1)} \cdot (\mathcal{B}^{\frac{7}{12}} - 1) + \frac{(q^5 - q^3)}{(q^4 - 1)} \cdot (\mathcal{B}^{\frac{1}{3}} - 1)$$

$$\mathcal{Z}_{1,5,\mathbb{F}_q(t)}(\mathcal{B}) \leq \frac{(q^{11} + q^{10} + q^9 + q^8 + q^7 - q^5 - q^4 - q^3 - q^2 - q^1)}{(q^6 - 1)} \cdot (\mathcal{B}^{\frac{1}{2}} - 1)$$

which is an equality when $\mathcal{B} = q^{12n}$ with $n \in \mathbb{N}$ implying that the upper bound is a sharp estimate, i.e., the upper bound is equal to the function at infinitely many values of $\mathcal{B} \in \mathbb{N}$.

Proof. The proof is at the end of §7.2. ∎

Through the global fields analogy, we formulate a heuristic for the analogous asymptotic counts over $\mathbb{Q}$ by passing the above sharp estimates where the bounded height $ht(\Delta)$ of the discriminant on the number field side is the cardinality of the ring of functions on $\mathrm{Spec}(\mathbb{Z}/(\Delta))$.

Conjecture 7.5 (Heuristic on $\mathcal{Z}_{1,m,\mathbb{Q}}(\mathcal{B})$)**.** The function $\mathcal{Z}_{1,m,\mathbb{Q}}(\mathcal{B})$, which counts the number of m-marked $(m-1)$-stable genus one curves for $2 \leq m \leq 5$ over $\mathbb{Z}$ ordered by $0 < ht(\Delta_1) \leq \mathcal{B}$, satisfies:

$$\mathcal{Z}_{1,2,\mathbb{Q}}(\mathcal{B}) \sim_{\mathcal{O}(1)} a \cdot \mathcal{B}^{\frac{3}{4}} + b \cdot \mathcal{B}^{\frac{1}{2}} + c$$
$$\mathcal{Z}_{1,3,\mathbb{Q}}(\mathcal{B}) \sim_{\mathcal{O}(1)} a \cdot \mathcal{B}^{\frac{2}{3}} + b \cdot \mathcal{B}^{\frac{1}{3}} + c$$
$$\mathcal{Z}_{1,4,\mathbb{Q}}(\mathcal{B}) \sim_{\mathcal{O}(1)} a \cdot \mathcal{B}^{\frac{7}{12}} + b \cdot \mathcal{B}^{\frac{1}{3}} + c$$
$$\mathcal{Z}_{1,5,\mathbb{Q}}(\mathcal{B}) \sim_{\mathcal{O}(1)} a \cdot \mathcal{B}^{\frac{1}{2}} + b$$

7.1. **Arithmetic of the moduli of generalized elliptic curves over $\mathbb{P}^1$ with level structures.** The essential geometrical idea in acquiring the sharp estimate is to consider the moduli stack of rational curves on a compactified modular curve as in [HP]. The various compactified modular curves $\overline{\mathcal{M}}_{1,1}[\Gamma]$ are isomorphic to the 1-dimensional weighted projective stacks $\mathcal{P}(a,b)$.

Proposition 7.6. *The moduli stack $\overline{\mathcal{M}}_{1,1}[\Gamma]$ of generalized elliptic curves with $[\Gamma]$-structures is isomorphic to the following when over a field K:*

(1) if $\mathrm{char}(K) \neq 2$, the tame Deligne–Mumford moduli stack of generalized elliptic curves with $[\Gamma_1(2)]$-structures is isomorphic to

$$(\overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])_K \cong [(\mathrm{Spec}\ K[a_2,a_4] - (0,0))/\mathbb{G}_m] = \mathcal{P}_K(2,4),$$

(2) if $\mathrm{char}(K) \neq 3$, the tame Deligne–Mumford moduli stack of generalized elliptic curves with $[\Gamma_1(3)]$-structures is isomorphic to

$$(\overline{\mathcal{M}}_{1,1}[\Gamma_1(3)])_K \cong [(\mathrm{Spec}\ K[a_1,a_3] - (0,0))/\mathbb{G}_m] = \mathcal{P}_K(1,3),$$

(3) if $\mathrm{char}(K) \neq 2$, the tame Deligne–Mumford moduli stack of generalized elliptic curves with $[\Gamma_1(4)]$-structures is isomorphic to

$$(\overline{\mathcal{M}}_{1,1}[\Gamma_1(4)])_K \cong [(\mathrm{Spec}\ K[a_1,a_2] - (0,0))/\mathbb{G}_m] = \mathcal{P}_K(1,2),$$

(4) if $\mathrm{char}(K) \neq 2$, the tame Deligne–Mumford moduli stack of generalized elliptic curves with $[\Gamma(2)]$-structures is isomorphic to

$$(\overline{\mathcal{M}}_{1,1}[\Gamma(2)])_K \cong [(\mathrm{Spec}\ K[a_2,a_2] - (0,0))/\mathbb{G}_m] = \mathcal{P}_K(2,2),$$

where $\lambda \cdot a_i = \lambda^i a_i$ for $\lambda \in \mathbb{G}_m$ and $i = 1,2,3,4$. Thus, the a_i's have degree i respectively. Moreover, the discriminant divisors of $(\overline{\mathcal{M}}_{1,1}[\Gamma])_K \cong \mathcal{P}_K(i,j)$ as above have degree 12.

Proof. Proof of the first, second, third and fourth equivalence can be found in [Behrens, §1.3], [HMe, Proposition 4.5], [Meier, Examples 2.1] and [Stojanoska, Proposition 7.1] respectively. By Proposition 3.3, the weighted projective stacks are tame Deligne–Mumford as well.

For the degree of the discriminant, it suffices to find the weight of the $\mathbb{G}_m$-action. First, the four papers cited above explicitly construct universal families of elliptic curves over the schematic covers $(\mathrm{Spec}\ K[a_i,a_j] - (0,0)) \to \mathcal{P}_K(i,j)$ of the corresponding moduli stacks. The explicit defining equation of the respective universal family implies that the $\lambda \in \mathbb{G}_m$ also acts on the discriminant of the universal family by multiplying λ^{12}. Therefore, the discriminant has degree 12. ■

We now consider the moduli stack $\mathcal{L}_{1,12n}^{[\Gamma]} := \mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma])$ of generalized elliptic curves over $\mathbb{P}^1$ with $[\Gamma]$-structures.

Proposition 7.7. *Assume $\mathrm{char}(K) = 0$ or $\neq 2$ for $[\Gamma] = [\Gamma_1(2)], [\Gamma_1(4)], [\Gamma(2)]$, and $\mathrm{char}(K) \neq 3$ for $[\Gamma] = [\Gamma_1(3)]$. Then, the moduli stack $\mathcal{L}_{1,12n}^{[\Gamma]}$ of generalized elliptic curves over $\mathbb{P}^1$ with discriminant degree $12n > 0$ and $[\Gamma]$-structures is the tame Deligne–Mumford stack $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma])$ parameterizing the K-morphisms $f : \mathbb{P}^1 \to \overline{\mathcal{M}}_{1,1}[\Gamma]$ such that $f^*\mathcal{O}_{\overline{\mathcal{M}}_{1,1}[\Gamma]}(1) \cong \mathcal{O}_{\mathbb{P}^1}(n)$.*

Proof. Without the loss of generality, we prove the $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])$ case over a field K with $\mathrm{char}(K) \neq 2$. The proof for the other cases are analogous. By the definition of the universal family p, any generalized elliptic curves $\pi : Y \to \mathbb{P}^1$ with $[\Gamma_1(2)]$-structures comes from a morphism $f : \mathbb{P}^1 \to \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)]$ and vice versa. As this correspondence also works in families, the moduli stack of generalized elliptic curves over $\mathbb{P}^1$ with $[\Gamma_1(2)]$-structures is isomorphic to $\mathrm{Hom}(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])$.

Since the discriminant degree of f is $12 \deg f^*\mathcal{O}_{\overline{\mathcal{M}}_{1,1}[\Gamma_1(2)]}(1)$ by Proposition 7.6, the substack $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])$ parametrizing such f's with $\deg f^*\mathcal{O}_{\overline{\mathcal{M}}_{1,1}[\Gamma_1(2)]}(1) = n$ is the desired moduli stack. Since $\deg f^*\mathcal{O}_{\overline{\mathcal{M}}_{1,1}[\Gamma_1(2)]}(1) = n$ is an open condition, $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])$ is an open substack of $\mathrm{Hom}(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])$, which is tame Deligne–Mumford by Proposition 3.6 as $\overline{\mathcal{M}}_{1,1}[\Gamma_1(2)]$ itself is tame Deligne–Mumford by Proposition 7.6. This shows that $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])$ satisfies the desired properties as well. ■

We now acquire the exact number $|\mathcal{L}_{1,12n}^{[\Gamma]}(\mathbb{F}_q)/\sim|$ of $\mathbb{F}_q$–isomorphism classes of $\mathbb{F}_q$–points (i.e., the non–weighted point count) of the moduli stack $\mathcal{L}_{1,12n}^{[\Gamma]} \cong \mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma])$ of generalized elliptic curves over $\mathbb{P}^1$ with discriminant degree $12n > 0$ and $[\Gamma]$-structures.

Proposition 7.8. *If* $\mathrm{char}(\mathbb{F}_q) \neq 2$*, then*

$$|\mathcal{L}_{1,12n}^{[\Gamma_1(2)]}(\mathbb{F}_q)/\sim| = 2 \cdot \#_q \left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(2,4))\right) = 2(q^{6n+1} - q^{6n-1})$$

$$|\mathcal{L}_{1,12n}^{[\Gamma_1(4)]}(\mathbb{F}_q)/\sim| = \#_q \left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(1,2))\right) = q^{3n+1} - q^{3n-1}$$

$$|\mathcal{L}_{1,12n}^{[\Gamma(2)]}(\mathbb{F}_q)/\sim| = 2 \cdot \#_q \left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(2,2))\right) = 2(q^{4n+1} - q^{4n-1})$$

If $\mathrm{char}(\mathbb{F}_q) \neq 3$*, then*

$$|\mathcal{L}_{1,12n}^{[\Gamma_1(3)]}(\mathbb{F}_q)/\sim| = \#_q \left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(1,3))\right) = q^{4n+1} - q^{4n-1}$$

Proof. Fix $n > 0$. Since any $\varphi_g \in \mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))$ is surjective, the generic stabilizer group $\mu_{\gcd(a,b)}$ of $\mathcal{P}(a,b)$ is the automorphism group of φ_g. Using the identification from Proposition 7.7 and then summing the weighted point counts of Hom stacks from Proposition 4.5 into Proposition 4.10 gives the desired formula. ■

Remark 7.9. For weighted projective lines $\mathcal{P}(a,b)$ as in the cases of $\mathcal{L}_{1,12n}^{[\Gamma]}$, $\left\{\mathcal{I}\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))\right)\right\}$ is a sum of $\left\{\mathrm{Hom}_n(\mathbb{P}^1_{\kappa(g)}, \mathcal{P}_{\kappa(g)}(a,b))\right\}$ for each closed point $g \in \mathbb{G}_m$ with $\mathrm{ord}(g) \mid \gcd(a,b)$, as the only possible generic stabilizer of positive dimensional substacks of $\mathcal{P}(a,b)$. On the other hand, the terms with division function $\delta(r, q-1)$ do not occur in $|\mathcal{L}_{1,12n}^{[\Gamma]}(\mathbb{F}_q)/\sim|$ as the characteristic condition required to identify $\overline{\mathcal{M}}_{1,1}[\Gamma]$ as a weighted projective line implies that $\gcd(a,b) \mid q-1$.

We now finally prove the Theorem 7.1 using the above arithmetic invariants as follows:

Proof of Theorem 7.1. Without the loss of the generality, we prove the $[\Gamma_1(2)]$–structures case over $\mathrm{char}(\mathbb{F}_q) \neq 2$. The proof for the other cases are analogous. By Proposition 7.7 and Proposition 7.8, we know the number of $\mathbb{F}_q$-isomorphism classes of generalized elliptic curves of discriminant degree $12n$ with $[\Gamma_1(2)]$-structures over $\mathbb{P}^1_{\mathbb{F}_q}$ is $|\mathcal{L}_{1,12n}^{[\Gamma_1(2)]}(\mathbb{F}_q)/\sim| = 2 \cdot (q^{6n+1} - q^{6n-1})$. Using this, we can explicitly compute the sharp bound on $\mathcal{Z}_{1,\mathbb{F}_q(t)}^{[\Gamma_1(2)]}(\mathcal{B})$ as the following,

$$\begin{aligned}\mathcal{Z}_{1,\mathbb{F}_q(t)}^{[\Gamma_1(2)]}(\mathcal{B}) = \sum_{n=1}^{\left\lfloor \frac{log_q \mathcal{B}}{12} \right\rfloor} |\mathcal{L}_{1,12n}^{[\Gamma_1(2)]}(\mathbb{F}_q)/\sim| &= \sum_{n=1}^{\left\lfloor \frac{log_q \mathcal{B}}{12} \right\rfloor} 2 \cdot (q^{6n+1} - q^{6n-1}) \\ &= 2 \cdot (q^1 - q^{-1}) \sum_{n=1}^{\left\lfloor \frac{log_q \mathcal{B}}{12} \right\rfloor} q^{6n} \leq 2 \cdot (q^1 - q^{-1}) \left(q^6 + \cdots + q^{6 \cdot (\frac{log_q \mathcal{B}}{12})}\right)\end{aligned}$$

$$= 2 \cdot (q^1 - q^{-1}) \frac{q^6(\mathcal{B}^{\frac{1}{2}} - 1)}{(q^6 - 1)} = 2 \cdot \frac{(q^7 - q^5)}{(q^6 - 1)} \cdot (\mathcal{B}^{\frac{1}{2}} - 1)$$

On the second line of the equations above, inequality becomes an equality if and only if $n := \frac{log_q \mathcal{B}}{12} \in \mathbb{N}$, i.e., $\mathcal{B} = q^{12n}$ with $n \in \mathbb{N}$. This implies that the acquired upper bound on $\mathcal{Z}^{[\Gamma_1(2)]}_{1,\mathbb{F}_q(t)}(\mathcal{B})$ is a sharp estimate. ∎

7.2. **Arithmetic of the moduli of m-marked genus one fibrations over $\mathbb{P}^1$.** We proceed to estimate the sharp bound on the number of m-marked $(m-1)$-stable genus one fibrations over $\mathbb{P}^1_{\mathbb{F}_q}$ for $2 \leq m \leq 5$. First, we state the definition of m-marked $(m-1)$-stability from [LP, Definition 1.5.3], which is a modification of the Deligne–Mumford stability [DM]:

Definition 7.10. Let K be a field and m be a positive integer. Then, a tuple $(C, p_1, \ldots, p_m)$, of a geometrically connected, geometrically reduced, and proper K-curve C of arithmetic genus one with m distinct K-rational points p_i in the smooth locus of C, is a $(m-1)$-stable m-marked curve of arithmetic genus one if the curve $C_{\overline{K}} := C \times_K \overline{K}$ and the divisor $\Sigma := \{p_1, \ldots, p_m\}$ satisfy the following properties, where $\overline{K}$ is the algebraic closure of K:

1. $C_{\overline{K}}$ has only nodes and elliptic u-fold points as singularities (see below), where $u < m$,
2. $C_{\overline{K}}$ has no disconnecting nodes, and
3. every irreducible component of $C_{\overline{K}}$ contains at least one marked point.

Remark 7.11. A singular point of a curve over $\overline{K}$ is an elliptic u-fold singular point if it is Gorenstein and étale locally isomorphic to a union of u general lines in $\mathbb{P}^{u-1}_{\overline{K}}$ passing through a common point.

Note that the name "$(m-1)$-stability" comes from [Smyth, §1.1], which is defined when $\mathrm{char}(K) \neq 2, 3$. By [LP, Proposition 1.5.4], the above definition (by [LP, Definition 1.5.3]) coincides with that of Smyth when $\mathrm{char}(K) \neq 2, 3$, hence we adapt Smyth's naming convention on Lekili and Polishchuk's definition. Regardless, we focus on the case when $\mathrm{char}(K) \neq 2, 3$, so that the moduli stack of such curves behaves reasonably.

By the work of Smyth [Smyth, Theorem 3.8], we are able to formulate the moduli stack of $(m-1)$-stable m-marked curves of arithmetic genus one over any field of characteristic $\neq 2, 3$:

Theorem 7.12. *There exists a proper irreducible Deligne–Mumford moduli stack $\overline{\mathcal{M}}_{1,m}(m-1)$ of $(m-1)$-stable m-marked curves arithmetic genus one over* $\mathrm{Spec}(\mathbb{Z}[1/6])$

Note that when $m = 1$, $\overline{\mathcal{M}}_{1,1}(0) \cong \overline{\mathcal{M}}_{1,1}$ is the Deligne–Mumford moduli stack of stable elliptic curves.

In fact, the construction of $\overline{\mathcal{M}}_{1,m}(m-1)$ extends to $\mathrm{Spec}\ \mathbb{Z}$ by [LP, Theorem 1.5.7] (called $\overline{\mathcal{M}}^{\infty}_{1,m}$ in loc.cit.) as an algebraic stack, which is proper over $\mathrm{Spec}\ \mathbb{Z}[1/N]$ where N depends on m:

- if $m \geq 3$, then $N = 1$,
- if $m = 2$, then $N = 2$, and
- if $m = 1$, then $N = 6$.

However, even with those assumptions above, $\overline{\mathcal{M}}_{1,m}(m-1)$ is not necessarily Deligne–Mumford. Nevertheless, by [LP, Theorem 1.5.7.], we obtain the explicit descriptions of $\overline{\mathcal{M}}_{1,m}(m-1)$:

Proposition 7.13. *The moduli stack $\overline{\mathcal{M}}_{1,m}(m-1)$ of m-marked $(m-1)$-stable curves of arithmetic genus one for $2 \leq m \leq 5$ is isomorphic to the following, for a field K:*

(1) if $\mathrm{char}(K) \neq 2, 3$, the tame Deligne–Mumford moduli stack of 2-marked 1-stable curves of arithmetic genus one is isomorphic to

$$(\overline{\mathcal{M}}_{1,2}(1))_K \cong [(\mathrm{Spec}\ K[a_2, a_3, a_4] - 0)/\mathbb{G}_m] = \mathcal{P}_K(2, 3, 4),$$

(2) *if* $\mathrm{char}(K) \neq 2, 3$*, the tame Deligne–Mumford moduli stack of 3-marked 2-stable curves of arithmetic genus one is isomorphic to*

$$(\overline{\mathcal{M}}_{1,3}(2))_K \cong [(\mathrm{Spec}\ K[a_1, a_2, a_2, a_3] - 0)/\mathbb{G}_m] = \mathcal{P}_K(1, 2, 2, 3),$$

(3) *if* $\mathrm{char}(K) \neq 2$*, the tame Deligne–Mumford moduli stack of 4-marked 3-stable curves of arithmetic genus one is isomorphic to*

$$(\overline{\mathcal{M}}_{1,4}(3))_K \cong [(\mathrm{Spec}\ K[a_1, a_1, a_1, a_2, a_2] - 0)/\mathbb{G}_m] = \mathcal{P}_K(1, 1, 1, 2, 2),$$

(4) *the moduli stack of 5-marked 4-stable curves of arithmetic genus one is isomorphic to a scheme*

$$(\overline{\mathcal{M}}_{1,5}(4))_K \cong [(\mathrm{Spec}\ K[a_1, a_1, a_1, a_1, a_1, a_1] - 0)/\mathbb{G}_m] = \mathbb{P}_K(1, 1, 1, 1, 1, 1) \cong \mathbb{P}^5_K,$$

where $\lambda \cdot a_i = \lambda^i a_i$ *for* $\lambda \in \mathbb{G}_m$ *and* $i = 1, 2, 3, 4$*. Thus, the* a_i*'s have degree* i *respectively. Furthermore, if* $\mathrm{char}(K) \neq 2, 3$*, then the discriminant divisors of such* $\overline{\mathcal{M}}_{1,m}(m-1)$ *have degree 12.*

Proof. Proof of [LP, Theorem 1.5.7.] gives the corresponding isomorphisms $\overline{\mathcal{M}}_{1,m}(m-1) \cong \mathcal{P}(\vec{\lambda})$. By Proposition 3.3, the weighted projective stacks are tame Deligne–Mumford as well, and in fact, smooth.

For the degree of the discriminant when $\mathrm{char}(K) \neq 2, 3$, it suffices to describe the discriminant divisor, the locus of singular curves in $\overline{\mathcal{M}}_{1,m}(m-1)$. First, [LP, Theorem 1.5.7.] shows that in the above case, where $\overline{\mathcal{M}}_{1,m}(m-1) \cong \mathcal{P}(\vec{\lambda})$, the line bundle $\mathcal{O}_{\mathcal{P}(\vec{\lambda})}(1)$ of degree one is isomorphic to $\lambda := \pi_* \omega_\pi$, where $\pi : \overline{\mathcal{C}}_{1,m}(m-1) \to \overline{\mathcal{M}}_{1,m}(m-1)$ is the universal family of $(m-1)$-stable m-marked curves of arithmetic genus one. Since $\overline{\mathcal{M}}_{1,m}(m-1)$ is smooth and the Picard rank is one (generated by λ), the discriminant divisor is Cartier. In fact, by [Smyth2, §3.1], it coincides with the locus Δ_{irr} of curves with non-disconnecting nodes or non-nodal singular points. Then [Smyth2, Remark 3.3] (which assumes $\mathrm{char}(K) \neq 2, 3$) implies that $\Delta_{irr} \sim 12\lambda$, thus the discriminant divisor has degree 12. ■

We now consider the moduli stacks of m-marked $(m-1)$-stable genus one fibrations over $\mathbb{P}^1_K$ for any field K of $\mathrm{char}(K) = 0$ or > 3:

Proposition 7.14. *Assume* $\mathrm{char}(K) = 0$ *or* > 3*. If* $2 \leq m \leq 5$*, then the moduli stack* $\mathcal{L}_{1,m,12n}$ *of* m*-marked* $(m-1)$*-stable genus one fibrations over* $\mathbb{P}^1_K$ *with discriminant degree* $12n$ *is the tame Deligne–Mumford stack* $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,m}(m-1))$ *parameterizing the* K*-morphisms* $f : \mathbb{P}^1 \to \overline{\mathcal{M}}_{1,m}(m-1)$ *such that* $f^*\mathcal{O}_{\mathcal{P}(\vec{\lambda})}(1) \cong \mathcal{O}_{\mathbb{P}^1}(n)$*.*

Proof. Without the loss of the generality, we prove the 2-marked 1-stable curves of arithmetic genus one case $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,2}(1))$ over $\mathrm{char}(\mathbb{F}_q) \neq 2, 3$. The proof for the other cases are analogous. By the definition of the universal family p, any 2-marked 1-stable arithmetic genus one curves $\pi : Y \to \mathbb{P}^1$ with discriminant degree $12n$ comes from a morphism $f : \mathbb{P}^1 \to \overline{\mathcal{M}}_{1,2}(1)$ and vice versa. As this correspondence also works in families, the moduli stack of 2-marked 1-stable curves of arithmetic genus one over $\mathbb{P}^1_K$ is isomorphic to $\mathrm{Hom}(\mathbb{P}^1, \overline{\mathcal{M}}_{1,2}(1))$.

Since the discriminant degree of f is $12 \deg f^*\mathcal{O}_{\overline{\mathcal{M}}_{1,2}(1)}(1)$ by Proposition 7.13, the substack $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,2}(1))$ parametrizing such f's with $\deg f^*\mathcal{O}_{\overline{\mathcal{M}}_{1,2}(1)}(1) = n$ is the desired moduli stack. Since $\deg f^*\mathcal{O}_{\overline{\mathcal{M}}_{1,2}(1)}(1) = n$ is an open condition, $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,2}(1))$ is an open substack of $\mathrm{Hom}(\mathbb{P}^1, \overline{\mathcal{M}}_{1,2}(1))$, which is tame Deligne–Mumford by Proposition 3.6 as $\overline{\mathcal{M}}_{1,2}(1)$ itself is tame Deligne–Mumford by Proposition 7.13. This shows that $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,2}(1))$ satisfies the desired properties as well. ■

We now acquire the exact number $|\mathcal{L}_{1,m,12n}(\mathbb{F}_q)/\sim|$ of $\mathbb{F}_q$–isomorphism classes of $\mathbb{F}_q$–points (i.e., the non–weighted point count) of the moduli stack $\mathcal{L}_{1,m,12n} \cong \mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,m}(m-1))$ of m-marked $(m-1)$-stable genus one fibrations over $\mathbb{P}^1$ with discriminant degree $12n > 0$.

Proposition 7.15. *If* $\mathrm{char}(\mathbb{F}_q) \neq 2, 3,$ *then*

$$\begin{aligned}|\mathcal{L}_{1,2,12n}(\mathbb{F}_q)/\sim| &= \#_q\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(2,3,4))\right) + \#_q\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(2,4))\right)\\ &= (q^{9n+2} + q^{9n+1} - q^{9n-1} - q^{9n-2}) + (q^{6n+1} - q^{6n-1})\end{aligned}$$

$$\begin{aligned}|\mathcal{L}_{1,3,12n}(\mathbb{F}_q)/\sim| &= \#_q\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(1,2,2,3))\right) + \#_q\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(2,2))\right)\\ &= (q^{8n+3} + q^{8n+2} + q^{8n+1} - q^{8n-1} - q^{8n-2} - q^{8n-3}) + (q^{4n+1} - q^{4n-1})\end{aligned}$$

$$\begin{aligned}|\mathcal{L}_{1,4,12n}(\mathbb{F}_q)/\sim| &= \#_q\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(1,1,1,2,2))\right) + \#_q\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(2,2))\right)\\ &= (q^{7n+4} + q^{7n+3} + q^{7n+2} + q^{7n+1} - q^{7n-1} - q^{7n-2} - q^{7n-3} - q^{7n-4})\\ &\quad + (q^{4n+1} - q^{4n-1})\\ |\mathcal{L}_{1,5,12n}(\mathbb{F}_q)/\sim| &= \#_q\left(\mathrm{Hom}_n(\mathbb{P}^1, \mathbb{P}(1,1,1,1,1,1) \cong \mathbb{P}^5)\right)\\ &= q^{6n+5} + q^{6n+4} + q^{6n+3} + q^{6n+2} + q^{6n+1} - q^{6n-1} - q^{6n-2} - q^{6n-3} - q^{6n-4} - q^{6n-5}\end{aligned}$$

Proof. Note that $\overline{\mathcal{M}}_{1,2}(1) \cong \mathcal{P}(2,3,4)$ has the substack $\mathcal{P}(2,4)$ with the generic stabilizer of order 2. This implies that the number of isomorphism classes of $\mathbb{F}_q$-points of $\mathcal{L}_{1,2,12n}$ with discriminant degree $12n$ is $|\mathcal{L}_{1,2,12n}(\mathbb{F}_q)/\sim| = (q^{9n+2} + q^{9n+1} - q^{9n-1} - q^{9n-2}) + (q^{6n+1} - q^{6n-1})$ by summing the weighted point counts of Hom stacks from Proposition 4.5 into Proposition 4.10. Similarly, $\overline{\mathcal{M}}_{1,3}(2) \cong \mathcal{P}(1,2,2,3)$ and $\overline{\mathcal{M}}_{1,4}(3) \cong \mathcal{P}(1,1,1,2,2)$ has the substack $\mathcal{P}(2,2)$ with the generic stabilizer of order 2. This implies that adding $(q^{4n+1} - q^{4n-1})$ to the corresponding weighted points count gives the desired non–weighted point counts. Finally, $\overline{\mathcal{M}}_{1,5}(4) \cong \mathbb{P}^5$, so that the non-weighted point count coincides with the weighted point count. ∎

We now finally prove the Theorem 7.4 using the above arithmetic invariants as follows:

Proof. Without the loss of the generality, we prove the 2-marked 1-stable curves of arithmetic genus one case $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,2}(1) \cong \mathcal{P}(2,3,4))$ over $\mathrm{char}(\mathbb{F}_q) \neq 2, 3$. The proof for the other cases are analogous. Knowing the number of $\mathbb{F}_q$-isomorphism classes of 1-stable arithmetic genus one curves over $\mathbb{P}^1$ with discriminant degree $12n$ and 2-marked Weierstrass sections over $\mathbb{F}_q$ is $|\mathcal{L}_{1,2,12n}(\mathbb{F}_q)/\sim| = (q^{9n+2} + q^{9n+1} - q^{9n-1} - q^{9n-2}) + (q^{6n+1} - q^{6n-1})$ by Proposition 7.15, we can explicitly compute the sharp bound on $\mathcal{Z}_{1,2,\mathbb{F}_q(t)}(\mathcal{B})$ as the following,

$$\begin{aligned}\mathcal{Z}_{1,2,\mathbb{F}_q(t)}(\mathcal{B}) = \sum_{n=1}^{\left\lfloor \frac{log_q \mathcal{B}}{12} \right\rfloor} |\mathcal{L}_{1,2,12n}(\mathbb{F}_q)/\sim| &= \sum_{n=1}^{\left\lfloor \frac{log_q \mathcal{B}}{12} \right\rfloor} (q^{9n+2} + q^{9n+1} - q^{9n-1} - q^{9n-2}) + (q^{6n+1} - q^{6n-1})\\ = (q^2 + q^1 - q^{-1} - q^{-2}) \sum_{n=1}^{\left\lfloor \frac{log_q \mathcal{B}}{12} \right\rfloor} q^{9n} + (q^1 - q^{-1}) \sum_{n=1}^{\left\lfloor \frac{log_q \mathcal{B}}{12} \right\rfloor} q^{6n}\\ \le (q^2 + q^1 - q^{-1} - q^{-2}) \left(q^9 + \cdots + q^{9 \cdot (\frac{log_q \mathcal{B}}{12})}\right) + (q^1 - q^{-1}) \left(q^6 + \cdots + q^{6 \cdot (\frac{log_q \mathcal{B}}{12})}\right)\end{aligned}$$

$$
\begin{aligned}
&= (q^2+q^1-q^{-1}-q^{-2})\cdot\frac{q^9(\mathcal{B}^{\frac{3}{4}}-1)}{(q^9-1)}+(q^1-q^{-1})\frac{q^6(\mathcal{B}^{\frac{1}{2}}-1)}{(q^6-1)}\\
&= \frac{(q^{11}+q^{10}-q^8-q^7)}{(q^9-1)}\cdot(\mathcal{B}^{\frac{3}{4}}-1)+\frac{(q^7-q^5)}{(q^6-1)}\cdot(\mathcal{B}^{\frac{1}{2}}-1)
\end{aligned}
$$

On the third line of the equations above, inequality becomes an equality if and only if $n := \frac{log_q \mathcal{B}}{12} \in \mathbb{N}$, i.e., $\mathcal{B} = q^{12n}$ with $n \in \mathbb{N}$. This implies that the acquired upper bound on $\mathcal{Z}_{1,2,\mathbb{F}_q(t)}(\mathcal{B})$ is a sharp estimate. ■

Acknowledgments. The authors thank Dori Bejleri, Benson Farb, Joe Harris, Ariyan Javanpeykar, Qing Liu, Hunter Spink, Craig Westerland and Jesse Wolfson for helpful discussions and Brian Conrad for a careful reading of earlier versions of the manuscript. Jun-Yong Park was supported by the Institute for Basic Science in Korea (IBS-R003-D1). Changho Han acknowledge the partial support of the Natural Sciences and Engineering Research Council of Canada (NSERC), [PGSD3-487436-2016], and of the Research and Training Group in Algebraic Geometry, Algebra and Number Theory, at the University of Georgia.

References

[AGV] D. Abramovich, T. Graber and A. Vistoli, *Gromov-Witten Theory of Deligne-Mumford Stacks*, American Journal of Mathematics, **130**, No. 5, (2008): 1337–1398.

[AOV] D. Abramovich, M. Olsson and A. Vistoli, *Tame stacks in positive characteristic*, Annales de l'Institut Fourier, **58**, No. 4, (2008): 1057–1091.

[AW] E. Artin and G. Whaples, *Axiomatic characterization of fields by the product formula for valuations*, Bulletin of the American Mathematical Society, **51**, No. 7, (1945): 469–492.

[AW2] E. Artin and G. Whaples, *A note on axiomatic characterization of fields*, Bulletin of the American Mathematical Society, **52**, No. 4, (1946): 245–247.

[Baier] S. Baier, *Elliptic curves with square-free* Δ, International Journal of Number Theory, **12**, No. 3, (2016): 737–764.

[BB] S. Baier and T. D. Browning, *Inhomogeneous cubic congruences and rational points on Del Pezzo surfaces*, Journal für die reine und angewandte Mathematik, **680**, (2013): 69–151.

[Behrend] K. A. Behrend, *The Lefschetz trace formula for algebraic stacks*, Inventiones mathematicae, **112**, No. 1, (1993): 127–149.

[Behrens] M. Behrens, *A modular description of the $K(2)$-local sphere at the prime 3*, Topology, **45**, No. 2, (2006): 343–402.

[BG] M. Bhargava and B. Gross, *The average size of the 2-Selmer group of Jacobians of hyperelliptic curves having a rational Weierstrass point*, Automorphic representations and L-functions, The Studies in Mathematics, **22**, Tata Institute of Fundamental Research, Mumbai (2013): 23—91.

[BM] V. V. Batyrev and Y. I. Manin, *Sur le nombre des points rationnels de hauteur borné des variétés algébriques*, Mathematische Annalen, **286**, No. 1-3, (1990): 27–43.

[BMc] A. Brumer and O. McGuinness, *The behavior of the Mordell-Weil group of elliptic curves*, Bulletin of the American Mathematical Society, **23**, No. 2, (1990): 375–382.

[Brumer] A. Brumer, *The average rank of elliptic curves I*, Inventiones mathematicae, **109**, No. 1, (1992): 445–472.

[de Jong] A. J. de Jong, *Counting elliptic surfaces over finite fields*, Moscow Mathematical Journal, **2**, No. 2, (2002): 281–311.

[DM] P. Deligne and D. Mumford, *The irreducibility of the space of curves of given genus*, Publications Mathématiques de l'I.H.É.S., **36**, (1969): 75–109.

[DR] P. Deligne and M. Rapoport, *Les Schémas de Modules de Courbes Elliptiques*, Modular Functions of One Variable II. Lecture Notes in Mathematics, **349**, (1973): 143–316.

[Duke] W. Duke, *Elliptic curves with no exceptional primes*, Comptes Rendus de l'Académie des Sciences - Series I - Mathematics, **325**, No. 8, (1997): 813–818.

[EH] D. Eisenbud and J. Harris, *The Kodaira dimension of the moduli space of curves of genus ≥ 23*, Inventiones mathematicae, **90**, No. 2, (1987): 359–387.

[Ekedahl] T. Ekedahl, *The Grothendieck group of algebraic stacks*, arXiv:0903.3143, (2009).

[Fedorchuk] M. Fedorchuk, *Moduli spaces of hyperelliptic curves with A and D singularities*, Mathematische Zeitschrift, **276**, No. 1-2, (2014): 299–328.

[FMT] J. Franke, Y. I. Manin and Y. Tschinkel, *Rational points of bounded height on Fano varieties*, Inventiones mathematicae, **95**, No. 2, (1989): 421–435.

[Fujino] O. Fujino, *Minimal Model Theory for Log Surfaces*, Publications of the Research Institute for Mathematical Sciences, **48**, No. 2, (2012): 339–371.

[FW] B. Farb and J. Wolfson, *Topology and arithmetic of resultants, I*, New York Journal of Mathematics, **22**, (2016): 801–821.

[GGW] C. Gómez-Gonzáles and J. Wolfson, *Problems in Arithmetic Topology*, Research in the Mathematical Sciences, Special Issue of PIMS 2019 Workshop on Arithmetic Topology, **8**, No. 2, #23 (2021).

[GHM] C. Galindo, F. Hernando and F. Monserrat, *The log-canonical threshold of a plane curve*, Mathematical Proceedings of the Cambridge Philosophical Society, **160**, No. 3, (2016): 513–535.

[Grant] D. Grant, *A Formula for the Number of Elliptic Curves with Exceptional Primes*, Compositio Mathematica, **122**, No. 2, (2000): 151–164.

[Hassett] B. Hassett, *Moduli spaces of weighted pointed stable curves*, Advances in Mathematics, **173**, No. 2, (2003): 316–352.

[Hassett2] B. Hassett, *Classical and minimal models of the moduli space of curves of genus two*, Geometric Methods in Algebra and Number Theory, Progress in Mathematics, **235**, Birkhäuser Boston, (2005): 169—192.

[HM] J. Harris and D. Mumford, *On the Kodaira dimension of the moduli space of curves*, Inventiones mathematicae, **67**, No. 1, (1982): 23–86.

[HMe] M. Hill and L. Meier, *The C_2-spectrum $Tmf_1(3)$ and its invertible modules*, Algebraic & Geometric Topology, **17**, No. 4, (2017): 1953–2011.

[Hortsch] R. Hortsch, *Counting elliptic curves of bounded Faltings height*, Acta Arithmetica, **173**, No. 3, (2016): 239–253.

[HP] C. Han and J. Park, *Arithmetic of the moduli of semistable elliptic surfaces*, Mathematische Annalen, **375**, No. 3–4, (2019): 1745–1760.

[HS] R. Harron and A. Snowden, *Counting elliptic curves with prescribed torsion*, Journal für die reine und angewandte Mathematik, **729**, (2017): 151–170.

[Järvilehto] T. Järvilehto, *Jumping numbers of a simple complete ideal in a two-dimensional regular local ring*, Memoirs of the American Mathematical Society, **214**, No. 1009, (2011).

[Känel] R. V. Känel, *An effective proof of the hyperelliptic Shafarevich conjecture*, Journal de Théorie des Nombres de Bordeaux, **26**, No. 2, (2014): 507–530.

[KM] N. Katz and B. Mazur, *Arithmetic Moduli of Elliptic Curves*, Annals of Mathematics Studies, **108**, Princeton University Press (1985).

[Knudsen] F. F. Knudsen, *The projectivity of the moduli space of stable curves II, III*, Mathematica Scandinavica, **52**, No. 2, (1983): 161–199, 200–212.

[Knutson] D. Knutson, *Algebraic Spaces*, Lecture Notes in Mathematics, **203**, Springer-Verlag, Berlin (1971).

[Landesman] A. Landesman, *The Torelli map restricted to the hyperelliptic locus*, arXiv:1911.02084, (2019).

[Liedtke] C. Liedtke, *Algebraic Surfaces in Positive Characteristic*, Birational Geometry, Rational Curves, and Arithmetic, Simons Symposia, Springer–Verlag New York (2013): 229–292.

[Liu] Q. Liu, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics, **6**, Oxford University Press (2002).

[Liu2] Q. Liu, *Modèles entiers des courbes hyperelliptiques sur un corps de valuation discrète*, Transactions of the American Mathematical Society, **348**, No. 11, (1996): 4577–4610.

[Liu3] Q. Liu, *Conducteur et discriminant minimal de courbes de genre 2*, Compositio Mathematica, **94**, No. 1, (1994): 51–79.

[LO] Y. Laszlo and M. Olsson *The six operations for sheaves on Artin stacks II: Adic coefficients*, Publications Mathématiques de l'I.H.É.S., **107**, (2008): 169–210.

[Lockhart] P. Lockhart, *On the discriminant of a hyperelliptic curve*, Transactions of the American Mathematical Society, **342**, No. 2, (1994): 729–752.

[LP] Y. Lekili and A. Polishchuk, *A modular compactification of $\mathcal{M}_{1,n}$ from $\mathcal{A}_\infty$-structures*, Journal für die reine und angewandte Mathematik, **755**, (2019): 151–189.

[Mazur] B. Mazur, *Modular curves and the Eisenstein ideal*, Publications Mathématiques de l'I.H.É.S., **47**, (1977): 33–186.

[Meier] L. Meier, *Additive decompositions for rings of modular forms*, arXiv:1710.03461, (2017).

[Milne] J. S. Milne, *Jacobian Varieties*, Arithmetic Geometry, (Storrs, Conn., 1984), Springer, New York, NY, (1986): 167–212.

[Niles] A. Niles, *Moduli of elliptic curves via twisted stable maps*, Algebra and Number Theory, **7**, No. 9, (2013): 2141–2202.

[Olsson] M. Olsson, *Hom-stacks and restriction of scalars*, Duke Mathematical Journal, **134**, No. 1, (2006): 139–164.

[Olsson2] M. Olsson, *Algebraic Spaces and Stacks*, Colloquium Publications, **62**, American Mathematical Society (2016).

[Oort] F. Oort, *Hyperelliptic curves over number fields*, Classification of Algebraic Varieties and Compact Complex Manifolds, Lecture Notes in Mathematics, **412**, Springer–Verlag Berlin Heidelberg (1974): 211–218.

[OS] F. Oort and J. Steenbrink, *The local Torelli problem for algebraic curves*, Journées de Géometrie Algébrique d'Angers, Juillet 1979 / Algebraic Geometry, Angers, Sijthoff & Noordhoff. Alphen aan den Rijn-Germantown Md. 1980, (1979): 157–204.

[OU] F. Oort and K. Ueno, *Principally polarized abelian varieties of dimension two or three are Jacobian varieties*, Journal of the Faculty of Science, University of Tokyo, Section IA Mathematics, **20**, (1973): 377–381.

[Ogg] A. P. Ogg, *Elliptic curves and wild ramification*, American Journal of Mathematics, **89**, No. 1, (1967): 1–21.

[OSr] A. Obus and P. Srinivasan, *Conductor-discriminant inequality for hyperelliptic curves in odd residue characteristic*, arXiv:1910.02589, (2019).

[Parshin] A. N. Parshin, *Minimal models of curves of genus 2 and homomorphisms of abelian varieties defined over a field of finite characteristic*, Izv. Akad. Nauk SSSR Ser. Mat., **36**, No. 1, (1972): 67–109. (Russian). Mathematics of the USSR-Izvestiya, **6**, No. 1, (1972): 65–108. (English).

[PS] J. Park and H. Spink, *Motive of the moduli stack of rational curves on a weighted projective stack*, Research in the Mathematical Sciences, Special Issue of PIMS 2019 Workshop on Arithmetic Topology, **8**, No. 1, #1 (2021).

[Saito] T. Saito, *Conductor, discriminant, and the Noether formula of arithmetic surfaces*, Duke Mathematical Journal, **57**, No. 1, (1988): 151–173.

[Smyth] D. I. Smyth, *Modular compactifications of the space of pointed elliptic curves I*, Compositio Mathematica, **147**, No. 3, (2011): 877–913.

[Smyth2] D. I. Smyth, *Modular compactifications of the space of pointed elliptic curves II*, Compositio Mathematica, **147**, No. 6, (2011): 1843–1884.

[Stacks] The Stacks Project Authors: Stacks Project (2021). `https://stacks.math.columbia.edu/tags`

[Stojanoska] V. Stojanoska, *Duality for topological modular forms*, Documenta Mathematica, **17**, (2012): 271–311.

[Sun] S. Sun, *L-series of Artin stacks over finite fields*, Algebra & Number Theory, **6**, No. 1, (2012): 47–122.

[Srinivasan] P. Srinivasan, *Conductors and minimal discriminants of hyperelliptic curves with rational Weierstrass points*, arXiv:1508.05172, (2015).

[Tanaka] H. Tanaka, *Minimal Models and Abundance for Positive Characteristic Log Surfaces*, Nagoya Mathematical Journal, **216**, (2014): 1–70.

[Weil] A. Weil, *Zum Beweis des Torellischen Satzes*, Nachrichten der Akademie der Wissenschaften in Göttingen. II. Mathematisch-Physikalische Klasse, (1957): 33–53.

Department of Mathematics, University of Georgia, Athens, GA 30602, USA
Email address: `Changho.Han@uga.edu`

Center for Geometry and Physics, Institute for Basic Science (IBS), Pohang 37673, Korea
Email address: `junepark@ibs.re.kr`