Totality of Rational points on Moduli stacks

Counting Families of Varieties : Lecture 1

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The University of Sydney

KIAS-LFANT Winter School on Number Theory

Rational Points on Projective Varieties over $\mathbb{Q}$

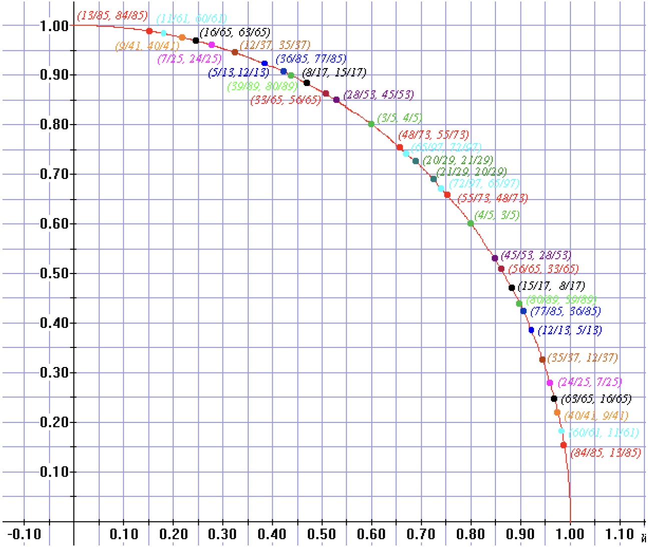


Figure 1: Rational points on $x^2 + y^2 = 1$ over $\mathbb{Q}$ - Pythagorean Triples

Why should we be happy?

1. Height of a rational number a/b with $\gcd(a,b)=1$ is $ht(a/b)=\max(|a|,|b|)$. Therefore, $ht(4/10)=5$ and $ht(1000000001/1000000000)=1000000001\neq 1$. Bigger height allows more possibilities for numerator or denominator thus more rational points that are *aritmetically complex*.

2. Geometry is enlightening and the quadratic formula is awesome as we have found / parametrized all rational points $(x,y)=\left(\frac{t^2-1}{t^2+1},\frac{2t}{t^2+1}\right)\in\mathbb{Q}^2$ on the unit circle over $\mathbb{Q}$

3. Integral points $[X:Y:Z]=[a^2-b^2:2ab:a^2+b^2]\in\mathbb{Z}^3$ on $C:=V(X^2+Y^2-Z^2)$ correspond to *"Pythagorean Triples"*

4. On **projective varieties**, the integral and the rational points coincide i.e., $X(\mathbb{Q})=X(\mathbb{Z})$. Bear in mind $\gcd(a,b)=1$.

Why should we be unhappy?

1. If we don't have a rational point to begin the process then we cannot apply quadratic formula. For example, $x^2 + y^2 = 3$, it turns out $X(\mathbb{Q}) = \emptyset$. We need *arithmetic* (Fermat's Method of Infinite Descent) to prove this.

2. Take $x^4 + y^4 = 1$ then we have *"Fermat's Last Theorem"* regarding $x^n + y^n = 1$ with $n = 4$. By Wiles-Taylor, we **know** it has only 4 rational points $X(\mathbb{Q}) = \{(\pm 1, 0), (0, \pm 1)\}$. Recalling Mordell-Faltings, we **know** it had $X(\mathbb{Q}) < \infty$

3. Take $y^2 = x^3 + Ax + B$ this is 1 polynomial in 2 variables of degree 3 (the Weierstrass cubic for an elliptic curve over $\mathbb{Q}$). **What are $E(\mathbb{Q})$?** Shockingly, *we still cannot answer this.*

4. Actually, we know there is at least 1 rational point, the point at $\infty = [0 : 1 : 0]$ for $E : V(Y^2Z - X^3 - AXZ^2 - BZ^3)$

Degree of countable infinity, the Rank

1. By Mordell-Weil, the set $E(\mathbb{Q})$ of rational points on $E/\mathbb{Q}$ has a finitely-generated abelian group structure $E(\mathbb{Q}) = \mathbb{Z}^r \oplus T$ with algebraic rank $r \in \mathbb{Z}_{\geq 0}$ and torsion subgroup T

2. The rank r of $E(\mathbb{Q})$ is **not** well understood.

   2.1 An algorithm that is guaranteed to correctly compute r?

   2.2 Which values of r can occur? *How often do they occur?*

   2.3 Is there an upper limit, or can r be arbitrarily large?

3. When r is small, computational methods exist but when r is large, often the best we can do is a lower bound; we now know, there is an $E/\mathbb{Q}$ with $r \geq 29$ by Elkies-Klagsbrun (2024). Assuming GRH we can show that $r = 29$.

Rank 29

https://web.math.pmf.unizg.hr/~duje/tors/z1.html

Trivial torsion group, rank ≥ 29

Elkies - Klagsbrun (2024)

$$y^2 + xy = x^3 - 270061832416309222184346521452974537847680546218363579547373 85x + 552580585513423764757366995911181918215210670325350796083724047791494132777161734256367214 97$$

Independent points of infinite order:

P_1 = [28911954742285371894582555366 34, 115993074809612470645983591072731867959342 5283]

P_2 = [3402542165322127811451484642234, 1661508223164691055862657623730465560755290883]

P_3 = [4298760026558467240422107564794, 4313142249890236204790986787384907722927474563]

P_4 = [3728756667770947009884455714554, 2530180219584734091116528693531660545660397443]

P_5 = [5991744132052078230511185130234, 10418901628842034362301273055728300669218858883]

P_6 = [3236493534632768520540227223034, 1324626796262167243658687198416201825373745283]

P_7 = [7822668613499117423238068938 6234, 69039421006275989650342965412551677999951255 4883]

P_8 = [11492605643548859374635605140234, 3553631691145095215546162423830845602961894088 3]

P_9 = [-5143303362384229804906088118566, 7622356511107986864120352355674305680222368483]

P_{10} = [4439856555750654352815684350 02, 6584468124388858623214803939643557365635620355]

P_{11} = [-9795650189042696807526297497 66, 8987348422104537684966706438714038633832170883]

P_{12} = [5184894285212178249566461261834, 7390536788003150201273204464695859875505480483]

P_{13} = [-44691710236871465020671796121 66, 93106588928414589341332211373920814034144556 83]

P_{14} = [3606405835110925482450522970234, 2183644666981703632482662193390480040127898883]

P_{15} = [16151744576785317732688993162234, 61908882092472338946519909276455831463747210883]

P_{16} = [3573684355943766387962362869754, 2094467155115749424853047283659077805560259203]

P_{17} = [-7593760499388581664364916441 66, 8679171135458197195914024161800061810952119683]

P_{18} = [-5328058719935886182106003119366, 69205881473794976332029355573674996762243500 83]

P_{19} = [5380268474895377355583039694554, 8105660240030025092450118297303424395856037443]

P_{20} = [17069233487425098088940203248484, 67583677272795299213867443505411893525786510633]

P_{21} = [5215432542403430758248050783794, 7501515746204716855921710958364078294243814643]

P_{22} = [2838942178046024039763692432122, 1212346280964590308944175800544505700108208003]

P_{23} = [2431468823953820159463664048081 54/81, 811625272160726332199288136187427505366582108107/729]

P_{24} = [2558229016839511149831260080762, 1706598395830079994387505244133382709649637123]

P_{25} = [2361253942905600810977556672634, 2157503396243552448798851089310708763298766083]

P_{26} = [2678312077644931683114439906234, 1462722361020796436741527433473386115047618883]

P_{27} = [3379397084927230910084852603902, 1608494167359575995485655188349208450365853755]

P_{28} = [3632407730870998917912491355514, 2255654937037700801978158381185619053396712963]

P_{29} = [2428778263277521959543043930234, 1998325023610603606161737305486867803334410883]

Previous record with rank ≥ 28

Demography of Elliptic Curves $E/\mathbb{Q}$

Trying to find / parametrize all the rational points on a given $E/\mathbb{Q}$ is a dead-end. Thus we rotate our entry. We would like to think about *the Question of Distribution and Proportion* over all $E/\mathbb{Q}$

Naive height for $E : y^2 = x^3 + Ax + B$ with no $p^4|A$ and $p^6|B$ (minimal Weierstrass model) is $ht(E) := \max(4|A|^3, 27B^2)$.

Conjecture (Minimality + Parity; Goldfeld and Katz-Sarnak)

Over any number field, 50% of all elliptic curves (when ordered by height) have Mordell-Weil rank $r = 0$ and the other 50% have Mordell-Weil rank $r = 1$. Moreover, higher Mordell-Weil ranks $r \geq 2$ constitute 0% of all elliptic curves, even though there may exist infinitely many such elliptic curves. Therefore, a suitably-defined average rank would be $\frac{1}{2}$.

What does this really mean? To talk about Average, we need the **"Total number of elliptic curves over $\mathbb{Q}$ up to isomorphism"**.

Triangle of Rational Dedekind Domains

Consider not only $E/\mathbb{Q}$ but also $E/\mathbb{F}_q(t)$ as well as $E/\mathbb{C}(z)$

1. The rational number field $\mathbb{Q}$ consisting of ratio of integer numbers in $\mathbb{Z}$ is **the rational global field of char** $= 0$

2. The rational function field $\mathbb{F}_q(t)$ with *coefficients* in $\mathbb{F}_q = \mathbb{F}_{p^r}$ consisting of ratio of polynomial functions in $\mathbb{F}_q[t]$ is **the rational global field of char** $= p > 0 \Leftrightarrow$ Projective line $\mathbb{P}^1_{\mathbb{F}_q}$

3. The meromorphic function field $\mathbb{C}(z)$ with *coefficients* in $\mathbb{C}$ consisting of ratio of holomorphic functions in $\mathbb{C}[z]$ is **NOT** the rational global field of char $= 0 \Leftrightarrow$ Riemann sphere $\mathbb{CP}^1$

   Let us count ALL elliptic curves over $K = \mathbb{F}_q(t)$ wrt height.

∞ $r = \frac{1}{2}$, $\mathcal{Z} = 4$

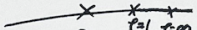
-1 $+i$ 1 $\mathrm{Vol}(\mathbb{CP}^1) = \pi$

0

$b_0 = 1$

$b_2 = 1$

$e = 2$



$t=0$ $t=1$ $t=\infty$

$[u:v]$ $t = u/v$

$|\mathbb{P}^1(\mathbb{F}_q)| = q+1$

$$\mathbb{CP}^1 \xrightarrow{\text{G.B.F}} S^2 \xrightarrow{\text{G.L.T.F}} \mathbb{P}^1_{\mathbb{F}_q}$$

Chow W.C. W.T.

$$\mathbb{C}(z) \dashrightarrow \overline{\mathbb{F}}_q(t) \xrightarrow{\infty} \mathbb{F}_q(t) - \mathrm{Poly}/\mathrm{poly}$$

Holo/Holo

G.F.A.

$\mathbb{Q}$ — Int/Int

AG/$\mathbb{C}$ AT/k AG/$\mathbb{F}_q$

KT

CV CA

NT

$\mathbb{Z} \sim \mathbb{F}_q[t]$ 'As integers so polynomials'

Let $\mathcal{C}$ be a suitable Cat. of sheaves then

$$\mathcal{C}(\mathrm{Spec}\,\mathbb{Z} \setminus T) \sim \mathcal{C}(\mathbb{P}^1_{\mathbb{F}_q} \setminus S)$$

Analogy

'Aware of each others'

Elliptic curve over a function field

Let K be the function field of a smooth, projective, absolutely irreducible curve C over the field of constants k. An elliptic curve over K is a smooth, projective, absolutely irreducible curve of genus 1 over K equipped with a K-rational point O (the origin).
$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$

Definition (Constant, Isotrivial and Non-isotrivial)

Let E be an elliptic curve over $K = k(C)$.

- We say E is *constant* if there is an elliptic curve E_0 defined over k such that $E \cong E_0 \times_k K$. Equivalently, E is constant if it can be defined by a Weierstrass cubic where the $a_i \in k$.
- We say E is *isotrivial* if there exists a finite extension K' of K such that E becomes constant over K'. A constant curve is isotrivial. Equivalently, E is isotrivial if and only if $j(E) \in k$.
- We say E is *non-isotrivial* if it is not isotrivial. We say E is *non-constant* if it is not constant.

Projective Elliptic K3 Surface of height $n = 2$

$$y^2 = x^3 + a_4(u:v)x + a_6(u:v)$$

Weierstrass data for elliptic fibration on algebraic K3 surface,

$$\begin{cases} a_4(u:v) & = -3u^4v^4, \\ a_6(u:v) & = u^5v^5(u^2+v^2). \end{cases}$$

Then we have $\Delta = 4a_4^3 + 27a_6^2$ and $j = 1728 \cdot 4a_4^3/\Delta$

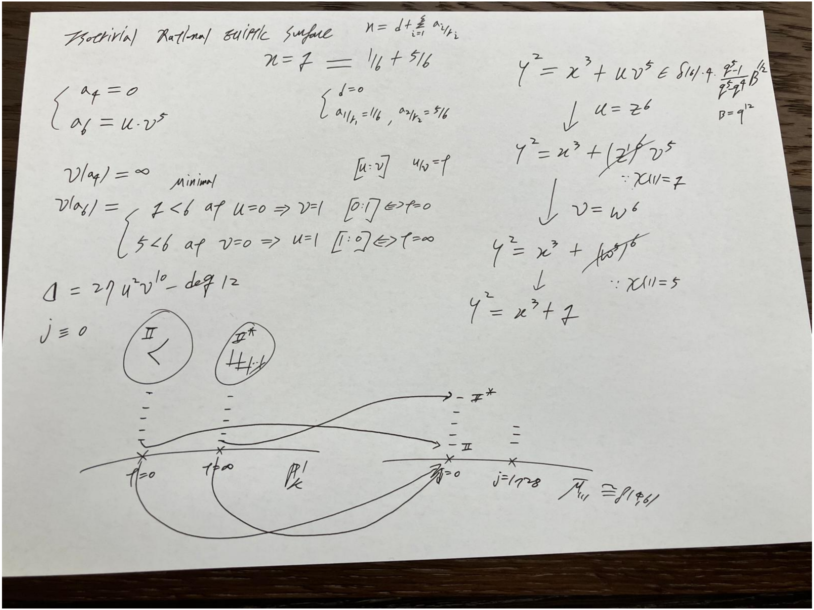
$$\begin{cases} \Delta & = 27u^{10}v^{10}(u-v)^2(u+v)^2, \\ j & = 1728 \cdot -\frac{4u^2v^2}{(u-v)^2(u+v)^2}. \end{cases}$$

Wait, the degree of j-map is 4 and NOT 24. Where did 20 go?

After all, we should have $j : \mathbb{P}^1 \to \overline{M}_{1,1} \cong \mathbb{P}^1$ of degree 24?

Well, it can get whole lot worse.

Isotrivial Rational Elliptic Surface of height $n = 1$



The Sharp Enumeration over Rational Function Field

Define *height of discriminant* Δ *over* $\mathbb{F}_q(t)$ as $ht(\Delta) := q^{\deg \Delta}$

- Elliptic case: $Deg(\Delta) = 12n \implies ht(\Delta) = q^{12n}$ for $n \in \mathbb{Z}_{\geq 0}$

We consider the counting function $\mathcal{N}(\mathbb{F}_q(t),\ B) :=$

$$\left|\left\{\text{Minimal elliptic curves over } \mathbb{P}^1_{\mathbb{F}_q} \text{ with } 0 < ht(\Delta) \leq B\right\}\right|$$

Theorem (Dori Bejleri–JP–Matthew Satriano; April 2024)

Let $\mathrm{char}(\mathbb{F}_q) > 3$ and $\delta(x) := \begin{cases} 1 & \text{if } x \text{ divides } q-1, \\ 0 & \text{otherwise.} \end{cases}$, then

$$\begin{aligned}\mathcal{N}(\mathbb{F}_q(t),\ B) = 2\left(\frac{q^9-1}{q^8-q^7}\right)B^{5/6} - 2B^{1/6} \\ + \delta(6)\cdot 4\left(\frac{q^5-1}{q^5-q^4}\right)B^{1/2} + \delta(4)\cdot 2\left(\frac{q^3-1}{q^3-q^2}\right)B^{1/3} \\ + \delta(6)\cdot 4 + \delta(4)\cdot 2\end{aligned}$$

Origins of Each Terms in $\mathcal{N}(\mathbb{F}_q(t),\ B)$

- $2\left(\frac{q^9-1}{q^8-q^7}\right) B^{5/6}$ comes from non-constant μ_2 twist families that are either non-isotrivial or isotrivial with $j \neq \infty$
- $-2B^{1/6}$ comes from non-constant μ_2 twist families of generically singular isotrivial elliptic curves with $j = \infty$
- $\delta(6) \cdot 4\left(\frac{q^5-1}{q^5-q^4}\right) B^{1/2}$ comes from non-constant μ_6 twist families of isotrivial elliptic curves with $j = 0$
- $\delta(4) \cdot 2\left(\frac{q^3-1}{q^3-q^2}\right) B^{1/3}$ comes from non-constant μ_4 twist families of isotrivial elliptic curves with $j = 1728$
- $\delta(6) \cdot 4$ comes from constant elliptic curves with $j = 0$
- $\delta(4) \cdot 2$ comes from constant elliptic curves with $j = 1728$

Precise proportions of E/K motivated by NT

Theorem (Generic Torsion Freeness; Phillips)

The set of torsion-free elliptic curves over global function fields has density 1. i.e., 'Most elliptic curves over K are torsion free'.

Theorem (Boundedness; Tate-Shafarevich & Ulmer)

The ranks of <u>non-constant</u> elliptic curves over $\mathbb{F}_q(t)$ are unbounded (in both the **isotrivial** *and the* **non-isotrivial** *cases).*

Ulmer's non-isotrivial elliptic curve of infinite rank

1. Start with $y^2 + xy = x^3 - t^d$, then *complete the square* via $y = y' - \frac{x}{2}$ and then *complete the cubic* via $x = x' - \frac{1}{12}$. We need $\mathrm{char}(k) \neq 2, 3$ to get to the short Weierstrass form.

2. We get $y^2 = x^3 - \frac{1}{48}x + \frac{1}{864} - t^d$. Coefficients should be integral thus we take $\lambda = 2 \cdot 3$ to multiply λ^4 to $-\frac{1}{48}$ and λ^6 to $+\frac{1}{864} - t^d$.

3. We arrive at $y^2 = x^3 - 27x + 54 - 2^6 \cdot 3^6 \cdot t^d$ thus $[-\frac{1}{48} : \frac{1}{864} - t^d] = [-27 : 54 - 2^6 \cdot 3^6 \cdot t^d]$.

4. Remember the isomorphism, for any $\lambda \in \mathbb{G}_m$

$$\left[y^2 = x^3 + Ax + B\right] \cong \left[y^2 = x^3 + \lambda^4 \cdot Ax + \lambda^6 \cdot B\right]$$

via $x \mapsto \lambda^{-2} \cdot x$ and $y \mapsto \lambda^{-3} \cdot y$ by the *Weighted homogeneous coordinate* of $\mathcal{P}(4, 6)$.

Magma Calculator

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Enter your code in the box below. Click on "Submit" to have it evaluated by Magma.

```
KK<t> := FunctionField(GF(4007));
E := EllipticCurve([-27, 54 - 2^6*3^6*t^11]);
E;
&*BadPlaces(E);
LocalInformation(E);
```

Cancel Submit

```
Elliptic Curve defined by y^2 = x^3 + 3980*x + (1428*t^11 + 54) over Univariate
rational function field over GF(4007)
t^11 + 1549
[ <(t^5 + 3335*t^4 + 2186*t^3 + 488*t^2 + 2393*t + 906), 1, 1, 1, I1, false>,
<(t^5 + 3337*t^4 + 2186*t^3 + 488*t^2 + 3369*t + 906), 1, 1, 1, I1, false>,
<(t), 11, 1, 11, I11, true>, <(1/t), 2, 2, 1, II, true>, <(t + 1342), 1, 1, 1,
I1, false> ]
```

1. The corresponding elliptic surface has a fiber of Kodaira type I_d at zero (at $t = 0$), while the fiber at infinity (at $1/t = 0$) is given by the congruence class $\overline{d}$ of d modulo 6 : $(\overline{d}, \Theta)$ $(\overline{0}, \mathrm{I}_0)$ $(\overline{1}, \mathrm{II}^*)$ $(\overline{2}, \mathrm{IV}^*)$ $(\overline{3}, \mathrm{I}_0^*)$ $(\overline{4}, \mathrm{IV})$ $(\overline{5}, \mathrm{II})$

2. Outside $\mathrm{char}\ 2, 3$, there are d fibres of type I_1 at the zeroes of $432t^d - 1$ (some of which may be merged if $\mathrm{char}(k)|d$).

The aim of this paper is to produce elliptic curves over $K = \mathbb{F}_p(t)$ which are nonisotrivial ($j \notin \mathbb{F}_p$) and which have arbitrarily large rank.

THEOREM 1.5. *Let p be an arbitrary prime number, $\mathbb{F}_p$ the field of p elements, and $\mathbb{F}_p(t)$ the rational function field in one variable over $\mathbb{F}_p$. Let E be the elliptic curve defined over $K = \mathbb{F}_p(t)$ by the Weierstrass equation*

$$y^2 + xy = x^3 - t^d$$

where $d = p^n + 1$ and n is a positive integer. Then $j(E) \notin \mathbb{F}_p$, the conjecture of Birch and Swinnerton-Dyer holds for E over K, and the rank of $E(K)$ is at least $(p^n - 1)/2n$.

By *the Shioda-Tate formula* and assuming maximal Picard number of $\rho = 10n$ for Faltings height n (while $b_2 = 12n - 2$), we know that $r = 10n - rk(T)$ where T is the trivial lattice. Ulmer's proof shows that as the height of Ulmer's curve goes up as $n = 1 + \left\lfloor \frac{d-1}{6} \right\rfloor \to \infty$, the algebraic/analytic rank r goes up to ∞.

Sketch of Ulmer's proof

1. Construct an elliptic surface $S \to \mathbb{P}^1$ over $\mathbb{F}_p$ with generic fiber $E : y^2 + xy = x^3 - t^d$ for $d = p^n + 1$ and $n \in \mathbb{Z}_+$.
2. Construct (and carefully study) a birational isomorphism between S and F_d/G, the quotient of a Fermat surface i.e. $V(x^d + y^d + z^d + w^d) \subset \mathbb{P}^3$ ($d = 4$ then it is K3 surface).
3. Using the fact that the Tate conjecture for surfaces is known for Fermat surfaces, one can deduce the Tate conjecture for S.
4. Use the fact that the Tate conjecture for S implies the Birch and Swinnerton-Dyer conjecture for E. Thus the ranks of the elliptic curves in the family all equal their analytic ranks.
5. The analytic ranks can be computed by relating the L-function of E to the zeta function of S, which can be related to the zeta function of F_d, which is known by Gauss sum computation of Weil. From this one is able to compute the analytic rank which is unbounded from below.

Precise proportions of E/K motivated by NT

We consider the counting function $\mathcal{N}_T^r(\mathbb{F}_q(t),\ B) :=$

$|\{\text{Minimal } E/\mathbb{F}_q(t) \text{ with algebraic rank } r, \text{ torsion } T \text{ and } ht(\Delta) \leq B\}|$

Quantitative Rank Distribution Conjecture over $K = \mathbb{F}_q(t)$

$$\mathcal{N}_{T=0}^{r=0}(\mathbb{F}_q(t),\ B) = \left(\frac{q^9-1}{q^8-q^7}\right) B^{5/6} + o(B^{\frac{5}{6}}),$$

$$\mathcal{N}_{T=0}^{r=1}(\mathbb{F}_q(t),\ B) = \left(\frac{q^9-1}{q^8-q^7}\right) B^{5/6} + o(B^{\frac{5}{6}}),$$

$$\mathcal{N}_{T}^{r\geq 2}(\mathbb{F}_q(t),\ B) = o(B^{\frac{5}{6}}),\ \text{where all o are little-o.}$$

† $|E(K)| = 1$ and $E(K) = \mathbb{Z}$ each corresponds to 50% of all elliptic curves over K ordered by discriminant height having *equal* main leading term $B^{5/6}$ with *identical* leading coefficient $\left(\frac{q^9-1}{q^8-q^7}\right)$. Furthermore, the exact counting formulas for $\mathcal{N}_{T=0}^{r=0}(\mathbb{F}_q(t),\ B)$ and $\mathcal{N}_{T=0}^{r=1}(\mathbb{F}_q(t),\ B)$ *do not coincide* since the respective counting functions have **distinct lower-order main terms**.

Totality of Rational points on Moduli stacks

Counting Families of Varieties : Lecture 2

June Park

The University of Sydney

KIAS-LFANT Winter School on Number Theory

Moduli Theory Sponsored by Fibered Categories

One of the fundamental principles of geometry is that to understand a given class of varieties, we should study their families. This perspective is formalized through moduli functors, which assign to each base scheme the **'set' of families of these varieties** parametrized by that base.

Recalling Functor-of-Points viewpoint and Yoneda lemma, the question comes down to the existence of representing object of moduli functor. If such fine moduli space exists then we can do usual AG-AT-NT on it such as Compactification (limits of families as boundary divisors), compute Cohomology or Chow rings, determine Motives over k or Point counts over $\mathbb{F}_q$, count the number of Rational points over Global fields such as $\mathbb{Q}$ or $\mathbb{F}_q(t)$.

Problem is that it is often impossible to find this representing object in the category of schemes precisely because varieties we would like to understand via moduli-theoretic-framework have automorphisms (yes, every torus has an involution $\tau^2 = 1$).

Thus, we declare there is a category fibered in groupoids over a base category of algebraic spaces / schemes with étale topology such that isomorphisms form a sheaf and every descent datum is effective. A stack is algebraic in the sense of Deligne–Mumford (resp. Artin) if it has an étale (resp. smooth) presentation.

We now have the category of algebraic stacks Stck_k as the fibered category over Sch_k with étale site.

We want fibered category since we would like to consider families-of-varieties over a base as objects where morphisms on the base category induce pullbacks of families. And we want groupoids since we would like points-on-stacks to have group-like dimension (Recall what is a groupoid of a single object). We want to glue local families to a unique global family. We want to use étale topology rather than Zariski topology since we want to distinguish étale torsors from Zariski torsors. We want étale presentation since we can work on stacks just like we would on schemes via fiber products and representability (morphisms with characters).

Algebraization of Moduli Functors as Artin Stacks

And the superpower of AG is that moduli functor we began with has representing objects in the category of Stck_k which means there is the **geometric 'space' that is entirely algebraic** which has its points as isomorphism classes and each point also has group attached to it for automoprhisms of that isomorphism class.

What's more, it naturally carries universal family over it which allows any pullback to acts as along representable classifying morphisms i.e. every family of varieties corresponds uniquely to a certain representable classifying morphisms and vice versa.

We can now do what we learned to do in algebraic topology with classifying spaces and $EG \to BG$ in algebraic geometry or number theory as well (yes, alien intellect of Grothendieck is awe-inspiring).

Algebraic Stacks 101 - Global Quotient Stacks

Let G be a linear algebraic k-group acting on a k-scheme of finite type X. We consider any action which is not necessarily free.

Then there exists a quotient stack $[X/G]$, and the projection map $X \to [X/G]$ is a G-torsor (i.e. a principal G-bundle).

If $X = Spec\ k$, we get the classifying stack $\mathcal{B}G := [Spec\ k/G]$.

Open/closed substacks of $[X/G]$ all have the form $[Y/G]$, where $Y \subset X$ is open/closed and G-invariant.

Theorem 1.1. *Let X be a normal noetherian algebraic stack (over $\mathbb{Z}$) whose stabilizer groups at closed points of X are affine. The following are equivalent.*

(1) *X has the resolution property: every coherent sheaf on X is a quotient of a vector bundle on X.*

(2) *X is isomorphic to the quotient stack of some quasi-affine scheme by an action of the group $\mathrm{GL}(n)$ for some n.*

For X of finite type over a field k, these are also equivalent to:

(3) *X is isomorphic to the quotient stack of some affine scheme over k by an action of an affine group scheme of finite type over k.*

What if our algebraic stack $\mathcal{X}$ is **NOT** a global quotient stack?

If $\mathcal{X}$ is an algebraic stack of finite type over a field with *affine stabilizer groups* then it admits a stratification by quotient stacks.

Stratification means disjoint union of locally closed substacks.

Proposition 3.5.9. *Let Y be a stack. Then Y admits a stratification by global quotient stacks if and only if for every geometric point* $x\colon \operatorname{Spec}\Omega \to Y$*, the stabilizer group* $\operatorname{Isom}_\Omega(x,x)$ *is affine.*

What kind of Artin stacks of finite type are we leaving out here?

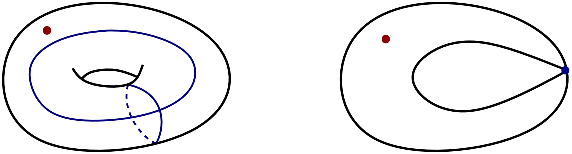
Q: How can stabilizers be non-affine? When do we ever quotient a scheme by a projective abelian variety really?

A: We don't since we get an Artin stack that is virtually impossible to work with like $\mathcal{M}_{1,0} = \mathcal{B}E$ the classifying stack of $G = E$.

All in all, the basic building blocks of algebraic stacks are the quotient stacks of the form $[Spec\ A/G]$ where $Spec\ A$ is an affine scheme and G is a linearly reductive group.

Deligne–Mumford stack $\overline{\mathcal{M}}_{1,1}$ of stable elliptic curves

Fine moduli stack $\overline{\mathcal{M}}_{1,1}$ parametrizes isomorphism classes $[E]$ of stable elliptic curves with the coarse moduli space $\overline{M}_{1,1} \cong \mathbb{P}^1$ parametrizing the j-invariant $j([E]) = 1728 \cdot 4a_4^3/(4a_4^3 + 27a_6^2)$



When the characteristic of the field k is not equal to 2 or 3, $(\overline{\mathcal{M}}_{1,1})_k \cong [(Spec\ k[a_4, a_6] - (0,0))/\mathbb{G}_m] =: \mathcal{P}_k(4,6)$ through the short Weierstrass equation: $y^2 = x^3 + a_4x + a_6$

Stabilizers are the orbifold points $[1:0]$ & $[0:1]$ with μ_4 & μ_6 respectively and the generic stacky points such as $[1:1]$ with μ_2

The fine moduli stack $\overline{\mathcal{M}}_{1,1}$ comes equipped with the universal family $p : \overline{\mathcal{E}}_{1,1} \to \overline{\mathcal{M}}_{1,1}$ of stable elliptic curves.

Boundary Divisor $\overline{\mathcal{M}}_{1,1} \setminus \mathcal{M}_{1,1} = [\infty]$ for I_1 nodal fiber

1. Take the nodal curve $y^2 = x^3 + x^2$, then *complete the cubic* via $x = x' - \frac{1}{3}$. This is why we require $\mathrm{char}(k) \neq 2, 3$.

2. We get $y^2 = x^3 - \frac{1}{3}x + \frac{2}{27}$. Coefficients should be integral thus we take $\lambda = 3$ to multiply λ^4 to $-\frac{1}{3}$ and λ^6 to $+\frac{2}{27}$. Notice here *weighted homogeneous coordinate* of $\mathcal{P}(4, 6)$.

3. We arrive at $y^2 = x^3 - 27x + 54$ thus $[-\frac{1}{3} : \frac{2}{27}] = [-27 : 54]$.

   Curve is singular $\Delta = 4(-27)^3 + 27(54)^2 = 0$ thus $j = \infty$. Stable nodal cubic written as I_1 in Kodaira notation.

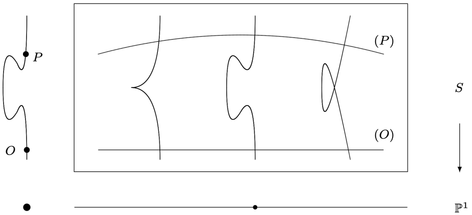
4. Remember the isomorphism, for any $\lambda \in \mathbb{G}_m$

$$\left[y^2 = x^3 + Ax + B\right] \cong \left[y^2 = x^3 + \lambda^4 \cdot Ax + \lambda^6 \cdot B\right]$$

   via $x \mapsto \lambda^{-2} \cdot x$ and $y \mapsto \lambda^{-3} \cdot y$.

Elliptic surfaces $/k$ = Families of elliptic curves $/K$

The study of **fibrations of algebraic curves** lies at the heart of the Enriques-Kodaira classification of algebraic surfaces.



We call an algebraic surface S to be an **elliptic surface**, if it admits an elliptic fibration $f : S \to C$ which is a flat proper morphism f from a nonsingular surface S to a nonsingular curve C, such that a generic fiber is a smooth curve of genus 1.

While this is the most general setup, it is natural to work with the case when the base curve is the smooth projective line $\mathbb{P}^1$ and there exists a section $O : \mathbb{P}^1 \hookrightarrow S$ coming from the identity points of the elliptic fibres and not passing through the singular points.

Moduli stack of stable elliptic fibrations

Thus, a stable elliptic fibration $g : Y \to \mathbb{P}^1$ is induced by a morphism $\varphi_f : \mathbb{P}^1 \to \overline{\mathcal{M}}_{1,1}$ and vice versa.

$$\begin{tikzcd} X \arrow[r, "\nu"] \arrow[d, "f"] & Y = \varphi_f^*(\overline{\mathcal{E}}_{1,1}) \arrow[r] \arrow[d, "g"] & \overline{\mathcal{E}}_{1,1} \arrow[d, "p"] \\ \mathbb{P}^1 \arrow[r, equal] & \mathbb{P}^1 \arrow[r, "\varphi_f"] & \overline{\mathcal{M}}_{1,1} \end{tikzcd}$$

X is the non-singular semistable elliptic surface; Y is the stable elliptic fibration; $\nu : X \to Y$ is the minimal resolution.

The moduli stack $\mathcal{L}_{12n}$ of stable elliptic fibrations over the $\mathbb{P}^1$ with $12n$ nodal singular fibers and a marked section **is** the Hom stack $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$ where $\varphi_f^*\mathcal{O}_{\overline{\mathcal{M}}_{1,1}}(1) \cong \mathcal{O}_{\mathbb{P}^1}(n)$.

A morphism $\varphi_f : \mathbb{P}^1 \to \overline{\mathcal{M}}_{1,1}$ consists of global sections (homogeneous polynomials in $[u : v]$) $\varphi_f = (a_4(u,v), a_6(u,v))$ where $\deg(a_4) = 4n$ and $\deg(a_6) = 6n$ **(!)** and $\mathrm{Res}(a_4, a_6) \neq 0$.

Grothendieck ring $K_0(\mathrm{Stck}_k)$ of k-algebraic stacks

Ekedahl in 2009 introduced the Grothendieck ring $K_0(\mathrm{Stck}_k)$ of algebraic stacks extending the classical Grothendieck ring $K_0(\mathrm{Var}_k)$ of varieties first defined by Grothendieck in 1964.

Definition

Fix a field k. Then the *Grothendieck ring $K_0(\mathrm{Stck}_k)$ of algebraic stacks of finite type over k all of whose stabilizer group schemes are affine* is an abelian group generated by isomorphism classes of algebraic stacks $\{\mathcal{X}\}$ modulo relations:

- $\{\mathcal{X}\} = \{\mathcal{Z}\} + \{\mathcal{X} \setminus \mathcal{Z}\}$ for $\mathcal{Z} \subset \mathcal{X}$ a closed substack,
- $\{\mathcal{E}\} = \{\mathcal{X} \times \mathbb{A}^n\}$ for $\mathcal{E}$ a vector bundle of rank n on $\mathcal{X}$.

Multiplication on $K_0(\mathrm{Stck}_k)$ is induced by $\{\mathcal{X}\}\{\mathcal{Y}\} := \{\mathcal{X} \times \mathcal{Y}\}$.
A distinguished element $\mathbb{L} := \{\mathbb{A}^1\}$ is called the *Lefschetz motive*.

$\{\mathbb{P}^1\} = \mathbb{L} + 1,\ \{\mathbb{P}^N\} = \mathbb{L}^N + \ldots + 1,\ \{\mathbb{G}_m\} = \mathbb{L} - 1,\ \{E\} = ?$

Universal Property for Additive Invariants

For any ring R and any function $\tilde{\nu} : \mathrm{Stck}_k \to R$ satisfying relations
1) $\tilde{\nu}(\mathcal{X}) = \tilde{\nu}(\mathcal{Y})$ whenever $\mathcal{X} \cong \mathcal{Y}$,
2) $\tilde{\nu}(\mathcal{X}) = \tilde{\nu}(\mathcal{U}) + \tilde{\nu}(\mathcal{X} \setminus \mathcal{U})$ for $\mathcal{U} \hookrightarrow \mathcal{X}$ an open immersion,
2) $\tilde{\nu}(\mathcal{X} \times \mathcal{Y}) = \tilde{\nu}(\mathcal{X}) \cdot \tilde{\nu}(\mathcal{Y})$,
there is a unique ring homomorphism $\nu : K_0(\mathrm{Stck}_k) \to R$

$$\begin{tikzcd} & \mathrm{Stck}_k \arrow[dl, "\{\ \}"'] \arrow[dr, "\tilde{\nu}"] & \\ K_0(\mathrm{Stck}_k) \arrow[rr, "\nu"'] & & R \end{tikzcd}$$

Such homomorphism ν are called **motivic measures**.

$\therefore$ When $k = \mathbb{F}_q$, the point counting measure $\{\mathcal{X}\} \mapsto \#_q(\mathcal{X})$ is a well-defined ring homomorphism $\#_q : K_0(\mathrm{Stck}_{\mathbb{F}_q}) \to \mathbb{Q}$ giving the weighted point count $\#_q(\mathcal{X})$ of $\mathcal{X}$ over $\mathbb{F}_q$.

$$|\mathbb{P}^N(\mathbb{F}_q)| = q^N + \ldots + 1, \ q + 1 - 2\sqrt{q} \leq |E(\mathbb{F}_q)| \leq q + 1 + 2\sqrt{q}$$

Accessing Cruder Level of Topology via Motives

A priori, point counts over $\mathbb{F}_q$ shouldn't know any topology.

In $\mathbb{A}^2_k$, cusp singular fiber II and affine line $\mathbb{A}^1$ have the same point counts (motives) i.e. $\{\mathrm{II} = V(y^2 = x^3)\} = \mathbb{L} = \{\mathbb{A}^1 = V(x)\}$ but they have very different *topology*.

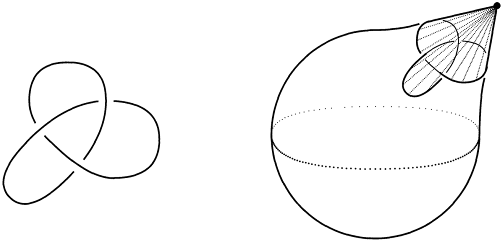
Same motive since we have a stratification of $\mathrm{II} = X_1 \cup X_2$ where $X_1 = \mathrm{II} - \{pt\}$ and $X_2 = \{pt\}$ and $\mathbb{A}^1 = Y_1 \cup Y_2$ where $Y_1 = \mathbb{A}^1 - \{pt\}$ and $Y_2 = \{pt\}$.

Indeed, $X_1 \cong Y_1$ (smooth complement) and $X_2 \cong Y_2$ (a singular point is just like a smooth point as $Spec(k)$) i.e. they are *cut-and-paste equivalent* and naturally $\{\mathrm{II}\} = \{\mathbb{A}^1\} = \mathbb{L}$

Same for nodal cubic $\{\mathrm{I}_1 = V(y^2 = x^3 + x^2)\} = \mathbb{L}$

Different topology since, II and I_1 have arithmetic genus 1 (they are singular elliptic curves) whereas $\mathbb{A}^1$ has arithmetic genus 0

Singular point on II is the tip of a cone over the trefoil knot whereas singular point on I_1 is the tip of a cone over the Hopf link. (Every isolated singularity of a complex curve in a complex surface can be described topologically as the tip of a cone on a link)



8.7. Trefoil knot, and cusp fiber

Miracle: When a variety is smooth projective then its point count over $\mathbb{F}_q$ knows topology via Frobenius weights and étale purity (the finite field analogue of RH) through the Grothendieck-Lefschetz trace formula under the Weil conjecture framework.

Thinking the other way around, this suggests that we can ignore finer topology if we are just interested in the arithmetic invariant.

Corollary (Dori Bejleri–JP–Matthew Satriano; April 2024)

The disjoint union of $\psi_{n,e}$

$$\psi_n : \bigsqcup_{e=0}^{n} \mathcal{W}_{n-e}^{min} \times \mathbb{P}(V_e^1) \to \mathcal{P}\left(\bigoplus_{i=0}^{N} V_n^{\lambda_j}\right)$$

is an isomorphism after stratifying the source and target.

If we want to point count X one way to do it is to find a stratification of Y (where we know $\{X\} = \{Y\}$ even though $X \not\cong Y$) into disjoint union of locally-closed subvarieties where we can compute its motivic classes and add them up. That is, utilize *cut-and-paste property* by stratifying source X and target Y.

Grothendieck ring $K_0(\mathrm{Stck}_k)$ of k-algebraic stacks allows us to this procedure motivically (free of particular choice of ground field k and also free of choice of additive invariant on Var_k or Stck_k)

V. Arnol'd, J. Milnor, M. Atiyah, G. Segal

1. Hom space $\mathrm{Hom}_n(\mathbb{P}^1_D, \mathbb{P}^1_T)$ is the moduli space of morphisms $f : \mathbb{P}^1_D \to \mathbb{P}^1_T$ of degree n as $f^*\mathcal{O}_{\mathbb{P}^1_T}(1) \cong L_{\mathbb{P}^1_D} \cong \mathcal{O}_{\mathbb{P}^1_D}(n)$.

2. A morphism $f : \mathbb{P}^1_D \to \mathbb{P}^1_T$ consists of global sections (global homogeneous polynomials) $f = (s_0(u:v), s_1(u:v))$ where $\deg(s_0) = \deg(s_1) = n$ and are coprime i.e. $\mathrm{Res}(s_0, s_1) \neq 0$.

3. Consider $f = (-27u^{12}v^{12}, 27u^{14}v^{10} - 54u^{12}v^{12} + 27u^{10}v^{14})$ is a **degree 4** morphism as the common factor is $27u^{10}v^{10}$

4. The rational maps and the morphisms coincide i.e. $f : \mathbb{P}^1_D \dashrightarrow \mathbb{P}^1_T = f : \mathbb{P}^1_D \to \mathbb{P}^1_T$ ($\mathbb{P}^1_D$ smooth $\mathbb{P}^1_T$ projective) **after cancellation of common factors** i.e. $\gcd(s_0, s_1) = 1$

5. $\mathbb{P}^1_T(k(t))_n = \mathbb{P}^1_T(k[t])_n$ for $\mathbb{P}^1_D$ with function field $k(t)$ and ring of integers $\mathcal{O}_{k(t)} = k[t] \sim \mathbb{P}^1_T(\mathbb{Q})_{ht(a/b)} = \mathbb{P}^1_T(\mathbb{Z})_{ht(a/b)}$

Projective Elliptic K3 Surface of height $n = 2$

$$y^2 = x^3 + a_4(u:v)x + a_6(u:v)$$

Weierstrass data for elliptic fibration on algebraic K3 surface,

$$\begin{cases} a_4(u:v) & = -3u^4v^4, \text{ degree } 8 = 4 \times 2, \\ a_6(u:v) & = u^5v^5(u^2+v^2), \text{ degree } 12 = 6 \times 2. \end{cases}$$

Then we have $\Delta = 4a_4^3 + 27a_6^2$ and $j = 1728 \cdot 4a_4^3/\Delta$

$$\begin{cases} \Delta & = 27u^{10}v^{10}(u-v)^2(u+v)^2, \text{ degree } 24 = 12 \times 2, \\ j & = \frac{27u^{10}v^{10}}{27u^{10}v^{10}} \cdot -\frac{1728 \cdot 4u^2v^2}{(u-v)^2(u+v)^2}, \text{ degree } 4! \text{ NOT } 24. \end{cases}$$

The j-map $j : \mathbb{P}^1 \to \overline{M}_{1,1} \cong \mathbb{P}^1$ is always a morphism but **lost the valuation data crucial for Tate's algorithm** to find out what are (additive) singular fibers at $[0:1]$ for $t = 0$ and $[1:0]$ for $t = \infty$.

Arithmetic of $X_n := \mathrm{Hom}_n(\mathbb{P}^1_D, \mathbb{P}^1_T)$

1. $X_n = \mathbb{P}^{2n+1} - V(\mathrm{Res}(s_0, s_1))$ is the open complement of **Resultant hypersurface** $\mathrm{Res}(s_0, s_1) = 0$ in $\mathbb{P}^{2n+1}$ thus it is an open quasiprojective variety of dimension $2n+1$

2. By Farb-Wolfson's seminal work (2016) $\{X_n\} = \mathbb{L}^{2n+1} - \mathbb{L}^{2n-1} \to |X_n(\mathbb{F}_q)| = q^{2n+1} - q^{2n-1}$

3. Both domain $\mathbb{P}^1_D$ and target $\mathbb{P}^1_T$ are **unparameterized** and the action of an element of PGL_2 on the homogeneous coordinates $[u : v]$ of $\mathbb{P}^1_D$ translates to an action on the global sections s_i of $\mathcal{O}_{\mathbb{P}^1_D}(n)$ for $i = 0, 1$ which are the homogeneous coordinates of $\mathbb{P}(V) = \mathcal{P}(\underbrace{1, \ldots, 1}_{n+1 \text{ times}}, \underbrace{1, \ldots, 1}_{n+1 \text{ times}}) = \mathbb{P}^{2n+1}$

4. $\mathbb{L}^{2n+1} - \mathbb{L}^{2n-1} = \mathbb{L}(\mathbb{L}^2 - 1) \cdot \mathbb{L}^{2n-2}$ as $\{\mathrm{PGL}_2\} = \mathbb{L}(\mathbb{L}^2 - 1)$

Topology of $X_n := \mathrm{Hom}_n(\mathbb{P}^1_D, \mathbb{P}^1_T)$

1. $\mathrm{Hom}^*_n(\mathbb{P}^1_D, \mathbb{P}^1_T) \hookrightarrow \mathrm{Hom}_n(\mathbb{P}^1_D, \mathbb{P}^1_T) \to \mathbb{P}^1_T$ via the evaluation morphism $\mathrm{ev}_\infty : \mathrm{Hom}_n(\mathbb{P}^1_D, \mathbb{P}^1_T) \to \mathbb{P}^1_T$ with $f \mapsto f(\infty) \in \mathbb{P}^1_T$
2. Fiber $\mathrm{Hom}^*_n(\mathbb{P}^1_D, \mathbb{P}^1_T)$ is the based mapping space which is identical to the space of coprime polynomials $\mathrm{Poly}_1^{(n,n)}$

Definition

Fix a field K with algebraic closure $\overline{K}$. Fix $k, l \geq 0$. Define $\mathrm{Poly}_1^{(k,l)}$ to be the set of pairs (u, v) of monic polynomials in $K[z]$ so that:

2.1 $\deg u = k$ and $\deg v = l$.

2.2 u and v have no common root in $\overline{K}$.

3. ev_∞ is a Zariski-locally trivial fibration via the transitive action of $Aut(\mathbb{P}^1_T) = \mathrm{PGL}_2$
4. $\mathbb{L}^{2n+1} - \mathbb{L}^{2n-1} = (\mathbb{L}+1) \cdot (\mathbb{L}^{2n} - \mathbb{L}^{2n-1})$ as $\{\mathrm{Hom}^*_n(\mathbb{P}^1_D, \mathbb{P}^1_T)\} = \{\mathrm{Poly}_1^{(n,n)}\} = \mathbb{L}^{2n} - \mathbb{L}^{2n-1}$

Arithmetic of Algebraic Stacks over Finite Fields

The weighted point count of $\mathcal{X}$ over $\mathbb{F}_q$ is defined as a sum: $\#_q(\mathcal{X}) := \sum_{x \in \mathcal{X}(\mathbb{F}_q)/\sim} \frac{1}{|\mathrm{Aut}(x)|}$ where $\mathcal{X}(\mathbb{F}_q)/\sim$ is the set of $\mathbb{F}_q$–isomorphism classes of $\mathbb{F}_q$–points of $\mathcal{X}$.

What we really need is the unweighted point count $|\mathcal{X}(\mathbb{F}_q)/\sim|$. But this is immune to the Grothendieck-Lefschetz trace formula.

We clarify the arithmetic role of the *inertia stack* $\mathcal{I}(\mathcal{X})$ of an algebraic stack $\mathcal{X}$ over $\mathbb{F}_q$ which parameterizes pairs $(x, Aut(x))$.

Theorem (Changho Han–JP)

Let $\mathcal{X}$ be an algebraic stack over $\mathbb{F}_q$ of finite type with affine diagonal. Then,

$$|\mathcal{X}(\mathbb{F}_q)/\sim| = \#_q(\mathcal{I}(\mathcal{X}))$$

Thus the weighted point count $\#_q(\mathcal{I}(\mathcal{X}))$ of the inertia stack $\mathcal{I}(\mathcal{X})$ is the unweighted point count $|\mathcal{X}(\mathbb{F}_q)/\sim|$ of $\mathcal{X}$ over $\mathbb{F}_q$.

How many elliptic curves over $k = \mathbb{F}_q$ upto isom?

The inertia stack $\mathcal{I}\overline{\mathcal{M}}_{1,1}$ parametrizes $[E]$ and automorphism groups $([E], \mathrm{Aut}[E])$. To keep track of the primitive roots of unity contained in $\mathbb{F}_q$, define function $\delta(x) := \begin{cases} 1 & \text{if } x \text{ divides } q-1, \\ 0 & \text{otherwise.} \end{cases}$

Grothendieck class in $K_0(\mathrm{Stck}_k)$ with $\mathrm{char}(k) \neq 2, 3$,

$$\{\mathcal{I}\overline{\mathcal{M}}_{1,1}\} = 2 \cdot (\mathbb{L} + 1) + 2 \cdot \delta(4) + 4 \cdot \delta(6)$$

Weighted point count over $\mathbb{F}_q$ with $\mathrm{char}(\mathbb{F}_q) \neq 2, 3$,

$$\#_q(\mathcal{I}\overline{\mathcal{M}}_{1,1}) = 2 \cdot (q + 1) + 2 \cdot \delta(4) + 4 \cdot \delta(6)$$

Exact number of $\mathbb{F}_q$–isomorphism classes with $\mathrm{char}(\mathbb{F}_q) \neq 2, 3$,

$$|\overline{\mathcal{M}}_{1,1}(\mathbb{F}_q)/\sim| = 2 \cdot (q + 1) + 2 \cdot \delta(4) + 4 \cdot \delta(6)$$

Motivic Analytic Number Theory Praxis

Moduli of minimal stable $E/\mathbb{F}_q(t)$ is $\mathcal{L}_{12n} = \mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$

Theorem (Changho Han–JP)

Grothendieck class in $K_0(\mathrm{Stck}_k)$ with $\mathrm{char}(k) \neq 2, 3$,

$$\{\mathcal{L}_{12n}\} = \mathbb{L}^{10n+1} - \mathbb{L}^{10n-1}$$

Weighted point count over $\mathbb{F}_q$ with $\mathrm{char}(\mathbb{F}_q) \neq 2, 3$,

$$\#_q(\mathcal{L}_{12n}) = q^{10n+1} - q^{10n-1}$$

Exact number of $\mathbb{F}_q$–isomorphism classes with $\mathrm{char}(\mathbb{F}_q) \neq 2, 3$,

$$|\mathcal{L}_{12n}(\mathbb{F}_q)/\sim| = \#_q(\mathcal{I}\mathcal{L}_{12n}) = 2 \cdot (q^{10n+1} - q^{10n-1})$$

$$\mathcal{Z}_{\mathbb{F}_q(t)}(\mathcal{B}) = \sum_{n=1}^{\left\lfloor \frac{log_q \mathcal{B}}{12} \right\rfloor} |\mathcal{L}_{1,12n}(\mathbb{F}_q)/\sim| = 2 \cdot \frac{(q^{11} - q^9)}{(q^{10} - 1)} \cdot \left(\mathcal{B}^{\frac{5}{6}} - 1\right)$$

Totality of Rational points on Moduli stacks

Counting Families of Varieties : Lecture 3

June Park

The University of Sydney

KIAS-LFANT Winter School on Number Theory

The Mordell's Conjecture, 1922

The Faltings' Theorem, 1983

Let X be a smooth projective curve of genus $g \geq 2$ defined over a number field K. Then the set $X(K)$ of rational points is finite set.

Truly inspiring, the arithmetic of the number of solutions is, in totality, determined by the topology of the underlying equation.

It is still a very difficult problem to determine what the actual solutions are or even the actual number of rational points.

Famous work of G. Faltings in 1983 proved the Mordell's conjecture as a direct corollary of Shafarevich's conjecture.

The overarching theme is the finiteness of families-of-varieties.

Fundamental Questions in Number Theory

Q: How many number fields are there?

A: Since there are so many, it is natural to refine the question.

Q: How many number fields are there of degree d over $\mathbb{Q}$ and discriminant $\Delta \leq B$?

A: The classical theorem of Hermite–Minkowski shows finiteness.

It leads us to fix the extension degree d and try to understand the counting function with respect to B (a positive real number) which is the bounded norm of Δ (i.e., the bounded "height" of Δ). Many remarkable results giving the upper bounds as in Schmidt, Ellenberg–Venkatesh and Couveignes.

One could consider a one-dimensional higher analogue of Shafarevich's Conjecture for Algebraic Curves.

Q: How many algebraic curves over a number field are there?

A: Since there are so many, it is natural to refine the question.

Q: Let S be a finite set of prime ideals in $\mathcal{O}_K$ the ring of integers of the number field K. How many distinct K-isomorphism classes of algebraic curves X/K are there, of genus g and possessing good reduction at all primes $P \notin S$?

A: This is the influential Shafarevich's conjecture for algebraic curves first called to attention by Igor R. Shafarevich in his 1962 address at the International Congress in Stockholm.

'Shafarevich's conjecture' is the assertion that there is only a finite number of families-of-algebraic curves for any given (K, g, S).

Remarkably, by the work of A. N. Parshin, Shafarevich's finiteness implies Mordell's conjecture in both the function field and the number field case through the Parshin's covering construction.

Summary of Faltings' Proof by H. Darmon

Faltings' proof of Mordell's conjecture is based on a sequence of maps (here X is a curve of genus g defined over K and having good reduction outside of the finite set S of primes of K):

$$\left\{ \begin{array}{l} K\text{-rational} \\ \text{points on } X \end{array} \right\} \xrightarrow{R_1} \left\{ \begin{array}{l} \text{Curves of genus } g' \text{ over } K' \\ \text{with good reduction outside } S' \end{array} \right\}$$

$$\xrightarrow{R_2} \left\{ \begin{array}{l} \text{Isomorphism classes of semistable} \\ \text{abelian varieties of dimension } g' \\ \text{with good reduction outside } S' \end{array} \right\}$$

$$\xrightarrow{R_3} \left\{ \begin{array}{l} \text{Isogeny classes of abelian varieties} \\ \text{of dimension } g' \\ \text{with good reduction outside } S' \end{array} \right\}$$

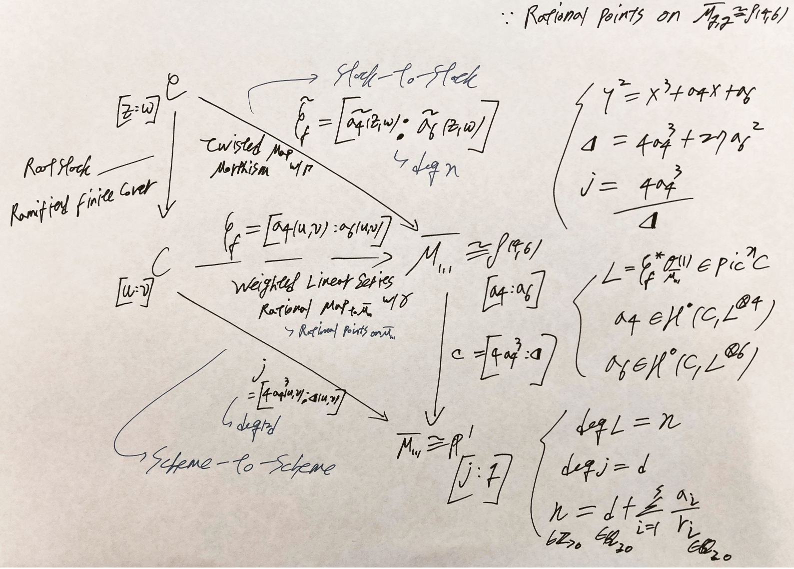
$$\xrightarrow{R_4} \left\{ \begin{array}{l} \text{Rational semisimple } \ell\text{-adic representations} \\ \text{of dimension } 2g' \text{ unramified outside } S'_\ell \end{array} \right\}$$

Construct the height moduli spaces of rational points on moduli stacks of algebraic curves, of principally-polarized abelian varieties, of characters over global fields and show they are *'of finite type'*.

1. The map R_1 is given by Parshin's construction, and is finite-to-one, by the geometric theorem of De Franchis.
2. The map R_2 is defined by passing to the jacobian of a curve, and is finite-to-one by Torelli's theorem.
3. The map R_3 is the obvious one, and is finite-to-one, by Falting's fundamental Theorem 2.11 on finiteness of abelian varieties in a given isogeny class.
4. The map R_4 is defined by passing to the Tate module, and is one-to-one, thanks to the Tate conjectures proved by Faltings. The proof of the Tate conjectures is obtained by combining a strategy of Tate with the finiteness Theorem 2.11. These ideas are also used to show that the Galois representations arising in the image of R_4 are *semisimple*.
5. The last set in this sequence of maps is finite by the finiteness principle for rational semisimple ℓ-adic representations, which is itself a consequence of the Chebotarev density theorem and the Hermite–Minkowski theorem.

Better yet, show *rationality of motivic height zeta functions* of height moduli spaces followed by extraction of coefficients.

Rational points on $\overline{\mathcal{M}}_{1,1} \cong \mathcal{P}(4,6)$ over $K = k(C)$



Stacky Heights on Algebraic Stacks wrt 'Ample' $\mathcal{V}$

Ellenberg, Zureick-Brown, and Satriano extends the rational point $x \in \mathcal{X}(K)$ to a stacky curve, called a *tuning stack* $(\mathcal{C}, \pi, \overline{x})$ for x.

$$
\begin{tikzcd}
Spec(K) \arrow[r] \arrow[rr, bend left, "x"] \arrow[dr] & \mathcal{C} \arrow[r, "\overline{x}"] \arrow[d, "\pi"] & \mathcal{X} \\
& C &
\end{tikzcd}
$$

$\mathcal{C}$ is a normal, π is a birational coarse space map.

Definition

If $\mathcal{V}$ is a vector bundle on $\mathcal{X}$ and $x \in \mathcal{X}(K)$, the *height of x with respect to $\mathcal{V}$* is defined as

$$\mathrm{ht}_{\mathcal{V}}(x) := -\deg(\pi_* \overline{x}^* \mathcal{V}^\vee)$$

for any choice of tuning stack $(\mathcal{C}, \pi, \overline{x})$.

8 Different Types of Additive Bad Reductions

Let $\kappa := \operatorname{lcm}\{\lambda_0, \ldots, \lambda_N\}$ and $\bar{\lambda}_j := \kappa/\lambda_j$ as usual.

Lemma (Dori Bejleri–JP–Matthew Satriano; April 2024)

Suppose $\kappa > 1$. Then the map

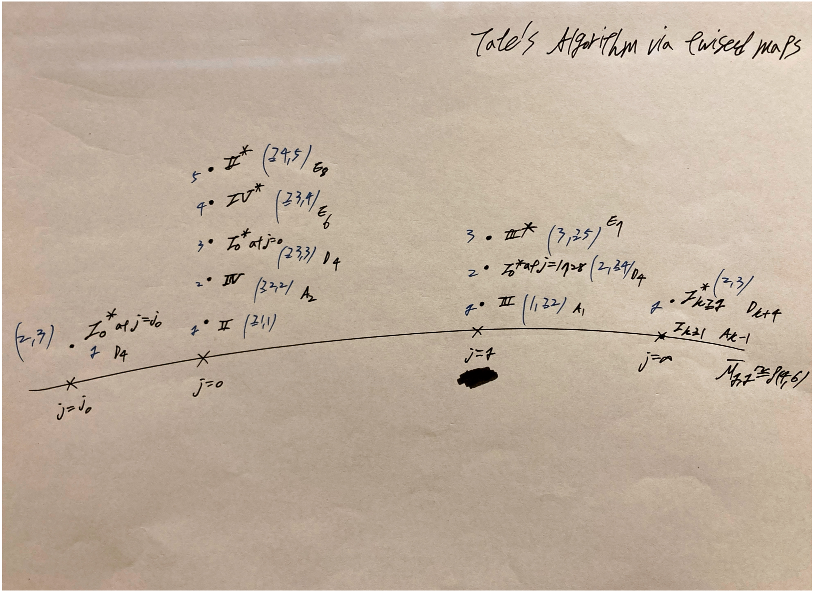
$$m \mapsto \left(\frac{\kappa}{\gcd(m, \kappa)}, \frac{m}{\gcd(m, \kappa)} \right)$$

induces a bijection from the set $\{1, \ldots, \kappa - 1\}$ to the set

$$\{(r, a) : 1 \leq a < r,\ r|\kappa,\ \gcd(r, a) = 1\}$$

For $\overline{\mathcal{M}}_{1,1} \cong \mathcal{P}(4, 6)$ we have $\kappa = 12$ which means we have $m \in \{2, 3, 4, 6, 8, 9, 10\}$ ($\{1, 5, 7, 11\}$ are excluded as prime) that corresponds to following rooting data $m = 2 \mapsto \frac{1}{6}$, $m = 3 \mapsto \frac{1}{4}$, $m = 4 \mapsto \frac{1}{3}$, $m = 6 \mapsto \frac{1}{2}$, $m = 8 \mapsto \frac{2}{3}$, $m = 9 \mapsto \frac{3}{4}$, $m = 10 \mapsto \frac{5}{6}$ which correspond to $7 + 1$ types of additive reductions. ($+1$ since ramification at $j = \infty$ for $\mathrm{I}^*_{k>0}$)

Geometric Tate's algorithm



Tate's Algorithm via Twisted Morphisms

Theorem (Dori Bejleri–JP–Matthew Satriano; April 2024)

If $\mathrm{char}(K) \neq 2, 3$*. Then the twisting condition* (r, a) *and the order of vanishing of* j *at* $j = \infty$ *determine the Kodaira fiber type, and* (r, a) *is in turn determined by* $m = \min\{3\nu(a_4), 2\nu(a_6)\}$*.*

$\gamma : (\nu(a_4),\ \nu(a_6))$	*Reduction type with* $j \in \overline{M}_{1,1}$	$\Gamma : (r, a)$
$(\geq 1, 1)$ $m = 2$	II *with* $j = 0$	$(6, 1)$
$(1, \geq 2)$ $m = 3$	III *with* $j = 1728$	$(4, 1)$
$(\geq 2, 2)$ $m = 4$	IV *with* $j = 0$	$(3, 1)$
$(2, 3)$ $m = 6$	$\mathrm{I}^*_{k>0}$ *with* $j = \infty$ I^*_0 *with* $j \neq 0, 1728$	$(2, 1)$
$(\geq 3, 3)$ $m = 6$	I^*_0 *with* $j = 0$	$(2, 1)$
$(2, \geq 4)$ $m = 6$	I^*_0 *with* $j = 1728$	$(2, 1)$
$(\geq 3, 4)$ $m = 8$	IV^* *with* $j = 0$	$(3, 2)$
$(3, \geq 5)$ $m = 9$	III^* *with* $j = 1728$	$(4, 3)$
$(\geq 4, 5)$ $m = 10$	II^* *with* $j = 0$	$(6, 5)$

Theorem (Dori Bejleri–JP–Matthew Satriano; April 2024)

Let $f\colon C \dashrightarrow \mathcal{P}(\vec{\lambda})$ be a rational map of smooth projective curve C, and let $P \in \mathcal{P}_C(\vec{\lambda})(K)$ denote the corresponding rational point over $K = k(C)$. Let $\{x_j\}$ be the indeterminacy points of f.

1. *Let $(L, s_0, \ldots, s_N)$ be any $\vec{\lambda}$-weighted linear series inducing f. Then the universal tuning stack $(\mathcal{C}, \pi, \overline{P})$ of P is the root stack of C obtained by taking the r_j-th root at x_j, where $r_j = r_{\min}(x_j; L, s_0, \ldots, s_N)$. Moreover, the induced morphism on stabilizers over x_j is given by the character $\chi_j^{-a_j}$ where $a_j = a_{\min}(x_j, L, s_0, \ldots, s_N)$.*
2. *A wls $(L, s_0, \ldots, s_N)$ is minimal if for each indeterminacy point $x \in C$, there exists an j such that $\nu_x(s_j) < \lambda_i$. There exists a unique minimal $\vec{\lambda}$-weighted linear series inducing f.*
3. *The stacky height $\mathrm{ht}_{\mathcal{O}(1)}(P)$ is equal to $\deg L$ where $(L, s_0, \ldots, s_N)$ is the unique minimal linear series. Moreover, the stable height is given by $\mathrm{ht}^{st}_{\mathcal{O}(1)}(P) = \deg \overline{P}^* \mathcal{O}(1)$ and the local contribution at x_j is given by $\delta_{x_j}(P) = \frac{a_j}{r_j}[k(x_j) : k]$.*

Height Moduli Space on Cyclotomic Stacks

There is a height moduli stack $\mathcal{M}_n(\mathcal{X}, \mathcal{L})$ parametrizing all rational points on general proper polarized cyclotomic stacks of stacky height n and that the spaces of twisted maps yield a stratification of $\mathcal{M}_n(\mathcal{X}, \mathcal{L})$ corresponding to fixing the local contributions to the stacky height. The fact that $\mathcal{M}_n(\mathcal{X}, \mathcal{L})$ is of finite type is a geometric incarnation of the Northcott property.

Theorem (Dori Bejleri–JP–Matthew Satriano; April 2024)

Let $(\mathcal{X}, \mathcal{L})$ be a proper polarized cyclotomic stack over a perfect field k. Fix a smooth projective curve C/k with function field $K = k(C)$ and $n, d \in \mathbb{Q}_{\geq 0}$.

1. *There exists a separated Deligne–Mumford stack $\mathcal{M}_{n,C}(\mathcal{X}, \mathcal{L})$ of finite type over k with a quasi-projective coarse space and a canonical bijection of k-points*

$$\mathcal{M}_{n,C}(\mathcal{X}, \mathcal{L})(k) = \left\{ P \in \mathcal{X}(K) \mid \mathrm{ht}_{\mathcal{L}}(P) = n \right\}.$$

1. There is a finite locally closed stratification

$$\bigsqcup_{\Gamma,d} \mathcal{H}^{\Gamma}_{d,C}(\mathcal{X},\mathcal{L})/S_\Gamma \to \mathcal{M}_{n,C}(\mathcal{X},\mathcal{L})$$

   where $\mathcal{H}^{\Gamma}_{d,C}$ are moduli spaces of twisted maps and the union runs over all possible admissible local conditions

$$\Gamma = (\{r_1, a_1\}, \ldots, \{r_s, a_s\})$$

   and degrees d for a twisted map to $(\mathcal{X},\mathcal{L})$ satisfying

$$n = d + \sum_{i=1}^{s} \frac{a_i}{r_i}$$

   and S_Γ is a subgroup of the symmetric group on s letters that permutes the stacky points of the twisted map.
2. Under the bijection in part (1), each k-point of $\mathcal{H}^{\Gamma}_{d,C}(\mathcal{X},\mathcal{L})/S_\Gamma$ corresponds to a K-point P with the stable height and local contributions given by

$$\mathrm{ht}^{st}_{\mathcal{L}}(P) = d \qquad \left\{\delta_i = \frac{a_i}{r_i}\right\}_{i=1}^{s}.$$

Specializing to the canonical case of $\overline{\mathcal{M}}_{1,1} \cong \mathcal{P}(4,6)$

1. The Hodge line bundle $\mathcal{L}$ of $\overline{\mathcal{M}}_{1,1}$ is $\nu = \mathcal{O}(1)$ on $\mathcal{P}(4,6)$

2. An elliptic curve E/K is a rational point $P \in \overline{\mathcal{M}}_{1,1}(K)$ which in turn corresponds to a weighted linear series on $K = k(C)$ of height n consisting of Weierstrass coefficients $a_4 \in H^0(C, \mathcal{O}(4n))$ and $a_6 \in H^0(C, \mathcal{O}(6n))$

3. The orders of vanishing at a point can be encoded in a vector $\gamma = (\nu_x(a_4), \nu_x(a_6))$ which corresponds to a certain twisting data $\Gamma = (r, a)$ of universal tuning stack, a twisted curve $\mathcal{C}$

4. The spaces $\mathcal{W}^{\gamma}_{n,C}$ and $\mathcal{H}^{\Gamma}_{d,C}$ can be identified with moduli of certain canonical models of elliptic surfaces with a specified fiber of additive bad reduction and the isomorphism between the two via Tate's algorithm can be understood in the context of the minimal model program.

Geometric Interpretation of Tate's Algorithm

$$
\begin{tikzcd}
 & & & & \mathcal{Y}=\varphi^*\,\overline{\mathcal{E}} \arrow[r] \arrow[dl, "\pi"'] \arrow[dd, "h"] & \overline{\mathcal{E}} \arrow[dd, "p"] \\
X \arrow[ddd, "f"] & X' \arrow[l, "\text{Weierstrass}"'] \arrow[ddd, "f'"] & \hat{X} \arrow[l, "\text{contraction}"'] \arrow[r, "\text{resolution}"] \arrow[ddd, "\hat{f}"] & Y \arrow[ddd, "g"] & & \\
 & & & & \mathcal{C} \arrow[r, "\varphi"] \arrow[ddl, "\rho"'] & \overline{\mathcal{M}}_{1,1} \arrow[dd] \\
 & & & & & \\
C \arrow[r, equal] & C \arrow[r, equal] & C \arrow[r, equal] & C \arrow[rr, "j"] \arrow[uurr, dashed, "\psi"] & & \overline{M}_{1,1}
\end{tikzcd}
$$

Here f is a Weierstrass model, ψ is the associated weighted linear series viewed as a rational map to $\overline{\mathcal{M}}_{1,1}$, φ is a twisted morphism from the universal tuning stack $\mathcal{C}$ which induces a stable stack-like model $h : \mathcal{Y} \to \mathcal{C}$ where $g : Y \to C$ is the twisted model via coarse moduli maps, $\hat{f}$ is a resolution of Y, and f' is the relative minimal model obtained by contracting relative (-1)-curves.

Suppose that normalized base multiplicity $m = 3$. This occurs if and only if $(\nu(a_4), \nu(a_6)) = (1, \geq 2)$. Then $r = 12/\gcd(3, 12) = 4$ and $a = 3/\gcd(3, 12) = 1$. Thus the stabilizer of the twisted curve acts on the central fiber of the twisted model via the character $\mu_4 \to \mu_4$, $\zeta_4 \mapsto \zeta_4^{-1}$. In particular, the central fiber E of $\mathcal{Y}$ has $j = 1728$. The μ_4 action on E has two fixed points, and there is an orbit of size two with stabilizer $\mu_2 \subset \mu_4$. Let E_0 be the image of E in the twisted model Y. As E appears with multiplicity 4, Y has $\frac{1}{4}(-1, -1)$ quotient singularities at the images of the the fixed points and a $\frac{1}{2}(-1, -1)$ singularity at the image of the orbit of size two. Each of these singularities is resolved by a single blowup to obtain $\hat{X}$ with central fiber $4\tilde{E}_0 + E_1 + E_2 + E_3$ where E_i are the exceptional divisors of the resolution for $i = 1, 2, 3$ and $E_1^2 = E_2^2 = -4$ with $E_3^2 = -2$. Then $\tilde{E}_0$ is a (-1)-curve so it needs to be contracted. After this contraction E_2 becomes a (-1) curve and must also be contracted. Since E_i for $i = 1, 2, 3$ are incident and pairwise transverse after blowing down $\tilde{E}_0$, then the images of E_1 and E_2 must be tangent after blowing down E_3. Moreover, they are now (-2)-curves and the relatively minimal model for type III.

Geometric Meaning of Height Moduli Framework

1. So one can run the resolution / minimal model. As these are *algebraic surfaces* it can be done over $\mathrm{char}(K) = p > 0$

2. A twisted morphism $\varphi : \mathcal{C} \to \overline{\mathcal{M}}_{1,1}$ with its twisting data Γ from the universal tuning stack $\mathcal{C}$ induces a stable stack-like model $h : \mathcal{Y} \to \mathcal{C}$ as a unique pullback of the universal family $p : \overline{\mathcal{E}} \to \overline{\mathcal{M}}_{1,1}$. All the ensuing birational geometry is natural.

3. True purpose of a **representable classifying morphism** is in the universal principle that φ intrinsically contains all the algebro-geometric data necessary to uniquely determine a fibration with singular fibers. This is the very essence of the inner arithmetic of rational points on moduli stacks over K.

1. Consider the fact that $\overline{\mathcal{M}}_{1,1}$ could have been any other algebraic stack $\mathcal{X}$ (such as $\overline{\mathcal{M}}_g$ or $\overline{\mathcal{A}}_g$) which is the representing object for certain moduli functor as the fine moduli stack together with the universal family $p : \overline{\mathcal{E}} \to \mathcal{X}$.

2. Representable classifying morphisms as twisted morphisms $\varphi : \mathcal{C} \to \mathcal{X}$ uniquely determines certain families of varieties (of algebraic curves or abelian varieties) with non-abelian stabilizers ($g \geq 2$). And they naturally have corresponding "Tate's algorithm", counting statements and so on.

3. Geometrizing $\mathcal{X}(K)$ leads to Height moduli space $\mathcal{M}_n(\mathcal{X}, \mathcal{V})$. Once we have a **space**, we compute its **invariants**, consider all invariants simultaneously via generating series and show the motivic height zeta function's **rationality**, naturally having various kinds of **consequences**.

Motivic Height Zeta Function as Generating Series

Definition

A $\vec{\lambda}$-weighted linear series $(L, s_0, \dots, s_N)$ is *minimal* if for each indeterminacy point $x \in C$, there exists an j such that $\nu_x(s_j) < \lambda_i$.

Definition

The motivic height zeta function of $\mathcal{P}(\lambda_0, \dots, \lambda_N)$ is the formal power series

$$Z_{\vec{\lambda}}(t) := \sum_{n \geq 0} \{\mathcal{W}_n^{min}\} \, t^n \in K_0(\mathrm{Stck})[\![t]\!]$$

where $\mathcal{W}_n^{min}$ is the space of minimal weighted linear series on $\mathbb{P}^1$ of height n. We also define the variant

$$\mathcal{I}Z_{\vec{\lambda}}(t) := \sum_{n \geq 0} \{\mathcal{I}\mathcal{W}_n^{min}\} \, t^n \in K_0(\mathrm{Stck}_k)[\![t]\!]$$

Stratification on Ambient Projective Stacks

Minimality defect e measures the degree of failure of a weighted linear series to be minimal (not a rational point of height n).

Definition

Let μ be the normalized base profile. We can divide each part μ_i by κ to obtain $\mu_i = \kappa q_i + r_i$. We define $q(\mu)$ and $r(\mu)$ to be the partitions with parts q_i and r_i respectively.
The minimality defect of μ is the size of the quotient $e = |q(\mu)|$.

Corollary (Dori Bejleri–JP–Matthew Satriano; April 2024)

The disjoint union of $\psi_{n,e}$

$$\psi_n : \bigsqcup_{e=0}^{n} \mathcal{W}_{n-e}^{min} \times \mathbb{P}(V_e^1) \to \mathcal{P}\left(\bigoplus_{i=0}^{N} V_n^{\lambda_j}\right)$$

is an isomorphism after stratifying the source and target.

1. We denote the usual motivic zeta function of $\mathbb{P}^1$ by

$$Z(t) = \sum \{\mathrm{Sym}^e \mathbb{P}^1\} t^e = \frac{1}{(1 - \mathbb{L}t)(1 - t)}$$

2. **We stratify by minimality defect** e to obtain an equality

$$\left\{ \mathcal{P}\left(\bigoplus_{i=0}^{N} V_n^{\lambda_i} \right) \right\} = \sum_{e=0}^{n} \{\mathcal{W}_{n-e}^{min}\}\{\mathrm{Sym}^e \mathbb{P}^1\}$$

which implies

$$\sum_{n \geq 0} \left\{ \mathcal{P}\left(\bigoplus_{i=0}^{N} V_n^{\lambda_i} \right) \right\} t^n = Z_{\vec{\lambda}}(t) \cdot Z(t) \qquad (1)$$

3. *Homogeneous polynomials* live in compact ambient stack!

$$\sum_{n \geq 0} \left\{ \mathcal{P}\left(\bigoplus_{i=0}^{N} V_n^{\lambda_i} \right) \right\} t^n = \frac{\{\mathbb{P}^N\} + \mathbb{L}^{N+1}\{\mathbb{P}^{|\vec{\lambda}|-N-2}\} t}{(1 - t)(1 - \mathbb{L}^{|\vec{\lambda}|} t)}$$

Rationality of Motivic Height Zeta Function

Fix weights $\vec{\lambda} = (\lambda_0, \ldots, \lambda_N)$ and let $|\vec{\lambda}| := \sum_{i=0}^{N} \lambda_i$. Suppose for simplicity that k contains all $\mathrm{lcm} = \mathrm{lcm}(\lambda_0, \ldots, \lambda_N)$ roots of unity.

Theorem (Dori Bejleri–JP–Matthew Satriano; April 2024)

For $k, \vec{\lambda}$ as above and $C = \mathbb{P}^1_k$, consider $\mathcal{W}_n^{min}$ and its inertia stack $\mathcal{I}\mathcal{W}_n^{min}$. We have the following formulas over $K_0(\mathrm{Stck}_k)$.

$$\sum_{n \geq 0} \{\mathcal{W}_n^{min}\} t^n = \frac{1 - \mathbb{L}t}{1 - \mathbb{L}^{|\vec{\lambda}|} t} \left(\{\mathbb{P}^N\} + \mathbb{L}^{N+1} \{\mathbb{P}^{|\vec{\lambda}| - N - 2}\} t \right)$$

$$\sum_{n \geq 0} \{\mathcal{I}\mathcal{W}_n^{min}\} t^n = \sum_{g \in \mu_{\mathrm{lcm}}(k)} \frac{1 - \mathbb{L}t}{1 - \mathbb{L}^{|\vec{\lambda}_g|} t} \left(\{\mathbb{P}^{N_g}\} + \mathbb{L}^{N_g+1} \{\mathbb{P}^{|\vec{\lambda}_g| - N_g - 2}\} t \right)$$

where g runs over the lcm roots of unity and $\vec{\lambda}_g$ is a subset of $\vec{\lambda}$ of size $N_g + 1$ depending explicitly on the order of g.

Motives of Moduli Stacks of Elliptic Surfaces

Theorem (Dori Bejleri–JP–Matthew Satriano)

Let $\mathrm{char}(k) \neq 2, 3$. *The motives (modulo* $\{\mathsf{PGL}_2\}$*) of moduli stacks* $\mathcal{W}^{\Theta}_{\min,n}$ *of minimal Weierstrass fibrations with a single Kodaira fiber* Θ *and at worst multiplicative reduction elsewhere is*

Reduction type Θ *with* $j \in \overline{M}_{1,1}$	$\lvert\gamma\rvert$	$\{\mathcal{W}^{\Theta}_{\min,n}\} \in K_0(\mathrm{Stck}_K)$
$\mathrm{I}_{k>0}$ *with* $j = \infty$	0	$\mathbb{L}^{10n-2}$
II *with* $j = 0$	2	$\mathbb{L}^{10n-3}$
III *with* $j = 1728$	3	$\mathbb{L}^{10n-4}$
IV *with* $j = 0$	4	$\mathbb{L}^{10n-5}$
$\mathrm{I}^*_{k>0}$ *with* $j = \infty$ I^*_0 *with* $j \neq 0, 1728$	5	$\mathbb{L}^{10n-6} - \mathbb{L}^{10n-7}$
I^*_0 *with* $j = 0, 1728$	6	$\mathbb{L}^{10n-7}$
IV^* *with* $j = 0$	7	$\mathbb{L}^{10n-8}$
III^* *with* $j = 1728$	8	$\mathbb{L}^{10n-9}$
II^* *with* $j = 0$	9	$\mathbb{L}^{10n-10}$

Theorem (Dori Bejleri–JP–Matthew Satriano; April 2024)

$$\left\{\mathcal{W}^{\min}_{n=1}(\mathcal{P}(\vec{\lambda}))\right\} = \{\mathbb{P}^N\}(\mathbb{L}^{|\vec{\lambda}|} - \mathbb{L}) + \mathbb{L}^{N+1}\{\mathbb{P}^{|\vec{\lambda}|-N-2}\}$$

$$\left\{\mathcal{W}^{\min}_{n\geq 2}(\mathcal{P}(\vec{\lambda}))\right\} = \mathbb{L}^{(n-2)|\vec{\lambda}|+N+2}(\mathbb{L}^{|\vec{\lambda}|-1} - 1)\{\mathbb{P}^{|\vec{\lambda}|-1}\}$$

Take $|\vec{\lambda}| = 10$ and $N = 1$ as $\overline{\mathcal{M}}_{1,1} \cong \mathcal{P}(4,6)$ over $\mathbb{Z}[1/6]$.

1. When $n = 1$, X is a **Rational elliptic surface**.

$$\left\{\mathcal{W}^{\min}_{1}\right\}=\mathbb{L}^{11}+\mathbb{L}^{10}+\mathbb{L}^{9}+\mathbb{L}^{8}+\mathbb{L}^{7}+\mathbb{L}^{6}+\mathbb{L}^{5}+\mathbb{L}^{4}+\mathbb{L}^{3}-\mathbb{L}$$

2. When $n = 2$, X is algebraic $K3$ surface with elliptic fibration (i.e., **Projective elliptic K3 surface with moduli dim. 18**).

$$\left\{\mathcal{W}^{\min}_{2}\right\}=\mathbb{L}^{21}+\mathbb{L}^{20}+\mathbb{L}^{19}+\mathbb{L}^{18}+\mathbb{L}^{17}+\mathbb{L}^{16}+\mathbb{L}^{15}+\mathbb{L}^{14}+\mathbb{L}^{13}-\mathbb{L}^{11}-\mathbb{L}^{10}-\mathbb{L}^{9}-\mathbb{L}^{8}-\mathbb{L}^{7}-\mathbb{L}^{6}-\mathbb{L}^{5}-\mathbb{L}^{4}-\mathbb{L}^{3}$$

$$=\mathbb{L}(\mathbb{L}^2-1)\left(\mathbb{L}^{18}+\mathbb{L}^{17}+2\mathbb{L}^{16}+2\mathbb{L}^{15}+3\mathbb{L}^{14}+3\mathbb{L}^{13}+4\mathbb{L}^{12}+4\mathbb{L}^{11}+5\mathbb{L}^{10}+4\mathbb{L}^{9}+4\mathbb{L}^{8}+3\mathbb{L}^{7}+3\mathbb{L}^{6}+2\mathbb{L}^{5}+2\mathbb{L}^{4}+\mathbb{L}^{3}+\mathbb{L}^{2}\right)$$