

RATIONALITY OF THE TRIVIAL LATTICE RANK WEIGHTED MOTIVIC HEIGHT ZETA FUNCTION FOR ELLIPTIC SURFACES

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ABSTRACT. Let k be a perfect field with $\text{char}(k) \neq 2, 3$, set $K = k(t)$, and let \mathcal{W}_n^{\min} be the moduli stack of minimal elliptic curves over K of Faltings height n from the height-moduli framework of [BPS22] applied to $\overline{\mathcal{M}}_{1,1} \simeq \mathcal{P}(4, 6)$. For $[E] \in \mathcal{W}_n^{\min}$, let $S \rightarrow \mathbb{P}_k^1$ be the associated elliptic surface with section. Motivated by the Shioda–Tate formula, we consider the trivariate motivic height zeta function

$$\mathcal{Z}(u, v; t) := \sum_{n \geq 0} \left(\sum_{[E] \in \mathcal{W}_n^{\min}} u^{T(S)} v^{\text{rk}(E/K)} \right) t^n \in K_0(\text{Stck}_k)[u, v][[t]],$$

which refines the height series by weighting each height stratum with the trivial lattice rank $T(S)$ and the Mordell–Weil rank $\text{rk}(E/K)$. We prove rationality for the trivial lattice specialization $Z_{\text{Triv}}(u; t) = \mathcal{Z}(u, 1; t)$ by giving an explicit finite Euler product. We conjecture irrationality for the Néron–Severi $Z_{\text{NS}}(w; t) = \mathcal{Z}(w, w; t)$ and the Mordell–Weil $Z_{\text{MW}}(v; t) = \mathcal{Z}(1, v; t)$ specializations.

1. INTRODUCTION

Let k be a perfect field with $\text{char}(k) \neq 2, 3$, and set $K := k(t)$. An elliptic curve E/K determines a relatively minimal elliptic surface with section

$$f : S \longrightarrow \mathbb{P}_k^1$$

unique up to isomorphism (see [Mir89, SS10] for background on elliptic surfaces).

The arithmetic of E/K is reflected in the geometry of S , and a basic organizing principle is the Shioda–Tate formula [Shi90]

$$(1) \quad \rho(S) = T(S) + \text{rk}(E/K),$$

where $\rho(S) = \text{rk NS}(S_{\bar{k}})$ is the *geometric Picard rank*, $T(S)$ is the *rank of the geometric trivial lattice* generated by the zero section, a fiber class, and the components of reducible fibers not meeting the zero section, and $\text{rk}(E/K)$ is the *Mordell–Weil rank*. For the elliptic surfaces arising from the height moduli considered in this paper (so that $p_g(S) = n - 1$), one has the standard bounds.

$$(2) \quad 2 \leq \rho(S) \leq 10n, \quad 2 \leq T(S) \leq 10n, \quad 0 \leq \text{rk}(E/K) \leq 10n - 2,$$

where $\rho(S) \leq 10n = h^{1,1}(S)$ is the Lefschetz bound over $k = \mathbb{C}$ (or in general the Igusa’s inequality $\rho(S) \leq b_2(S) = 12n - 2$).

In [BPS22], Bejleri–Park–Satriano construct height-moduli stacks of rational points on proper polarized cyclotomic stacks. In the fundamental modular curve example

$$\overline{\mathcal{M}}_{1,1} \simeq \mathcal{P}(4, 6), \quad \lambda \simeq \mathcal{O}_{\mathcal{P}(4,6)}(1),$$

a minimal elliptic curve over K can be viewed as a rational point of λ –height n on $\overline{\mathcal{M}}_{1,1}$ over K . This yields a separated Deligne–Mumford stack of finite type

$$\mathcal{W}_n^{\min} := \mathcal{W}_{n, \mathbb{P}_k^1}^{\min}(\mathcal{P}(4, 6), \mathcal{O}(1))$$

parametrizing minimal elliptic curves over K of discriminant degree $12n$. Here a K -rational point of $\overline{\mathcal{M}}_{1,1}$ of λ -height n means the stacky height n with respect to the Hodge line bundle λ , in the sense of [ESZB23]. Under the identification $\overline{\mathcal{M}}_{1,1} \cong \mathcal{P}(4,6)$ one has $\lambda \simeq \mathcal{O}_{\mathcal{P}(4,6)}(1)$, and this height agrees with the Faltings height of the corresponding elliptic curve by [BPS22, Cor. 7.6].

Guided by (1), we introduce the following motivic generating series (see [Eke25] for background on the Grothendieck ring of stacks) refining the height generating series in [BPS22, §8] by weighting each height stratum with the *lattice ranks* of the associated relatively minimal elliptic surface.

Definition 1.1. Let k be a perfect field of characteristic $\neq 2, 3$, and consider the height moduli stack

$$\mathcal{W}_n^{\min} = \mathcal{W}_{n, \mathbb{P}_k^1}^{\min}(\mathcal{P}(4,6), \mathcal{O}(1))$$

parametrizing minimal elliptic curves over $K = k(t)$ of discriminant height $12n$. The *trivariate height zeta function* is

$$\mathcal{Z}(u, v; t) := \sum_{n \geq 0} \left(\sum_{[E] \in \mathcal{W}_n^{\min}} u^{T(S)} v^{\text{rk}(E/K)} \right) t^n \in K_0(\text{Stck}_k)[u, v][[t]],$$

where for each $[E] \in \mathcal{W}_n^{\min}$ we write $S \rightarrow \mathbb{P}_k^1$ for the associated minimal elliptic surface, and:

- $T(S)$ is the rank of the trivial lattice of S ;
- $\text{rk}(E/K)$ is the Mordell–Weil rank.

The following specializations are the associated *bivariate height zeta functions*:

$$(3) \quad Z_{\text{Triv}}(u; t) := \mathcal{Z}(u, 1; t),$$

$$(4) \quad Z_{\text{MW}}(v; t) := \mathcal{Z}(1, v; t),$$

$$(5) \quad Z_{\text{NS}}(w; t) := \mathcal{Z}(w, w; t).$$

Remark 1.2. Setting $u = v = 1$ forgets the lattice-rank grading and specializes to the *univariate motivic height zeta function* $Z_{\bar{\lambda}}(t) = \mathcal{Z}(1, 1; t) \in K_0(\text{Stck}_k)[[t]]$ and likewise to its inertial refinement $\mathcal{I}Z_{\bar{\lambda}}(t)$ which encodes the totality of rational points on $\overline{\mathcal{M}}_{1,1}$ over $K = k(t)$. Theorem [BPS22, Thm. 8.9] shows that both series are in fact rational in t , i.e. lie in $K_0(\text{Stck}_k)(t)$, and gives explicit formulas.

In this paper we focus on $Z_{\text{Triv}}(u; t)$. The key point is that the trivial lattice is governed by *local bad reduction*: its rank is determined by the geometric Kodaira fiber configuration of $\pi_{\bar{k}}: S_{\bar{k}} \rightarrow \mathbb{P}_{\bar{k}}^1$. Writing $\text{Triv}(S) \subset \text{NS}(S_{\bar{k}})$ for the geometric trivial lattice and $T(S) := \text{rk}(\text{Triv}(S))$, we have the following explicit formula.

Lemma 1.3. Let $\pi: S \rightarrow \mathbb{P}_k^1$ be a relatively minimal elliptic surface with section, and let \mathfrak{f} be the multiset of singular fibers of $\pi_{\bar{k}}: S_{\bar{k}} \rightarrow \mathbb{P}_{\bar{k}}^1$. If m_v denotes the number of irreducible components of the fiber at v , then

$$T(S) = 2 + \sum_{\substack{v \in \mathfrak{f} \\ 2}} (m_v - 1).$$

Definition 1.4. Fix $n \geq 1$. For a geometric fiber configuration \mathfrak{f} , let $\mathcal{W}_n^{\min,(\mathfrak{f})} \subset \mathcal{W}_n^{\min}$ denote the locus parametrizing those $[E] \in \mathcal{W}_n^{\min}$ whose associated surface $S_k \rightarrow \mathbb{P}_k^1$ has singular fiber configuration \mathfrak{f} .

Definition 1.5. Fix $n \geq 0$. By Proposition 2.1, \mathcal{W}_n^{\min} admits a *finite* constructible stratification by Kodaira data, and $T(S)$ is constant on each stratum. For $n \geq 1$ and each T with $2 \leq T \leq 10n$, let

$$\mathcal{W}_n^{\min}(T) \subset \mathcal{W}_n^{\min}$$

be the finite union of those Kodaira strata on which $T(S) = T$ (hence a finite union of locally closed substacks). For $n = 0$, set $\mathcal{W}_0^{\min} := \mathcal{W}_0^{\min}(2)$.

The trivial–lattice–rank–weighted motivic height zeta function is

$$Z_{\text{Triv}}(u; t) := \sum_{n \geq 0} \sum_{T \geq 2} u^T \{ \mathcal{W}_n^{\min}(T) \} t^n \in K_0(\text{Stck}_k)[u][[t]].$$

We prove that $Z_{\text{Triv}}(u; t)$ is rational after inverting \mathbb{L} , and we give an explicit finite Euler product (Theorem 2.8).

Theorem 1.6. Let k be a perfect field with $\text{char}(k) \neq 2, 3$ and put $s = t^{1/12}$. Then

$$Z_{\text{Triv}}(u; t) \in K_0(\text{Stck}_k)[\mathbb{L}^{-1}][u](s).$$

Moreover, $Z_{\text{Triv}}(u; t)$ admits an explicit finite Euler product in s .

The proof is a motivic local-to-global factorization argument [CLL16, Bil23], implemented on the twisted-map stratification of the height moduli \mathcal{W}_n^{\min} via the evaluation morphisms [BM96, GP06]. Throughout we use the Bejleri–Park–Satriano correspondence [BPS22, Thm. 3.3] between rational points, minimal weighted linear series, and twisted morphisms; in particular, local reduction conditions are encoded by representable twisted morphisms to $\overline{\mathcal{M}}_{1,1} \simeq \mathcal{P}(4, 6)$, yielding a moduli-theoretic Tate’s algorithm compatible with the minimal model program [BPS22, Thm. 7.12]. The power structure on $K_0(\text{Stck}_k)$ packages unordered collections of local packets into Euler factors, and the finiteness of packet shapes yields a finite Euler product after inverting \mathbb{L} [GZLMH13]. The only unbounded discrete parameter is the cusp contact order in the two families \mathbf{I}_k and \mathbf{I}_k^* , which is collapsed by geometric resummation. Finally, specializing $x_\alpha = u^{m(\alpha)-1}$ (for $\alpha \in \mathcal{A}_{\text{nc}}$) together with the cusp substitutions produces the Euler product expression for $Z_{\text{Triv}}(u; t)$ as an element of $K_0(\text{Stck}_k)[\mathbb{L}^{-1}][u](s)$ with $s^{12} = t$.

Remark 1.7. Replacing $\mathcal{W}_n^{\min}(T)$ by its inertia stack (see [HP23, §2] for background on the inertia stack $\mathcal{I}(\mathcal{X})$ of an algebraic stack \mathcal{X}) gives

$$\mathcal{I}Z_{\text{Triv}}(u; t) := \sum_{n \geq 0} \sum_{T \geq 2} u^T \{ \mathcal{I}\mathcal{W}_n^{\min}(T) \} t^n \in K_0(\text{Stck}_k)[u][[t]].$$

After inverting \mathbb{L} , the same argument yields a finite Euler product for $\mathcal{I}Z_{\text{Triv}}(u; t)$.

2. RATIONALITY OF THE TRIVIAL LATTICE RANK WEIGHTED CASE

Throughout, let k be a perfect field with $\text{char}(k) \neq 2, 3$, set $K = k(t)$, and let $\pi: S \rightarrow \mathbb{P}_k^1$ be the relatively minimal elliptic surface with section associated to E/K . Write $\text{Triv}(S) \subset \text{NS}(S_{\bar{k}})$ for the geometric trivial lattice and $T(S) = \text{rk Triv}(S)$.

Proposition 2.1. *Fix $n \geq 1$. The discriminant-degree constraint $\sum_v e(F_v) = 12n$ implies that only finitely many geometric fiber configurations \mathfrak{f} occur among surfaces parametrized by \mathcal{W}_n^{\min} . Consequently,*

$$\mathcal{W}_n^{\min} = \bigsqcup_{\mathfrak{f}} \mathcal{W}_n^{\min,(\mathfrak{f})}$$

is a finite constructible stratification. Moreover, the trivial-lattice rank $T(S)$ is constant on each stratum $\mathcal{W}_n^{\min,(\mathfrak{f})}$.

Proof. Fix $n \geq 1$ and let $S \rightarrow \mathbb{P}_k^1$ be a surface parametrized by \mathcal{W}_n^{\min} . For any relatively minimal elliptic surface with section one has $\sum_{v \in \mathbb{P}_k^1} e(F_v) = e(S_{\bar{k}})$ and in our height- n locus this total Euler number equals $12n$ (equivalently, the discriminant has degree $12n$). For each singular fiber F_v , the Kodaira–Néron classification [Kod63, N64] gives the types I_k, I_k^* ($k \geq 1$) and II, III, IV, I_0^*, IV^*, III^*, II^* . Their Euler numbers satisfy $e(I_k) = k$, $e(I_k^*) = k + 6$ while the remaining types have Euler number $e(F_v) \in \{2, 3, 4, 6, 8, 9, 10\}$ (see [Her91, Table 1]). Since $\sum_v e(F_v) = 12n$, the integers k occurring in fibers of type I_k and I_k^* are bounded in terms of n . Hence there are only finitely many multisets of Kodaira symbols (equivalently, fiber configurations \mathfrak{f}) whose Euler numbers sum to $12n$. Therefore only finitely many configurations occur, and $\mathcal{W}_n^{\min} = \bigsqcup_{\mathfrak{f}} \mathcal{W}_n^{\min,(\mathfrak{f})}$ is a finite stratification by locally closed substacks as in [BPS22, Prop. 7.8]. Finally, on a fixed stratum $\mathcal{W}_n^{\min,(\mathfrak{f})}$ the multiset \mathfrak{f} (hence the integers m_v) is constant, so Lemma 1.3 implies that $T(S) = 2 + \sum_{v \in \mathfrak{f}} (m_v - 1)$ is constant on that stratum. ■

A multivariate height series. We briefly recall the local indexing used in the twisted-maps description of height moduli. By [BPS22, Thm. 5.1] the height- n moduli stack $\mathcal{M}_{n,C}(\mathcal{X}, \mathcal{L})$ on a proper polarized cyclotomic stack \mathcal{X} with polarizing line bundle \mathcal{L} admits a finite stratification by locally closed substacks indexed by admissible local conditions and degrees: there is a finite disjoint union of morphisms

$$\bigsqcup_{\Gamma, d} \mathcal{H}_{d,C}^{\Gamma}(\mathcal{X}, \mathcal{L}) / S_{\Gamma} \longrightarrow \mathcal{M}_{n,C}(\mathcal{X}, \mathcal{L}),$$

where $\mathcal{H}_{d,C}^{\Gamma}(\mathcal{X}, \mathcal{L})$ is the moduli stack of representable twisted morphisms of stable height d to $(\mathcal{X}, \mathcal{L})$ with and local twisting conditions

$$\Gamma = (\{r_1, a_1\}, \dots, \{r_s, a_s\}),$$

recording the stabilizer orders r_i and the corresponding characters a_i at the stacky marked points of the source root stack. The indices (Γ, d) range over those satisfying the height decomposition formula

$$n = d + \sum_{i=1}^s \frac{a_i}{r_i}.$$

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Here $S_\Gamma \subset S_s$ is the subgroup permuting stacky marked points of the same local type.

Definition 2.2. For the Euler-product argument it is useful to distinguish *packet shapes* from *elementary local patterns*. Let $\mathcal{IP}(4, 6)$ be the cyclotomic inertia stack.

(1) Packet shapes. Let J denote the finite set of packet shapes occurring in the Tate-algorithm stratification via twisted maps (see [BPS22, §7]); concretely one may take

$$J = \{\text{II}, \text{III}, \text{IV}, \text{II}^*, \text{III}^*, \text{IV}^*, \text{I}_0^*(j \neq 0, 1728), \text{I}_0^*(j \in \{0, 1728\}), \text{I}_\bullet, \text{I}_\bullet^*\},$$

where I_\bullet and I_\bullet^* are the two cusp shapes over $j = \infty$.

(2) Elementary local patterns. Let \mathcal{A} denote the set of elementary local patterns used to index evaluation conditions, i.e. the inertia components in which the evaluation maps land. Away from the cusp $j = \infty$, the inertia label determines the Kodaira symbol, so the non-cusp patterns form a finite set

$$\mathcal{A}_{\text{nc}} = \{\text{II}, \text{III}, \text{IV}, \text{II}^*, \text{III}^*, \text{IV}^*, \text{I}_0^*(j \neq 0, 1728), \text{I}_0^*(j \in \{0, 1728\})\}.$$

At the cusp $j = \infty$, the inertia label records only the cusp shape (I_\bullet or I_\bullet^*); the additional integer $k \geq 1$ (contact order with the boundary, equivalently the pole order of j) is part of the admissible local data on a twisted-maps chart and is treated as a multiplicity parameter within the cusp shape. Accordingly we set

$$\mathcal{A} := \mathcal{A}_{\text{nc}} \sqcup \{\text{I}_\bullet, \text{I}_\bullet^*\}.$$

For $\alpha \in \mathcal{A}_{\text{nc}}$, let $m(\alpha) \in \mathbb{Z}_{\geq 1}$ be the number of irreducible components of the corresponding Kodaira fiber, so that $m(\alpha) - 1$ is its contribution to the trivial lattice. For the cusp shapes I_\bullet and I_\bullet^* , the component number depends on the contact order $k \geq 1$ (of the corresponding I_k or I_k^* fiber); this k -dependence will be incorporated later by geometric resummation (Proposition 2.7). In summary, J indexes the *packet shapes* (basic chart types) that become Euler factors under the power structure on $K_0(\text{Stck}_k)$, whereas \mathcal{A} indexes the evaluation labels, i.e. exactly what inertia can see; in particular, over $j = \infty$ inertia distinguishes only the two cusp shapes and not the contact order k .

Remark 2.3. When an evaluation condition lands over the cusp $j = \infty$, the corresponding component of the cyclotomic inertia stack $\mathcal{IP}(4, 6)$ records only the *cusp shape* (multiplicative I_\bullet or additive I_\bullet^*); it does *not* record the *multiplicity* $k \geq 1$. Equivalently, inertia detects that j has a pole, but not its pole order. The missing discrete datum is the *contact order with the boundary*. Geometrically, it is visible on the log canonical model obtained by contracting, in each reducible fiber, the components not meeting the zero section.

(1) The multiplicative family I_k . If the fiber at $t \in \mathbb{P}^1$ is of type I_k ($k \geq 1$), then the contraction produces an A_{k-1} surface singularity. Étale locally one has

$$xy = u^k,$$

where u is a local parameter at t . Since an étale neighbourhood of the universal nodal fiber over the cusp $[\infty] \in \overline{\mathcal{M}}_{1,1} \simeq \mathcal{P}(4,6)$ is given by $xy = s$ (with s a parameter at the cusp), the classifying map $\varphi_g : \mathbb{P}^1 \rightarrow \overline{\mathcal{M}}_{1,1}$ satisfies $s = u^k$. Thus φ_g meets the boundary with contact order k , and $v(\Delta) = k$ for type I_k .

(2) The additive family I_k^* . If the fiber at t is of type I_k^* ($k \geq 1$), then the contraction produces a D_{k+4} surface singularity. The classifying map still lands at $j = \infty$ with boundary contact order k (so locally $s = u^k$), while the discriminant valuation is shifted by the starred contribution: $v(\Delta) = k + 6$ for type I_k^* .

For Z_{Triv} one has

$$m(I_k) - 1 = k - 1, \quad m(I_k^*) - 1 = k + 4,$$

so the trivial-lattice exponent depends linearly on k in each cusp family.

Definition 2.4. Fix an auxiliary variable s with $s^{12} = t$. Introduce variables $\{x_\alpha\}_{\alpha \in \mathcal{A}}$ and define

$$(6) \quad \mathcal{H}(s; \mathbf{x}) := \sum_{n \geq 0} \sum_{\mathfrak{f}} \left(\prod_{v \in \mathfrak{f}} x_{\alpha_v} \right) \{ \mathcal{W}_n^{\min,(\mathfrak{f})} \} s^{12n} \in K_0(\text{Stck}_k)[\mathbf{x}][[s]],$$

where for fixed n the inner sum ranges over the finitely many geometric fiber configurations \mathfrak{f} occurring in height n (Proposition 2.1).

For each singular fiber F_v in \mathfrak{f} , let $\alpha_v \in \mathcal{A}$ denote the corresponding inertia/evaluation label (Definition 2.2). Away from the cusp $j = \infty$ this label is the Kodaira symbol, while over $j = \infty$ it records only the cusp shape I_\bullet or I_\bullet^* . The additional contact order $k \geq 1$ at the cusp (Remark 2.3) is part of the twisted-maps chart data and is *not* recorded by the variables x_α .

Remark 2.5. The local conditions defining the strata are imposed via evaluation maps ev_i to $\mathcal{IP}(4,6)$, hence are naturally indexed by connected components of the inertia stack. In particular, the same Kodaira symbol may correspond to distinct inertia components. For example, I_0^* splits into distinct inertia components according to whether $j \in \{0, 1728\}$ or $j \notin \{0, 1728\}$. Accordingly we index local conditions by inertia labels, not by Kodaira symbols alone.

Lemma 2.6. Let \mathcal{A} be the finite set of elementary local patterns from Definition 2.2, and let $\mathcal{H}(s; \mathbf{x})$ be the multivariate height series defined in (6). After inverting \mathbb{L} , the series $\mathcal{H}(s; \mathbf{x})$ is a rational function of s with coefficients in $K_0(\text{Stck}_k)[\mathbb{L}^{-1}][\mathbf{x}]$.

More precisely, there exist:

- a finite index set J of packet shapes,
- motivic classes $A_j \in K_0(\text{Stck}_k)[\mathbb{L}^{-1}]$,
- integers $c_j \geq 1$, recording the discriminant-degree (equivalently height) increment contributed by one packet of type j ,
- and exponents $\beta_{j,\alpha} \in \mathbb{Z}_{\geq 0}$ for $\alpha \in \mathcal{A}$, recording how many markings of inertia type α occur in a packet of type j ,

such that

$$\mathcal{H}(s; \mathbf{x}) = \prod_{j \in J} \frac{1}{1 - A_j \left(\prod_{\alpha \in \mathcal{A}} x_{\alpha}^{\beta_{j,\alpha}} \right) s^{c_j}} \in K_0(\text{Stck}_k)[\mathbb{L}^{-1}][\mathbf{x}](s).$$

Moreover, for $\alpha \in \{\mathbf{I}_{\bullet}, \mathbf{I}_{\bullet}^*\}$ the exponent $\beta_{j,\alpha}$ counts only the number of cusp markings of the given cusp shape in packet type j ; it does not record the contact order $k \geq 1$.

Proof. By [BPS22, Thm. 5.1], for each n the stack \mathcal{W}_n^{\min} admits a finite locally closed stratification by charts of the form

$$\mathcal{H}_{d, \mathbb{P}_k^1}^{\Gamma}(\mathcal{P}(4, 6), \mathcal{O}(1)) / S_{\Gamma},$$

where $\mathcal{H}_{d, \mathbb{P}_k^1}^{\Gamma}(\mathcal{P}(4, 6), \mathcal{O}(1))$ parametrizes representable twisted morphisms equipped with an *ordered* list of stacky markings and admissible local data. We write the admissible local condition as

$$\Gamma = (\Gamma_{\text{nc}}, (\mathbf{k}^{\mathbf{I}}, \mathbf{k}^{\mathbf{I}^*})), \quad \mathbf{k}^{\mathbf{I}} = (k_1^{\mathbf{I}}, \dots, k_{m_1}^{\mathbf{I}}), \quad \mathbf{k}^{\mathbf{I}^*} = (k_1^{\mathbf{I}^*}, \dots, k_{m_1^*}^{\mathbf{I}^*}),$$

where Γ_{nc} records the non-cusp inertia labels and $\mathbf{k}^{\mathbf{I}}, \mathbf{k}^{\mathbf{I}^*}$ record the contact orders at cusp markings of shape \mathbf{I}_{\bullet} and \mathbf{I}_{\bullet}^* (equivalently, one may list the cusp markings as pairs (α_j, k_j) with $\alpha_j \in \{\mathbf{I}_{\bullet}, \mathbf{I}_{\bullet}^*\}$ and $k_j \geq 1$). Thus the charts are indexed by full local data, including the individual contact orders k_j .

For each marking there is an evaluation morphism

$$\text{ev}_i : \mathcal{H}_{d, \mathbb{P}_k^1}^{\Gamma}(\mathcal{P}(4, 6), \mathcal{O}(1)) \rightarrow \mathcal{IP}(4, 6), \quad (\varphi, \sigma_1, \dots, \sigma_s) \mapsto \varphi(\sigma_i),$$

and prescribing an inertia label $\alpha \in \mathcal{A}$ is equivalent to requiring ev_i to land in the corresponding connected component of $\mathcal{IP}(4, 6)$ (cf. [BPS22, Thm. 7.12]). Over the cusp $j = \infty$, the inertia label records only the cusp *shape* $\alpha \in \{\mathbf{I}_{\bullet}, \mathbf{I}_{\bullet}^*\}$; the contact orders k_j are additional admissible boundary-contact data on the chart (Remark 2.3). The variables $\mathbf{x} = \{x_{\alpha}\}_{\alpha \in \mathcal{A}}$ record only the inertia labels, so the series $\mathcal{H}(s; \mathbf{x})$ is obtained from the stratification by forgetting the extra contact-order data.

Passing to the quotient by S_{Γ} forgets the ordering among markings of the same inertia type. Consequently, the incidence data are governed by unordered configurations of marked points on the coarse curve $\pi(\mathcal{C}) \simeq \mathbb{P}^1$, hence by symmetric powers $\text{Sym}^N(\mathbb{P}^1)$. The power structure on $K_0(\text{Stck}_k)$ packages these symmetric powers into Euler factors (after inverting \mathbb{L}): each packet shape j contributes a factor

$$\frac{1}{1 - A_j \left(\prod_{\alpha \in \mathcal{A}} x_{\alpha}^{\beta_{j,\alpha}} \right) s^{c_j}}, \quad A_j \in K_0(\text{Stck}_k)[\mathbb{L}^{-1}],$$

where $c_j \geq 1$ records the increment of $\deg(\Delta)$ contributed by one packet of type j (so the s -grading is integral, with $s^{12} = t$). Multiplying over the finitely many packet shapes yields the claimed product formula.

Finally, for $\alpha \in \{\mathbf{I}_{\bullet}, \mathbf{I}_{\bullet}^*\}$ the exponent $\beta_{j,\alpha}$ counts only the number of cusp markings of the given cusp *shape* in packet type j ; the individual contact orders k_j are handled separately by geometric resummation. \blacksquare

Proposition 2.7. *Let R be a commutative ring.*

(1) Geometric resummation Fix $A \in R$. For integers $a, c \geq 1$ and $b, d \geq 0$, one has in $R[[u, t]]$

$$(7) \quad \sum_{k \geq 1} A u^{ak+b} t^{ck+d} = A u^{a+b} t^{c+d} \cdot \frac{1}{1 - u^a t^c}.$$

Moreover, if $k_1, \dots, k_M \geq 1$ are independent and contribute multiplicatively with the same step (a, c) , then

$$(8) \quad \sum_{k_1, \dots, k_M \geq 1} A \prod_{i=1}^M u^{ak_i+b} t^{ck_i+d} = A (u^{a+b} t^{c+d})^M \cdot \frac{1}{(1 - u^a t^c)^M}.$$

Equivalently, each marking contributes one factor $(1 - u^a t^c)^{-1}$, so M such markings contribute the power $(1 - u^a t^c)^{-M}$, up to the monomial shift $(u^{a+b} t^{c+d})^M$.

(2) Cusp shapes for Z_{Triv} Assume $\text{char}(k) \neq 2, 3$ and work in $R = K_0(\text{Stck}_k)[\mathbb{L}^{-1}]$. Introduce an auxiliary variable s with $t = s^{12}$, so that t^n corresponds to $\deg(\Delta) = 12n$, while s records the integral discriminant degree $\deg(\Delta)$.

After specializing $x_\beta = u^{m(\beta)-1}$ for $\beta \in \mathcal{A}_{\text{nc}}$, a cusp marking of shape \mathbf{I}_\bullet (resp. \mathbf{I}_\bullet^*) with contact order $k \geq 1$ contributes weight $u^{k-1} s^k$ (resp. $u^{k+4} s^{k+6}$), since

$$m(\mathbf{I}_k) - 1 = k - 1, \quad v(\Delta) = k, \quad m(\mathbf{I}_k^*) - 1 = k + 4, \quad v(\Delta) = k + 6$$

(cf. Remark 2.3). Hence summing over $k \geq 1$ at a single cusp marking gives, in $R[[u, s]]$,

$$(9) \quad x_{\mathbf{I}_\bullet} = \sum_{k \geq 1} u^{k-1} s^k = \frac{s}{1 - us}, \quad x_{\mathbf{I}_\bullet^*} = \sum_{k \geq 1} u^{k+4} s^{k+6} = \frac{u^5 s^7}{1 - us}.$$

In particular, each cusp marking of either shape contributes one factor $(1 - us)^{-1}$ after resummation. Thus a packet type j with $\beta_{j, \mathbf{I}_\bullet}$ markings of shape \mathbf{I}_\bullet and $\beta_{j, \mathbf{I}_\bullet^*}$ markings of shape \mathbf{I}_\bullet^* contributes the cusp factor

$$(1 - us)^{-(\beta_{j, \mathbf{I}_\bullet} + \beta_{j, \mathbf{I}_\bullet^*})},$$

together with the monomial shift

$$u^{5\beta_{j, \mathbf{I}_\bullet^*}} s^{\beta_{j, \mathbf{I}_\bullet} + 7\beta_{j, \mathbf{I}_\bullet^*}}$$

coming from (9).

Proof. For (7), factor out the $k = 1$ term and apply the geometric-series identity:

$$\sum_{k \geq 1} A u^{ak+b} t^{ck+d} = A u^{a+b} t^{c+d} \sum_{k \geq 0} (u^a t^c)^k = A u^{a+b} t^{c+d} \cdot \frac{1}{1 - u^a t^c}.$$

Equation (8) follows because the sum over (k_1, \dots, k_M) factorizes as a product of M copies of (7). Part (2) is (7) with $(a, c) = (1, 1)$ applied in $R[[u, s]]$ to the two affine-linear weights $u^{k-1} s^k$ and $u^{k+4} s^{k+6}$, yielding (9) and the stated denominator power. \blacksquare

We now prove the Main Theorem.

Theorem 2.8 (Rationality and finite Euler product for Z_{Triv}). *Let k be a perfect field of characteristic $\neq 2, 3$, and set $s = t^{1/12}$ (so $t = s^{12}$). Then*

$$Z_{\text{Triv}}(u; t) \in K_0(\text{Stck}_k)[\mathbb{L}^{-1}][u](s).$$

More precisely, let J , A_j , c_j , and $\beta_{j,\alpha}$ be as in Lemma 2.6. Put

$$\Delta(s) := 1 - us, \quad b_j := \sum_{\beta \in \mathcal{A}_{\text{nc}}} \beta_{j,\beta} (m(\beta) - 1), \quad m_j := \beta_{j,\mathbf{I}_\bullet} + \beta_{j,\mathbf{I}_\bullet^*},$$

and define

$$B_j := b_j + 5\beta_{j,\mathbf{I}_\bullet^*}, \quad C_j := c_j + \beta_{j,\mathbf{I}_\bullet} + 7\beta_{j,\mathbf{I}_\bullet^*}.$$

Then one has the finite Euler product

$$Z_{\text{Triv}}(u; t) = u^2 \cdot \prod_{j \in J} \frac{1}{1 - A_j u^{B_j} s^{C_j} \Delta(s)^{-m_j}}, \quad (s^{12} = t).$$

Moreover, all dependence on $k \geq 1$ in the cusp families \mathbf{I}_k and \mathbf{I}_k^ (over $j = \infty$) is absorbed by the single geometric-series denominator $\Delta(s)^{-1} = (1 - us)^{-1}$.*

Proof. Work in the localized ring $K_0(\text{Stck}_k)[\mathbb{L}^{-1}]$. By Lemma 2.6 we have

$$\mathcal{H}(s; \mathbf{x}) = \prod_{j \in J} \frac{1}{1 - A_j \left(\prod_{\alpha \in \mathcal{A}} x_\alpha^{\beta_{j,\alpha}} \right) s^{c_j}} \in K_0(\text{Stck}_k)[\mathbb{L}^{-1}][\mathbf{x}](s).$$

Non-cusp specialization. For $\beta \in \mathcal{A}_{\text{nc}}$ specialize $x_\beta = u^{m(\beta)-1}$. Then the contribution of the non-cusp markings in packet type j becomes the monomial u^{b_j} .

Cusp resummation. Over the cusp $j = \infty$, the contact order $k \geq 1$ varies. In the discriminant-degree grading (with $t = s^{12}$), the two cusp families contribute weights $u^{k-1}s^k$ and $u^{k+4}s^{k+6}$, hence

$$x_{\mathbf{I}_\bullet} = \sum_{k \geq 1} u^{k-1}s^k = \frac{s}{1 - us}, \quad x_{\mathbf{I}_\bullet^*} = \sum_{k \geq 1} u^{k+4}s^{k+6} = \frac{u^5 s^7}{1 - us}.$$

Substituting these into the Euler product for \mathcal{H} yields, for each packet type j , the denominator factor $\Delta(s)^{-m_j}$ and the numerator shifts encoded in B_j and C_j , giving the displayed finite Euler product for $Z_{\text{Triv}}(u; t)$ in $K_0(\text{Stck}_k)[\mathbb{L}^{-1}][u](s)$. \blacksquare

Remark 2.9. Assume $\text{char}(k) \neq 2, 3$. For each Kodaira type Θ and $n \geq 1$, let

$$\mathcal{W}_{n, \mathbb{P}^1}^\Theta$$

be the moduli stack of minimal elliptic fibrations over \mathbb{P}_k^1 of discriminant degree $12n$ having exactly one singular fiber of type Θ over a varying degree-one point and semistable everywhere else.¹ Then [BPS22, Thm. 1.6] and [HP19, Cor. 2] determine the following normalized one-fiber classes.

These one-fiber classes should be viewed as *local building blocks* for the packet-shape Euler product in Theorem 2.8. For each non-cusp type II, III, IV, II*, III*, IV*

¹The *unparameterized* \mathbb{P}_k^1 corresponds to taking the $\text{Aut}(\mathbb{P}^1) = \text{PGL}_2$ quotient; motivically this factors out $\{\text{PGL}_2\} = \mathbb{L}(\mathbb{L}^2 - 1)$, thereby treating the base as a smooth conic. See [PS25] for a comprehensive treatment.

Reduction type Θ	(r, a)	$m_v - 1$	$\frac{\{\mathcal{W}_{n, \mathbb{P}^1}^\Theta\}}{\{\mathrm{PGL}_2\} \mathbb{L}^{10n-18}} \in K_0(\mathrm{Stck}_k)$
$I_{k \geq 1} (j = \infty)$	$(0, 0)$	$k - 1$	\mathbb{L}^{16}
$II (j = 0)$	$(6, 1)$	0	\mathbb{L}^{15}
$III (j = 1728)$	$(4, 1)$	1	\mathbb{L}^{14}
$IV (j = 0)$	$(3, 1)$	2	\mathbb{L}^{13}
$I_{k \geq 1}^* (j = \infty)$	$(2, 1)$	$k + 4$	$\mathbb{L}^{12} - \mathbb{L}^{11}$
$I_0^* (j \neq 0, 1728)$	$(2, 1)$	4	$\mathbb{L}^{12} - \mathbb{L}^{11}$
$I_0^* (j = 0, 1728)$	$(2, 1)$	4	\mathbb{L}^{11}
$IV^* (j = 0)$	$(3, 2)$	6	\mathbb{L}^{10}
$III^* (j = 1728)$	$(4, 3)$	7	\mathbb{L}^9
$II^* (j = 0)$	$(6, 5)$	8	\mathbb{L}^8

and the two I_0^* cases, the corresponding local packet shapes contribute k -independent Euler factors

$$(1 - A_j u^{b_j} t^{c_j})^{-1}$$

after the specialization $x_\alpha = u^{m(\alpha)-1}$ for $\alpha \in \mathcal{A}_{\mathrm{nc}}$. In this way, c_j records the height increment contributed by one packet of type j , b_j records the corresponding trivial-lattice-rank increment from its non-cusp markings, and A_j records the normalized motivic classes of the packet locus.

For the cusp families I_k and I_k^* , the table gives the one-fiber contribution for each contact order $k \geq 1$. In the packet-shape Euler product for $\mathcal{H}(t; \mathbf{x})$, the exponents $\beta_{j, \bullet}$ and $\beta_{j, \bullet}^*$ record only the number of cusp markings of each cusp *shape* in packet type j ; the individual contact orders are not part of the inertia label. The infinite k -variation is collapsed by the geometric resummations

$$x_{I_\bullet} = \sum_{k \geq 1} u^{k-1} t^k = \frac{t}{1-ut}, \quad x_{I_\bullet^*} = \sum_{k \geq 1} u^{k+4} t^{k+6} = \frac{u^5 t^7}{1-ut},$$

so that each cusp marking contributes one factor $(1-ut)^{-1}$. Consequently, packet type j contributes the cusp factor

$$(1-ut)^{-(\beta_{j, \bullet} + \beta_{j, \bullet}^*)},$$

together with the monomial shift $u^{5\beta_{j, \bullet}^*} t^{\beta_{j, \bullet} + 7\beta_{j, \bullet}^*}$ coming from the cusp substitutions.

3. IRRATIONALITY OF THE NÉRON–SEVERI AND MORDELL–WEIL SPECIALIZATIONS

The rationality of $Z_{\mathrm{Triv}}(u; t)$ reflects the fact that the trivial lattice rank $T(S)$ is governed by *local* reduction data. Indeed, by Lemma 1.3 it depends only on the

multiset of fiber component numbers m_v , hence is constant on each Kodaira stratum $\mathcal{W}_n^{\min,(\mathfrak{f})}$, in the finite constructible stratification of Proposition 2.1. This locality is exactly what makes the evaluation-map factorization and the power structure on $K_0(\text{Stck}_k)$ applicable: unordered collections of local packets assemble into a *finite* Euler product.

In contrast, the Mordell–Weil rank $\text{rk}(E/K)$ is not determined by the fiber configuration. Even on a fixed Kodaira stratum $\mathcal{W}_n^{\min,(\mathfrak{f})}$, the rank typically varies, reflecting genuinely *global* constraints rather than local reduction data. Since $T(S)$ is constant on $\mathcal{W}_n^{\min,(\mathfrak{f})}$, the Shioda–Tate formula (1) shows that variation of $\text{rk}(E/K)$ is equivalent to variation of the Néron–Severi rank $\rho(S)$. Thus any refinement of the height series by $\text{rk}(E/K)$, or equivalently by $\rho(S)$, necessarily detects global jump phenomena invisible to the local packet stratification used for Z_{Triv} .

One way to organize this global complexity is via Néron–Severi jump loci. Fix a fiber configuration \mathfrak{f} and write $\text{Triv}^{(\mathfrak{f})} \subset \text{NS}(S_{\bar{k}})$ for the sublattice generated by the zero section, a fiber class, and the components of reducible fibers in the configuration \mathfrak{f} . Inside $\mathcal{W}_n^{\min,(\mathfrak{f})}$, imposing that $\text{NS}(S_{\bar{k}})$ contain additional algebraic classes *independent of* $\text{Triv}^{(\mathfrak{f})}$ (equivalently, that $\rho(S)$, hence $\text{rk}(E/K)$, jump) is an algebraic condition. Over \mathbb{C} , these conditions are naturally modeled by Noether–Lefschetz (Hodge) loci for the variation of Hodge structure coming from the family of elliptic surfaces over $\mathcal{W}_n^{\min,(\mathfrak{f})}$, and the theorem of Cattani–Deligne–Kaplan [CDK95] shows that such loci are, in general, only a *countable union of closed algebraic subsets*. In particular, unlike the Kodaira stratification at fixed height n (which is finite by Proposition 2.1), refinements by Néron–Severi *lattice type* are not expected to admit a finite constructible stratification.

This suggests that the local-to-global factorization mechanism producing a finite Euler product for Z_{Triv} should structurally fail for the Mordell–Weil and Néron–Severi specializations, and motivates the following conjecture.

Conjecture 3.1. Let $k = \mathbb{C}$ and $K = \mathbb{C}(z)$. The specializations

$$Z_{\text{MW}}(v; t) := \mathcal{Z}(1, v; t), \quad Z_{\text{NS}}(w; t) := \mathcal{Z}(w, w; t)$$

do not become rational functions of t after inverting \mathbb{L} , i.e.

$$Z_{\text{MW}}(v; t) \notin K_0(\text{Stck}_{\mathbb{C}})[\mathbb{L}^{-1}][v](t), \quad Z_{\text{NS}}(w; t) \notin K_0(\text{Stck}_{\mathbb{C}})[\mathbb{L}^{-1}][w](t).$$

In other words, after inverting \mathbb{L} neither bivariate height zeta series is rational in the height variable t .

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