COUNTING ISOMORPHISM CLASSES OF ELLIPTIC CURVES OVER $\mathbb{F}_q(t)$

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ABSTRACT. We determine the precise number of isomorphism classes of elliptic curves over $\mathbb{F}_q(t)$ with $\mathrm{char}(\mathbb{F}_q) = 3, 2$. The key idea is to obtain the exact unweighted number of rational points on the classifying stacks $\mathcal{B}\underline{Q_{12}}$, $\mathcal{B}\underline{T_{24}}$ and $\mathcal{B}\underline{\mathbb{Z}/2}$, where $\underline{Q_{12}}$ and $\underline{T_{24}}$ denote the constant group schemes associated to the dicyclic group of order 12 and the binary tetrahedral group of order 24, respectively, and $\underline{\mathbb{Z}/2}$ denotes the constant group scheme of order 2. This computation, inspired by the classical work of [dJ02] and performed via motivic height zeta functions of height moduli spaces constructed in [BPS22], establishes a complete determination of the total number of isomorphism classes of rational points on $\overline{\mathcal{M}}_{1,1}$ over any rational function field $k(t)$, where k is a perfect field of $\mathrm{char}(k) \geq 0$.

1. INTRODUCTION

The families of abelian varieties defined over a global field are fundamental. In this paper, we study families of elliptic curves over function fields. By a family, we mean specifically an *elliptic fibration*, that is an algebraic surface X that admits a proper flat morphism $f : X \to C$ to a smooth projective curve C/k over a field k such that a general fiber is a smooth curve of genus one. Such an X is sometimes called an *elliptic surface* in other literature. It is natural to work with the case when there exists a distinguished section $s : C \hookrightarrow X$ coming from the identity points on each of the elliptic fibers. An elliptic fibration is called *relatively-minimal* if none of the fibers contain any (-1)-curves.

It is natural to ask how many elliptic fibrations $f : X \to C$ exist. This question is equivalent to determining the total number of rational points on $\overline{\mathcal{M}}_{1,1}$ over a function field $K = k(C)$. For proper stacks, unlike schemes, there is a distinction between rational and integral points. Moreover, rational points have extra automorphism groups. In the case where $K = \mathbb{F}_q(t)$ with $\mathrm{char}(\mathbb{F}_q) > 3$, the exact number of isomorphism classes of elliptic curves over K was established in [BPS22, Theorem 9.7]. The proof relies on the height moduli framework developed in [BPS22, Theorem 1.2] by Bejleri, Satriano, and the author. Specifically, the method involves extracting the coefficients of rational motivic height zeta functions $Z_{\vec{\lambda}}(t)$ associated to the height moduli spaces and their variants on the corresponding inertia stacks $\mathcal{I}Z_{\vec{\lambda}}(t)$ as described in [BPS22, Theorem 8.9].

In the present work, we extend the enumerations to the remaining cases $\mathrm{char}(\mathbb{F}_q) = 3, 2$ inspired by the classical work of [dJ02]. Specifically, we establish the following sharp enumeration of elliptic curves over a global function field $K = \mathbb{F}_q(t)$ with precise lower order main terms. Recall that the height of the discriminant of an elliptic curve E over K is given by $ht(\Delta) := q^{\deg \Delta} = q^{12n}$ for some integer n (also called the Faltings height of E).

Theorem 1.1. *Let $n \in \mathbb{Z}_{\geq 0}$. The counting function $\mathcal{N}^w(\mathbb{F}_q(t),\, B)$ (resp. $\mathcal{N}(\mathbb{F}_q(t),\, B)$), which gives the weighted count (resp. unweighted count) of the number of isomorphism classes of minimal elliptic curves over $\mathbb{P}^1_{\mathbb{F}_q}$ ordered by the multiplicative height of the discriminant $ht(\Delta) = q^{12n} \leq B$, is given by the following.*

$$\mathcal{N}^w(\mathbb{F}_q(t),\, B) = \left(\frac{q^9-1}{q^8-q^7}\right) B^{5/6} - B^{1/6}$$

(1) For $q = 3^r$

- *r is odd :*

$$\begin{aligned}\mathcal{N}(\mathbb{F}_q(t),\, B) = 2\left(\frac{q^9-1}{q^8-q^7}\right) & B^{5/6} - 2B^{1/6} \\ & + 2\left(\frac{q^7-1}{q^7-q^6}\right) B^{2/3} - 2\left(\frac{q^3-1}{q^4-q^3}\right) B^{1/3}\end{aligned}$$

- *r is even :*

$$\begin{aligned}\mathcal{N}(\mathbb{F}_q(t),\, B) = 2\left(\frac{q^9-1}{q^8-q^7}\right) & B^{5/6} - 2B^{1/6} \\ & + 4\left(\frac{q^7-1}{q^7-q^6}\right) B^{2/3} - 4\left(\frac{q^3-1}{q^4-q^3}\right) B^{1/3}\end{aligned}$$

(2) For $q = 2^r$

- *r is odd :*

$$\begin{aligned}\mathcal{N}(\mathbb{F}_q(t),\, B) = 2\left(\frac{q^9-1}{q^8-q^7}\right) & B^{5/6} - 2B^{1/6} \\ & + \left(\frac{q^8-1}{q^8-q^7}\right) B^{3/4} - \left(\frac{q^5-1}{q^6-q^5}\right) B^{1/2} \\ & - 2q + 4\end{aligned}$$

- *r is even :*

$$\begin{aligned}\mathcal{N}(\mathbb{F}_q(t),\, B) = 2\left(\frac{q^9-1}{q^8-q^7}\right) & B^{5/6} - 2B^{1/6} \\ & + 5\left(\frac{q^8-1}{q^8-q^7}\right) B^{3/4} - 5\left(\frac{q^5-1}{q^6-q^5}\right) B^{1/2} \\ & - 2q + 4\end{aligned}$$

Remark 1.2. The lower order main term of order $B^{1/6}$ present in both the weighted and unweighted counts comes from subtracting the quadratic twist families of generically singular (i.e. isotrivial $j = \infty$) elliptic curves. The lower order main terms of order $B^{2/3}$ & $B^{1/3}$ for $\mathrm{char}(\mathbb{F}_q) = 3$ and $B^{3/4}$ & $B^{1/2}$ for $\mathrm{char}(\mathbb{F}_q) = 2$ in the unweighted count $\mathcal{N}(\mathbb{F}_q(t), B)$ arise from counting the non-constant families of isotrivial elliptic curves having strictly additive bad reductions with extra automorphisms concentrated at the supersingular j-invariant $j = 0$.

1.1. **Outline of the paper.** In Section 2, we compute the motivic classes in the Grothendieck ring of stacks of the inertia stack $\mathcal{I}\mathcal{M}_{1,1}$ for $\mathrm{char}(k) \geq 0$ and height moduli in the case of weighted projective stacks. In Section 3, we count the exact weighted number of rational points, following the work of [dJ02], on the classifying stacks $\mathcal{B}\underline{Q_{12}}$, $\mathcal{B}\underline{T_{24}}$ and $\mathcal{B}\underline{\mathbb{Z}/2}$, where $\underline{Q_{12}}$ and $\underline{T_{24}}$ denote the constant group schemes associated to the dicyclic group of order 12 and the binary tetrahedral group of order 24, respectively, and $\underline{\mathbb{Z}/2}$ denotes the constant group scheme of order 2. We then recall the number of twists, following the work of [KST17], which leads to the enumeration of elliptic curves over $\mathbb{F}_q(t)$ with $\mathrm{char}(\mathbb{F}_q) = 3, 2$ and prove Theorem 1.1.

2. Motives & Point counts over finite fields

In this section, we briefly review the arithmetic of algebraic stacks over a perfect field k, with a focus on the case when $k = \mathbb{F}_q$ is a finite field. Afterward, we introduce motivic invariants of moduli stacks via the Grothendieck ring $K_0(\mathrm{Stck}_k)$ of k-stacks. We compute the motives of $\mathcal{M}_{1,1}$ and its inertia stack $\mathcal{I}\mathcal{M}_{1,1}$.

Due to the presence of automorphisms, point counts of an algebraic stack $\mathcal{X}$ over finite fields are weighted.

Definition 2.1. The weighted point count of an algebraic stack $\mathcal{X}$ with finite inertia over $\mathbb{F}_q$ is defined as a sum

$$\#_q(\mathcal{X}) := \sum_{x \in \mathcal{X}(\mathbb{F}_q)/\sim} \frac{1}{|\mathrm{Aut}(x)|},$$

where $\mathcal{X}(\mathbb{F}_q)/\sim$ is the set of $\mathbb{F}_q$–isomorphism classes of $\mathbb{F}_q$–points of $\mathcal{X}$.

The main advantage of the weighted point count is that it is *algebro-topological* as it depends only on the cohomology of $\mathcal{X}$ and is equal to the usual point count of the coarse moduli space via the Grothendieck-Lefschetz trace formula for algebraic stacks proven by classical works of [Beh93, Sun12].

It is important to note that the above sum runs over the set $\mathcal{X}(\mathbb{F}_q)/\sim$ of isomorphism classes over $\mathbb{F}_q$ thus the weighted point count $\#_q(\mathcal{X})$ is *not equal* to the number $|\mathcal{X}(\mathbb{F}_q)/\sim|$ of $\mathbb{F}_q$–isomorphism classes when there is a non-trivial automorphism $|\mathrm{Aut}(x)| \neq 1$ for some stacky point $x \in \mathcal{X}(\mathbb{F}_q)/\sim$. Because of this, for enumeration purposes, it is important to consider the unweighted count of isomorphism classes. The following result of [HP23] shows that the unweighted point count is also natural and depends on the arithmetic of the inertia stack of $\mathcal{X}$.

Theorem 2.2 (Theorem 1.1. of [HP23]). *Let $\mathcal{X}$ be an algebraic stack over $\mathbb{F}_q$ of finite type with quasi-separated finite type diagonal and let $\mathcal{I}(\mathcal{X})$ be the inertia stack of $\mathcal{X}$. Then,*

$$|\mathcal{X}(\mathbb{F}_q)/\sim| = \#_q(\mathcal{I}(\mathcal{X}))$$

In this paper, we study enumerations via motivic classes in the Grothendieck ring $K_0(\mathrm{Stck}_k)$ of k-stacks. We review some properties of the Grothendieck ring of stacks introduced in [Eke25].

Definition 2.3. [Eke25, §1] The *Grothendieck ring of stacks* $K_0(\mathrm{Stck}_k)$ is the abelian group generated by classes $\{\mathcal{X}\}_k$ for each algebraic stack $\mathcal{X}$ of finite type over k with affine inertia modulo the relations

- $\{\mathcal{X}\}_k = \{\mathcal{Z}\}_k + \{\mathcal{X} \setminus \mathcal{Z}\}_k$ for $\mathcal{Z} \subset \mathcal{X}$ a closed substack,
- $\{\mathcal{E}\}_k = \{\mathcal{X} \times_k \mathbb{A}^n\}_k$ for $\mathcal{E}$ a vector bundle of rank n on $\mathcal{X}$.

Multiplication on $K_0(\mathrm{Stck}_k)$ is induced by $\{\mathcal{X}\}_k\{\mathcal{Y}\}_k := \{\mathcal{X} \times_k \mathcal{Y}\}$. There is a distinguished element $\mathbb{L} := \{\mathbb{A}^1\}_k \in K_0(\mathrm{Stck}_k)$, called the *Lefschetz motive*. We drop the subscript if k is clear.

We denote by $K_0'(\mathrm{Stck}_k)$ the ring obtained by imposing only the cut-and-paste relation but not the vector bundle relation and denote the class of a stack in this ring by $\{\mathcal{X}\}'$. The Grothendieck ring is universal among all additive and multiplicative invariants. For instance, when $k = \mathbb{F}_q$, the point counting measure $\{\mathcal{X}\} \mapsto \#_q(\mathcal{X})$ is a well-defined ring homomorphism $\#_q : K_0(\mathrm{Stck}_{\mathbb{F}_q}) \to \mathbb{Q}$ giving the weighted point count $\#_q(\mathcal{X})$ of $\mathcal{X}$ over $\mathbb{F}_q$. When $\{\mathcal{X}\}$ is mixed Tate i.e. a polynomial in the Lefschetz motive $\mathbb{L} := \{\mathbb{A}^1_k\}$, the weighted point count is a polynomial in q.

Recall that an algebraic group G is *special* in the sense of [Ser58] and [Gro58], if every G-torsor is Zariski-locally trivial; for example $\mathbb{G}_a$, GL_d, SL_d are special and PGL_2, PGL_3 are non-special. If $\mathcal{X} \to \mathcal{Y}$ is a G-torsor and G is special, then we have $\{\mathcal{X}\} = \{G\} \cdot \{\mathcal{Y}\}$ ([Eke25, Prop. 1.1 iii)]).

Finally, we can use the following result to access unweighted point counts.

Proposition 2.4. [dFLNU07, Prop. 5.3] *The association $\mathcal{X} \mapsto \mathcal{I}\mathcal{X}$ extends to a unique ring homomorphism*

$$\mathcal{I} : K_0'(\mathrm{Stck}_k) \to K_0'(\mathrm{Stck}_k)$$

which we call the inertia operator.

Note that $\mathcal{I}$ does not descend to a well defined operator on $K_0(\mathrm{Stck}_k)$ as in [Eke25, Prop. 1.1 iii)]. In order to keep track of the primitive roots of unity contained in $\mathbb{F}_q$, we define the following auxiliary function.

$$\delta(x) := \begin{cases} 1 & \text{if } x \text{ divides } q-1, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5. *Let k be a perfect field with $\mathrm{char}(k) \nmid a, b$. The motivic classes of the weighted projective stack $\mathcal{P}(a,b)$ and its inertia stack $\mathcal{I}\mathcal{P}(a,b)$ in $K_0(\mathrm{Stck}_k)$ are equal to*

$$\{\mathcal{P}(a,b)\} = \mathbb{L} + 1$$

$$\{\mathcal{IP}(a,b)\} = \gcd(a,b)\cdot(\mathbb{L}+1)+\delta(a)\cdot(a-\gcd(a,b))+\delta(b)\cdot(b-\gcd(a,b))$$

Proof. As $\mathcal{P}(a,b) := \left[(\mathbb{A}^2 \smallsetminus \{0\})/\mathbb{G}_m\right]$ we have $\{\mathcal{P}(a,b)\} = \frac{\mathbb{L}^2-1}{\mathbb{L}-1} = \mathbb{L}+1$ which holds as $\mathbb{G}_m = \mathrm{GL}_1$ is a special group. For the inertia stack $\mathcal{IP}(a,b)$, note that

$$\mathcal{IP}(a,b) = \sqcup_{\gcd(a,b)}\mathcal{P}(a,b) \sqcup_{a-\gcd(a,b)} \mathcal{P}(a) \sqcup_{b-\gcd(a,b)} \mathcal{P}(b)$$

by [HP23, Proposition 3.5] which translates to

$$\{\mathcal{IP}(a,b)\} = \gcd(a,b)\cdot\{\mathcal{P}(a,b)\}+\delta(a)\cdot(a-\gcd(a,b))\cdot\{\mathcal{P}(a)\}+\delta(b)\cdot(b-\gcd(a,b))\cdot\{\mathcal{P}(b)\}$$

As $\mathcal{P}(r) := \left[(\mathbb{A}^1 \smallsetminus \{0\})/\mathbb{G}_m\right]$ hence $\{\mathcal{P}(r)\} = \frac{\mathbb{L}-1}{\mathbb{L}-1} = 1$ we get the desired formula. ■

Note that computing the exact weighted point count $\#_q(\mathcal{I}(\mathcal{X}))$ of the inertia stack via algebro-topological method is useful as we are able to deduce the exact unweighted point count of the underlying stack $|\mathcal{X}(\mathbb{F}_q)/\sim|$.

Let us illustrate this important discrepancy with an example. One can ask how many isomorphism classes of elliptic curves are there over $\mathbb{F}_q$.

Proposition 2.6. *Let k be a perfect field. The motivic class of the fine modular curve $\mathcal{M}_{1,1}$ of smooth elliptic curves in $K_0(\mathrm{Stck}_k)$ is equal to*

$$\left\{\mathcal{M}_{1,1}\right\} = \mathbb{L}$$

The motivic class of the inertia stack $\mathcal{IM}_{1,1}$ for $\mathrm{char}(k) \neq 2,3$ is equal to

$$\left\{\mathcal{IM}_{1,1}\right\} = 2\mathbb{L}+\delta(6)\cdot 4+\delta(4)\cdot 2$$

which translates to the following weighted point counts for $\mathrm{char}(\mathbb{F}_q) \neq 2,3$

$$\begin{aligned}\#_q(\mathcal{IM}_{1,1}) &= 2q+6, \text{ if } q \equiv 1 \mod 12,\\ &= 2q+2, \text{ if } q \equiv 5 \mod 12,\\ &= 2q+4, \text{ if } q \equiv 7 \mod 12,\\ &= 2q, \text{ if } q \equiv 11 \mod 12.\end{aligned}$$

The weighted point count of the inertia stack $\mathcal{IM}_{1,1}$ for $\mathrm{char}(\mathbb{F}_q) = 2,3$ is equal to

$$\begin{aligned}\#_q(\mathcal{IM}_{1,1}) &= 2q+1, \text{ if } q = 2^r \text{ with } r \text{ odd},\\ &= 2q+5, \text{ if } q = 2^r \text{ with } r \text{ even},\\ &= 2q+2, \text{ if } q = 3^r \text{ with } r \text{ odd},\\ &= 2q+4, \text{ if } q = 3^r \text{ with } r \text{ even}.\end{aligned}$$

Proof. The weighted point count is identical over any field which follows from the coarse moduli space $\overline{M}_{1,1} \cong \mathbb{P}^1$ over $\operatorname{Spec}\mathbb{Z}$ thus we have $\{\mathcal{M}_{1,1}\} = \{\overline{M}_{1,1} - \{j=\infty\}\} = (\mathbb{L}+1)-1 = \mathbb{L}$. The classes $\{\mathcal{I}(\mathcal{M}_{1,1} - \{j=0\} - \{j=1728\})\} = 2(\mathbb{L}-2)$ of the inertia stack for $\operatorname{char}(k) \neq 3,2$ follows as the automorphism group of a geometric point of $\mathcal{M}_{1,1}$ is of order 2 away from $j=0$ and $j=1728$ (c.f. [Sil09, Ap. A, Prop. 1.2]). It remains to find out how many $\mathbb{F}_q$–points $j=0,1728$ have respectively depending on the primitive roots of unity contained in $\mathbb{F}_q$. This is $\delta(6)\cdot(6-2)$ for $j=0$ and $\delta(4)\cdot(4-2)$ for $j=1728$ where -2 comes from taking into account the hyperelliptic involution i.e. $\delta(2)\cdot 2$ is always present.

For the classes of the inertia stack for $\operatorname{char}(\mathbb{F}_q)=3,2$, the classes $\{\mathcal{I}(\mathcal{M}_{1,1} - \{j=0\}\} = 2(\mathbb{L}-1)$ of the inertia stack follows as the automorphism group of a geometric point of $\mathcal{M}_{1,1}$ is of order 2 away from $j=0$ (c.f. [Sil09, Ap. A, Prop. 1.2]). For $q=3^r$, It remains to find out how many $\mathbb{F}_q$–points $j=0$ has which is the number of $\mathbb{F}_q$–isomorphism classes of supersingular elliptic curves. For r odd we have 4 isomorphism classes and for r even we have 6 isomorphism classes from [CJ04, Thm. 3.5]. Thus we have $\{\mathcal{I}\mathcal{M}_{1,1}\} = 2(\mathbb{L}-1)+4$ for r odd and $\{\mathcal{I}\mathcal{M}_{1,1}\} = 2(\mathbb{L}-1)+6$ for r even. Similarly for the number of $\mathbb{F}_q$–isomorphism classes of supersingular elliptic curves for $q=2^r$, for r odd we have 3 isomorphism class and for r even we have 7 isomorphism classes from [Men93, Thm. 3.6 & 3.7]. Thus we have $\{\mathcal{I}\mathcal{M}_{1,1}\} = 2(\mathbb{L}-1)+1$ for r odd and $\{\mathcal{I}\mathcal{M}_{1,1}\} = 2(\mathbb{L}-1)+5$ for r even.

■

Let us briefly recall the formulation of height moduli spaces on weighted projective stack $\mathcal{P}(\vec{\lambda}) := \mathcal{P}(\lambda_0,\dots,\lambda_N)$ and ample line bundle $\mathcal{L} = \mathcal{O}(1)$. By [BPS22, Theorem 4.28] the height moduli space $\mathcal{M}_{n,C}(\mathcal{P}(\vec{\lambda}),\mathcal{O}(1))$ was constructed as a moduli space of $\vec{\lambda}$-weighted linear series $(L,s_0,\dots,s_N)$ on the curve C.

Definition 2.7. A *$\vec{\lambda}$-weighted linear series* on C is a tuple $(L,s_0,\dots,s_N)$ where $s_i \in H^0(C, L^{\otimes \lambda_i})$. The tuple is *minimal* if for all $x \in C(k^{sep})$, there exists j such that $\nu_x(s_j) < \lambda_j$ where ν_x is the order of vanishing at x.

Particularly, $\overline{\mathcal{M}}_{1,1} \cong \mathcal{P}(4,6)$ over $\mathbb{Z}\left[\frac{1}{6}\right]$ with $\mathcal{L}$ the Hodge line bundle by the short Weierstrass equation $y^2 = x^3 + a_4 x + a_6$, where $\zeta \cdot a_i = \zeta^i a_i$ for $\zeta \in \mathbb{G}_m$ and $i = 4,6$. The minimal weighted linear series on the smooth projective curve C are Weierstrass data which are rational points on $\overline{\mathcal{M}}_{1,1}$ over $K = k(C)$. And in [BPS22, §7], the moduli stacks of elliptic surfaces over k with $\operatorname{char}(k) \neq 2,3$ of stacky height n with fixed singular fibers are identified with the height moduli spaces. As $\operatorname{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2$, it is also natural to consider the PGL_2 stack quotient $\left[\mathcal{W}_n^{\min}/\mathrm{PGL}_2\right]$ as was done in [PS25, Main Theorem 1.2]. Note that the 12th root of the minimal discriminant of an elliptic curve is known to be the Faltings height (c.f. [Lan24, Remark 1.2]). In the cited paper, Landesman showed that the Faltings height on $\overline{\mathcal{M}}_{1,1}$ in characteristic 3 is not a height function in the sense of [ESZB23]. The presentations of $\mathcal{M}_{1,1}$ as a quotient stack in characteristic 2 and 3 are given in [LP24, §7].

We now consider the motives $\left\{\mathcal{W}^{\min}_{n,\mathbb{P}^1_k}\right\}$ of the height moduli space $\mathcal{M}_{n,\mathbb{P}^1_k}(\mathcal{P}(a,b),\mathcal{O}(1)) = \mathcal{W}^{\min}_n$ in the Grothendieck ring $K_0(\mathrm{Stck}_k)$. In [BPS22, §8 and §9], the exact motivic classes in the Grothendieck ring $K_0(\mathrm{Stck}_k)$ of stacks of height moduli spaces (together with their inertia stacks) of $k(t)$-points on weighted projective stacks $\mathcal{P}(\vec{\lambda}) := \mathcal{P}(\lambda_0,\dots,\lambda_N)$ with $\mathcal{L} = \mathcal{O}(1)$ were determined as follows.

Theorem 2.8 (Theorems 9.1 and 9.2 of [BPS22])**.** *The classes $\left\{\mathcal{W}^{\min}_n\right\}$ are given by the following formulas:*

$$\begin{aligned}
\left\{\mathcal{W}^{\min}_0\right\} &= \{\mathbb{P}^N\}\\
\left\{\mathcal{W}^{\min}_1\right\} &= \{\mathbb{P}^N\}(\mathbb{L}^{|\vec{\lambda}|} - \mathbb{L}) + \mathbb{L}^{N+1}\{\mathbb{P}^{|\vec{\lambda}|-N-2}\}\\
\left\{\mathcal{W}^{\min}_{n\geq 2}\right\} &= \mathbb{L}^{(n-2)|\vec{\lambda}|+N+2}(\mathbb{L}^{|\vec{\lambda}|-1} - 1)\{\mathbb{P}^{|\vec{\lambda}|-1}\}
\end{aligned}$$

This is proven via extracting the coefficients of rational motivic height zeta functions $Z_{\vec{\lambda}}(t)$ of height moduli spaces with its variant on the inertia stacks $\mathcal{I}Z_{\vec{\lambda}}(t)$ in [BPS22, Theorem 8.9]. The rationality follows from establishing the stratification by minimality defect in [BPS22, Corollary 6.2] where we stratify the complement of $\mathcal{W}^{min}_n$ inside $\mathcal{W}_n \smallsetminus \mathcal{W}^{\infty}_n$ into strata corresponding to minimal weighted linear series of smaller height.

For later counting purposes, it is important to work out the $\left\{\mathcal{W}^{\min}_{n,\mathbb{P}^1_k}\right\}$ for 0-dimensional weighted projective stack $\mathcal{P}(a)$.

Corollary 2.9. *The classes $\left\{\mathcal{W}^{\min}_n\right\}$ for $\mathcal{P}(a)$ are given by the following formulas:*

$$\begin{aligned}
\left\{\mathcal{W}^{\min}_0\right\} &= 1\\
\left\{\mathcal{W}^{\min}_1\right\} &= \mathbb{L}^a + \mathbb{L}^{a-1} + \ldots + \mathbb{L}^2\\
\left\{\mathcal{W}^{\min}_{n\geq 2}\right\} &= \mathbb{L}^{(n-2)a+2}(\mathbb{L}^{a-1} - 1)\{\mathbb{P}^{a-1}\}
\end{aligned}$$

We also need slightly modified motive formula for $\left\{\mathcal{W}^{\min}_{n,\mathbb{P}^1_k}\right\}$ for $\mathcal{P}(\check{b})$ which behaves as a virtual (-1)-dimensional weighted projective stack. We will define this motive formally to facilitate subsequent operations involving formal addition and subtraction in the Grothendieck ring of stacks.

Definition 2.10. The classes $\left\{\mathcal{W}^{\min}_n\right\}$ for $\mathcal{P}(\check{b})$ are given by the following formulas:

$$\begin{aligned}
\left\{\mathcal{W}^{\min}_1\right\} &= \mathbb{L}^{b-1} + \mathbb{L}^{b-2} + \ldots + \mathbb{L} + 1\\
\left\{\mathcal{W}^{\min}_{n\geq 2}\right\} &= \mathbb{L}^{(n-2)b+1}(\mathbb{L}^{b-1} - 1)\{\mathbb{P}^{b-1}\}
\end{aligned}$$

3. Counting isomorphism classes of $E/\mathbb{F}_q(t)$

In counting the exact unweighted number of isomorphism classes of $E/\mathbb{F}_q(t)$, it is vital to understand the families of elliptic curves that are non-constant (i.e. height $n \geq 1$) and isotrivial (i.e. a fixed j-invariant) with extra automorphisms (i.e. more twists than generic quadratic twists) which leads to various lower order main terms that are hidden from the exact weighted number of isomorphism classes of $E/\mathbb{F}_q(t)$ (c.f. [BPS22, Rmk 9.8]).

In this regard, to pass from the weighted count of rational points to the unweighted count, it is necessary to account for the number of twists. When $\mathrm{char}(\mathbb{F}_q) = 3, 2$, it is important to note that the number of twists does not coincide with the number of automorphisms—unlike the case $\mathrm{char}(\mathbb{F}_q) \neq 3, 2$, where the two quantities are equal.

Proposition 3.1 (Proposition 2.2 of [KST17]). *Let $q = 3^r$ and suppose $E/\mathbb{F}_q$ is an elliptic curve. Then*

$$\#\,\mathrm{Twist}(E/\mathbb{F}_q) = \begin{cases} 2 & \text{if } j(E) \neq 0, \\ 4 & \text{if } j(E) = 0 \text{ and } r \text{ is odd}, \\ 6 & \text{if } j(E) = 0 \text{ and } r \text{ is even}. \end{cases}$$

Proposition 3.2 (Proposition 3.2 of [KST17]). *Let $q = 2^r$ and suppose $E/\mathbb{F}_q$ is an elliptic curve. Then*

$$\#\,\mathrm{Twist}(E/\mathbb{F}_q) = \begin{cases} 2 & \text{if } j(E) \neq 0, \\ 3 & \text{if } j(E) = 0 \text{ and } r \text{ is odd}, \\ 7 & \text{if } j(E) = 0 \text{ and } r \text{ is even}. \end{cases}$$

A given generalized Weierstrass form $F = x^3 + a_2x^2 + a_4x + a_6 - y^2 - a_1xy - a_3y$, where $a_i \in H^0(\mathcal{O}_{\mathbb{P}^1}(in))$ defines an $E/\mathbb{F}_q(t)$ and if F defines an $E/\mathbb{F}_q(t)$ with a fixed j-invariant then as explained in [Sil09, Ap. A, Prop. 1.1] there exists a coordinate change of the form $x \mapsto x + b_1$ and $y \mapsto y + b_1x + b_3$ with $b_i \in H^0(\mathcal{O}_{\mathbb{P}^1}(in))$ such that for $\mathrm{char}(\mathbb{F}_q) = 3$ and $j = 0$ has the following normal Weierstrass form

$$y^2 = x^3 + a_4x + a_6$$

As explained in [dJ02, §4.13 (b2)], there exists a unique choice of the pair (b_1, b_3) so that we get

$$y^2 = x^3 + a'_4x + a'_6$$

where a'_4 is a nonzero section of $\mathcal{O}(4n)$ and a'_6 is any section of $\mathcal{O}(6n)$. As in the paragraph before [dJ02, Prop. 4.14], we have $(q^{4n+1} - 1)q^{6n+1}q^{4n+2}$ as the total number of Weierstrass polynomials which are non-constant and isotrivial with $j = 0$ where $(q^{4n+1} - 1)$ corresponds to counting a nonzero section of $\mathcal{O}(4n)$ and q^{6n+1} corresponds to counting any section of $\mathcal{O}(6n)$ and q^{4n+2} corresponds to counting any sections of $\mathcal{O}(n)$ and $\mathcal{O}(3n)$ due to (b_1, b_3) worth of freedom.

After dividing out the coordinate change factor of $q^{6n+3}(q - 1)$ consisting of (b_1, b_2, b_3) worth of freedom and a nonzero scalar $\lambda \in k^*$, we have

$$(q^{4n} + \ldots + 1) \cdot q^{4n} = (q^{8n} + \ldots + q^{4n}) = (q^{8n} + \ldots + 1) - (q^{4n-1} + \ldots + 1)$$

This expression is the total number M_n of minimal Weierstrass forms that are isotrivial with $j = 0$ upto height n. Now, let J_n denote the weighted number of isomorphism classes of elliptic curves $E/\mathbb{F}_q(t)$ of height n whose j-invariant is constant. Using the recursive relation as in [dJ02, (4.13.1)]

$$M_n = J_n + (q + 1)J_{n-1} + \ldots + (q^n + \ldots + q + 1)J_0,$$

we see that counting the weighted number of rational points of height $n \geq 1$ on the classifying stacks $\mathcal{B}\underline{Q_{12}}$ (the stacky point of $\mathcal{M}_{1,1}$ at $j=0$) has the same cardinality as counting the weighted number of rational points on $\mathcal{P}(8) - \mathcal{P}(\check{4})$.

Proposition 3.3. *The classes $\left\{\mathcal{W}_n^{\min}\right\}$ for the classifying stack $\mathcal{B}\underline{Q_{12}}$ of the constant group scheme associated to the dicyclic group of order 12 are given by the following formulas:*

$$\begin{aligned}
&\left\{\mathcal{W}_0^{\min}\right\} = 1 \\
&\left\{\mathcal{W}_1^{\min}\right\} = (\mathbb{L}^8 + \ldots + \mathbb{L}^2) - (\mathbb{L}^3 + \mathbb{L}^2 + \mathbb{L} + 1) \\
&\left\{\mathcal{W}_{n\geq 2}^{\min}\right\} = \mathbb{L}^{(n-2)8+2}(\mathbb{L}^7 - 1)\{\mathbb{P}^7\} - \mathbb{L}^{(n-2)4+1}(\mathbb{L}^3 - 1)\{\mathbb{P}^3\}
\end{aligned}$$

Proof. At height $n=0$ we only have a constant map to $j=0$ which means $\mathcal{W}_0^{\min}$ has the motive of $\left\{\mathbb{P}^0\right\} = 1$. For height $n=1$, we have the motive of $\left\{\mathcal{W}_1^{\min}(\mathcal{P}(8), \mathcal{O}(1))\right\} = \mathbb{L}^8 + \mathbb{L}^7 + \mathbb{L}^6 + \mathbb{L}^5 + \mathbb{L}^4 + \mathbb{L}^3 + \mathbb{L}^2$ from this we need to subtract $\left\{\mathcal{W}_1^{\min}(\mathcal{P}(\check{4}), \mathcal{O}(1))\right\} = \mathbb{L}^3 + \mathbb{L}^2 + \mathbb{L} + 1$. For height $n \geq 2$, we have the motive of $\left\{\mathcal{W}_{n\geq 2}^{\min}(\mathcal{P}(8), \mathcal{O}(1))\right\} = \mathbb{L}^{8n-14}(\mathbb{L}^7 - 1)\{\mathbb{P}^7\}$ from this we subtract $\left\{\mathcal{W}_{n\geq 2}^{\min}(\mathcal{P}(\check{4}), \mathcal{O}(1))\right\} = \mathbb{L}^{4n-7}(\mathbb{L}^3 - 1)\{\mathbb{P}^3\}$. ■

As for $\mathrm{char}(\mathbb{F}_q) = 2$ and $j \neq 0$ has the following normal Weierstrass form

$$y^2 + a_1 xy = x^3 + a_2 x^2 + a_6$$

As explained in [dJ02, §4.13 (c1)], there exists a unique choice of the pair (b_2, b_3) so that we get

$$y^2 + a_1' xy = x^3 + a_2' x^2 + (a_1')^6$$

where a_1' is a nonzero section of $\mathcal{O}(n)$ and a_2' is any section of $\mathcal{O}(2n)$. As in the paragraph before [dJ02, Prop. 4.14], we have $(q-1)(q^{n+1}-1)q^{2n+1}q^{5n+2}$ as the total number of Weierstrass polynomials which are non-constant and isotrivial with a certain fixed $j \neq 0$ where $(q-1)$ corresponds to $\mathcal{M}_{1,1} - \{j=0\}$ and $(q^{n+1}-1)$ corresponds to counting a nonzero section of $\mathcal{O}(n)$ and q^{2n+1} corresponds to counting any section of $\mathcal{O}(2n)$ and q^{5n+2} corresponds to counting any sections of $\mathcal{O}(2n)$ and $\mathcal{O}(3n)$ due to (b_2, b_3) worth of freedom.

After dividing out the coordinate change factor of $q^{6n+3}(q-1)$ consisting of (b_1, b_2, b_3) worth of freedom and a nonzero scalar $\lambda \in k^*$, we have

$$(q^n + \ldots + 1) \cdot q^n = (q^{2n} + \ldots + q^n) = (q^{2n} + \ldots + 1) - (q^{n-1} + \ldots + 1)$$

This expression is the total number M_n of minimal Weierstrass forms that are isotrivial with $j \neq 0$ upto height n. Now, let J_n denote the weighted number of isomorphism classes of elliptic curves $E/\mathbb{F}_q(t)$ of height n whose j-invariant is constant. Using the recursive relation as in [dJ02, (4.13.1)]

$$M_n = J_n + (q+1)J_{n-1} + \ldots + (q^n + \ldots + q + 1)J_0,$$

we see that counting the weighted number of rational points of height $n \geq 1$ on the classifying stack $\mathcal{B}\underline{\mathbb{Z}/2}$ (the generic stacky point of $\mathcal{M}_{1,1}$ at $j \neq 0$) has the same cardinality as counting the weighted number of rational points on $\mathcal{P}(2) - \mathcal{P}(\check{1})$.

Proposition 3.4. *The classes $\{\mathcal{W}_n^{\min}\}$ for the classifying stack $\mathcal{B}\underline{\mathbb{Z}/2}$ of the constant group scheme of order 2 are given by the following formulas:*

$$\begin{aligned}\{\mathcal{W}_0^{\min}\} &= 1\\ \{\mathcal{W}_1^{\min}\} &= \mathbb{L}^2 - 1\\ \{\mathcal{W}_{n\geq 2}^{\min}\} &= \mathbb{L}^{(n-2)2+2}(\mathbb{L}-1)\{\mathbb{P}^1\}\end{aligned}$$

Proof. At height $n=0$ we only have a constant map to $j=0$ which means $\mathcal{W}_0^{\min}$ has the motive of $\{\mathbb{P}^0\} = 1$. For height $n=1$, we have the motive of $\{\mathcal{W}_1^{\min}(\mathcal{P}(2), \mathcal{O}(1))\} = \mathbb{L}^2$ from this we need to subtract $\{\mathcal{W}_1^{\min}(\mathcal{P}(\check{1}), \mathcal{O}(1))\} = 1$. For height $n \geq 2$, we have the motive of $\{\mathcal{W}_{n\geq 2}^{\min}(\mathcal{P}(2), \mathcal{O}(1))\} = \mathbb{L}^{2n-2}(\mathbb{L}-1)\{\mathbb{P}^1\}$ from this we subtract nothing as $\{\mathcal{W}_{n\geq 2}^{\min}(\mathcal{P}(\check{1}), \mathcal{O}(1))\} = 0$.

■

Lastly, as for $\mathrm{char}(\mathbb{F}_q) = 2$ and $j=0$ has the following normal Weierstrass form

$$y^2 + a_3 y = x^3 + a_4 x + a_6$$

As explained in [dJ02, §4.13 (c2)], there exists a unique choice of b_2 so that we get

$$y^2 + a_3' y = x^3 + a_4' x + a_6'$$

where a_3' is a nonzero section of $\mathcal{O}(3n)$ and a_4' (resp. a_6') is any section of $\mathcal{O}(4n)$ (resp. $\mathcal{O}(6n)$). As in the paragraph before [dJ02, Prop. 4.14], we have $(q^{3n+1} - 1)q^{4n+1}q^{6n+1}q^{2n+1}$ as the total number of Weierstrass polynomials which are nonconstant and isotrivial with $j=0$ where $(q^{3n+1}-1)$ corresponds to counting a nonzero section of $\mathcal{O}(3n)$ and q^{4n+1} (resp. q^{6n+1}) corresponds to counting any section of $\mathcal{O}(4n)$ (resp. $\mathcal{O}(6n)$) and q^{2n+1} corresponds to counting any sections of $\mathcal{O}(2n)$ due to b_2 worth of freedom.

After dividing out the coordinate change factor of $q^{6n+3}(q-1)$ consisting of (b_1, b_2, b_3) worth of freedom and a nonzero scalar $\lambda \in k^*$, we have

$$(q^{3n} + \ldots + 1) \cdot q^{6n} = (q^{9n} + \ldots + q^{6n}) = (q^{9n} + \ldots + 1) - (q^{6n-1} + \ldots + 1)$$

This expression is the total number M_n of minimal Weierstrass forms that are isotrivial with $j=0$ upto height n. Now, let J_n denote the weighted number of isomorphism classes of elliptic curves $E/\mathbb{F}_q(t)$ of height n whose j-invariant is constant. Using the recursive relation as in [dJ02, (4.13.1)]

$$M_n = J_n + (q+1)J_{n-1} + \ldots + (q^n + \ldots + q + 1)J_0,$$

we see that counting the weighted number of rational points of height $n \geq 1$ on the classifying stacks $\mathcal{B}\underline{T_{24}}$ (the stacky point of $\mathcal{M}_{1,1}$ at $j=0$) has the same cardinality as counting the weighted number of rational points on $\mathcal{P}(9) - \mathcal{P}(\check{6})$.

Proposition 3.5. *The classes $\{\mathcal{W}_n^{\min}\}$ for the classifying stack $\mathcal{B}\underline{T_{24}}$ of the constant group scheme associated to the binary tetrahedral group of order 24 are given by the* *following formulas:*

$$\begin{aligned}
\left\{\mathcal{W}_0^{\min}\right\} &= 1\\
\left\{\mathcal{W}_1^{\min}\right\} &= (\mathbb{L}^9+\ldots+\mathbb{L}^2)-(\mathbb{L}^5+\ldots+1)\\
\left\{\mathcal{W}_{n\geq 2}^{\min}\right\} &= \mathbb{L}^{(n-2)9+2}(\mathbb{L}^8-1)\{\mathbb{P}^8\}-\mathbb{L}^{(n-2)6+1}(\mathbb{L}^5-1)\{\mathbb{P}^5\}
\end{aligned}$$

Proof. At height $n=0$ we only have a constant map to $j=0$ which means $\mathcal{W}_0^{\min}$ has the motive of $\left\{\mathbb{P}^0\right\}=1$. For height $n=1$, we have the motive of $\left\{\mathcal{W}_1^{\min}(\mathcal{P}(9),\mathcal{O}(1))\right\}=\mathbb{L}^9+\mathbb{L}^8+\mathbb{L}^7+\mathbb{L}^6+\mathbb{L}^5+\mathbb{L}^4+\mathbb{L}^3+\mathbb{L}^2$ from this we need to subtract $\left\{\mathcal{W}_1^{\min}(\mathcal{P}(\check{6}),\mathcal{O}(1))\right\}=\mathbb{L}^5+\mathbb{L}^4+\mathbb{L}^3+\mathbb{L}^2+\mathbb{L}+1$. For height $n\geq 2$, we have the motive of $\left\{\mathcal{W}_{n\geq 2}^{\min}(\mathcal{P}(9),\mathcal{O}(1))\right\}=\mathbb{L}^{9n-16}(\mathbb{L}^8-1)\{\mathbb{P}^8\}$ from this we subtract $\left\{\mathcal{W}_{n\geq 2}^{\min}(\mathcal{P}(\check{6}),\mathcal{O}(1))\right\}=\mathbb{L}^{6n-11}(\mathbb{L}^5-1)\{\mathbb{P}^5\}$. ■

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. We prove for $q=3^r$ case first. We recall that the weighted count $\mathcal{N}^w(\mathbb{F}_q(t),\ B)$ in [BPS22, Theorem 9.7] is the same for all positive characteristic as shown in [dJ02, Proposition 4.12]. As for the unweighted count, we multiply 2 to the weighted count $\mathcal{N}^w(\mathbb{F}_q(t),\ B)$ since every elliptic curves have the quadratic twists as in Proposition 3.1.

For $j=0$, we take the weighted count in Proposition 3.3 and multiply appropriate factors which are $(4-2)=2$ for r odd case and $(6-2)=4$ for r even case depending on the parity of prime power of 3 as there are 3 non-trivial twists (resp. 5 non-trivial twists) of supersingular elliptic curves for r odd case (resp. r even case) (c.f. Proposition 3.1) and -2 comes from taking into account the quadratic twist which exists for both even and odd powers [KST17, §2].

Thus for the unweighted count, we would like to compute the following.

When r is odd:

$$\mathcal{N}(\mathbb{F}_q(t),B)=2\cdot\mathcal{N}^w(\mathbb{F}_q(t),\ B)+2\cdot(\#_q\mathcal{W}_{n,\mathbb{P}^1}(8)-\#_q\mathcal{W}_{n,\mathbb{P}^1}(\check{4}))$$

When r is even:

$$\mathcal{N}(\mathbb{F}_q(t),B)=2\cdot\mathcal{N}^w(\mathbb{F}_q(t),\ B)+4\cdot(\#_q\mathcal{W}_{n,\mathbb{P}^1}(8)-\#_q\mathcal{W}_{n,\mathbb{P}^1}(\check{4}))$$

We first compute the following by summing over $n\geq 2$

$$\sum_{n=2}^{\left\lfloor\frac{log_q B}{12}\right\rfloor}\#_q\mathcal{W}_{n,\mathbb{P}^1}(8)=\frac{q^7-1}{q^7-q^6}\cdot(B^{2/3}-q^8)$$

$$\sum_{n=2}^{\left\lfloor\frac{log_q B}{12}\right\rfloor}\#_q\mathcal{W}_{n,\mathbb{P}^1}(\check{4})=-\left(\frac{q^3-1}{q\cdot(q^3-q^2)}\right)\cdot(B^{1/3}-q^4)$$

which leads to the following as we sum over all $n\geq 0$:

$$\begin{aligned}\#_q\mathcal{W}_{n,\mathbb{P}^1}(8)-\#_q\mathcal{W}_{n,\mathbb{P}^1}(\check{4}) &= \left(\frac{q^7-1}{q^7-q^6}\right)\cdot(B^{2/3}-q^8)-\left(\frac{q^3-1}{q^4-q^3}\right)\cdot(B^{1/3}-q^4)\\ &\quad+(q^8+\ldots+q^2)-(q^3+q^2+q+1)+1\\ &= \left(\frac{q^7-1}{q^7-q^6}\right)B^{2/3}-\left(\frac{q^3-1}{q^4-q^3}\right)B^{1/3}\end{aligned}$$

from which the result follows.

We now prove for $q=2^r$ case. We recall that the weighted count $\mathcal{N}^w(\mathbb{F}_q(t),\ B)$ in [BPS22, Theorem 9.7] is the same for all positive characteristic as shown in [dJ02, Proposition 4.12]. As for the unweighted count, when r is odd, we multiply 2 to the weighted count $\big(\mathcal{N}^w(\mathbb{F}_q(t),\ B)-(\mathcal{P}(9)-\mathcal{P}(\check{6}))\big)$ since any elliptic curve away from $j=0$ has the quadratic twists as in Proposition 3.2 but there is no quadratic twist at $j=0$ as in [KST17, §3]. For r even, we multiply 2 to the weighted count $\mathcal{N}^w(\mathbb{F}_q(t),\ B)$ since every elliptic curves have the quadratic twists as in [KST17, §3]. There needs to be an adjustment, however, as the weighted number of rational points on $\mathcal{P}(2)$ and $\mathcal{P}(2)-\mathcal{P}(\check{1})$ differ at height $n=1$. The difference of $\big\{\mathcal{W}_1^{\min}(\mathcal{P}(2))\big\}=\mathbb{L}^2$ and $\big\{\mathcal{W}_1^{\min}(\mathcal{P}(2)-\mathcal{P}(\check{1}))\big\}=\mathbb{L}^2-1$ is -1.

Thus taking account of all points for every $j\neq 0$, we need to subtract $(q-1)$ from the total number of weighted count corresponding to $\mathcal{M}_{1,1}-\{j=0\}$ for a certain fixed $j\neq 0$. And then we need to add 1 from the total number of weighted count as we subtract the rational points landing on $j=\infty$ since we do not want to count the generically singular $j=\infty$ isotrivial elliptic curves.

Lastly for $j=0$, we take the weighted count in Proposition 3.5 and multiply appropriate factors which are 3 for r odd case and $(7-2)=5$ for r even case depending on the parity of prime power of 2 as there are 2 non-trivial twists (resp. 6 non-trivial twists) of supersingular elliptic curves for r odd case (resp. r even case) (c.f. Proposition 3.2) and -2 comes from taking into account the quadratic twists. For r odd case, in the end, we have $(3-2)=1$ as the factor.

Thus for the unweighted count, we would like to compute the following.

When r is odd:

$$\mathcal{N}(\mathbb{F}_q(t),B)=2\cdot(\mathcal{N}^w(\mathbb{F}_q(t),\ B)-(q-1)+1)+(\#_q\mathcal{W}_{n,\mathbb{P}^1}(9)-\#_q\mathcal{W}_{n,\mathbb{P}^1}(\check{6}))$$

When r is even:

$$\mathcal{N}(\mathbb{F}_q(t),B)=2\cdot(\mathcal{N}^w(\mathbb{F}_q(t),\ B)-(q-1)+1)+5\cdot(\#_q\mathcal{W}_{n,\mathbb{P}^1}(9)-\#_q\mathcal{W}_{n,\mathbb{P}^1}(\check{6}))$$

We first compute the following by summing over $n \geq 2$

$$\sum_{n=2}^{\left\lfloor \frac{log_q B}{12} \right\rfloor} \#_q \mathcal{W}_{n,\mathbb{P}^1}(9) = \frac{q^8-1}{q^8-q^7} \cdot (B^{3/4} - q^9)$$

$$\sum_{n=2}^{\left\lfloor \frac{log_q B}{12} \right\rfloor} \#_q \mathcal{W}_{n,\mathbb{P}^1}(\check{6}) = -\left(\frac{q^5-1}{q \cdot (q^5-q^4)}\right) \cdot (B^{1/2} - q^6)$$

which leads to the following as we sum over all $n \geq 0$:

$$\begin{aligned}
\#_q \mathcal{W}_{n,\mathbb{P}^1}(9) - \#_q \mathcal{W}_{n,\mathbb{P}^1}(\check{6}) &= \frac{q^8-1}{q^8-q^7} \cdot (B^{3/4} - q^9) - \left(\frac{q^5-1}{q^6-q^5}\right) \cdot (B^{1/2} - q^6) \\
&\quad + (q^9 + \ldots + q^2) - (q^5 + q^4 + q^3 + q^2 + q + 1) + 1 \\
&= \left(\frac{q^8-1}{q^8-q^7}\right) B^{3/4} - \left(\frac{q^5-1}{q^6-q^5}\right) B^{1/2}
\end{aligned}$$

from which the result follows.

■

Acknowledgements

Warm thanks to Dori Bejleri, Andrea Di Lorenzo, Scott Mullane, Tristan Phillips and Matthew Satriano for numerous helpful discussions. The author was partially supported by the ARC grant DP210103397 and the Sydney Mathematical Research Institute.

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