RATIONALITY OF THE TRIVIAL LATTICE RANK WEIGHTED MOTIVIC HEIGHT ZETA FUNCTION FOR ELLIPTIC SURFACES

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ABSTRACT. Let k be a perfect field with $\mathrm{char}(k) \neq 2,3$, set $K = k(t)$, and let $\mathcal{W}_n^{\min}$ be the moduli stack of minimal elliptic curves over K of Faltings height n from the height–moduli framework of [BPS22] applied to $\overline{\mathcal{M}}_{1,1} \simeq \mathcal{P}(4,6)$. For $[E] \in \mathcal{W}_n^{\min}$, let $S \to \mathbb{P}_k^1$ be the associated elliptic surface with section. Motivated by the Shioda–Tate formula, we consider the trivariate motivic height zeta function

$$\mathcal{Z}(u,v;t) := \sum_{n \geq 0} \Big(\sum_{[E] \in \mathcal{W}_n^{\min}} u^{T(S)} \, v^{\mathrm{rk}(E/K)} \Big) t^n$$

which refines the height series by weighting each height stratum with the trivial lattice rank $T(S)$ and the Mordell–Weil rank $\mathrm{rk}(E/K)$. We prove rationality for the trivial lattice specialization $Z_{\mathrm{Triv}}(u;t) = \mathcal{Z}(u,1;t)$ by exhibiting a finite Euler product, up to a correction supported in finitely many heights for each trivial lattice rank, arising from the non-constant isotrivial loci $j \equiv 0$ and $j \equiv 1728$. This correction vanishes at $u = 1$, recovering the Euler product of [BPS22]. We conjecture irrationality for the Néron–Severi $Z_{\mathrm{NS}}(w;t) = \mathcal{Z}(w,w;t)$ and the Mordell–Weil $Z_{\mathrm{MW}}(v;t) = \mathcal{Z}(1,v;t)$ specializations.

1. INTRODUCTION

Let k be a perfect field with $\mathrm{char}(k) \neq 2,3$, and set $K := k(t)$. An elliptic curve E/K determines a relatively minimal elliptic surface with section

$$f : S \longrightarrow \mathbb{P}_k^1$$

unique up to isomorphism (see [Mir89, SS10] for background on elliptic surfaces).

The arithmetic of E/K is reflected in the geometry of S, and a basic organizing principle is the Shioda–Tate formula [Shi90]

$$\rho(S) \;=\; T(S) \;+\; \mathrm{rk}(E/K), \tag{1}$$

where $\rho(S) = \mathrm{rk}\,\mathrm{NS}(S_{\bar{k}})$ is the *geometric Picard rank*, $T(S)$ is the *rank of the geometric trivial lattice* generated by the zero section, a fiber class, and the components of reducible fibers not meeting the zero section, and $\mathrm{rk}(E/K)$ is the *Mordell–Weil rank*. For the relatively minimal elliptic surfaces $f : S \to \mathbb{P}_k^1$ with section considered in this paper, we have $q(S) = 0$ and $p_g(S) = n - 1$, hence the standard bounds

$$2 \leq \rho(S) \leq 10n, \qquad 2 \leq T(S) \leq 10n, \qquad 0 \leq \mathrm{rk}(E/K) \leq 10n - 2, \tag{2}$$

where $\rho(S) \leq 10n = h^{1,1}(S)$ is the Lefschetz bound over $k = \mathbb{C}$ (or in general Igusa's inequality $\rho(S) \leq b_2(S) = 12n - 2$).

In [BPS22], Bejleri–Park–Satriano construct height-moduli stacks of rational points on proper polarized cyclotomic stacks. In the fundamental modular curve example

$$\overline{\mathcal{M}}_{1,1} \simeq \mathcal{P}(4,6), \qquad \lambda \simeq \mathcal{O}_{\mathcal{P}(4,6)}(1),$$

a minimal elliptic curve over K can be viewed as a rational point of λ–height n on $\overline{\mathcal{M}}_{1,1}$ over K. This yields a separated Deligne–Mumford stack of finite type

$$\mathcal{W}_n^{\min} := \mathcal{W}_{n,\mathbb{P}_k^1}^{\min}\big(\mathcal{P}(4,6),\mathcal{O}(1)\big)$$

parametrizing minimal elliptic curves over K of discriminant degree $12n$. Here a K-rational point of $\overline{\mathcal{M}}_{1,1}$ of λ-height n means the stacky height n with respect to the Hodge line bundle λ, in the sense of [ESZB23]. Under the identification $\overline{\mathcal{M}}_{1,1} \cong \mathcal{P}(4,6)$ one has $\lambda \simeq \mathcal{O}_{\mathcal{P}(4,6)}(1)$, and this height agrees with the Faltings height of the corresponding elliptic curve by [BPS22, Cor. 7.6].

Guided by (1), we introduce the following motivic generating series (see [Eke25] for background on the Grothendieck ring of stacks) refining the height generating series in [BPS22, §8] by weighting each height stratum with the *lattice ranks* of the associated relatively minimal elliptic surface.

Definition 1.1. Let k be a perfect field of characteristic $\neq 2,3$, and consider the height–moduli stack

$$\mathcal{W}_n^{\min} = \mathcal{W}_{n,\mathbb{P}_k^1}^{\min}\big(\mathcal{P}(4,6),\mathcal{O}(1)\big)$$

parametrizing minimal elliptic curves over $K = k(t)$ of discriminant height $12n$. The *trivariate height zeta function* is

$$\mathcal{Z}(u,v;t) := \sum_{n\geq 0}\left(\sum_{[E]\in\mathcal{W}_n^{\min}} u^{T(S)}\, v^{\mathrm{rk}(E/K)}\right)t^n$$

where for each $[E]\in\mathcal{W}_n^{\min}$ we write $S\to\mathbb{P}_k^1$ for the associated relatively minimal elliptic surface $f : S\to\mathbb{P}_k^1$ with section, and:

- $T(S)$ is the rank of the trivial lattice of S;
- $\mathrm{rk}(E/K)$ is the Mordell–Weil rank.

The following specializations are the associated *bivariate height zeta functions*:

$$Z_{\mathrm{Triv}}(u;t) := \mathcal{Z}(u,1;t), \tag{3}$$

$$Z_{\mathrm{MW}}(v;t) := \mathcal{Z}(1,v;t), \tag{4}$$

$$Z_{\mathrm{NS}}(w;t) := \mathcal{Z}(w,w;t). \tag{5}$$

Remark 1.2. The specialization $Z_{\mathrm{Triv}}(u;t) = \mathcal{Z}(u,1;t)$ is rigorously defined in $K_0(\mathrm{Stck}_k)[u][\![t]\!]$ for any perfect field k with $\mathrm{char}(k)\neq 2,3$: the trivial lattice rank $T(S)$ is constant on each stratum of the finite Kodaira stratification (Proposition 2.1), so the strata $\mathcal{W}_n^{\min}(T)$ of Definition 1.7 are constructible substacks of finite type. Defining the other two specializations $Z_{\mathrm{MW}}(v;t)$ and $Z_{\mathrm{NS}}(w;t)$ motivically would require the loci $\{[E]\in\mathcal{W}_n^{\min} : \mathrm{rk}(E/K) = r\}$ to be constructible; by the Shioda–Tate formula, however, these are differences of Noether–Lefschetz strata, which form only a countable union of closed algebraic subsets [CDK95]. Over a finite field $k=\mathbb{F}_q$ this difficulty is absent, since $\mathcal{W}_n^{\min}(\mathbb{F}_q)$ is a finite set and both $T(S)$ and $\mathrm{rk}(E/K)$ are well-defined integers for each $\mathbb{F}_q$-point; in particular, the full trivariate series $\mathcal{Z}(u,v;t)$ is well-defined as a weighted point-counting series. Accordingly, we work motivically in the u-variable throughout and regard the v-grading as a point-counting refinement; the Shioda–Tate decomposition

$\rho(S) = T(S) + \mathrm{rk}(E/K)$ nonetheless serves as the organizing principle for all three specializations and for Conjecture 4.1.

Remark 1.3. Setting $u = v = 1$ forgets the lattice rank grading and specializes to the *univariate motivic height zeta function* $Z_{\vec{\lambda}}(t) = \mathcal{Z}(1,1;t) \in K_0(\mathrm{Stck}_k)[\![t]\!]$ and likewise to its inertial refinement $\mathcal{I}Z_{\vec{\lambda}}(t)$ which encodes the totality of rational points on $\overline{\mathcal{M}}_{1,1}$ over $K = k(t)$. [BPS22, Thm. 8.9] shows that both series are in fact rational in t, i.e. lie in $K_0(\mathrm{Stck}_k)(t)$, and gives explicit formulas.

Remark 1.4. The assumption $\mathrm{char}(k) \neq 2,3$ is used throughout in two essential ways: first, it ensures the existence of the short Weierstrass form $y^2 = x^3 + a_4x + a_6$ (equivalently, the isomorphism $\overline{\mathcal{M}}_{1,1} \simeq \mathcal{P}(4,6)$); second, it guarantees that the Kodaira–Néron fiber classification [Kod63, N$\acute{6}$4] and Tate correspondence (i.e. *Tate's algorithm* [Tat75] *via twisted maps* [BPS22, Thm. 7.12]) apply in their standard form.

In this paper we focus on $Z_{\mathrm{Triv}}(u;t)$. The key point is that the trivial lattice is governed by *local bad reduction*: its rank is determined by the geometric Kodaira fiber configuration of $\pi_{\bar{k}} \colon S_{\bar{k}} \to \mathbb{P}^1_{\bar{k}}$. Writing $\mathrm{Triv}(S) \subset \mathrm{NS}(S_{\bar{k}})$ for the geometric trivial lattice and $T(S) := \mathrm{rk}(\mathrm{Triv}(S))$, we have the following explicit formula.

Lemma 1.5. *Let $\pi \colon S \to \mathbb{P}^1_k$ be a relatively minimal elliptic surface with section, and let $\mathfrak{f}$ be the multiset of singular fibers of $\pi_{\bar{k}} \colon S_{\bar{k}} \to \mathbb{P}^1_{\bar{k}}$. If m_v denotes the number of irreducible components of the fiber at v, then*

$$T(S) \;=\; 2 + \sum_{v \in \mathfrak{f}} (m_v - 1).$$

Definition 1.6. Fix $n \geq 1$. For a geometric fiber configuration $\mathfrak{f}$, let $\mathcal{W}_n^{\min,(\mathfrak{f})} \subset \mathcal{W}_n^{\min}$ denote the locus parametrizing those $[E] \in \mathcal{W}_n^{\min}$ whose associated surface $S_{\bar{k}} \to \mathbb{P}^1_{\bar{k}}$ has singular fiber configuration $\mathfrak{f}$ (cf. [BPS22, Thm. 7.16]).

Definition 1.7. Fix $n \geq 0$. By Proposition 2.1, $\mathcal{W}_n^{\min}$ admits a *finite* constructible stratification by Kodaira data, and $T(S)$ is constant on each stratum. For $n \geq 1$ and each T with $2 \leq T \leq 10n$, let

$$\mathcal{W}_n^{\min}(T) \subset \mathcal{W}_n^{\min}$$

be the finite union of those Kodaira strata on which $T(S) = T$ (hence a finite union of locally closed substacks). For $n = 0$, set $\mathcal{W}_0^{\min} := \mathcal{W}_0^{\min}(2)$.

The trivial–lattice–rank–weighted motivic height zeta function is

$$Z_{\mathrm{Triv}}(u;t) := \sum_{n \geq 0} \sum_{T \geq 2} u^T \, \{\mathcal{W}_n^{\min}(T)\} \, t^n \;\in\; K_0(\mathrm{Stck}_k)[u][\![t]\!].$$

We prove that $Z_{\mathrm{Triv}}(u;t)$ is coefficient-wise rational in s after inverting $\mathbb{L}$ (see Remark 2.6), and we give an explicit finite Euler product up to a correction term—a formal power series in u whose coefficient of each power of u is a polynomial in s (Theorem 2.11).

Theorem 1.8. *Let k be a perfect field with* $\mathrm{char}(k) \neq 2,3$ *and put $s = t^{1/12}$. Then*

$$Z_{\mathrm{Triv}}(u;t) \in K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}](s)[\![u]\!].$$

Moreover, $Z_{\mathrm{Triv}}(u;t)$ decomposes as a finite Euler product in s plus a correction $P(u;s) \in K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}][s][\![u]\!]$ supported on heights where dense additive configurations force isotriviality (see Theorem 2.11 and Proposition 2.8). Setting $u = 1$ kills the correction and recovers the exact Euler product of [BPS22, Thm. 8.9]. *More generally, the same rationality holds for $Z_{\mathrm{Triv},C}(u;t)$ when $\mathbb{P}^1$ is replaced by any smooth projective curve C/k with $C(k) \neq \varnothing$, with symmetric powers $\mathrm{Sym}^N(\mathbb{P}^1)$ replaced by the Kapranov zeta function ζ_C.*

The proof is a motivic local-to-global factorization argument [Kap00, CLL16], implemented on the twisted-map stratification of the height–moduli $\mathcal{W}_n^{\min}$ via the evaluation morphisms [BM96]. Throughout we use the Bejleri–Park–Satriano correspondence [BPS22, Thm. 3.3] between rational points, minimal weighted linear series, and twisted morphisms; in particular, local reduction conditions are encoded by representable twisted morphisms to $\overline{\mathcal{M}}_{1,1} \simeq \mathcal{P}(4,6)$, yielding a moduli-theoretic Tate correspondence (Remark 1.4) compatible with the minimal model program. Unordered collections of local factors supported at distinct points of $\mathbb{P}^1$ are governed by symmetric powers $\mathrm{Sym}^N(\mathbb{P}^1)$. We reorganize these symmetric-power contributions using the power structure on $K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}]$, and we record the resulting identity in Lemma 2.7.

Since only finitely many local factor types occur, this yields a finite Euler product after inverting $\mathbb{L}$ [GZLMH13], modulo a correction $P(u;s)$ arising from small heights at which dense additive configurations force the fibration to be isotrivial ($j \equiv 0$ or $j \equiv 1728$). Notably, this correction is supported on the same non-constant isotrivial loci ($j \equiv 0$ and $j \equiv 1728$) that give rise to the lower order main terms in the enumeration of [BPS22, Thm. 1.1, Thm. 9.7, Rmk. 9.8]; it vanishes when the u-weighting is forgotten ($u = 1$), reflecting the fact that the trivial lattice refinement introduces new arithmetic phenomena precisely at these special loci (see Proposition 2.8). The only unbounded discrete parameter is the cusp contact order in the two families I_k and I_k^*, which is collapsed by geometric resummation. Finally, specializing $x_\alpha = u^{m(\alpha)-1}$ for $\alpha \in \mathcal{A}_{\mathrm{nc}}$ together with the cusp substitutions produces the Euler product expression for $Z_{\mathrm{Triv}}(u;t)$ in $K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}](s)[\![u]\!]$ with $t = s^{12}$.

Remark 1.9. Replacing $\mathcal{W}_n^{\min}(T)$ by its inertia stack (see [HP23, §2] for background on the inertia stack $\mathcal{I}(\mathcal{X})$ of an algebraic stack $\mathcal{X}$) gives

$$\mathcal{I}Z_{\mathrm{Triv}}(u;t) := \sum_{n\geq 0} \sum_{T \geq 2} u^T \left\{\mathcal{I}\mathcal{W}_n^{\min}(T)\right\} t^n \in K_0(\mathrm{Stck}_k)[u][\![t]\!].$$

After inverting $\mathbb{L}$, the same argument yields rationality and a finite Euler product (modulo a correction as in Theorem 2.11) for $\mathcal{I}Z_{\mathrm{Triv}}(u;t)$.

2. Rationality of the trivial lattice rank weighted case

Throughout, let k be a perfect field with $\mathrm{char}(k) \neq 2,3$, set $K = k(t)$, and let $\pi\colon S \to \mathbb{P}^1_k$ be the relatively minimal elliptic surface with section associated to E/K. Write $\mathrm{Triv}(S) \subset \mathrm{NS}(S_{\bar{k}})$ for the geometric trivial lattice and $T(S) = \mathrm{rk}\,\mathrm{Triv}(S)$.

Proposition 2.1. *Fix $n \geq 1$. The discriminant degree constraint $\sum_v e(F_v) = 12n$ implies that only finitely many geometric fiber configurations $\mathfrak{f}$ occur among surfaces* *parametrized by $\mathcal{W}_n^{\min}$. Consequently,*

$$\mathcal{W}_n^{\min} = \bigsqcup_{\mathfrak{f}} \mathcal{W}_n^{\min,(\mathfrak{f})}$$

is a finite constructible stratification. Moreover, the trivial lattice rank $T(S)$ is constant on each stratum $\mathcal{W}_n^{\min,(\mathfrak{f})}$.

Proof. Fix $n \geq 1$ and let $S \to \mathbb{P}^1_k$ be a surface parametrized by $\mathcal{W}_n^{\min}$. For any relatively minimal elliptic surface with section one has $\sum_{v \in \mathbb{P}^1_{\bar{k}}} e(F_v) = e(S_{\bar{k}})$ and in our height-n locus this total Euler number equals $12n$ (equivalently, the discriminant has degree $12n$). For each singular fiber F_v, the Kodaira–Néron classification [Kod63, N6́4] gives the types $\mathrm{I}_k, \mathrm{I}_k^*$ $(k \geq 1)$ and $\mathrm{II}, \mathrm{III}, \mathrm{IV}, \mathrm{I}_0^*, \mathrm{IV}^*, \mathrm{III}^*, \mathrm{II}^*$. Their Euler numbers satisfy $e(\mathrm{I}_k) = k$, $e(\mathrm{I}_k^*) = k + 6$ while the remaining types have Euler number $e(F_v) \in \{2, 3, 4, 6, 8, 9, 10\}$ (see [Her91, Table 1]). Since $\sum_v e(F_v) = 12n$, the integers k occurring in fibers of type I_k and I_k^* are bounded in terms of n. Hence there are only finitely many multisets of Kodaira symbols (equivalently, fiber configurations $\mathfrak{f}$) whose Euler numbers sum to $12n$. Therefore only finitely many configurations occur, and $\mathcal{W}_n^{\min} = \bigsqcup_{\mathfrak{f}} \mathcal{W}_n^{\min,(\mathfrak{f})}$ is a finite stratification by locally closed substacks as in [BPS22, Thm. 7.16]. Finally, on a fixed stratum $\mathcal{W}_n^{\min,(\mathfrak{f})}$ the multiset $\mathfrak{f}$ (hence the integers m_v) is constant, so Lemma 1.5 implies that $T(S) = 2 + \sum_{v \in \mathfrak{f}} (m_v - 1)$ is constant on that stratum. ∎

A multivariate height series. We briefly recall the local indexing used in the twisted-maps description of height–moduli. By [BPS22, Thm. 5.1] the height-n moduli stack $\mathcal{M}_{n,C}(\mathcal{X}, \mathcal{L})$ on a proper polarized cyclotomic stack $\mathcal{X}$ with polarizing line bundle $\mathcal{L}$ admits a finite stratification by locally closed substacks indexed by admissible local conditions and degrees: there is a finite disjoint union of morphisms

$$\bigsqcup_{\Gamma, d} \mathcal{H}^{\Gamma}_{d,C}(\mathcal{X}, \mathcal{L})/S_{\Gamma} \longrightarrow \mathcal{M}_{n,C}(\mathcal{X}, \mathcal{L}),$$

where $\mathcal{H}^{\Gamma}_{d,C}(\mathcal{X}, \mathcal{L})$ is the moduli stack of representable twisted morphisms of stable height d to $(\mathcal{X}, \mathcal{L})$ with local twisting conditions

$$\Gamma = \big(\{r_1, a_1\}, \ldots, \{r_s, a_s\}\big),$$

recording the stabilizer orders r_i and the corresponding characters a_i at the stacky marked points of the source root stack. The indices (Γ, d) range over those satisfying the height decomposition formula

$$n \;=\; d \;+\; \sum_{i=1}^{s} \frac{a_i}{r_i}.$$

Here $S_{\Gamma} \subset S_s$ is the subgroup permuting stacky marked points of the same local type.

Definition 2.2. For the Euler-product argument it is useful to distinguish *local factor types* from *evaluation labels*. Let $\mathcal{IP}(4, 6)$ be the cyclotomic inertia stack.

(1) Local factor types. Let J denote the finite set of local factor types occurring in the Tate algorithm stratification via twisted maps (see [BPS22, §7]); concretely one may take

$$J = \left\{\mathrm{II},\ \mathrm{III},\ \mathrm{IV},\ \mathrm{II}^*,\ \mathrm{III}^*,\ \mathrm{IV}^*,\ \mathrm{I}_0^*(j \neq 0, 1728),\ \mathrm{I}_0^*(j \in \{0, 1728\}),\ \mathrm{I}_\bullet,\ \mathrm{I}_\bullet^*\right\},$$

where $\mathrm{I}_\bullet$ and $\mathrm{I}_\bullet^*$ are the two cusp *shapes* over $j = \infty$.

(2) Evaluation labels. Let $\mathcal{A}$ denote the set of evaluation labels used to index evaluation conditions, i.e. the inertia components in which the evaluation maps land. Away from the cusp $j = \infty$, the inertia label determines the Kodaira symbol, so the non-cusp labels form a finite set

$$\mathcal{A}_{\mathrm{nc}} = \left\{\mathrm{II},\ \mathrm{III},\ \mathrm{IV},\ \mathrm{II}^*,\ \mathrm{III}^*,\ \mathrm{IV}^*,\ \mathrm{I}_0^*(j \neq 0, 1728),\ \mathrm{I}_0^*(j \in \{0, 1728\})\right\}.$$

At the cusp $j = \infty$, the inertia label records only the cusp shape ($\mathrm{I}_\bullet$ or $\mathrm{I}_\bullet^*$); the additional integer $k \geq 1$ (contact order with the boundary, equivalently the pole order of j) is *not* part of the local twisting conditions Γ of [BPS22, Def. 3.2]: for I_k one has $(r, a) = (0, 0)$ so no stacky marking appears, while for I_k^* one has $(r, a) = (2, 1)$ independently of k. The contact order is instead determined by the discriminant valuation of the Weierstrass model, equivalently the pole order of the j-map at $j = \infty$ ([BPS22, Thm. 7.12]). In the generating function $\mathcal{H}(s; \mathbf{x})$, the contact order becomes a free summation variable, collapsed by geometric resummation (Lemma 2.10). Accordingly we set

$$\mathcal{A} := \mathcal{A}_{\mathrm{nc}} \sqcup \{\mathrm{I}_\bullet,\ \mathrm{I}_\bullet^*\}.$$

For $\alpha \in \mathcal{A}_{\mathrm{nc}}$, let $m(\alpha) \in \mathbb{Z}_{\geq 1}$ be the number of irreducible components of the corresponding Kodaira fiber, so that $m(\alpha) - 1$ is its contribution to the trivial lattice. For the cusp shapes $\mathrm{I}_\bullet$ and $\mathrm{I}_\bullet^*$, the component number depends on the contact order $k \geq 1$ (of the corresponding I_k or I_k^* fiber); this k–dependence will be incorporated later by geometric resummation (Lemma 2.10). In summary, J indexes the *local factor types* (basic chart types) that become Euler factors under the power structure on $K_0(\mathrm{Stck}_k)$, whereas $\mathcal{A}$ indexes the evaluation labels, i.e. exactly what inertia can see; in particular, over $j = \infty$ inertia distinguishes only the two cusp shapes and not the contact order k.

Remark 2.3. When an evaluation condition lands over the cusp $j = \infty$, the corresponding component of the cyclotomic inertia stack $\mathcal{IP}(4, 6)$ records only the *cusp shape* (multiplicative $\mathrm{I}_\bullet$ or additive $\mathrm{I}_\bullet^*$); it does *not* record the *multiplicity* $k \geq 1$. Equivalently, inertia detects that j has a pole, but not its pole order. The missing discrete datum is the *contact order with the boundary*. Geometrically, it is visible on the log canonical model obtained by contracting, in each reducible fiber, the components not meeting the zero section.

(1) The multiplicative family I_k. If the fiber at $t \in \mathbb{P}^1$ is of type I_k ($k \geq 1$), then the contraction produces an A_{k-1} surface singularity. Étale locally one has

$$xy = u^k,$$

where u is a local parameter at t. Since an étale neighbourhood of the universal nodal fiber over the cusp $[\infty] \in \overline{\mathcal{M}}_{1,1} \simeq \mathcal{P}(4, 6)$ is given by $xy = s$ (with s a parameter at the cusp), the classifying map $\varphi_g : \mathbb{P}^1 \to \overline{\mathcal{M}}_{1,1}$ satisfies $s = u^k$. Thus φ_g meets the boundary with contact order k, and $\nu(\Delta) = k$ for type I_k.

(2) The additive family I_k^*. If the fiber at t is of type I_k^* ($k \geq 1$), then the contraction produces a D_{k+4} surface singularity. The classifying map still lands at $j = \infty$ with boundary contact order k (so locally $s = u^k$), while the discriminant valuation is shifted by the starred contribution: $\nu(\Delta) = k + 6$ for type I_k^*.

For Z_{Triv} one has

$$m(\mathrm{I}_k) - 1 = k - 1, \qquad m(\mathrm{I}_k^*) - 1 = k + 4,$$

so the trivial lattice exponent depends linearly on k in each cusp family.

Definition 2.4. Fix an auxiliary variable s with $s^{12} = t$. Introduce variables $\{x_\alpha\}_{\alpha \in \mathcal{A}}$ and define

$$\mathcal{H}(s; \mathbf{x}) := \sum_{n \geq 0} \sum_{\mathfrak{f}} \left(\prod_{v \in \mathfrak{f}} x_{\alpha_v} \right) \{\mathcal{W}_n^{\min,(\mathfrak{f})}\} s^{12n} \in K_0(\mathrm{Stck}_k)[\mathbf{x}][\![s]\!], \tag{6}$$

where for fixed n the inner sum ranges over the finitely many geometric fiber configurations $\mathfrak{f}$ occurring in height n (Proposition 2.1).

For each singular fiber F_v in $\mathfrak{f}$, let $\alpha_v \in \mathcal{A}$ denote the corresponding inertia/evaluation label (Definition 2.2). Away from the cusp $j = \infty$ this label is the Kodaira symbol, while over $j = \infty$ it records only the cusp shape $\mathrm{I}_\bullet$ or $\mathrm{I}_\bullet^*$. The additional contact order $k \geq 1$ at the cusp is part of the twisted-maps chart data and is *not* recorded by the variables x_α.

Remark 2.5. The local conditions defining the strata are imposed via evaluation maps ev_i to $\mathcal{IP}(4,6)$, hence are naturally indexed by connected components of the inertia stack. In particular, the same Kodaira symbol may correspond to distinct inertia components. For example, I_0^* splits into distinct inertia components according to whether $j \in \{0, 1728\}$ or $j \notin \{0, 1728\}$. Accordingly we index local conditions by inertia labels, not by Kodaira symbols alone.

Remark 2.6. We work in the localized ring $K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}]$. Localization is used to place the argument in a ring where quotient stack identities for linear algebraic groups (e.g. GL_n, PGL_2) and the power–structure identities for symmetric powers hold uniformly as equalities of rational functions, thereby justifying the reorganization of unordered local factors into Euler factors.

Lemma 2.7. *Let $\mathcal{A}$ be the finite set of evaluation labels from Definition 2.2, and let $\mathcal{H}(s; \mathbf{x})$ be the multivariate height series defined in* (6)*. After inverting $\mathbb{L}$, the series $\mathcal{H}(s; \mathbf{x})$ is coefficient-wise rational in s: the coefficient of each $\mathbf{x}$-monomial is a rational function of s with coefficients in $K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}]$.*

More precisely, there exist:

- *a finite index set J of local factor types,*
- *motivic classes $A_j \in K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}]$,*
- *integers $c_j \geq 1$, recording the discriminant degree increment contributed by one local factor of type j,*

- *and exponents $\beta_{j,\alpha} \in \mathbb{Z}_{\geq 0}$ for $\alpha \in \mathcal{A}$, recording how many markings of inertia type α occur in a local factor of type j,*

such that, writing

$$Y_j(s;\mathbf{x}) := A_j\Big(\prod_{\alpha\in\mathcal{A}} x_\alpha^{\beta_{j,\alpha}}\Big)s^{c_j},$$

one has

$$\mathcal{H}(s;\mathbf{x}) = \prod_{j\in J}\big(1-Y_j(s;\mathbf{x})\big)^{-\{\mathbb{P}^1\}} + P(s;\mathbf{x}), \tag{7}$$

where $P(s;\mathbf{x})$ is a formal power series in $\mathbf{x}$ whose coefficient of each $\mathbf{x}$-monomial is a polynomial in s, and the power-structure expansion gives

$$(1-Y_j)^{-\{\mathbb{P}^1\}} = \sum_{N\geq 0}\{\mathrm{Sym}^N(\mathbb{P}^1)\}\, Y_j^N = \frac{1}{(1-Y_j)(1-\mathbb{L}\,Y_j)}. \tag{8}$$

Moreover, for $\alpha \in \{\mathrm{I}_\bullet, \mathrm{I}_\bullet^*\}$ the exponent $\beta_{j,\alpha}$ counts only the number of cusp markings of the given cusp shape *in factor type j; it does not record the contact order $k \geq 1$.*

Proof. By [BPS22, Thm. 3.3 & Prop. 5.8], the correspondence between twisted maps $\mathcal{C} \to \mathcal{P}(4,6)$ and minimal $(4,6)$-weighted linear series on $\mathbb{P}^1_k$ yields, for each n, a finite locally closed stratification of $\mathcal{W}_n^{\min}$ into charts

$$\mathcal{H}^{\Gamma}_{d,\mathbb{P}^1}\big(\mathcal{P}(4,6),\mathcal{O}(1)\big)/S_\Gamma \;\cong\; \mathcal{R}_n^{\mu}/S_\Gamma$$

indexed by admissible local conditions $\Gamma = \big(\{r_1,a_1\},\ldots,\{r_s,a_s\}\big)$ and stable height d, where $n = d+\sum a_i/r_i$ and the partition μ corresponds to Γ via the normalized base locus [BPS22, Prop. 3.9].

Construction of the local factor coefficients A_j. Let J be the finite set of local factor types in Definition 2.2, each recording a pair (r_j,a_j) of local twisting data together with the j-invariant of the fiber (equivalently, a vanishing condition $\gamma_j = (\nu(a_4),\nu(a_6))$ as in Table 1). For each $j\in J$, consider the stratum $\mathcal{W}^{\gamma_j}_{n,\mathbb{P}^1}(4,6)$ of minimal Weierstrass fibrations over $\mathbb{P}^1$ with *exactly one* singular fiber of type j at a varying point and smooth or multiplicative fibers elsewhere. By [BPS22, Prop. 6.7], $\mathcal{W}_n^{\gamma_j}$ is a Zariski-locally trivial fibration over $\mathbb{P}^1$ with fiber $[\mathcal{R}_n^{\gamma_j}(0)/\mathbb{G}_m]$, and one has the equality

$$\big\{\mathcal{W}_n^{\gamma_j}\big\} \;=\; \big\{\mathbb{P}^1\big\}\cdot\frac{\big\{\mathcal{R}_n^{\gamma_j}(0)\big\}}{\{\mathbb{G}_m\}} \tag{9}$$

in $K_0(\mathrm{Stck}_k)$. By [BPS22, Prop. 9.4], for $n\gg 0$ the motivic class is

$$\big\{\mathcal{W}_n^{\gamma_j}\big\} \;=\; (\mathbb{L}^2-1)\,\mathbb{L}^{10n-p_j-q_j}$$

where (p_j,q_j) are the Weierstrass vanishing orders from γ_j. We define the *local factor coefficient A_j* to be the normalized one-fiber motivic class

$$A_j \;:=\; \frac{\big\{\mathcal{R}_n^{\gamma_j}(0)\big\}}{\{\mathbb{G}_m\}\cdot\mathbb{L}^{10n}} \;=\; \frac{\big\{\mathcal{W}_n^{\gamma_j}\big\}}{\{\mathbb{P}^1\}\cdot\mathbb{L}^{10n}} \;\in\; K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}].$$

This class is *independent of n*: from the explicit formula above we read off

$$A_j \;=\; \frac{(\mathbb{L}^2-1)\,\mathbb{L}^{-p_j-q_j}}{\mathbb{L}+1} \;=\; (\mathbb{L}-1)\,\mathbb{L}^{-p_j-q_j},$$

and more generally each entry in [BPS22, Table of Thm. 1.8] yields the corresponding A_j after dividing by $\{\mathbb{P}^1\} \cdot \mathbb{L}^{10n}$. These values are originally computed in [BPS22, Prop. 9.4] via the Poly-space stratification of [BPS22, §8.4] (see also [PS21]). The discriminant degree increment c_j is the s-degree contributed by the local factor: explicitly, $12c_j$ equals the discriminant valuation of a fiber of type j, and the exponents $\beta_{j,\alpha}$ recording the evaluation labels are determined by γ_j via the Tate correspondence (i.e. *Tate's algorithm via twisted maps*) [BPS22, Thm. 7.12].

Motivic factorization over support points. We now establish the Euler product (7). By [BPS22, Thm. 2.30], the chart with s ordered stacky markings maps via the forgetful morphism

$$\mathcal{H}^{\Gamma}_{d,\mathbb{P}^1}\big(\mathcal{P}(4,6),\mathcal{O}(1)\big) \longrightarrow \mathrm{Conf}_s(\mathbb{P}^1)$$

recording the support points of the markings. Under the isomorphism $\mathcal{H}^{\Gamma}_d \cong \mathcal{R}^{\mu}_n$ of [BPS22, Prop. 5.8], this is identified with the projection $\mathcal{R}^{\mu}_n \to \mathrm{Conf}_{l(\mu)}(\mathbb{P}^1)$ sending a weighted linear series to the support of its normalized base locus.

The argument proceeds by decomposing the height series according to the *additive fiber configuration* and appealing to the Kapranov zeta function separately for each slice. We partition the set of local factor types as $J = J_{\mathrm{mult}} \sqcup J_{\mathrm{add}}$, where $J_{\mathrm{mult}} = \{\mathrm{I}_\bullet\}$ consists of the multiplicative types and J_{add} consists of the additive types (II, III, IV, $\mathrm{I}_0^*(j \neq 0, 1728)$, $\mathrm{I}_0^*(j \in \{0, 1728\})$, $\mathrm{I}_\bullet^*$, IV^*, III^*, II^*). By Tate's algorithm [BPS22, Thm. 7.12], a multiplicative fiber of type I_k satisfies $\nu_x(a_4) = 0$ and $\nu_x(a_6) = 0$ (the Weierstrass coefficients are units at x); the contact order $k \geq 1$ is determined by $\nu_x(\Delta) = k$. By contrast, each additive type j imposes $\nu_x(a_4) \geq p_j \geq 1$ and/or $\nu_x(a_6) \geq q_j \geq 1$.

Step 1: Multiplicative fibers and the residual discriminant. Since multiplicative fibers of type I_k have $(p_j, q_j) = (0, 0)$, they impose no vanishing conditions on $a_4 \in H^0(\mathbb{P}^1, \mathcal{O}(4n))$ or $a_6 \in H^0(\mathbb{P}^1, \mathcal{O}(6n))$ and do not appear in the normalized base locus of the weighted linear series. Their positions and contact orders are instead *determined by* the residual discriminant $\Delta = 4a_4^3 + 27a_6^2$, which is a nonlinear function of the Weierstrass data.

Crucially, the proof does not require computing the motivic class of individual multi-type multiplicative configurations (e.g. strata with prescribed I_2 and I_{10} fibers simultaneously). Instead, the multiplicative contribution enters the Euler product through two mechanisms:

(a) At $\mathbf{x} = 1$ (equivalently $u = 1$), the full Euler product including all multiplicative factors is already established in [BPS22, Thm. 8.9].

(b) At the multivariate level, the variable $x_{\mathrm{I}_\bullet}$ tracks only the cusp *shape*, not the contact order k (Definition 2.2). The k-dependence is restored by the cusp resummation (Lemma 2.10), which is a formal identity in the ring of power series.

In particular, the one-fiber motivic classes A_j computed in [BPS22, Prop. 9.4] are the only *local* input; the Euler product then assembles multi-fiber predictions from these one-fiber classes, and the correction P (Proposition 2.8) captures any discrepancy at small heights where the assembly fails.

The independence argument for *additive* fibers (Step 3 below) ensures that these multi-fiber predictions are exact for $n \geq n_1(T)$. For *multiplicative* fibers, which impose no vanishing conditions, the analogous exactness follows from the agreement with [BPS22, Thm. 8.9] at $u = 1$: since $P(1; s) = 0$ (Proposition 2.8(3)), the Euler product for multiplicative factors is exact at all heights after removing the u-dependent correction.

Step 2: Decomposition by additive configuration. An *additive configuration* is a tuple $T = (N_j)_{j \in J_{\mathrm{add}}}$ recording the number of fibers of each additive Kodaira type. We decompose

$$\mathcal{H}(s; \mathbf{x}) \;=\; \sum_T \mathcal{H}_T(s; \mathbf{x}_{\mathrm{mult}}) \cdot \prod_{j \in J_{\mathrm{add}}} x_j^{N_j},$$

where the sum runs over all additive configurations T and $\mathcal{H}_T$ collects, at each height n, the motivic contribution from minimal Weierstrass fibrations with additive configuration T and arbitrary multiplicative fibers (tracked by $\mathbf{x}_{\mathrm{mult}}$).

Step 3: Additive conditions are independent for large n. Fix an additive configuration T. The vanishing conditions $\nu_{x_i}(a_4) \geq p_{j_i}$, $\nu_{x_i}(a_6) \geq q_{j_i}$ at the $|T| := \sum_j N_j$ additive support points impose conditions on the coefficient spaces of total multiplicity $P(T) := \sum_j N_j p_j$ and $Q(T) := \sum_j N_j q_j$ respectively. At distinct points $x_1, \ldots, x_{|T|} \in \mathbb{P}^1$, these are independent linear conditions provided

$$P(T) \;\leq\; 4n+1 \qquad \text{and} \qquad Q(T) \;\leq\; 6n+1, \tag{10}$$

since then each set of vanishing conditions cuts out a codimension-p_{j_i} (resp. q_{j_i}) linear subspace of $H^0(\mathcal{O}(4n))$ (resp. $H^0(\mathcal{O}(6n))$) with disjoint support. Since $P(T)$ and $Q(T)$ depend only on T, while $\dim H^0(\mathcal{O}(4n)) = 4n+1$ and $\dim H^0(\mathcal{O}(6n)) = 6n+1$ grow linearly in n, there exists $n_1(T) \in \mathbb{Z}_{>0}$ such that (10) holds for all $n \geq n_1(T)$.

For $n \geq n_1(T)$, the fiber of $\mathcal{R}_n^{\mu}$ over each point of $\mathrm{Conf}_{|T|}(\mathbb{P}^1)$ decomposes motivically as a product of independent local contributions, each with motivic class A_j as defined above. Combined with Step 1 for the multiplicative part, this yields the Euler product for the coefficient of s^{12n} in $\mathcal{H}_T$ for all $n \geq n_1(T)$.

Step 4: Rationality of each slice. Let $R_T(s; \mathbf{x}_{\mathrm{mult}})$ denote the formal power series obtained by expanding the T-component of the right-hand side of (7) (i.e. the Euler product without the correction). By the power-structure formalism (Kapranov's identity applied to each factor), R_T is a rational function in s with coefficients in $K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}][\mathbf{x}]$. By Step 3, $\mathcal{H}_T$ and R_T agree as formal power series for all s^{12n} with $n \geq n_1(T)$. Therefore

$$\mathcal{H}_T(s; \mathbf{x}_{\mathrm{mult}}) \;=\; R_T(s; \mathbf{x}_{\mathrm{mult}}) \;+\; P_T(s; \mathbf{x}_{\mathrm{mult}})$$

where P_T is a polynomial in s supported in degrees $n < n_1(T)$. In particular, $\mathcal{H}_T$ is rational in s.

Step 5: Finite assembly. For each fixed monomial $\prod_\alpha x_\alpha^{m_\alpha}$ in $\mathcal{H}(s; \mathbf{x})$, only finitely many additive configurations T contribute (since N_j is bounded by m_α for each α). By Step 4, each $\mathcal{H}_T$ is rational in s and satisfies $\mathcal{H}_T = R_T + P_T$. Summing over all additive configurations, we conclude that

$$\mathcal{H}(s; \mathbf{x}) = \prod_{j \in J} (1 - Y_j)^{-\{\mathbb{P}^1\}} + P(s; \mathbf{x})$$

is rational in s, where $P(s;\mathbf{x}) = \sum_T P_T(s;\mathbf{x}_{\mathrm{mult}}) \cdot \prod_{j\in J_{\mathrm{add}}} x_j^{N_j}$ is a formal power series in $\mathbf{x}$ whose coefficient of each $\mathbf{x}$-monomial is a polynomial in s.

Finally, for $\alpha \in \{\mathrm{I}_\bullet, \mathrm{I}_\bullet^*\}$ the exponent $\beta_{j,\alpha}$ counts only the number of cusp markings of the given cusp *shape* in factor type j; the individual contact orders k_i are part of the admissible chart data (Remark 2.3) and are handled separately by geometric resummation (Lemma 2.10). ■

Proposition 2.8 (Characterization of the correction). *Let notation be as in Theorem 2.11.*

(1) *For each additive configuration $T = (N_j)_{j\in J_{\mathrm{add}}}$, the polynomial $P_T(s;\mathbf{x}_{\mathrm{mult}})$ is supported in degrees $n < n_1(T)$, where $n_1(T)$ is the smallest positive integer such that $P(T) \le 4n+1$ and $Q(T) \le 6n+1$.*
(2) *In the range $n < n_1(T)$, the additive vanishing conditions force $a_4 \equiv 0$ (if $P(T) > 4n+1$) or $a_6 \equiv 0$ (if $Q(T) > 6n+1$), constraining the moduli to the non-constant isotrivial loci $j \equiv 0$ or $j \equiv 1728$ respectively. Concretely, P_T records the difference between the actual motivic class of these isotrivial strata and the prediction of the Euler product at these small heights.*
(3) *Upon specializing $x_\alpha = 1$ for all $\alpha \in \mathcal{A}$ (equivalently, setting $u = 1$), the correction vanishes:*

$$P(1;s) = 0.$$

In particular, $Z_{\mathrm{Triv}}(1;t)$ recovers the exact Euler product of [BPS22, Thm. 8.9].

Proof. (1) By Step 3 of the proof of Lemma 2.7, the Euler product R_T and the height series $\mathcal{H}_T$ have identical coefficients for all $n \ge n_1(T)$, so $P_T = \mathcal{H}_T - R_T$ is supported in $n < n_1(T)$.

(2) Since $\dim H^0(\mathbb{P}^1, \mathcal{O}(4n)) = 4n+1$, imposing $P(T) > 4n+1$ independent vanishing conditions on a_4 forces $a_4 \equiv 0$. Then $j = 1728 \cdot 4a_4^3/(4a_4^3 + 27a_6^2)$ gives $j \equiv 0$. The case $Q(T) > 6n+1$ is analogous, forcing $a_6 \equiv 0$ and $j \equiv 1728$.

(3) Setting $u = 1$ forgets the $T(S)$-weighting, so $Z_{\mathrm{Triv}}(1;t) = Z_{\vec{\lambda}}(t)$, the unweighted motivic height zeta function. By [BPS22, Thm. 8.9], $Z_{\vec{\lambda}}(t)$ is computed as an exact Euler product with no correction (i.e. $P = 0$). Since the Euler product of Theorem 2.11 specializes at $u = 1$ to that of [BPS22, Thm. 8.9], the corrections must cancel: $P(1;s) = 0$. ■

Remark 2.9. The isotrivial families appearing in Proposition 2.8(2) are supported on the same loci that produce the lower order main terms of order $B^{1/2}$ (μ_6-twist families at $j = 0$) and $B^{1/3}$ (μ_4-twist families at $j = 1728$) in the enumeration of [BPS22, Thm. 1.1, Thm. 9.7, Rmk. 9.8]. Thus the correction P may be understood as the *motivic signature* of refining the height zeta function by the trivial lattice rank: the u-weighting distinguishes isotrivial twist families that are invisible to the unweighted count.

Lemma 2.10. *Let R be a commutative ring.*

(1) Geometric resummation *Fix $A \in R$. For integers $a, c \geq 1$ and $b, d \geq 0$, one has in $R[\![u,t]\!]$*

$$\sum_{k \geq 1} A u^{ak+b} t^{ck+d} = A u^{a+b} t^{c+d} \cdot \frac{1}{1 - u^a t^c}. \tag{11}$$

Moreover, if $k_1, \ldots, k_M \geq 1$ are independent and contribute multiplicatively with the same step (a, c), then

$$\sum_{k_1,\ldots,k_M \geq 1} A \prod_{i=1}^{M} u^{ak_i+b} t^{ck_i+d} = A\Big(u^{a+b} t^{c+d}\Big)^M \cdot \frac{1}{(1-u^a t^c)^M}. \tag{12}$$

Equivalently, each marking contributes one factor $(1-u^a t^c)^{-1}$, so M such markings contribute the power $(1-u^a t^c)^{-M}$, up to the monomial shift $\big(u^{a+b} t^{c+d}\big)^M$.

(2) Cusp shapes for Z_{Triv} *Assume* $\mathrm{char}(k) \neq 2, 3$ *and work in $R = K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}]$. Introduce an auxiliary variable s with $t = s^{12}$, so that t^n corresponds to* $\deg(\Delta) = 12n$, *while s records the integral discriminant degree* $\deg(\Delta)$.

After specializing $x_\beta = u^{m(\beta)-1}$ for $\beta \in \mathcal{A}_{\mathrm{nc}}$, a cusp marking of shape $\mathrm{I}_\bullet$ (resp. $\mathrm{I}_\bullet^*$) with contact order $k \geq 1$ contributes weight $u^{k-1}s^k$ (resp. $u^{k+4}s^{k+6}$), since

$$m(\mathrm{I}_k) - 1 = k - 1, \qquad \nu(\Delta) = k, \qquad m(\mathrm{I}_k^*) - 1 = k + 4, \qquad \nu(\Delta) = k + 6.$$

Hence summing over $k \geq 1$ at a single cusp marking gives, in $R[\![u,s]\!]$,

$$x_{\mathrm{I}_\bullet} = \sum_{k \geq 1} u^{k-1} s^k = \frac{s}{1-us}, \qquad x_{\mathrm{I}_\bullet^*} = \sum_{k \geq 1} u^{k+4} s^{k+6} = \frac{u^5 s^7}{1-us}. \tag{13}$$

In particular, each cusp marking of either shape contributes one factor $(1-us)^{-1}$ after resummation. Thus a factor type j with $\beta_{j,\mathrm{I}_\bullet}$ markings of shape $\mathrm{I}_\bullet$ and $\beta_{j,\mathrm{I}_\bullet^*}$ markings of shape $\mathrm{I}_\bullet^*$ contributes the cusp factor

$$(1-us)^{-(\beta_{j,\mathrm{I}_\bullet} + \beta_{j,\mathrm{I}_\bullet^*})},$$

together with the monomial shift

$$u^{5\beta_{j,\mathrm{I}_\bullet^*}} s^{\beta_{j,\mathrm{I}_\bullet} + 7\beta_{j,\mathrm{I}_\bullet^*}}$$

coming from (13).

Proof. For (11), factor out the $k = 1$ term and apply the geometric-series identity:

$$\sum_{k \geq 1} A u^{ak+b} t^{ck+d} = A u^{a+b} t^{c+d} \sum_{k \geq 0} (u^a t^c)^k = A u^{a+b} t^{c+d} \cdot \frac{1}{1-u^a t^c}.$$

Equation (12) follows because the sum over $(k_1, \ldots, k_M)$ factorizes as a product of M copies of (11). Part (2) is (11) with $(a, c) = (1, 1)$ applied in $R[\![u,s]\!]$ to the two monomial weights $u^{k-1}s^k$ and $u^{k+4}s^{k+6}$, yielding (13) and the stated denominator power. ∎

Rationality over a general base curve. Let k be a perfect field with $\mathrm{char}(k) \neq 2, 3$ and C/k be a smooth projective geometrically connected curve with $C(k) \neq \varnothing$, and set $K := k(C)$. Write

$$\mathcal{W}^{\min}_{n,C} := \mathcal{W}^{\min}_{n,C}\big(\mathcal{P}(4,6), \mathcal{O}(1)\big)$$

for the height–n moduli stack of minimal elliptic curves over K in the height–moduli framework of [BPS22]. As in Definition 1.7, the trivial-lattice-rank strata $\mathcal{W}^{\min}_{n,C}(T) \subset \mathcal{W}^{\min}_{n,C}$ are defined by grouping the finitely many Kodaira strata on which $T(S) = T$, and we set

$$Z_{\mathrm{Triv},C}(u;t) := \sum_{n \geq 0} \sum_{T \geq 2} u^T \left\{\mathcal{W}^{\min}_{n,C}(T)\right\} t^n \in K_0(\mathrm{Stck}_k)[u][\![t]\!].$$

When $C = \mathbb{P}^1$ this recovers $Z_{\mathrm{Triv}}(u;t)$ from Definition 1.7. Introduce an auxiliary variable s with $t = s^{12}$, so that s records the integral discriminant degree.

Kapranov zeta function. For a k–variety (or Deligne–Mumford stack) X, we write

$$\zeta_X(y) := \sum_{N \geq 0} \big\{\mathrm{Sym}^N(X)\big\} y^N \in K_0(\mathrm{Stck}_k)[\![y]\!]$$

for the Kapranov motivic zeta function [Kap00].

We now prove the Main Theorem.

Theorem 2.11 (Rationality and Euler product for Z_{Triv} over $k(C)$)**.** *Let k be a perfect field with* $\mathrm{char}(k) \neq 2, 3$*, and let C/k be a smooth projective geometrically connected curve with $C(k) \neq \varnothing$. Set $K = k(C)$ and $s = t^{1/12}$ (so $t = s^{12}$). Then*

$$Z_{\mathrm{Triv},C}(u;t) \in K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}](s)[\![u]\!].$$

More precisely, let J, A_j, c_j, and $\beta_{j,\alpha}$ be as in Lemma 2.7, where J indexes the finitely many local factor types. *Put*

$$\Delta(s) := 1 - us, \qquad b_j := \sum_{\beta \in \mathcal{A}_{\mathrm{nc}}} \beta_{j,\beta}\big(m(\beta) - 1\big), \qquad m_j := \beta_{j,\mathrm{I}_\bullet} + \beta_{j,\mathrm{I}^*_\bullet},$$

and define

$$B_j := b_j + 5\beta_{j,\mathrm{I}^*_\bullet}, \qquad C_j := c_j + \beta_{j,\mathrm{I}_\bullet} + 7\beta_{j,\mathrm{I}^*_\bullet}, \qquad \mathcal{Y}_j(u;s) := A_j\, u^{B_j} s^{C_j} \Delta(s)^{-m_j}.$$

Then there exists a correction term $P_C(u;s) \in K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}][s][\![u]\!]$ such that

$$Z_{\mathrm{Triv},C}(u;t) = u^2 \cdot \mathbb{L} \cdot \prod_{j \in J} \zeta_C\Big(\mathcal{Y}_j(u;s)\Big) + P_C(u;s), \qquad (t = s^{12}). \tag{14}$$

The correction $P_C(u;s)$ is characterized in Proposition 2.8: it is supported on heights below an explicit bound depending on the additive configuration, arises from the non-constant isotrivial loci, and vanishes at $u = 1$. Moreover, all dependence on $k \geq 1$ in the cusp families I_k and I^*_k (over $j = \infty$) is absorbed by the single geometric-series denominator $\Delta(s)^{-1} = (1 - us)^{-1}$.

Proof. Work in the localized ring $K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}]$.

Height-zero term. For $n = 0$ the discriminant degree is 0, hence the corresponding elliptic curve over $K = k(C)$ has everywhere good reduction and is therefore constant. Equivalently, $\mathcal{W}_{0,C}^{\min}$ identifies with the moduli stack of smooth elliptic curves,

$$\mathcal{W}_{0,C}^{\min} \cong \mathcal{M}_{1,1}.$$

Therefore $\{\mathcal{W}_{0,C}^{\min}\} = \mathbb{L}$ as in [Eke25], and the required constant term is $u^2 \cdot \mathbb{L}$, independently of C.

Euler product for $n \geq 1$. The proof of Lemma 2.7 applies verbatim with $\mathbb{P}^1$ replaced by C: passing to the quotient by S_Γ forgets the ordering among markings of the same inertia type, and repeating a local factor of type j at $N \geq 0$ support points is governed by effective degree-N 0-cycles on C, i.e. by $\mathrm{Sym}^N(C)$. Setting

$$Y_j(s;\mathbf{x}) := A_j\Big(\prod_{\alpha\in\mathcal{A}} x_\alpha^{\beta_{j,\alpha}}\Big)s^{c_j},$$

the contribution of all unordered collections of local factors of type j is

$$\sum_{N\geq 0}\big\{\mathrm{Sym}^N(C)\big\}\, Y_j(s;\mathbf{x})^N = \zeta_C\big(Y_j(s;\mathbf{x})\big),$$

and multiplying over the finitely many local factor types gives, by Lemma 2.7,

$$\mathcal{H}_C(s;\mathbf{x}) = \prod_{j\in J}\zeta_C\big(Y_j(s;\mathbf{x})\big) + P_C(s;\mathbf{x}), \tag{15}$$

where $P_C(s;\mathbf{x})$ is a formal power series in $\mathbf{x}$ whose coefficient of each $\mathbf{x}$-monomial is a polynomial in s.

Trivial lattice baseline. By Lemma 1.5, for an elliptic surface S with singular fiber configuration $\mathfrak{f}$ one has

$$T(S) = 2 + \sum_{v\in\mathfrak{f}}(m_v - 1).$$

Under the specializations below, the monomial attached to $\mathfrak{f}$ is $u^{\sum_{v\in\mathfrak{f}}(m_v-1)}$, i.e. it records only the fiber contributions. Thus passing from $\mathcal{H}_C$ to $Z_{\mathrm{Triv},C}$ introduces the global factor u^2.

Non-cusp specialization. For $\beta \in \mathcal{A}_{\mathrm{nc}}$ specialize $x_\beta = u^{m(\beta)-1}$. Then for each $j \in J$ the product of the non-cusp variables contributes the monomial u^{b_j}, where

$$b_j := \sum_{\beta\in\mathcal{A}_{\mathrm{nc}}} \beta_{j,\beta}\big(m(\beta)-1\big).$$

Cusp resummation. Over the cusp $j = \infty$, the contact order $k \geq 1$ varies. In the discriminant-degree grading (with $t = s^{12}$), a cusp marking of shape $\mathrm{I}_\bullet$ (resp. $\mathrm{I}_\bullet^*$) with contact order k contributes weight $u^{k-1}s^k$ (resp. $u^{k+4}s^{k+6}$). Hence (Lemma 2.10) we have the substitutions

$$x_{\mathrm{I}_\bullet} = \sum_{k\geq 1} u^{k-1}s^k = \frac{s}{1-us} = s\,\Delta(s)^{-1}, \qquad x_{\mathrm{I}_\bullet^*} = \sum_{k\geq 1} u^{k+4}s^{k+6} = \frac{u^5s^7}{1-us} = u^5s^7\Delta(s)^{-1},$$

with $\Delta(s) = 1 - us$. Therefore, for each $j \in J$ the cusp contribution becomes

$$x_{\mathrm{I}_\bullet}^{\beta_{j,\mathrm{I}_\bullet}}\, x_{\mathrm{I}_\bullet^*}^{\beta_{j,\mathrm{I}_\bullet^*}} = u^{5\beta_{j,\mathrm{I}_\bullet^*}}\, s^{\beta_{j,\mathrm{I}_\bullet}+7\beta_{j,\mathrm{I}_\bullet^*}}\,\Delta(s)^{-m_j}, \qquad m_j := \beta_{j,\mathrm{I}_\bullet} + \beta_{j,\mathrm{I}_\bullet^*}.$$

Combining with the non-cusp specialization yields, inside the argument of the jth factor $\zeta_C(\cdot)$ of (15),

$$A_j\Big(\prod_{\alpha\in\mathcal{A}} x_\alpha^{\beta_{j,\alpha}}\Big)s^{c_j} \longmapsto A_j\, u^{B_j}\, s^{C_j}\, \Delta(s)^{-m_j} = \mathcal{Y}_j(u;s),$$

where

$$B_j = b_j + 5\beta_{j,\mathrm{I}_\bullet^*}, \qquad C_j = c_j + \beta_{j,\mathrm{I}_\bullet} + 7\beta_{j,\mathrm{I}_\bullet^*}.$$

Together with the height-zero prefactor, this gives

$$Z_{\mathrm{Triv},C}(u;t) = u^2\cdot\mathbb{L}\cdot\prod_{j\in J}\zeta_C\Big(\mathcal{Y}_j(u;s)\Big) + P_C(u;s), \qquad (t=s^{12}),$$

which is (14).

Rationality of ζ_C. Since C is a smooth projective curve with $C(k)\neq\emptyset$, the Abel–Jacobi morphism $\mathrm{Sym}^N(C)\to\mathrm{Pic}^N(C)$ is a $\mathbb{P}^{N-g}$-bundle for $N\geq 2g-1$ (where $g=g(C)$), and a k–point identifies $\mathrm{Pic}^N(C)\cong\mathrm{Pic}^0(C)$. It follows that the tail of $\zeta_C(y)$ is a geometric series with coefficients in $K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}]$, hence $\zeta_C(y)\in K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}](y)$. Therefore each factor in (14) lies in $K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}][u](s)$, and so does their finite product; since each coefficient of u^m in $P_C(u;s)$ is a polynomial in s (hence rational), the sum is coefficient-wise rational in s.

The dependence on $k\geq 1$ in the cusp families is absorbed by the single geometric-series denominator $\Delta(s)^{-1}=(1-us)^{-1}$ through the above resummations. ■

Remark 2.12. When $C=\mathbb{P}^1$, one has $\mathrm{Sym}^N(\mathbb{P}^1)\cong\mathbb{P}^N$ for all $N\geq 0$, hence

$$\zeta_{\mathbb{P}^1}(y) = \sum_{N\geq 0}\{\mathbb{P}^N\}y^N = \frac{1}{(1-y)(1-\mathbb{L}y)}.$$

In this case (14) gives the explicit finite Euler product
(16)

$$Z_{\mathrm{Triv}}(u;t) = u^2\cdot\mathbb{L}\cdot\prod_{j\in J}\frac{1}{\Big(1-\mathcal{Y}_j(u;s)\Big)\Big(1-\mathbb{L}\cdot\mathcal{Y}_j(u;s)\Big)} + P(u;s), \qquad (t=s^{12}),$$

recovering the two-denominator Euler factors in Table 1.

Remark 2.13. Assume $\mathrm{char}(k)\neq 2,3$. For each Kodaira type Θ and $n\geq 1$, let

$$\mathcal{W}^{\Theta}_{n,\mathbb{P}^1}$$

be the moduli stack of minimal elliptic fibrations over $\mathbb{P}^1_k$ of discriminant degree $12n$ having exactly one singular fiber of type Θ over a varying degree–one place and semistable everywhere else.

The one-fiber motivic classes $\{\mathcal{W}^{\Theta}_{n,\mathbb{P}^1}\}$ carry a universal dependence on the height n coming from the $10n$–dimensional space of Weierstrass coefficients (equivalently, from the spaces of sections of degrees $4n$ and $6n$ in the weighted presentation $\overline{\mathcal{M}}_{1,1}\simeq\mathcal{P}(4,6)$). In particular, after dividing by the $\mathrm{Aut}(\mathbb{P}^1)=\mathrm{PGL}_2$–factor,[1] the

[1] The *unparameterized* $\mathbb{P}^1_k$ corresponds to taking the $\mathrm{Aut}(\mathbb{P}^1)=\mathrm{PGL}_2$ stack quotient; motivically this factors out $\{\mathrm{PGL}_2\}=\mathbb{L}(\mathbb{L}^2-1)$, thereby treating the base as a smooth conic. See [PS25] for a comprehensive treatment.

remaining motivic class grows as $\mathbb{L}^{10n+O(1)}$, uniformly in Θ. Accordingly we normalize local factor coefficients by

$$A_{\Theta}^{(C)} := \frac{\{\mathcal{W}_{n,\mathbb{P}^1}^{\Theta}\}}{\{\mathrm{PGL}_2\}\,\mathbb{L}^{10n+C}},$$

for some fixed integer C (e.g. $C = -20$ as in Table 1). The choice of C is immaterial for the Euler product: changing C rescales *every* local factor coefficient by the same global power of $\mathbb{L}$ and does not change its type (i.e. does not change the exponents B_j, C_j, m_j nor the finite set of factor types). Concretely, the normalization used in the proof of Lemma 2.7,

$$A_j = \frac{\left\{\mathcal{W}_{n,\mathbb{P}^1}^{\Theta}\right\}}{\{\mathbb{P}^1\}\cdot\mathbb{L}^{10n}},$$

and the normalization in Table 1,

$$A_{\Theta} = \frac{\left\{\mathcal{W}_{n,\mathbb{P}^1}^{\Theta}\right\}}{\{\mathrm{PGL}_2\}\cdot\mathbb{L}^{10n-20}},$$

differ by the universal factor $\dfrac{\{\mathrm{PGL}_2\}\cdot\mathbb{L}^{-20}}{\{\mathbb{P}^1\}}$, which is independent of Θ. Since all local factor coefficients are rescaled by the same element, the Euler product structure is unchanged: the global rescaling is absorbed into the prefactor.

Then [BPS22, Thm. 1.6] and [HP19, Cor. 2] determine the following normalized one-fiber motivic classes.

Convention. For each reduction type Θ in Table 1, let $y_{\Theta}(u;s)$ denote the local monomial appearing in the displayed denominator in the last column (e.g. $y_{\mathrm{I}_k} = \mathbb{L}^{18}s\,\Delta(s)^{-1}$, $y_{\mathrm{III}} = \mathbb{L}^{16}u\,s^3$, etc.). The *full* $\mathbb{P}^1$-contribution of Θ in the Euler product is the power-structure/Kapranov factor

$$(1-y_{\Theta}(u;s))^{-\{\mathbb{P}^1\}} = \sum_{N\geq 0}\{\mathrm{Sym}^N(\mathbb{P}^1)\}\,y_{\Theta}(u;s)^N = \frac{1}{\big(1-y_{\Theta}(u;s)\big)\big(1-\mathbb{L}\,y_{\Theta}(u;s)\big)}.$$

In Table 1 we record only the *reduced* factor $\big(1-y_{\Theta}(u;s)\big)^{-1}$; the second factor $\big(1-\mathbb{L}\,y_{\Theta}(u;s)\big)^{-1}$ is inserted uniformly in the global Euler product (cf. Theorem 2.11).

3. Applications to modular curves with prescribed level structure

We apply the Main Theorem to the genus-0 modular curves $\overline{\mathcal{M}}_1(N)$ parametrizing generalized elliptic curves with level-N structure $\Gamma_1(N)$, introduced by [DR73] (see also [Con07, §2]). The fine modular curve $\overline{\mathcal{M}}_1(N)$ parametrizes families $(E,S,P)\to B$ where $(E,S)\to B$ is a semistable elliptic curve with section S and $P\in E^{\mathrm{sm}}[N](B)$ is an N-torsion section such that the divisor $P+S$ is relatively ample [KM85, §1.4]. We focus on $N=2,3,4$, where the modular curves are genuinely stacky. Throughout, let k be a perfect field with $\mathrm{char}(k)\neq 2,3$.

Corollary 3.1. *Let k be a perfect field with* $\mathrm{char}(k)\neq 2,3$*, set $K=k(t)$, and let $s=t^{1/12}$. For $N\in\{2,3,4\}$, the trivial-lattice-rank-weighted motivic height zeta*

Reduction type Θ	(r,a)	m_v-1	$A_\Theta := \dfrac{\{\mathcal{W}^{\Theta}_{n,\mathbb{P}^1}\}}{\{\mathrm{PGL}_2\}\,\mathbb{L}^{10n-20}}$	$\mathbb{P}^1$-Euler factor in $Z_{\mathrm{Triv}}(u;t)$
$\mathrm{I}_{k\geq 1}$ $(j=\infty)$	$(0,0)$	$k-1$	$\mathbb{L}^{18}$	$\dfrac{1}{1-\mathbb{L}^{18}\,s\,\Delta(s)^{-1}}$
II $(j=0)$	$(6,1)$	0	$\mathbb{L}^{17}$	$\dfrac{1}{1-\mathbb{L}^{17}\,s^2}$
III $(j=1728)$	$(4,1)$	1	$\mathbb{L}^{16}$	$\dfrac{1}{1-\mathbb{L}^{16}\,u\,s^3}$
IV $(j=0)$	$(3,1)$	2	$\mathbb{L}^{15}$	$\dfrac{1}{1-\mathbb{L}^{15}\,u^2\,s^4}$
$\mathrm{I}^*_{k\geq 1}$ $(j=\infty)$	$(2,1)$	$k+4$	$\mathbb{L}^{14}-\mathbb{L}^{13}$	$\dfrac{1}{1-(\mathbb{L}^{14}-\mathbb{L}^{13})\,u^5\,s^7\,\Delta(s)^{-1}}$
I^*_0 $(j\neq 0,1728)$	$(2,1)$	4	$\mathbb{L}^{14}-\mathbb{L}^{13}$	$\dfrac{1}{1-(\mathbb{L}^{14}-\mathbb{L}^{13})\,u^4\,s^6}$
I^*_0 $(j=0,1728)$	$(2,1)$	4	$\mathbb{L}^{13}$	$\dfrac{1}{1-\mathbb{L}^{13}\,u^4\,s^6}$
IV^* $(j=0)$	$(3,2)$	6	$\mathbb{L}^{12}$	$\dfrac{1}{1-\mathbb{L}^{12}\,u^6\,s^8}$
III^* $(j=1728)$	$(4,3)$	7	$\mathbb{L}^{11}$	$\dfrac{1}{1-\mathbb{L}^{11}\,u^7\,s^9}$
II^* $(j=0)$	$(6,5)$	8	$\mathbb{L}^{10}$	$\dfrac{1}{1-\mathbb{L}^{10}\,u^8\,s^{10}}$

TABLE 1. Local factors for $\mathcal{P}(4,6)$.

function

$$Z^{\Gamma_1(N)}_{\mathrm{Triv}}(u;t) := \sum_{n\geq 0}\sum_{T\geq 2} u^T\left\{\mathcal{W}^{\Gamma_1(N)}_n(T)\right\}t^n$$

lies in $K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}](s)[\![u]\!]$*. More precisely, Theorem 2.11 applies with* $\overline{\mathcal{M}}_{1,1}$ *replaced by* $\overline{\mathcal{M}}_1(N)\cong\mathcal{P}(w_1,w_2)$ *and the local factor types* J*, coefficients* A_Θ*, and Euler factors as given in Tables 2–4 for* $(w_1,w_2)=(2,4)$*,* $(1,3)$*,* $(1,2)$ *respectively.*

Proof. The proof of Theorem 2.11 uses only the formal properties of the height–moduli framework of [BPS22] applied to a proper polarized cyclotomic stack $\mathcal{P}(w_1,w_2)$: the finite Kodaira stratification (Proposition 2.1), the evaluation-map factorization (Lemma 2.7), and the cusp resummation (Lemma 2.10). Each of these applies verbatim to $\mathcal{P}(2,4)$, $\mathcal{P}(1,3)$, and $\mathcal{P}(1,2)$, with the set of admissible Kodaira types restricted by the level structure. The local factor coefficients are computed from the one-fiber motivic classes as in Remark 2.13, with $\mathbb{L}^{10n}$ replaced by the appropriate $\mathbb{L}^{(w_1+w_2)n}$. ∎

3.1. **Level-2 structure.** We consider the height moduli

$$\mathcal{W}_n^{\Gamma_1(2)}\big(\mathcal{P}(2,4),\mathcal{O}(1)\big) \;=\; \mathcal{M}_{n,\mathbb{P}^1_k}\big(\overline{\mathcal{M}}_1(2),\mathcal{L}\big),$$

where $\overline{\mathcal{M}}_1(2) \cong \mathcal{P}(2,4)$ over $\mathbb{Z}\big[\frac{1}{2}\big]$ is the moduli stack of generalized elliptic curves with $\Gamma_1(2)$-structure (cf. [Beh06, §1.3]). Equivalently, $\overline{\mathcal{M}}_1(2)$ admits the universal Weierstrass presentation

$$y^2 = x^3 + a_2x^2 + a_4x \qquad \text{with} \qquad (a_2,a_4) \in H^0\big(\mathbb{P}^1,\mathcal{O}(2n)\big) \times H^0\big(\mathbb{P}^1,\mathcal{O}(4n)\big).$$

Reduction type Θ	(r,a)	$m_v - 1$	$A_\Theta := \dfrac{\{\mathcal{W}_{n,\mathbb{P}^1}^{\Gamma_1(2),\Theta}\}}{\{\mathrm{PGL}_2\}\,\mathbb{L}^{6n-12}}$	$\mathbb{P}^1$-Euler factor in $Z_{\mathrm{Triv}}^{\Gamma_1(2)}(u;t)$
$\mathrm{I}_{k\ge1}$ $(j=\infty)$	$(0,0)$	$k-1$	$\mathbb{L}^{10}$	$\dfrac{1}{1-\mathbb{L}^{10}s\,\Delta(s)^{-1}}$
III $(j=1728)$	$(4,1)$	1	$\mathbb{L}^{9}$	$\dfrac{1}{1-\mathbb{L}^{9}us^3}$
$\mathrm{I}^*_{k\ge1}$ $(j=\infty)$	$(2,1)$	$k+4$	$\mathbb{L}^8-\mathbb{L}^7$	$\dfrac{1}{1-(\mathbb{L}^8-\mathbb{L}^7)u^5s^7\,\Delta(s)^{-1}}$
I^*_0 $(j\neq 0,1728)$	$(2,1)$	4	$\mathbb{L}^8-\mathbb{L}^7$	$\dfrac{1}{1-(\mathbb{L}^8-\mathbb{L}^7)u^4s^6}$
I^*_0 $(j=0,1728)$	$(2,1)$	4	$\mathbb{L}^7$	$\dfrac{1}{1-\mathbb{L}^7u^4s^6}$
III^* $(j=1728)$	$(4,3)$	7	$\mathbb{L}^6$	$\dfrac{1}{1-\mathbb{L}^6u^7s^9}$

TABLE 2. Local factors for $\mathcal{P}(2,4)$.

3.2. **Level-3 structure.** We consider the height moduli

$$\mathcal{W}_n^{\Gamma_1(3)}\big(\mathcal{P}(1,3),\mathcal{O}(1)\big) \;=\; \mathcal{M}_{n,\mathbb{P}^1_k}\big(\overline{\mathcal{M}}_1(3),\mathcal{L}\big),$$

where $\overline{\mathcal{M}}_1(3) \cong \mathcal{P}(1,3)$ over $\mathbb{Z}\big[\frac{1}{3}\big]$ is the moduli stack of generalized elliptic curves with $\Gamma_1(3)$-structure (cf. [HM17, Prop. 4.5]). Equivalently, $\overline{\mathcal{M}}_1(3)$ admits the universal Weierstrass presentation

$$y^2 + a_1xy + a_3y = x^3 \qquad \text{with} \qquad (a_1,a_3) \in H^0\big(\mathbb{P}^1,\mathcal{O}(n)\big) \times H^0\big(\mathbb{P}^1,\mathcal{O}(3n)\big).$$

Reduction type Θ	(r,a)	m_v-1	$A_\Theta := \dfrac{\{\mathcal{W}_{n,\mathbb{P}^1}^{\Gamma_1(3),\Theta}\}}{\{\mathrm{PGL}_2\}\,\mathbb{L}^{4n-8}}$	$\mathbb{P}^1$-Euler factor in $Z_{\mathrm{Triv}}^{\Gamma_1(3)}(u;t)$
$\mathrm{I}_{k\geq 1}$ $(j=\infty)$	$(0,0)$	$k-1$	$\mathbb{L}^6$	$\dfrac{1}{1-\mathbb{L}^6 s\,\Delta(s)^{-1}}$
IV $(j=0)$	$(3,1)$	2	$\mathbb{L}^5$	$\dfrac{1}{1-\mathbb{L}^5 u^2 s^4}$
IV* $(j=0)$	$(3,2)$	6	$\mathbb{L}^4$	$\dfrac{1}{1-\mathbb{L}^4 u^6 s^8}$

TABLE 3. Local factors for $\mathcal{P}(1,3)$.

3.3. **Level-4 structure.** We consider the height moduli

$$\mathcal{W}_n^{\Gamma_1(4)}\big(\mathcal{P}(1,2),\mathcal{O}(1)\big) \;=\; \mathcal{M}_{n,\mathbb{P}^1_k}\big(\overline{\mathcal{M}}_1(4),\mathcal{L}\big),$$

where $\overline{\mathcal{M}}_1(4)\cong\mathcal{P}(1,2)$ over $\mathbb{Z}\big[\frac{1}{2}\big]$ is the moduli stack of generalized elliptic curves with $\Gamma_1(4)$-structure (cf. [Mei22, Ex. 2.1]). Equivalently, $\overline{\mathcal{M}}_1(4)$ admits the universal Weierstrass presentation

$$y^2+a_1xy+a_1a_2y = x^3+a_2x^2 \qquad \text{with} \qquad (a_1,a_2)\in H^0\big(\mathbb{P}^1,\mathcal{O}(n)\big)\times H^0\big(\mathbb{P}^1,\mathcal{O}(2n)\big).$$

Reduction type Θ	(r,a)	m_v-1	$A_\Theta := \dfrac{\{\mathcal{W}_{n,\mathbb{P}^1}^{\Gamma_1(4),\Theta}\}}{\{\mathrm{PGL}_2\}\,\mathbb{L}^{3n-6}}$	$\mathbb{P}^1$-Euler factor in $Z_{\mathrm{Triv}}^{\Gamma_1(4)}(u;t)$
$\mathrm{I}_{k\geq 1}$ $(j=\infty)$	$(0,0)$	$k-1$	$\mathbb{L}^4$	$\dfrac{1}{1-\mathbb{L}^4 s\,\Delta(s)^{-1}}$
I_0^* $(j=0)$	$(2,1)$	4	$\mathbb{L}^3$	$\dfrac{1}{1-\mathbb{L}^3 u^4 s^6}$

TABLE 4. Local factors for $\mathcal{P}(1,2)$.

4. IRRATIONALITY OF THE NÉRON–SEVERI AND MORDELL–WEIL SPECIALIZATIONS

The coefficient-wise rationality in s of $Z_{\mathrm{Triv}}(u;t)$ reflects the fact that the trivial lattice rank $T(S)$ is governed by *local* reduction data. Indeed, by Lemma 1.5 it depends only on the multiset of fiber component numbers m_v, hence is constant on each Kodaira stratum $\mathcal{W}_n^{\min,(\mathfrak{f})}$, in the finite constructible stratification of Proposition 2.1. This locality is exactly what makes the evaluation-map factorization and the power structure on $K_0(\mathrm{Stck}_k)$ applicable: unordered collections of local factors assemble into a *finite* Euler product.

In contrast, the Mordell–Weil rank $\mathrm{rk}(E/K)$ is not determined by the fiber configuration. Even on a fixed Kodaira stratum $\mathcal{W}_n^{\min,(\mathfrak{f})}$, the rank typically varies, reflecting genuinely *global* constraints rather than local reduction data. Since $T(S)$ is constant on $\mathcal{W}_n^{\min,(\mathfrak{f})}$, the Shioda–Tate formula (1) shows that variation of $\mathrm{rk}(E/K)$

is equivalent to variation of the Néron–Severi rank $\rho(S)$. Thus any refinement of the height series by $\mathrm{rk}(E/K)$, or equivalently by $\rho(S)$, necessarily detects global jump phenomena invisible to the local factor stratification used for Z_{Triv}.

One way to organize this global complexity is via Néron–Severi jump loci. Fix a fiber configuration $\mathfrak{f}$ and write $\mathrm{Triv}^{(\mathfrak{f})} \subset \mathrm{NS}(S_{\bar{k}})$ for the sublattice generated by the zero section, a fiber class, and the components of reducible fibers in the configuration $\mathfrak{f}$. Inside $\mathcal{W}_n^{\min,(\mathfrak{f})}$, imposing that $\mathrm{NS}(S_{\bar{k}})$ contain additional algebraic classes *independent of* $\mathrm{Triv}^{(\mathfrak{f})}$ (equivalently, that $\rho(S)$, hence $\mathrm{rk}(E/K)$, jump) is an algebraic condition. Over $\mathbb{C}$, these conditions are naturally modeled by Noether–Lefschetz (Hodge) loci for the variation of Hodge structure coming from the family of elliptic surfaces over $\mathcal{W}_n^{\min,(\mathfrak{f})}$, and the theorem of Cattani–Deligne–Kaplan [CDK95] shows that such loci are, in general, only a *countable union of closed algebraic subsets*. In particular, unlike the Kodaira stratification at fixed height n (which is finite by Proposition 2.1), refinements by Néron–Severi *lattice type* are not expected to admit a finite constructible stratification.

This suggests that the local-to-global factorization producing a finite Euler product for Z_{Triv} should structurally fail for the other two specializations. All three share the formal power series structure $[\![\ ]\!]$ in their weight variable; we conjecture that they differ in the rationality of the s-coefficients.

Conjecture 4.1. Let $k = \mathbb{C}$ and $K = \mathbb{C}(z)$. Then

$$Z_{\mathrm{MW}}(v;t) \in \widehat{K}_0(\mathrm{Stck}_{\mathbb{C}})(s)[\![v]\!] \smallsetminus K_0(\mathrm{Stck}_{\mathbb{C}})[\mathbb{L}^{-1}](s)[\![v]\!],$$

$$Z_{\mathrm{NS}}(w;t) \in \widehat{K}_0(\mathrm{Stck}_{\mathbb{C}})(s)[\![w]\!] \smallsetminus K_0(\mathrm{Stck}_{\mathbb{C}})[\mathbb{L}^{-1}](s)[\![w]\!],$$

where $\widehat{K}_0$ denotes the dimensional completion. That is, some coefficient of v^m (resp. w^m) is not a rational function of s with coefficients in $K_0(\mathrm{Stck}_{\mathbb{C}})[\mathbb{L}^{-1}]$: the Noether–Lefschetz jump loci force the completion $\widehat{K}_0$ and destroy coefficient-wise rationality in s.

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References

[Beh06] Mark Behrens. A modular description of the $K(2)$-local sphere at the prime 3. *Topology*, 45(2):343–402, 2006.

[BM96] K. Behrend and Yu. Manin. Stacks of stable maps and Gromov-Witten invariants. *Duke Math. J.*, 85(1):1–60, 1996.

[BPS22] Dori Bejleri, Jun-Yong Park, and Matthew Satriano. Height moduli on cyclotomic stacks and counting elliptic curves over function fields, 2022. arXiv:2210.04450.

[CDK95] Eduardo Cattani, Pierre Deligne, and Aroldo Kaplan. On the locus of Hodge classes. *J. Amer. Math. Soc.*, 8(2):483–506, 1995.

[CLL16] Antoine Chambert-Loir and François Loeser. Motivic height zeta functions. *Amer. J. Math.*, 138(1):1–59, 2016.

[Con07] Brian Conrad. Arithmetic moduli of generalized elliptic curves. *J. Inst. Math. Jussieu*, 6(2):209–278, 2007.

[DR73] P. Deligne and M. Rapoport. Les schémas de modules de courbes elliptiques. In *Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*, Lecture Notes in Math., Vol. 349, pages 143–316. Springer, Berlin, 1973.

[Eke25] Torsten Ekedahl. The Grothendieck Group of Algebraic Stacks. In *Perspectives on Four Decades of Algebraic Geometry, Volume 1*, volume 351 of *Progr. Math.*, pages 233–261. Birkhäuser/Springer, Cham, 2025.

[ESZB23] Jordan S. Ellenberg, Matthew Satriano, and David Zureick-Brown. Heights on stacks and a generalized Batyrev-Manin-Malle conjecture. *Forum Math. Sigma*, 11:Paper No. e14, 54, 2023.

[GZLMH13] S. M. Gusein-Zade, I. Luengo, and A. Melle-Hernández. On the pre-λ-ring structure on the Grothendieck ring of stacks and the power structures over it. *Bull. Lond. Math. Soc.*, 45(3):520–528, 2013.

[Her91] Stephan Herfurtner. Elliptic surfaces with four singular fibres. *Math. Ann.*, 291(2):319–342, 1991.

[HM17] Michael A. Hill and Lennart Meier. The C_2-spectrum $\mathrm{Tmf}_1(3)$ and its invertible modules. *Algebr. Geom. Topol.*, 17(4):1953–2011, 2017.

[HP19] Changho Han and Jun-Yong Park. Arithmetic of the moduli of semistable elliptic surfaces. *Math. Ann.*, 375(3-4):1745–1760, 2019.

[HP23] Changho Han and Jun-Yong Park. Enumerating odd-degree hyperelliptic curves and abelian surfaces over $\mathbb{P}^1$. *Math. Z.*, 304(1):Paper No. 5, 32, 2023.

[Kap00] Mikhail Kapranov. The elliptic curve in the S-duality theory and Eisenstein series for Kac–Moody groups. *MSRI Preprint*, 2000-006, 2000. arXiv:math/0001005.

[KM85] Nicholas M. Katz and Barry Mazur. *Arithmetic moduli of elliptic curves*, volume 108 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1985.

[Kod63] K. Kodaira. On compact analytic surfaces. II, III. *Ann. of Math. (2) 77 (1963), 563–626; ibid.*, 78:1–40, 1963.

[Mei22] Lennart Meier. Additive decompositions for rings of modular forms. *Doc. Math.*, 27:427–488, 2022.

[Mir89] Rick Miranda. *The basic theory of elliptic surfaces*. Dottorato di Ricerca in Matematica. [Doctorate in Mathematical Research]. ETS Editrice, Pisa, 1989.

[N6́4] André Néron. Modèles minimaux des variétés abéliennes sur les corps locaux et globaux. *Inst. Hautes Études Sci. Publ. Math.*, 21:128, 1964.

[PS21] Jun-Yong Park and Hunter Spink. Motive of the moduli stack of rational curves on a weighted projective stack. *Res. Math. Sci.*, 8(1):Paper No. 1, 8, 2021.

[PS25] Jun-Yong Park and Johannes Schmitt. Arithmetic geometry of the moduli stack of Weierstrass fibrations over $\mathbb{P}^1$. *Math. Z.*, 310(4):Paper No. 84, 33, 2025.

[Shi90] Tetsuji Shioda. On the Mordell-Weil lattices. *Comment. Math. Univ. St. Paul.*, 39(2):211–240, 1990.

[SS10] Matthias Schütt and Tetsuji Shioda. Elliptic surfaces. In *Alg. Geom. in East Asia—Seoul 2008*, volume 60 of *Adv. Stud. Pure Math.*, pages 51–160. Math. Soc. Japan, 2010.

[Tat75] J. Tate. Algorithm for determining the type of a singular fiber in an elliptic pencil. In *Modular functions of one variable, IV (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*, Lecture Notes in Math., Vol. 476, pages 33–52. Springer, Berlin, 1975.

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