

# **Totality of Rational points on Moduli stacks**

## **Counting Families of Varieties : Lecture 1**

June Park

The University of Sydney

KIAS-LFANT Winter School on Number Theory

# Rational Points on Projective Varieties over $\mathbb{Q}$

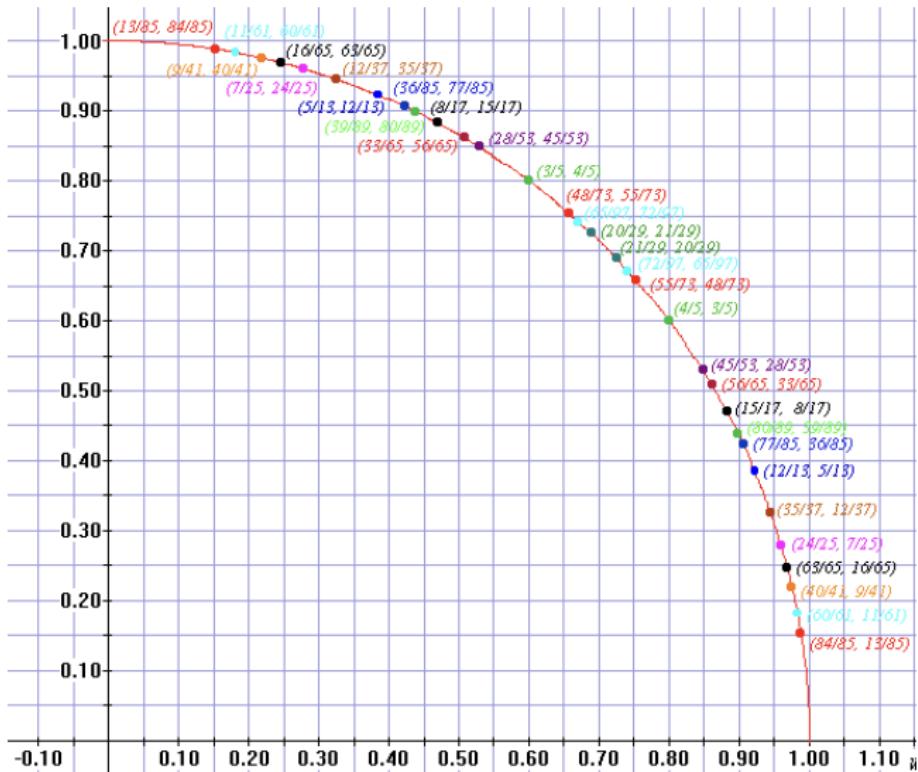


Figure 1: Rational points on  $x^2 + y^2 = 1$  over  $\mathbb{Q}$  - Pythagorean Triples

# Why should we be happy?

1. Height of a rational number  $a/b$  with  $\gcd(a, b) = 1$  is  $ht(a/b) = \max(|a|, |b|)$ . Therefore,  $ht(4/10) = 5$  and  $ht(1000000001/1000000000) = 1000000001 \neq 1$ . Bigger height allows more possibilities for numerator or denominator thus more rational points that are *aritmetically complex*.
2. Geometry is enlightening and the quadratic formula is awesome as we have found / parametrized all rational points  $(x, y) = \left(\frac{t^2-1}{t^2+1}, \frac{2t}{t^2+1}\right) \in \mathbb{Q}^2$  on the unit circle over  $\mathbb{Q}$
3. Integral points  $[X : Y : Z] = [a^2 - b^2 : 2ab : a^2 + b^2] \in \mathbb{Z}^3$  on  $C := V(X^2 + Y^2 - Z^2)$  correspond to “*Pythagorean Triples*”
4. On **projective varieties**, the integral and the rational points coincide i.e.,  $X(\mathbb{Q}) = X(\mathbb{Z})$ . Bear in mind  $\gcd(a, b) = 1$ .

# Why should we be unhappy?

1. If we don't have a rational point to begin the process then we cannot apply quadratic formula. For example,  $x^2 + y^2 = 3$ , it turns out  $X(\mathbb{Q}) = \emptyset$ . We need *arithmetic* (Fermat's Method of Infinite Descent) to prove this.
2. Take  $x^4 + y^4 = 1$  then we have "*Fermat's Last Theorem*" regarding  $x^n + y^n = 1$  with  $n = 4$ . By Wiles-Taylor, we **know** it has only 4 rational points  $X(\mathbb{Q}) = \{(\pm 1, 0), (0, \pm 1)\}$ . Recalling Mordell-Faltings, we **know** it had  $X(\mathbb{Q}) < \infty$
3. Take  $y^2 = x^3 + Ax + B$  this is 1 polynomial in 2 variables of degree 3 (the Weierstrass cubic for an elliptic curve over  $\mathbb{Q}$ ). **What are  $E(\mathbb{Q})$ ?** Shockingly, we *still cannot answer this*.
4. Actually, we know there is at least 1 rational point, the point at  $\infty = [0 : 1 : 0]$  for  $E : V(Y^2Z - X^3 - AXZ^2 - BZ^3)$

# Degree of countable infinity, the Rank

1. By Mordell-Weil, the set  $E(\mathbb{Q})$  of rational points on  $E/\mathbb{Q}$  has a finitely-generated abelian group structure  $E(\mathbb{Q}) = \mathbb{Z}^r \oplus T$  with algebraic rank  $r \in \mathbb{Z}_{\geq 0}$  and torsion subgroup  $T$
2. The rank  $r$  of  $E(\mathbb{Q})$  is **not** well understood.
  - 2.1 An algorithm that is guaranteed to correctly compute  $r$ ?
  - 2.2 Which values of  $r$  can occur? How often do they occur?
  - 2.3 Is there an upper limit, or can  $r$  be arbitrarily large?
3. When  $r$  is small, computational methods exist but when  $r$  is large, often the best we can do is a lower bound; we now know, there is an  $E/\mathbb{Q}$  with  $r \geq 29$  by Elkies-Klagsbrun (2024). Assuming GRH we can show that  $r = 29$ .

Rank 29

https://web.math.pmf.unizg.hr/~duje/tors/z1.html

Trivial torsion group, rank  $\geq 29$

Elkies - Klagsbrun (2024)

$$y^2 + xy = x^3 - 27006183241630922218434652145297453784768054621836357954737385x + 55258058551342376475736699591118191821521067032535079608372408779149413277716173425636721497$$

Independent points of infinite order:

- $P_1 = [2891195474228537189458255536634, 1159930748896124706459835910727318679593425283]$   
 $P_2 = [340254216532212781145148462234, 1661508223164691055862657623730465560755290883]$   
 $P_3 = [4298760026558467240422107564794, 4313142249890236204790986787384987722927474563]$   
 $P_4 = [372875666770947009884455714554, 2530180219584734091116528693531660545668397443]$   
 $P_5 = [5991744132052078230511185130234, 104189016228842034362301273055728300669218858883]$   
 $P_6 = [3236493534632768520540227223034, 1324626796262167243658687198416201825373745283]$   
 $P_7 = [78226686134991174232380689386234, 698394210862759896503429654125516779999512554883]$   
 $P_8 = [11492605643548859374635605140234, 355363169114508952155461624238308456029618940883]$   
 $P_9 = [-51433033623842980496088118566, 7622356511107896864120352355674305680222368483]$   
 $P_{10} = [443985655575065435281568435002, 658446812438858623214803939643557365635620355]$   
 $P_{11} = [-979565018904269688752629749766, 8987348422105376894667064387140386338321708883]$   
 $P_{12} = [51849428521217824956461261834, 739853678800315020127328446469585987505480483]$   
 $P_{13} = [-4469171023687146502067179612166, 9310658892841458934133221137392081403414455683]$   
 $P_{14} = [360640583310925480522978234, 2183644666981097363248266219339048800401278998883]$   
 $P_{15} = [16151744576785311732688993162234, 61988882092472338946519909276455831463747210883]$   
 $P_{16} = [35736843559437663878962362869754, 2094467155115749424853047283659077805560259203]$   
 $P_{17} = [-759376049938858166436491644166, 8679171135458197195914024161800061810952119683]$   
 $P_{18} = [-53280587199386182106003119366, 692058814737949763320293557367499676224350083]$   
 $P_{19} = [5388268474895377355583039694554, 81056602400308025892450118297303424395868037443]$   
 $P_{20} = [17069233487425098808940203248484, 67583677227299213867443585411893525786510633]$   
 $P_{21} = [5215432542403430758248050783794, 7501515746204716855921710958364078294243814643]$   
 $P_{22} = [283894217804624039763692432122, 1212346288964590308944175880544505700180280003]$   
 $P_{23} = [243146882395382015946366404808154/81, 811625272160726332199288136187427505366582108107/729]$   
 $P_{24} = [2558229016839511149831260088762, 170659839583087994387505244133328709649637123]$   
 $P_{25} = [2361253942905600810977556672634, 215750339624355244879851089310708763298760883]$   
 $P_{26} = [2678312077644931683114439986234, 1462722361029796436741527433473386115047618883]$   
 $P_{27} = [3379397084927230910084852603902, 1608494167359575995485655188349208450365853755]$   
 $P_{28} = [363240773087098917912491355514, 225565493703770081978158381185619053396712963]$   
 $P_{29} = [2428778263277521959543043930234, 19983250236106366161737305486867803334410883]$

Previous record with [rank  \$\geq 28\$](#)

# Demography of Elliptic Curves $E/\mathbb{Q}$

Trying to find / parametrize all the rational points on a given  $E/\mathbb{Q}$  is a dead-end. Thus we rotate our entry. We would like to think about *the Question of Distribution and Proportion* over all  $E/\mathbb{Q}$

Naive height for  $E : y^2 = x^3 + Ax + B$  with no  $p^4|A$  and  $p^6|B$  (minimal Weierstrass model) is  $ht(E) := \max(4|A|^3, 27B^2)$ .

## Conjecture (Minimality + Parity; Goldfeld and Katz-Sarnak)

Over any number field, 50% of all elliptic curves (when ordered by height) have Mordell-Weil rank  $r = 0$  and the other 50% have Mordell-Weil rank  $r = 1$ . Moreover, higher Mordell-Weil ranks  $r \geq 2$  constitute 0% of all elliptic curves, even though there may exist infinitely many such elliptic curves. Therefore, a suitably-defined average rank would be  $\frac{1}{2}$ .

What does this really mean? To talk about Average, we need the **Total number of elliptic curves over  $\mathbb{Q}$  up to isomorphism**.

# Triangle of Rational Dedekind Domains

Consider not only  $E/\mathbb{Q}$  but also  $E/\mathbb{F}_q(t)$  as well as  $E/\mathbb{C}(z)$

1. The rational number field  $\mathbb{Q}$  consisting of ratio of integer numbers in  $\mathbb{Z}$  is **the rational global field of char = 0**
2. The rational function field  $\mathbb{F}_q(t)$  with *coefficients* in  $\mathbb{F}_q = \mathbb{F}_{p^r}$  consisting of ratio of polynomial functions in  $\mathbb{F}_q[t]$  is **the rational global field of char =  $p > 0 \Leftrightarrow$  Projective line  $\mathbb{P}_{\mathbb{F}_q}^1$**
3. The meromorphic function field  $\mathbb{C}(z)$  with *coefficients* in  $\mathbb{C}$  consisting of ratio of holomorphic functions in  $\mathbb{C}[z]$  is **NOT** the rational global field of char = 0  $\Leftrightarrow$  Riemann sphere  $\mathbb{CP}^1$

Let us count ALL elliptic curves over  $K = \mathbb{F}_q(t)$  wrt height.

$$r = \frac{1}{2}, z = 4$$

○

-

$$\begin{array}{c} b_0 = 1 \\ b_2 = 1 \\ c = 2 \end{array}$$

$$\frac{p=0 \text{ at } r=\infty}{[u:v]} \quad r = u/v \quad \left| P/(P_2) \right| = q+1$$

$$GP^1 \xrightarrow{G.B.F} S^2 \xrightarrow{G.L.T.F} P^1$$

$$\begin{array}{c} \text{Chow} \quad \quad \quad \text{W.C.} \quad \quad \quad \text{P}_q^1 \\ \swarrow \quad \quad \quad \downarrow \quad \quad \quad \searrow \\ H_0/\bar{H}_0 \quad \quad \quad S^2 \dashrightarrow \overline{P_q}(T) \quad \quad \quad \overline{P_q}(T) - \text{poly/poly} \end{array}$$

$$\begin{array}{c} \text{Adic} \quad AT_k \quad AB/E \\ KT \\ SV \quad CA \end{array}$$

NT

$$Q \xrightarrow{\quad \quad G.F.A. \quad \quad}$$

$$-z^p/z^p$$

$\mathbb{Z} \sim \overline{P_q}(T)$  'As integers so polynomials'  
 let  $\mathcal{E}$  be a suitable cat. of sheaves  
 plan  $\mathcal{E}(\text{spec } T) \sim \mathcal{E}(P_{\overline{E}} \setminus S)$   
 anal.  $\mathcal{E}$   
 'Aware of each others'

# Elliptic curve over a function field

Let  $K$  be the function field of a smooth, projective, absolutely irreducible curve  $C$  over the field of constants  $k$ . An elliptic curve over  $K$  is a smooth, projective, absolutely irreducible curve of genus 1 over  $K$  equipped with a  $K$ -rational point  $O$  (the origin).

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$$

## Definition (Constant, Isotrivial and Non-isotrivial)

Let  $E$  be an elliptic curve over  $K = k(C)$ .

- ▶ We say  $E$  is *constant* if there is an elliptic curve  $E_0$  defined over  $k$  such that  $E \cong E_0 \times_k K$ . Equivalently,  $E$  is constant if it can be defined by a Weierstrass cubic where the  $a_i \in k$ .
- ▶ We say  $E$  is *isotrivial* if there exists a finite extension  $K'$  of  $K$  such that  $E$  becomes constant over  $K'$ . A constant curve is isotrivial. Equivalently,  $E$  is isotrivial if and only if  $j(E) \in k$ .
- ▶ We say  $E$  is *non-isotrivial* if it is not isotrivial. We say  $E$  is *non-constant* if it is not constant.

# Projective Elliptic K3 Surface of height $n = 2$

$$y^2 = x^3 + a_4(u:v)x + a_6(u:v)$$

Weierstrass data for elliptic fibration on algebraic K3 surface,

$$\begin{cases} a_4(u:v) &= -3u^4v^4, \\ a_6(u:v) &= u^5v^5(u^2 + v^2). \end{cases}$$

Then we have  $\Delta = 4a_4^3 + 27a_6^2$  and  $j = 1728 \cdot 4a_4^3/\Delta$

$$\begin{cases} \Delta &= 27u^{10}v^{10}(u-v)^2(u+v)^2, \\ j &= 1728 \cdot -\frac{4u^2v^2}{(u-v)^2(u+v)^2}. \end{cases}$$

Wait, the degree of  $j$ -map is 4 and NOT 24. Where did 20 go?

After all, we should have  $j : \mathbb{P}^1 \rightarrow \overline{M}_{1,1} \cong \mathbb{P}^1$  of degree 24?

Well, it can get whole lot worse.

## Isotrivial Rational Elliptic Surface of height $n = 1$

$$\text{Frictional Radial Sipple Surface} \quad n = d + \sum_{i=1}^k a_i x_i$$

$$n=7 = 116 + 516$$

$$\begin{cases} \alpha_4 = 0 \\ \alpha_6 = u \cdot v^5 \end{cases}$$

$$\left\{ \begin{array}{l} d=0 \\ a_{1/k_1}=1/6, \quad a_{2/k_2}=5/6 \end{array} \right.$$

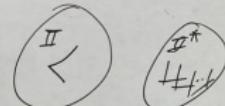
$$V(a_f) = \infty \quad \text{minimal}$$

$$[u:v] \quad u/v = f$$

$$v(\alpha_6) = \begin{cases} 2 < 6 \text{ af } u=0 \Rightarrow v=1 & [0:1] \Leftrightarrow t=0 \\ 5 < 6 \text{ af } v=0 \Rightarrow u=1 & [1:0] \Leftrightarrow t=\infty \end{cases}$$

$$A = 27 u^2 v^{10} - \deg 12$$

$$j \equiv 0$$



$$y^2 = x^3 + uv^5 \in S_{161 \cdot 4} \cdot \frac{q^5 - 1}{q^5 + q^4} B^{12}$$

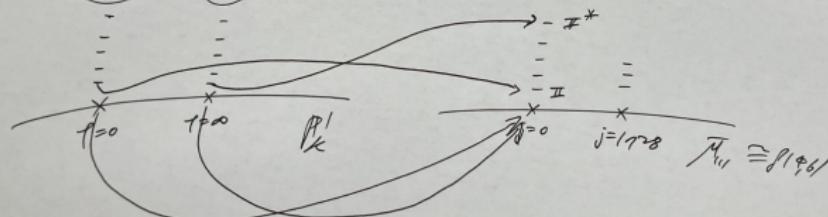
$$\downarrow \quad u = z^6 \quad \quad \quad v = q^{12}$$

$$y^2 = x^3 + 12x^2 + 25$$

$$\downarrow v = w^6$$

$$y^2 = x^3 + \frac{105}{6}$$

$$y^2 = x^3 + 1 \quad \downarrow \quad \because x^{(1)} = 5$$



# The Sharp Enumeration over Rational Function Field

Define *height of discriminant*  $\Delta$  over  $\mathbb{F}_q(t)$  as  $ht(\Delta) := q^{\deg \Delta}$

- ▶ Elliptic case:  $Deg(\Delta) = 12n \implies ht(\Delta) = q^{12n}$  for  $n \in \mathbb{Z}_{\geq 0}$

We consider the counting function  $\mathcal{N}(\mathbb{F}_q(t), B) :=$

$$\left| \left\{ \text{Minimal elliptic curves over } \mathbb{P}_{\mathbb{F}_q}^1 \text{ with } 0 < ht(\Delta) \leq B \right\} \right|$$

**Theorem (Dori Bejleri–JP–Matthew Satriano; April 2024)**

Let  $\text{char}(\mathbb{F}_q) > 3$  and  $\delta(x) := \begin{cases} 1 & \text{if } x \text{ divides } q - 1, \\ 0 & \text{otherwise.} \end{cases}$ , then

$$\begin{aligned} \mathcal{N}(\mathbb{F}_q(t), B) &= 2 \left( \frac{q^9 - 1}{q^8 - q^7} \right) B^{5/6} - 2B^{1/6} \\ &\quad + \delta(6) \cdot 4 \left( \frac{q^5 - 1}{q^5 - q^4} \right) B^{1/2} + \delta(4) \cdot 2 \left( \frac{q^3 - 1}{q^3 - q^2} \right) B^{1/3} \\ &\quad + \delta(6) \cdot 4 + \delta(4) \cdot 2 \end{aligned}$$

## Precise proportions of $E/K$ motivated by NT

### Theorem (Generic Torsion Freeness; Phillips)

*The set of torsion-free elliptic curves over global function fields has density 1. i.e., 'Most elliptic curves over  $K$  are torsion free'.*

### Theorem (Boundedness; Tate-Shafarevich & Ulmer)

*The ranks of non-constant elliptic curves over  $\mathbb{F}_q(t)$  are unbounded (in both the **isotrivial** and the **non-isotrivial** cases).*

## Ulmer's non-isotrivial elliptic curve of infinite rank

1. Start with  $y^2 + xy = x^3 - t^d$ , then *complete the square* via  $y = y' - \frac{x}{2}$  and then *complete the cubic* via  $x = x' - \frac{1}{12}$ . We need  $\text{char}(k) \neq 2, 3$  to get to the short Weierstrass form.
2. We get  $y^2 = x^3 - \frac{1}{48}x + \frac{1}{864} - t^d$ . Coefficients should be integral thus we take  $\lambda = 2 \cdot 3$  to multiply  $\lambda^4$  to  $-\frac{1}{48}$  and  $\lambda^6$  to  $+\frac{1}{864} - t^d$ .
3. We arrive at  $y^2 = x^3 - 27x + 54 - 2^6 \cdot 3^6 \cdot t^d$  thus  $[-\frac{1}{48} : \frac{1}{864} - t^d] = [-27 : -2^6 \cdot 3^6 \cdot t^d]$ .
4. Remember the isomorphism, for any  $\lambda \in \mathbb{G}_m$

$$[y^2 = x^3 + Ax + B] \cong [y^2 = x^3 + \lambda^4 \cdot Ax + \lambda^6 \cdot B]$$

via  $x \mapsto \lambda^{-2} \cdot x$  and  $y \mapsto \lambda^{-3} \cdot y$  by the *Weighted homogeneous coordinate* of  $\mathcal{P}(4, 6)$ .

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```
KK<t> := FunctionField(GF(4007));
E := EllipticCurve([-27, 54 - 2^6*3^6*t^11]);
E;
&*BadPlaces(E);
LocalInformation(E);
```

Cancel Submit

```
Elliptic Curve defined by  $y^2 = x^3 + 3980*x + (1428*t^{11} + 54)$  over Univariate rational function field over GF(4007)
 $t^{11} + 1549$ 
[ <(t^5 + 3335*t^4 + 2186*t^3 + 488*t^2 + 2393*t + 906), 1, 1, 1, I1, false>,
  <(t^5 + 3337*t^4 + 2186*t^3 + 488*t^2 + 3369*t + 906), 1, 1, 1, I1, false>,
  <(t), 11, 1, 11, I11, true>, <(1/t), 2, 2, 1, II, true>, <(t + 1342), 1, 1, 1, I1, false> ]
```

- The corresponding elliptic surface has a fiber of Kodaira type  $I_d$  at zero (at  $t = 0$ ), while the fiber at infinity (at  $1/t = 0$ ) is given by the congruence class  $\bar{d}$  of  $d$  modulo 6 :  $(\bar{d}, \Theta)$   
 $(\bar{0}, I_0)$   $(\bar{1}, II^*)$   $(\bar{2}, IV^*)$   $(\bar{3}, I_0^*)$   $(\bar{4}, IV)$   $(\bar{5}, II)$
- Outside char 2, 3, there are  $d$  fibres of type  $I_1$  at the zeroes of  $432t^d - 1$  (some of which may be merged if  $\text{char}(k)|d$ ).

The aim of this paper is to produce elliptic curves over  $K = \mathbb{F}_p(t)$  which are nonisotrivial ( $j \notin \mathbb{F}_p$ ) and which have arbitrarily large rank.

**THEOREM 1.5.** *Let  $p$  be an arbitrary prime number,  $\mathbb{F}_p$  the field of  $p$  elements, and  $\mathbb{F}_p(t)$  the rational function field in one variable over  $\mathbb{F}_p$ . Let  $E$  be the elliptic curve defined over  $K = \mathbb{F}_p(t)$  by the Weierstrass equation*

$$y^2 + xy = x^3 - t^d$$

*where  $d = p^n + 1$  and  $n$  is a positive integer. Then  $j(E) \notin \mathbb{F}_p$ , the conjecture of Birch and Swinnerton-Dyer holds for  $E$  over  $K$ , and the rank of  $E(K)$  is at least  $(p^n - 1)/2n$ .*

By the Shioda-Tate formula and assuming maximal Picard number of  $\rho = 10n$  for Faltings height  $n$  (while  $b_2 = 12n - 2$ ), we know that  $r = 10n - rk(T)$  where  $T$  is the trivial lattice. Ulmer's proof shows that as the height of Ulmer's curve goes up as  $n = 1 + \lfloor \frac{d-1}{6} \rfloor \rightarrow \infty$ , the algebraic/analytic rank  $r$  goes up to  $\infty$ .

## Sketch of Ulmer's proof

1. Construct an elliptic surface  $S \rightarrow \mathbb{P}^1$  over  $\mathbb{F}_p$  with generic fiber  $E : y^2 + xy = x^3 - t^d$  for  $d = p^n + 1$  and  $n \in \mathbb{Z}_+$ .
2. Construct (and carefully study) a birational isomorphism between  $S$  and  $F_d/G$ , the quotient of a Fermat surface i.e.  $V(x^d + y^d + z^d + w^d) \subset \mathbb{P}^3$  ( $d = 4$  then it is K3 surface).
3. Using the fact that the Tate conjecture for surfaces is known for Fermat surfaces, one can deduce the Tate conjecture for  $S$ .
4. Use the fact that the Tate conjecture for  $S$  implies the Birch and Swinnerton-Dyer conjecture for  $E$ . Thus the ranks of the elliptic curves in the family all equal their analytic ranks.
5. The analytic ranks can be computed by relating the L-function of  $E$  to the zeta function of  $S$ , which can be related to the zeta function of  $F_d$ , which is known by Gauss sum computation of Weil. From this one is able to compute the analytic rank which is unbounded from below.

# Precise proportions of $E/K$ motivated by NT

We consider the counting function  $\mathcal{N}_T^r(\mathbb{F}_q(t), B) :=$

$|\{\text{Minimal } E/\mathbb{F}_q(t) \text{ with algebraic rank } r, \text{torsion } T \text{ and } ht(\Delta) \leq B\}|$

## Quantitative Rank Distribution Conjecture over $K = \mathbb{F}_q(t)$

$$\mathcal{N}_{T=0}^{r=0}(\mathbb{F}_q(t), B) = \left( \frac{q^9 - 1}{q^8 - q^7} \right) B^{5/6} + o(B^{5/6}),$$

$$\mathcal{N}_{T=0}^{r=1}(\mathbb{F}_q(t), B) = \left( \frac{q^9 - 1}{q^8 - q^7} \right) B^{5/6} + o(B^{5/6}),$$

$$\mathcal{N}_{T=0}^{r \geq 2}(\mathbb{F}_q(t), B) = o(B^{5/6}), \text{ where all o are little-o.}$$

†  $|E(K)| = 1$  and  $E(K) = \mathbb{Z}$  each corresponds to 50% of all elliptic curves over  $K$  ordered by discriminant height having *equal* main leading term  $B^{5/6}$  with *identical* leading coefficient  $\left( \frac{q^9 - 1}{q^8 - q^7} \right)$ .

Furthermore, the exact counting formulas for  $\mathcal{N}_{T=0}^{r=0}(\mathbb{F}_q(t), B)$  and  $\mathcal{N}_{T=0}^{r=1}(\mathbb{F}_q(t), B)$  do not coincide since the respective counting functions have **distinct lower-order main terms**.

# **Totality of Rational points on Moduli stacks**

## **Counting Families of Varieties : Lecture 2**

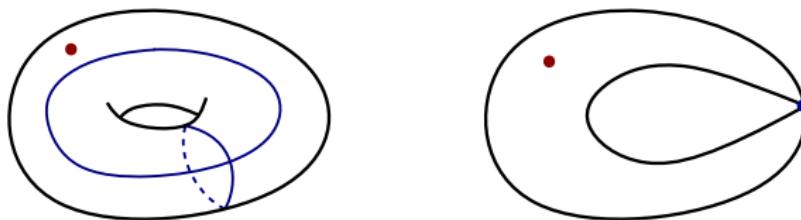
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# Deligne–Mumford stack $\overline{\mathcal{M}}_{1,1}$ of stable elliptic curves

Fine moduli stack  $\overline{\mathcal{M}}_{1,1}$  parametrizes isomorphism classes  $[E]$  of stable elliptic curves with the coarse moduli space  $\overline{\mathcal{M}}_{1,1} \cong \mathbb{P}^1$  parametrizing the  $j$ -invariant  $j([E]) = 1728 \cdot 4a_4^3 / (4a_4^3 + 27a_6^2)$



When the characteristic of the field  $k$  is not equal to 2 or 3,  
 $(\overline{\mathcal{M}}_{1,1})_k \cong [(Spec \ k[a_4, a_6] - (0, 0)) / \mathbb{G}_m] =: \mathcal{P}_k(4, 6)$  through the short Weierstrass equation:  $y^2 = x^3 + a_4x + a_6$

Stabilizers are the orbifold points  $[1 : 0]$  &  $[0 : 1]$  with  $\mu_4$  &  $\mu_6$  respectively and the generic stacky points such as  $[1 : 1]$  with  $\mu_2$

The fine moduli stack  $\overline{\mathcal{M}}_{1,1}$  comes equipped with the universal family  $p : \overline{\mathcal{E}}_{1,1} \rightarrow \overline{\mathcal{M}}_{1,1}$  of stable elliptic curves.

## Boundary Divisor $\overline{\mathcal{M}}_{1,1} \setminus \mathcal{M}_{1,1} = [\infty]$ for I<sub>1</sub> nodal fiber

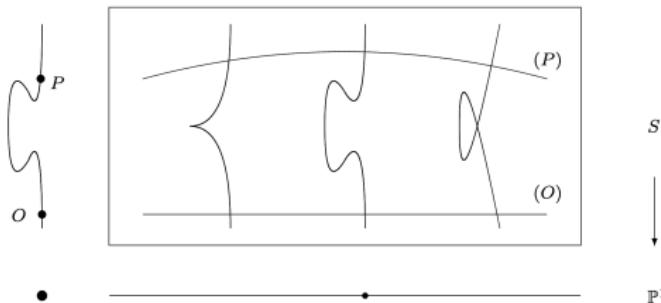
1. Take the nodal curve  $y^2 = x^3 + x^2$ , then *complete the cubic* via  $x = x' - \frac{1}{3}$ . This is why we require  $\text{char}(k) \neq 2, 3$ .
2. We get  $y^2 = x^3 - \frac{1}{3}x + \frac{2}{27}$ . Coefficients should be integral thus we take  $\lambda = 3$  to multiply  $\lambda^4$  to  $-\frac{1}{3}$  and  $\lambda^6$  to  $+\frac{2}{27}$ . Notice here *weighted homogeneous coordinate* of  $\mathcal{P}(4, 6)$ .
3. We arrive at  $y^2 = x^3 - 27x + 54$  thus  $[-\frac{1}{3} : \frac{2}{27}] = [-27 : 54]$ . Curve is singular  $\Delta = 4(-27)^3 + 27(54)^2 = 0$  thus  $j = \infty$ . Written as I<sub>1</sub> multiplicative reduction in Kodaira notation.
4. Remember the isomorphism, for any  $\lambda \in \mathbb{G}_m$

$$[y^2 = x^3 + Ax + B] \cong [y^2 = x^3 + \lambda^4 \cdot Ax + \lambda^6 \cdot B]$$

via  $x \mapsto \lambda^{-2} \cdot x$  and  $y \mapsto \lambda^{-3} \cdot y$ .

# Elliptic surfaces / $k$ = Families of elliptic curves / $K$

The study of **fibrations of algebraic curves** lies at the heart of the Enriques-Kodaira classification of algebraic surfaces.



We call an algebraic surface  $S$  to be an **elliptic surface**, if it admits an elliptic fibration  $f : S \rightarrow C$  which is a flat proper morphism  $f$  from a nonsingular surface  $S$  to a nonsingular curve  $C$ , such that a generic fiber is a smooth curve of genus 1.

While this is the most general setup, it is natural to work with the case when the base curve is the smooth projective line  $\mathbb{P}^1$  and there exists a section  $O : \mathbb{P}^1 \hookrightarrow S$  coming from the identity points of the elliptic fibres and not passing through the singular points.

# Moduli stack of stable elliptic fibrations

Thus, a stable elliptic fibration  $g : Y \rightarrow \mathbb{P}^1$  is induced by a morphism  $\varphi_f : \mathbb{P}^1 \rightarrow \overline{\mathcal{M}}_{1,1}$  and vice versa.

$$\begin{array}{ccccc} X & \xrightarrow{\nu} & Y = \varphi_f^*(\overline{\mathcal{E}}_{1,1}) & \longrightarrow & \overline{\mathcal{E}}_{1,1} \\ \downarrow f & & \downarrow g & & \downarrow p \\ \mathbb{P}^1 & \xlongequal{\quad} & \mathbb{P}^1 & \xrightarrow{\varphi_f} & \overline{\mathcal{M}}_{1,1} \end{array}$$

$X$  is the non-singular semistable elliptic surface;  $Y$  is the stable elliptic fibration;  $\nu : X \rightarrow Y$  is the minimal resolution.

The moduli stack  $\mathcal{L}_{12n}$  of stable elliptic fibrations over the  $\mathbb{P}^1$  with  $12n$  nodal singular fibers and a marked section is the Hom stack  $\text{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$  where  $\varphi_f^* \mathcal{O}_{\overline{\mathcal{M}}_{1,1}}(1) \cong \mathcal{O}_{\mathbb{P}^1}(n)$ .

A morphism  $\varphi_f : \mathbb{P}^1 \rightarrow \overline{\mathcal{M}}_{1,1}$  consists of global sections (homogeneous polynomials in  $[u : v]$ )  $\varphi_f = (a_4(u, v), a_6(u, v))$  where  $\deg(a_4) = 4n$  and  $\deg(a_6) = 6n$  (!) and  $\text{Res}(a_4, a_6) \neq 0$ .

# Grothendieck ring $K_0(\mathrm{Stck}_k)$ of $k$ -algebraic stacks

Ekedahl in 2009 introduced the Grothendieck ring  $K_0(\mathrm{Stck}_k)$  of algebraic stacks extending the classical Grothendieck ring  $K_0(\mathrm{Var}_k)$  of varieties first defined by Grothendieck in 1964.

## Definition

Fix a field  $k$ . Then the *Grothendieck ring  $K_0(\mathrm{Stck}_k)$  of algebraic stacks of finite type over  $k$  all of whose stabilizer group schemes are affine* is an abelian group generated by isomorphism classes of algebraic stacks  $\{\mathcal{X}\}$  modulo relations:

- ▶  $\{\mathcal{X}\} = \{\mathcal{Z}\} + \{\mathcal{X} \setminus \mathcal{Z}\}$  for  $\mathcal{Z} \subset \mathcal{X}$  a closed substack,
- ▶  $\{\mathcal{E}\} = \{\mathcal{X} \times \mathbb{A}^n\}$  for  $\mathcal{E}$  a vector bundle of rank  $n$  on  $\mathcal{X}$ .

Multiplication on  $K_0(\mathrm{Stck}_k)$  is induced by  $\{\mathcal{X}\}\{\mathcal{Y}\} := \{\mathcal{X} \times \mathcal{Y}\}$ . A distinguished element  $\mathbb{L} := \{\mathbb{A}^1\}$  is called the *Lefschetz motive*.

$$\{\mathbb{P}^1\} = \mathbb{L} + 1, \quad \{\mathbb{P}^N\} = \mathbb{L}^N + \dots + 1, \quad \{\mathbb{G}_m\} = \mathbb{L} - 1, \quad \{E\} = ?$$

# Universal Property for Additive Invariants

For any ring  $R$  and any function  $\tilde{\nu} : \text{Stck}_k \rightarrow R$  satisfying relations

- 1)  $\tilde{\nu}(\mathcal{X}) = \tilde{\nu}(\mathcal{Y})$  whenever  $\mathcal{X} \cong \mathcal{Y}$ ,
- 2)  $\tilde{\nu}(\mathcal{X}) = \tilde{\nu}(\mathcal{U}) + \tilde{\nu}(\mathcal{X} \setminus \mathcal{U})$  for  $\mathcal{U} \hookrightarrow \mathcal{X}$  an open immersion,
- 2)  $\tilde{\nu}(\mathcal{X} \times \mathcal{Y}) = \tilde{\nu}(\mathcal{X}) \cdot \tilde{\nu}(\mathcal{Y})$ ,

there is a unique ring homomorphism  $\nu : K_0(\text{Stck}_k) \rightarrow R$

$$\begin{array}{ccc} & \text{Stck}_k & \\ \{ \} \swarrow & & \searrow \tilde{\nu} \\ K_0(\text{Stck}_k) & \xrightarrow{\nu} & R \end{array}$$

Such homomorphisms  $\nu$  are called **motivic measures**.

$\therefore$  When  $k = \mathbb{F}_q$ , the point counting measure  $\{\mathcal{X}\} \mapsto \#_q(\mathcal{X})$  is a well-defined ring homomorphism  $\#_q : K_0(\text{Stck}_{\mathbb{F}_q}) \rightarrow \mathbb{Q}$  giving the weighted point count  $\#_q(\mathcal{X})$  of  $\mathcal{X}$  over  $\mathbb{F}_q$ .

$$|\mathbb{P}^N(\mathbb{F}_q)| = q^N + \dots + 1, \quad q + 1 - 2\sqrt{q} \leq |E(\mathbb{F}_q)| \leq q + 1 + 2\sqrt{q}$$

# Accessing Cruder Level of Topology via Motives

A priori, point counts over  $\mathbb{F}_q$  shouldn't know any topology.

In  $\mathbb{A}^2_k$ , cusp singular fiber II and affine line  $\mathbb{A}^1$  have the same point counts (motives) i.e.  $\{\text{II} = V(y^2 = x^3)\} = \mathbb{L} = \{\mathbb{A}^1 = V(x)\}$  but they have very different *topology*.

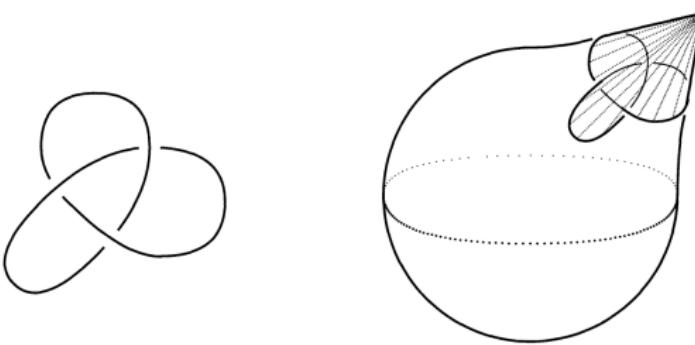
Same motive since we have a stratification of  $\text{II} = X_1 \cup X_2$  where  $X_1 = \text{II} - \{pt\}$  and  $X_2 = \{pt\}$  and  $\mathbb{A}^1 = Y_1 \cup Y_2$  where  $Y_1 = \mathbb{A}^1 - \{pt\}$  and  $Y_2 = \{pt\}$ .

Indeed,  $X_1 \cong Y_1$  (smooth complement) and  $X_2 \cong Y_2$  (a singular point is just like a smooth point as  $\text{Spec}(k)$ ) i.e. they are *cut-and-paste equivalent* and naturally  $\{\text{II}\} = \{\mathbb{A}^1\} = \mathbb{L}$

Same for nodal cubic  $\{\text{I}_1 = V(y^2 = x^3 + x^2)\} = \mathbb{L}$

Different topology since,  $\text{II}$  and  $\text{I}_1$  have arithmetic genus 1 (they are singular elliptic curves) whereas  $\mathbb{A}^1$  has arithmetic genus 0

Singular point on  $\text{II}$  is the tip of a cone over the trefoil knot whereas singular point on  $\text{I}_1$  is the tip of a cone over the Hopf link. (Every isolated singularity of a complex curve in a complex surface can be described topologically as the tip of a cone on a link)



8.7. Trefoil knot, and cusp fiber

**Miracle:** When a variety is smooth projective then its point count over  $\mathbb{F}_q$  knows topology via Frobenius weights and étale purity (the finite field analogue of RH) through the Grothendieck-Lefschetz trace formula under the Weil conjecture framework.

Thinking the other way around, this suggests that we can ignore finer topology if we are just interested in the arithmetic invariant.

### Corollary (Dori Bejleri–JP–Matthew Satriano; April 2024)

The disjoint union of  $\psi_{n,e}$

$$\psi_n : \bigsqcup_{e=0}^n \mathcal{W}_{n-e}^{\min} \times \mathbb{P}(V_e^1) \rightarrow \mathcal{P} \left( \bigoplus_{i=0}^N V_n^{\lambda_j} \right)$$

is an isomorphism after stratifying the source and target.

If we want to point count  $X$  one way to do it is to find a stratification of  $Y$  (where we know  $\{X\} = \{Y\}$  even though  $X \not\cong Y$ ) into disjoint union of locally-closed subvarieties where we can compute its motivic classes and add them up. That is, utilize *cut-and-paste property* by stratifying source  $X$  and target  $Y$ .

Grothendieck ring  $K_0(\text{Stck}_k)$  of  $k$ -algebraic stacks allows us to this procedure motivically (free of particular choice of ground field  $k$  and also free of choice of additive invariant on  $\text{Var}_k$  or  $\text{Stck}_k$ )

## V. Arnol'd, J. Milnor, M. Atiyah, G. Segal

1. Hom space  $\text{Hom}_n(\mathbb{P}^1_D, \mathbb{P}^1_T)$  is the moduli space of morphisms  $f : \mathbb{P}^1_D \rightarrow \mathbb{P}^1_T$  of degree  $n$  as  $f^*\mathcal{O}_{\mathbb{P}^1_T}(1) \cong L_{\mathbb{P}^1_D} \cong \mathcal{O}_{\mathbb{P}^1_D}(n)$ .
2. A morphism  $f : \mathbb{P}^1_D \rightarrow \mathbb{P}^1_T$  consists of global sections (global homogeneous polynomials)  $f = (s_0(u:v), s_1(u:v))$  where  $\deg(s_0) = \deg(s_1) = n$  and are coprime i.e.  $\text{Res}(s_0, s_1) \neq 0$ .
3. Consider  $f = (-27u^{12}v^{12}, 27u^{14}v^{10} - 54u^{12}v^{12} + 27u^{10}v^{14})$  is a **degree 4** morphism as the common factor is  $27u^{10}v^{10}$
4. The rational maps and the morphisms coincide i.e.  
 $f : \mathbb{P}^1_D \dashrightarrow \mathbb{P}^1_T = f : \mathbb{P}^1_D \rightarrow \mathbb{P}^1_T$  ( $\mathbb{P}^1_D$  smooth  $\mathbb{P}^1_T$  projective)  
**after cancellation of common factors** i.e.  $\text{gcd}(s_0, s_1) = 1$
5.  $\mathbb{P}^1_T(k(t))_n = \mathbb{P}^1_T(k[t])_n$  for  $\mathbb{P}^1_D$  with function field  $k(t)$  and ring of integers  $\mathcal{O}_{k(t)} = k[t] \sim \mathbb{P}^1_T(\mathbb{Q})_{ht(a/b)} = \mathbb{P}^1_T(\mathbb{Z})_{ht(a/b)}$

# Projective Elliptic K3 Surface of height $n = 2$

$$y^2 = x^3 + a_4(u:v)x + a_6(u:v)$$

Weierstrass data for elliptic fibration on algebraic K3 surface,

$$\begin{cases} a_4(u:v) = -3u^4v^4, \text{ degree } 8 = 4 \times 2, \\ a_6(u:v) = u^5v^5(u^2 + v^2), \text{ degree } 12 = 6 \times 2. \end{cases}$$

Then we have  $\Delta = 4a_4^3 + 27a_6^2$  and  $j = 1728 \cdot 4a_4^3/\Delta$

$$\begin{cases} \Delta = 27u^{10}v^{10}(u-v)^2(u+v)^2, \text{ degree } 24 = 12 \times 2, \\ j = \frac{27u^{10}v^{10}}{27u^{10}v^{10}} \cdot -\frac{1728 \cdot 4u^2v^2}{(u-v)^2(u+v)^2}, \text{ degree } 4! \text{ NOT } 24. \end{cases}$$

The  $j$ -map  $j : \mathbb{P}^1 \rightarrow \overline{M}_{1,1} \cong \mathbb{P}^1$  is always a morphism but **lost the valuation data crucial for Tate's algorithm** to find out what are (additive) singular fibers at  $[0:1]$  for  $t = 0$  and  $[1:0]$  for  $t = \infty$ .

## Arithmetic of $X_n := \text{Hom}_n(\mathbb{P}_D^1, \mathbb{P}_T^1)$

1.  $X_n = \mathbb{P}^{2n+1} - V(\text{Res}(s_0, s_1))$  is the open complement of **Resultant hypersurface**  $\text{Res}(s_0, s_1) = 0$  in  $\mathbb{P}^{2n+1}$  thus it is an open quasiprojective variety of dimension  $2n + 1$
2. By Farb-Wolfson's seminal work (2016)  
$$\{X_n\} = \mathbb{L}^{2n+1} - \mathbb{L}^{2n-1} \rightarrow |X_n(\mathbb{F}_q)| = q^{2n+1} - q^{2n-1}$$
3. Both domain  $\mathbb{P}_D^1$  and target  $\mathbb{P}_T^1$  are **unparameterized** and the action of an element of  $\text{PGL}_2$  on the homogeneous coordinates  $[u : v]$  of  $\mathbb{P}_D^1$  translates to an action on the global sections  $s_i$  of  $\mathcal{O}_{\mathbb{P}_D^1}(n)$  for  $i = 0, 1$  which are the homogeneous coordinates of  $\mathbb{P}(V) = \mathbb{P}(\underbrace{1, \dots, 1}_{n+1 \text{ times}}, \underbrace{1, \dots, 1}_{n+1 \text{ times}}) = \mathbb{P}^{2n+1}$
4.  $\mathbb{L}^{2n+1} - \mathbb{L}^{2n-1} = \mathbb{L}(\mathbb{L}^2 - 1) \cdot \mathbb{L}^{2n-2}$  as  $\{\text{PGL}_2\} = \mathbb{L}(\mathbb{L}^2 - 1)$

# Topology of $X_n := \text{Hom}_n(\mathbb{P}_D^1, \mathbb{P}_T^1)$

1.  $\text{Hom}_n^*(\mathbb{P}_D^1, \mathbb{P}_T^1) \hookrightarrow \text{Hom}_n(\mathbb{P}_D^1, \mathbb{P}_T^1) \rightarrow \mathbb{P}_T^1$  via the evaluation morphism  $\text{ev}_\infty : \text{Hom}_n(\mathbb{P}_D^1, \mathbb{P}_T^1) \rightarrow \mathbb{P}_T^1$  with  $f \mapsto f(\infty) \in \mathbb{P}_T^1$
2. Fiber  $\text{Hom}_n^*(\mathbb{P}_D^1, \mathbb{P}_T^1)$  is the based mapping space which is identical to the space of coprime polynomials  $\text{Poly}_1^{(n,n)}$

## Definition

Fix a field  $K$  with algebraic closure  $\overline{K}$ . Fix  $k, l \geq 0$ . Define  $\text{Poly}_1^{(k,l)}$  to be the set of pairs  $(u, v)$  of monic polynomials in  $K[z]$  so that:

- 2.1  $\deg u = k$  and  $\deg v = l$ .
- 2.2  $u$  and  $v$  have no common root in  $\overline{K}$ .
3.  $\text{ev}_\infty$  is a Zariski-locally trivial fibration via the transitive action of  $\text{Aut}(\mathbb{P}_T^1) = \text{PGL}_2$
4.  $\mathbb{L}^{2n+1} - \mathbb{L}^{2n-1} = (\mathbb{L} + 1) \cdot (\mathbb{L}^{2n} - \mathbb{L}^{2n-1})$  as  $\{\text{Hom}_n^*(\mathbb{P}_D^1, \mathbb{P}_T^1)\} = \{\text{Poly}_1^{(n,n)}\} = \mathbb{L}^{2n} - \mathbb{L}^{2n-1}$

# Arithmetic of Algebraic Stacks over Finite Fields

The weighted point count of  $\mathcal{X}$  over  $\mathbb{F}_q$  is defined as a sum:

$\#_q(\mathcal{X}) := \sum_{x \in \mathcal{X}(\mathbb{F}_q)/\sim} \frac{1}{|\text{Aut}(x)|}$  where  $\mathcal{X}(\mathbb{F}_q)/\sim$  is the set of  $\mathbb{F}_q$ -isomorphism classes of  $\mathbb{F}_q$ -points of  $\mathcal{X}$ .

What we really need is the unweighted point count  $|\mathcal{X}(\mathbb{F}_q)/\sim|$ .  
But this is immune to the Grothendieck-Lefschetz trace formula.

We clarify the arithmetic role of the *inertia stack*  $\mathcal{I}(\mathcal{X})$  of an algebraic stack  $\mathcal{X}$  over  $\mathbb{F}_q$  which parameterizes pairs  $(x, \text{Aut}(x))$ .

## Theorem (Changho Han–JP)

Let  $\mathcal{X}$  be an algebraic stack over  $\mathbb{F}_q$  of finite type with affine diagonal. Then,

$$|\mathcal{X}(\mathbb{F}_q)/\sim| = \#_q(\mathcal{I}(\mathcal{X}))$$

Thus the weighted point count  $\#_q(\mathcal{I}(\mathcal{X}))$  of the inertia stack  $\mathcal{I}(\mathcal{X})$  is the unweighted point count  $|\mathcal{X}(\mathbb{F}_q)/\sim|$  of  $\mathcal{X}$  over  $\mathbb{F}_q$ .

## How many elliptic curves over $k = \mathbb{F}_q$ upto isom?

The inertia stack  $\overline{\mathcal{IM}}_{1,1}$  parametrizes  $[E]$  and automorphism groups  $([E], \text{Aut}[E])$ . To keep track of the primitive roots of unity contained in  $\mathbb{F}_q$ , define function  $\delta(x) := \begin{cases} 1 & \text{if } x \text{ divides } q-1, \\ 0 & \text{otherwise.} \end{cases}$

Grothendieck class in  $K_0(\text{Stck}_k)$  with  $\text{char}(k) \neq 2, 3$ ,

$$\{\overline{\mathcal{IM}}_{1,1}\} = 2 \cdot (\mathbb{L} + 1) + 2 \cdot \delta(4) + 4 \cdot \delta(6)$$

Weighted point count over  $\mathbb{F}_q$  with  $\text{char}(\mathbb{F}_q) \neq 2, 3$ ,

$$\#_q(\overline{\mathcal{IM}}_{1,1}) = 2 \cdot (q + 1) + 2 \cdot \delta(4) + 4 \cdot \delta(6)$$

Exact number of  $\mathbb{F}_q$ -isomorphism classes with  $\text{char}(\mathbb{F}_q) \neq 2, 3$ ,

$$|\overline{\mathcal{M}}_{1,1}(\mathbb{F}_q)/\sim| = 2 \cdot (q + 1) + 2 \cdot \delta(4) + 4 \cdot \delta(6)$$

# Motivic Analytic Number Theory Praxis

Moduli of minimal stable  $E/\mathbb{F}_q(t)$  is  $\mathcal{L}_{12n} = \text{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$

## Theorem (Changho Han–JP)

Grothendieck class in  $K_0(\text{Stck}_k)$  with  $\text{char}(k) \neq 2, 3$ ,

$$\{\mathcal{L}_{12n}\} = \mathbb{L}^{10n+1} - \mathbb{L}^{10n-1}$$

Weighted point count over  $\mathbb{F}_q$  with  $\text{char}(\mathbb{F}_q) \neq 2, 3$ ,

$$\#_q(\mathcal{L}_{12n}) = q^{10n+1} - q^{10n-1}$$

Exact number of  $\mathbb{F}_q$ -isomorphism classes with  $\text{char}(\mathbb{F}_q) \neq 2, 3$ ,

$$|\mathcal{L}_{12n}(\mathbb{F}_q)/\sim| = \#_q(\mathcal{IL}_{12n}) = 2 \cdot (q^{10n+1} - q^{10n-1})$$

$$\mathcal{Z}_{\mathbb{F}_q(t)}(\mathcal{B}) = \sum_{n=1}^{\left\lfloor \frac{\log_q \mathcal{B}}{12} \right\rfloor} |\mathcal{L}_{1,12n}(\mathbb{F}_q)/\sim| = 2 \cdot \frac{(q^{11} - q^9)}{(q^{10} - 1)} \cdot \left( \mathcal{B}^{\frac{5}{6}} - 1 \right)$$

# **Totality of Rational points on Moduli stacks**

## **Counting Families of Varieties : Lecture 3**

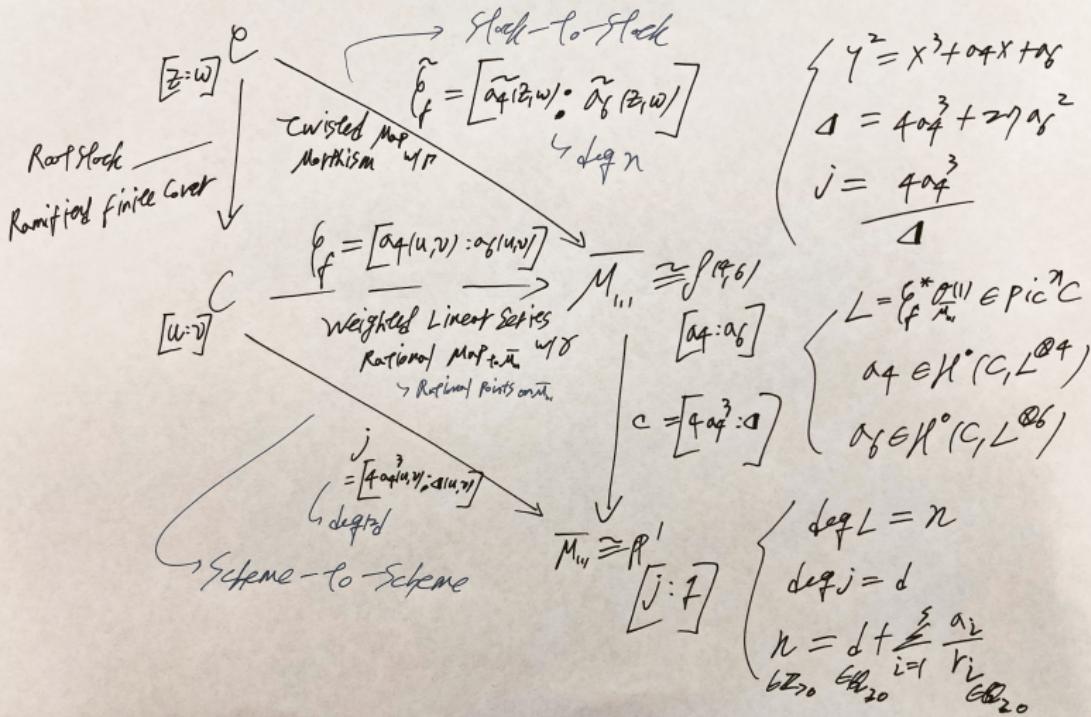
June Park

The University of Sydney

KIAS-LFANT Winter School on Number Theory

# Rational points on $\overline{\mathcal{M}}_{1,1} \cong \mathcal{P}(4,6)$ over $K = k(C)$

$\therefore$  Rational points on  $\overline{\mathcal{M}}_{1,1} \cong \mathcal{P}(4,6)$



# Stacky Heights on Algebraic Stacks wrt ‘Ample’ $\mathcal{V}$

Ellenberg, Zureick-Brown, and Satriano extends the rational point  $x \in \mathcal{X}(K)$  to a stacky curve, called a *tuning stack*  $(\mathcal{C}, \pi, \bar{x})$  for  $x$ .

$$\begin{array}{ccccc} & & x & & \\ & \nearrow & \curvearrowright & \searrow & \\ Spec(K) & \longrightarrow & \mathcal{C} & \xrightarrow{\bar{x}} & \mathcal{X} \\ & \searrow & \downarrow \pi & & \\ & & C & & \end{array}$$

$\mathcal{C}$  is a normal,  $\pi$  is a birational coarse space map.

## Definition

If  $\mathcal{V}$  is a vector bundle on  $\mathcal{X}$  and  $x \in \mathcal{X}(K)$ , the *height of  $x$  with respect to  $\mathcal{V}$*  is defined as

$$ht_{\mathcal{V}}(x) := -\deg(\pi_* \bar{x}^* \mathcal{V}^\vee)$$

for any choice of tuning stack  $(\mathcal{C}, \pi, \bar{x})$ .

## 8 Different Types of Additive Bad Reductions

Let  $\kappa := \text{lcm}\{\lambda_0, \dots, \lambda_N\}$  and  $\bar{\lambda}_j := \kappa/\lambda_j$  as usual.

**Lemma (Dori Bejleri–JP–Matthew Satriano; April 2024)**

Suppose  $\kappa > 1$ . Then the map

$$m \mapsto \left( \frac{\kappa}{\gcd(m, \kappa)}, \frac{m}{\gcd(m, \kappa)} \right)$$

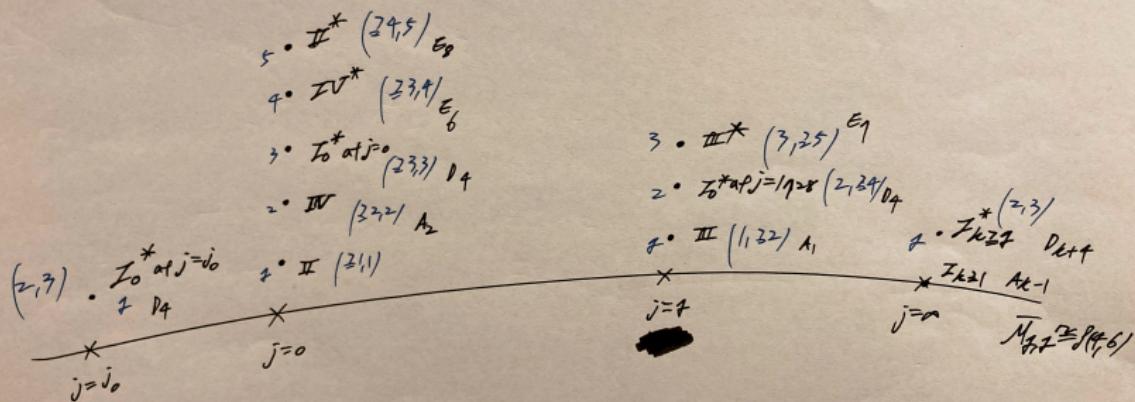
induces a bijection from the set  $\{1, \dots, \kappa - 1\}$  to the set

$$\{(r, a) : 1 \leq a < r, r|\kappa, \gcd(r, a) = 1\}$$

For  $\overline{\mathcal{M}}_{1,1} \cong \mathcal{P}(4, 6)$  we have  $\kappa = 12$  which means we have  $m \in \{2, 3, 4, 6, 8, 9, 10\}$  ( $\{1, 5, 7, 11\}$  are excluded as prime) that corresponds to following rooting data  $m = 2 \mapsto \frac{1}{6}$ ,  $m = 3 \mapsto \frac{1}{4}$ ,  $m = 4 \mapsto \frac{1}{3}$ ,  $m = 6 \mapsto \frac{1}{2}$ ,  $m = 8 \mapsto \frac{2}{3}$ ,  $m = 9 \mapsto \frac{3}{4}$ ,  $m = 10 \mapsto \frac{5}{6}$  which correspond to 7 + 1 types of additive reductions.  
(+1 since ramification at  $j = \infty$  for  $I_{k>0}^*$ )

# Geometric Tate's algorithm

Tate's Algorithm via raised maps



# Tate's Algorithm via Twisted Morphisms

**Theorem (Dori Bejleri–JP–Matthew Satriano; April 2024)**

If  $\text{char}(K) \neq 2, 3$ . Then the twisting condition  $(r, a)$  and the order of vanishing of  $j$  at  $j = \infty$  determine the Kodaira fiber type, and  $(r, a)$  is in turn determined by  $m = \min\{3\nu(a_4), 2\nu(a_6)\}$ .

$\gamma : (\nu(a_4), \nu(a_6))$	Reduction type with $j \in \overline{M}_{1,1}$	$\Gamma : (r, a)$
$(\geq 1, 1) m = 2$	II with $j = 0$	$(6, 1)$
$(1, \geq 2) m = 3$	III with $j = 1728$	$(4, 1)$
$(\geq 2, 2) m = 4$	IV with $j = 0$	$(3, 1)$
$(2, 3) m = 6$	$I_{k>0}^*$ with $j = \infty$ $I_0^*$ with $j \neq 0, 1728$	$(2, 1)$
$(\geq 3, 3) m = 6$	$I_0^*$ with $j = 0$	$(2, 1)$
$(2, \geq 4) m = 6$	$I_0^*$ with $j = 1728$	$(2, 1)$
$(\geq 3, 4) m = 8$	IV* with $j = 0$	$(3, 2)$
$(3, \geq 5) m = 9$	III* with $j = 1728$	$(4, 3)$
$(\geq 4, 5) m = 10$	II* with $j = 0$	$(6, 5)$

## Theorem (Dori Bejleri–JP–Matthew Satriano; April 2024)

Let  $f : C \dashrightarrow \mathcal{P}(\vec{\lambda})$  be a rational map of smooth projective curve  $C$ , and let  $P \in \mathcal{P}_C(\vec{\lambda})(K)$  denote the corresponding rational point over  $K = k(C)$ . Let  $\{x_j\}$  be the indeterminacy points of  $f$ .

1. Let  $(L, s_0, \dots, s_N)$  be any  $\vec{\lambda}$ -weighted linear series inducing  $f$ . Then the universal tuning stack  $(\mathcal{C}, \pi, \overline{P})$  of  $P$  is the root stack of  $C$  obtained by taking the  $r_j$ -th root at  $x_j$ , where  $r_j = r_{\min}(x_j; L, s_0, \dots, s_N)$ . Moreover, the induced morphism on stabilizers over  $x_j$  is given by the character  $\chi_j^{-a_j}$  where  $a_j = a_{\min}(x_j, L, s_0, \dots, s_N)$ .
2. A wls  $(L, s_0, \dots, s_N)$  is minimal if for each indeterminacy point  $x \in C$ , there exists an  $j$  such that  $\nu_x(s_j) < \lambda_i$ . There exists a unique minimal  $\vec{\lambda}$ -weighted linear series inducing  $f$ .
3. The stacky height  $\text{ht}_{\mathcal{O}(1)}(P)$  is equal to  $\deg L$  where  $(L, s_0, \dots, s_N)$  is the unique minimal linear series. Moreover, the stable height is given by  $\text{ht}_{\mathcal{O}(1)}^{\text{st}}(P) = \deg \overline{P}^* \mathcal{O}(1)$  and the local contribution at  $x_j$  is given by  $\delta_{x_j}(P) = \frac{a_j}{r_j} [k(x_j) : k]$ .

# Height Moduli Space on Cyclotomic Stacks

There is a height moduli stack  $\mathcal{M}_n(\mathcal{X}, \mathcal{L})$  parametrizing all rational points on general proper polarized cyclotomic stacks of stacky height  $n$  and that the spaces of twisted maps yield a stratification of  $\mathcal{M}_n(\mathcal{X}, \mathcal{L})$  corresponding to fixing the local contributions to the stacky height. The fact that  $\mathcal{M}_n(\mathcal{X}, \mathcal{L})$  is of finite type is a geometric incarnation of the Northcott property.

**Theorem (Dori Bejleri–JP–Matthew Satriano; April 2024)**

Let  $(\mathcal{X}, \mathcal{L})$  be a proper polarized cyclotomic stack over a perfect field  $k$ . Fix a smooth projective curve  $C/k$  with function field  $K = k(C)$  and  $n, d \in \mathbb{Q}_{\geq 0}$ .

1. There exists a separated Deligne–Mumford stack  $\mathcal{M}_{n,C}(\mathcal{X}, \mathcal{L})$  of finite type over  $k$  with a quasi-projective coarse space and a canonical bijection of  $k$ -points

$$\mathcal{M}_{n,C}(\mathcal{X}, \mathcal{L})(k) = \{P \in \mathcal{X}(K) \mid \text{ht}_{\mathcal{L}}(P) = n\}.$$

1. There is a finite locally closed stratification

$$\bigsqcup_{\Gamma, d} \mathcal{H}_{d,C}^{\Gamma}(\mathcal{X}, \mathcal{L}) / S_{\Gamma} \rightarrow \mathcal{M}_{n,C}(\mathcal{X}, \mathcal{L})$$

where  $\mathcal{H}_{d,C}^{\Gamma}$  are moduli spaces of twisted maps and the union runs over all possible admissible local conditions

$$\Gamma = (\{r_1, a_1\}, \dots, \{r_s, a_s\})$$

and degrees  $d$  for a twisted map to  $(\mathcal{X}, \mathcal{L})$  satisfying

$$n = d + \sum_{i=1}^s \frac{a_i}{r_i}$$

and  $S_{\Gamma}$  is a subgroup of the symmetric group on  $s$  letters that permutes the stacky points of the twisted map.

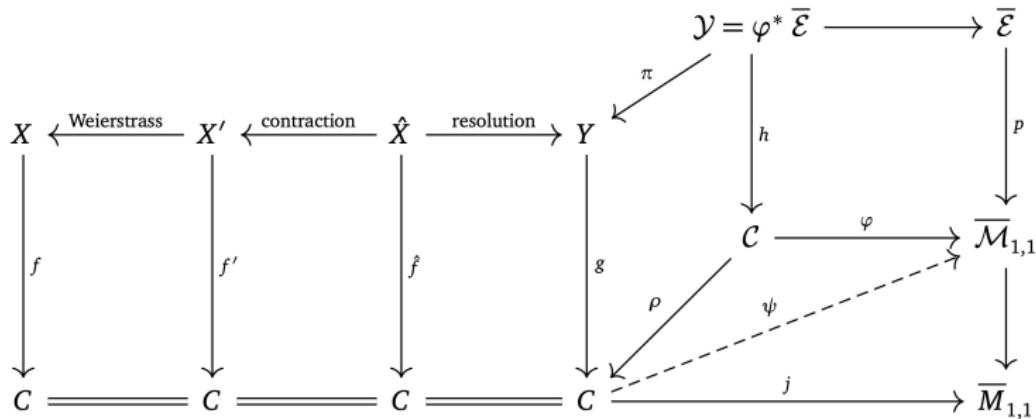
2. Under the bijection in part (1), each  $k$ -point of  $\mathcal{H}_{d,C}^{\Gamma}(\mathcal{X}, \mathcal{L}) / S_{\Gamma}$  corresponds to a  $K$ -point  $P$  with the stable height and local contributions given by

$$\text{ht}_{\mathcal{L}}^{st}(P) = d \quad \left\{ \delta_i = \frac{a_i}{r_i} \right\}_{i=1}^s .$$

## Specializing to the canonical case of $\overline{\mathcal{M}}_{1,1} \cong \mathcal{P}(4,6)$

1. The Hodge line bundle  $\mathcal{L}$  of  $\overline{\mathcal{M}}_{1,1}$  is  $\nu = \mathcal{O}(1)$  on  $\mathcal{P}(4,6)$
2. An elliptic curve  $E/K$  is a rational point  $P \in \overline{\mathcal{M}}_{1,1}(K)$  which in turn corresponds to a weighted linear series on  $K = k(C)$  of height  $n$  consisting of Weierstrass coefficients  $a_4 \in H^0(C, \mathcal{O}(4n))$  and  $a_6 \in H^0(C, \mathcal{O}(6n))$
3. The orders of vanishing at a point can be encoded in a vector  $\gamma = (\nu_x(a_4), \nu_x(a_6))$  which corresponds to a certain twisting data  $\Gamma = (r, a)$  of universal tuning stack, a twisted curve  $\mathcal{C}$
4. The spaces  $\mathcal{W}_{n,C}^\gamma$  and  $\mathcal{H}_{d,C}^\Gamma$  can be identified with moduli of certain canonical models of elliptic surfaces with a specified fiber of additive bad reduction and the isomorphism between the two via Tate's algorithm can be understood in the context of the minimal model program.

# Geometric Interpretation of Tate's Algorithm



Here  $f$  is a Weierstrass model,  $\psi$  is the associated weighted linear series viewed as a rational map to  $\bar{\mathcal{M}}_{1,1}$ ,  $\varphi$  is a twisted morphism from the universal tuning stack  $\mathcal{C}$  which induces a stable stack-like model  $h : \mathcal{Y} \rightarrow \mathcal{C}$  where  $g : Y \rightarrow C$  is the twisted model via coarse moduli maps,  $\hat{f}$  is a resolution of  $Y$ , and  $f'$  is the relative minimal model obtained by contracting relative  $(-1)$ -curves.

Suppose that normalized base multiplicity  $m = 3$ . This occurs if and only if  $(\nu(a_4), \nu(a_6)) = (1, \geq 2)$ . Then  $r = 12/\gcd(3, 12) = 4$  and  $a = 3/\gcd(3, 12) = 1$ . Thus the stabilizer of the twisted curve acts on the central fiber of the twisted model via the character  $\mu_4 \rightarrow \mu_4$ ,  $\zeta_4 \mapsto \zeta_4^{-1}$ . In particular, the central fiber  $E$  of  $\mathcal{Y}$  has  $j = 1728$ . The  $\mu_4$  action on  $E$  has two fixed points, and there is an orbit of size two with stabilizer  $\mu_2 \subset \mu_4$ . Let  $E_0$  be the image of  $E$  in the twisted model  $Y$ . As  $E$  appears with multiplicity 4,  $Y$  has  $\frac{1}{4}(-1, -1)$  quotient singularities at the images of the the fixed points and a  $\frac{1}{2}(-1, -1)$  singularity at the image of the orbit of size two. Each of these singularities is resolved by a single blowup to obtain  $\hat{X}$  with central fiber  $4\tilde{E}_0 + E_1 + E_2 + E_3$  where  $E_i$  are the exceptional divisors of the resolution for  $i = 1, 2, 3$  and

$E_1^2 = E_2^2 = -4$  with  $E_3^2 = -2$ . Then  $\tilde{E}_0$  is a  $(-1)$ -curve so it needs to be contracted. After this contraction  $E_2$  becomes a  $(-1)$  curve and must also be contracted. Since  $E_i$  for  $i = 1, 2, 3$  are incident and pairwise transverse after blowing down  $\tilde{E}_0$ , then the images of  $E_1$  and  $E_2$  must be tangent after blowing down  $E_3$ . Moreover, they are now  $(-2)$ -curves and the relatively minimal model for type III.

# Geometric Meaning of Height Moduli Framework

1. So one can run the resolution / minimal model. As these are *algebraic surfaces* it can be done over  $\text{char}(K) = p > 0$
2. A twisted morphism  $\varphi : \mathcal{C} \rightarrow \overline{\mathcal{M}}_{1,1}$  with its twisting data  $\Gamma$  from the universal tuning stack  $\mathcal{C}$  induces a stable stack-like model  $h : \mathcal{Y} \rightarrow \mathcal{C}$  as a unique pullback of the universal family  $p : \overline{\mathcal{E}} \rightarrow \overline{\mathcal{M}}_{1,1}$ . All the ensuing birational geometry is natural.
3. True purpose of a **representable classifying morphism** is in the universal principle that  $\varphi$  intrinsically contains all the algebro-geometric data necessary to uniquely determine a fibration with singular fibers. This is the very essence of the inner arithmetic of rational points on moduli stacks over  $K$ .

# Algebraic Geometry $\cap$ Topology $\Longleftrightarrow$ Arithmetic

1. Consider the fact that  $\overline{\mathcal{M}}_{1,1}$  could have been any other algebraic stack  $\mathcal{X}$  (such as  $\overline{\mathcal{M}}_g$  or  $\overline{\mathcal{A}}_g$ ) which is the representing object for certain moduli functor as the fine moduli stack together with the universal family  $p : \overline{\mathcal{E}} \rightarrow \mathcal{X}$ .
2. Representable classifying morphisms as twisted morphisms  $\varphi : \mathcal{C} \rightarrow \mathcal{X}$  uniquely determines certain families of varieties (of algebraic curves or abelian varieties) with non-abelian stabilizers ( $g \geq 2$ ). And they naturally have corresponding “Tate’s algorithm”, counting statements and so on.
3. Geometrizing  $\mathcal{X}(K)$  leads to Height moduli space  $\mathcal{M}_n(\mathcal{X}, \mathcal{V})$ . Once we have a **space**, we compute its **invariants**, consider all invariants simultaneously via generating series and show the motivic height zeta function’s **rationality**, naturally having various kinds of **consequences**.

# Motivic Height Zeta Function as Generating Series

## Definition

A  $\vec{\lambda}$ -weighted linear series  $(L, s_0, \dots, s_N)$  is *minimal* if for each indeterminacy point  $x \in C$ , there exists an  $j$  such that  $\nu_x(s_j) < \lambda_i$ .

## Definition

The motivic height zeta function of  $\mathcal{P}(\lambda_0, \dots, \lambda_N)$  is the formal power series

$$Z_{\vec{\lambda}}(t) := \sum_{n \geq 0} \{\mathcal{W}_n^{\min}\} t^n \in K_0(\text{Stck})[[t]]$$

where  $\mathcal{W}_n^{\min}$  is the space of minimal weighted linear series on  $\mathbb{P}^1$  of height  $n$ . We also define the variant

$$\mathcal{IZ}_{\vec{\lambda}}(t) := \sum_{n \geq 0} \{\mathcal{IW}_n^{\min}\} t^n \in K_0(\text{Stck}_k)[[t]]$$

# Stratification on Ambient Projective Stacks

Minimality defect  $e$  measures the degree of failure of a weighted linear series to be minimal (not a rational point of height  $n$ ).

## Definition

Let  $\mu$  be the normalized base profile. We can divide each part  $\mu_i$  by  $\kappa$  to obtain  $\mu_i = \kappa q_i + r_i$ . We define  $q(\mu)$  and  $r(\mu)$  to be the partitions with parts  $q_i$  and  $r_i$  respectively.

The minimality defect of  $\mu$  is the size of the quotient  $e = |q(\mu)|$ .

## Corollary (Dori Bejleri–JP–Matthew Satriano; April 2024)

*The disjoint union of  $\psi_{n,e}$*

$$\psi_n : \bigsqcup_{e=0}^n \mathcal{W}_{n-e}^{\min} \times \mathbb{P}(V_e^1) \rightarrow \mathcal{P} \left( \bigoplus_{i=0}^N V_n^{\lambda_j} \right)$$

*is an isomorphism after stratifying the source and target.*

- 1.** We denote the usual motivic zeta function of  $\mathbb{P}^1$  by

$$Z(t) = \sum \{\text{Sym}^e \mathbb{P}^1\} t^e = \frac{1}{(1 - \mathbb{L}t)(1 - t)}$$

- 2.** We stratify by minimality defect  $e$  to obtain an equality

$$\left\{ \mathcal{P} \left( \bigoplus_{i=0}^N V_n^{\lambda_i} \right) \right\} = \sum_{e=0}^n \{ \mathcal{W}_{n-e}^{\min} \} \{ \text{Sym}^e \mathbb{P}^1 \}$$

which implies

$$\sum_{n \geq 0} \left\{ \mathcal{P} \left( \bigoplus_{i=0}^N V_n^{\lambda_i} \right) \right\} t^n = Z_{\vec{\lambda}}(t) \cdot Z(t) \quad (1)$$

- 3.** *Homogeneous polynomials* live in compact ambient stack!

$$\sum_{n \geq 0} \left\{ \mathcal{P} \left( \bigoplus_{i=0}^N V_n^{\lambda_i} \right) \right\} t^n = \frac{\{\mathbb{P}^N\} + \mathbb{L}^{N+1}\{\mathbb{P}^{|\vec{\lambda}| - N - 2}\}t}{(1 - t)(1 - \mathbb{L}^{|\vec{\lambda}|}t)}$$

# Rationality of Motivic Height Zeta Function

Fix weights  $\vec{\lambda} = (\lambda_0, \dots, \lambda_N)$  and let  $|\vec{\lambda}| := \sum_{i=0}^N \lambda_i$ . Suppose for simplicity that  $k$  contains all  $\text{lcm} = \text{lcm}(\lambda_0, \dots, \lambda_N)$  roots of unity.

**Theorem (Dori Bejleri–JP–Matthew Satriano; April 2024)**

For  $k, \vec{\lambda}$  as above and  $C = \mathbb{P}_k^1$ , consider  $\mathcal{W}_n^{\min}$  and its inertia stack  $\mathcal{IW}_n^{\min}$ . We have the following formulas over  $K_0(\text{Stck}_k)$ .

$$\sum_{n \geq 0} \{\mathcal{W}_n^{\min}\} t^n = \frac{1 - \mathbb{L}t}{1 - \mathbb{L}^{|\vec{\lambda}|}t} \left( \{\mathbb{P}^N\} + \mathbb{L}^{N+1} \{\mathbb{P}^{|\vec{\lambda}| - N - 2}\} t \right)$$

$$\sum_{n \geq 0} \{\mathcal{IW}_n^{\min}\} t^n = \sum_{g \in \mu_{\text{lcm}}(k)} \frac{1 - \mathbb{L}t}{1 - \mathbb{L}^{|\vec{\lambda}_g|}t} \left( \{\mathbb{P}^{N_g}\} + \mathbb{L}^{N_g+1} \{\mathbb{P}^{|\vec{\lambda}_g| - N_g - 2}\} t \right)$$

where  $g$  runs over the lcm roots of unity and  $\vec{\lambda}_g$  is a subset of  $\vec{\lambda}$  of size  $N_g + 1$  depending explicitly on the order of  $g$ .

# Motives of Moduli Stacks of Elliptic Surfaces

## Theorem (Dori Bejleri–JP–Matthew Satriano)

Let  $\text{char}(k) \neq 2, 3$ . The motives (modulo  $\{\text{PGL}_2\}$ ) of moduli stacks  $\mathcal{W}_{\min,n}^\Theta$  of minimal Weierstrass fibrations with a single Kodaira fiber  $\Theta$  and at worst multiplicative reduction elsewhere is

Reduction type $\Theta$ with $j \in \overline{M}_{1,1}$	$ \gamma $	$\{\mathcal{W}_{\min,n}^\Theta\} \in K_0(\text{Stck}_K)$
I <sub>k&gt;0</sub> with $j = \infty$	0	$\mathbb{L}^{10n-2}$
II with $j = 0$	2	$\mathbb{L}^{10n-3}$
III with $j = 1728$	3	$\mathbb{L}^{10n-4}$
IV with $j = 0$	4	$\mathbb{L}^{10n-5}$
I <sub>k&gt;0</sub> <sup>*</sup> with $j = \infty$	5	$\mathbb{L}^{10n-6} - \mathbb{L}^{10n-7}$
I <sub>0</sub> <sup>*</sup> with $j \neq 0, 1728$		
I <sub>0</sub> <sup>*</sup> with $j = 0, 1728$	6	$\mathbb{L}^{10n-7}$
IV <sup>*</sup> with $j = 0$	7	$\mathbb{L}^{10n-8}$
III <sup>*</sup> with $j = 1728$	8	$\mathbb{L}^{10n-9}$
II <sup>*</sup> with $j = 0$	9	$\mathbb{L}^{10n-10}$

## Theorem (Dori Bejleri–JP–Matthew Satriano; April 2024)

$$\left\{ \mathcal{W}_{n=1}^{\min}(\mathcal{P}(\vec{\lambda})) \right\} = \{\mathbb{P}^N\}(\mathbb{L}^{|\vec{\lambda}|} - \mathbb{L}) + \mathbb{L}^{N+1}\{\mathbb{P}^{|\vec{\lambda}|-N-2}\}$$

$$\left\{ \mathcal{W}_{n \geq 2}^{\min}(\mathcal{P}(\vec{\lambda})) \right\} = \mathbb{L}^{(n-2)|\vec{\lambda}|+N+2}(\mathbb{L}^{|\vec{\lambda}|-1} - 1)\{\mathbb{P}^{|\vec{\lambda}|-1}\}$$

Take  $|\vec{\lambda}| = 10$  and  $N = 1$  as  $\overline{\mathcal{M}}_{1,1} \cong \mathcal{P}(4, 6)$  over  $\mathbb{Z}[1/6]$ .

- When  $n = 1$ ,  $X$  is a **Rational elliptic surface**.

$$\{\mathcal{W}_1^{\min}\} = \mathbb{L}^{11} + \mathbb{L}^{10} + \mathbb{L}^9 + \mathbb{L}^8 + \mathbb{L}^7 + \mathbb{L}^6 + \mathbb{L}^5 + \mathbb{L}^4 + \mathbb{L}^3 - \mathbb{L}$$

- When  $n = 2$ ,  $X$  is algebraic  $K3$  surface with elliptic fibration (i.e., **Projective elliptic K3 surface with moduli dim. 18**).

$$\{\mathcal{W}_2^{\min}\} = \mathbb{L}^{21} + \mathbb{L}^{20} + \mathbb{L}^{19} + \mathbb{L}^{18} + \mathbb{L}^{17} + \mathbb{L}^{16} + \mathbb{L}^{15} + \mathbb{L}^{14} + \mathbb{L}^{13} - \mathbb{L}^{11} - \mathbb{L}^{10} - \mathbb{L}^9 - \mathbb{L}^8 - \mathbb{L}^7 - \mathbb{L}^6 - \mathbb{L}^5 - \mathbb{L}^4 - \mathbb{L}^3$$

$$= \mathbb{L}(\mathbb{L}^2 - 1) \left( \mathbb{L}^{18} + \mathbb{L}^{17} + 2\mathbb{L}^{16} + 2\mathbb{L}^{15} + 3\mathbb{L}^{14} + 3\mathbb{L}^{13} + 4\mathbb{L}^{12} + 4\mathbb{L}^{11} + 5\mathbb{L}^{10} + 4\mathbb{L}^9 + 4\mathbb{L}^8 + 3\mathbb{L}^7 + 3\mathbb{L}^6 + 2\mathbb{L}^5 + 2\mathbb{L}^4 + \mathbb{L}^3 + \mathbb{L}^2 \right)$$