RATIONALITY OF THE TRIVIAL LATTICE RANK WEIGHTED MOTIVIC HEIGHT ZETA FUNCTION FOR ELLIPTIC SURFACES

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ABSTRACT. Let k be a perfect field with $\mathrm{char}(k) \neq 2,3$, set $K = k(t)$, and let $\mathcal{W}_n^{\min}$ be the moduli stack of minimal elliptic curves over K of Faltings height n from the height–moduli framework of [BPS22] applied to $\overline{\mathcal{M}}_{1,1} \simeq \mathcal{P}(4,6)$. For $[E] \in \mathcal{W}_n^{\min}$, let $S \to \mathbb{P}_k^1$ be the associated elliptic surface with section. Inspired by the Shioda–Tate formula, we introduce the trivariate motivic height zeta function

$$\mathcal{Z}(u,v;t) := \sum_{n\geq 1}\Big(\sum_{[E]\in\mathcal{W}_n^{\min}} u^{T(S)}\, v^{\mathrm{rk}(E/K)}\Big)t^n \in K_0(\mathrm{Stck}_k)[u,v][\![t]\!],$$

which refines the height series by the trivial lattice rank $T(S)$ and the Mordell–Weil rank $\mathrm{rk}(E/K)$. Using the finite constructible stratification of $\mathcal{W}_n^{\min} = \bigsqcup_{T\geq 0} \mathcal{W}_n^{\min}(T)$ into loci where $T(S)$ is constant, we prove rationality for the trivial lattice specialization $Z_{\mathrm{Triv}}(u;t) = \mathcal{Z}(u,1;t)$ by giving an explicit finite Euler product for its normalization $\widetilde{Z}_{\mathrm{Triv}}(u;t)$. We conjecture irrationality for the Néron–Severi $Z_{\mathrm{NS}}(w;t) = \mathcal{Z}(w,w;t)$ and the Mordell–Weil $Z_{\mathrm{MW}}(v;t) = \mathcal{Z}(1,v;t)$ specializations.

1. INTRODUCTION

Let k be a perfect field with $\mathrm{char}(k) \neq 2,3$, and set $K := k(t)$. An elliptic curve E/K determines a relatively minimal elliptic surface with section

$$f : S \longrightarrow \mathbb{P}_k^1$$

unique up to isomorphism (see [Mir89, SS10] for background on elliptic surfaces).

The arithmetic of E/K is reflected in the geometry of S, and a basic organizing principle is the Shioda–Tate formula [Shi90]

$$\rho(S) \;=\; T(S) \;+\; \mathrm{rk}(E/K), \tag{1}$$

where $\rho(S) = \mathrm{rk}\,\mathrm{NS}(S_{\bar{k}})$ is the *geometric Picard rank*, $T(S)$ is the *rank of the geometric trivial lattice* generated by the zero section, a fiber class, and the components of reducible fibres not meeting the zero section, and $\mathrm{rk}(E/K)$ is the *Mordell–Weil rank*. For the elliptic surfaces arising from the height moduli considered in this paper (so that $p_g(S) = n-1$), one has the standard bounds (see [SS19])

$$2 \leq \rho(S) \leq 10n, \qquad 2 \leq T(S) \leq 10n, \qquad 0 \leq \mathrm{rk}(E/K) \leq 10n-2, \tag{2}$$

where $\rho(S) \leq 10n = h^{1,1}(S)$ is Shioda's bound.

In [BPS22], Bejleri–Park–Satriano construct height-moduli stacks of rational points on proper polarized cyclotomic stacks. In the fundamental modular curve example

$$\overline{\mathcal{M}}_{1,1} \simeq \mathcal{P}(4,6), \qquad \lambda \simeq \mathcal{O}_{\mathcal{P}(4,6)}(1),$$

a minimal elliptic curve over K can be viewed as a rational point of λ–height n on $\overline{\mathcal{M}}_{1,1}$ over K. This yields a separated Deligne–Mumford stack of finite type

$$\mathcal{W}_n^{\min} \;:=\; \mathcal{W}_{n,\mathbb{P}_k^1}^{\min}\big(\mathcal{P}(4,6), \mathcal{O}(1)\big)$$

parametrizing minimal elliptic curves over K of discriminant degree $12n$. Here a K-rational point of $\overline{\mathcal{M}}_{1,1}$ of λ-height n means stacky height n with respect to the Hodge line bundle λ. Under the identification $\overline{\mathcal{M}}_{1,1} \cong \mathcal{P}(4,6)$ one has $\lambda \simeq \mathcal{O}_{\mathcal{P}(4,6)}(1)$, and this height agrees with the Faltings height of the corresponding elliptic curve by [BPS22, Cor. 7.6].

Guided by (1), we introduce the following motivic generating series refining the height generating series in [BPS22, §8] by *lattice ranks* attached to the associated relatively minimal elliptic surface.

Definition 1.1. Let k be a perfect field of characteristic $\neq 2,3$, and consider the height moduli stack

$$\mathcal{W}_n^{\min} = \mathcal{W}^{\min}_{n,\mathbb{P}^1_k}\big(\mathcal{P}(4,6),\mathcal{O}(1)\big)$$

parametrizing minimal elliptic curves over $K = k(t)$ of discriminant height $12n$. The *trivariate height zeta function* is

$$\mathcal{Z}(u,v;t) := \sum_{n\geq 1}\left(\sum_{[E]\in\mathcal{W}_n^{\min}} u^{T(S)}\, v^{\mathrm{rk}(E/K)}\right)t^n \in K_0(\mathrm{Stck}_k)[u,v][\![t]\!],$$

where for each $[E] \in \mathcal{W}_n^{\min}$ we write $S \to \mathbb{P}^1_k$ for the associated minimal elliptic surface, and:

- $T(S)$ is the rank of the trivial lattice of S;
- $\mathrm{rk}(E/K)$ is the Mordell–Weil rank.

The following specializations are the associated *bivariate height zeta functions*:

$$Z_{\mathrm{Triv}}(u;t) := \mathcal{Z}(u,1;t), \tag{3}$$

$$Z_{\mathrm{MW}}(v;t) := \mathcal{Z}(1,v;t), \tag{4}$$

$$Z_{\mathrm{NS}}(w;t) := \mathcal{Z}(w,w;t). \tag{5}$$

Remark 1.2. The variables u and v record the two independent contributions in the Shioda–Tate formula (1), namely the trivial lattice rank $T(S)$ and the Mordell–Weil rank $\mathrm{rk}(E/K)$, while t records the height n. In particular, $Z_{\bar{\lambda}}(t) = \mathcal{Z}(1,1;t)$ is the *univariate motivic height zeta function* obtained by forgetting the lattice-rank weights. Its rationality, and that of the inertial refinement $\mathcal{I}Z_{\bar{\lambda}}(t)$ which encodes the totality of rational points on $\overline{\mathcal{M}}_{1,1}$ over $k(t)$, is proved in [BPS22, Thm. 8.9].

In this paper we focus on $Z_{\mathrm{Triv}}(u;t)$. The key point is that the trivial lattice is governed by *local bad reduction*: its rank is determined by the geometric Kodaira fibre configuration of $\pi_{\bar{k}}\colon S_{\bar{k}} \to \mathbb{P}^1_{\bar{k}}$. Writing $\mathrm{Triv}(S) \subset \mathrm{NS}(S_{\bar{k}})$ for the geometric trivial lattice and $T(S) := \mathrm{rk}(\mathrm{Triv}(S))$, we have the following explicit formula.

Lemma 1.3 (Trivial lattice rank from fibre components)**.** *Let $\pi\colon S \to \mathbb{P}^1_k$ be a relatively minimal elliptic surface with section, and let $\mathfrak{f}$ be the multiset of singular fibres of $\pi_{\bar{k}}\colon S_{\bar{k}} \to \mathbb{P}^1_{\bar{k}}$. If m_v denotes the number of irreducible components of the fibre at v, then*

$$T(S) = 2 + \sum_{v\in\mathfrak{f}}(m_v - 1).$$

Definition 1.4 (Kodaira strata)**.** Fix $n \geq 1$. For a geometric fibre configuration $\mathfrak{f}$, let $\mathcal{W}_n^{\min,(\mathfrak{f})} \subset \mathcal{W}_n^{\min}$ denote the locus parametrizing those $[E] \in \mathcal{W}_n^{\min}$ whose associated surface $S_{\bar{k}} \to \mathbb{P}^1_{\bar{k}}$ has singular fibre configuration $\mathfrak{f}$.

Definition 1.5 (Trivial-lattice rank loci and weighted series)**.** Fix $n \geq 1$. By Proposition 2.1, $\mathcal{W}_n^{\min}$ admits a *finite* constructible stratification by Kodaira data, and $T(S)$ is constant on each stratum. For each integer T with $2 \leq T \leq 10n$, let

$$\mathcal{W}_n^{\min}(T) \subset \mathcal{W}_n^{\min}$$

be the finite union of those Kodaira strata on which $T(S) = T$ (hence a finite union of locally closed substacks).

The trivial-lattice-rank–weighted motivic height zeta function is

$$Z_{\mathrm{Triv}}(u;t) := \sum_{n\geq 1} \sum_{T=2}^{10n} u^T \{\mathcal{W}_n^{\min}(T)\}\, t^n \ \in\ K_0(\mathrm{Stck}_k)[u][\![t]\!],$$

Our main result is a rationality theorem: the normalized trivial lattice series $\widetilde{Z}_{\mathrm{Triv}}(u;t)$ lies in $K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}][u](t)$ and admits a finite Euler-product expression.

Theorem 1.6 (Rationality of Z_{Triv})**.** *Let k be a perfect field of characteristic $\neq 2,3$. Then the normalized series $\widetilde{Z}_{\mathrm{Triv}}(u;t)$ is a rational function of t with coefficients in $K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}][u]$, i.e.*

$$\widetilde{Z}_{\mathrm{Triv}}(u;t) \ \in\ K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}][u](t) \subset K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}][u][\![t]\!].$$

More precisely, there exist a finite index set J and elements

$$A_j \in K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}], \qquad b_j \in \mathbb{Z}_{\geq 0}, \qquad c_j \in \mathbb{Z}_{\geq 1} \qquad (j \in J)$$

such that

$$\widetilde{Z}_{\mathrm{Triv}}(u;t) = u^2 \prod_{j\in J} \frac{1}{1 - A_j\, u^{b_j} t^{c_j}} \ \in\ K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}][u](t).$$

Equivalently,

$$Z_{\mathrm{Triv}}(u;t) = u^2 \prod_{j\in J} \frac{1}{1 - A_j\, u^{b_j} t^{c_j}} - u^2.$$

The proof is a motivic local-to-global factorization argument [CLL16, Bil23], implemented on the twisted-map charts of the height moduli $\mathcal{W}_n^{\min}$ via the evaluation morphisms [BM96, GP06]. Throughout we use the Bejleri–Park–Satriano correspondence [BPS22, Thm. 3.3] between rational points, minimal weighted linear series, and twisted morphisms; in particular, local reduction conditions are encoded by representable twisted morphisms to $\overline{\mathcal{M}}_{1,1} \simeq \mathcal{P}(4,6)$, yielding a moduli-theoretic form of Tate's algorithm compatible with the minimal model program [BPS22, Thm. 7.12]. The power structure on $K_0(\mathrm{Stck}_k)$ packages symmetric powers of the base into Euler factors, and the finiteness of the allowed elementary local patterns yields a finite Euler product after inverting $\mathbb{L}$ [GZLMH13]. Specializing $x_\alpha = u^{m(\alpha)-1}$ and multiplying by the global prefactor u^2 then gives $\widetilde{Z}_{\mathrm{Triv}}(u;t)$.

Remark 1.7 (Inertial refinements)**.** One may form an inertial analogue of the trivial–lattice–rank–weighted series by replacing each moduli stack by its inertia stack (see [HP23, §3]). Concretely, define

$$\mathcal{IZ}_{\mathrm{Triv}}(u;t) := \sum_{n\geq 1}\sum_{T=2}^{10n} u^T \left\{\mathcal{IW}_n^{\min}(T)\right\} t^n \in K_0(\mathrm{Stck}_k)[u][\![t]\!],$$

and similarly an inertial normalization

$$\widetilde{\mathcal{IZ}}_{\mathrm{Triv}}(u;t) := u^2\cdot \mathcal{IH}\big(t;\, x_\alpha = u^{m(\alpha)-1}\big),$$

where $\mathcal{IH}(t;\mathbf{x})$ is defined from the same local-pattern stratification as $\mathcal{H}(t;\mathbf{x})$, but with motivic classes $\{\cdot\}$ replaced by inertia motivic classes $\{\mathcal{I}(\cdot)\}$. The same evaluation-map factorization and power-structure argument yields a finite Euler product for $\widetilde{\mathcal{IZ}}_{\mathrm{Triv}}(u;t)$ after inverting $\mathbb{L}$.

2. Rationality of the trivial lattice rank weighted case

Throughout, let k be a perfect field with $\mathrm{char}(k)\neq 2,3$, set $K=k(t)$, and let $\pi\colon S\to\mathbb{P}^1_k$ be the relatively minimal elliptic surface with section associated to E/K. Write $\mathrm{Triv}(S)\subset \mathrm{NS}(S_{\bar{k}})$ for the geometric trivial lattice and $T(S)=\mathrm{rk}\,\mathrm{Triv}(S)$.

Proposition 2.1 (Finite Kodaira stratification)**.** *Fix $n\geq 1$. The discriminant-degree constraint $\sum_v e(F_v)=12n$ implies that only finitely many geometric fibre configurations $\mathfrak{f}$ occur among surfaces parametrized by $\mathcal{W}_n^{\min}$. Consequently,*

$$\mathcal{W}_n^{\min} = \bigsqcup_{\mathfrak{f}} \mathcal{W}_n^{\min,(\mathfrak{f})}$$

is a finite constructible stratification. Moreover, the trivial-lattice rank $T(S)$ is constant on each stratum $\mathcal{W}_n^{\min,(\mathfrak{f})}$.

Proof. Fix $n\geq 1$ and let $S\to\mathbb{P}^1_k$ be a surface parametrized by $\mathcal{W}_n^{\min}$. For any relatively minimal elliptic surface with section one has

$$\sum_{v\in\mathbb{P}^1_{\bar{k}}} e(F_v) = e(S_{\bar{k}}),$$

and in our height-n locus this total Euler number equals $12n$ (equivalently, the discriminant has degree $12n$). For each singular fibre F_v, the Kodaira–Néron classification [Kod63, N64] gives a finite list of possible fibre types, and each type has a positive integer Euler number $e(F_v)\in\{1,2,\ldots,10\}$ (see [Her91, Table 1]). Hence there are only finitely many multisets of Kodaira symbols (equivalently, fibre configurations $\mathfrak{f}$) whose Euler numbers sum to $12n$. Therefore only finitely many configurations occur, and

$$\mathcal{W}_n^{\min} = \bigsqcup_{\mathfrak{f}} \mathcal{W}_n^{\min,(\mathfrak{f})}$$

is a finite stratification by locally closed substacks.

Finally, on a fixed stratum $\mathcal{W}_n^{\min,(\mathfrak{f})}$ the multiset $\mathfrak{f}$ (hence the integers m_v) is constant, so Lemma 1.3 implies that $T(S)=2+\sum_{v\in\mathfrak{f}}(m_v-1)$ is constant on that stratum. ∎

A multivariate height series. We briefly recall the local indexing used in the twisted maps description of height moduli. By [BPS22, Thm. 5.1] the height-n moduli stack $\mathcal{M}_{n,C}(\mathcal{X},\mathcal{L})$ admits a finite stratification by locally closed substacks indexed by admissible local conditions and degrees: there is a finite disjoint union of morphisms

$$\bigsqcup_{\Gamma,d} \mathcal{H}^{\Gamma}_{d,C}(\mathcal{X},\mathcal{L})/S_{\Gamma} \longrightarrow \mathcal{M}_{n,C}(\mathcal{X},\mathcal{L}),$$

where $\mathcal{H}^{\Gamma}_{d,C}(\mathcal{X},\mathcal{L})$ is the moduli stack of representable twisted morphisms of degree d to $(\mathcal{X},\mathcal{L})$ with admissible local condition

$$\Gamma = \big(\{r_1,a_1\},\ldots,\{r_s,a_s\}\big),$$

and the indices (Γ,d) range over those satisfying the height decomposition formula

$$n \;=\; d \;+\; \sum_{i=1}^{s} \frac{a_i}{r_i}.$$

Here $S_\Gamma \subset S_s$ is the subgroup permuting stacky marked points of the same local type.

Definition 2.2 (Elementary local patterns)**.** In the twisted-maps stratification, the local correction data are encoded by admissible local conditions

$$\Gamma = (\{r_1,a_1\},\ldots,\{r_s,a_s\}),$$

recording stabilizer orders r_i and characters a_i at stacky marked points of the source root stack.

For our purposes it is convenient to use a slightly finer indexing adapted to evaluation morphisms: instead of recording only the Kodaira symbol, we record the cyclotomic inertia component of $\mathcal{IP}(4,6)$ hit by the evaluation map at each marked gerbe (so that distinct components can refine the same Kodaira type by j-behaviour/automorphism data).

Let $\mathcal{A}$ denote the finite set of connected components of $\mathcal{IP}(4,6)$ with cyclotomic stabilizers that occur in the Tate-algorithm stratification via twisted maps (see [BPS22, §7]). We refer to elements $\alpha \in \mathcal{A}$ as *elementary local patterns*. For each $\alpha \in \mathcal{A}$, let $m(\alpha) \in \mathbb{Z}_{\geq 1}$ be the number of irreducible components of the associated geometric fibre. Thus $m(\alpha)$ refines the Kodaira symbol; distinct α may correspond to the same Kodaira type but different j-behaviour.

Remark 2.3 (Why inertia refinement is used)**.** Our strata are defined via evaluation morphisms landing in connected components of the cyclotomic inertia stack $\mathcal{IP}(4,6)$. A single Kodaira symbol may correspond to several such components (distinguished, for instance, by the special values $j = 0, 1728, \infty$ and the associated stabilizer/character data). For this reason we index local patterns by inertia components rather than by Kodaira symbols.

Definition 2.4 (Multivariate height series)**.** Introduce variables $\{x_\alpha\}_{\alpha\in\mathcal{A}}$. Define

$$\mathcal{H}(t;\mathbf{x}) \;:=\; 1 + \sum_{n\geq 1} \sum_{\mathfrak{f}} \Bigg(\prod_{v\in\mathfrak{f}} x_{\alpha_v}\Bigg) \big\{\mathcal{W}_n^{\min,(\mathfrak{f})}\big\} t^n \;\in\; K_0(\mathrm{Stck}_k)[\mathbf{x}][\![t]\!], \tag{6}$$

where for fixed n the inner sum ranges over the finitely many geometric Kodaira fibre configurations $\mathfrak{f}$ occurring in height n (cf. Proposition 2.1), and $\alpha_v \in \mathcal{A}$ is the elementary local pattern of the fibre at v. The term 1 corresponds to the empty configuration (i.e. no reducible fibers).

Lemma 2.5 (Evaluation factorization and finite Euler product)**.** *Let $\mathcal{A}$ be the finite set of elementary local patterns from Definition 2.2, and let $\mathcal{H}(t;\mathbf{x})$ be as in* (6). *After inverting $\mathbb{L}$, the series $\mathcal{H}(t;\mathbf{x})$ is a rational function of t with coefficients in $K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}][\mathbf{x}]$. More precisely, there exist a finite index set J, elements $A_j \in K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}]$, integers $c_j \in \mathbb{Z}_{\geq 1}$, and exponents $\beta_{j,\alpha} \in \mathbb{Z}_{\geq 0}$ such that*

$$\mathcal{H}(t;\mathbf{x}) = \prod_{j\in J} \frac{1}{1 - A_j\Big(\prod_{\alpha\in\mathcal{A}} x_\alpha^{\beta_{j,\alpha}}\Big)t^{c_j}} \in K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}][\mathbf{x}](t).$$

Proof. By [BPS22, Thm. 5.1], for each n the stack $\mathcal{W}_n^{\min}$ admits a finite locally closed stratification by charts of the form $\mathcal{H}^{\Gamma}_{d,\mathbb{P}^1_k}\big(\mathcal{P}(4,6),\mathcal{O}(1)\big)/S_\Gamma$, where $\mathcal{H}^{\Gamma}_{d,\mathbb{P}^1_k}$ is a moduli stack of representable twisted morphisms with ordered stacky marked gerbes and admissible local conditions Γ. On $\mathcal{H}^{\Gamma}_{d,\mathbb{P}^1_k}$ the i-th marked gerbe defines an evaluation morphism

$$\mathrm{ev}_i : \mathcal{H}^{\Gamma}_{d,\mathbb{P}^1_k}\big(\mathcal{P}(4,6),\mathcal{O}(1)\big) \to \mathcal{IP}(4,6),$$

and prescribing that the i-th marking has elementary local pattern $\alpha \in \mathcal{A}$ is equivalent to requiring ev_i to land in the corresponding connected component of $\mathcal{IP}(4,6)$; this is exactly the inertia-level encoding of Tate data via twisted maps, cf. [BPS22, Thm. 7.12].

Unordered collections of marked gerbes on the source root stack $\mathcal{C}$ are parametrized by symmetric powers of $\mathcal{C}$. The power structure on $K_0(\mathrm{Stck}_k)$ packages motivic classes of symmetric powers into Euler factors, turning the contribution of a fixed basic local condition into a geometric series in a monomial $\Big(\prod_{\alpha\in\mathcal{A}} x_\alpha^{\beta_{j,\alpha}}\Big)t^{c_j}$; see e.g. [GZLMH13, §3]. After inverting $\mathbb{L}$, each such basic local condition contributes an Euler factor of the form

$$\frac{1}{1 - A_j\Big(\prod_{\alpha\in\mathcal{A}} x_\alpha^{\beta_{j,\alpha}}\Big)t^{c_j}} \qquad \text{with } A_j \in K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}].$$

Multiplying over the finitely many basic local conditions yields the stated product formula for $\mathcal{H}(t;\mathbf{x})$.

Finally, finiteness holds because $\mathcal{A}$ is finite by definition and only finitely many basic local conditions occur in the stratification for minimal Weierstrass models (equivalently, only finitely many cyclotomic inertia components arise in the twisted-maps/Tate stratification), hence the index set J is finite. ∎

Remark 2.6 (Packets)**.** A *packet* is one Euler factor in Lemma 2.5. It is determined by $(A_j, c_j, (\beta_{j,\alpha})_{\alpha\in\mathcal{A}})$ and has the form

$$\frac{1}{1 - A_j\Big(\prod_{\alpha\in\mathcal{A}} x_\alpha^{\beta_{j,\alpha}}\Big)t^{c_j}}.$$

Rationality of Z_{Triv}. Define the *normalized* trivial lattice series

$$(7)\qquad \widetilde{Z}_{\mathrm{Triv}}(u;t) \;:=\; u^2\cdot\mathcal{H}\big(t;\; x_\alpha = u^{m(\alpha)-1}\big) \;\in\; K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}][u][\![t]\!],$$

so that

$$(8)\qquad Z_{\mathrm{Triv}}(u;t) \;=\; \widetilde{Z}_{\mathrm{Triv}}(u;t) \;-\; u^2.$$

(Thus Z_{Triv} starts at t^1, whereas $\widetilde{Z}_{\mathrm{Triv}}$ includes the constant term u^2 coming from the empty configuration in (6).)

Theorem 2.7 (Rationality and Euler product for Z_{Triv})**.** *Let k be a perfect field of characteristic $\neq 2,3$. Then $\widetilde{Z}_{\mathrm{Triv}}(u;t)$ is a rational function of t with coefficients in $K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}][u]$. More precisely, there exist a finite index set J, elements*

$$A_j \in K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}],\qquad b_j\in\mathbb{Z}_{\geq 0},\qquad c_j\in\mathbb{Z}_{\geq 1}\qquad (j\in J)$$

such that

$$(9)\qquad \widetilde{Z}_{\mathrm{Triv}}(u;t) = u^2\prod_{j\in J}\frac{1}{1-A_j\,u^{b_j}\,t^{c_j}} \;\in\; K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}][u](t).$$

Equivalently,

$$Z_{\mathrm{Triv}}(u;t) = u^2\prod_{j\in J}\frac{1}{1-A_j\,u^{b_j}\,t^{c_j}} - u^2.$$

Let $\mathcal{A}$ be the finite set of elementary local patterns (Definition 2.2), and for each $\alpha\in\mathcal{A}$ let $m(\alpha)\geq 1$ be the number of irreducible components of the corresponding geometric fibre. Choose data $(A_j, c_j, \beta_{j,\alpha})$ as in Lemma 2.5, so that

$$(10)\qquad \mathcal{H}(t;\mathbf{x}) = \prod_{j\in J}\frac{1}{1-A_j\Big(\prod_{\alpha\in\mathcal{A}} x_\alpha^{\beta_{j,\alpha}}\Big)t^{c_j}}\qquad \text{in } K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}][\mathbf{x}][\![t]\!].$$

Then, after the specialisation $x_\alpha = u^{m(\alpha)-1}$, the exponent b_j in (9) *may be taken to be*

$$(11)\qquad b_j = \sum_{\alpha\in\mathcal{A}}\beta_{j,\alpha}\big(m(\alpha)-1\big)\in\mathbb{Z}_{\geq 0}.$$

With this choice, the triple (A_j, b_j, c_j) records:

(i) Height step. *The exponent c_j is the height increment contributed by one copy of the local packet j (equivalently, inserting one copy increases the discriminant degree by* $12c_j$*).*

(ii) Trivial-lattice increment. *The exponent b_j is the increment in the trivial lattice rank contributed by one copy of the packet j, in the sense that* (11) *is exactly the contribution obtained from the fibre formula* $T(S) = 2+\sum_v(m_v-1)$ *under the identification $x_\alpha = u^{m(\alpha)-1}$.*

(iii) Motivic weight. *The coefficient A_j is the motivic class of the corresponding basic evaluation/configuration locus (after inverting $\mathbb{L}$), as it appears in the Euler factor in* (10)*.*

Consequently, each factor $(1-A_ju^{b_j}t^{c_j})^{-1}$ is the generating function for unordered collections of copies of the packet j, and the denominator in (9) *is the resulting finite u–refined Euler product.*

Proof. By Lemma 2.5, we have

$$\mathcal{H}(t;\mathbf{x})=\prod_{j\in J}\frac{1}{1-A_j\Big(\prod_{\alpha\in\mathcal{A}}x_\alpha^{\beta_{j,\alpha}}\Big)t^{c_j}}\qquad\text{in } K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}][\mathbf{x}][\![t]\!].$$

Specializing $x_\alpha\mapsto u^{m(\alpha)-1}$ yields

$$\mathcal{H}\big(t;\,x_\alpha=u^{m(\alpha)-1}\big)=\prod_{j\in J}\frac{1}{1-A_ju^{b_j}t^{c_j}},\qquad b_j=\sum_{\alpha\in\mathcal{A}}\beta_{j,\alpha}\big(m(\alpha)-1\big).$$

Multiplying by u^2 gives (9), and (8) gives the stated formula for $Z_{\mathrm{Triv}}(u;t)$. ■

Remark 2.8 (One–fibre motives and the origin of (A_j,b_j,c_j))**.** Assume $\mathrm{char}(k)\neq 2,3$. For each Kodaira type Θ and each $n\geq 1$, let

$$\mathcal{W}^{\Theta}_{n,\mathbb{P}^1}$$

denote the moduli stack of minimal elliptic fibrations over $\mathbb{P}^1_k$ with[1] discriminant degree $12n$, having exactly one specified singular fibre of type Θ over a varying degree–one point and being semistable everywhere else. Then [BPS22, Thm. 1.6], [HP19, Cor. 2] and [PS21, Prop. 3.1] yield the following table of motives:

Reduction type Θ with $j\in\overline{M}_{1,1}$	(r,a)	m_v-1	$\dfrac{\{\mathcal{W}^{\Theta}_{n,\mathbb{P}^1}\}}{\{\mathrm{PGL}_2\}\,\mathbb{L}^{10n-18}}\in K_0(\mathrm{Stck}_k)$
$\mathrm{I}_{k\geq 0}$ with $j=\infty$	$(0,0)$	$k-1$	$\mathbb{L}^{16}$
II with $j=0$	$(6,1)$	0	$\mathbb{L}^{15}$
III with $j=1728$	$(4,1)$	1	$\mathbb{L}^{14}$
IV with $j=0$	$(3,1)$	2	$\mathbb{L}^{13}$
$\mathrm{I}^*_{k>0}$ with $j=\infty$	$(2,1)$	$4+k$	$\mathbb{L}^{12}-\mathbb{L}^{11}$
I^*_0 with $j\neq 0,1728$	$(2,1)$	4	$\mathbb{L}^{12}-\mathbb{L}^{11}$
I^*_0 with $j=0,1728$	$(2,1)$	4	$\mathbb{L}^{11}$
IV^* with $j=0$	$(3,2)$	6	$\mathbb{L}^{10}$
III^* with $j=1728$	$(4,3)$	7	$\mathbb{L}^{9}$
II^* with $j=0$	$(6,5)$	8	$\mathbb{L}^{8}$

These one–fibre motives are the basic building blocks for the Euler factors in Theorem 2.7.

[1] The *unparameterized* $\mathbb{P}^1_k$ corresponds to taking the $\mathrm{Aut}(\mathbb{P}^1)=\mathrm{PGL}_2$ stack quotient on $\mathcal{W}^{\min}_n=\mathcal{M}_{n,\mathbb{P}^1_k}(\overline{\mathcal{M}}_{1,1},\mathcal{L})$ motivically factoring out $\{\mathrm{PGL}_2\}=\mathbb{L}(\mathbb{L}^2-1)$, thereby treating the base as a smooth conic curve. For a comprehensive treatment, we refer the reader to [PS25].

3. IRRATIONALITY OF THE NÉRON–SEVERI AND MORDELL–WEIL SPECIALIZATIONS

The rationality of $Z_{\mathrm{Triv}}(u;t)$ reflects the fact that the trivial lattice rank $T(S)$ is governed by *local* reduction data. Indeed, by Lemma 1.3 it depends only on the multiset of fibre component numbers m_v, hence is constant on each Kodaira stratum $\mathcal{W}_n^{\min,(\mathfrak{f})}$, in the finite constructible stratification of Proposition 2.1. This locality is exactly what makes the evaluation-map factorization and the power structure on $K_0(\mathrm{Stck}_k)$ applicable: unordered collections of local packets assemble into a *finite* Euler product.

In contrast, the Mordell–Weil rank $\mathrm{rk}(E/K)$ is not determined by the fibre configuration. Even on a fixed Kodaira stratum $\mathcal{W}_n^{\min,(\mathfrak{f})}$, the rank typically varies, reflecting genuinely *global* constraints rather than local reduction data. Since $T(S)$ is constant on $\mathcal{W}_n^{\min,(\mathfrak{f})}$, the Shioda–Tate formula (1) shows that variation of $\mathrm{rk}(E/K)$ is equivalent to variation of the Néron–Severi rank $\rho(S)$. Thus any refinement of the height series by $\mathrm{rk}(E/K)$, or equivalently by $\rho(S)$, necessarily detects global jump phenomena invisible to the local packet stratification used for Z_{Triv}.

One way to organize this global complexity is via Néron–Severi jump loci. Fix a fibre configuration $\mathfrak{f}$ and write $\mathrm{Triv}^{(\mathfrak{f})} \subset \mathrm{NS}(S_{\bar{k}})$ for the sublattice generated by the zero section, a fibre class, and the components of reducible fibres in the configuration $\mathfrak{f}$. Inside $\mathcal{W}_n^{\min,(\mathfrak{f})}$, imposing that $\mathrm{NS}(S_{\bar{k}})$ contain additional algebraic classes *independent of* $\mathrm{Triv}^{(\mathfrak{f})}$ (equivalently, that $\rho(S)$, hence $\mathrm{rk}(E/K)$, jump) is an algebraic condition. Over $\mathbb{C}$, these conditions are naturally modeled by Noether–Lefschetz (Hodge) loci for the variation of Hodge structure coming from the family of elliptic surfaces over $\mathcal{W}_n^{\min,(\mathfrak{f})}$, and the theorem of Cattani–Deligne–Kaplan [CDK95] implies that such loci are, in general, only a *countable union of closed algebraic subsets*. In particular, unlike the Kodaira stratification at fixed height n (which is finite by Proposition 2.1), refinements by Néron–Severi *lattice type* are not expected to admit a finite constructible stratification.

This suggests that the local-to-global factorization mechanism producing a finite Euler product for Z_{Triv} should fail for the Mordell–Weil and Néron–Severi specializations, and motivates the following conjecture.

Conjecture 3.1 (Irrationality over $\mathbb{C}$)**.** Let $k = \mathbb{C}$ and $K = \mathbb{C}(t)$. The specializations

$$Z_{\mathrm{MW}}(v;t) := \mathcal{Z}(1,v;t), \qquad Z_{\mathrm{NS}}(w;t) := \mathcal{Z}(w,w;t)$$

do not become rational functions of t after inverting $\mathbb{L}$, i.e.

$$Z_{\mathrm{MW}}(v;t) \notin K_0(\mathrm{Stck}_{\mathbb{C}})[\mathbb{L}^{-1}][v](t), \qquad Z_{\mathrm{NS}}(w;t) \notin K_0(\mathrm{Stck}_{\mathbb{C}})[\mathbb{L}^{-1}][w](t).$$

In other words, after inverting $\mathbb{L}$ neither bivariate height zeta series is rational in the height variable t.

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