RATIONALITY OF THE TRIVIAL LATTICE RANK WEIGHTED MOTIVIC HEIGHT ZETA FUNCTION FOR ELLIPTIC SURFACES

JUN–YONG PARK

ABSTRACT. Let k be a perfect field with $\mathrm{char}(k) \neq 2, 3$, set $K = k(t)$, and let $\mathcal{W}_n^{\min}$ be the moduli stack of minimal elliptic curves over K of Faltings height n from the height–moduli framework of [BPS22] applied to $\overline{\mathcal{M}}_{1,1} \simeq \mathcal{P}(4,6)$. For $[E] \in \mathcal{W}_n^{\min}$, let $S \to \mathbb{P}^1_k$ be the associated elliptic surface with section. Motivated by the Shioda–Tate formula, we consider the trivariate motivic height zeta function

$$\mathcal{Z}(u, v; t) := \sum_{n \geq 0} \Big(\sum_{[E] \in \mathcal{W}_n^{\min}} u^{T(S)} \, v^{\mathrm{rk}(E/K)} \Big) t^n \in K_0(\mathrm{Stck}_k)[u, v][\![t]\!],$$

which refines the height series by weighting each height stratum with the trivial lattice rank $T(S)$ and the Mordell–Weil rank $\mathrm{rk}(E/K)$. We prove rationality for the trivial lattice specialization $Z_{\mathrm{Triv}}(u; t) = \mathcal{Z}(u, 1; t)$ by giving an explicit finite Euler product. We conjecture irrationality for the Néron–Severi $Z_{\mathrm{NS}}(w; t) = \mathcal{Z}(w, w; t)$ and the Mordell–Weil $Z_{\mathrm{MW}}(v; t) = \mathcal{Z}(1, v; t)$ specializations.

1. INTRODUCTION

Let k be a perfect field with $\mathrm{char}(k) \neq 2, 3$, and set $K := k(t)$. An elliptic curve E/K determines a relatively minimal elliptic surface with section

$$f : S \longrightarrow \mathbb{P}^1_k$$

unique up to isomorphism (see [Mir89, SS10] for background on elliptic surfaces).

The arithmetic of E/K is reflected in the geometry of S, and a basic organizing principle is the Shioda–Tate formula [Shi90]

$$\rho(S) \;=\; T(S) \;+\; \mathrm{rk}(E/K), \tag{1}$$

where $\rho(S) = \mathrm{rk}\,\mathrm{NS}(S_{\bar{k}})$ is the *geometric Picard rank*, $T(S)$ is the *rank of the geometric trivial lattice* generated by the zero section, a fiber class, and the components of reducible fibers not meeting the zero section, and $\mathrm{rk}(E/K)$ is the *Mordell–Weil rank*. For the relatively minimal elliptic surfaces $f : S \to \mathbb{P}^1_k$ with section considered in this paper, we have $q(S) = 0$ and $p_g(S) = n - 1$, hence the standard bounds

$$2 \leq \rho(S) \leq 10n, \qquad 2 \leq T(S) \leq 10n, \qquad 0 \leq \mathrm{rk}(E/K) \leq 10n - 2, \tag{2}$$

where $\rho(S) \leq 10n = h^{1,1}(S)$ is the Lefschetz bound over $k = \mathbb{C}$ (or in general the Igusa's inequality $\rho(S) \leq b_2(S) = 12n - 2$).

In [BPS22], Bejleri–Park–Satriano construct height-moduli stacks of rational points on proper polarized cyclotomic stacks. In the fundamental modular curve example

$$\overline{\mathcal{M}}_{1,1} \simeq \mathcal{P}(4,6), \qquad \lambda \simeq \mathcal{O}_{\mathcal{P}(4,6)}(1),$$

a minimal elliptic curve over K can be viewed as a rational point of λ–height n on $\overline{\mathcal{M}}_{1,1}$ over K. This yields a separated Deligne–Mumford stack of finite type

$$\mathcal{W}_n^{\min} \;:=\; \mathcal{W}^{\min}_{n, \mathbb{P}^1_k}\big(\mathcal{P}(4,6), \mathcal{O}(1)\big)$$

parametrizing minimal elliptic curves over K of discriminant degree $12n$. Here a K-rational point of $\overline{\mathcal{M}}_{1,1}$ of λ-height n means the stacky height n with respect to the Hodge line bundle λ, in the sense of [ESZB23]. Under the identification $\overline{\mathcal{M}}_{1,1} \cong \mathcal{P}(4,6)$ one has $\lambda \simeq \mathcal{O}_{\mathcal{P}(4,6)}(1)$, and this height agrees with the Faltings height of the corresponding elliptic curve by [BPS22, Cor. 7.6].

Guided by (1), we introduce the following motivic generating series (see [Eke25] for background on the Grothendieck ring of stacks) refining the height generating series in [BPS22, §8] by weighting each height stratum with the *lattice ranks* of the associated relatively minimal elliptic surface.

Definition 1.1. Let k be a perfect field of characteristic $\neq 2, 3$, and consider the height–moduli stack

$$\mathcal{W}_n^{\min} = \mathcal{W}^{\min}_{n,\mathbb{P}^1_k}\big(\mathcal{P}(4,6), \mathcal{O}(1)\big)$$

parametrizing minimal elliptic curves over $K = k(t)$ of discriminant height $12n$. The *trivariate height zeta function* is

$$\mathcal{Z}(u,v;t) := \sum_{n\geq 0}\left(\sum_{[E]\in\mathcal{W}_n^{\min}} u^{T(S)}\, v^{\operatorname{rk}(E/K)}\right) t^n \in K_0(\mathrm{Stck}_k)[u,v][\![t]\!],$$

where for each $[E] \in \mathcal{W}_n^{\min}$ we write $S \to \mathbb{P}^1_k$ for the associated relatively minimal elliptic surfaces $f : S \to \mathbb{P}^1_k$ with section, and:

- $T(S)$ is the rank of the trivial lattice of S;
- $\operatorname{rk}(E/K)$ is the Mordell–Weil rank.

The following specializations are the associated *bivariate height zeta functions*:

$$Z_{\mathrm{Triv}}(u;t) := \mathcal{Z}(u,1;t), \tag{3}$$

$$Z_{\mathrm{MW}}(v;t) := \mathcal{Z}(1,v;t), \tag{4}$$

$$Z_{\mathrm{NS}}(w;t) := \mathcal{Z}(w,w;t). \tag{5}$$

Remark 1.2. Setting $u = v = 1$ forgets the lattice rank grading and specializes to the *univariate motivic height zeta function* $Z_{\bar{\lambda}}(t) = \mathcal{Z}(1,1;t) \in K_0(\mathrm{Stck}_k)[\![t]\!]$ and likewise to its inertial refinement $\mathcal{I}Z_{\bar{\lambda}}(t)$ which encodes the totality of rational points on $\overline{\mathcal{M}}_{1,1}$ over $K = k(t)$. Theorem [BPS22, Thm. 8.9] shows that both series are in fact rational in t, i.e. lie in $K_0(\mathrm{Stck}_k)(t)$, and gives explicit formulas.

In this paper we focus on $Z_{\mathrm{Triv}}(u;t)$. The key point is that the trivial lattice is governed by *local bad reduction*: its rank is determined by the geometric Kodaira fiber configuration of $\pi_{\bar{k}} : S_{\bar{k}} \to \mathbb{P}^1_{\bar{k}}$. Writing $\mathrm{Triv}(S) \subset \mathrm{NS}(S_{\bar{k}})$ for the geometric trivial lattice and $T(S) := \operatorname{rk}(\mathrm{Triv}(S))$, we have the following explicit formula.

Lemma 1.3. *Let $\pi : S \to \mathbb{P}^1_k$ be a relatively minimal elliptic surface with section, and let $\mathfrak{f}$ be the multiset of singular fibers of $\pi_{\bar{k}} : S_{\bar{k}} \to \mathbb{P}^1_{\bar{k}}$. If m_v denotes the number of irreducible components of the fiber at v, then*

$$T(S) = 2 + \sum_{v\in\mathfrak{f}} (m_v - 1).$$

Definition 1.4. Fix $n \geq 1$. For a geometric fiber configuration $\mathfrak{f}$, let $\mathcal{W}_n^{\min,(\mathfrak{f})} \subset \mathcal{W}_n^{\min}$ denote the locus parametrizing those $[E] \in \mathcal{W}_n^{\min}$ whose associated surface $S_{\bar{k}} \to \mathbb{P}^1_{\bar{k}}$ has singular fiber configuration $\mathfrak{f}$ (c.f. [BPS22, Thm. 7.16]).

Definition 1.5. Fix $n \geq 0$. By Proposition 2.1, $\mathcal{W}_n^{\min}$ admits a *finite* constructible stratification by Kodaira data, and $T(S)$ is constant on each stratum. For $n \geq 1$ and each T with $2 \leq T \leq 10n$, let

$$\mathcal{W}_n^{\min}(T) \subset \mathcal{W}_n^{\min}$$

be the finite union of those Kodaira strata on which $T(S) = T$ (hence a finite union of locally closed substacks). For $n = 0$, set $\mathcal{W}_0^{\min} := \mathcal{W}_0^{\min}(2)$.

The trivial–lattice–rank–weighted motivic height zeta function is

$$Z_{\mathrm{Triv}}(u;t) := \sum_{n \geq 0} \sum_{T \geq 2} u^T \{\mathcal{W}_n^{\min}(T)\}\, t^n \;\in\; K_0(\mathrm{Stck}_k)[u][\![t]\!].$$

We prove that $Z_{\mathrm{Triv}}(u;t)$ is rational after inverting $\mathbb{L}$ (see Remark 2.6), and we give an explicit finite Euler product (Theorem 2.9).

Theorem 1.6. *Let k be a perfect field with* $\mathrm{char}(k) \neq 2,3$ *and put $s = t^{1/12}$. Then*

$$Z_{\mathrm{Triv}}(u;t) \in K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}][u](s).$$

Moreover, $Z_{\mathrm{Triv}}(u;t)$ admits an explicit finite Euler product in s.

The proof is a motivic local-to-global factorization argument [CLL16], implemented on the twisted-map stratification of the height–moduli $\mathcal{W}_n^{\min}$ via the evaluation morphisms [GP06]. Throughout we use the Bejleri–Park–Satriano correspondence [BPS22, Thm. 3.3] between rational points, minimal weighted linear series, and twisted morphisms; in particular, local reduction conditions are encoded by representable twisted morphisms to $\overline{\mathcal{M}}_{1,1} \simeq \mathcal{P}(4,6)$, yielding a moduli-theoretic Tate's algorithm [Tat75] compatible with the minimal model program [BPS22, Thm. 7.12]. Unordered collections of local factors supported at distinct points of $\mathbb{P}^1$ are governed by symmetric powers $\mathrm{Sym}^N(\mathbb{P}^1)$. We reorganize these symmetric-power contributions using the power structure on $K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}]$, and we record the resulting identity explicitly in Lemma 2.7. Since only finitely many local factor types occur, this yields a finite Euler product after inverting $\mathbb{L}$ [Kap00, GZLMH13]. The only unbounded discrete parameter is the cusp contact order in the two families I_k and I_k^*, which is collapsed by geometric resummation. Finally, specializing $x_\alpha = u^{m(\alpha)-1}$ for $\alpha \in \mathcal{A}_{\mathrm{nc}}$ together with the cusp substitutions produces the Euler product expression for $Z_{\mathrm{Triv}}(u;t)$ in $K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}][u](s)$ with $t = s^{12}$.

Remark 1.7. Replacing $\mathcal{W}_n^{\min}(T)$ by its inertia stack (see [HP23, §2] for background on the inertia stack $\mathcal{I}(\mathcal{X})$ of an algebraic stack $\mathcal{X}$) gives

$$\mathcal{I}Z_{\mathrm{Triv}}(u;t) := \sum_{n \geq 0} \sum_{T \geq 2} u^T \left\{\mathcal{I}\mathcal{W}_n^{\min}(T)\right\} t^n \in K_0(\mathrm{Stck}_k)[u][\![t]\!].$$

After inverting $\mathbb{L}$, the same argument yields a finite Euler product for $\mathcal{I}Z_{\mathrm{Triv}}(u;t)$.

2. Rationality of the trivial lattice rank weighted case

Throughout, let k be a perfect field with $\mathrm{char}(k) \neq 2,3$, set $K = k(t)$, and let $\pi \colon S \to \mathbb{P}^1_k$ be the relatively minimal elliptic surface with section associated to E/K. Write $\mathrm{Triv}(S) \subset \mathrm{NS}(S_{\bar{k}})$ for the geometric trivial lattice and $T(S) = \mathrm{rk}\,\mathrm{Triv}(S)$.

Proposition 2.1. *Fix $n \geq 1$. The discriminant degree constraint $\sum_v e(F_v) = 12n$ implies that only finitely many geometric fiber configurations $\mathfrak{f}$ occur among surfaces parametrized by $\mathcal{W}_n^{\min}$. Consequently,*

$$\mathcal{W}_n^{\min} = \bigsqcup_{\mathfrak{f}} \mathcal{W}_n^{\min,(\mathfrak{f})}$$

is a finite constructible stratification. Moreover, the trivial lattice rank $T(S)$ is constant on each stratum $\mathcal{W}_n^{\min,(\mathfrak{f})}$.

Proof. Fix $n \geq 1$ and let $S \to \mathbb{P}^1_k$ be a surface parametrized by $\mathcal{W}_n^{\min}$. For any relatively minimal elliptic surface with section one has $\sum_{v \in \mathbb{P}^1_{\bar{k}}} e(F_v) = e(S_{\bar{k}})$ and in our height-n locus this total Euler number equals $12n$ (equivalently, the discriminant has degree $12n$). For each singular fiber F_v, the Kodaira–Néron classification [Kod63, N64́] gives the types $\mathrm{I}_k, \mathrm{I}_k^*$ ($k \geq 1$) and $\mathrm{II}, \mathrm{III}, \mathrm{IV}, \mathrm{I}_0^*, \mathrm{IV}^*, \mathrm{III}^*, \mathrm{II}^*$. Their Euler numbers satisfy $e(\mathrm{I}_k) = k$, $e(\mathrm{I}_k^*) = k + 6$ while the remaining types have Euler number $e(F_v) \in \{2,3,4,6,8,9,10\}$ (see [Her91, Table 1]). Since $\sum_v e(F_v) = 12n$, the integers k occurring in fibers of type I_k and I_k^* are bounded in terms of n. Hence there are only finitely many multisets of Kodaira symbols (equivalently, fiber configurations $\mathfrak{f}$) whose Euler numbers sum to $12n$. Therefore only finitely many configurations occur, and $\mathcal{W}_n^{\min} = \bigsqcup_{\mathfrak{f}} \mathcal{W}_n^{\min,(\mathfrak{f})}$ is a finite stratification by locally closed substacks as in [BPS22, Thm. 7.16]. Finally, on a fixed stratum $\mathcal{W}_n^{\min,(\mathfrak{f})}$ the multiset $\mathfrak{f}$ (hence the integers m_v) is constant, so Lemma 1.3 implies that $T(S) = 2 + \sum_{v \in \mathfrak{f}} (m_v - 1)$ is constant on that stratum. ∎

A multivariate height series. We briefly recall the local indexing used in the twisted-maps description of height–moduli. By [BPS22, Thm. 5.1] the height-n moduli stack $\mathcal{M}_{n,C}(\mathcal{X}, \mathcal{L})$ on a proper polarized cyclotomic stack $\mathcal{X}$ with polarizing line bundle $\mathcal{L}$ admits a finite stratification by locally closed substacks indexed by admissible local conditions and degrees: there is a finite disjoint union of morphisms

$$\bigsqcup_{\Gamma,d} \mathcal{H}^{\Gamma}_{d,C}(\mathcal{X}, \mathcal{L})/S_\Gamma \longrightarrow \mathcal{M}_{n,C}(\mathcal{X}, \mathcal{L}),$$

where $\mathcal{H}^{\Gamma}_{d,C}(\mathcal{X}, \mathcal{L})$ is the moduli stack of representable twisted morphisms of stable height d to $(\mathcal{X}, \mathcal{L})$ with and local twisting conditions

$$\Gamma = \big(\{r_1, a_1\}, \ldots, \{r_s, a_s\}\big),$$

recording the stabilizer orders r_i and the corresponding characters a_i at the stacky marked points of the source root stack. The indices (Γ, d) range over those satisfying the height decomposition formula

$$n = d + \sum_{i=1}^{s} \frac{a_i}{r_i}.$$

Here $S_\Gamma \subset S_s$ is the subgroup permuting stacky marked points of the same local type.

Definition 2.2. For the Euler-product argument it is useful to distinguish *local factor types* from *elementary local patterns*. Let $\mathcal{IP}(4,6)$ be the cyclotomic inertia stack.

(1) local factor types. Let J denote the finite set of local factor types occurring in the Tate-algorithm stratification via twisted maps (see [BPS22, §7]); concretely one may take

$$J = \Big\{\mathrm{II},\ \mathrm{III},\ \mathrm{IV},\ \mathrm{II}^*,\ \mathrm{III}^*,\ \mathrm{IV}^*,\ \mathrm{I}_0^*(j \neq 0, 1728),\ \mathrm{I}_0^*(j \in \{0, 1728\}),\ \mathrm{I}_\bullet,\ \mathrm{I}_\bullet^*\Big\},$$

where $\mathrm{I}_\bullet$ and $\mathrm{I}_\bullet^*$ are the two cusp *shapes* over $j = \infty$.

(2) Elementary local patterns. Let $\mathcal{A}$ denote the set of elementary local patterns used to index evaluation conditions, i.e. the inertia components in which the evaluation maps land. Away from the cusp $j = \infty$, the inertia label determines the Kodaira symbol, so the non-cusp patterns form a finite set

$$\mathcal{A}_{\mathrm{nc}} = \Big\{\mathrm{II},\ \mathrm{III},\ \mathrm{IV},\ \mathrm{II}^*,\ \mathrm{III}^*,\ \mathrm{IV}^*,\ \mathrm{I}_0^*(j \neq 0, 1728),\ \mathrm{I}_0^*(j \in \{0, 1728\})\Big\}.$$

At the cusp $j = \infty$, the inertia label records only the cusp shape ($\mathrm{I}_\bullet$ or $\mathrm{I}_\bullet^*$); the additional integer $k \geq 1$ (contact order with the boundary, equivalently the pole order of j) is part of the admissible local data on a twisted-maps chart and is treated as a multiplicity parameter within the cusp shape. Accordingly we set

$$\mathcal{A} := \mathcal{A}_{\mathrm{nc}} \sqcup \{\mathrm{I}_\bullet,\ \mathrm{I}_\bullet^*\}.$$

For $\alpha \in \mathcal{A}_{\mathrm{nc}}$, let $m(\alpha) \in \mathbb{Z}_{\geq 1}$ be the number of irreducible components of the corresponding Kodaira fiber, so that $m(\alpha)-1$ is its contribution to the trivial lattice. For the cusp shapes $\mathrm{I}_\bullet$ and $\mathrm{I}_\bullet^*$, the component number depends on the contact order $k \geq 1$ (of the corresponding I_k or I_k^* fiber); this k–dependence will be incorporated later by geometric resummation (Lemma 2.8). In summary, J indexes the *local factor types* (basic chart types) that become Euler factors under the power structure on $K_0(\mathrm{Stck}_k)$, whereas $\mathcal{A}$ indexes the evaluation labels, i.e. exactly what inertia can see; in particular, over $j = \infty$ inertia distinguishes only the two cusp shapes and not the contact order k.

Remark 2.3. When an evaluation condition lands over the cusp $j = \infty$, the corresponding component of the cyclotomic inertia stack $\mathcal{IP}(4,6)$ records only the *cusp shape* (multiplicative $\mathrm{I}_\bullet$ or additive $\mathrm{I}_\bullet^*$); it does *not* record the *multiplicity* $k \geq 1$. Equivalently, inertia detects that j has a pole, but not its pole order. The missing discrete datum is the *contact order with the boundary*. Geometrically, it is visible on the log canonical model obtained by contracting, in each reducible fiber, the components not meeting the zero section.

(1) The multiplicative family I_k. If the fiber at $t \in \mathbb{P}^1$ is of type I_k ($k \geq 1$), then the contraction produces an A_{k-1} surface singularity. Étale locally one has

$$xy = u^k,$$

where u is a local parameter at t. Since an étale neighbourhood of the universal nodal fiber over the cusp $[\infty] \in \overline{\mathcal{M}}_{1,1} \simeq \mathcal{P}(4,6)$ is given by $xy = s$ (with s a parameter at the cusp), the classifying map $\varphi_g : \mathbb{P}^1 \to \overline{\mathcal{M}}_{1,1}$ satisfies $s = u^k$. Thus φ_g meets the boundary with contact order k, and $v(\Delta) = k$ for type I_k.

(2) The additive family I_k^*. If the fiber at t is of type I_k^* ($k \geq 1$), then the contraction produces a D_{k+4} surface singularity. The classifying map still lands at $j = \infty$ with boundary contact order k (so locally $s = u^k$), while the discriminant valuation is shifted by the starred contribution: $v(\Delta) = k + 6$ for type I_k^*.

For Z_{Triv} one has

$$m(\mathrm{I}_k) - 1 = k - 1, \qquad m(\mathrm{I}_k^*) - 1 = k + 4,$$

so the trivial lattice exponent depends linearly on k in each cusp family.

Definition 2.4. Fix an auxiliary variable s with $s^{12} = t$. Introduce variables $\{x_\alpha\}_{\alpha \in \mathcal{A}}$ and define

$$\mathcal{H}(s;\mathbf{x}) := \sum_{n \geq 0} \sum_{\mathfrak{f}} \Big(\prod_{v \in \mathfrak{f}} x_{\alpha_v} \Big) \{\mathcal{W}_n^{\min,(\mathfrak{f})}\} s^{12n} \in K_0(\mathrm{Stck}_k)[\mathbf{x}][\![s]\!], \tag{6}$$

where for fixed n the inner sum ranges over the finitely many geometric fiber configurations $\mathfrak{f}$ occurring in height n (Proposition 2.1).

For each singular fiber F_v in $\mathfrak{f}$, let $\alpha_v \in \mathcal{A}$ denote the corresponding inertia/evaluation label (Definition 2.2). Away from the cusp $j = \infty$ this label is the Kodaira symbol, while over $j = \infty$ it records only the cusp shape $\mathrm{I}_\bullet$ or $\mathrm{I}_\bullet^*$. The additional contact order $k \geq 1$ at the cusp (Remark 2.3) is part of the twisted-maps chart data and is *not* recorded by the variables x_α.

Remark 2.5. The local conditions defining the strata are imposed via evaluation maps ev_i to $\mathcal{IP}(4,6)$, hence are naturally indexed by connected components of the inertia stack. In particular, the same Kodaira symbol may correspond to distinct inertia components. For example, I_0^* splits into distinct inertia components according to whether $j \in \{0, 1728\}$ or $j \notin \{0, 1728\}$. Accordingly we index local conditions by inertia labels, not by Kodaira symbols alone.

Remark 2.6. We work in the localized ring $K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}]$. Localization is used to place the argument in a ring where quotient stack identities for linear algebraic groups (e.g. GL_n, PGL_2) and the power–structure identities for symmetric powers hold uniformly as equalities of rational functions, thereby justifying the reorganization of unordered local factors into Euler factors.

Lemma 2.7. *Let $\mathcal{A}$ be the finite set of elementary local patterns from Definition 2.2, and let $\mathcal{H}(s;\mathbf{x})$ be the multivariate height series defined in* (6). *After inverting $\mathbb{L}$, the series $\mathcal{H}(s;\mathbf{x})$ is a rational function of s with coefficients in $K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}][\mathbf{x}]$.*

More precisely, there exist:

- *a finite index set J of local factor types,*
- *motivic classes $A_j \in K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}]$,*
- *integers $c_j \geq 1$, recording the discriminant degree increment contributed by one local factor of type j,*

- *and exponents $\beta_{j,\alpha} \in \mathbb{Z}_{\geq 0}$ for $\alpha \in \mathcal{A}$, recording how many markings of inertia type α occur in a local factor of type j,*

such that

$$\mathcal{H}(s;\mathbf{x}) = \prod_{j\in J}\left(1 - A_j\Big(\prod_{\alpha\in\mathcal{A}} x_\alpha^{\beta_{j,\alpha}}\Big)s^{c_j}\right)^{-\{\mathbb{P}^1\}} \in K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}][\mathbf{x}](s). \tag{7}$$

Equivalently, writing

$$Y_j(s;\mathbf{x}) := A_j\Big(\prod_{\alpha\in\mathcal{A}} x_\alpha^{\beta_{j,\alpha}}\Big)s^{c_j},$$

one has the explicit specialization

$$(1 - Y_j)^{-\{\mathbb{P}^1\}} = \sum_{N\geq 0}\{\mathrm{Sym}^N(\mathbb{P}^1)\}\, Y_j^N = \frac{1}{(1-Y_j)(1-\mathbb{L}\,Y_j)}. \tag{8}$$

Moreover, for $\alpha \in \{\mathrm{I}_\bullet, \mathrm{I}_\bullet^*\}$ the exponent $\beta_{j,\alpha}$ counts only the number of cusp markings of the given cusp shape *in factor type j; it does not record the contact order $k \geq 1$.*

Proof. By [BPS22, Thm. 7.16], for each n the stack $\mathcal{W}_n^{\min}$ admits a finite locally closed stratification by charts of the form

$$\mathcal{H}^{\Gamma}_{d,\mathbb{P}^1_k}\big(\mathcal{P}(4,6),\mathcal{O}(1)\big)/S_\Gamma,$$

where $\mathcal{H}^{\Gamma}_{d,\mathbb{P}^1_k}(\mathcal{P}(4,6),\mathcal{O}(1))$ parametrizes representable twisted morphisms with an *ordered* list of stacky markings and admissible local data. We write the admissible local condition as

$$\Gamma = \big(\Gamma_{\mathrm{nc}}, (\mathbf{k}^{\mathrm{I}}, \mathbf{k}^{\mathrm{I}^*})\big), \qquad \mathbf{k}^{\mathrm{I}} = (k_1^{\mathrm{I}}, \ldots, k_{m_{\mathrm{I}}}^{\mathrm{I}}),\ \ \mathbf{k}^{\mathrm{I}^*} = (k_1^{\mathrm{I}^*}, \ldots, k_{m_{\mathrm{I}^*}}^{\mathrm{I}^*}),$$

where Γ_{nc} records the ordered list of non-cusp inertia labels, and $\mathbf{k}^{\mathrm{I}}, \mathbf{k}^{\mathrm{I}^*}$ record the contact orders at cusp markings of shape $\mathrm{I}_\bullet$ and $\mathrm{I}_\bullet^*$.

For each marking there is an evaluation morphism

$$\mathrm{ev}_i : \mathcal{H}^{\Gamma}_{d,\mathbb{P}^1_k}\big(\mathcal{P}(4,6),\mathcal{O}(1)\big) \to \mathcal{IP}(4,6), \qquad (\varphi, \Sigma_1, \ldots, \Sigma_s) \longmapsto \varphi(\Sigma_i),$$

and prescribing an inertia label $\alpha \in \mathcal{A}$ is equivalent to requiring ev_i to land in the corresponding connected component of $\mathcal{IP}(4,6)$. Over the cusp $j = \infty$, the inertia label records only the cusp *shape* $\alpha \in \{\mathrm{I}_\bullet, \mathrm{I}_\bullet^*\}$; the contact orders k_j are extra admissible boundary contact data on the chart (Remark 2.3). The variables $\mathbf{x} = \{x_\alpha\}_{\alpha\in\mathcal{A}}$ therefore record only inertia labels, i.e. only what can be read off from the evaluations ev_i, and $\mathcal{H}(s;\mathbf{x})$ is obtained from the stratification by forgetting the extra contact-order data.

Passing to the quotient by S_Γ forgets the ordering among markings of the same inertia type. Fix a local factor type j. Repeating this local factor N times is governed by unordered configurations of N support points on the coarse curve $\mathbb{P}^1$, hence by the symmetric power $\mathrm{Sym}^N(\mathbb{P}^1)$. Set

$$Y_j(s;\mathbf{x}) := A_j\Big(\prod_{\alpha\in\mathcal{A}} x_\alpha^{\beta_{j,\alpha}}\Big)s^{c_j}.$$

Accordingly, the contribution of all unordered collections of local factors of type j sums to

$$\sum_{N\geq 0}\{\mathrm{Sym}^N(\mathbb{P}^1)\}\cdot Y_j(s;\mathbf{x})^N.$$

By the power-structure/Kapranov zeta-function identity on $K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}]$ as in [GZLMH13, §1],

$$\sum_{N\geq 0}\{\mathrm{Sym}^N(X)\}\cdot y^N=(1-y)^{-\{X\}},$$

this equals $(1-Y_j)^{-\{\mathbb{P}^1\}}$. Since $\{\mathbb{P}^1\}=1+\mathbb{L}$, we obtain the explicit rational form

$$(1-Y_j)^{-\{\mathbb{P}^1\}}=\frac{1}{(1-Y_j)(1-\mathbb{L}\cdot Y_j)}. \tag{9}$$

Since only finitely many local factor types occur (Definition 2.2), multiplying over $j\in J$ yields (7) and hence rationality of $\mathcal{H}(s;\mathbf{x})$ in s after inverting $\mathbb{L}$.

Finally, for $\alpha\in\{\mathrm{I}_\bullet,\mathrm{I}_\bullet^*\}$ the exponent $\beta_{j,\alpha}$ counts only the number of cusp markings of the given cusp *shape* in factor type j; the individual contact orders k_j are handled separately by geometric resummation. ■

Lemma 2.8. *Let R be a commutative ring.*

(1) Geometric resummation *Fix $A\in R$. For integers $a,c\geq 1$ and $b,d\geq 0$, one has in $R[\![u,t]\!]$*

$$\sum_{k\geq 1}Au^{ak+b}t^{ck+d}=Au^{a+b}t^{c+d}\cdot\frac{1}{1-u^at^c}. \tag{10}$$

Moreover, if $k_1,\dots,k_M\geq 1$ are independent and contribute multiplicatively with the same step (a,c), then

$$\sum_{k_1,\dots,k_M\geq 1}A\prod_{i=1}^{M}u^{ak_i+b}t^{ck_i+d}=A\Big(u^{a+b}t^{c+d}\Big)^M\cdot\frac{1}{(1-u^at^c)^M}. \tag{11}$$

Equivalently, each marking contributes one factor $(1-u^at^c)^{-1}$, so M such markings contribute the power $(1-u^at^c)^{-M}$, up to the monomial shift $\big(u^{a+b}t^{c+d}\big)^M$.

(2) Cusp shapes for Z_{Triv} *Assume $\mathrm{char}(k)\neq 2,3$ and work in $R=K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}]$. Introduce an auxiliary variable s with $t=s^{12}$, so that t^n corresponds to $\deg(\Delta)=12n$, while s records the integral discriminant degree $\deg(\Delta)$.*

After specializing $x_\beta=u^{m(\beta)-1}$ for $\beta\in\mathcal{A}_{\mathrm{nc}}$, a cusp marking of shape $\mathrm{I}_\bullet$ (resp. $\mathrm{I}_\bullet^*$) with contact order $k\geq 1$ contributes weight $u^{k-1}s^k$ (resp. $u^{k+4}s^{k+6}$), since

$$m(\mathrm{I}_k)-1=k-1,\qquad v(\Delta)=k,\qquad m(\mathrm{I}_k^*)-1=k+4,\qquad v(\Delta)=k+6$$

(cf. Remark 2.3). Hence summing over $k\geq 1$ at a single cusp marking gives, in $R[\![u,s]\!]$,

$$x_{\mathrm{I}_\bullet}=\sum_{k\geq 1}u^{k-1}s^k=\frac{s}{1-us},\qquad x_{\mathrm{I}_\bullet^*}=\sum_{k\geq 1}u^{k+4}s^{k+6}=\frac{u^5s^7}{1-us}. \tag{12}$$

In particular, each cusp marking of either shape contributes one factor $(1-us)^{-1}$ *after resummation. Thus a factor type* j *with* $\beta_{j,\mathrm{I}_\bullet}$ *markings of shape* $\mathrm{I}_\bullet$ *and* $\beta_{j,\mathrm{I}_\bullet^*}$ *markings of shape* $\mathrm{I}_\bullet^*$ *contributes the cusp factor*

$$(1-us)^{-(\beta_{j,\mathrm{I}_\bullet}+\beta_{j,\mathrm{I}_\bullet^*})},$$

together with the monomial shift

$$u^{5\beta_{j,\mathrm{I}_\bullet^*}}\, s^{\beta_{j,\mathrm{I}_\bullet}+7\beta_{j,\mathrm{I}_\bullet^*}}$$

coming from (12).

Proof. For (10), factor out the $k=1$ term and apply the geometric-series identity:

$$\sum_{k\geq 1} A u^{ak+b} t^{ck+d} = A u^{a+b} t^{c+d} \sum_{k\geq 0} (u^a t^c)^k = A u^{a+b} t^{c+d} \cdot \frac{1}{1-u^a t^c}.$$

Equation (11) follows because the sum over $(k_1,\dots,k_M)$ factorizes as a product of M copies of (10). Part (2) is (10) with $(a,c)=(1,1)$ applied in $R[\![u,s]\!]$ to the two monomial weights $u^{k-1}s^k$ and $u^{k+4}s^{k+6}$, yielding (12) and the stated denominator power. ■

Note that although the resummations sum over all $k\geq 1$, for any fixed height n only finitely many contact orders can occur: since $\sum_v \nu(\Delta)_v = 12n$ and $\nu(\Delta)=k$ for I_k and $\nu(\Delta)=k+6$ for I_k^*, one has $k\leq 12n$ (resp. $k\leq 12n-6$) on the height-n stratum. Thus the "infinite" cusp sum is merely a generating function device, and each coefficient $[t^n]$ (equiv. $[s^{12n}]$) receives contributions from finitely many k.

We now prove the Main Theorem.

Theorem 2.9 (Rationality and finite Euler product for Z_{Triv}). *Let* k *be a perfect field of characteristic* $\neq 2,3$*, and set* $s=t^{1/12}$ *(so* $t=s^{12}$*). Then*

$$Z_{\mathrm{Triv}}(u;t)\in K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}][u](s).$$

More precisely, let J*,* A_j*,* c_j*, and* $\beta_{j,\alpha}$ *be as in Lemma 2.7. Put*

$$\Delta(s) := 1-us, \qquad b_j := \sum_{\beta\in\mathcal{A}_{\mathrm{nc}}} \beta_{j,\beta}\big(m(\beta)-1\big), \qquad m_j := \beta_{j,\mathrm{I}_\bullet}+\beta_{j,\mathrm{I}_\bullet^*},$$

and define

$$B_j := b_j + 5\beta_{j,\mathrm{I}_\bullet^*}, \qquad C_j := c_j+\beta_{j,\mathrm{I}_\bullet}+7\beta_{j,\mathrm{I}_\bullet^*}, \qquad \mathcal{Y}_j(u;s) := A_j\, u^{B_j} s^{C_j}\, \Delta(s)^{-m_j}.$$

Then one has the finite Euler product

$$Z_{\mathrm{Triv}}(u;t) = u^2\cdot\mathbb{L}\cdot\prod_{j\in J}\frac{1}{\big(1-\mathcal{Y}_j(u;s)\big)\big(1-\mathbb{L}\cdot\mathcal{Y}_j(u;s)\big)}, \qquad (t=s^{12}). \tag{13}$$

Moreover, all dependence on $k\geq 1$ *in the cusp families* I_k *and* I_k^* *(over* $j=\infty$*) is absorbed by the single geometric-series denominator* $\Delta(s)^{-1}=(1-us)^{-1}$.

Proof. Work in the localized ring $K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}]$. By Lemma 2.7 we have

$$\mathcal{H}(s;\mathbf{x}) = \prod_{j\in J}\left(1-A_j\Big(\prod_{\alpha\in\mathcal{A}} x_\alpha^{\beta_{j,\alpha}}\Big)s^{c_j}\right)^{-\{\mathbb{P}^1\}} \in\ K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}][\mathbf{x}](s). \tag{14}$$

Trivial lattice baseline. By Lemma 1.3, for an elliptic surface S with singular fiber configuration $\mathfrak{f}$ one has

$$T(S) = 2 + \sum_{v \in \mathfrak{f}} (m_v - 1).$$

Under the specializations below, the monomial attached to $\mathfrak{f}$ is $u^{\sum_{v\in\mathfrak{f}}(m_v-1)}$, i.e. it records only the fiber contributions. Thus passing from $\mathcal{H}$ to Z_{Triv} introduces the global factor u^2.

Height-zero term. By Definition 1.5 one has $\mathcal{W}_0^{\min} = \mathcal{W}_0^{\min}(2)$, so

$$[t^0]\, Z_{\mathrm{Triv}}(u;t) = u^2\{\mathcal{W}_0^{\min}\}.$$

For $n = 0$ the discriminant degree is 0, hence the corresponding elliptic curve over $K = k(t)$ has everywhere good reduction and is therefore constant. Equivalently, $\mathcal{W}_0^{\min}$ identifies with the moduli stack of smooth elliptic curves,

$$\mathcal{W}_0^{\min} \;\cong\; \mathcal{M}_{1,1}.$$

Therefore $\{\mathcal{W}_0^{\min}\} = \mathbb{L}$, and the required constant term is $u^2 \cdot \mathbb{L}$.

Non-cusp specialization. For $\beta \in \mathcal{A}_{\mathrm{nc}}$ specialize $x_\beta = u^{m(\beta)-1}$. Then for each $j \in J$ the product of the non-cusp variables contributes the monomial u^{b_j}, where

$$b_j := \sum_{\beta \in \mathcal{A}_{\mathrm{nc}}} \beta_{j,\beta}\big(m(\beta) - 1\big).$$

Cusp resummation. Over the cusp $j = \infty$, the contact order $k \geq 1$ varies. In the discriminant-degree grading (with $t = s^{12}$), a cusp marking of shape $\mathrm{I}_\bullet$ (resp. $\mathrm{I}_\bullet^*$) with contact order k contributes weight $u^{k-1}s^k$ (resp. $u^{k+4}s^{k+6}$). Hence (Lemma 2.8) we have the substitutions

$$x_{\mathrm{I}_\bullet} = \sum_{k\geq 1} u^{k-1}s^k = \frac{s}{1-us} = s\,\Delta(s)^{-1}, \qquad x_{\mathrm{I}_\bullet^*} = \sum_{k\geq 1} u^{k+4}s^{k+6} = \frac{u^5s^7}{1-us} = u^5s^7\Delta(s)^{-1},$$

with $\Delta(s) = 1 - us$. Therefore, for each $j \in J$ the cusp contribution becomes

$$x_{\mathrm{I}_\bullet}^{\beta_{j,\mathrm{I}_\bullet}}\, x_{\mathrm{I}_\bullet^*}^{\beta_{j,\mathrm{I}_\bullet^*}} = u^{5\beta_{j,\mathrm{I}_\bullet^*}}\, s^{\beta_{j,\mathrm{I}_\bullet} + 7\beta_{j,\mathrm{I}_\bullet^*}}\, \Delta(s)^{-m_j}, \qquad m_j := \beta_{j,\mathrm{I}_\bullet} + \beta_{j,\mathrm{I}_\bullet^*}.$$

Combining with the non-cusp specialization yields, inside the jth factor of (14),

$$A_j\Big(\prod_{\alpha\in\mathcal{A}} x_\alpha^{\beta_{j,\alpha}}\Big)s^{c_j} \longmapsto A_j\, u^{B_j}\, s^{C_j}\, \Delta(s)^{-m_j} = \mathcal{Y}_j(u;s),$$

where

$$B_j = b_j + 5\beta_{j,\mathrm{I}_\bullet^*}, \qquad C_j = c_j + \beta_{j,\mathrm{I}_\bullet} + 7\beta_{j,\mathrm{I}_\bullet^*}.$$

Thus

$$Z_{\mathrm{Triv}}(u;t) = u^2 \cdot \{\mathcal{M}_{1,1}\} \cdot \prod_{j\in J}\Big(1 - \mathcal{Y}_j(u;s)\Big)^{-\{\mathbb{P}^1\}}, \qquad (t = s^{12}).$$

Finally, since $\{\mathbb{P}^1\} = 1 + \mathbb{L}$ in $K_0(\mathrm{Stck}_k)$, we may expand

$$\Big(1 - \mathcal{Y}_j(u;s)\Big)^{-\{\mathbb{P}^1\}} = \frac{1}{\Big(1 - \mathcal{Y}_j(u;s)\Big)\Big(1 - \mathbb{L}\cdot\mathcal{Y}_j(u;s)\Big)},$$

which gives (13). The dependence on $k \geq 1$ in the cusp families is absorbed by the single geometric-series denominator $\Delta(s)^{-1} = (1-us)^{-1}$ through the above resummations. ■

Remark 2.10. Assume $\mathrm{char}(k) \neq 2, 3$. For each Kodaira type Θ and $n \geq 1$, let

$$\mathcal{W}^{\Theta}_{n,\mathbb{P}^1}$$

be the moduli stack of minimal elliptic fibrations over $\mathbb{P}^1_k$ of discriminant degree $12n$ having exactly one singular fiber of type Θ over a varying degree–one place and semistable everywhere else.

The one-fiber motivic classes $\{\mathcal{W}^{\Theta}_{n,\mathbb{P}^1}\}$ carry a universal dependence on the height n coming from the $10n$–dimensional space of Weierstrass coefficients (equivalently, from the spaces of sections of degrees $4n$ and $6n$ in the weighted presentation $\overline{\mathcal{M}}_{1,1} \simeq \mathcal{P}(4,6)$). In particular, after dividing by the $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2$–factor,[1] the remaining motivic class grows as $\mathbb{L}^{10n+O(1)}$, uniformly in Θ. Accordingly we normalize local factor coefficients by

$$A^{(C)}_{\Theta} := \frac{\{\mathcal{W}^{\Theta}_{n,\mathbb{P}^1}\}}{\{\mathrm{PGL}_2\}\,\mathbb{L}^{10n+C}},$$

for some fixed integer C (e.g. $C = -18$ as in Table 2.10). The choice of C is immaterial for the Euler product: changing C rescales *every* local factor coefficient by the same global power of $\mathbb{L}$ and does not change its type (i.e. does not change the exponents B_j, C_j, m_j nor the finite set of factor types).

Then [BPS22, Thm. 1.6] and [HP19, Cor. 2] determine the following normalized one-fiber motivic classes.

Convention. For each reduction type Θ in Table 2.10, let $y_\Theta(u;s)$ denote the local monomial appearing in the displayed denominator in the last column (e.g. $y_{\mathrm{I}_k} = \mathbb{L}^{16} s\,\Delta(s)^{-1}$, $y_{\mathrm{III}} = \mathbb{L}^{14} u s^3$, etc.). The *full* $\mathbb{P}^1$-contribution of Θ in the Euler product is the power-structure/Kapranov factor

$$(1-y_\Theta(u;s))^{-\{\mathbb{P}^1\}} = \sum_{N\geq 0} \{\mathrm{Sym}^N(\mathbb{P}^1)\}\, y_\Theta(u;s)^N = \frac{1}{\big(1-y_\Theta(u;s)\big)\big(1-\mathbb{L}\,y_\Theta(u;s)\big)}.$$

In Table 2.10 we record only the *reduced* factor $\big(1-y_\Theta(u;s)\big)^{-1}$; the second factor $\big(1-\mathbb{L}\,y_\Theta(u;s)\big)^{-1}$ is inserted uniformly in the global Euler product (cf. Theorem 2.9).

These one-fiber motivic classes should be viewed as *local building blocks* for the factor–type Euler product in Theorem 2.9. For each non-cusp type $\mathrm{II}, \mathrm{III}, \mathrm{IV}, \mathrm{IV}^*, \mathrm{III}^*, \mathrm{II}^*$ and the two distinct cases $\mathrm{I}_0^*(j \neq 0, 1728)$ and $\mathrm{I}_0^*(j \in \{0, 1728\})$, the corresponding local factor types contribute k–independent reduced local factors in the s–grading (so $t = s^{12}$), namely $\big(1-y_\Theta(u;s)\big)^{-1}$ after the specialization $x_\alpha = u^{m(\alpha)-1}$ for $\alpha \in \mathcal{A}_{\mathrm{nc}}$. In this way, the s–exponent in $y_\Theta(u;s)$ records the discriminant s–degree

[1] The *unparameterized* $\mathbb{P}^1_k$ corresponds to taking the $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2$ stack quotient; motivically this factors out $\{\mathrm{PGL}_2\} = \mathbb{L}(\mathbb{L}^2 - 1)$, thereby treating the base as a smooth conic. See [PS25] for a comprehensive treatment.

Reduction type Θ	(r,a)	m_v-1	$A_\Theta := \dfrac{\{\mathcal{W}^{\Theta}_{n,\mathbb{P}^1}\}}{\{\mathrm{PGL}_2\}\,\mathbb{L}^{10n-18}}$	$\mathbb{P}^1$-Euler factor in $Z_{\mathrm{Triv}}(u;t)$
$\mathrm{I}_{k\geq 1}$ $(j=\infty)$	$(0,0)$	$k-1$	$\mathbb{L}^{16}$	$\dfrac{1}{1-\mathbb{L}^{16}s\,\Delta(s)^{-1}}$
II $(j=0)$	$(6,1)$	0	$\mathbb{L}^{15}$	$\dfrac{1}{1-\mathbb{L}^{15}s^2}$
III $(j=1728)$	$(4,1)$	1	$\mathbb{L}^{14}$	$\dfrac{1}{1-\mathbb{L}^{14}u\,s^3}$
IV $(j=0)$	$(3,1)$	2	$\mathbb{L}^{13}$	$\dfrac{1}{1-\mathbb{L}^{13}u^2s^4}$
$\mathrm{I}^*_{k\geq 1}$ $(j=\infty)$	$(2,1)$	$k+4$	$\mathbb{L}^{12}-\mathbb{L}^{11}$	$\dfrac{1}{1-(\mathbb{L}^{12}-\mathbb{L}^{11})\,u^5s^7\,\Delta(s)^{-1}}$
I^*_0 $(j\neq 0,1728)$	$(2,1)$	4	$\mathbb{L}^{12}-\mathbb{L}^{11}$	$\dfrac{1}{1-(\mathbb{L}^{12}-\mathbb{L}^{11})\,u^4s^6}$
I^*_0 $(j=0,1728)$	$(2,1)$	4	$\mathbb{L}^{11}$	$\dfrac{1}{1-\mathbb{L}^{11}u^4s^6}$
IV^* $(j=0)$	$(3,2)$	6	$\mathbb{L}^{10}$	$\dfrac{1}{1-\mathbb{L}^{10}u^6s^8}$
III^* $(j=1728)$	$(4,3)$	7	$\mathbb{L}^{9}$	$\dfrac{1}{1-\mathbb{L}^{9}u^7s^9}$
II^* $(j=0)$	$(6,5)$	8	$\mathbb{L}^{8}$	$\dfrac{1}{1-\mathbb{L}^{8}u^8s^{10}}$

increment of the local factor, the u–exponent records the corresponding trivial-lattice increment from its non-cusp markings, and A_Θ records the normalized motivic class of the one-fiber locus.

For the cusp families I_k and I^*_k, the table gives the one-fiber motivic contribution for each contact order $k\geq 1$. In the factor–type Euler product for $\mathcal{H}(s;\mathbf{x})$, the exponents $\beta_{j,\mathrm{I}_\bullet}$ and $\beta_{j,\mathrm{I}^*_\bullet}$ record only the number of cusp markings of each cusp *shape* in factor type j; the individual contact orders are not part of the inertia label. The infinite k–variation is collapsed by the geometric resummations

$$x_{\mathrm{I}_\bullet} = \sum_{k\geq 1} u^{k-1}s^k = \frac{s}{1-us}, \qquad x_{\mathrm{I}^*_\bullet} = \sum_{k\geq 1} u^{k+4}s^{k+6} = \frac{u^5s^7}{1-us},$$

so that each cusp marking contributes one factor $\Delta(s)^{-1} = (1-us)^{-1}$. Consequently, factor type j contributes the cusp factor

$$\Delta(s)^{-(\beta_{j,\mathrm{I}_\bullet}+\beta_{j,\mathrm{I}^*_\bullet})},$$

together with the monomial prefactor

$$u^{5\beta_{j,\mathrm{I}^*_\bullet}} s^{\beta_{j,\mathrm{I}_\bullet}+7\beta_{j,\mathrm{I}^*_\bullet}}$$

coming from the cusp substitutions.

3. Applications to modular curves with prescribed level structure

We apply the Main Theorem to the genus-0 modular curves $\overline{\mathcal{M}}_1(N)$ parametrizing generalized elliptic curves with level-N structure $\Gamma_1(N)$, introduced by [DR73] (see also [Con07, §2]). The fine modular curve $\overline{\mathcal{M}}_1(N)$ parametrizes families $(E,S,P) \to B$ where $(E,S) \to B$ is a semistable elliptic curve with section S and $P \in E^{\mathrm{sm}}[N](B)$ is an N-torsion section such that the divisor $P+S$ is relatively ample [KM85, §1.4]. We focus on $N = 2,3,4$, where the modular curves are genuinely stacky. Throughout, let k be a perfect field with $\mathrm{char}(k) \neq 2,3$.

3.1. Level-2 structure. We consider the height moduli

$$\mathcal{W}_n^{\Gamma_1(2)}\big(\mathcal{P}(2,4),\mathcal{O}(1)\big) \;=\; \mathcal{M}_{n,\mathbb{P}^1_k}\big(\overline{\mathcal{M}}_1(2),\mathcal{L}\big),$$

where $\overline{\mathcal{M}}_1(2) \cong \mathcal{P}(2,4)$ over $\mathbb{Z}\big[\frac{1}{2}\big]$ is the moduli stack of generalized elliptic curves with $\Gamma_1(2)$-structure (cf. [Beh06, §1.3]). Equivalently, $\overline{\mathcal{M}}_1(2)$ admits the universal Weierstrass presentation

$$y^2 = x^3 + a_2x^2 + a_4x \qquad \text{with} \qquad (a_2,a_4) \in H^0\big(\mathbb{P}^1,\mathcal{O}(2n)\big) \times H^0\big(\mathbb{P}^1,\mathcal{O}(4n)\big).$$

3.2. Level-3 structure. We consider the height moduli

$$\mathcal{W}_n^{\Gamma_1(3)}\big(\mathcal{P}(1,3),\mathcal{O}(1)\big) \;=\; \mathcal{M}_{n,\mathbb{P}^1_k}\big(\overline{\mathcal{M}}_1(3),\mathcal{L}\big),$$

where $\overline{\mathcal{M}}_1(3) \cong \mathcal{P}(1,3)$ over $\mathbb{Z}\big[\frac{1}{3}\big]$ is the moduli stack of generalized elliptic curves with $\Gamma_1(3)$-structure (cf. [HM17, Prop. 4.5]). Equivalently, $\overline{\mathcal{M}}_1(3)$ admits the universal Weierstrass presentation

$$y^2 + a_1xy + a_3y = x^3 \qquad \text{with} \qquad (a_1,a_3) \in H^0\big(\mathbb{P}^1,\mathcal{O}(n)\big) \times H^0\big(\mathbb{P}^1,\mathcal{O}(3n)\big).$$

3.3. Level-4 structure. We consider the height moduli

$$\mathcal{W}_n^{\Gamma_1(4)}\big(\mathcal{P}(1,2),\mathcal{O}(1)\big) \;=\; \mathcal{M}_{n,\mathbb{P}^1_k}\big(\overline{\mathcal{M}}_1(4),\mathcal{L}\big),$$

where $\overline{\mathcal{M}}_1(4) \cong \mathcal{P}(1,2)$ over $\mathbb{Z}\big[\frac{1}{2}\big]$ is the moduli stack of generalized elliptic curves with $\Gamma_1(4)$-structure (cf. [Mei22, Ex. 2.1]). Equivalently, $\overline{\mathcal{M}}_1(4)$ admits the universal Weierstrass presentation

$$y^2 + a_1xy + a_1a_2y = x^3 + a_2x^2 \qquad \text{with} \qquad (a_1,a_2) \in H^0\big(\mathbb{P}^1,\mathcal{O}(n)\big) \times H^0\big(\mathbb{P}^1,\mathcal{O}(2n)\big).$$

Reduction type Θ	(r,a)	$m_v - 1$	$A_\Theta := \dfrac{\{\mathcal{W}_{n,\mathbb{P}^1}^{\Gamma_1(2),\Theta}\}}{\{\mathrm{PGL}_2\}\,\mathbb{L}^{6n-10}}$	$\mathbb{P}^1$-Euler factor in $Z_{\mathrm{Triv}}^{\Gamma_1(2)}(u;t)$
$\mathrm{I}_{k\geq 1}$ $(j=\infty)$	$(0,0)$	$k-1$	$\mathbb{L}^8$	$\dfrac{1}{1-\mathbb{L}^8\, s\, \Delta(s)^{-1}}$
III $(j=1728)$	$(4,1)$	1	$\mathbb{L}^7$	$\dfrac{1}{1-\mathbb{L}^7\, u\, s^3}$
$\mathrm{I}^*_{k\geq 1}$ $(j=\infty)$	$(2,1)$	$k+4$	$\mathbb{L}^6-\mathbb{L}^5$	$\dfrac{1}{1-(\mathbb{L}^6-\mathbb{L}^5)\, u^5\, s^7\, \Delta(s)^{-1}}$
I^*_0 $(j\neq 0,1728)$	$(2,1)$	4	$\mathbb{L}^6-\mathbb{L}^5$	$\dfrac{1}{1-(\mathbb{L}^6-\mathbb{L}^5)\, u^4\, s^6}$
I^*_0 $(j=0,1728)$	$(2,1)$	4	$\mathbb{L}^5$	$\dfrac{1}{1-\mathbb{L}^5\, u^4\, s^6}$
III^* $(j=1728)$	$(4,3)$	7	$\mathbb{L}^4$	$\dfrac{1}{1-\mathbb{L}^4\, u^7\, s^9}$

Reduction type Θ	(r,a)	$m_v - 1$	$A_\Theta := \dfrac{\{\mathcal{W}_{n,\mathbb{P}^1}^{\Gamma_1(3),\Theta}\}}{\{\mathrm{PGL}_2\}\,\mathbb{L}^{4n-6}}$	$\mathbb{P}^1$-Euler factor in $Z_{\mathrm{Triv}}^{\Gamma_1(3)}(u;t)$
$\mathrm{I}_{k\geq 1}$ $(j=\infty)$	$(0,0)$	$k-1$	$\mathbb{L}^4$	$\dfrac{1}{1-\mathbb{L}^4\, s\, \Delta(s)^{-1}}$
IV $(j=0)$	$(3,1)$	2	$\mathbb{L}^3$	$\dfrac{1}{1-\mathbb{L}^3\, u^2\, s^4}$
IV^* $(j=0)$	$(3,2)$	6	$\mathbb{L}^2$	$\dfrac{1}{1-\mathbb{L}^2\, u^6\, s^8}$

Reduction type Θ	(r,a)	$m_v - 1$	$A_\Theta := \dfrac{\{\mathcal{W}_{n,\mathbb{P}^1}^{\Gamma_1(4),\Theta}\}}{\{\mathrm{PGL}_2\}\,\mathbb{L}^{3n-4}}$	$\mathbb{P}^1$-Euler factor in $Z_{\mathrm{Triv}}^{\Gamma_1(4)}(u;t)$
$\mathrm{I}_{k\geq 1}$ $(j=\infty)$	$(0,0)$	$k-1$	$\mathbb{L}^2$	$\dfrac{1}{1-\mathbb{L}^2\, s\, \Delta(s)^{-1}}$
I^*_0 $(j=0)$	$(2,1)$	4	$\mathbb{L}$	$\dfrac{1}{1-\mathbb{L}\, u^4\, s^6}$

4. IRRATIONALITY OF THE NÉRON–SEVERI AND MORDELL–WEIL SPECIALIZATIONS

The rationality of $Z_{\mathrm{Triv}}(u;t)$ reflects the fact that the trivial lattice rank $T(S)$ is governed by *local* reduction data. Indeed, by Lemma 1.3 it depends only on the multiset of fiber component numbers m_v, hence is constant on each Kodaira stratum $\mathcal{W}_n^{\min,(\mathfrak{f})}$, in the finite constructible stratification of Proposition 2.1. This locality is exactly what makes the evaluation-map factorization and the power structure on $K_0(\mathrm{Stck}_k)$ applicable: unordered collections of local factors assemble into a *finite* Euler product.

In contrast, the Mordell–Weil rank $\mathrm{rk}(E/K)$ is not determined by the fiber configuration. Even on a fixed Kodaira stratum $\mathcal{W}_n^{\min,(\mathfrak{f})}$, the rank typically varies, reflecting genuinely *global* constraints rather than local reduction data. Since $T(S)$ is constant on $\mathcal{W}_n^{\min,(\mathfrak{f})}$, the Shioda–Tate formula (1) shows that variation of $\mathrm{rk}(E/K)$ is equivalent to variation of the Néron–Severi rank $\rho(S)$. Thus any refinement of the height series by $\mathrm{rk}(E/K)$, or equivalently by $\rho(S)$, necessarily detects global jump phenomena invisible to the local factor stratification used for Z_{Triv}.

One way to organize this global complexity is via Néron–Severi jump loci. Fix a fiber configuration $\mathfrak{f}$ and write $\mathrm{Triv}^{(\mathfrak{f})} \subset \mathrm{NS}(S_{\bar{k}})$ for the sublattice generated by the zero section, a fiber class, and the components of reducible fibers in the configuration $\mathfrak{f}$. Inside $\mathcal{W}_n^{\min,(\mathfrak{f})}$, imposing that $\mathrm{NS}(S_{\bar{k}})$ contain additional algebraic classes *independent of* $\mathrm{Triv}^{(\mathfrak{f})}$ (equivalently, that $\rho(S)$, hence $\mathrm{rk}(E/K)$, jump) is an algebraic condition. Over $\mathbb{C}$, these conditions are naturally modeled by Noether–Lefschetz (Hodge) loci for the variation of Hodge structure coming from the family of elliptic surfaces over $\mathcal{W}_n^{\min,(\mathfrak{f})}$, and the theorem of Cattani–Deligne–Kaplan [CDK95] shows that such loci are, in general, only a *countable union of closed algebraic subsets*. In particular, unlike the Kodaira stratification at fixed height n (which is finite by Proposition 2.1), refinements by Néron–Severi *lattice type* are not expected to admit a finite constructible stratification.

This suggests that the local-to-global factorization mechanism producing a finite Euler product for Z_{Triv} should structurally fail for the Mordell–Weil and Néron–Severi specializations, and motivates the following conjecture.

Conjecture 4.1. Let $k = \mathbb{C}$ and $K = \mathbb{C}(z)$. The specializations

$$Z_{\mathrm{MW}}(v;t) := \mathcal{Z}(1,v;t), \qquad Z_{\mathrm{NS}}(w;t) := \mathcal{Z}(w,w;t)$$

are not rational in t with coefficients in $K_0(\mathrm{Stck}_{\mathbb{C}})[\mathbb{L}^{-1}][v]$ (resp. $[w]$); i.e.

$$Z_{\mathrm{MW}}(v;t) \notin K_0(\mathrm{Stck}_{\mathbb{C}})[\mathbb{L}^{-1}][v](t), \qquad Z_{\mathrm{NS}}(w;t) \notin K_0(\mathrm{Stck}_{\mathbb{C}})[\mathbb{L}^{-1}][w](t).$$

ACKNOWLEDGEMENTS

Warm thanks to Dori Bejleri, Klaus Hulek, Scott Mullane and Matthew Satriano for helpful discussions. The author was partially supported by the ARC grant DP210103397 and the Sydney Mathematical Research Institute.

References

[Beh06] Mark Behrens. A modular description of the $K(2)$-local sphere at the prime 3. *Topology*, 45(2):343–402, 2006.

[BPS22] Dori Bejleri, Jun-Yong Park, and Matthew Satriano. Height moduli on cyclotomic stacks and counting elliptic curves over function fields, 2022. arXiv:2210.04450.

[CDK95] Eduardo Cattani, Pierre Deligne, and Aroldo Kaplan. On the locus of Hodge classes. *J. Amer. Math. Soc.*, 8(2):483–506, 1995.

[CLL16] Antoine Chambert-Loir and François Loeser. Motivic height zeta functions. *Amer. J. Math.*, 138(1):1–59, 2016.

[Con07] Brian Conrad. Arithmetic moduli of generalized elliptic curves. *J. Inst. Math. Jussieu*, 6(2):209–278, 2007.

[DR73] P. Deligne and M. Rapoport. Les schémas de modules de courbes elliptiques. In *Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*, Lecture Notes in Math., Vol. 349, pages 143–316. Springer, Berlin, 1973.

[Eke25] Torsten Ekedahl. The Grothendieck Group of Algebraic Stacks. In *Perspectives on Four Decades of Algebraic Geometry, Volume 1*, volume 351 of *Progr. Math.*, pages 233–261. Birkhäuser/Springer, Cham, 2025.

[ESZB23] Jordan S. Ellenberg, Matthew Satriano, and David Zureick-Brown. Heights on stacks and a generalized Batyrev-Manin-Malle conjecture. *Forum Math. Sigma*, 11:Paper No. e14, 54, 2023.

[GP06] Ezra Getzler and Rahul Pandharipande. The Betti numbers of $\overline{\mathcal{M}}_{0,n}(r,d)$. *J. Algebraic Geom.*, 15(4):709–732, 2006.

[GZLMH13] S. M. Gusein-Zade, I. Luengo, and A. Melle-Hernández. On the pre-λ-ring structure on the Grothendieck ring of stacks and the power structures over it. *Bull. Lond. Math. Soc.*, 45(3):520–528, 2013.

[Her91] Stephan Herfurtner. Elliptic surfaces with four singular fibres. *Math. Ann.*, 291(2):319–342, 1991.

[HM17] Michael A. Hill and Lennart Meier. The C_2-spectrum $\mathrm{Tmf}_1(3)$ and its invertible modules. *Algebr. Geom. Topol.*, 17(4):1953–2011, 2017.

[HP19] Changho Han and Jun-Yong Park. Arithmetic of the moduli of semistable elliptic surfaces. *Math. Ann.*, 375(3-4):1745–1760, 2019.

[HP23] Changho Han and Jun-Yong Park. Enumerating odd-degree hyperelliptic curves and abelian surfaces over $\mathbb{P}^1$. *Math. Z.*, 304(1):Paper No. 5, 32, 2023.

[Kap00] Mikhail Kapranov. The elliptic curve in the S-duality theory and Eisenstein series for Kac–Moody groups. *MSRI Preprint*, 2000-006, 2000. arXiv:0001005.

[KM85] Nicholas M. Katz and Barry Mazur. *Arithmetic moduli of elliptic curves*, volume 108 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1985.

[Kod63] K. Kodaira. On compact analytic surfaces. II, III. *Ann. of Math. (2) 77 (1963), 563–626; ibid.*, 78:1–40, 1963.

[Mei22] Lennart Meier. Additive decompositions for rings of modular forms. *Doc. Math.*, 27:427–488, 2022.

[Mir89] Rick Miranda. *The basic theory of elliptic surfaces*. Dottorato di Ricerca in Matematica. [Doctorate in Mathematical Research]. ETS Editrice, Pisa, 1989.

[Ń64] André Néron. Modèles minimaux des variétés abéliennes sur les corps locaux et globaux. *Inst. Hautes Études Sci. Publ. Math.*, 21:128, 1964.

[PS25] Jun-Yong Park and Johannes Schmitt. Arithmetic geometry of the moduli stack of Weierstrass fibrations over $\mathbb{P}^1$. *Math. Z.*, 310(4):Paper No. 84, 33, 2025.

[Shi90] Tetsuji Shioda. On the Mordell-Weil lattices. *Comment. Math. Univ. St. Paul.*, 39(2):211–240, 1990.

[SS10] Matthias Schütt and Tetsuji Shioda. Elliptic surfaces. In *Alg. Geom. in East Asia—Seoul 2008*, volume 60 of *Adv. Stud. Pure Math.*, pages 51–160. Math. Soc. Japan, 2010.

[Tat75] J. Tate. Algorithm for determining the type of a singular fiber in an elliptic pencil. In *Modular functions of one variable, IV (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*, Lecture Notes in Math., Vol. 476, pages 33–52. Springer, Berlin, 1975.

Jun–Yong Park – june.park@sydney.edu.au
School of Mathematics and Statistics, University of Sydney, Australia