Math 353-05: Elementary Differential Equations & Boundary Value Problems

Instructor: Anna Nelson

September 23, 2021

Exam #1

Instructions:

- Do not open exam until you are directed to do so.
- You have 75 minutes to complete the exam.
- You are allowed one 8.5" by 11" sheet of notes, both sides.
- This is an individual exam, no group work.
- Calculators, cell phones, and books are not allowed.
- Please show all your work and justify your answer, as partial credit will be given where appropriate.
- **Circle your final answer!**

By signing below, you acknowledge having read the above instructions, and you agree to abide by the Duke Community Standard.

Name: _____

Signature: _____

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Question	Points	Score
1	12	
2	14	
3	13	
4	20	
5	24	
6	22	
Total:	105	

1. Consider the following differential equation. Circle true or false.

$$\frac{dy}{dx} = e^3(x + xy)^{1/3}$$

(a) (1 point) This is a linear differential equation. True False

(b) (1 point) This is a separable differential equation. True False

(c) (1 point) This is a first-order differential equation. True False

(d) (1 point) This is an autonomous differential equation. True False

(e) (8 points) Solve for the general implicit solution.

2. A 100 gal (gallon) tank initially contains 60 gal of pure water. Brine containing 1 lb/gal of salt flows into the tank at a rate of 4 gal/minute. The well-stirred mixture in the tank flows out at a rate of 3 gal/minute.

   (a) (6 points) Write down a differential equation and initial condition for the amount of salt in the tank, $x(t)$.

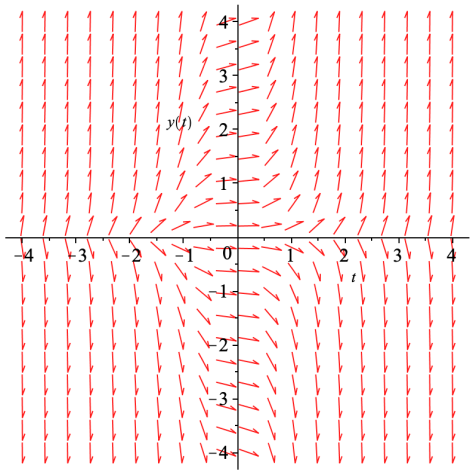
   (b) (8 points) Find the explicit solution to the differential equation.

3. Consider the following initial value problem.

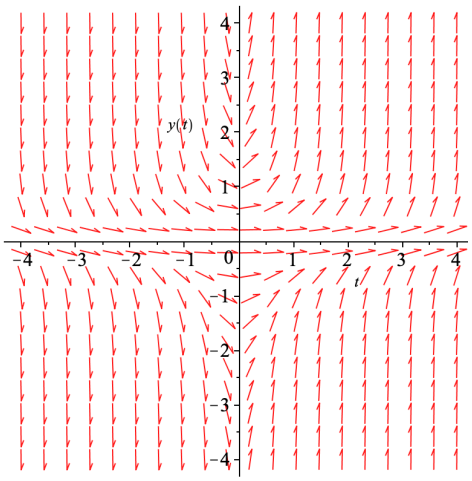
$$\frac{dy}{dx} = 2xy^2, \quad y(0) = y_0$$

(a) (10 points) Solve the general initial value problem and determine how the interval in which the solution exists depends on the initial value, y_0.

(b) (3 points) Identify the slope field that matches the given equation. Explain your reasoning.



(a)



(b)

Explanation:

4. Solve the following differential equations.

(a) (10 points)

$$(1 + e^x y + xe^x y)dx + (xe^x + 2)dy = 0$$

(b) (10 points)

$$y'' - 4y' + 7y = x + \cos x$$

5. (a) (8 points) Find two values of m such that $y(t) = t^m$ solves the following homogeneous equation.

$$t^2y'' - 5ty' + 8y = 0, \qquad t > 0$$

(b) (4 points) Determine if the functions found above are linearly independent. Explain your reasoning.

(c) (12 points) Given that $y_1(x) = \cos(\ln x)$ and $y_2(x) = \sin(\ln x)$, use variation of parameters to find a particular solution to the differential equation

$$x^2y'' + xy' + y = \ln x, \qquad x > 0$$

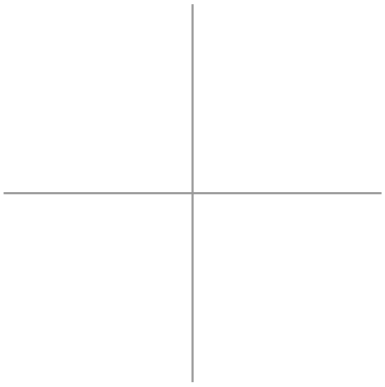
6. Consider the following differential equation.

$$\frac{dP}{dt} = P^2 - hP$$

(a) (4 points) Find all the critical points.

(b) (8 points) Draw phase lines and determine the stability of the critical points as a function of h.

(c) (6 points) Draw the bifurcation diagram in the hP-plane for the differential equation.



(d) (4 points) Suppose $h = 4$ and $P(0) = 1$. Find the following limits.

$$\lim_{t \to -\infty} P(t) = \qquad\qquad \lim_{t \to \infty} P(t) =$$

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