1) (a)

color 1 is picked, etc

$$X = \mathbb{1}_1 + \mathbb{1}_2 + \dots \mathbb{1}_{12}$$

$$E(\mathbb{1}_1) = P(\text{color 1 is picked}) = 1 - \left(\frac{11}{12}\right)^N$$

$$E(X) = \sum_{i=1}^{12} \mathbb{1}_i = 12(\mathbb{1}_i) = 12\left(1 - \left(\frac{11}{12}\right)^N\right)$$

(b) As N gets larger, X will approach 12
Conceptually, this is b/c there are 12 colors that could be represented, & as N gets larger there's more of a chance to represent all colors

Yes, this matches my answer, b/c

$$\lim_{N \to \infty} 12\left(1 - \left(\frac{11}{12}\right)^N\right) = 12$$

← $\left(\frac{11}{12}\right)^N$ converges to 0 as N approaches ∞,

so $E(X)$ would be 12 as $n \to \infty$

(c)

$$X = \mathbb{1}_1 + \mathbb{1}_2 + \dots \mathbb{1}_{12}$$

$$E(\mathbb{1}_1) = P(\text{color 1 represented}) = 1 - \left(\frac{55}{60} \cdot \frac{54}{59} \cdot \dots \cdot \frac{25}{30}\right)$$

same as

$$E(\mathbb{1}_1) = P(\text{color 1 represented}) = 1 - \frac{\binom{55}{30}\binom{5}{0}}{\binom{60}{30}} =$$

$$E(X) = \sum_{i=1}^{12} E(\mathbb{1}_i) = 12\left(1 - \frac{\binom{55}{30}\binom{5}{0}}{\binom{60}{30}}\right)$$

$$= 12\left(\frac{4367}{4484}\right)$$

close to 12

$$\approx 11.686887$$

2 $P(A)P(B) = 0.25 \cdot 0.3 = 0.075 \neq P(A \cap B) = 0.2$

Thus, A & B are dependent.

$$E(\mathbb{1}_A \cdot \mathbb{1}_B) = \sum_{all(a,b)} ab \, P(\mathbb{1}_A = a, \mathbb{1}_B = b)$$

$$= \underbrace{\cancel{(0)(0)\, P(\mathbb{1}_A=0, \mathbb{1}_B=0)}}_{\overset{0}{} \; P(A^c \cap B^c)} + \underbrace{\cancel{(0)(1)\, P(\mathbb{1}_A=0, \mathbb{1}_B=1)}}_{\overset{0}{} \; P(A^c \cap B)}$$

$$+ \underbrace{\cancel{(1)(0)\, P(\mathbb{1}_A=1, \mathbb{1}_B=0)}}_{\overset{0}{} \; P(A \cap B^c)} + \underbrace{(1)(1)\, P(\mathbb{1}_A=1, \mathbb{1}_B=1)}_{\overset{1}{} \; P(A \cap B)}$$

$$= 1 \cdot P(A \cap B)$$

$$= 0.2$$

3 (a)

$$X = \underbrace{\mathbb{1}_1}_{\substack{\text{Equals 1 if} \\ \text{You draw} \\ \text{an ace}}} + \mathbb{1}_2 + \dots + \underbrace{\mathbb{1}_{13}}_{\substack{\text{Equals 1 if} \\ \text{You draw} \\ \text{a king}}}$$

$$E(X) = 1 \cdot \overset{1/13}{P(\text{Drawing } 1)} + 2 \cdot \overset{1/13}{P(\text{Drawing } 2)} + \dots$$
$$12 \cdot \overset{1/13}{P(\text{Drawing } 12)} +$$
$$13 \cdot \overset{1/13}{P(\text{Drawing } 13)}$$

$$= \frac{1}{13}\left(\frac{1+13}{2}\right)13$$

$$= 7$$

(b)

$$E(\overset{\text{value of}}{\text{1st card}} + \overset{\text{value of}}{\text{2nd card}} + \dots + \overset{\text{value of}}{\text{13rd card}})$$

$$= E(\overset{\text{value of}}{\text{1st}}) + E(\overset{\text{value of}}{\text{2nd}}) + \dots + E(\overset{\text{value of}}{\text{13 rd}})$$

7

assume $E(\overset{\text{value}}{\text{1st}}) = E(\overset{\text{value}}{\text{2nd}})$, etc

$$= \sum_{i=1}^{13} E(i^{th}) = 13\, E(i^{th}) = 13 \cdot 7 = 91$$

(b) Alt method

let a = heart, b = spade, c = diamond, d = club

$$X = 1_{1a} + 1_{1b} + 1_{1c} + 1_{1d} + 2_{1a} + \dots$$
$$\dots + 1_{13a} + 1_{13b} + 1_{13c} + 1_{13d}$$

$$E(X) = 4\left[E(1_{1a}) + 2E(1_{2a}) + 3E(1_{3a}) \cdot 13E(1_{13a})\right]$$

$$= 4 \cdot \frac{13}{52}[1+2+\dots+13)$$

$$= 7 \cdot 13 = 91$$

calculation for $E(1ia)$

$P(1ia) = P(ia \text{ appears}) = 1 - P(ia \text{ doesn't appear})$

$P(ia \text{ not appear}) = \underbrace{\frac{51}{52} \cdot \frac{50}{51} \cdots \frac{39}{40}}_{13 \text{ times}} = \frac{39}{52}$

$1 - \frac{39}{52} = \frac{13}{52}$

(C) Without Replacement

drawing king $\rightarrow X = 1_1 + 1_2 + \dots + 1_{13}$ ← king on ith term

$E(X) = P(\text{king on 1st card}) + \dots + P(\text{king on 13th card})$

$P(\text{King on 1st}) = \frac{4}{52} = \frac{1}{13}$

$P(\text{king on 2nd}) = P(2nd \mid 1st)P(1st) + P(2nd \mid 1st^c)P(1st^c)$

$= \frac{3}{51} \cdot \frac{1}{13} + \frac{4}{51} \cdot \frac{48}{52}$

$= \frac{4}{52} = \frac{1}{13}$

same principle holds for rest of elements.

So, $P(\text{king on ith card}) = \frac{1}{13}$

$E(X) = \frac{1}{13} + \frac{1}{13} + \dots + \frac{1}{13} = 13\left(\frac{1}{13}\right)$

$= 1$

(d) with replacement

$$Y = \mathbb{1}_1 + \mathbb{1}_2 + \dots + \mathbb{1}_{13}$$

$$E(Y) = P\left(\text{king on 1st card}\right) + \dots + P\left(\text{king on 13th card}\right)$$

$$P(\text{king on ith card}) = \frac{1}{13}$$

$$\frac{1}{13} + \frac{1}{13} + \dots + \frac{1}{13} = 13\left(\frac{1}{13}\right) = 1$$

Thus, $E(Y)$ is same as $E(X)$

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