Honor Code: I have completed this exam in the spirit of the Duke Community Standard and have neither given nor received aid.

Signature:

1. (15 pts.) Solve the equation

$$y' = e^{-y}\left(\frac{1}{2x^2} + \frac{1}{1+x^2}\right),\ y(1) = \ln(\pi/4 - 1/2).$$

Describe the long time behavior of the solution, i.e., $\lim_{x\to\infty} y(x) =?$

2. (10 pts.) Consider the initial value problem

$$y' - y = 1 + 3\sin t, \; y(0) = y_0$$

a) Find the solution to the given problem.

b) Find the value of y_0 such that $y(t)$ remains bounded as $t \to \infty$.

3. (20 pts.) a) Determine whether the equation is exact. And if it is exact, solve it.

$$(ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x) + (xe^{xy}\cos 2x - 3)y' = 0.$$

b) Find the integration factor of the equation (Hint: The integration factor has the form $v(x, y) = v(x)$. To get it, you need to solve the equation $(vM)_y = (vN)_x \Rightarrow v_y M + v M_y = v_x N + v N_x$ (What is M and N in this setting?). Since the $\partial_y v$ vanishes, calculation yields that the equation is an ODE.):

$$y' = e^{2x} + y - 1.$$

4. (10 pts.) Apply the reduction of order to determine the other homogeneous solution to the equation

$$t^2y'' - 4ty' + 6y = 0, t > 0; \quad y_1(t) = t^2.$$

5. (10 pts.) Solve the homogeneous equation:

$$\frac{dy}{dx} = \frac{y^2 + x^2 + xy}{x^2}.$$

6. (20 pts.) a) Find the solution to the following inhomogeneous equation using the method of variation of parameters.

$$(1-t)y'' + ty' - y = 2(t-1)^2 e^{-t}, \quad 0 < t < 1, \quad y_1(t) = e^t, \quad y_2(t) = t. \tag{1}$$

b) Explain why $y' = \sqrt{y}$ does not have unique solution initiating at $y(0) = 0$. Give an example of ODE which blows up in finite time. .

7. (15 pts.) Consider the ODE

$$\frac{d}{dx}y = y(a - y^2).$$

Find the equilibrium point and draw the phase line in the cases $a > 0$, $a = 0$ and $a < 0$.