AUTOMATIC CONTROL THEORY

PGS. TS. Nguyễn H. Nam

Department of Automation Engineering
School of Electrical and Electronic Engineering

Email: nam.nguyenhoai@hust.edu.vn
Website: https://sites.google.com/view/masc-lab

Office: E416-C7

Chapter 2 Fundamental of analysis and design for control system

- 2.1 Laplace transform
- 2.2 Properties of Laplace transform
- 2.3 Application

2.1. Phép biến đổi Laplace

Pierre Simon Laplace (1749–1827)

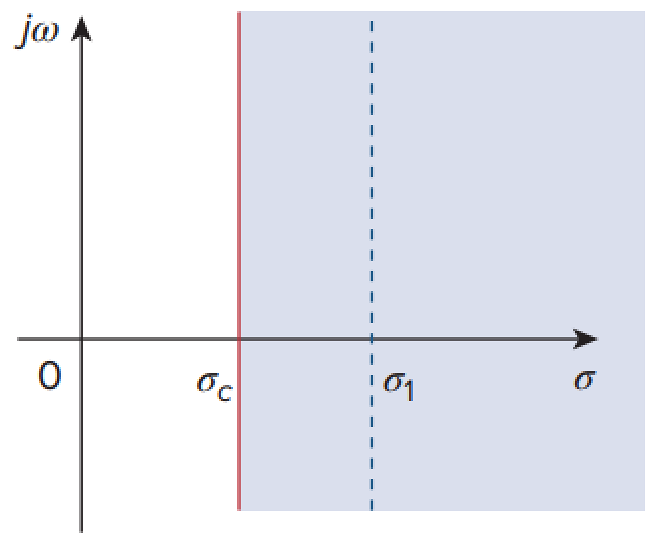
$$X(s) = \int_{0}^{\infty} x(t)e^{-st}dt$$

$$s = \sigma + j\omega, \qquad j^2 = -1$$

- Condition for existence:

$$\int_0^{\infty} |x(t)|e^{-\sigma t}dt < \infty,$$

with $\sigma = \sigma_c$



2.1 Laplace Transform

Pierre Simon Laplace (1749–1827)

$$X(s) = \int_{0}^{\infty} x(t)e^{-st}dt$$

$$s = \sigma + j\omega, \qquad j^2 = -1$$

Example:

- $x(t) = 1(t) = \begin{cases} 1, & if\ t \geq 0 \\ 0, & if\ t < 0 \end{cases}$
- $X(s) = \int_0^{\infty} x(t)e^{-st}dt = \left.\frac{e^{-st}}{-s}\right|_0^{\infty} = \left.\frac{e^{-(\sigma+j\omega)t}}{-s}\right|_0^{\infty} = \frac{1}{s}$
- with $\sigma > 0$

2.1 Laplace Transform

Example:

- $x(t) = e^{-at}$
- $X(s) = \int_0^{\infty} x(t)e^{-st}dt = \left.\frac{e^{-(s+a)t}}{-s}\right|_0^{\infty} = \frac{1}{\text{s+a}}$

with $\sigma \succ a$

2.1 Laplace Transform

- Tool in Matlab: laplace, ilaplace

2.2 Properties of Laplace transform

- Linearity: $\mathrm{L}\{ax(t) + by(t)\} = aX(s) + bY(s)$
- $a,\ b$: constant.
- Example: Find $L\{\sin(\omega t)\}, L\{cos(\omega t)\}$

2.2 Properties of Laplace transform

- Time shifting: $\mathrm{L}\{x(t-T)\} = X(s)e^{-sT}$

- Example: $L\{1(t-2) - 1(t-3)\}$

2.2 Properties of Laplace transform

- Frequency shifting: $\mathrm{L}\{x(t)e^{-\alpha t}\} = X(s+\alpha)$

- Example: $L\{\sin(2t)e^{-3t}\}$

2.2 Properties of Laplace transform

- Differentiation: $y(t) = \frac{dx(t)}{dt}$

$$Y(s) = sX(s) - x(0)$$

- Example: $L\left\{\frac{d}{dt}(\sin(2t))e^{-t}\right\}$

2.2 Properties of Laplace transform

- Integration: $y(t) = \int_0^t x(\tau)d\,\tau$

$$Y(s) = \frac{X(s)}{s}$$

$$L\{y(t)\} = \int_0^\infty \int_0^t x(\tau)d\,\tau e^{-st}dt.$$

- Let $u = \int_0^t x(\tau)d\,\tau$ and $dv = e^{-st}dt$.
- → $du = x(t)dt$ and $v = \frac{e^{-st}}{-s}$

$$L\{y(t)\} = uv\Big|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}x(t)dt = \frac{X(s)}{s}.$$

2.2 Properties of Laplace transform

- Example: $L\{t\}$

2.2 Properties of Laplace transform

- Differentiation in frequency: $y(t) = t^n x(t)$

$$Y(s) = (-1)^n \frac{d^n X(s)}{ds^n}$$

$$\frac{dX(s)}{ds} = \int_0^{\infty} -tx(t)e^{-st}dt = L\{-tx(t)\}$$

- Example: $L\{te^{-t}\}$

2.2 Properties of Laplace transform

- Initial value:

$$x(0) = \lim_{s\to\infty} sX(s)$$

$$sX(s) - x(0) = \int_0^\infty \frac{dx(t)}{dt} e^{-st} dt = \int_0^\infty e^{-st}\, dx(t)$$

$$\lim_{s\to\infty} [sX(s) - x(0)] = 0$$

- Example: Find $x(0)$ given $X(s) = \frac{1}{s+2}$

2.2 Properties of Laplace transform

- Final value:

$$x(\infty) = \lim_{s\to 0} sX(s)$$

$$\lim_{s\to 0}[sX(s) - x(0)] = \lim_{s\to 0}\int_{0}^{\infty} e^{-st}\,dx(t) = x(\infty) - x(0)$$

- Example: Find $x(\infty)$ with $X(s) = \frac{1}{(s+1)(s+2)}$

2.2 Properties of Laplace transform

- Convolution: $x(t) * y(t) = \int_0^\infty x(\tau)y(t-\tau)d\tau$

$$\mathrm{L}\{x(t) * y(t)\} = X(s)Y(s)$$

$$X(s) = \int_0^\infty x(\tau)\tau d\tau$$

→ $X(s)Y(s) = \int_0^\infty x(\tau)Y(s)e^{-s\tau}d\tau$

$$= \int_0^\infty x(\tau) \int_0^\infty y(t-\tau)1(t-\tau)e^{-st}dt\, d\tau$$

$$= \int_0^\infty \left[\int_0^t x(\tau)y(t-\tau)d\tau\right]e^{-st}dt = \mathrm{L}\{x(t) * y(t)\}$$

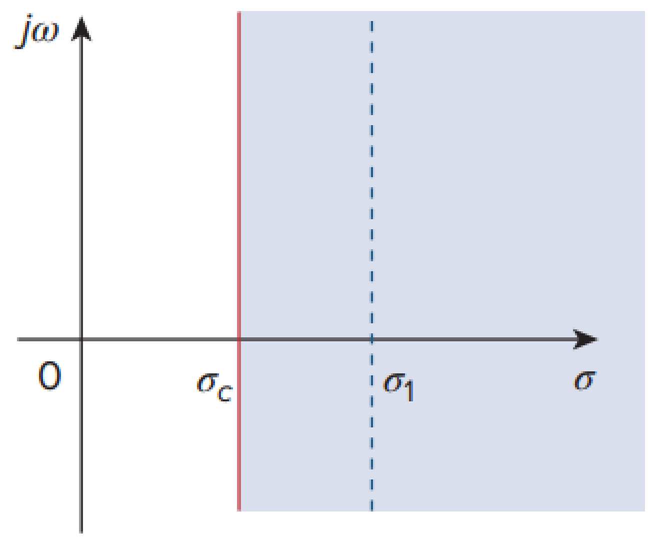
2.2 Properties of Laplace transform

- Example: L$\{1(t) * e^{-2t}\}$

Inverse Laplace Transform

$$x(t) = L^{-1}\{X(s)\}$$
$$= \frac{1}{2\pi j}\int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} X(s)e^{st}ds$$

- Along the line $\sigma_1 + j\omega, -\infty < \omega < \infty$ with $\sigma_1 > \sigma_c$.



Property	$f(t)$	$F(s)$
Linearity	$a_1f_1(t) + a_2f_2(t)$	$a_1F_1(s) + a_2F_2(s)$
Scaling	$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
Time shift	$f(t-a)u(t-a)$	$e^{-as}F(s)$
Frequency shift	$e^{-at}f(t)$	$F(s+a)$
Time differentiation	$\frac{df}{dt}$	$sF(s) - f(0^-)$
	$\frac{d^2f}{dt^2}$	$s^2F(s) - sf(0^-) - f'(0^-)$
	$\frac{d^3f}{dt^3}$	$s^3F(s) - s^2f(0^-) - sf'(0^-) - f''(0^-)$
	$\frac{d^nf}{dt^n}$	$s^nF(s) - s^{n-1}f(0^-) - s^{n-2}f'(0^-) - \cdots - f^{(n-1)}(0^-)$
Time integration	$\int_0^t f(x)\,dx$	$\frac{1}{s}F(s)$
Frequency differentiation	$tf(t)$	$-\frac{d}{ds}F(s)$
Frequency integration	$\frac{f(t)}{t}$	$\int_s^\infty F(s)\,ds$
Time periodicity	$f(t) = f(t+nT)$	$\frac{F_1(s)}{1-e^{-sT}}$
Initial value	$f(0)$	$\lim_{s\to\infty} sF(s)$
Final value	$f(\infty)$	$\lim_{s\to 0} sF(s)$
Convolution	$f_1(t) * f_2(t)$	$F_1(s)F_2(s)$

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
te^{-at}	$\frac{1}{(s+a)^2}$
t^ne^{-at}	$\frac{n!}{(s+a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t + \theta)$	$\frac{s\sin\theta + \omega\cos\theta}{s^2+\omega^2}$
$\cos(\omega t + \theta)$	$\frac{s\cos\theta - \omega\sin\theta}{s^2+\omega^2}$
$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

*Defined for $t \geq 0$; $f(t) = 0$, for $t < 0$.

Inverse Laplace Transform

a) Simple poles:

$$X(s) = \frac{N(s)}{(s + p_1)(s + p_2)\dots(s + p_n)}$$

Assume $p_i \neq p_j$ with all i $\neq j$.

$$X(s) = \frac{k_1}{s+p_1} + \frac{k_2}{s+p_2} + \dots + \frac{k_n}{s+p_n}$$

$$k_i = X(s)(s + p_i)\Big|_{s = -p_i}$$

$$x(t) = (k_1 e^{-p_1 t} + k_2 e^{-p_2 t} + \dots + k_n e^{-p_n t})1(t)$$

Inverse Laplace Transform

b) Repeated poles:

$$X(s) = X_1(s) + \frac{k_1}{s+p} + \frac{k_2}{(s+p)^2} + \ldots + \frac{k_n}{(s+p)^n}$$

$X_1(s)$ does not have a pole at $s = -p$.

$$k_{n-m} = \frac{1}{m!}\frac{d^m}{ds^m}\left[X(s)(s+p)^n\right]\Bigg|_{s=-p}$$

$$L^{-1}\left\{\frac{1}{(s+a)^n}\right\} = \frac{t^{n-1}e^{-at}}{(n-1)!}u(t)$$

$$x(t) = \left[x_1(t) + \sum_{i=1}^{n} k_i \frac{t^{i-1}e^{-pt}}{(i-1)!}\right]1(t)$$

Inverse Laplace Transform

c) Complex poles:

$$X(s) = X_1(s) + \frac{A_1 s + A_2}{s^2 + as + b}$$

$X_1(s)$ does not have the same complex poles.

$$s^2 + as + b = (s + \alpha)^2 + \beta^2$$

$$A_1 s + A_2 = A_1(s + \alpha) + B_1\beta$$

$$x(t) = [x_1(t) + (A_1 cos(\beta t) + B_1 sin(\beta t))e^{-\alpha t}]1(t)$$

Differential equation with constant coefficients

First order DE:

$$\dot{x} + ax = by$$

Differential equation with constant coefficients

Second-order DE:

$$\ddot{x} + a\dot{x} + bx = cy$$