AUTOMATIC CONTROL THEORY

Associate Professor. Nguyễn H. Nam

Department of Automation Engineering
School of Electrical and Electronic Engineering

Email: nam.nguyenhoai@hust.edu.vn
Website: https://sites.google.com/view/masc-lab

Office: E416-C7

Chapter 3 Linear system in frequency domain

- 3.1. SISO system
- 3.2. Modeling a nonlinear system by linearization
- 3.3. Typical linear dynamic systems
- 3.4. System stability and stability criteria
- 3.5. Transient and steady- state responses
- 3.6. Overshoot, settling time, steady-state error.
- 3.7. Open-loop controller design
- 3.8. PID controller
- 3.9. PID controller design methods

3.1. SISO system

- $T: \boldsymbol{u}(t) \rightarrow \boldsymbol{y}(t)$
- $\boldsymbol{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_r(t) \end{bmatrix}; \boldsymbol{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_s(t) \end{bmatrix}$
- System is linear if the superposition holds:

$$\mathrm{T}\left\{\sum_{i}^{n} a_i \boldsymbol{u}_i(t)\right\} = \sum_{i}^{n} a_i T\{\boldsymbol{u}_i(t)\}$$

3.1. SISO system

- SISO system: $r = s = 1$
- MIMO system: $r > 1; s > 1$
- MISO system : $r > 1; s = 1$
- SISO system can be represented as:
  - ❑Differential equations
  - ❑Transfer function $\mathrm{G}(s)$
  - ❑State-space model

3.1. SISO system

- ***Differential equation***

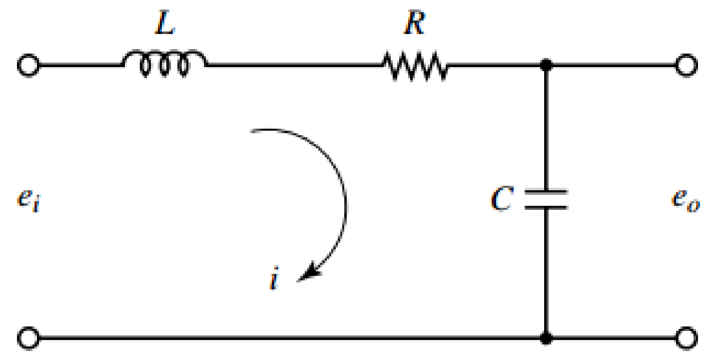
$$a_0 y + a_1 \frac{dy}{dt} + \cdots + a_n \frac{d^n y}{dt^n} = b_0 u + b_1 \frac{du}{dt} + \cdots + b_m \frac{d^m u}{dt^m}$$

- a_i, b_j: constant
- $n > m$ integer number
- u input
- y output

3.1. SISO system

- ***Electrical system:*** *Kirchoff Law 1 & 2*
  - ❑Sum of currents is zero for a node.
  - ❑Sum of voltages is zero for a loop.
- Resistor: $u = Ri$
- Capacitor: $i_C(t) = C\frac{du_C(t)}{dt}$;
- Inductor: $u_L(t) = L\frac{di_L(t)}{dt}$
- u voltage, i current.

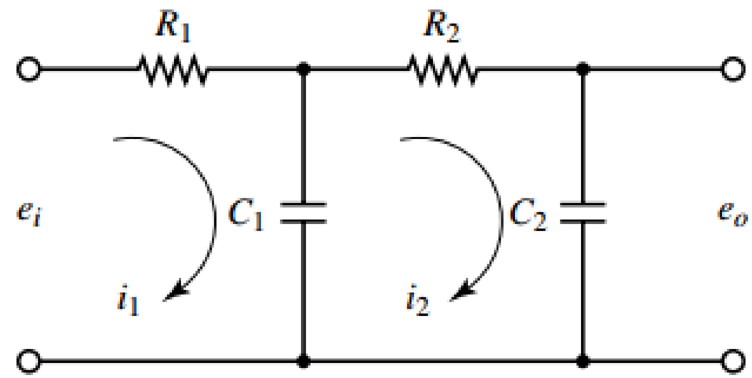
3.1. SISO system



$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = e_i$$

$$\frac{1}{C}\int i\,dt = e_o$$

3.1. SISO system



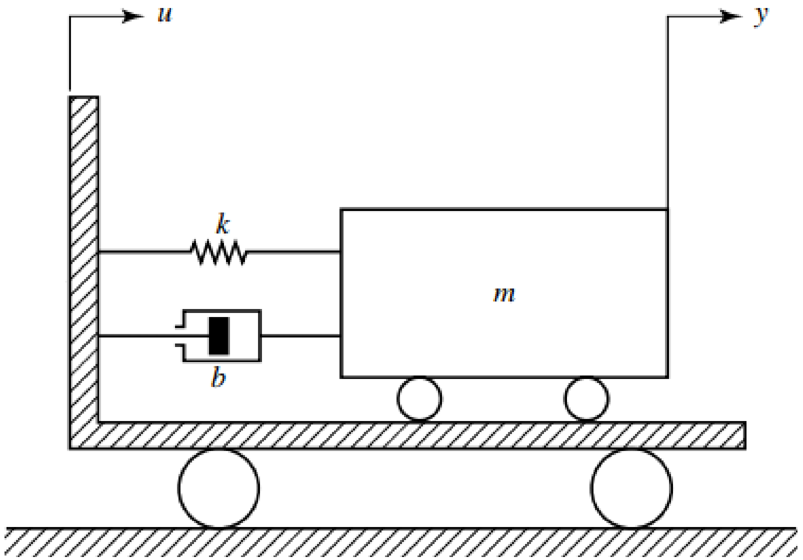
$$\frac{1}{C_1}\int (i_1 - i_2)\,dt + R_1 i_1 = e_i$$

$$\frac{1}{C_1}\int (i_2 - i_1)\,dt + R_2 i_2 + \frac{1}{C_2}\int i_2\,dt = 0$$

$$\frac{1}{C_2}\int i_2\,dt = e_o$$

3.1. SISO system

- ***Mechanical system:***
- *Newton's second law*:
- $\sum F_i = ma$
- F_i force
- m mass
- a acceleration



$$m\frac{d^2y}{dt^2} = -b\left(\frac{dy}{dt} - \frac{du}{dt}\right) - k(y-u)$$

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = b\frac{du}{dt} + ku$$

3.2. Modeling a nonlinear system by linearization

$$\dot{x} = f(x, u)$$
$$y = g(x)$$

Working point:

$$u = u_0$$
$$f(x_0, u_0) = 0$$

$$\dot{z} = Az + Bv$$

$$A = \frac{df}{dx}\bigg|_{x = x_0, u = u_0} \quad B = \frac{df}{du}\bigg|_{x = x_0, u = u_0}$$
$$v = u - u_0$$
$$z = x - x_0$$

3.2. Modeling a nonlinear system by linearization

$$\dot{x} = -x^3 - x + u$$
$$y = x$$

Working point:

$$u = 0$$
$$f(x_0, u_0) = 0$$
$$x_0 = 0$$
$$A = -1; B = 1$$

$$\dot{z} = -z + v$$
$$v = u - u_0$$
$$z = x - x_0$$

3.3 Typical Linear Systems

- ***Unit Impulse:***
- ***Impulse Response:***

$$g(t) = L^{-1}\{G(s)\}$$

3.3 Typical Linear Systems

- ***Unit step function:***
- ***Unit Step Response:***

$$h(t) = L^{-1}\left\{\frac{G(s)}{s}\right\}$$

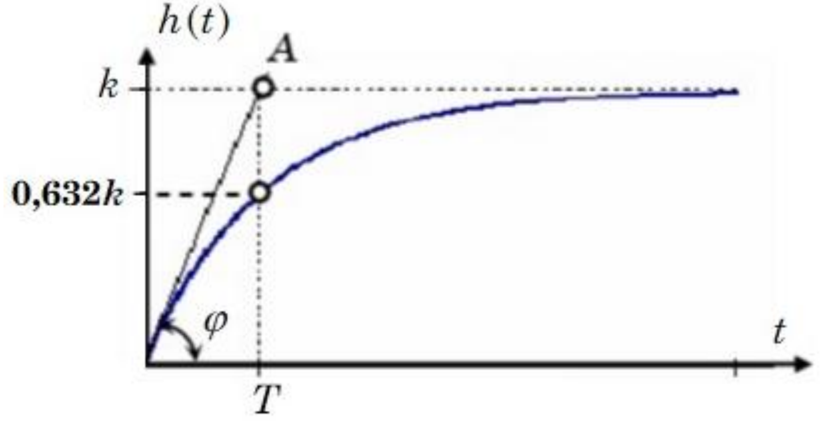
3.3 Typical Linear Systems

- ***First-order system:***

$$G(s) = \frac{k}{1 + Ts}$$

- Step response:

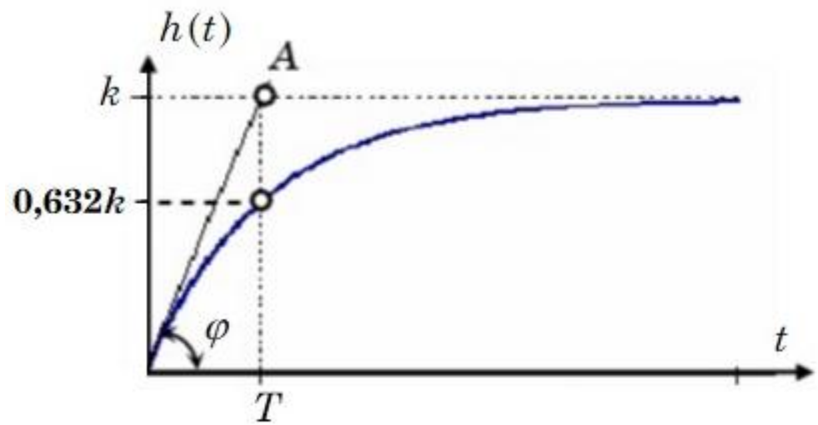
$$h(t) = L^{-1}\left\{\frac{G(s)}{s}\right\} = k(1 - e^{-\frac{t}{T}})$$



3.3 Typical Linear Systems

$$\frac{dh(0)}{dt} = \lim_{s\to\infty} s[sH(s) - h(0)] = \frac{k}{T} = \tan(\varphi)$$

- Determine k và T:
  - ❑Apply $1(t)$ to the system.
  - ❑k steady state output
  - ❑T t-coordinate of A



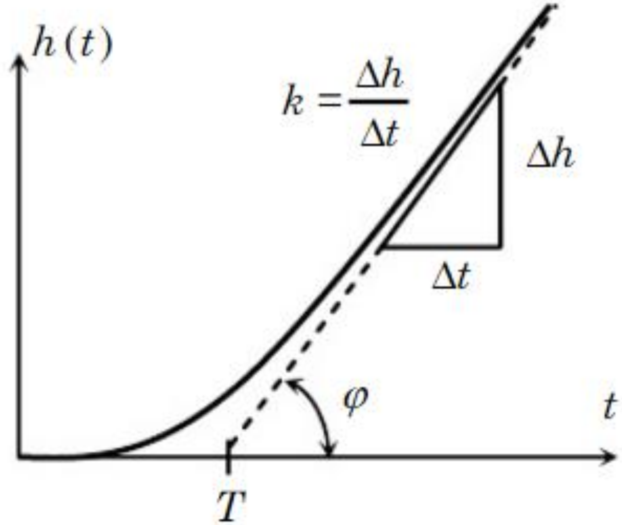
3.3 Typical Linear Systems

- ***First-order system with integrator***

$$G(s) = \frac{k}{s(1+Ts)}$$

$$\mathrm{H(s)} = \frac{G(s)}{s} = \frac{k}{s^2(1+Ts)}$$

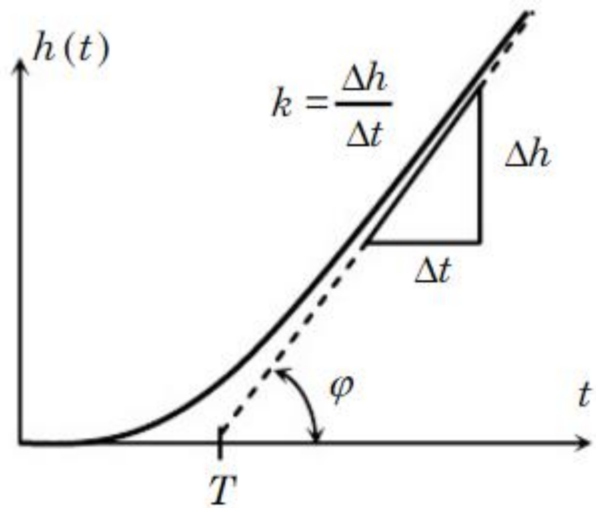
$$h(t) = k[t - T(1 - e^{-\frac{t}{T}})]$$



3.3 Typical Linear Systems

$$h(t) = k[t - T(1 - e^{-\frac{t}{T}})]$$

- Steady-state, $h(t) = k(t - T) = h_{tc}(t)$
- $\mathrm{k} = \tan(\varphi)$
- Asymptote at ∞.
- T: t-coordinate of the intersection between the asymptote and the time axis.
- $\mathrm{k} = \tan(\varphi)$.

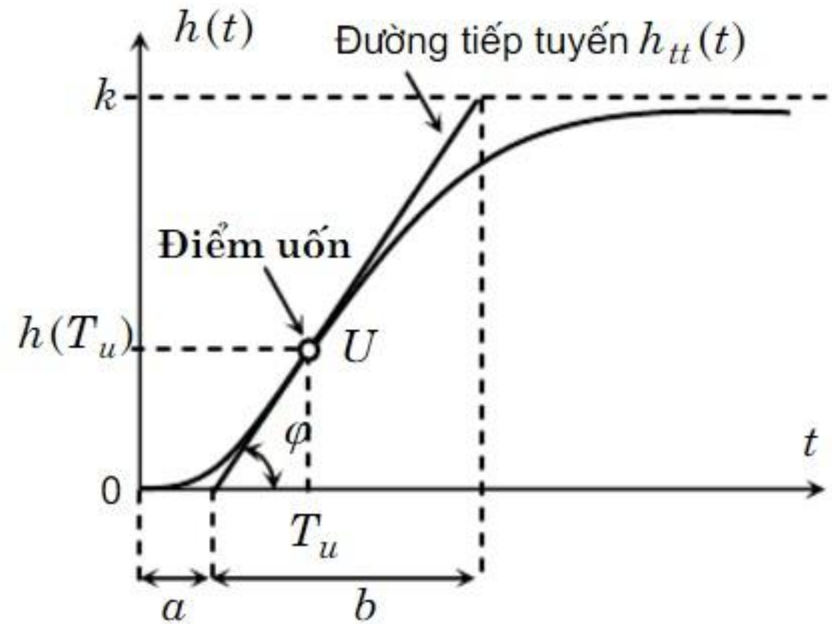


3.3 Typical Linear Systems

- ***Second-order system***

$$G(s) = \frac{k}{(1+sT_1)(1+sT_2)}, T_2 < T_1, x = \frac{T_2}{T_1}$$

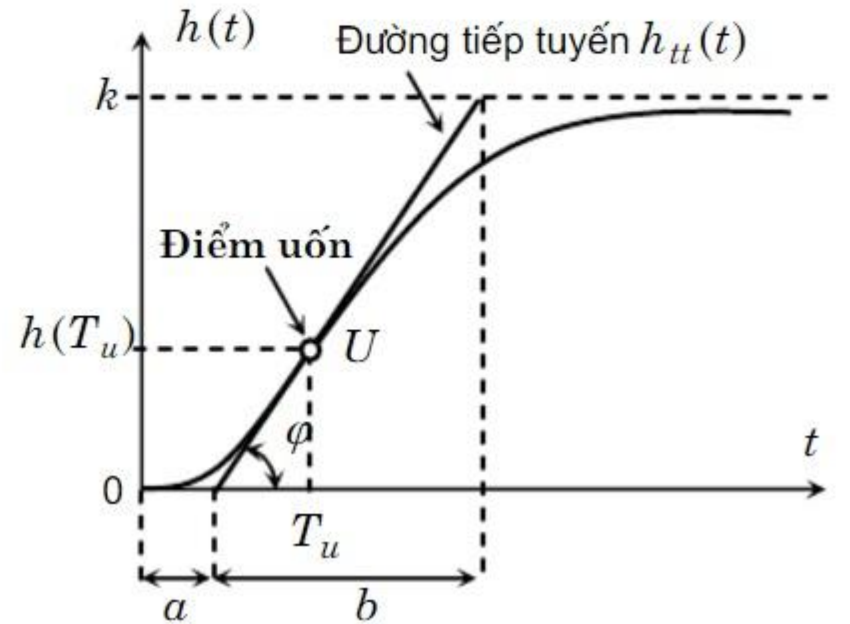
- $H(s) = \frac{k}{s(1+sT_1)(1+sT_2)}$
- $h(t) = k\left[1 - \frac{T_1 e^{-\frac{t}{T_1}} - T_2 e^{-\frac{t}{T_2}}}{T_1 - T_2}\right]$



- Điểm uốn (inflection point)
- Đường tiếp tuyến (tangent line)

3.3 Typical Linear Systems

- $\frac{dh(t)}{\mathrm{dt}} = k\frac{e^{-\frac{t}{T_1}} - e^{-\frac{t}{T_2}}}{T_1 - T_2}$
- $\frac{d^2h(t)}{dt^2} = k\left[\frac{e^{-\frac{t}{T_1}}}{T_1(T_2 - T_1)} + \frac{e^{-\frac{t}{T_2}}}{T_2(T_1 - T_2)}\right]$
- $T_u = \frac{T_1T_2}{T_2 - T_1} ln\frac{T_2}{T_1}$
- $v^* = \frac{dh(T_u)}{\mathrm{dt}} = \frac{k}{T_1}\left(\frac{T_2}{T_1}\right)^{\frac{T_2}{T_1 - T_2}}$



3.3 Typical Linear Systems

- Let x $= \frac{T_2}{T_1}$, $\quad v^* = \frac{k}{T_1} x^{\frac{x}{1-x}}$
- But $v^* = \frac{k}{b}$,
- So $\frac{\mathrm{b}}{T_1} = x^{\frac{x}{x-1}}$
- $a = T_u - \frac{h(T_u)}{v^*}$

$$\frac{a}{b} = x^{\frac{x}{1-x}} \frac{x\ln x + x^2 - 1}{x - 1} - 1 = g(x)$$

$$x = g^{-1}\left(\frac{a}{b}\right)$$

$$0 < \frac{a}{b} < 0{,}103$$

3.3 Typical Linear Systems

- ***Second-order system with oscillation***

$$G(s) = \frac{k}{1 + 2DTs + T^2 s^2}, \qquad 0 < D < 1$$

$$H(s) = \frac{G(s)}{s}$$

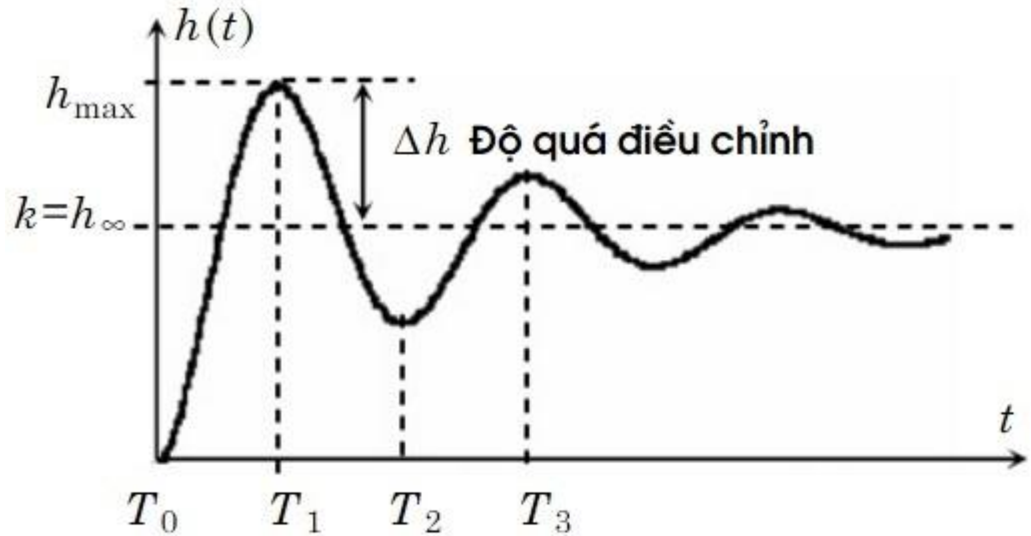
$$h(t) = k\left(1 - \frac{e^{-\frac{D}{T}t}}{\sqrt{1 - D^2}} \sin\left(\frac{\sqrt{1 - D^2}}{T} t + arcos(D)\right)\right)$$

3.3 Typical Linear Systems

$$\frac{dh(t)}{dt} = k\,\frac{e^{-\frac{D}{T}t}}{T\sqrt{1-D^2}}\sin\left(\frac{\sqrt{1-D^2}}{T}t\right);\ T_i = \frac{i\pi T}{\sqrt{1-D^2}}$$

$$h_{max} = h(T_1) = k[1 + e^{-\frac{\pi D}{\sqrt{1-D^2}}}]$$

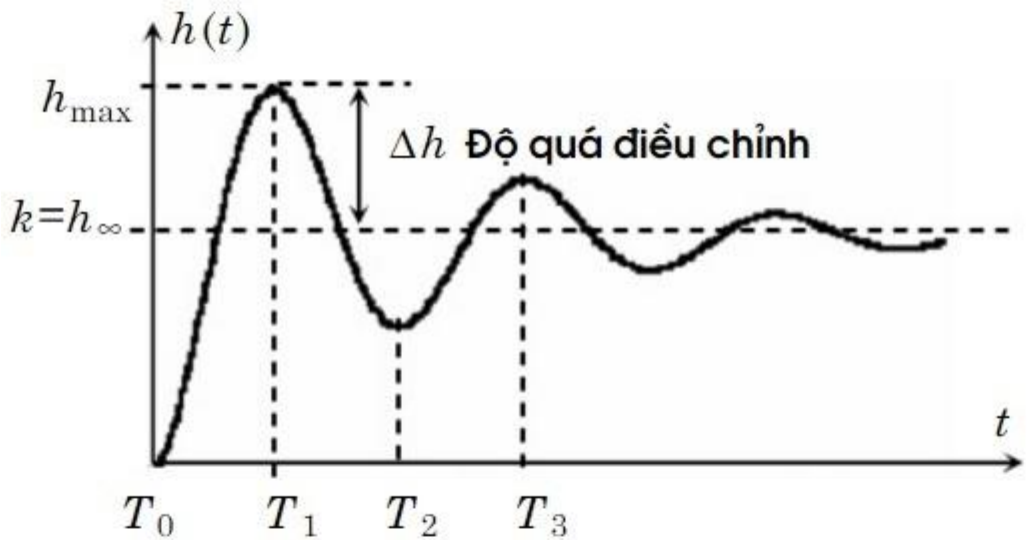
- $\Delta h = h_{max} - k = ke^{-\frac{\pi D}{\sqrt{1-D^2}}}$
- $D = \frac{1}{\sqrt{1+\frac{\pi^2}{ln^2\frac{\Delta h}{k}}}}$



3.3 Typical Linear Systems

$$T_1 = \frac{\pi T}{\sqrt{1 - D^2}}$$

$$T = \frac{T_1\sqrt{1 - D^2}}{\pi}$$



3.3 Typical Linear Systems

Frequency characteristics

$$G(s) = \frac{Y(s)}{X(s)} = \frac{b_0 + b_1 s + \cdots + b_m s^m}{a_0 + a_1 s + \cdots + s^n} \qquad (m \le n)$$

$$G(j\omega) = G(s)\Big|_{s = j\omega}$$

$$G(s) = \frac{1}{1+s}$$

$$G(j\omega) = \frac{1}{1+j\omega} = \frac{1}{1+\omega^2} - \frac{j\omega}{1+\omega^2}$$

3.3 Typical Linear Systems

$$Re\{G(j\omega)\} = \frac{1}{1+\omega^2} \rightarrow Re \geq 0$$

$$Im\{G(j\omega)\} = \frac{-\omega}{1+\omega^2} \rightarrow Im \leq 0$$

$$(Re - \frac{1}{2})^2 + Im^2 = \frac{1}{4}$$

3.3 Typical Linear Systems

- ***Bode plot***

$$L(\omega) = 20\lg|G(j\omega)| \text{ - biên tần}$$

$$\varphi(\omega) = arctan\left(\frac{Im\{G_f(j\omega)\}}{Re\{G_f(j\omega)\}}\right)$$

- Gain: $G(s) = k, k > 0$
- First-order system: $G(s) = \frac{k}{1+Ts}$

3.3 Typical Linear Systems

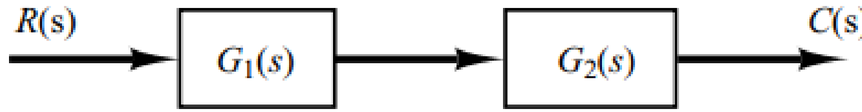
- PD: $G(s) = 1 + Ts$
- I: $G(s) = \frac{1}{T_I s}$
- D: $G(s) = T_D s$
- Second-order system:

$$G(s) = \frac{k}{1 + 2DTs + T^2 s^2}, \qquad 0 < D < 1$$

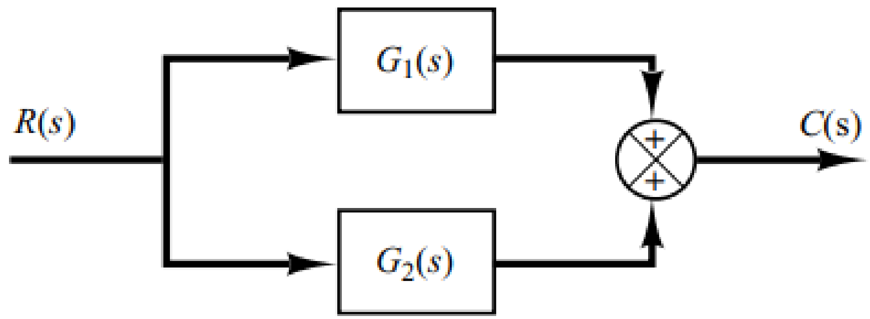
3.3 Typical Linear Systems

- ***Conversion of diagram:***

❑Series



❑Parallel



❑Feedback