Vector Calculus-3

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Numerator layout → expand num. along column

Denominator layout → expand denom along column

① y is a scalar, x is a scalar

Numerator layout: $\frac{\partial y}{\partial x}$

Denominator layout: $\frac{\partial y}{\partial x}$

② $\bar{y}$ is a vector, x is a scalar

Numerator layout:

$$\frac{\partial \bar{y}}{\partial x} = \begin{bmatrix} \partial y_1/\partial x \\ \vdots \\ \partial y_m/\partial x \end{bmatrix} \downarrow$$

column vector

Denominator layout:

row vector

$$\frac{\partial \bar{y}}{\partial x} = \begin{bmatrix} \frac{\partial y_1}{\partial x} & \cdots & \frac{\partial y_m}{\partial x} \end{bmatrix}$$

③ Y is a matrix, x is a scalar

Numerator layout:

$$\frac{\partial Y}{\partial x} = \begin{array}{c|c} & \xrightarrow{n} \\ \hline \downarrow m & \begin{bmatrix} \frac{\partial y_{11}}{\partial x} & \cdots & \frac{\partial y_{1n}}{\partial x} \\ \vdots & & \vdots \\ \frac{\partial y_{m1}}{\partial x} & \cdots & \frac{\partial y_{mn}}{\partial x} \end{bmatrix} \end{array}$$

Denominator layout:

$$\frac{\partial Y^T}{\partial x} = \begin{array}{c|c} & \xrightarrow{m} \\ \hline \downarrow n & \begin{bmatrix} \frac{\partial y_{11}}{\partial x} & \cdots & \frac{\partial y_{1m}}{\partial x} \\ \vdots & & \\ \frac{\partial y_{n1}}{\partial x} & \cdots & \frac{\partial y_{nm}}{\partial x} \end{bmatrix} \end{array}$$

④ y is a scalar, x is a vector

Numerator layout:

$$\frac{\partial y}{\partial \bar{x}} = \begin{bmatrix} \frac{\partial y}{\partial x_1} & \cdots & \frac{\partial y}{\partial x_n} \end{bmatrix}$$

→ row vector

Denominator layout:

$$\frac{\partial y}{\partial \bar{x}} = \begin{bmatrix} \partial y/\partial x_1 \\ \vdots \\ \partial y/\partial x_n \end{bmatrix} \downarrow$$

column vector

⑤ $\bar{y}$ is a vector ↗ m, $\bar{x}$ is a vector ↘ n

Numerator layout:

→ expand denom along row

$$\frac{\partial \bar{y}}{\partial \bar{x}} = \begin{bmatrix} \partial y_1/\partial x_1 & \cdots & \partial y_1/\partial x_n \\ \vdots & & \\ \partial y_m/\partial x_1 & \cdots & \partial y_m/\partial x_n \end{bmatrix}_{m \times n}$$

↓ expand num along column

Denominator layout:

→ expand num. along row

$$\frac{\partial \bar{y}^T}{\partial \bar{x}} = \begin{bmatrix} \partial y_1/\partial x_1 & \cdots & \partial y_m/\partial x_1 \\ \vdots & & \\ \partial y_1/\partial x_n & \cdots & \partial y_m/\partial x_n \end{bmatrix}$$

expand denom along column

→ Jacobian ←

Kronecker Product

$A : m \times n$

$B : p \times q$

$A \otimes B = pm \times qn$ ↖ Block matrix } Bigger than constituent matrices.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$A \otimes B = \begin{bmatrix} a_{11}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} & a_{12}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\ a_{21}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} & a_{22}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \end{bmatrix}$$

→ one element of A × B matrix.

$$= \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{21}b_{11} & a_{21}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{bmatrix}_{4\times 4}$$

Hadamard product

$A : m \times n$

$B : m \times n$

$A \circ B = m \times n$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$A \circ B = \begin{bmatrix} a_{11}b_{11} & a_{12}b_{12} \\ a_{21}b_{21} & a_{22}b_{22} \end{bmatrix}$$

- strictly element wise.
- one element of A × corresponding element of B

Important stuff

- Hadamard needs matrices of same dimⁿ → Produces matrix of same dimⁿ

undefined for higher dimⁿ

$$\begin{bmatrix} a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{bmatrix}_{4\times4}$$

- multiply each element in A by the whole matrix B, element by element.

- each product will be a new element of the kronecker matrix.

Produces matrix of same dimⁿ

- Kronecker can work w/ diff dimⁿ $\rightarrow$ Produces matrix of higher dimⁿ

same dimⁿ as $\bar{x}$

kronecker product of $\partial/\partial x$ with $\bar{y}$

	Scalar	Vector	Matrix
Scalar	$\frac{\partial y}{\partial x}$ = scalar $\frac{\partial}{\partial x}(\overset{y}{2x}) = 2$	$\frac{\partial \bar{y}}{\partial x}$ = colᵐ vector of same dimⁿ (num layout) $\frac{\partial}{\partial x} \otimes \overset{\bar{y}}{\begin{vmatrix} x \\ \cos x \\ 2x^2 \end{vmatrix}} = \begin{vmatrix} 1 \\ -\sin x \\ 4x \end{vmatrix}$	$\frac{\partial Y}{\partial x}$ = matrix of same dimension $\frac{\partial}{\partial x} \otimes \begin{vmatrix} x^2+1 & \cos x \\ \sin x & x-1 \end{vmatrix} = \begin{vmatrix} 2x & -\sin x \\ \cos x & 1 \end{vmatrix}$
Vector $\bar{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$	$\frac{\partial y}{\partial \bar{x}}$ = row vector (gradient) $\frac{\partial}{\partial \bar{x}}(xyz)$ $= \left[\frac{\partial}{\partial x}\ \frac{\partial}{\partial y}\ \frac{\partial}{\partial z}\right] \otimes xyz$ $= (yz)\cdot(xz)\cdot(yx)$	$\frac{\partial \bar{y}}{\partial \bar{x}}$ = matrix$_{m\times n}$ (jacobian) $\frac{\partial}{\partial \bar{x}} \begin{bmatrix} e^{xy^2} \\ x^2z \\ xy \end{bmatrix} = \left[\frac{\partial}{\partial x}\ \frac{\partial}{\partial y}\ \frac{\partial}{\partial z}\right]_{1\times n} \otimes \begin{bmatrix} e^{xyz} \\ x^2z \\ xy \end{bmatrix}_{m\times1}$ $= [\quad]_{(1\times m)\times(n\times1)}$	$\frac{\partial Y}{\partial \bar{x}}$ = matrix (kronecker) $\frac{\partial}{\partial \bar{x}} \begin{bmatrix} x^2yz & xy^2z \\ xyz^2 & \ln(xyz) \end{bmatrix} = \left[\frac{\partial}{\partial x}\ \frac{\partial}{\partial y}\ \frac{\partial}{\partial z}\right] \otimes Y$ $= [\quad]_{\text{bigger}}$.
Matrix $X = \begin{bmatrix} x_0 & x_1 \\ x_2 & x_3 \end{bmatrix}$ $\frac{\partial}{\partial X} = \begin{vmatrix} \frac{\partial}{\partial x_0} & \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \end{vmatrix}$	$\frac{\partial y}{\partial X}$ = matrix of Same dimⁿ $\frac{\partial}{\partial X}(\underset{y}{4x_3+3x_2+2x_1+x_0})$ $= \begin{bmatrix} \frac{\partial}{\partial x_0}(y) & \frac{\partial}{\partial x_1}(y) \\ \frac{\partial}{\partial x_2}(y) & \frac{\partial}{\partial x_3}(y) \end{bmatrix}$	$\frac{\partial \bar{y}}{\partial X}$ = larger matrix (kronecker) $\bar{y} = [y_0\ \ y_1]$ $\frac{\partial \bar{y}}{\partial X} = X \otimes \bar{y} = \begin{bmatrix} \frac{\partial}{\partial x_0} & \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \end{bmatrix}_{2\times2} \otimes [y_0\ y_1]_{1\times2}$ $= [\quad]_{(2\times1)\times(2\times2)}$	$\frac{\partial Y}{\partial X}$ = larger matrix (kronecker) $Y = \begin{bmatrix} y_0 & y_1 \\ y_2 & y_3 \end{bmatrix}$ $\frac{\partial Y}{\partial X} = \begin{bmatrix} \frac{\partial}{\partial x_0} & \frac{\partial}{\partial x_1} \\ \partial/\partial x_2 & \frac{\partial}{\partial x_3} \end{bmatrix}_{2\times2} \otimes \begin{bmatrix} y_0 & y_1 \\ y_2 & y_3 \end{bmatrix}_{2\times2} = [\quad]_{4\times4}$

Identity fⁿ

$$\bar{y} = f(\bar{x}) = \bar{x} \qquad f_i(x) = x_i$$

$$|\bar{y}| = m, \quad |\bar{x}| = n$$

$$\frac{\partial \bar{y}}{\partial \bar{x}} = \begin{bmatrix} \frac{\partial}{\partial x_1} f_1(\bar{x}) & \cdots & \frac{\partial}{\partial x_n} f_1(\bar{x}) \\ \vdots & & \\ \frac{\partial}{\partial x_1} f_m(\bar{x}) & \cdots & \frac{\partial}{\partial x_n} f_m(\bar{x}) \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdot & \cdots & 0 & 1 \end{bmatrix} = I$$

$$f_1(\bar{x}) = [0 + x_1 + \cdots 0]$$

$$\frac{\partial}{\partial x_j}(x_i) = 0 \text{ if } i \neq j$$

Element wise binary operator.

Let '$\circ$' denote element wise operation.

Vector of fⁿ w/ binary operator $\rightarrow$ $\boxed{y = f(\bar{w}) \circ f(\bar{x})}$ $\rightarrow$ $f_i(\bar{w}) \circ g_i(\bar{x}) = y_i$

$\underbrace{\text{Vector of f}^n \text{ w/ binary operator}}_{I/P}$

$$\frac{\partial}{\partial w_j} f_i(\bar{w}) = 0 \quad [i \neq j]$$

$\Rightarrow$ diagonal matrix

$$\frac{\partial}{\partial w_j} f_i(x) = 0 \quad [i \neq j]$$

$\Rightarrow$ diagonal matrix

$$\frac{\partial \bar{y}}{\partial \bar{w}} = \begin{bmatrix} \frac{\partial}{\partial w_1}(f_1(w_1) \circ g_1(w_1)) & & \\ & \ddots & \\ & & \frac{\partial}{\partial w_n}(f_n(w_n) \circ g_n(w_n)) \end{bmatrix}$$

Operation	Partial	Result
+ : $\frac{\partial}{\partial \bar{w}}(\bar{w} + \bar{x})$	$\text{diag}\left(\ldots \frac{\partial(w_i + x_i)}{\partial w_i}\right)$	I
: $\frac{\partial}{\partial \bar{x}}(\bar{w} + \bar{x})$	$\text{diag}\left(\cdots \frac{\partial(w_i + x_i)}{\partial x_i}\right)$	I
− : $\frac{\partial}{\partial \bar{w}}(\bar{w} - \bar{x})$	$\text{diag}\left(\cdots \frac{\partial(w_i - x_i)}{\partial w_i} \ldots\right)$	I
: $\frac{\partial}{\partial \bar{x}}(\bar{w} - \bar{x})$	$\text{diag}\left(\cdots \frac{\partial(w_i - x_i)}{\partial x_i} \ldots\right)$	$-I$

$\partial \bar{w}$ $\qquad\qquad$ ∂w_i)

: $\frac{\partial}{\partial \bar{x}} (\bar{w} - \bar{x})$ $\qquad$ $\mathrm{diag}\left(\cdots \frac{\partial (w_i - x_i)}{\partial x_i} \cdots \right)$ $\qquad$ $-I$

$\otimes$: $\frac{\partial}{\partial \bar{w}} (\bar{w} \times \bar{x})$ $\qquad$ $\mathrm{diag}\left(\cdots \frac{\partial (w_i \times x_i)}{\partial w_i} \right)$ $\qquad$ $\mathrm{diag}(\bar{x})$

: $\frac{\partial}{\partial \bar{x}} (\bar{w} \times \bar{x})$ $\qquad$ $\mathrm{diag}\left(\cdots \frac{\partial (w_i \times x_i)}{\partial x_i} \cdots \right)$ $\qquad$ $\mathrm{diag}(\bar{w})$

$\oslash$: $\frac{\partial}{\partial \bar{w}} (\bar{w} / \bar{x})$ $\qquad$ $\mathrm{diag}\left(\cdots \frac{\partial (w_i / x_i)}{\partial w_i} \cdots \right) = \mathrm{diag}\left(\cdots \frac{1}{x_i} \cdots \right)$

: $\frac{\partial}{\partial \bar{x}} (\bar{w} / \bar{x})$ $\qquad$ $\mathrm{diag}\left(\cdots \frac{\partial (w_i / x_i)}{\partial w_i} \cdots \right) = \mathrm{diag}\left(\cdots \frac{-w_i}{x_i^2} \cdots \right)$