CHAPTER

1

Introduction to Graph Theory

INTRODUCTION

It is no coincidence that graph theory has been independently discovered many times, since it may quite properly be regarded as an area of **applied mathematics.**

The basic combinatorial nature of graph theory and a clue to its wide applicability are indicated in the words of Sylvester, **''The theory of ramification is one of pure colligation, for it takes no account of magnitude or position ; geometrical lines are used, but have no more real bearing on the matter than those employed in genealogical tables have in explaining the laws of procreation.''**

Indeed, the earliest recorded mention of the subject occurs in the works of Euler, and although the original problem he was considering might be regarded as a somewhat frivolous puzzle, it did arise from the physical world. Subsequent rediscoveries of graph theory by Kirchhoff and Cayley also had their roots in the physical world.

Kirchhoff's investigations of electric networks led to his development of the basic concepts and theorems concerning trees in graphs, while Cayley considered trees arising from the enumeration of organic chemical isomers. Another puzzle approach to graphs was proposed by Hamilton. After this, the celebrated four colour conjecture came into prominence and has been notorious ever since.

In the **present century,** there have already been a great many rediscoveries of graph theory which we can only mention most briefly in this chronological account.

Euler (1707–1782) became the father of graph theory as well as topology. Graph theory is considered to have begun in 1736 with the publication of Euler's solution of the Königsberg bridge problem. The graph theory is one of the few fields of mathematics with a definite birth date by ore.

1.1 WHAT IS A GRAPH ? DEFINITION

A graph G consists of a set of objects V = $\{v_1, v_2, v_3, ...\}$ called **vertices** (also called **points** or **nodes**) and other set E = $\{e_1, e_2, e_3, ...\}$ whose elements are called **edges** (also called **lines** or **arcs**).

The set V(G) is called the **vertex set** of G and E(G) is the **edge set.**

Usually the graph is denoted as **G = (V, E)**

Let G be a graph and $\{u, v\}$ an edge of G. Since $\{u, v\}$ is 2-element set, we may write $\{v, u\}$ instead of $\{u, v\}$. It is often more convenient to represent this edge by *uv* or *vu.*

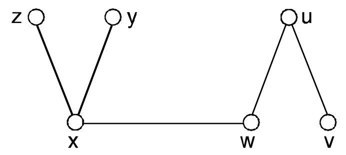
If $e = uv$ is an edge of a graph G, then we say that u and v are **adjacent** in G and that e joins u and v. (We may also say that each that of u and v is adjacent to or with the other).

For example :

A graph G is defined by the sets

$$V(G) = \{u, v, w, x, y, z\} \text{ and } E(G) = \{uv, uw, wx, xy, xz\}.$$

Now we have the following graph by considering these sets.



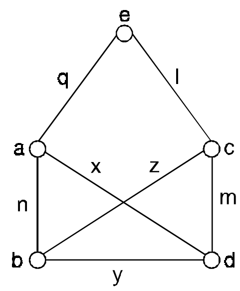
Every graph has a diagram associated with it. The vertex *u* and an edge *e* are **incident** with each other as are *v* and *e*. If two distinct edges say *e* and *f* are **incident** with a common vertex, then they are adjacent edges.

A graph with *p*-vertices and *q*-edges is called a ***(p, q)* graph.**

The (1, 0) graph is called **trivial graph.**

In the following figure the vertices *a* and *b* are adjacent but *a* and *c* are not. The edges *x* and *y* are adjacent but *x* and *z* are not.

Although the edges *x* and *z* intersect in the diagram, their intersection is not a vertex of the graph.

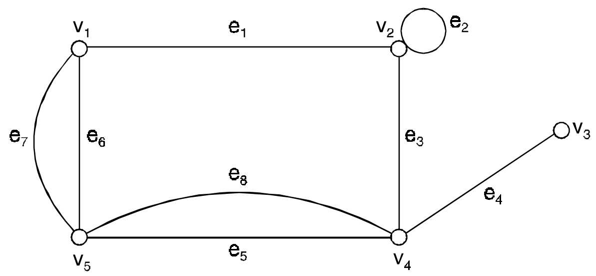


Examples :

(1) Let V = {1, 2, 3, 4} and E = {{1, 2}, {1, 3}, {3, 2}. {4, 4}}.
Then G(V, E) is a graph.

(2) Let V = {1, 2, 3, 4} and E = {{1, 5}, {2, 3}}.
Then G(V, E) is not a graph, as 5 is not in V.

(3)



A graph with 5-vertices and 8-edges is called a **(5, 8) graph.**

1.2 DIRECTED AND UNDIRECTED GRAPHS

1.2.1. Directed graph

A directed graph or digraph G consists of a set V of vertices and a set E of edges such that $e \in$ E is associated with an ordered pair of vertices.

In other words, if each edge of the graph G has a direction then the graph is called **directed graph.**

In the diagram of directed graph, each edge $e = (u, v)$ is represented by an arrow or directed curve from initial point u of e to the terminal point v.

Figure 1(a) is an example of a directed graph.

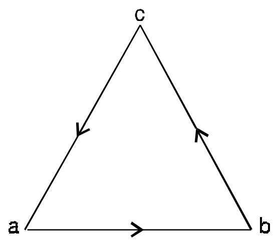


Fig. 1(a). Directed graph.

Suppose $e = (u, v)$ is a directed edge in a digraph, then (i) u is called the **initial vertex** of e and v is the terminal vertex of e

(ii) e is said to be **incident** from u and to be incident to v.

(iii) u is adjacent to v, and v is adjacent from u.

1.2.2. Un-directed graph

An un-directed graph G consists of set V of vertices and a set E of edges such that each edge $e \in$ E is associated with an unordered pair of vertices.

In other words, if each edge of the graph G has no direction then the graph is called **un-directed graph.**

Figure 1(b) is an example of an undirected graph.

We can refer to an edge joining the vertex pair i and j as either (i, j) or (j, i).

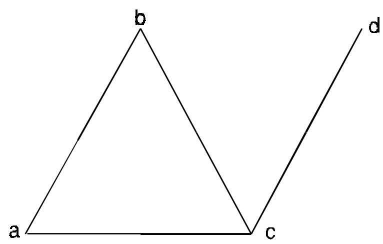


Figure 1(b). Un-directed graph.

1.3 BASIC TERMINOLOGIES

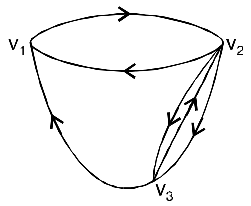
1.3.1 Loop : An edge of a graph that joins a node to itself is called **loop** or **self loop.**

i.e., a loop is an edge (v_i, v_j) where $v_i = v_f$.

1.3.2. Multigraph

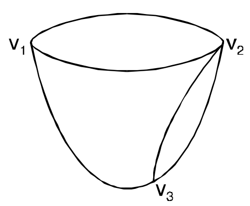
In a multigraph no loops are allowed but more than one edge can join two vertices, these edges are called **multiple edges** or parallel edges and a graph is called **multigraph.**

Two edges (v_i, v_j) and (v_f, v_r) are parallel edges if $v_i = v_r$ and v_j, v_f.



Directed multigraph

Fig. 2(*a*)



Un-directed multigraph

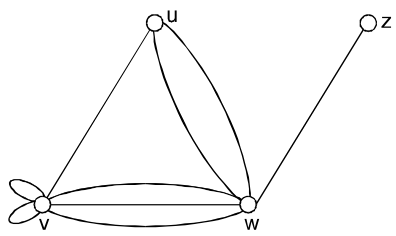
Fig. 2(*b*)

In Figure 1.2(*a*), there are two parallel edges associated with v_2 and v_3.

In Figure 1.2(*b*), there are two parallel edges joining nodes v_1 and v_2 and v_2 and v_3.

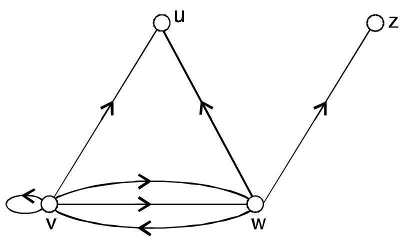
1.3.3. Pseudo graph

A graph in which loops and multiple edges are allowed, is called a pseudo graph.



Un-directed Pseudo graph

Fig. 3(*a*)



Directed Pseudo graph

Fig. 3(*b*)

1.3.4. Simple graph

A graph which has neither loops nor multiple edges. *i.e.,* where each edge connects two distinct vertices and no two edges connect the same pair of vertices is called a **simple graph.**

Figure 1.1(*a*) and (*b*) represents simple undirected and directed graph because the graphs do not contain loops and the edges are all distinct.

1.3.5. Finite and Infinite graphs

A graph with finite number of vertices as well as a finite number of edges is called a **finite graph.** Otherwise, it is an **infinite graph.**

1.4 DEGREE OF A VERTEX

The number of edges incident on a vertex v_i with **self-loops counted twice** (is called the **degree of a vertex** v_i and is denoted by **$\deg_G(v_i)$** or **deg v_i** or **$d(v_i)$.**

The degrees of vertices in the graph G and H are shown in Figure 4(*a*) and 4(*b*).

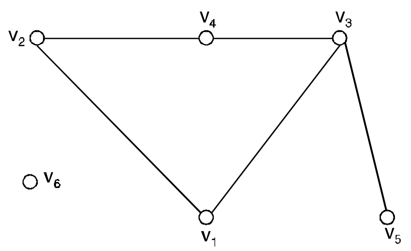


Fig. 4(*a*)

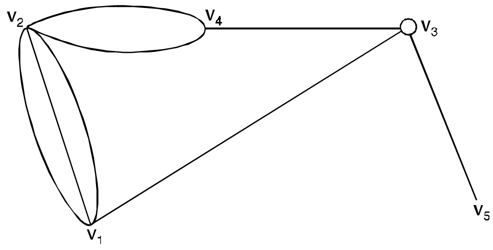


Fig. 4(*b*)

In G as shown in Figure 4(*a*),

$\deg_G(v_2) = 2 = \deg_G(v_4) = \deg_G(v_1)$, $\deg_G(v_3) = 3$ and $\deg_G(v_5) = 1$ and

In H as shown in Figure 4(*b*),

$\deg_H(v_2) = 5$, $\deg_H(v_4) = 3$, $\deg_H(v_3) = 5$, $\deg_H(v_1) = 4$ and $\deg_H(v_5) = 1$.

The degree of a vertex is some times also referred to as its **valency.**

1.5 ISOLATED AND PENDENT VERTICES

1.5.1. Isolated vertex

A vertex having **no incident edge** is called an **isolated vertex.**

In other words, isolated vertices are those with zero degree.

1.5.2. Pendent or end vertex

A vertex of **degree one,** is called a **pendent vertex** or an **end vertex.**

In the above Figure, v_5 is a pendent vertex.

1.5.3. In degree and out degree

In a graph G, the out degree of a vertex v_i of G, denoted by out $\deg_G(v_i)$ or $\deg_G^+(v_i)$, is the number of edges beginning at v_i and the in degree of v_i, denoted by in $\deg_G$ **(v_i)** or $\deg_G^{-1}$ **(v_i),** is the number of edges ending at v_i.

The sum of the in degree and out degree of a vertex is called the **total degree** of the vertex. A vertex with zero in degree is called a **source** and a vertex with zero out degree **is called a sink.** Since each edge has an initial vertex and terminal vertex.

1.6 THE HANDSHAKING THEOREM 1.1

If G = (v, E) be an undirected graph with e edges.

Then $\sum_{v \in V} \deg_G(v) = 2e$

i.e., the sum of degrees of the vertices is an undirected graph is even.

Proof : Since the degree of a vertex is the number of edges incident with that vertex, the sum of the degree counts the total number of times an edge is incident with a vertex.

Since every edge is incident with exactly two vertices, each edge gets counted twice, once at each end.

Thus the sum of the degrees equal twice the number of edges.

Note : This theorem applies even if multiple edges and loops are present. The above theorem holds this rule that if several people shake hands, the total number of hands shake must be even that is why the theorem is called handshaking theorem.

Corollary : In a non directed graph, the total number of odd degree vertices is even.

Proof : Let G = (V, E) a non directed graph.

Let U denote the set of even degree vertices in G and W denote the set of odd degree vertices.

Then $\sum_{v_i \in V} \deg_G (v_i) = \sum_{v_i \in U} \deg_G (v_i) + \sum_{v_i \in W} \deg_G (v_i)$

$$\Rightarrow \quad 2e - \sum_{v_i \in U} \deg_G (v_1) = \sum_{v_i \in W} \deg_G (v_1) \qquad \text{...(1)}$$

Now $\sum_{v_i \in W} \deg_G (v_i)$ is also even

Therefore, from (1) $\sum_{v_i \in W} \deg_G (v_i)$ is even

∴ The no. of odd vertices in G is even.

Theorem 1.2. If V = $\{v_1, v_2, \ldots\ldots v_n\}$ is the vertex set of a non directed graph G,

$$\text{then } \sum_{i=1}^{n} \deg (v_i) = 2\,|\,E\,|$$

If G is a directed graph, then $\sum_{i=1}^{n} \deg^+ (v_i) = \sum_{i=1}^{n} \deg^- (v_i) = |\,E\,|$

Proof : Since when the degrees are summed.

Each edge contributes a count of one to the degree of each of the two vertices on which the edge is incident.

Corollary (1) : In any non directed graph there is an even number of vertices of odd degree.

Proof : Let W be the set of vertices of odd degree and let U be the set of vertices of even degree.

Then $\sum_{v \in V(G)} \deg (v) = \sum_{v \in W} \deg (v) + \sum_{v \in U} \deg (v) = 2\,|\,E\,|$

Certainly, $\sum_{v \in U} \deg (v)$ is even, Hence $\sum_{v \in W} \deg(v)$ is even,

Implying that | W | is even.

Corollary (2) : If $k = \delta(G)$ is the minimum degree of all the vertices of a non directed graph G, then

$$k\,|V| \le \sum_{v \in V(G)} \deg(v) = 2\,|E|$$

In particular, if G is a k-regular graph, then

$$k\,|V| = \sum_{v \in V(G)} \deg(v) = 2\,|E|.$$

Problem 1.1. *Show that, in any gathering of six people, there are either three people who all know each other or three people none of whom knows either of the other two (six people at a party).*

Solution. To solve this problem, we draw a graph in which we represent each person by a vertex and join two vertices by a solid edge if the corresponding people know each other, and by a dotted edge if not. We must show that there is always a solid triangle or a dotted triangle.

Let v be any vertex. Then there must be exactly five edges incident with v, either solid or dashed, and so at least three of these edges must be of the same type.

Let us assume that there are three solid edges (see figure 5) ; the case of atleast three dashed edges is similar.

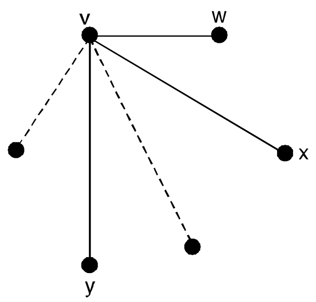


Fig. 5.

If the people corresponding to the vertices w and x know each other, then v, w and x form a solid triangle, as required.

Similarly, if the people corresponding to the vertices w and y, or to the vertices x and y, know each other, then we again obtain a solid triangle.

These three cases are shown in Figure (6).

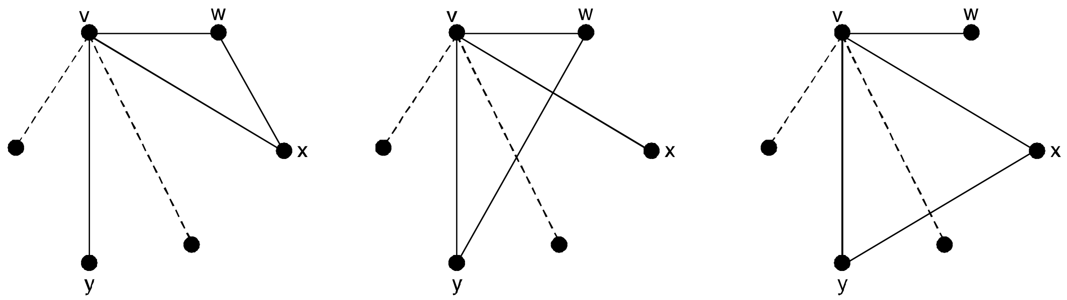


Fig. 6.

Finally, if no two of the people corresponding to the vertices *w*, *x* and *y* know each other, then *w*, *x* and *y* from a dotted triangle, as required (see figure (7).

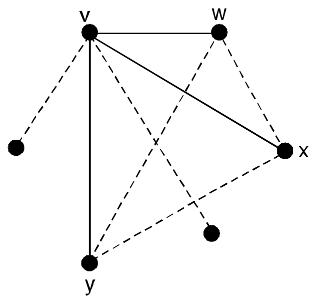


Fig. 7.

Problem 1.2. *Place the letters A, B, C, D, E, F, G, H into the eight circles in Figure (8), in such a way that no letter is adjacent to a letter that is next to it in the alphabet.*

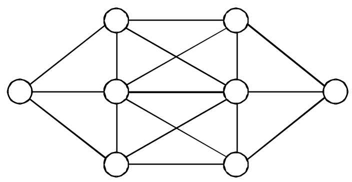


Fig. 8.

Solution. First note that trying all the possibilities is not a practical proposition, as there are 8 ! = 40320 ways of placing eight letters into eight circles.

Note that (*i*) the easiest letters to place are A and H, because each has only one letter to which it cannot be adjacent, namely, B and G, respectively.

(*ii*) the hardest circles to fill are those in the middle, as each is adjacent to six others.

This suggests that we place A and H in the middle circles. If we place A to the left of H, then the only possible positions for B and G are shown in Figure (9).

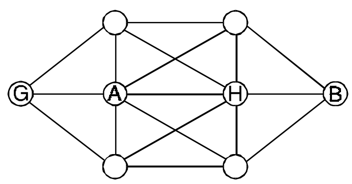


Fig. 9.

The letter C must now be placed on the left-hand side of the diagram, and F must be placed on the right-hand side.

It is then a simple matter to place the remaining letters, as shown in Figure (10).

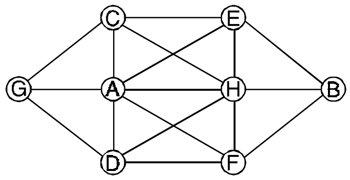


Fig. 10.

Problem 1.3. *Determine the number of edges in a graph with 6 vertices, 2 of degree 4 and 4 of degree 2. Draw two such graphs.*

Solution. Suppose the graph with 6 vertices has e number of edges. Therefore by Handshaking lemma

$$\sum_{i=1}^{6} \deg (v_i) = 2e$$

$\Rightarrow$ $d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5) + d(v_6) = 2e$

Now, given 2 vertices are of degree 4 and 4 vertices are of degree 2.

Hence the above equation,

$$(4 + 4) + (2 + 2 + 2 + 2) = 2e$$

$\Rightarrow$ $16 = 2e$ $\Rightarrow$ $e = 8.$

Hence the number of edges in a graph with 6 vertices with given condition is 8.

Two such graphs are shown below in Figure (11).

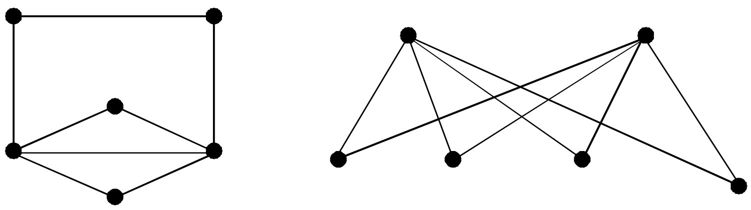


Fig. 11.

Problem 1.4. *How many vertices are needed to construct a graph with 6 edges in which each vertex is of degree 2.*

Solution. Suppose these are P vertices in the graph with 6 degree. Also given the degree of each vertex is 2.

By handshaking lemma,

$$\sum_{i=1}^{P} \deg (v_i) = 2q = 2 \times 6$$

$\Rightarrow$ $d(v_1) + d(v_2) + \ldots\ldots + d(v_n) = 12$

$\Rightarrow$ $2 + 2 + \ldots\ldots + 2 = 12$

$\Rightarrow$ $2P = 12$ $\Rightarrow$ P = 6 vertices are needed.

Problem 1.5. *It is possible to construct a graph with 12 vertices such that 2 of the vertices have degree 3 and the remaining vertices have degree 4.*

Solution. Suppose it is possible to construct a graph with 12 vertices out of which 2 of them are having degree 3 and remaining vertices are having degree 4.

Hence by handshaking lemma,

$$\sum_{i=1}^{12} d(v_i) = 2e \text{ where } e \text{ is the number of edges}$$

According to given conditions

$$(2 \times 3) + (10 \times 4) = 2e$$

$\Rightarrow \quad 6 + 40 = 2e$

$\Rightarrow \quad 2e = 46 \qquad \Rightarrow \quad e = 23$

It is possible to construct a graph with 23 edges and 12 vertices which satisfy given conditions.

Problem 1.6. *It is possible to draw a simple graph with 4 vertices and 7 edges ? Justify.*

Solution. In a simple graph with P-vertices, the maximum number of edges will be $\frac{P(P-1)}{2}$.

Hence a simple graph with 4 vertices will have at most $\frac{4 \times 3}{2}$ = 6 edges.

Therefore, the simple graph with 4 vertices cannot have 7 edges.

Hence such a graph does not exist.

Problem 1.7. *Show that the maximum degree of any vertex in a simple graph with P vertices is (P – 1).*

Solution. Let G be a simple graph with P-vertices. Consider any vertex v of G. Since the graph is simple (*i.e.,* without self loops and parallel edges), the vertex v can be adjacent to atmost remaining (P – 1) vertices.

Hence the maximum degree of any vertex in a simple graph with P vertices is (P – 1).

Problem 1.8. *Write down the vertex set and edge set of the following graphs shown in Figure 12(a) and 12(b).*

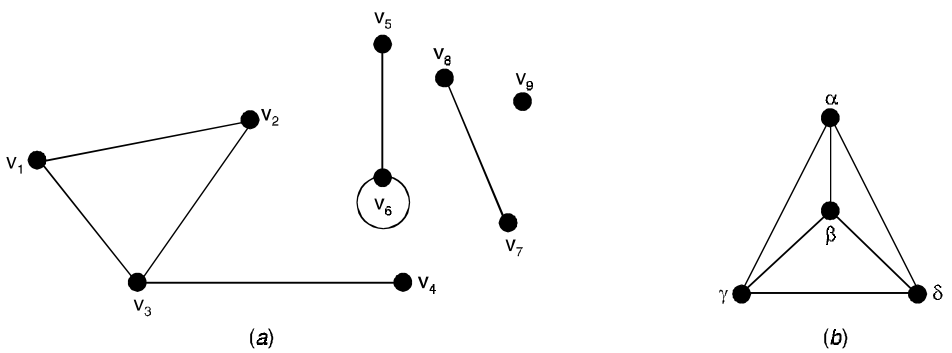


Fig. 12.

Solution. (*a*) V(G) = $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$

E(G) = $\{v_1v_2, v_1v_3, v_2v_3, v_3v_4, v_5v_6, v_6v_6, v_7v_8\}$

(*b*) V(G) = $\{\alpha, \beta, \gamma, \delta\}$

E(G) = $\{\alpha\beta, \alpha\gamma, \alpha\delta, \beta\delta, \beta\gamma, \gamma\delta\}$.

Problem 1.9. *Show that the size of a simple graph of order n cannot exceed* $^nC_2$.

Solution. Let G be a graph of order *n*.

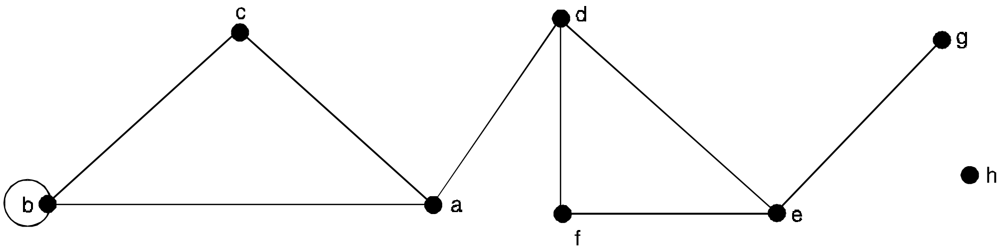
Let V be a vertex set of G.

Then cardinality of V is *n* and elements of E are distinct two elements subsets of V.

The number of ways we can choose two elements from a set V of *n* elements is $^nC_2$.

Thus, E may not have more than $^nC_2$ elements (edges).

Problem 1.10. *Find the degree sequence of the following graph.*



Solution. $\deg_G(a) = 3$, $\deg_G(b) = 4$, $\deg_G(c) = 2$

$\deg_G(d) = 3$, $\deg_G(e) = 3$, $\deg_G(f) = 2$

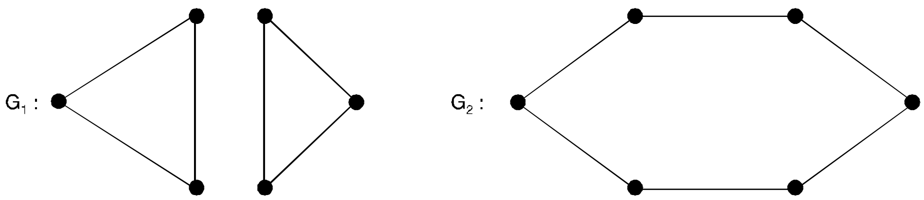
$\deg_G(g) = 1$, $\deg_G(h) = 0$.

Therefore, the degree sequence of the graph is 0, 1, 2, 2, 3, 3, 4.

Problem 1.11. *Construct two graphs having same degree sequence.*

Solution. The following two graphs have the same degree sequence.

The degree sequence of the graphs is 2, 2, 2, 2, 2, 2.



Problem 1.12. *Show that there exists no simple graph corresponds to the following degree sequence :*

(i) 0, 2, 2, 3, 4 *(ii) 1, 1, 2, 3* *(iii) 2, 2, 3, 4, 5, 5* *(iv) 2, 2, 4, 6.*

Solution. (*i*) to (*iii*) :

There are odd number of odd degree vertices in the graph.

Hence there exists no graph corresponds to this degree sequence.

(*iv*) Number of vertices in the graph is four and the maximum degree of a vertex is 6, which is not possible as the maximum degree cannot exceed one less than the number of vertices.

Problem 1.13. *Show that the total number of odd degree vertices of a (p, q)-graph is always even.*

Solution. Let v_1, v_2 ...... v_k be the odd degree vertices in G. Then, we have

$$\sum_{i=1}^{P} \deg_G (v_i) = 2q$$

i.e.,
$$\sum_{i=1}^{k} \deg_G (v_i) + \sum_{i=k+1}^{P} \deg_G (v_i) = 2q = \text{even number}$$

$$\Rightarrow \quad \sum_{i=1}^{k} \deg_G (v_i) = \text{even number} - \sum_{i=k+1}^{P} \deg_G (v_i)$$

$$\Rightarrow \quad \sum_{i=1}^{k} (\text{odd number}) = \text{even number} - \sum_{i=k+1}^{P} (\text{even number})$$

$$= \text{even number} - \text{even number}$$

$$= \text{even number.}$$

$\Rightarrow$ This implies that number of terms in the left-hand side of the equation is even.

Therefore, k is an even number.

Problem 1.14. *Show that the sequence 6, 6, 6, 6, 4, 3, 3, 0 is not graphical.*

Solution. To prove that the sequence is not graphical.

The given sequence is 6, 6, 6, 6, 4, 3, 3, 0

Resulting the sequence 5, 5, 5, 3, 2, 2, 0

Again consider the sequence 4, 4, 2, 1, 1, 0

Repeating the same 3, 1, 0, 0, 0

Since there exists no simple graph having one vertex of degree three and other vertex of degree one.

The last sequence is not graphical.

Hence the given sequence is also not graphical.

Problem 1.15. *Show that the following sequence is graphical. Also find a graph corresponding to the sequence 6, 5, 5, 4, 3, 3, 2, 2, 2.*

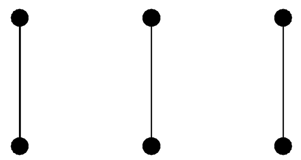
Solution. We can reduce the sequence as follows :

Given sequence	6, 5, 5, 4, 3, 3, 2, 2, 2
Reducing first 6 terms by 1 counting from second term	4, 4, 3, 2, 2, 1, 2, 2.
Writing in decreasing order	4, 4, 3, 2, 2, 2, 2, 1
Reducing first 4 terms by 1 counting from second	3, 2, 1, 1, 2, 2, 1
Writing in decending order	3, 2, 2, 2, 1, 1, 1
Reducing first 3 terms by 1, counting from second	1, 1, 1, 1, 1, 1

Sequence 1, 1, 1, 1, 1, 1 is graphical.

Hence the given sequence is also graphical.

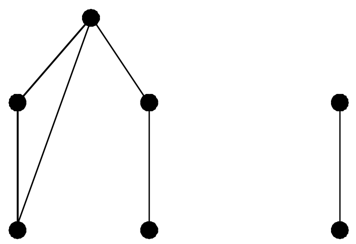
The graph corresponding to the sequence 1, 1, 1, 1, 1, 1 is given below



To obtain a graph corresponding to the given sequence, add a vertex to each of the vertices whose degrees are $t_1 - 1$, $t_2 - 1$, ... $t_s - 1$.

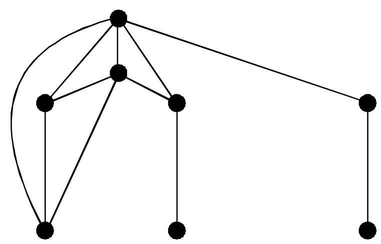
And repeat the process.

Step 1 :



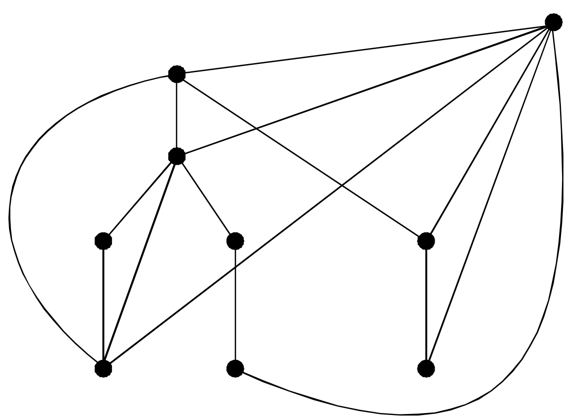
Degree sequence of this graph is 3, 2, 2, 2, 1, 1, 1

Step 2 :



Degree sequence of this graph is 4, 4, 3, 2, 2, 2, 2, 1.

Step 3 : Final graph



Degree sequence of this graph is 6, 5, 5, 4, 3, 3, 2, 2, 2.

Problem 1.16. *Show that no simple graph has all degrees of its vertices are distinct.*

(*i.e., in a degree sequence of a graph atleast one number should repeat.*)

Solution. Let G be a graph of order n.

Then there are n terms in the degree sequence of G. If no number (integer) in the degree sequence repeats, then only possible case it is of the form

$$0, 1, 2, 3, 4, ..., n-1$$

Since maximum degree cannot exceed $n-1$. But the last vertex of degree $n-1$ should be adjacent to every other vertex of G, since G is simple.

Thus minimum degree of every vertex is one.

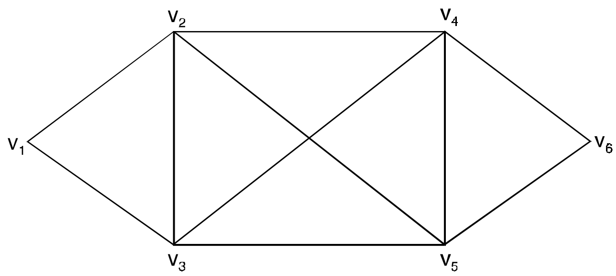
A contradiction to the fact that the degree of one vertex is zero.

Problem 1.17. *Is there a simple graph with degree sequence (1, 1, 3, 3, 3, 4, 6, 7) ?*

Solution. Assume there is such a graph. Then the vertex of degree 7 is adjacent to all other vertices, so in particular it must be adjacent to both vertices of degree 1.

Hence, the vertex v of degree 6 cannot be adjacent to either of the two vertices of degree 1.

Problem 1.18. *Find the degree of each vertex of the following graph :*

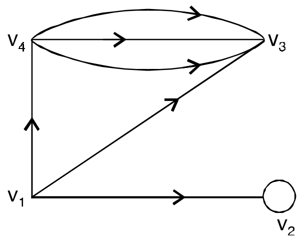


Solution. It is an undirected graph. Then

deg (v_1) = 2, deg (v_2) = 4, deg (v_3) = 4

deg (v_4) = 4, deg (v_5) = 4, deg (v_6) = 2.

Problem 1.19. *Find the in degree out degree and of total degree of each vertex of the following graph.*



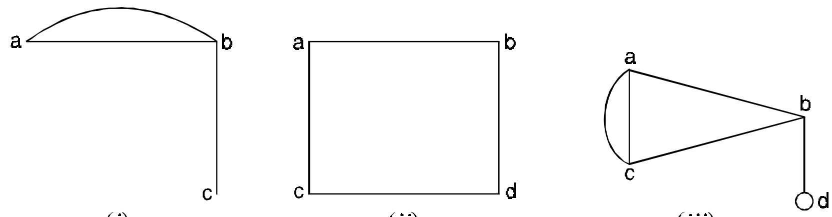
Solution. It is a directed graph

in deg (v_1) = 0, out deg (v_1) = 3, total deg (v_1) = 4

in deg (v_2) = 2, out get (v_2) = 1, total deg (v_2) = 3 in deg (v_3) = 4, out deg (v_3) = 0, total deg (v_3) = 4

in deg (v_4) = 1, out deg (v_4) = 3, total deg (v_4) = 4.

Problem 1.20. *State which of the following graphs are simple ?*



(*i*) (*ii*) (*iii*)

Solution. (*i*) The graph is not a simple graph, since it contains parallel edge between two vertices *a* and *b*.

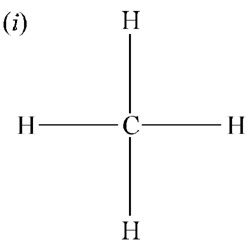
(*ii*) The graph is a simple graph, it does not contain loop and parallel edge.

(*iii*) The graph is not a simple graph, since it contains parallel edge and a loop.

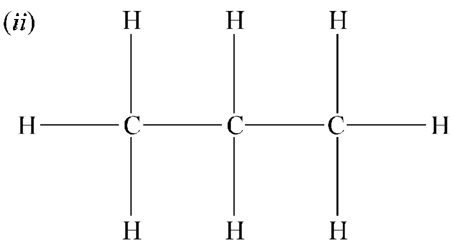
Problem 1.21. *Draw the graphs of the chemical molecules of*

(i) Methane (CH_4) *(ii) Propane* (C_3H_8).

Solution. (*i*)



(*ii*)



Problem 1.22. *Show that the degree of a vertex of a simple graph G on n vertices cannot exceed n – 1.*

Solution. Let *v* be a vertex of G, since G is simple, no multiple edges or loops are allowed in G.

Thus *v* can be adjacent to atmost all the remaining $n - 1$ vertices of G.

Hence *v* may have maximum degree $n - 1$ in G.

i.e., $$0 \le \deg_G(v) \le n - 1 \text{ for all } v \in V(G).$$

Problem 1.23. *Does there exists a simple graph with seven vertices having degrees (1, 3, 3, 4, 5, 6, 6) ?*

Solution. Suppose there exists a graph with seven vertices satisfying the given properties.

Since two vertices have degree 6, each of these two vertices is adjacent with every other vertex.

Hence the degree of each vertex is at least 2, so that the graph has no vertex of degree 1, which is a contradiction.

Hence there does not exist a simple graph with the given properties.

Problem 1.24. *Is there a simple graph corresponding to the following degree sequences ?*

(i) (1, 1, 2, 3) *(ii) (2, 2, 4, 6).*

Solution. (*i*) There are odd number (3) of odd degree vertices, 1, 1 and 3.

Hence there exist no graph corresponding to this degree sequence.

(*ii*) Number of vertices in the graph sequence is 4, and the maximum degree of a vertex is 6, which is not possible as the maximum degree cannot exist on less than the number of vertices.

Problem 1.25. *Show that the maximum number of edges in a simple graph with n vertices is* $\frac{n(n-1)}{2}$.

Solution. By the handshaking theorem,

$$\sum_{i=1}^{n} d(v_i) = 2e$$

where e is the number of edges with n vertices in the graph G.

$\Rightarrow \quad d(v_1) + d(v_2) + \ldots\ldots + d(v_n) = 2e$ ...(1)

Since we know that the maximum degree of each vertex in the graph G can be $(n - 1)$.

Therefore, equation (1) reduces

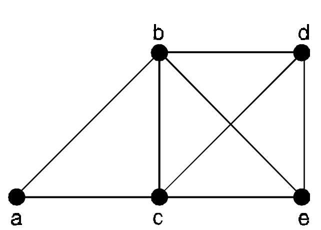
$$(n - 1) + (n - 1) + \ldots\ldots \text{ to } n \text{ terms} = 2e$$

$\Rightarrow \quad n(n - 1) = 2e$

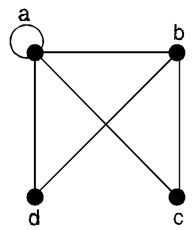
$$\Rightarrow \quad e = \frac{n(n-1)}{2}.$$

Hence the maximum number of edges in any simple graph with n vertices is $\frac{n(n-1)}{2}$.

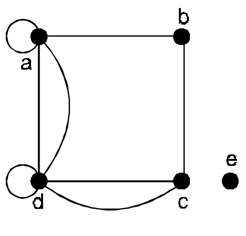
Problem 1.26. *Consider the following graphs and determine the degree of each vertex :*



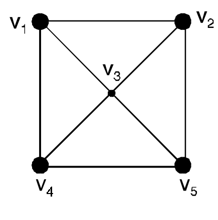
(*i*)



(*ii*)



(*iii*)



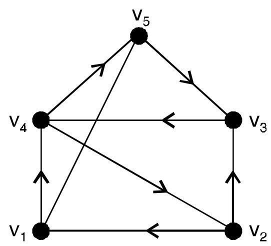
(*iv*)

Solution. (*i*) deg (a) = 2, deg (b) = 4, deg (c) = 4, deg (d) = 3, deg (e) = 3 (*ii*) deg (a) = 5, deg (b) = 2, deg (c) = 3, deg (d) = 6, deg (e) = 0

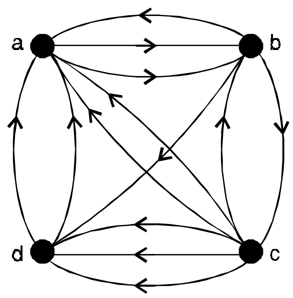
(*iii*) deg (a) = 5, deg (b) = 3, deg (c) = 2, deg (d) = 2,

(*iv*) Every vertex has degree 4.

Problem 1.27. *Find the in-degree and out-degree of each vertex of the following directed graphs :*



(*i*)



(*ii*)

Solution. (*i*) in-degree v_1 = 2, out-degree v_1 = 1

in-degree v_2 = 2, out-degree v_2 = 2

in-degree v_3 = 2, out-degree v_3 = 1

in-degree v_4 = 2, out-degree v_4 = 2

in-degree v_5 = 0, out-degree v_5 = 3

(*ii*) in-degree a = 6, out-degree a = 1

in-degree b = 1, out-degree b = 5

in-degree c = 2, out-degree c = 5

in-degree d = 2, out-degree d = 2.

Problem 1.28. *Draw a graph having the given properties or explain why no such graph exists.*

(i) Graph with four vertices of degree 1, 1, 2 and 3.

(ii) Graph with four vertices of degree 1, 1, 3 and 3

(iii) Simple graph with four vertices of degree 1, 1, 3 and 3

(iv) Graph with six vertices each of degree 3

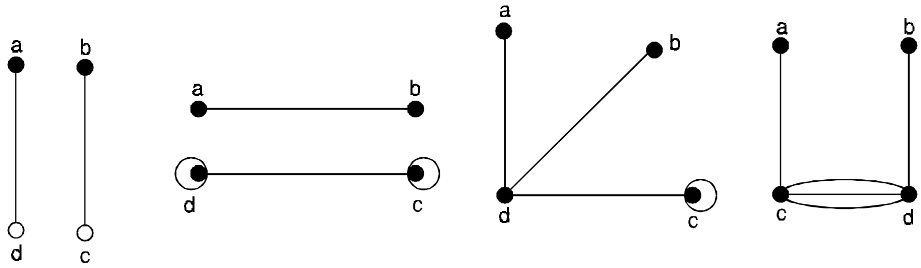
(v) Graph with six vertices and four edges

(vi) Graph with five vertices of degree 3, 3, 3, 3, 2

(vii) Graph with five vertices of degree 0, 1, 2, 2, 3.

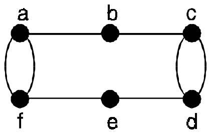
Solution. (*i*) No such graphs exists, total degree is odd.

(*ii*)

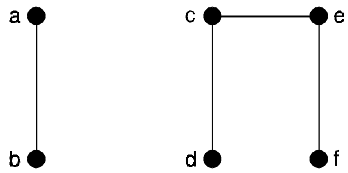


(*iii*) No simple graph.

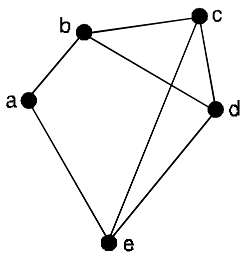
(*iv*)



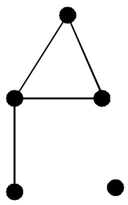
(*v*)



(*vi*)



(*vii*)

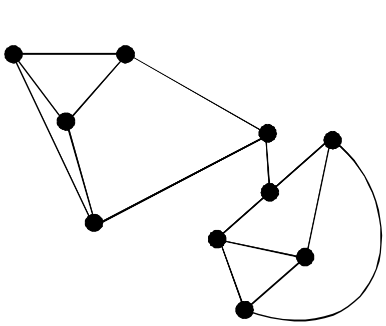


Problem 1.29. *If the simple graph G has V vertices and e edges, how many edges does G′ (complement of G) have ?*

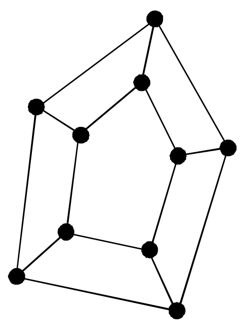
Solution. $\dfrac{v(v-1)}{2-e}$.

Problem 1.30. *Construct a 3-regular graph on 10 vertices.*

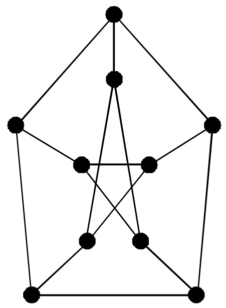
Solution. The following graphs are some examples of 3-regular graphs on 10 vertices.



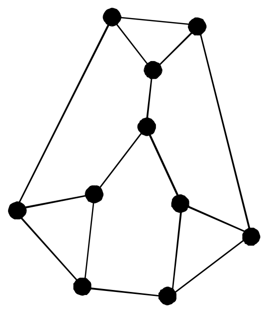
(*i*)



(*ii*)



(*iii*)

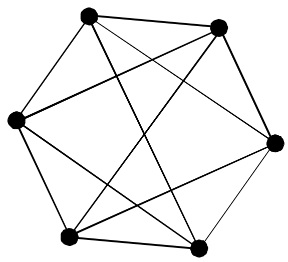


(*iv*)

Problem 1.31. *Does there exists a 4-regular graph on 6 vertices ? If so construct a graph.*

Solution. We have $q = \dfrac{P \times r}{2} = \dfrac{6 \times 4}{2} = 12$

Hence 4-regular graph on 6-vertices is possible and it contains 12 edges. One of the graph is shown below.



Every 4-regular graph contains a 3-regular graph.

Problem 1.32. *What is the size of an r-regular (p, q)-graph.*

Solution. Since G is an r-regular graph.

By the definition of regularity of G.

We have $\deg_G (v_i) = r$ for all $v_i \in$ V(G)

But $2q = \sum_{i=1}^{P} \deg_G (v_i)$

$$2q = \sum_{i=1}^{P} r = P \times r$$

$\Rightarrow \quad q = \dfrac{P \times r}{2}$.

Problem 1.33. *Does a 3-regular graph on 14 vertices exist ? What can you say on 17 vertices ?*

Solution. We have $q = \dfrac{P \times r}{2}$

given $r = 3$, P = 14

Now $q = \dfrac{14 \times 3}{2} = 21$, is a positive integer.

Hence 3-regular graphs on 14 vertices exist.

Further, if P = 17, then $q = \dfrac{P \times r}{2} = \dfrac{17 \times 3}{2} = \dfrac{51}{2}$ is not a positive integer.

Hence no 3-regular graphs on 17 vertices exist.

1.7 TYPES OF GRAPHS

Some important types of graph are introduced here.

1.7.1. Null graph

A graph which contains only **isolated node,** is called a null graph.

i.e., the set of edges in a null graph is empty.

Null graph is denoted on n vertices by N_n

N_4 is shown in Figure (13), Note that each vertex of a null graph is isolated.

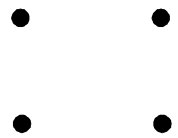


Fig. 13.

1.7.2. Complete graph

A simple graph G is said to be **complete** if every vertex in G is connected with every other vertex.

i.e., if G contains exactly one edge between each pair of distinct vertices.

A comple graph is usually denoted by $\mathbf{K}_{\boldsymbol{n}}$**.** It should be noted that K_n has exactly $\dfrac{\boldsymbol{n(n-1)}}{\mathbf{2}}$ edges.

The graphs K_n for n = 1, 2, 3, 4, 5, 6 are show in Figure 14.

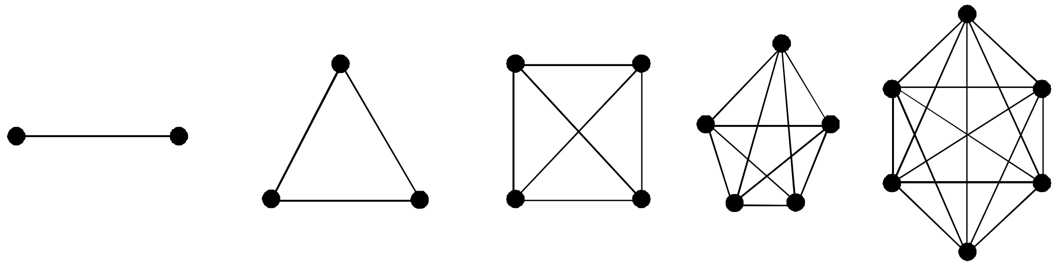


Fig. 14.

1.7.3. Regular graph

A graph in which all vertices are of **equal degree,** is called a **regular graph.**

If the degree of each vertex is r, then the graph is called a regular **graph of degree *r*.**

Note that every null graph is regular of degree zero, and that the complete graph K_n is a regular of degree $n - 1$. Also, note that, if G has n vertices and is regular of degree r, then G has $\left(\frac{1}{2}\right) r\, n$ edges.

1.7.4. Cycles

The cycle C_n, $n \geq 3$, consists of n vertices $v_1, v_2, \ldots\ldots, v_n$ and edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots\ldots, \{v_{n-1}, v_n\}$, and $\{v_n, v_1\}$.

The cyles c_3, c_4, c_5 and c_6 are shown in Figure 15.

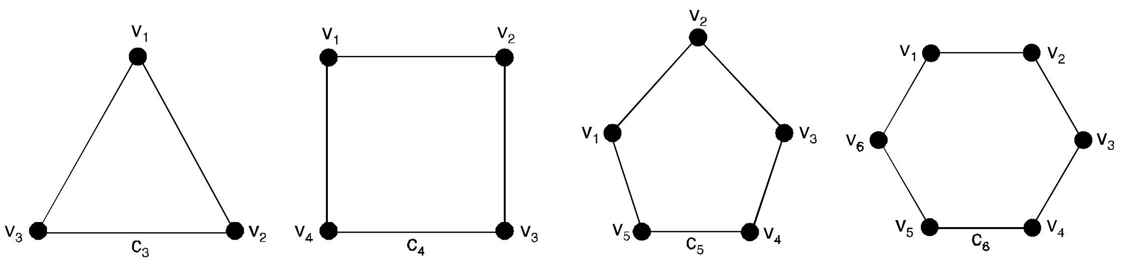


Fig. 15. Cycles C_3, C_4, C_5 and C_6.

1.7.5. Wheels

The wheel W_n is obtained when an additional vertex to the cycle c_n, for $n \geq 3$, and connect this new vertex to each of the n vertices in c_n, by new edges. The wheels W_3, W_4, W_5 and W_6 are displayed in Figure 16.

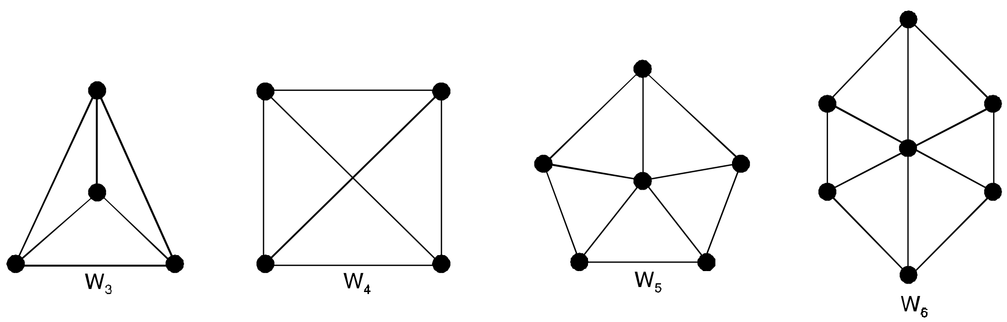


Figure 16. The wheels W_3, W_4, W_5 and W_6

1.7.6. Platonic graph

The graph formed by the vertices and edges of the five regular (platonic) solids—The tetrahedron, octahedron, cube, dodecahedron and icosahedron.

The graphs are shown in Figure 17.

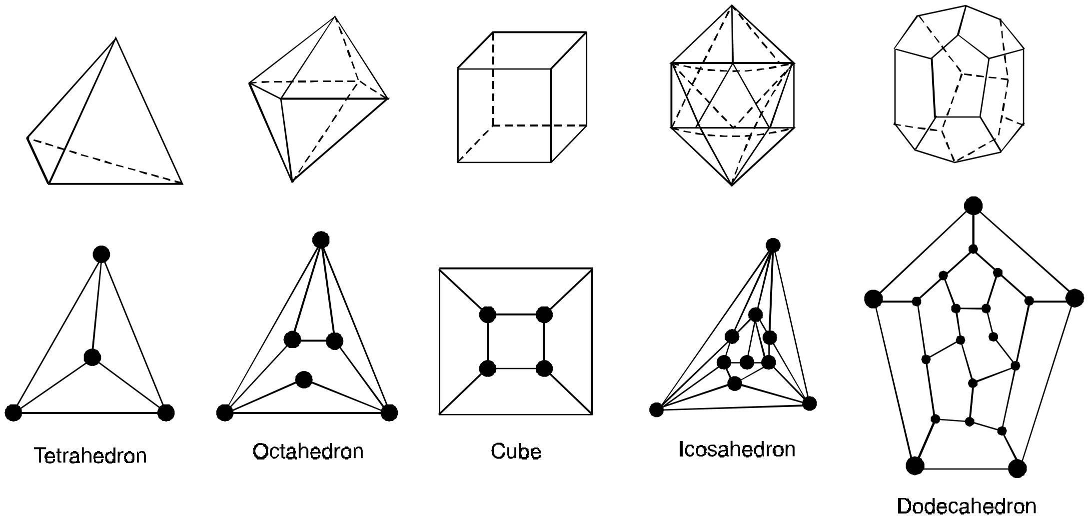


Fig. 17.

1.7.7. N-cube

The N-cube denoted by Q_n, is the graph that has vertices representing the 2^n bit strings of length n. Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position. The graphs Q_1, Q_2, Q_3 are displayed in Figure 18. Thus Q_n has 2^n vertices and $n \cdot 2^{n-1}$ edges, and is regular of degree n.

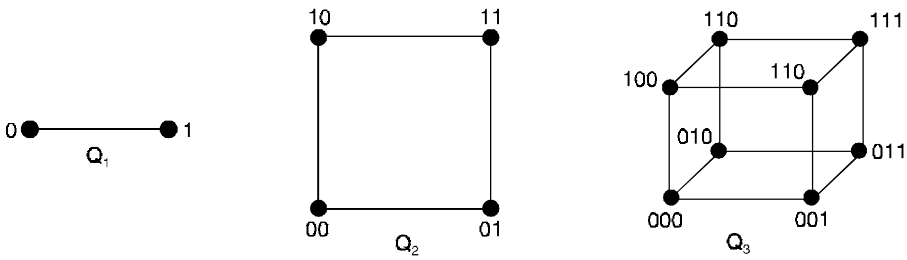
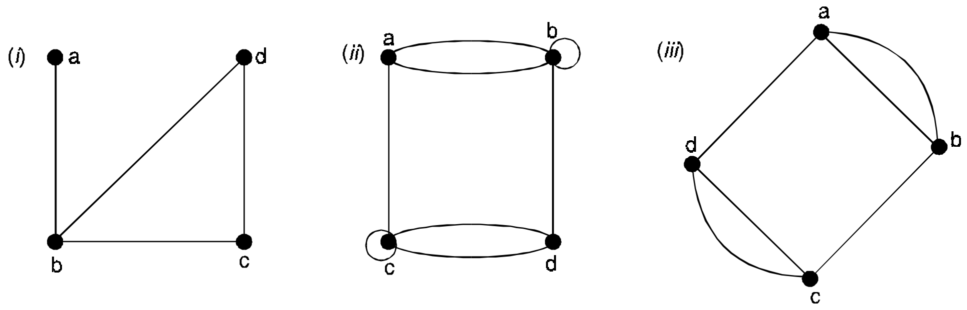


Fig. 18. The *n*-cube Q_n for n = 1, 2, 3.

Problem 1.34. *Determine whether the graphs shown is a simple graph, a multigraph, a pseudograph.*



Solution. (*i*) Simple graph

(*ii*) Pseudograph

(*iii*) Multigraph.

Problem 1.35. *Consider the following directed graph G : V(G) = {a, b, c, d, e, f, g}*

E(G) = {(a, a), (b, e), (a, e), (e, b), (g, c), (a, e), (d, f), (d, b), (g, g)}.

(*i*) *Identify any loops or parallel edges.*

(*ii*) *Are there any sources in G ?*

(*iii*) *Are there any sinks in G ?*

(*iv*) *Find the subgraph H of G determined by the vertex set V′ = {a, b, c, d}.*

Solution. (*i*) (*a*, *a*) and (*g*, *g*) are loops

(*a*, *a*) and (*a*, *e*) are parallel edges.

(*ii*) No sources

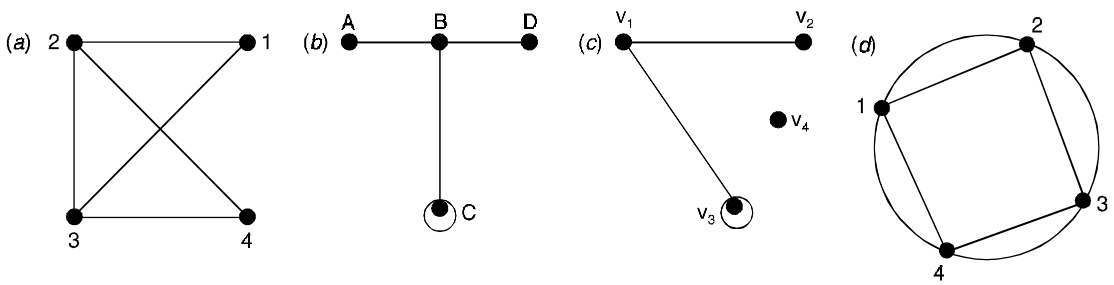
(*iii*) No sinks

(*iv*) V′ = {*a*, *b*, *c*, *d*}

E′ = {(*a*, *a*), (*d*, *b*)}

H = H(V′, E′).

Problem 1.36. *Consider the following graphs, determine the (i) vertex set and (ii) edge set.*



Solution. (*a*) (*i*) Vertex set V = {1, 2, 3, 4},

(*ii*) Edge set E = {(1, 2), (1, 3), (2, 3), (2, 4), (3, 4)}

(*b*) (*i*) Vertex set V = {A, B, C, D}

(*ii*) Edge set E = {(A, B), (B, C), (B, D), (C, C)} (*c*) (*i*) Vertex set V = $\{v_1, v_2, v_3, v_4\}$

(*ii*) Edge set E = $\{(v_1, v_2), (v_1, v_3), (v_3, v_3)\}$

(*d*) (*i*) Vertex set V = {1, 2, 3, 4}

(*ii*) Edge set E = {(1, 2), (2, 3), (3, 4), (4, 1)}

All edges are double edges.

Problem 1.37. *How many vertices and how many edges do the following graphs have ?*

(*i*) K_n (*ii*) C_n (*iii*) W_n (*iv*) $K_{m,\,n}$ (*v*) Q_n.

Solution. (*i*) n vertices and $\dfrac{n(n-1)}{2}$ edges.

(*ii*) n vertices and n edges

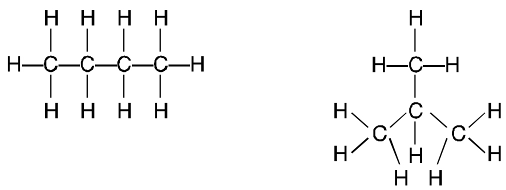
(*iii*) $n + 1$ vertices and $2n$ edges

(*iv*) $m + n$ vertices and mn edges

(*v*) 2^n vertices and $n \cdot 2^{n-1}$ edges.

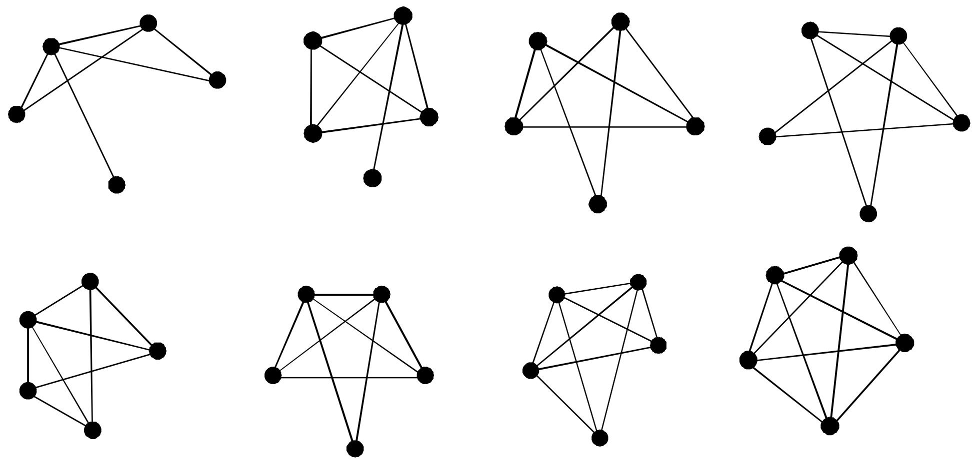
Problem 1.38. *There are two different chemical molecules with formula C_4H_{10} (isobutane). Draw the graphs corresponding to these molecules.*

Solution.



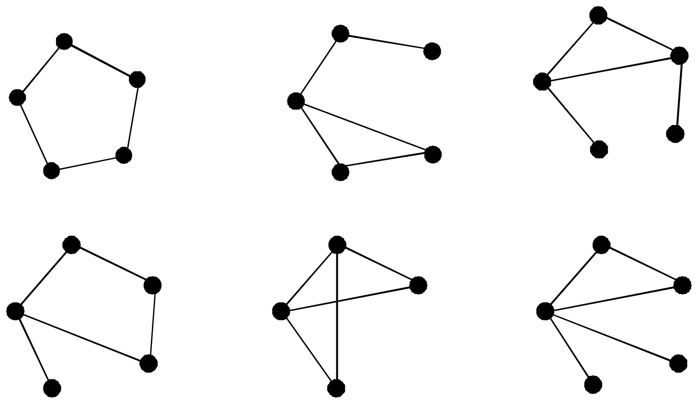
Problem 1.39. *Draw all eight graphs with five vertices and seven or more edges.*

Solution.



Problem 1.40. *Draw all six graphs with five vertices and five edges.*

Solution.



1.8 SUBGRAPH

A subgraph of G is a graph having all of its vertices and edges in G. If G_1 is a subgraph of G, then G is a super graph of G_1.

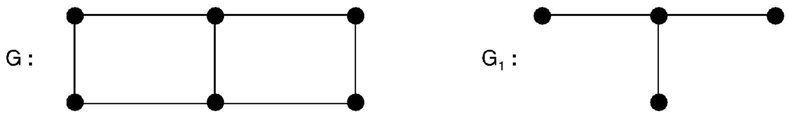


Fig. 19. G_1 is a subgraph of G.

In other words. If G and H are two graphs with vertex sets V(H), V(G) and edge sets E(H) and E(G) respectively such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ then we call H as a subgraph of G or G as a supergraph of H.

1.8.1. Spanning subgraph

A spanning subgraph is a subgraph containing all the vertices of G.

In other words, if $V(H) \subset V(G)$ and $E(H) \subseteq E(G)$ then H is a proper subgraph of G and if $V(H) = V(G)$ then we say that H is a spanning subgraph of G.

A spanning subgraph need not contain all the edges in G.

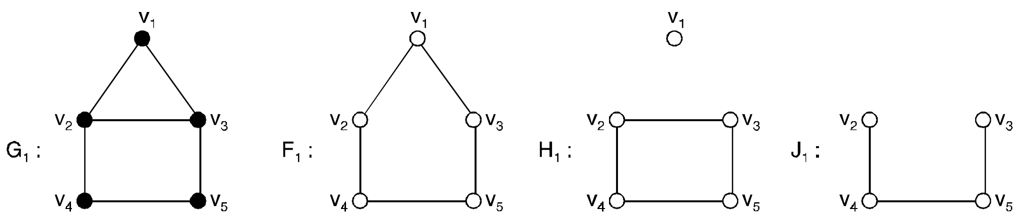


Fig. 20.

The graphs F_1 and H_1 of the above Fig. 20 are spanning subgraphs of G_1, but J_1 is not a spanning subgraph of G_1.

Since $V_1 \in V(G_1) - V(J_1)$. If E is a set of edges of a graph G, then G – E is a spanning subgraph of G obtained by deleting the edges in E from E(G).

In fact, H is a spanning subgraph of G if and only if H = G – E, where E = E(G) – E(H). If e is an edge of a graph G, then we write G – e instead of G – $\{e\}$. For the graphs G_1, F_1 and H_1 of the Fig. 20, we have $F_1 = G_1 - v_2v_3$ and $H_1 = G_1 - \{v_1v_2, v_2v_3\}$.

1.8.2. Removal of a vertex and an edge

The removal of a vertex v_i from a graph G result in that subgraph G – v_i of G containing of all vertices in G except v_i and all edges not incident with v_i. Thus G – v_i is the maximal subgraph of G not containing v_i. On the otherhand, the removal of an edge x_j from G yields the spanning subgraph G – x_j containing all edges of G except x_j.

Thus G – x_j is the maximal subgraph of G not containing x_j.

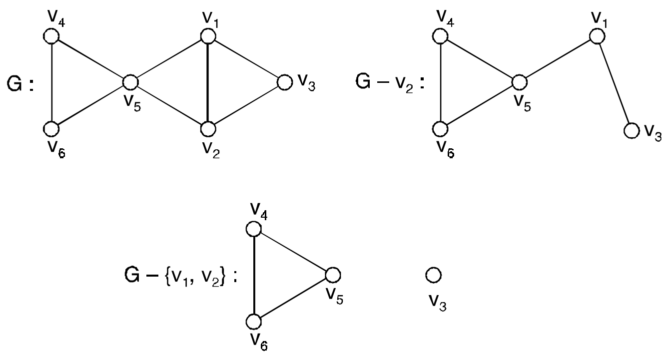


Fig. 21(*a*). Deleting vertices from a graph.

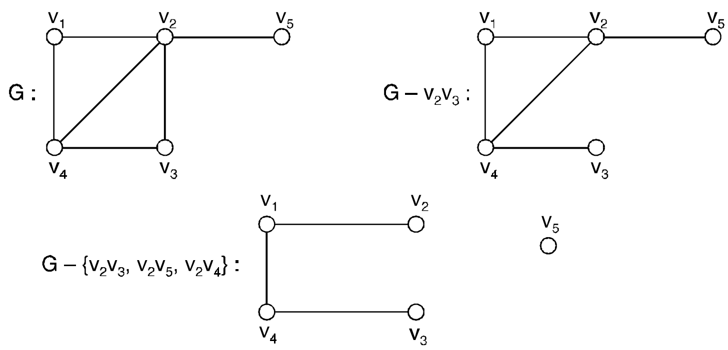
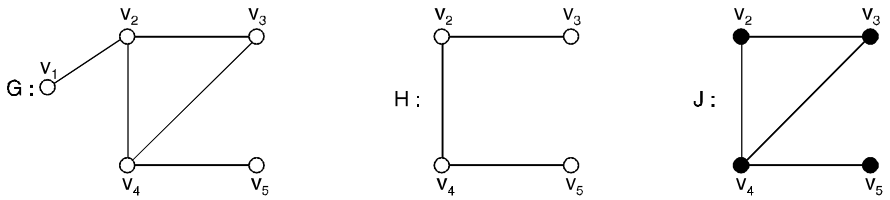


Fig. 21(*b*). Deleting edges from a graph.

1.8.3. Induced subgraph

For any set S of vertices of G, the vertex induced subgraph or simply an induced subgraph <S> is the maximal subgraph of G with vertex set S. Thus two vertices of S are adjacent in <S> if and only if they are adjacent in G.

In other words, if G is a graph with vertex set V and U is a subset of V then the subgraph G(U) of G whose vertex set is U and whose edge set comprises exactly the edges of E which join vertices in U is termed as induced subgraph of G.



Here H is not an induced subgraph since $v_4v_1 \in$ E(G), but $v_4v_3 \notin$ E(H).

On the otherhand the graph J is an induced subgraph of G. Thus every induced subgraph of a graph G is obtained by deleting a subset of vertices from G.

Note : Let | V | = m and | E | = n. The total non-empty subsets of V is $2^m - 1$ and total subsets of E is 2^n.

Thus, number of subgraphs is equal to $(2^m - 1) \times 2^n$.

The number of spanning subgraphs is equal to 2^n.

1.9 GRAPHS ISOMORPHISM

Let $G_1 = (v_1, E_1)$ and $G_2 = (v_2, E_2)$ be two graphs. A function $f : v_1 \to v_1$ is called a graphs isomorphism if

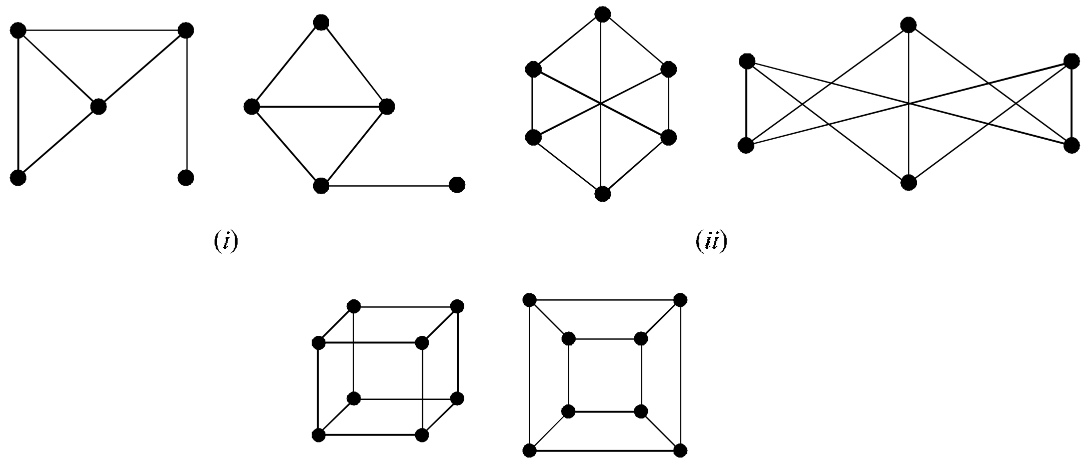
(i) f is one-to-one and onto.

(ii) for all $a, b \in v_1$, $\{a, b\} \in E_1$ if and only if $\{f(a), f(b)\} \in E_2$ when such a function exists, G_1 and G_2 are called isomorphic graphs and is written as $G_1 \cong G_2$.

In other words, two graphs G_1 and G_2 are said to be isomorphic to each other if there is a one-to-one correspondence between their vertices and between edges such that incidence relationship is preserve. Written as $G_1 \cong G_2$ or $G_1 = G_2$.

The necessary conditions for two graphs to be isomorphic are

1. Both must have the **same number of vertices**
2. Both must have the **same number of edges**
3. Both must have **equal number of vertices** with the **same degree.**
4. They must have the same degree sequence and same cycle vector $(c_1, ..., c_n)$, where c_i is the number of cycles of length i.



(iii)

Fig. 22(i), (ii) (iii) Isomorphic pair of graphs

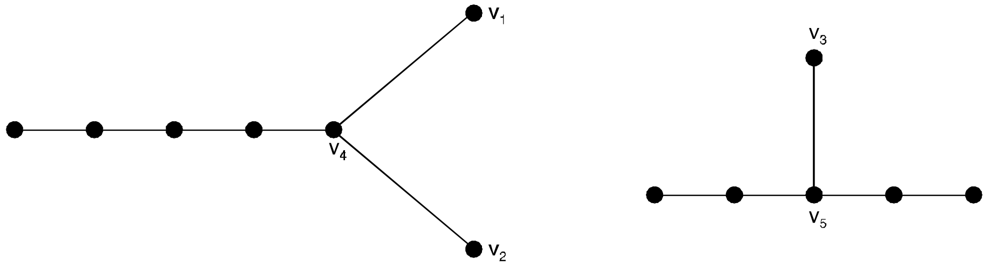
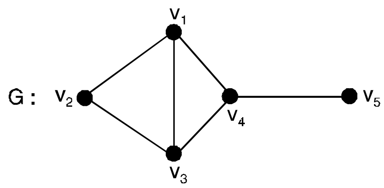
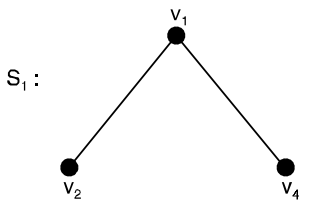


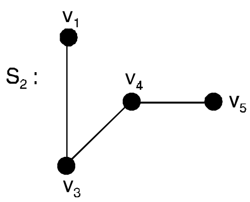
Fig. 23. Two graphs that are not isomorphic.

Problem 1.41. *Construct two edge-disjoint subgraphs and two vertex disjoint subgraphs of a graph G shown below*

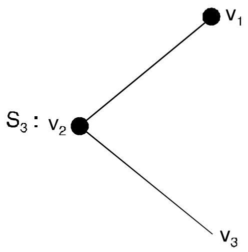


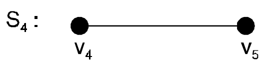
Solution.





The graphs S_1 and S_2 are edge-disjoint subgraphs of G.





S_3 and S_4 are vertex disjoint subgraphs of G which are also edge-disjoint subgraphs of G.

Problem 1.42. *Does there exist a proper subgraph S of G whose size is equal to the size of the graph ?*

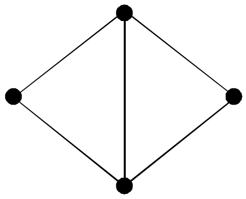
Solution. Yes, consider the graph G shown in Figure below.

The graph S is a subgraph of G with $V(S) \subset E(G)$ and $E(S) = E(G)$.

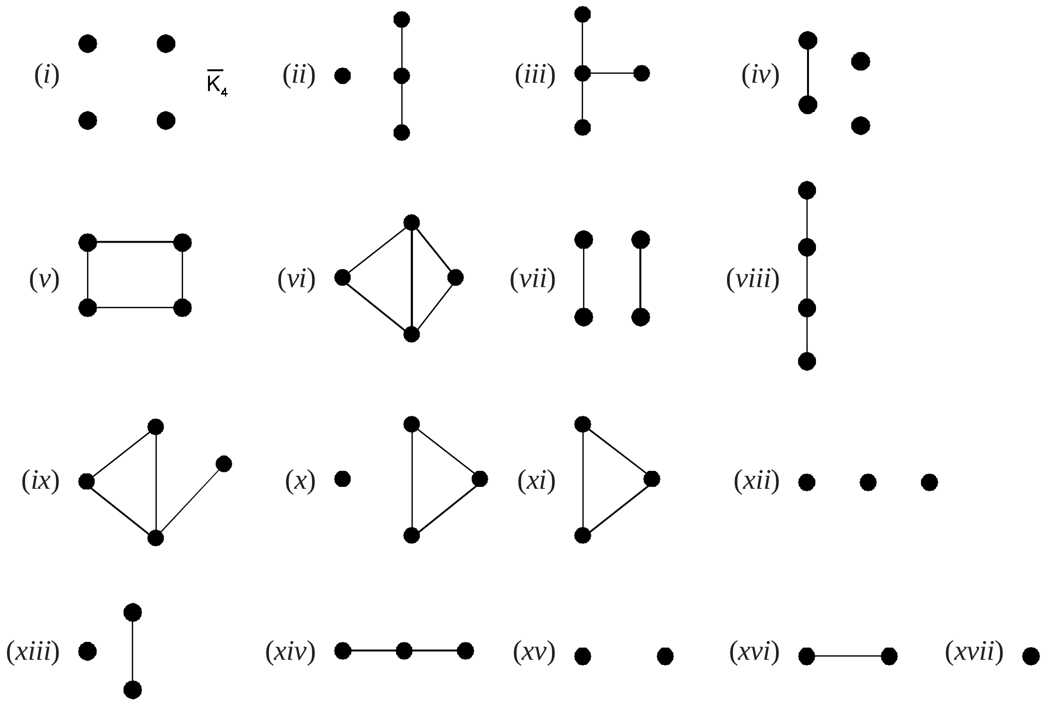
The graph G Subgraph S

Problem 1.43. *Write down all possible non-isomorphic subgraphs of the following graphs G. How many of they are spanning subgraphs ?*

G :



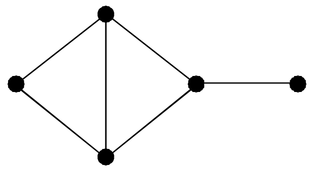
Solution. Its possible all (non-isomorphic) subgraphs are



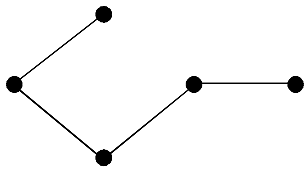
of these graphs (*i*) to (*x*) are spanning subgraphs of G.

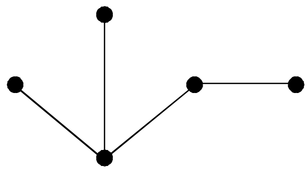
All the graphs except (*vi*) are proper subgraphs of G.

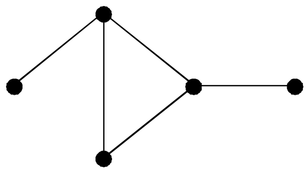
Problem 1.44. *Construct three non-isomorphic spanning subgraphs of the graph G shown below :*

G :


Solution. Three non-isomorphic subgraphs are

(*i*)


(*ii*)


(*iii*)


Problem 1.45. *Find the total number of subgraphs and spanning subgraphs in K_6, L_5 and Q_3.*

Solution. In graph K_6, we have | V | = 6 and | E | = 15

Thus, total number of subgraph is

$$(2^6 - 1) \times 2^{15} = 63 \times 32768 = 2064384$$

The total number of spanning subgraph is : $2^{15} = 32768$.

In the linear graph L_5, we have | V | = 5 and | E | = 4

Thus, total number of subgraph is

$$(2^5 - 1) \times 2^4 = 31 \times 16 = 496.$$

The total number of spanning subgraph is : $2^4 = 16$.

In the 3-cube graph Q_3, we have | V | = 8 and | E | = 12

Thus, total number of subgraph is

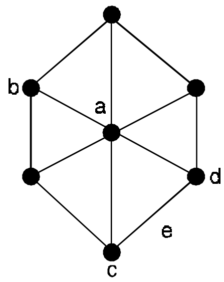
$$(2^8 - 1) \times 2^{12} = 127 \times 4096 = 520192$$

The total number of spanning subgraphs is

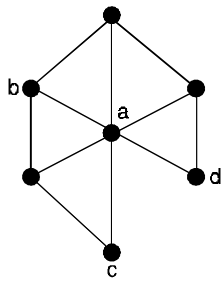
$$2^{12} = 4096.$$

Problem 1.46. *For the graph G shown below, draw the subgraphs*

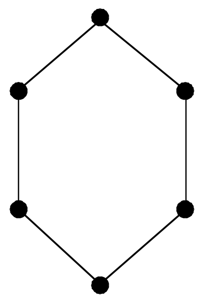
(*i*) *G – e* (*ii*) *G – a* (*iii*) *G – b*.



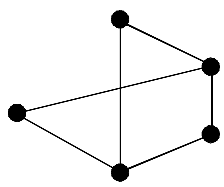
Solution. (*i*) After deleting the edge $e = (c, d)$ from the graph G, we get a subgraph G – e as shown below



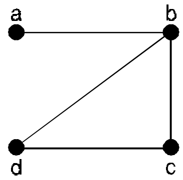
(*ii*) After deleting the vertex a from the graph G, and all edges incident on this vertex, we set the subgraph G – a as shown below :



(*iii*) The subgraph is obtained after deleting the vertex b.



Problem 1.47. *Consider the graph G(V, E) shown below, determine whether or not* $H(V_1, E_1)$ *is a subgraph of G, where*



(i) $V_1 = \{a, b, d\}$ $E_1 = \{(a, b), (a, d)\}$

(ii) $V_1 = \{a, b, c, d\}$ $E_1 = \{(b, c), (b, d)\}$

Solution. (i) H is not a subgraph because (a, d) is not an edge in G.

(ii) H is a subgraph because it satisfies condition for a subgraph of the given graph G.

Problem 1.48. *Find all possible non-isomorphic induced subgraphs of the following graph G corresponding to the three element subsets of the vertex set of G*

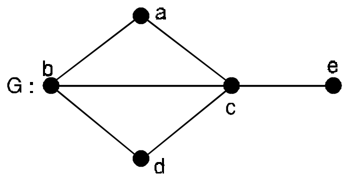
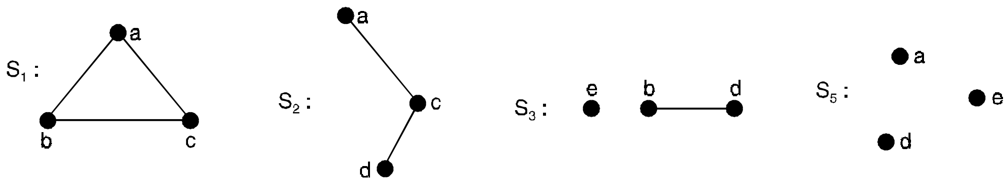


Fig. 24.

Solution.



The subgraph S shown in Figure (25) of the above graph G shown in Figure 24 is not a induced subgraph of G.

For the edge (a, d) of G can be added to S. The graph obtained by adding this edge is again a subgraph of

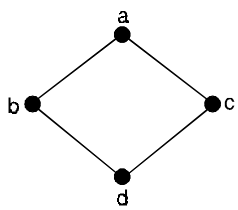
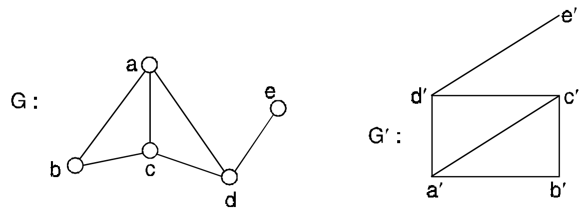


Fig. 25.

Note : The graph G is itself a maximal subgraph of G.

Problem 1.49. *Show that the following graphs are isomorphic*



Solution. Let $f : G \to G'$ be any function defined between two graphs degrees of the graph G and G′ are as follows :

deg (G)	deg (G′)
deg (a) = 3	deg (a') = 3
deg (b) = 2	deg (b') = 2
deg (c) = 3	deg (c') = 3
deg (d) = 3	deg (d') = 3
deg (e) = 1	deg (e') = 1

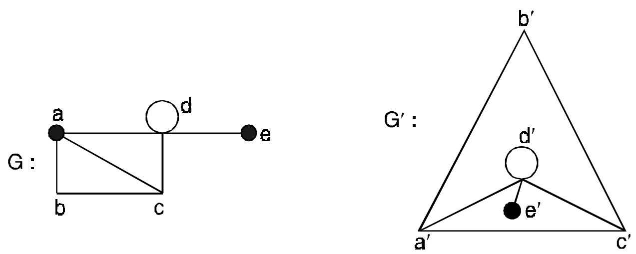
Each has 5-vertices and 6-edges.

$$d(a) = d(a') = 3$$
$$d(b) = d(b') = 2$$
$$d(c) = d(c') = 3$$
$$d(d) = d(d') = 3$$
$$d(e) = d(e') = 1$$

Hence the correspondence is $a - a'$, $b - b'$, ..., $e - e'$.

Therefore, the given two graphs are isomorphic.

Problem 1.50. *Show that the following graphs are isomorphic.*



Solution. Let $f : G \to G'$ be any function defined between two graphs degrees of the graphs G and G′ are as follows :

deg (G)	deg (G′)
deg (a) = 3	deg (a') = 3
deg (b) = 2	deg (b') = 2
deg (c) = 3	deg (c') = 3

deg (d) = 5 deg (d') = 5

deg (e) = 1 deg (e') = 1

Each has 5-vertices, 6-edges and 1-circuit.

deg(a) = deg(a') = 3

deg(b) = deg(b') = 2

deg(c) = deg(c′) = 3

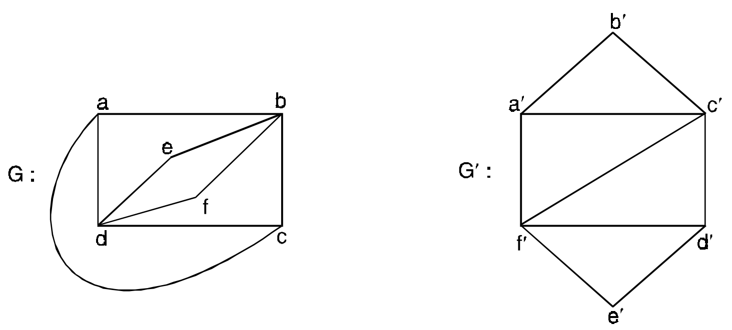
deg(d) = deg(d') = 5

deg(e) = deg(e') = 1

Hence the correspondence is $a - a'$, $b - b'$, ..., $e - e'$.

Therefore, the given two graphs G and G′ are isomorphic.

Problem 1.51. *Are the 2-graphs, is given below, is isomorphic ? Give a reason.*



Solution. Let us enumerate the degree of the vertices

Vertices of degree 4 : $b - f'$

$d - c'$

Vertices of degree 3 : $a - a'$

$c - d'$

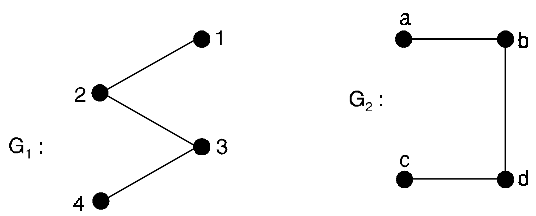
Vertices of degree 2 : $e - b'$

$f - e'$

Now the vertices of degree 3, in G are a and c and they are adjacent in G′, while these are a' and d' which are not adjacent in G′.

Hence the 2-graphs are not isomorphic.

Problem 1.52. *Show that the two graphs shown in Figure are isomorphic.*



Solution. Here, $V(G_1) = \{1, 2, 3, 4\}$, $V(G_2) = \{a, b, c, d\}$

$E(G_1) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$ and $E(G_2) = \{\{a, b\}, \{b, d\}, \{d, c\}\}$

Define a function $f : V(G_1) \to V(G_2)$ as

$$f(1) = a,\ f(2) = b,\ f(3) = d, \text{ and } f(4) = c$$

f is clearly one-one and onto, hence an isomorphism.

Further, $\{1, 2\} \in E(G_1)$ and $\{f(1), f(2)\} = \{a, b\} \in E(G_2)$

$\{2, 3\} \in E(G_1)$ and $\{f(2), f(3)\} = \{b, d\} \in E(G_2)$

$\{3, 4\} \in E(G_1)$ and $\{f(3), f(4)\} = \{d, c\} \in E(G_2)$

and $\{1, 3\} \notin E(G_1)$ and $\{f(1), f(3)\} = \{a, d\} \notin E(G_2)$

$\{1, 4\} \notin E(G_1)$ and $\{f(1), f(4)\} = \{a, c\} \notin E(G_2)$

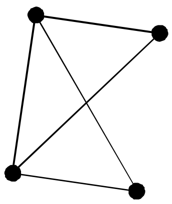
$\{2, 4\} \notin E(G_1)$ and $\{f(2), f(4)\} = \{b, c\} \notin E(G_2)$.

Hence f preserves adjacency as well as non-adjacency of the vertices.

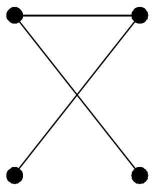
Therefore, G_1 and G_2 are isomorphic.

Problem 1.53. *For each pair of graphs shown, either label the graphs so as to exhibit an isomorphism or explain why the graphs are not isomorphic.*

(*i*)

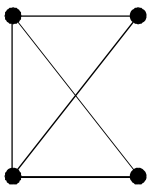


(a)

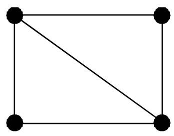


(b)

(*ii*)

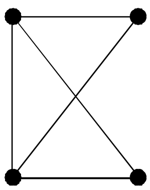


(a)

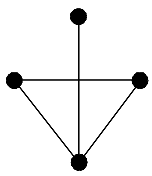


(b)

(*iii*)

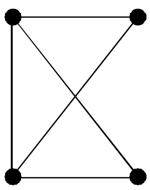


(a)

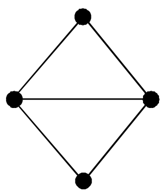


(b)

(*iv*)



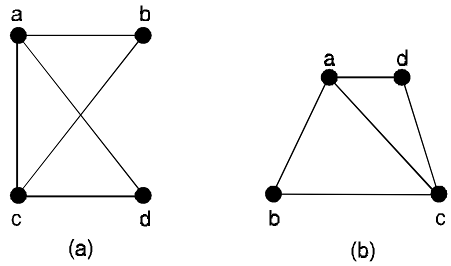
(a)



(b)

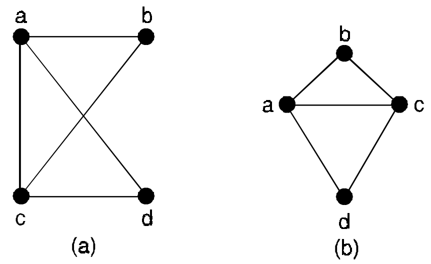
Solution. (*i*) The graphs are not isomorphic because (*a*) has 5-edges and (*b*) has 4-edges.

(*ii*) The graphs are isomorphic, as shown by the labelling



(*iii*) The graphs are not isomorphic because (*b*) has a vertex of degree 1 and (*a*) does not have.

(*iv*) The graphs are isomorphic, as shown by the labelling



Problem 1.54. *Whether the following pair of non-directed graphs in figure (26) are isomorphic or not ? Justify your answer ?*

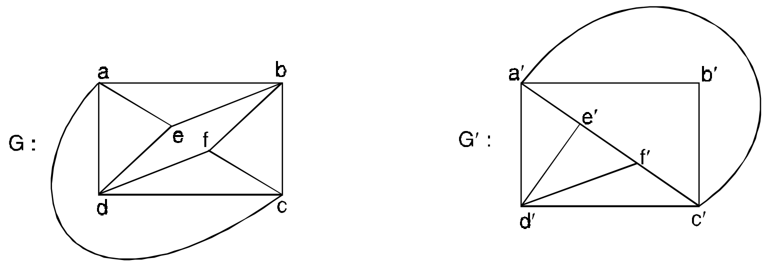


Fig. 26.

Solution. Here, G′ has vertex *b*′ of degree 2, while G has no vertex of degree 2.

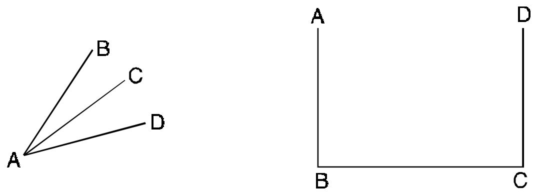
Hence, they are not isomorphic.

Problem 1.55. *How many different non-isomorphic trees are possible for a graph of order 4 ? Draw all of them.*

Solution. The sum of the degrees of the 4-vertices equals

$$2(e) = 2(n - 1) = 2n - 2 = 8 - 2 = 6$$

Hence, the degree of 4-vertices are (2, 2, 1, 1) or (3, 1, 1, 1), they are drawn as shown in Figure below



Problem 1.56. *Draw a cycle graph which is isomorphic to its complement.*

Solution. First we draw G and the complement of G denoted G′, by drawing edges between vertices which are non-adjacent in G.

The vertices in G′ are labelled so as to corresponds to those of G as follows :

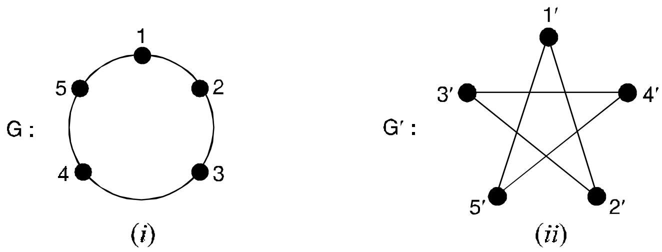


Fig. 27.

From Figure (27)

Vertices in G	*Vertices in G′*
1	1′
2	2′
3	3′
4	4′
5	5′

This labelling ensures that 5′ and 2′ are adjacent to 1′ in G′, while 5 and 2 are adjacent to 1 in G, 3′ and 1′ are adjacent to 2′ in G′, while 3 and 1 are adjacent to 2 in G.

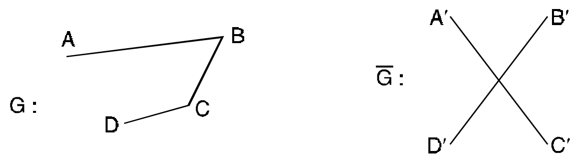
Also $d(i') = d(i)$ for all i.

Hence G and G′ are isomorphic.

Problem 1.57. *If a simple graph with n-vertices is isomorphic with its complement, how many vertices will that have ? Draw the corresponding graph.*

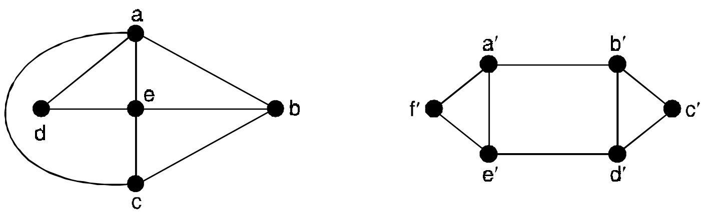
Solution. If e is the number of edges of G and $\bar{e}$ the number of edges in the complement $\overline{G}$, then

$e = \bar{e} = \dfrac{n(n+1)}{4}$. Hence n or $n + 1$ must be divisible by 4.

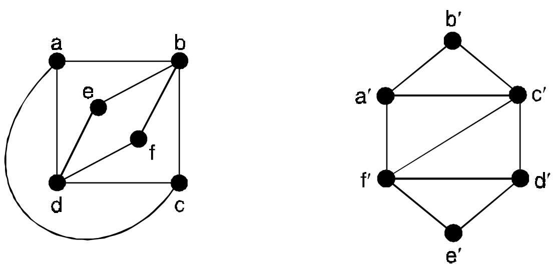


Problem 1.58. *Determine whether the following pairs of graphs are isomorphic. If the graphs are not isomorphic, give an invariant that the graphs do not share.*

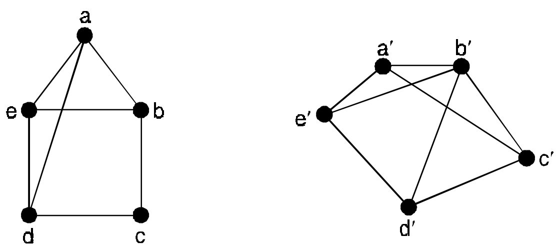
(*i*)



(*ii*)



(*iii*)



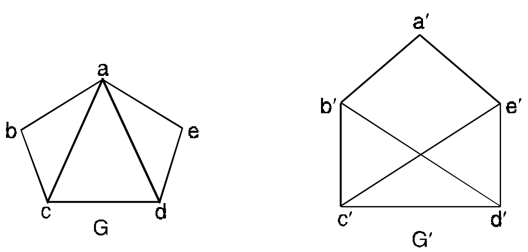
Solution. (*i*) Non isomorphic, they do not have the same number of vertices.

(*ii*) Non isomorphic, vertices of degree 3 are adjacent in one graph, non adjacent in the other.

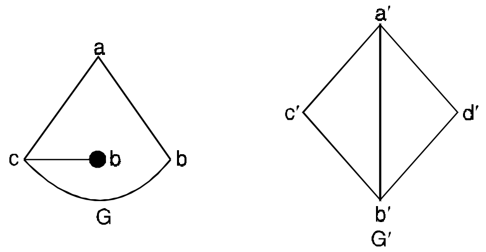
(*iii*) Non isomorphic, one has a vertex of degree 2 but other does not.

Problem 1.59. *Find whether the following pairs of graphs are isomorphic or not.*

(*i*)



(*ii*)



Solution. (*i*) Not isomorphic.

G has 2 nodes *b* and *e* of degree 2 while G′ has one node *a*′ of degree 2.

(*ii*) Not isomorphic.

G has 4 edges, and G′ has edges.

Problem 1.60. *If a graph G of n vertices is isomorphic to its complement* $\overline{G}$, *show that n or (n – 1) must be a multiple of 4.*

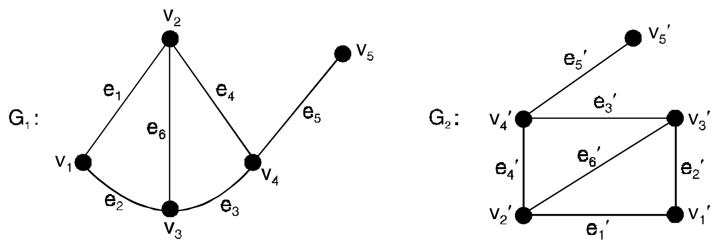
Solution. Since $G \approx \overline{G}$, both of G and $\overline{G}$ have the same number of edges.

Also, the total number of edges in G and $\overline{G}$ taken together must be equal to the number of edges in K_n.

Since K_n has $\dfrac{n(n-1)}{2}$ edges, it follows that each of G and $\overline{G}$ has $\dfrac{n(n-1)}{4}$ edges.

Thus, $\dfrac{n(n-1)}{4}$ must be a positive integer, as such, *n* or (*n* – 1) must be a multiple of 4.

Problem 1.61. *Consider two graphs G_1 and G_2 as shown below, show that the graphs G_1 and G_2 are isomorphic.*



Solution. The correspondence between the graphs is as follows :

The vertices (v_1, v_2, v_3, v_4, v_5) in G_1 correspond to (v_1', v_2', v_3', v_4', v_5') respectively in G_2.

The edges (e_1, e_2, e_3, e_4, e_5, e_6) in G_1 correspond to (e_1', e_2', e_3', e_4', e_5', e_6') respectively in G_2.

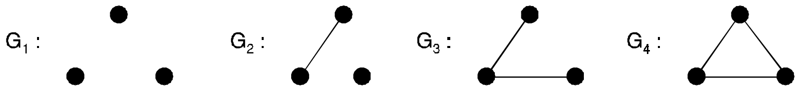
Here the incidence property is preserved.

Therefore the graphs G_1 and G_2 are isomorphic to each other.

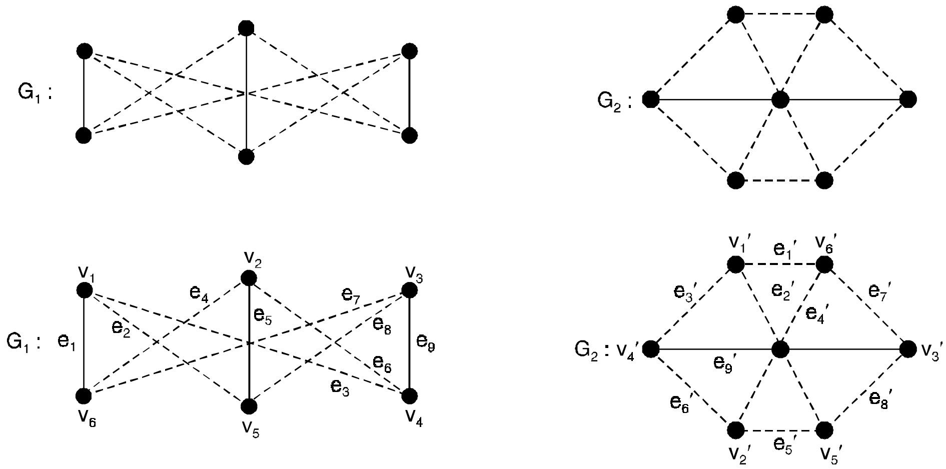
Problem 1.62. *Draw all non-isomorphic graphs on 2 and 3 vertices.*

Solution. All non-isomorphic graphs on 2 vertices are

All non-isomorphic graphs on 3 vertices are



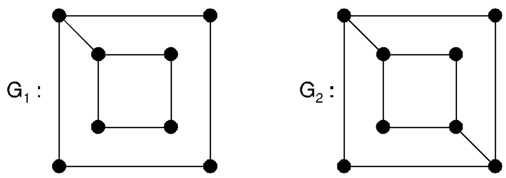
Problem 1.63. *Show that the following graphs are isomorphic.*



Solution. There is one-to-one correspondence between vertices and one-to-one correspodence between edges. Further incidence property is preserved.

Therefore G_1 is isomorphic to G_2,

Problem 1.64. *Determine whether the following graphs are isomorphic or not*



Solution. Here both the graphs G_1 and G_2 contains 8 vertices and 10 edges.

The number of vertices of degree 2 in both the graphs are four.

Also the number of vertices of degree 3 in both the graphs are four.

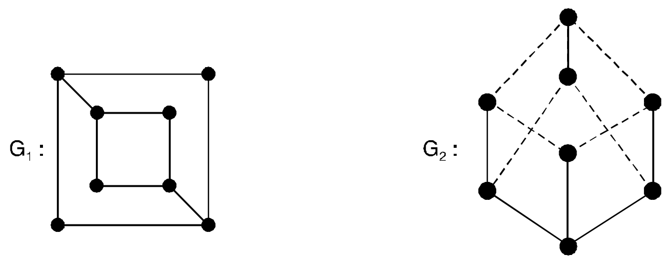
For adjacency, consider the vertex of degree 3 in G_1. It is adjacent to two vertices of degree 3 and one vertex of degree 2.

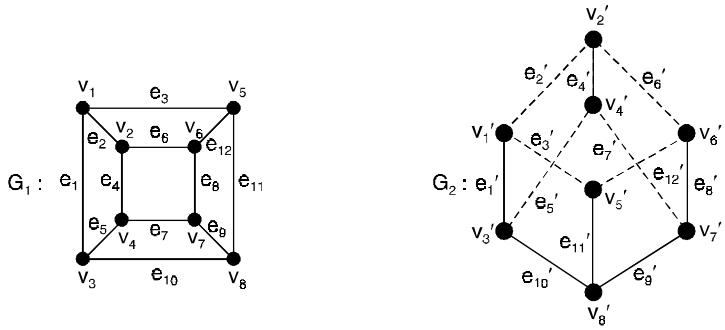
But in G_2 there does not exist any vertex of degree 3, which is adjacent to two vertices of degree 3 and one vertex of degree 2.

i.e., adjacency is not preserved.

Hence, given graphs are not isomorphic.

Problem 1.65. *Show that the following graphs are isomorphic.*

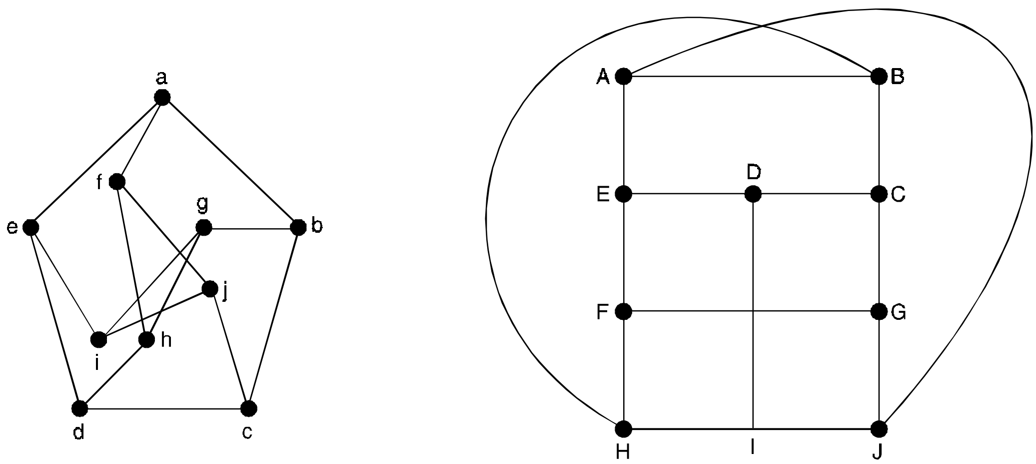




Solution. There are one-to-one correspondence between the vertices as well as between edges. Further, the incidence property is preserved.

Therefore, G_1 is isomorphic to G_2.

Problem 1.66. *Establish a one-one correspondence between the vertices and edges to show that the following graphs are isomorphic.*



Graph G_1 Graph G_2

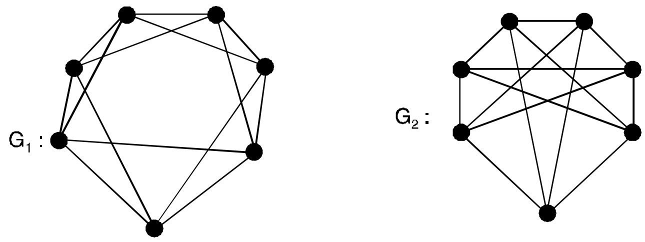
Solution. Define $\phi : V(G_1) \to V(G_2)$ by $\phi(a) = A$, $\phi(b) = B$

$\phi(c) = C$, $\phi(d) = D$, $\phi(e) = E$

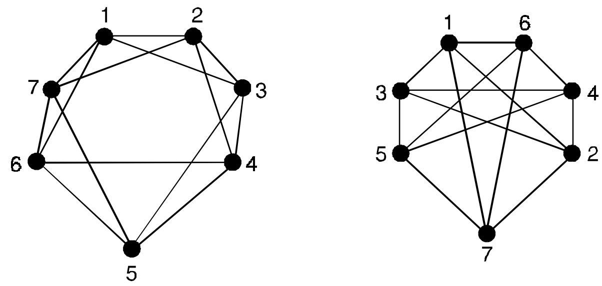
$\phi(f) = J$, $\phi(g) = H$, $\phi(h) = I$

$\phi(i) = F$, $\phi(j) = G$.

Problem 1.67. *Show that the following graphs are isomorphic.*



Solution. We first label the vertices of the graph as follows :



Define an isomorphism $\phi : V(G_1) \to V(G_2)$ by $\phi(i) = i$, we observe that ϕ preserves the adjacency and non-adjacency of the vertices.

Hence G_1 and G_2 are isomorphic to each other.

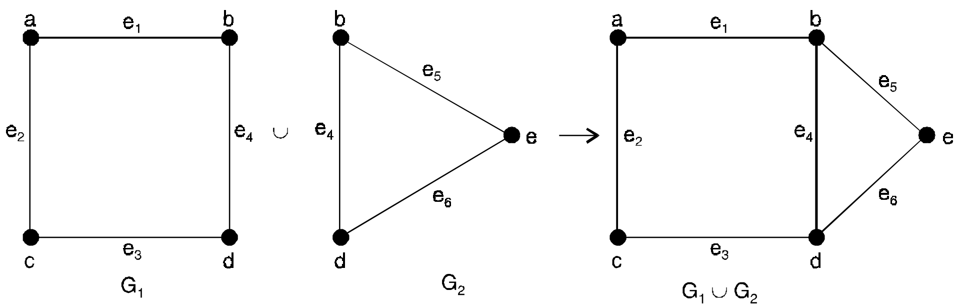
1.10 OPERATIONS OF GRAPHS

1.10.1. Union

Given two graphs G_1 and G_2, their union will be a graph such that

$$V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$$

and $$E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$$



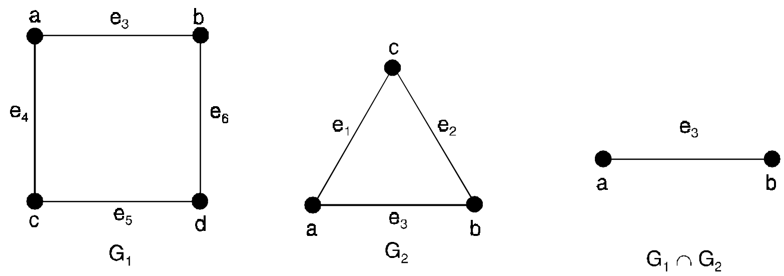
1.10.2. Intersection

Given two graphs G_1 and G_2 with at least one vertex in common then their intersection will be a graph such that

$$V(G_1 \cap G_2) = V(G_1) \cap V(G_2)$$

and

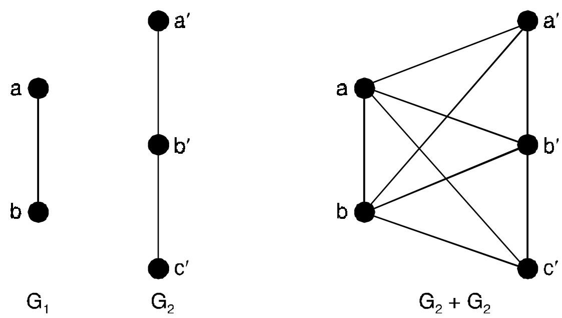
$$E(G_1 \cap G_2) = E(G_1) \cap E(G_2)$$



1.10.3. Sum of two graphs

If the graphs G_1 and G_2 such that $V(G_1) \cap V(G_2) = \phi$, then the sum $G_1 + G_2$ is defined as the graph whose vertex set is $V(G_1) + V(G_2)$ and the edge set is consisting those edges, which are in G_1 and in G_2 and the edges obtained, by joining each vertex of G_1 to each vertex of G_2.

For example,



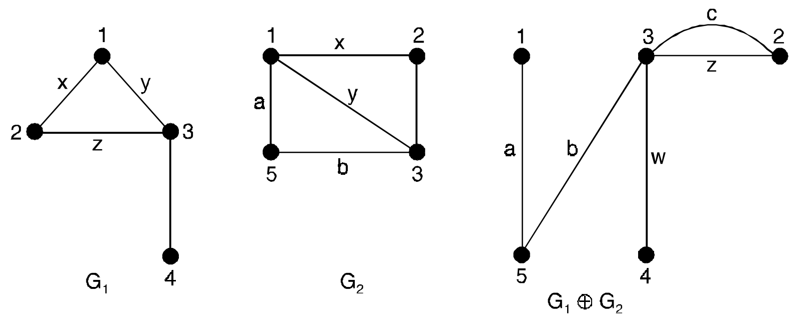
1.10.4. Ring sum

Let G_1 (V_1, E_1) and G_2 (V_2, E_2) be two graphs. Then the ring sum of G_1 and G_2, denoted by $G_1 \oplus G_2$ is defined as the graph G such that :

(*i*) $V(G) = V(G_1) \cup V(G_2)$

(*ii*) $E(G) = E(G_1) \cup E(G_2) - E(G_1) \cap E(G_2)$

i.e., the edges that either in G_1 or G_2 but not in both. The ring sum of two graphs G_1 and G_2 is shown below.



1.10.5. Product of graphs

To define the product $G_1 \times G_2$ of two graphs consider any two points $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $V = V_1 \times V_2$. Then u and v are adjacent in $G_1 \times G_2$ whenever [$u_1 = v_1$ and u_2 adj. v_2] or [$u_2 = v_1$ and u_1 adj. v_1]

For example,

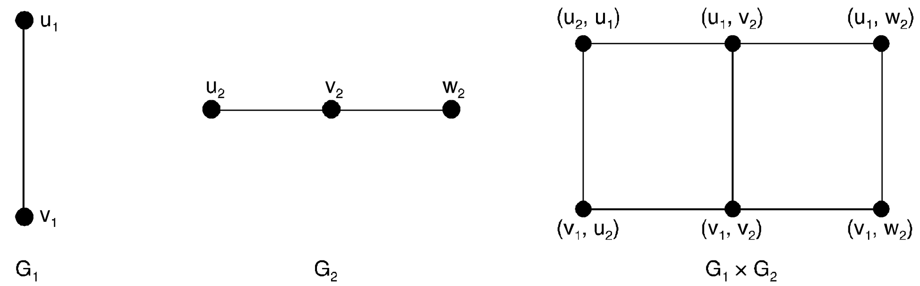


Fig. (*a*). The product of two graphs.

1.10.6. Composition

The composition $G = G_1[G_2]$ also has $V = V_1 \times V_2$ as its point set, and $u = (u_1, u_2)$ is adjacent with $v = (v_1, v_2)$ whenever (u_1 adj. v_1) or ($u_1 = v_1$ and u_2 adj. v_1)

For the graphs G_1 and G_2 of Figure. (*a*), both compositions $G_1[G_2]$ and $G_2[G_1]$ are shown in Figure (*b*).

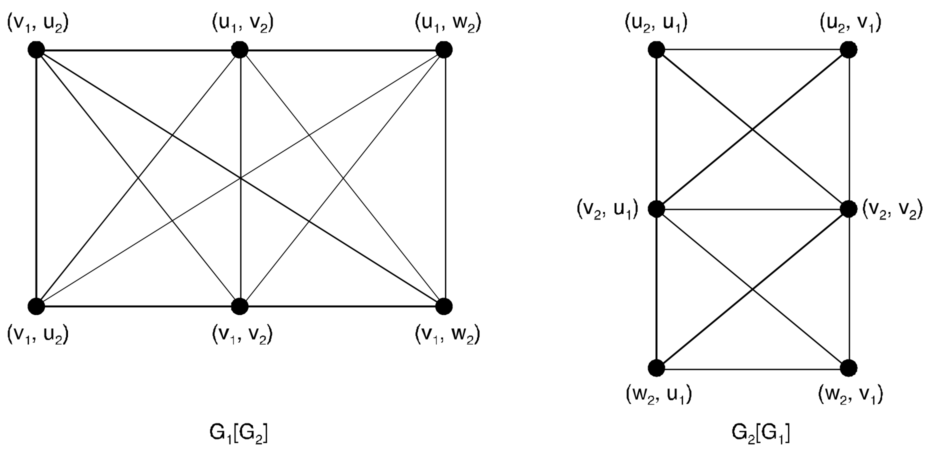
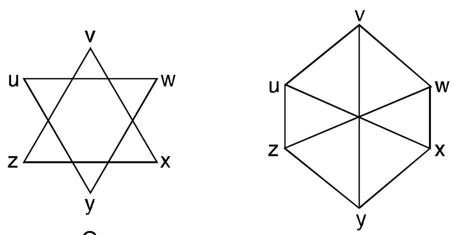


Fig. (*b*). Two compositions of graphs

1.10.7. Complement

The complement G′ of G is defined as a simple graph with the same vertex set as G and where two vertices *u* and *v* adjacent only when they are not adjacent in G.

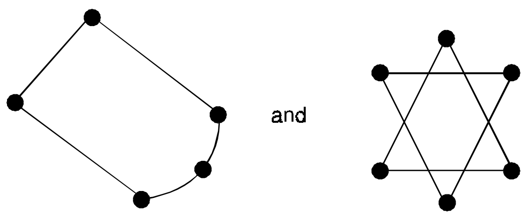
For example,



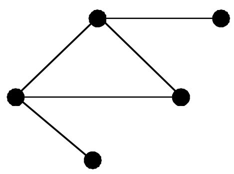
G Complement of G.

A graph G is self-complementary if it is isomorphic to its complement.

For example, the graphs



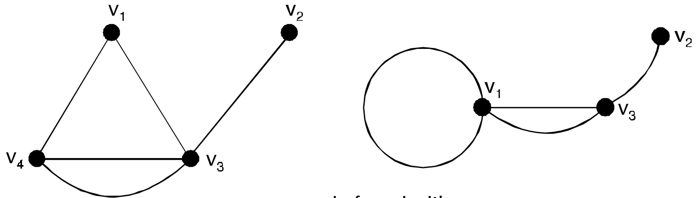
Self-complementary. The other self-complementary graph with five vertices is



1.10.8. Fusion

A pair of vertices v_1 and v_2 in graph G is said to be 'fused' if these two vertices are replaced by a single new vertex *v* such that every edge that was adjacent to either v_1 or v_2 or both is adjacent *v*.

Thus we observe that the fusion of two vertices does not alter the number of edges of graph but reduced the vertices by one.



v_1 is fused with v_4

Theorem 1.3. *For any graph G with six points, G or $\overline{G}$ contains a triangle.*

Proof. Let v be a point of a graph G with six points. Since v is adjacent either in G or in $\overline{G}$ to the other five points of G.

We can assume without loss of generality that there are three points u_1, u_2, u_3 adjacent to v in G.

If any two of these points are adjacent, then they are two points of a triangle whose third point is v.

If no two of them are adjacent in G, then u_1, u_2 and u_3 are the points of a triangle in $\overline{G}$.

1.11 THE PROBLEM OF RAMSEY : 1.4

Prove that at any party with six people, there are three mutual acquaintances or three mutual nonacquaintances.

Solution. This situation may be represented by a graph G with six points standing for people, in which adjacency indicates acquaintance.

Then the problem is to demonstrate that G has three mutually adjacent points or three mutually nonadjacent ones.

The complement $\overline{G}$ of a graph G also has V(G) as its point set, but two points are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

In Figure 28, G has no triangles, while $\overline{G}$ consists of exactly two triangles.

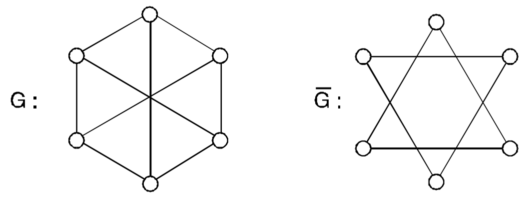


Fig. 28. A graph and its complement

In figure 29 : A self-complementary graph is isomorphic with its complement.

The complete graph K_p has every pair of its P points adjacent. Since V is not empty, $P \geq 1$.

Thus K_P has $\binom{P}{2}$ lines and is regular of degree $P - 1$.

As we have seen, K_3 is called a triangle. The graphs $\overline{K}_P$ are totally disconnected, and are regular of degree 0.

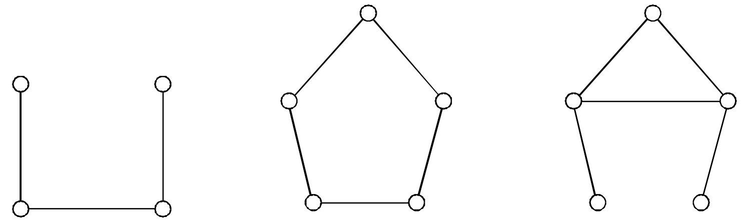


Figure 29. The smallest nontrivial self-complementary graphs.

Theorem 1.5. *The maximum number of lines among all P point graphs with no triangles is* $\left[\dfrac{P^2}{4}\right]$.

Proof. The statement is obvious for small values of P. An inductive proof may be given separately for odd P and for even P.

Suppose the statement is true for all even $P \le 2n$.

We then prove it for $P = 2n + 2$

Thus, let G be a graph with $P = 2n + 2$ points and no triangles.

Since G is not totally disconnected, there are adjacent points u and v.

The subgraph $G' = G - \{u, v\}$ has $2n$ points and no triangles, so that by the inductive hypothesis G' has at most $\left[\dfrac{4n^2}{4}\right] = n^2$ lines.

There can be no point W such that u and v are both adjacent to W, for then u, v and w would be points of a triangle in G.

Thus if u is adjacent to K points of G', v can be adjacent to at most $2n - K$ points.

Then G has at most

$$n^2 + K + (2n - K) + 1 = n^2 + 2n + 1 = \frac{P^2}{4} = \left[\frac{P^2}{4}\right] \text{ lines.}$$

Theorem 1.6. *Every graph is an intersection graph.*

Proof. For each point v_i of G

Let S_i be the union of $\{v_i\}$ with the set of lines incident with v_i.

Then it is immediate that G is isomorphic with Ω (F) where $F = \{S_i\}$.

Note : The intersection number ω' (G) of a given graph G is the minimum number of elements in a set S such that G is an intersection on S.

Corollary (1)

If G is connected and $P \ge 3$, then $\omega(G) \le q$.

Proof. In this case, the points can be omitted from the sets S_i used in the proof of the theorem, so that $S = X(G)$.

Corollary (2)

If G has P_0 isolated points and no K_2 components, then $\omega(G) \le q + P_0$.

Theorem 1.7. *Let G be a connected graph with P > 3 points. Then $\omega(G) = q$ if and only if G has no triangles.*

Proof. We first prove the sufficiency.

To show that $\omega(G) \ge q$ for any connected G with atleast 4 points having no triangles.

By definition of the intersection number, G is isomorphic with an intersection graph $\Omega(F)$ on a set S with $| S | = \omega(G)$.

For each point v_i of G, let S_i be the corresponding set.

Because G has no triangles, no element of S can belong to more than two of the sets S_i, and $S_i \cap S_j \ne \phi$ if and only if v_iv_j as a line of G.

Thus we can form a 1 – 1 correspondence between the lines of G and those elements of S which belong to exactly two sets S_i.

Therefore $\omega(G) = | S | \ge q$ so that $\omega(G) = q$.

To prove necessity :

Let $\omega(G) = q$ and assume that G has a triangle then let G_1 be a maximal triangle-free spanning subgraph of G. $\omega(G_1) = q_1 = | X(G_1) |$.

Suppose that $G_1 = \Omega(F)$, where F is a family of subsets of some set S with cardinality q_1.

Let x be a line of G not in G_1 and consider $G_2 = G_1 + x$. Since G_1 is a maximal triangle-free, G_2 must have some triangle, say u_1, u_2, u_3 where $x = u_1u_3$.

Denote by S_1, S_2, S_3 the subsets of S corresponding to u_1, u_2, u_3. Now if u_2 is adjacent to only u_1 and u_3 in G_1, replace S_2 by a singleton chosen from $S_1 \cap S_2$ and add that element to S_3.

Otherwise, replace S_3 by the union of S_3 and any element in $S_1 \cap S_2$.

In either case this gives a famly F′ of distinct subsets of S such that $G_2 = \Omega(F')$.

Thus $\omega(G_2) \le q_1$ while $| X(G_2) | = q_1 + 1$

If $G_2 \cong G$ there is nothing to prove.

But if $G_2 \ne G$, then let $| X(G) | - | X(G_2) | = q_0$

It follows that G is an intersection graph on a set with $q_1 + q_0$ elements.

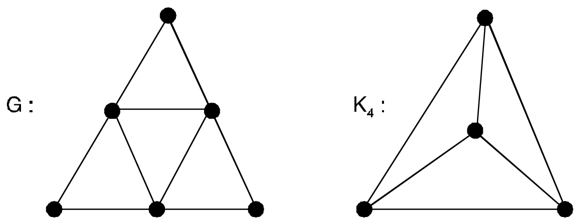
However, $q_1 + q_0 = q - 1$

Thus $\omega(G) < q$

Hence the proof.

Theorem 1.8. *For any graph G with $P \ge 4$ points, $\omega(G) \le \left[\frac{P^2}{4}\right]$.*

Theorem 1.9. *A graph G is a clique graph if and only if it contains a family F of complete subgraphs, whose union in G, such that whenever every pair of such complete graphs in some subfamily F ′ have a non empty intersection, the intersection of all the members of F ′ is non empty.*



A graph and its clique graph.

1.12 CONNECTED AND DISCONNECTED GRAPHS

A graph G is said to be a **connected** if every pair of vertices in G are connected. Otherwise, G is called a **disconnected** graph. Two vertices in G are said to be connected if there is at least one path from one vertex to the other.

In other words, a graph G is said to be connected if there is at least one path between every two vertices in G and disconnected if G has at least one pair of vertices between which there is no path.

A graph is **connected** if we can reach any vertex from any other vertex by travelling along the edges and disconnected otherwise.

For example, the graphs in Figure 30(*a*, *b*, *c*, *d*, *e*) are connected whereas the graphs in Figure 31(*a*, *b*, *c*) are disconnected.

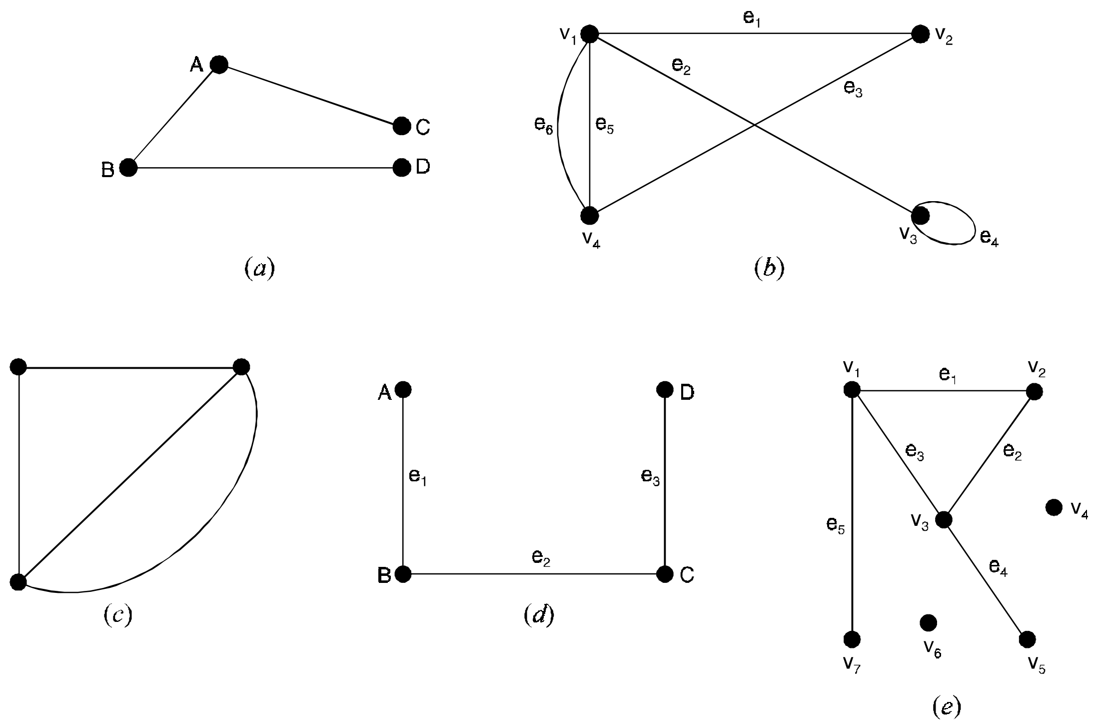


Fig. 30.

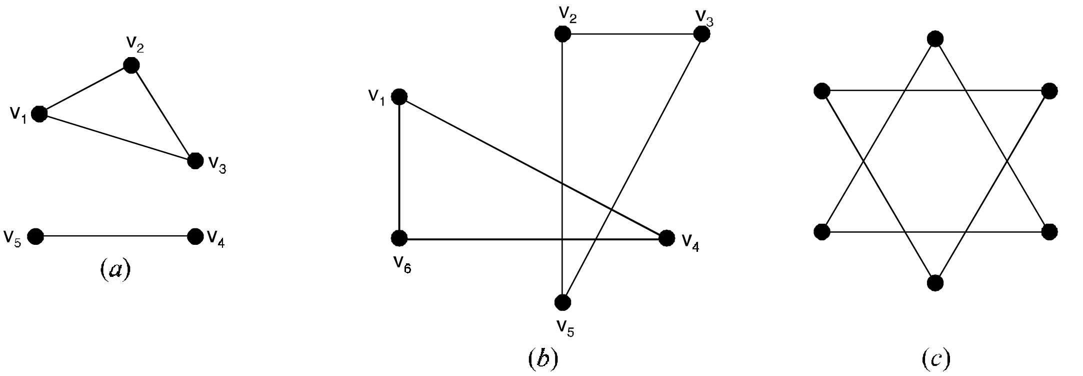


Fig. 31.

A complete graph is always connected, also, a null graph of more than one vertex is disconnected (see Fig. 32). All paths and circuits in a graph G are connected subgraphs of G.

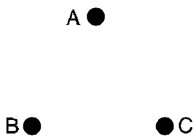


Fig. 32.

Every graph G consists of one or more connected graphs, each such connected graph is a subgraph of G and is called a component of G. A connected graph has only one component and a disconnected graph has two or more components.

For example, the graphs in Figure 31(*a, b*) have two components each.

1.12.1. Path graphs and cycle graphs

A connected graph that is 2-regular is called a cycle graph. Denote the cycle graph of n vertices by Γ_n. A circuit in a graph, if it exists, is a cycle subgraph of the graph.

The graph obtained from Γ_n by removing an edge is called the path graph of n vertices, it is denoted by P_n.

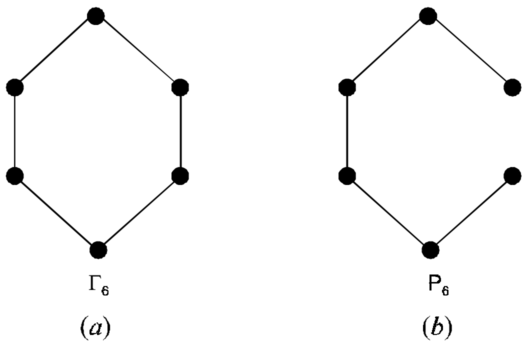


Fig. 33.

The graphs Γ_6 and P_6 are shown in Figure 33(*a*) and 33(*b*) respectively.

1.12.2. Rank and nullity

For a graph G with n vertices, m edges and k components we define the rank of G and is denoted by $\rho(G)$ and the nullity of G is denoted by $\mu(G)$ as follows.

$$\rho(G) = \text{Rank of G} = n - k$$

$$\mu(G) = \text{Nullity of G} = m - \rho(G) = m - n + k$$

If G is connected, then we have

$$\rho(G) = n - 1 \text{ and } \mu(G) = m - n + 1.$$

Problem 1.68. *Prove that a simple graph with n vertices must be connected if it has more than* $\frac{(n-1)(n-2)}{2}$ *edges.*

Solution. Consider a simple graph on n vertices.

Choose $n - 1$ vertices $v_1, v_2, ..., v_{n-1}$ of G.

We have maximum ${}^{n-1}C_2 = \frac{(n-1)(n-2)}{2}$ number of edges only can be drawn between these vertices.

Thus if we have more than $\frac{(n-1)(n-2)}{2}$ edges atleast one edge should be drawn between the nth vertex v_n to some vertex v_i, $1 \le i \le n - 1$ of G.

Hence G must be connected.

Problem 1.69. *Show that if a and b are the only two odd degree vertices of a graph G, then a and b are connected in G.*

Solution. If G is connected, nothing to prove.

Let G be disconnected.

If possible assume that a and b are not connected.

Then a and b lie in the different components of G.

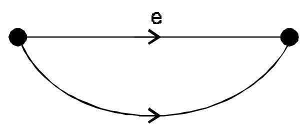
Hence the component of G containing a (similarly containing b) contains only one odd degree vertex a, which is not possible as each component of G is itself a connected graph and in a graph number of odd degree vertices should be even.

Therefore a and b lie in the same component of G.

Hence they are connected.

Problem 1.70. *Prove that a connected graph G remains connected after removing an edge e from G if and only if e lie in some circuit in G.*

Solution. If an edge e lies in a circuit C of the graph G then between the end vertices of e, there exist atleast two paths in G.



Hence removal of such an edge e from the connected graph G will not effect the connectivity of G.

Conversely, if e does not lies in any circuit of G then removal of e disconnects the end vertices of e.

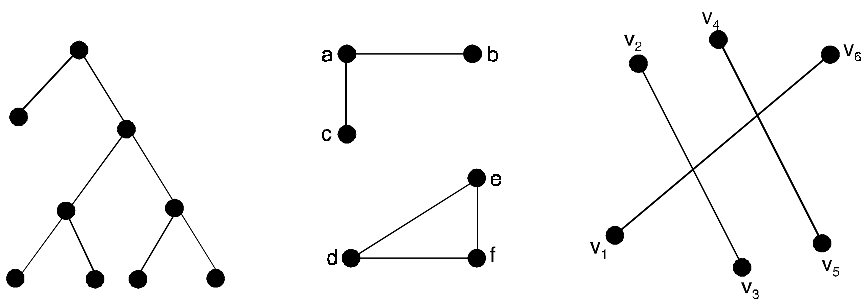
Hence G is disconnected.

Problem 1.71. *If G_1 and G_2 are (edge) decomposition of a connected graph G, then prove that $V(G_1) \cap V(G_2) \neq \phi$.*

Solution. If $V(G_1) \cap V(G_2) = \phi$ then $V(G_1)$ and $V(G_2)$ are the vertex partition of V(G) (there exists no edges left in G to include between vertex of $V(G_1)$ and $V(G_2)$ as G_1 and G_2 are edge partition of G).

Hence, G is disconnected, a contradiction to the fact that G is connected.

Problem 1.72. *Which of the graphs below are connected :*

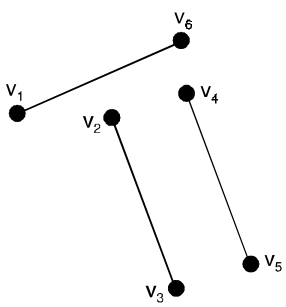


(*a*) (*b*) (*c*)

Solution. The graph shown in Figure (*a*) is connected graph since for every pair of distinct vertices there is a path between them.

The graph shown in Figure (*b*) is not connected since there is no path in the graph between vertices *b* and *d*.

The graph shown in Figure (*c*) is not connected. In drawing a graph two edges may cross at a point which is not a vertex. The graph can be redrawn as :



Theorem 1.10. *If a graph (connected or disconnected) has exactly two vertices of odd degree, there must be a path joining these two vertices.*

Proof. Let G be a graph with all even vertices except vertices v_1 and v_2, which are odd.

From theorem, which holds for every graph and therefore for every component of a diconnected graph, No graph can have an odd number of odd vertices.

Therefore, in graph G, v_1 and v_2 must belong to the same component and hence must have a path between them.

Theorem 1.11. *A simple graph with n vertices and k components cannot have more than* $\dfrac{(n-k)(n-k+1)}{2}$ *edges.*

Proof. Let n_i = the number of vertices in component i,

$$1 \le i \le k, \qquad \text{then } \sum_{i=1}^{k} n_i = n.$$

A component with n_i vertices will have the maximum possible number of edges when it is complete.

That is, it will contain $\frac{1}{2} n_i(n_i - 1)$ edges.

Hence the maximum number of edges is :

$$\begin{aligned}
\frac{1}{2}\sum_{i=1}^{k} n_i\,(n_i - 1) &= \frac{1}{2}\sum_{i=1}^{k} n_i^2 - \frac{1}{2}\sum_{i=1}^{k} n_i \\
&\le \frac{1}{2}\,[n^2 - (k-1)(2n-k)] - \frac{1}{2}\,n \\
&= \frac{1}{2}\,[n^2 - 2nk + k^2 + n - k] \\
&= \frac{1}{2}\,(n-k)(n-k+1).
\end{aligned}$$

Corollary :

If $m > \frac{1}{2}(n-1)(n-2)$ then a simple graph with n vertices and m edges is connected.

Proof. Suppose the graph is disconnected. Then it has at least two components, therefore by theorem.

$$\begin{aligned}
m &\le \frac{1}{2}\,(n-k)(n-k+1) \text{ for } k \ge 2 \\
&\le \frac{1}{2}(n-2)(n-1)
\end{aligned}$$

This contradicts the assumption that $m > \frac{1}{2}(n-1)(n-2)$.

Therefore, the graph should be connected.

Theorem 1.12. *A graph G is disconnected if and only if its vertex set V can be partitioned into two subsets V_1 and V_2 such that there exists no edge in G whose one end vertex is in the subset V_1 and the other in the subset V_2.*

Proof. Let G be disconnected. Then we have by the definition that there exists a vertex x in G and a vertex y in G such that there is no path between x and y in G

Let $V_1 = \{Z \in V : z \text{ is connected to } x\}$. Then V_1 is the set of all vertices of G which are connected to x.

Let $V_2 = V - V_1$. Then $V_1 \cap V_2 = \phi$ and $V_1 \cup V_2 = V$.

Hence V_1 and V_2 are the partition of V(G). Let a be any vertex of V_1.

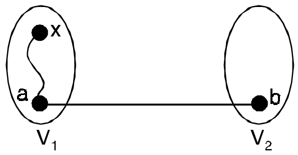
To prove that 'a' is not adjacent to any vertex of V_2.

If possible let $b \in V_2$ such that $ab \in E(G)$. Then $a \in V_1$ there exist a path P_1 : from x to a.

This path can be extended to the path $P_2 = P_1, ab, b$.

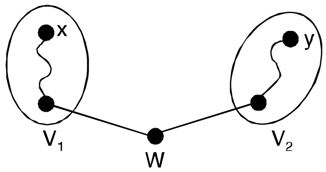
P_2 is a path from x to b in G.

Therefore x and b are connected. This implies that $b \in V_1$ which is contradiction to the fact $V_1 \cap V_2 = \phi$.



Conversely, let us assume that V can be partitioned into two subsets V_1 and V_2 such that no vertex of V_1 is adjacent to a vertex of V_2.

Let x be any vertex in V_1 and y be any vertex in V_2.



To prove that G is disconnected, if possible, suppose G is connected. Then x and y are connected.

Therefore, there exists a path between x and y in G. But this path is possible only through a vertex W in G which is not either in V_1 or V_2.

Hence $V_1 \cup V_2 \neq V$, a contradiction.

Theorem 1.13. *Show that a simple (p, q)-graph is connected then $P \leq q + 1$.*

Proof. The proof is by induction on the number of edges in G. If G has only one or two edges then the theorem is true. Assume that the theorem is true for each graph with fewer than n edges.

Let G be given connected (p, q) graph.

Case (*i*) : G contains a circuit.

Let S be a graph obtained by G by removing an edge from a circuit of G. Then S is a connected graph having $q - 1$ edges. The number of vertices of S and G are same, hence by inductive hypothesis $p \leq q - 1 + 1$.

Thus $p \leq q$, hence certainly $p \leq q + 1$.

Case (*ii*) : G does not contain a circuit.

Let p be a longest path in G. Let a and b be the end vertices of the path. The vertex a must be of degree 1, otherwise the path could be made longer, or there would be a circuit in G.

Remove the vertex a and the edge incident with the vertex a.

Let H be the graph so obtained. Then H contains exactly one vertex and one edge less than that of G.

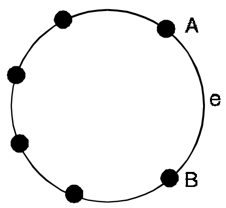
Further H is connected, hence by inductive hypothesis $p - 1 \leq (q - 1) + 1$.

Hence $p \leq q + 1$.

Problem 1.73. *Prove that a connected graph G remains connected after removing an edge e from G if and only if e belongs to some circuit in G.*

Solution. Suppose e belongs to some circuit C in G. Then the end vertices of e, say, A and B are joined by atleast two paths, one of which is e and the other C – e.

Hence the removal of e from G will not affect the connectivity of G ; even after the removal of e the end vertices of e. (*i.e.,* A and B) remain connected.



Conversely, suppose e does not belong to any circuit in G. Then the end vertices of e are connected by atmost one path.

Hence the removal of e from G disconnects these end points. This means that G – e is a disconnected graph.

Thus, if e does not belong to any circuit in G then G – e is disconnected.

This is equivalent to saying that if G – e is connected then e belongs to some circuit in G.

Problem 1.74. *Let G be a disconnected graph with n vertices where n is even. If G has two components each of which is complete, prove that G has a minimum of* $\frac{n(n-2)}{4}$ *edges.*

Solution. Let x be the number of vertices in one of the components.

Then the other component has $n - x$ number of vertices since both components are complete graphs, the number of edges they have are $\frac{x(x-1)}{2}$ and $\frac{(n-x)(n-x-1)}{2}$ respectively.

Therefore, the total number of edges in G is

$$m = \frac{x(x-1)}{2} + \frac{(n-x)(n-x-1)}{2}$$

$$= x^2 - nx + \frac{n}{2}(n-1)$$

$\Rightarrow \quad m' = 2x - n,\ m'' = 2 > 0,$ $\qquad \left(m' = \frac{dm}{dx} \text{ and } m'' \frac{d^2m}{dx^2}\right)$

Therefore, m is minimum when $2x - n = 0$

$$\Rightarrow \quad x = \frac{n}{2}$$

$$\text{Min. } m = \left(\frac{n}{2}\right)^2 - n\left(\frac{n}{2}\right) + \frac{n}{2}(n-1)$$

$$= \frac{n(n-2)}{4}.$$

Problem 1.75. *Find the rank and nullity of the complete graph k_n.*

Solution. k_n is a connected graph with n vertices and

$$m = \frac{n(n-1)}{2} \text{ edges.}$$

Therefore, by the definitions of rank and nullity, we have

Rank of $k_n = n - 1$

$$\text{Nullity of } k_n = m - n + 1 = \frac{1}{2}n(n-1) - n + 1$$

$$= \frac{1}{2}(n-1)(n-2).$$

1.13 WALKS, PATHS AND CIRCUITS

1.13.1. Walk

A **walk** is defined as a finite **alternative sequence** of vertices and edges, of the form

$$v_i e_j,\ v_{i+1}\ e_{j+1},\ v_{i+2},\ \ldots\ldots,\ e_k v_m$$

which **begins** and **ends** with **vertices,** such that

(*i*) each edge in the sequence is incident on the vertices preceding and following it in the sequence.

(*ii*) no edge appears more than once in the sequence, such a sequence is called a walk or a trial in G.

For example, in the graph shown in Figure 34, the sequences

$$v_2e_4v_6e_5v_4e_3v_3 \text{ and } v_1e_8v_2e_4v_6e_6v_5e_7v_5 \text{ are walks.}$$

Note that in the first of these, each vertex and each edge appears only once whereas in the second each edge appears only once but the vertex v_5 appears twice.

These walks may be denoted simply as $v_2v_6v_4v_3$ and $v_7v_2v_6v_5v_5$ respectively.

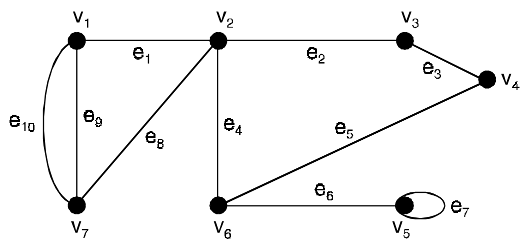


Fig. 34.

The vertex with which a walk begins is called the initial vertex and the vertex with which a walk ends is called the final vertex of the walk. The initial vertex and the final vertex are together called terminal vertices. Non-terminal vertices of a walk are called its internal vertices.

A walk having u as the initial vertex and v as the final vertex is called a walk from u to v or briefly a ***u* – *v* walk.** A walk that begins and ends at the same vertex is called a **closed walk. In other words,** a closed walk is a walk in which the terminal vertices are coincident.

A walk that is not closed is called an **open walk.**

In other words, an open walk is a walk that begins and ends at two different vertices.

For example, in the graph shown in Figure 34.

$v_1e_9v_7e_8v_2e_1v_1$ is a closed walk and $v_5e_7v_5e_6v_6e_5v_4$ is an open walk.

1.13.2. Path

In a walk, a vertex can appear more than once. An open walk in which no vertex appears more than once is called a **simple path** or a **path.**

For example, in the graph shown in Figure 34.

$v_6e_5v_4e_3v_3e_2v_2$ is a path whereas $v_5e_7v_5e_6v_6$ is an open walk but not a path.

1.13.3. Circuit

A closed walk with atleast one edge in which no vertex except the terminal vertices appears more than once is called **a circuit** or a cycle.

For example, in the graph shown in Figure 34,

$v_1e_1v_2e_8v_7e_9v_1$ and $v_2e_4v_6e_5v_4e_3v_3e_2v_2$ are circuits.

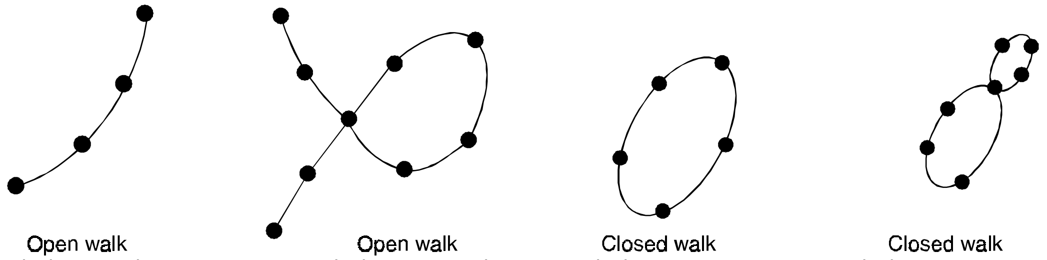
But $v_1e_9v_7e_8v_2e_4v_6e_5v_4e_3v_3e_2v_2e_1v_1$ is a closed walk but not a circuit.

Note : (*i*) In walks, path and circuit, no edge can appears more than once.

(*ii*) A vertex can appear more than once in a walk but not in a path.

(*iii*) A path is an open walk, but an open walk need not be a path.

(*iv*) A circuit is a closed walk, but a closed walk need not be a circuit.



Open walk which is a path | Open walk which is not a path | Closed walk which is a circuit | Closed walk which is not a circuit.

1.13.4. Length

The number of edges in a walk is called its length. Since paths and circuits are walks, it follows that the length of a path is the number of edges in the path and the length of a circuit is the number of edges in the circuit.

A circuit or cycle of length *k*, (with *k* edges) is called a *k*-circuit or a *k*-cycle. A *k*-circuit is called odd or even according as *k* is odd or even. A 3-cycle is called a triangle.

For example, in the graph shown in Figure 34,

The length of the open walk $v_6e_6v_5e_7v_5$ is 2

The length of the closed walk $v_1e_9v_7e_8v_2e_1v_1$ is 3

The length of the circuit $v_2e_4v_6e_5v_4e_3v_3e_2v_2$ is 4

The length of the path $v_6e_5v_4e_3v_3e_2v_2e_1v_1$ is 4

The circuit $v_1e_1v_2e_8v_7e_{10}v_1$ is a triangle.

Note : (*i*) A self-loop is a 1-cycle.

(*ii*) A pair of parallel edges form a cycle of length 2.

(*iii*) The edges in a 2-cycle are parallel edges.

Problem 1.76. *Write down all possible*

(*i*) *paths from v_1 to v_8* (*ii*) *Circuits of G and* (*iii*) *trails of length three.*

in G from v_3 to v_5 of the graph shown in Figure (35).

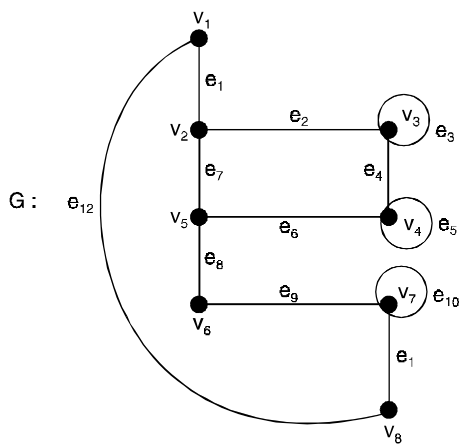


Fig. 35.

Solution.

(*i*) $P_1 : v_1e_{12}v_8$, $l(P_1) = 1$

$P_2 : v_1e_1v_2e_7v_5e_8v_6e_9v_7e_{11}v_8$, $l(P_2) = 5$

$P_3 : v_1e_1v_2e_2v_3e_4v_4e_6v_5e_8v_6e_9v_7e_{11}v_8$, $l(P_3) = 7$

These are the only possible paths from v_1 to v_8 in G.

(*ii*) $C_1 : v_1e_1v_2e_7v_5e_8v_6e_9v_7e_{11}v_8e_{12}v_1$, $l(C_1) = 6$

$C_2 : v_1e_1v_2e_2v_3e_4v_4e_6v_5e_8v_6e_9v_7e_{11}v_8e_{12}v_1$, $l(C_2) = 8$

$C_3 : v_2e_2v_3e_4v_4e_6v_5e_7v_2$, $l(C_3) = 4$

$C_4 : v_3e_3v_3$, $l(C_4) = 1$

$C_5 : v_4e_5v_4$, $l(C_5) = 1$

$C_6 : v_7e_7v_{10}$, $l(C_6) = 1$

These are the only possible circuits of G.

$W_1 : v_3e_3v_3e_2v_2e_7v_5$, $l(W_1) = 3$

$W_2 : v_3e_3v_3e_4v_4e_6v_5$, $l(W_2) = 3$

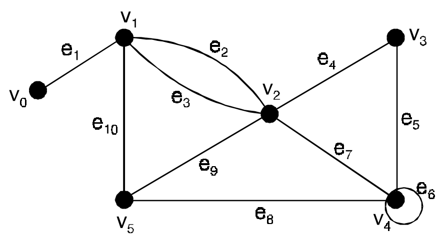
$W_3 : v_3e_4v_4e_5v_4e_6v_5$, $l(W_3) = 3$.

These are the only possible trails of length three from v_3 to v_5.

Problem 1.77. *In the graph below, determine whether the following are paths, simple paths, trails, circuits or simple circuits,*

(*i*) $v_0e_1v_1e_{10}v_5e_9v_2e_2v_1$ (*ii*) $v_4e_7v_2e_9v_5e_{10}v_1e_3v_2e_9v_5$

(*iii*) v_2 (*iv*) $v_5v_2v_3v_4v_4v_4v_5$.



Solution. (*i*) The sequence has a repeated vertex v_1 but does not have a repeated edge so it is a trail. It is not cycle or circuit.

(*ii*) The sequence has a repeated vertex v_2 and repeated edge e_9. Hence it is a path. It is not cycle or circuit.

(*iii*) It has no repeated edge, no repeated vertex, starts and ends at same vertex. Hence it is a simple circuit.

(*iv*) It is a circuit since it has no repeated edge, starts and ends at same vertex. It is not a simple circuit since vertex v_4 is repeated.

Theorem 1.14. *In a graph (directed or undirected) with n vertices, if there is a path from vertex u to vertex v then the path cannot be of length greater than (n – 1).*

Proof. Let $\pi : u, v_1, v_2, v_3, \ldots.. v_k, v$ be the sequence of vertices in a path u and v.

If there are m edges in the path then there are $(m + 1)$ vertices in the sequence.

If $m < n$, then the theorem is proved by default. Otherwise, if $m \geq n$ then there exists a vertex v_j in the path such that it appears more than once in the sequence

$$(u, v_1, \ldots\ldots., v_j \ldots\ldots, v_j , \ldots\ldots v_k , v).$$

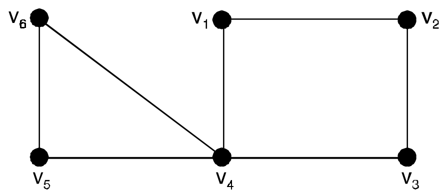
Deleting the sequence of vertices that leads back to the node v_j, all the cycles in the path can be removed.

The process when completed yields a path with all distinct nodes. Since there are n nodes in the graph, there cannot be more than n distinct nodes and hence $n - 1$ edges.

Problem 1.78. *For the graph shown in Figure, indicate the nature of the following sequences of vertices*

(*a*) $v_1v_2v_3v_2$ (*b*) $v_4v_1v_2v_3v_4v_5$ (*c*) $v_1v_2v_3v_4v_5$

(*d*) $v_1v_2v_3v_4v_1$ (*e*) $v_6v_5v_4v_3v_2v_1v_4v_6$



Solution. (*a*) Not a walk

(*b*) Open walk but not a path

(*c*) Open walk which is a path

(*d*) Closed walk which is a circuit

(*e*) Closed walk which is not a circuit.

Theorem 1.15. *Let G = (V, E) be an undirected graph, with $a, b \in V$, $a \neq b$. If there exists a trail (in G) from a to b, then there is a path (in G) from a to b.*

Proof. Since there is an trail from a to b.

We select one of shortest length, say$\{a, x_1\}$, $\{x_1, x_2\}$, ..., $\{x_n, b\}$.

If this trail is not a path, we have the situation $\{a, x_1\}$, $\{x_1, x_2\}$, ..., $\{x_{k-1}, x_k\}$, $\{x_k, x_{k+1}\}$, $\{x_{k+1}, x_{k+2}\}$, ..., $\{x_{m-1}, x_m\}$, $(x_m, x_{m+1}\}$, ..., $\{x_n, b\}$,

where $k < m$ and $x_k = x_m$, possibly with $k = 0$ and $a(= x_0) = x_m$, or $m = n + 1$ and $x_k = b$ $(= x_{n+1})$

But then we have a contradiction, because

$\{a,x_1\}$, $\{x_1, x_2\}$, ..., $\{x_{k-1}, x_k\}$, $\{x_m , x_{m+1}\}$, ..., $\{x_n, b\}$ is a shortest trail from a to b.

Problem 1.79. *Let G = (V, E) be a loop-free connected undirected graph, and let {a, b} be an edge of G. Prove that {a, b} is part of a cycle if and only if its removal (the vertices a and b are left) does not disconnect G.*

Solution. If $\{a, b\}$ is not part of a cycle, then its removal disconnects a and b (and G).

If not, there is a path P from a to b, and P together with $\{a, b\}$ provides a cycle containing $\{a, b\}$.

Conversely, if the removal of $\{a, b\}$ from G disconnects G, there exist $x, y \in$ V such that the only path P from x to y contains $e = \{a, b\}$. If e were part of a cycle C, then the edges in $(P - \{e\}) \cup (C - \{e\})$ would contain a second path connecting x to y.

Theorem 1.16. *In a graph G, every $u - v$ path contains a simple $u - v$ path.*

Proof. If a path is a closed path, then it certainly contains the trivial path.

Assume, then, that P is an open $u - v$ path.

We complete the proof by induction on the length n of P.

If P has length one, then P is itself a simple path.

Suppose that all open $u - v$ paths of length k. Where $1 \le k \le n$, contains a simple $u - v$ path. Now suppose that P is the open $u - v$ path

$\{v_0, v_1\}$, ... $\{v_n, v_{n+1}\}$, where $u = v_0$ and $v = v_{n+1}$ of course, it may be that P has repeated vertices, but if not, then P is a simple $u - v$ path.

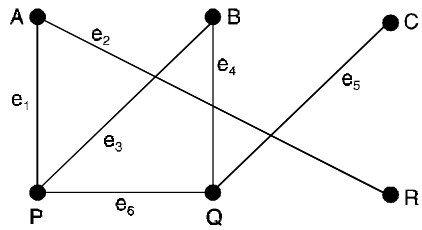
If, on the other hand, there are repeated vertices in P.

Let i and j be distinct positive integers where $i < j$ and $v_i = v_j$.

If the closed path $v_i - v_j$ is removed from P, an open path P′ is obtained having length $\le n$, since at least the edge $\{v_i, v_{i+1}\}$ was deleted from P.

Thus, by the inductive hypothesis, P′ contains a simple $u - v$ path and, thus, so does P.

Problem 1.80. *Find all circuits in the graph shown below :*



Solution. There are no circuits beginning and ending with the vertices A, C and R.

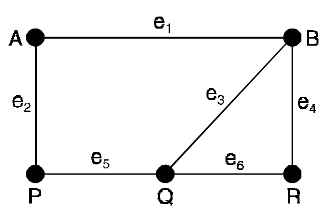
The circuits beginning and ending with the vertices

B, P, Q are $Be_3Pe_6Qe_4B$, $Pe_6Qe_4Be_3P$, $Qe_4Be_3Pe_6Q$

But all of these represent one and the same circuit.

Thus, there is only one circuit in the graph.

Problem 1.81. *Consider the graph shown in Figure, find all paths from vertex A to vertex R. Also, indicate their lengths.*



Solution. There are four paths from A to R.

These are Ae_1Be_4R, $Ae_1Be_3Qe_6R$, $Ae_2Pe_5Qe_6R$, $Ae_2Pe_5Qe_3Be_4R$.

These paths contain, two, three the and four edges.

Their lengths are two, three, three and four respectively.

Problem 1.82. *Prove the following :*

(*a*) *A path with n vertices is of length n – 1*

(*b*) *If a circuit has n vertices, it has n edges*

(*c*) *The degree of every vertex in a circuit is two.*

Solution. (*a*) In a path, every vertex except the last is followed by precisely one edge.

Therefore, if a path has n vertices, it must have $n - 1$ edges. Its length is $n - 1$.

(*b*) In a circuit, every vertex is followed by precisely one edge.

Therefore, if a circuit has n vertices, it must have n edges.

(*c*) In a circuit, exactly two edges are incident on every vertex.

Therefore, the degree of every vertex in a circuit is two.

Problem 1.83. *If G is a simple graph in which every vertex has degree at least k, prove that G contains a path of length at least k. Deduce that if $k \geq 2$ then G also contains a circuit of length at least $k + 1$.*

Solution. Consider a path P in G, which has a maximum number of vertices. Let u be an end vertex of P. Then every neighbour of u belongs to P. Since u has at least k neighbours and since G is simple, P must have at least k vertices other than u.

Thus, P is a path of length at least k

If $k \geq 2$ then P and the edge from u to its farthest neighbour v constitute a circuit of length at least $k + 1$.

1.14 EULERIAN GRAPHS

1.14.1. Euler path

A path in a graph G is called Euler path if it includes every edges exactly once. Since the path contains every edge exactly once, it is also called Euler trail.

1.14.2. Euler circuit

An Euler path that is circuit is called Euler circuit. A graph which has a Eulerian circuit is called an Eulerian graph.

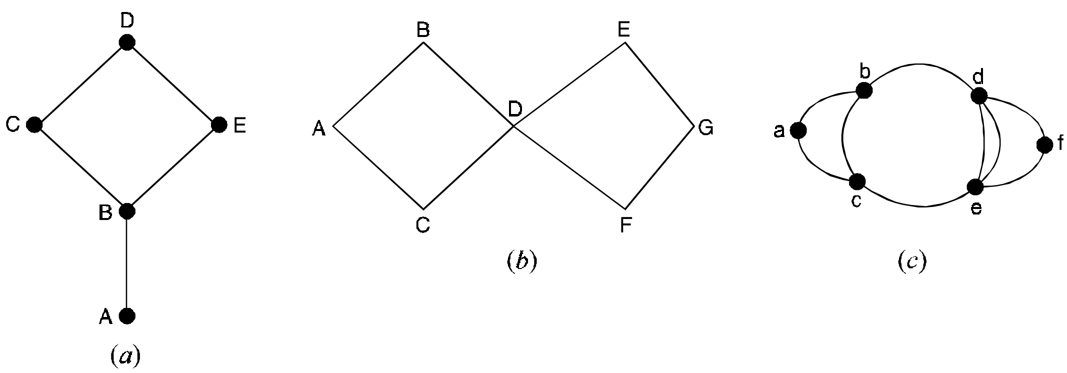


Fig. 36.

The graph of Figure 36(*a*) has an Euler path but no Euler circuit. Note that two vertices A and B are of odd degrees 1 and 3 respectively. That means AB can be used to either arrive at vertex A or leave vertex A but not for both.

Thus an Euler path can be found if we start either from vertex A or from B.

ABCDEB and BCDEBA are two Euler paths. Starting from any vertex no Euler circuit can be found.

The graph of Figure 36(*b*) has both Euler circuit and Euler path. ABDEGFDCA is an Euler path and circuit. Note that all vertices of even degree.

No Euler path and circuit is possible in Figure 36(*c*).

Note that all vertices are not even degree and more than two vertices are of odd degree.

The existence of Euler path and circuit depends on the degree of vertices.

Note : To determine whether a graph G has an Euler circuit, we note the following points :

(*i*) List the degree of all vertices in the graph.

(*ii*) If any value is zero, the graph is not connected and hence it cannot have Euler path or Euler circuit.

(*iii*) If all the degrees are even, then G has both Euler path and Euler circuit.

(*iv*) If exactly two vertices are odd degree, then G has Euler path but no Euler circuit.

Theorem 1.17. *The following statements are equivalent for a connected graph G :*

(*i*) *G is Eulerian*

(*ii*) *Every point of G has even degree*

(*iii*) *The set of lines of G be partitioned into cycles.*

Proof. (*i*) implies (*ii*)

Let T be an Eulerian trail in G.

Each occurrence of a given point in T contributes 2 to the degree of that point, and since each line of G appears exactly once in T, every point must have even degree.

(*ii*) implies (*iii*)

Since G is connected and non trivial, every point has degree at least 2, so G contains a cycle Z.

The removal of the lines of Z results in a spanning subgraph G_1 in which every point still has even degree.

If G_1 has no lines, then (*iii*) already holds ; otherwise, repetition of the argument applied to G_1 results in a graph G_2 in which again all points are even, etc.

When a totally disconnected graph G_n is obtained, we have a partition of the lines of G into n cycles.

(*iii*) implies (*i*)

Let Z_1 be one of the cycles of this partition.

If G consists only of this cycle, then G is obviously Eulerian.

Otherwise, there is another cycle Z_2 with a point v in common with Z_1.

The walk beginning at v and consisting of the cycles Z_1 and Z_2 in succession is a closed trail containing the lines of these two cycles.

By continuing this process, we can construct a closed trail containing all lines of G.

Hence G is Eulerian.

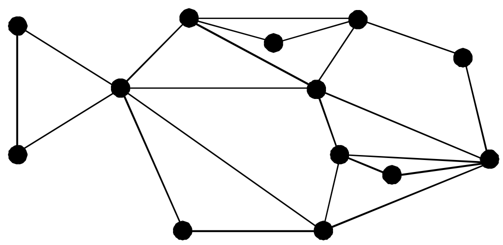


Fig. 37. An Eulerian graph.

For example, the connected graph of Figure 37 in which every point has even degree has an Eulerian trail, and the set of lines can be partitioned into cycles.

Corollary (1) :

Let G be a connected graph with exactly $2n$ odd points, $n \geq 1$, then the set of lines of G can be partitioned into n open trails.

Corollary (2) :

Let G be a connected graph with exactly two odd points. Then G has an open trail containing all the points and lines of G (which begins at one of the odd points and ends at the other).

Problem 1.18. *A non empty connected graph G is Eulerian if and only if its vertices are all of even degree.*

Proof. Let G be Eulerian.

Then G has an Eulerian trail which begins and ends at u, say.

If we travel along the trail then each time we visit a vertex we use two edges, one in and one out.

This is also true for the start vertex because we also ends there.

Since an Eulerian trial uses every edge once, each occurrence of v represents a contribution of 2 to its degree.

Thus deg(v) is even.

Conversely, suppose that G is connected and every vertex is even.

We construct an Eulerian trail. We begin a trail T_1 at any edge e. We extend T_1 by adding an edge after the other.

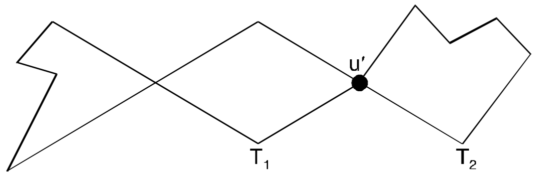
If T_1 is not closed at any step, say T_1 begins at u but ends at $v \neq u$, then only an odd number of the edges incident on v appear in T_1.

Hence we can extend T_1 by another edge incident on v.

Thus we can continue to extend T_1 until T_1 returns to its initial vertex u.

i.e., until T_1 is closed.

If T_1 includes all the edges of G then T_1 is an Eulerian trail.



Suppose T_1 does not include all edges of G.

Consider the graph H obtained by deleting all edges of T_1 from G.

H may not be connected, but each vertex of H has even degree since T_1 contains an even number of the edges incident on any vertex.

Since G is connected, there is an edge e' of H which has an end point u' in T_1.

We construct a trail T_2 in H beginning at u' and using e'. Since all vertices in H have even degree.

We can continue to extent T_2 until T_2 returns to u' as shown in Figure.

We can clearly put T_1 and T_2 together to form a larger closed trail in G.

We continue this process until all the edges of G are used.

We finally obtain an Eulerian trail, and so G is Eulerian.

Theorem 1.18. *A connected graph G has an Eulerian trail if and only if it has at most two odd vertices.*

i.e., it has either no vertices of odd degree or exactly two vertices of odd degree.

Proof. Suppose G has an Eulerian trail which is not closed. Since each vertex in the middle of the trail is associated with two edges and since there is only one edge associated with each end vertex of the trail, these end vertices must be odd and the other vertices must be even.

Conversely, suppose that G is connected with atmost two odd vertices.

If G has no odd vertices then G is Euler and so has Eulerian trail.

The leaves us to treat the case when G has two odd vertices (G cannot have just one odd vertex since in any graph there is an even number of vertices with odd degree).

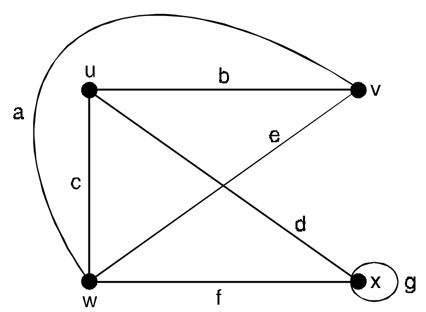
Corollary (1) :

A directed multigraph G has an Euler path if and only if it is unilaterally connected and the in degree of each vertex is equal to its out degree with the possible exception of two vertices, for which it may be that the in degree of is larger than its out degree and the in degree of the other is oneless than its out degree.

Corollary (2) :

A directed multigraph G has an Euler circuit if and only if G is unilaterally connected and the indegree of every vertex in G is equal to its out degree.

Problem 1.84. *Show that the graph shown in Figure has no Eulerian circuit but has a Eulerian trail.*

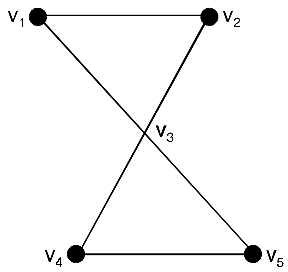


Solution. Here deg (u) = deg (v) = 3 and deg (w) = 4, deg (x) = 4

Since u and v have only two vertices of odd degree, the graph shown in Figure, does not contain Eulerian circuit, but the path.

$b - a - c - d - g - f - e$ is an Eulerian path.

Problem 1.85. *Let G be a graph of Figure. Verify that G has an Eulerian circuit.*



Solution. We observe that G is connected and all the vertices are having even degree

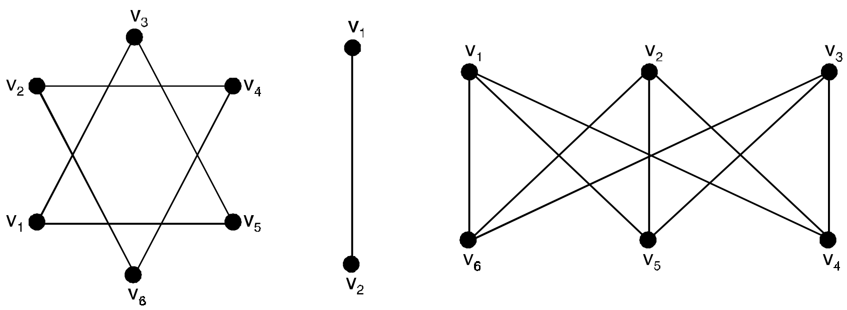
$$\deg(v_1) = \deg(v_2) = \deg(v_4) = \deg(v_5) = 2.$$

Thus G has a Eulerian circuit.

By inspection, we find the Eulerian circuit

$$v_1 - v_3 - v_5 - v_4 - v_3 - v_2 - v_1.$$

Problem 1.86. *Show that the graphs in Figure below contain no Eulerian circuit.*



(a) (b) (c)

Solution. The graph shown in Figure (a) does not contain Eulerian circuit since it is not connected.

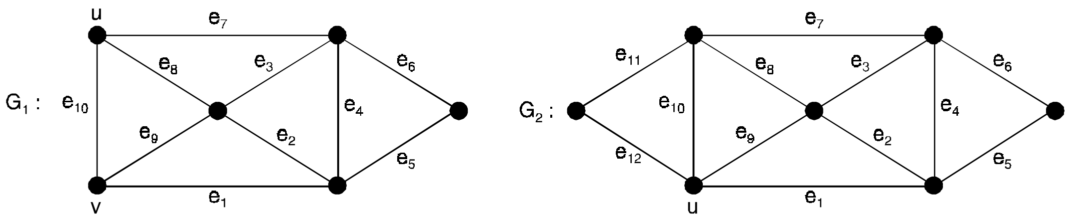
The graph shown in Figure (b) is connected but vertices v_1 and v_2 are of degree 1.

Hence it does not contain Eulerian circuit.

All the vertices of the graph shown in Figure (c) are of degree 3.

Hence it does not contain Eulerian circuit.

Problem 1.87. *Which of the following graphs have Eulerian trail and Eulerian circuit.*



Solution. In G_1 an Eulerian trail from *u* to *v* is given by the sequence of edges e_1, e_2, ..., e_{10}.

While in G_2 an Eulerian cycle (circuit) from *u* to *v* is given by e_1, e_2, ... e_{11}, e_{12}.

Problem 1.88. *Show that a connected graph with exactly two odd vertices is a unicursal graph.*

Solution. Suppose A and B be the only two odd vertices in a connected graph G.

Join these vertices by an edge *e*.

Then A and B become even vertices.

Since all other vertices in G are of even degree, the graph G ∪ *e* is an Eulerian graph.

Therefore, it has an Euler line which must include. The open walk got by deleting *e* from this Euler line is a semi-Euler line in G.

Hence G is a unicursal graph.

Problem 1.89. (*i*) *Is there is an Euler graph with even number of vertices and odd number of edges ?*

(*ii*) *Is there an Euler graph with odd number of vertices and even number of edges ?*

Solution. (*i*) Yes. Suppose C is a circuit with even number of vertices.

Let *v* be one of these vertices.

Consider a circuit C′ with odd number of vertices passing through *v* such that C and C′ have no edge in common.

The closed walk *q* that consists of the edges of C and C′ is an Eulerian graph of the desired type.

(*ii*) Yes, in (*i*), suppose C and C′ are circuits with odd number of vertices.

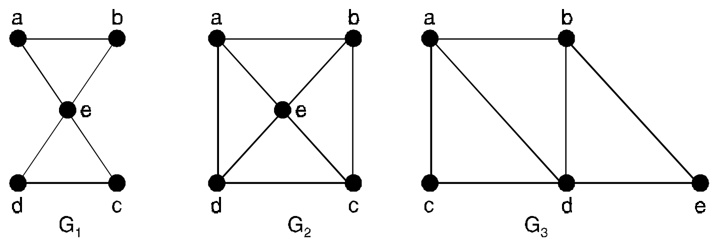
Then *q* is an Eulerian graph of the desired type.

Problem 1.90. *Find all positive integers n such that the complete graph k_n is Eulerian.*

Solution. In the complete graph k_n, the degree of every vertex is $n - 1$.

Therefore, k_n is Eulerian if and only if $n - 1$ is even, *i.e.*, if and only if *n* is odd.

Problem 1.91. *Which of the undirected graph in Figure have an Euler circuit ? Of those that do not, which have an Euler path ?*



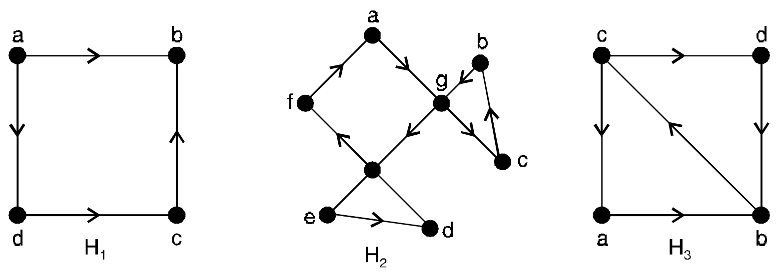
The undirected graphs G_1, G_2 and G_3.

Solution. The graph G_1 has an Euler circuit.

For example, *a, e, c, d, e, b, a*. Neither of the graphs G_2 or G_3 has an Euler circuit. However, G_3 has an Euler path, namely *a, c, d, e, b, d, a, b.*

G_2 does not have an Euler path.

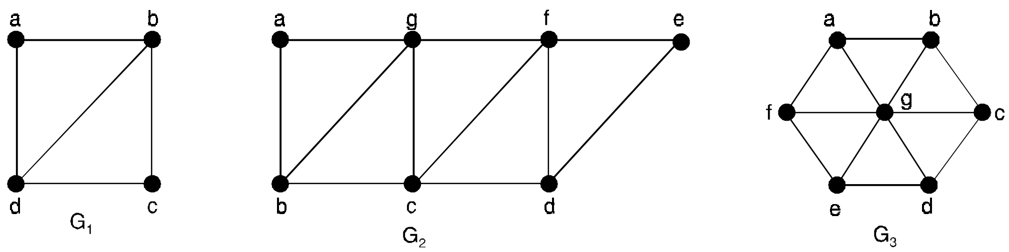
Problem 1.92. *Which of the directed graphs in Figure have an Euler circuit ? Of those that do not, which have an Euler path ?*



The directed graphs H_1, H_2, H_3

Solution. The graph H_2 has an Euler circuit, for example *a, g, c, b, g, e, d, f, a*. Neither H_1 nor H_3 has an Euler circuit. H_3 has an Euler path, namely *e, a, b, c, d, b* but H_1 does not.

Problem 1.93. *Which graphs shown in Figure have an Euler path ?*



Three undirected graphs.

Solution. G_1 contains exactly two vertices of odd degree, namely, *b* and *d*.

Hence it has an Euler path that must have *b* and *d* as its end points. One such Euler path is *d, a, b, c, d, b.* Similarly, G_2 has exactly two vertices of odd degree, namely, *b* and *d*. So it has an Euler path that must have *b* and *d* as enpoints. One such Euler path is *b, a, g, f, e, d, c, g, b, c, f, d.*

G_3 has no Euler path since it has six vertices of odd.

Lemma

If G is a graph in which the degree of each vertex is at least 2, then G contains a cycle.

Proof. If G has any loops or multiple edges, the result is trivial.

Suppose that G is a simple graph.

Let *v* be any vertex of G.

We construct a walk $v \to v_1 \to v_2 \to$ ...... inductively by choosing v_1 to be any vertex adjacent to v and for each $i > 1$.

Choosing v_{i+1} to be any vertex adjacent to v_i except v_{i-1}, the existence of such a vertex is guaranteed by our hypothesis.

Since G has only finitely many vertices, we must eventually choose a vertex that has been chosen before.

If v_k is the first such vertex, then that part of the walk lying between the two occurrences of v_k is the required cycle.

Theorem 1.19. *A connected graph G is Eulerian if and only if the degree of each vertex of G is even.*

Proof. Suppose that P is an Eulerian trail of G. Whenever P passes through a vertex, there is a contradiction of 2 towards the degree of that vertex.

Since each edge occurs exactly once in P, each vertex must have even degree.

The proof is by induction on the number of edges of G.

Suppose that the degree of each vertex is even.

Since G is connected, each vertex has degree at least 2 and so by lemma, G contains a cycle C.

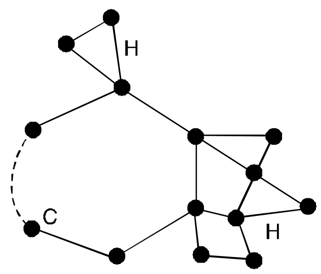
If C contains every edge of G, the proof is complete.

If not, we remove from G the edges of C to form a new, possibly disconnected, graph H with fewer edges that G and in which each vertex still has even degree.

By the induction hypothesis, each component of H has an Eulerian trail.

Since each component of H has at least one vertex in common with C, by connectedness, we obtain the required Eulerian trail of G by following the edges of C until a non-isolated vertex of H is reached, tracing the Eulerian trail of the component of H that contains that vertex, and then continuing along the edges of C until we reach a vertex belonging to another component of H and so on.

The whole process terminates when we return to the initial vertex (see Figure below)



Corollary (1) :

A connected graph is Eulerian if and only if its set of edges can be split up into disjoint cycles.

Corollary (2) :

A connected graph is semi-Eulerian if and only if it has exactly two vertices of odd degree.

Theorem 1.20. *Let G be an Eulerian graph. Then the following construction is always possible, and produces an Eulerian trail of G.*

Start at any vertex u and traverse the edges in an arbitrary manner, subject only to the following results :

(*i*) *erase the edges as they are traversed, and if any isolated vertices result, erase them too ;*

(*ii*) *at each stage, use a bridge only if there is no alternative.*

Proof. We show first that the construction can be carried out at each stage.

Suppose that we have just reached a vertex v.

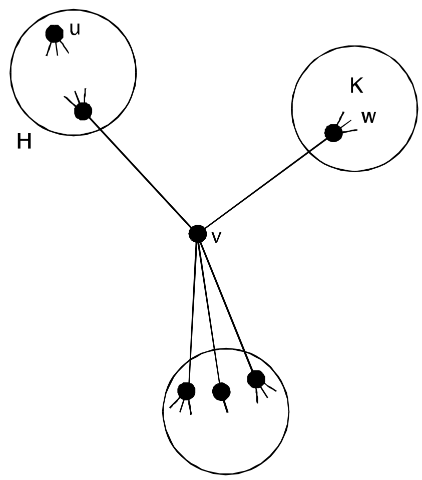
If $v \neq u$ then the subgraph H that remains is connected and contains only two vertices of odd degree u and v.

To show that the construction can be carried out, we must show that the removal of the next edge does not disconnected H or equivalently, that v is incident with atmost one bridge.

But if this is not the case, then there exists a bridge vw such that the component K of H – vw containing w does not contain u (see Figure, below).

Since the vertex w has odd degree in K, some other vertex of K must also have odd degree, giving the required contradiction.

If $v = u$, the proof is almost identical, as long as there are still edges incident with u.



Figure

It remains only to show that this construction always yields a complete Eulerian trail.

But this is clear, since there can be no edges of G remaining untraversed when the last edge incident to u is used, since otherwise the removal of some earlier edge adjacent to one of these edges would have disconnected the graph, contradicting (*ii*).

Theorem 1.21. *(a) If a graph G has more than two vertices of odd degree, then there can be no Euler path in G.*

(b) If G is connected and has exactly two vertices of odd degree, there is an Euler path in G. Any Euler path in G must begin at one vertex of odd degree and end at the other.

Proof. (*a*) Let v_1, v_2, v_3 be vertices of odd degree.

Any possible Euler path must leave (or arrive at) each of v_1, v_2, v_3 with no way to return (or leave) since each of these vertices has odd degree.

One vertex of these three vertices may be the beginning of the Euler path and another the end, but this leaves the third vertex at one end of an untraveled edge.

Thus there is no Euler path.

(*b*) Let u and v be the two vertices of odd degree. Adding the edge $\{u, v\}$ to G produces a connected graph G′ all of whose vertices has even degree. There is an Euler circuit π' in G′.

Omitting $\{u, v\}$ from π' produces an Euler path that begins at u (or v) and ends at v (or u).

Theorem 1.22. *(a) If a graph G has a vertex of odd degree, there can be no Euler circuit in G.*

(b) If G is a connected graph and every vertex has even degree, then there is an Euler circuit in G.

Proof. (*b*) Suppose that there are connected graphs where every vertex has even degree, but there is no Euler circuit. Choose such a G with the smallest number of edges.

G must have more than one vertex since, if there were only one vertex of even degree, there is clearly in Euler circuit. We show first that G must have atleast one circuit. If v is a fixed vertex of G, then since G is connected and has more than one vertex, there must be an edge between v and some other vertex v_1.

This is a simple path (of length 1) and so simple paths exists. Let π_0 be a simple path in G having the longest possible length, and let its vertex sequence be v_1, v_2, ... v_s. Since v_s has even degree and π_0 uses only one edge that has v_s as a vertex, there must be an edge e not in π_0 that also has v_s as a vertex.

If the other vertex of e is not one of the v_i, then we could construct a simple path longer than π_0. Which is a contradiction.

Thus e has some v_i as its other vertex, and therefore we can construct a circuit.

$$v_i, v_{i+1}, \ldots\ldots v_s, v_i \text{ in G.}$$

Since we know that G has circuits, we may choose a circuit π in G that has the longest possible length. Since we assumed that G has no Euler circuits, π cannot contain all the edges of G.

Let G_1 be the graph formed from G by deleting all edges in π (but not vertices).

Since π is a circuit, deleting its edges will reduce the degree of every vertex by 0 or 2, so G_1 is also a graph with all vertices of even degree.

The graph G_1 may not be connected, but we can choose a largest connected component (piece) and call this graph G_2 (G_2 may be G_1).

Now G_2 has fewer edges than G, and so (because of the way G was chosen), G_2 must have an Euler path π'.

If π' passes through all the vertices on G, then π and π' clearly have vertices in common.

If not, then these must be an edge in G between some vertex v' in π', and some vertex v not in π'.

Otherwise we could not get from vertices in π' to the other vertices in G, and G would not be connected.

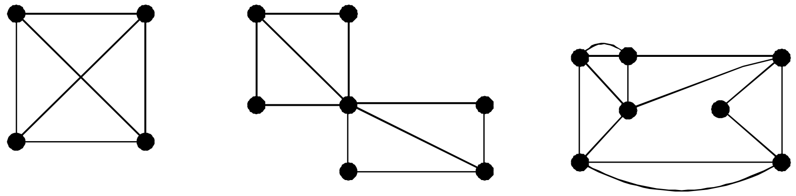
Since e is not in π', it must have been deleted when G_1 was created from G, and so must be an edge in π.

Then v' is also in the vertex sequence of π, and so in any case π and π' have atleast one vertex v' in common. We can then construct a circuit in G that is longer than π by combining π and π' at v'.

This is a contradiction, since π was chosen to be the longest possible circuit in G.

Hence the existence of the graph G always produces a contradiction, and so no such graph is possible.

Problem 1.94. *Which of the graphs in Figure (a), (b), (c) have an Euler circuit, an Euler path but not an Euler circuit, or neither ?*



(*a*) (*b*) (*c*)

Solution. (*i*) In Figure (*a*), each of the four vertices has degree 3 ; thus, there is neither an Euler circuit nor an Euler path.

(*ii*) The graph in Figure (*b*) has exactly two vertices of odd degree. There is no Euler circuit, but there must be an Euler path.

(*iii*) In Figure (*c*), every vertex has even degree ; thus the graph must have an Euler circuit.

1.15 FLEURY'S ALGORITHM

Let G = (V, E) be a connected graph with each vertex of even degree.

Step 1. Select an edge e_1 that is not a bridge in G.

Let its vertices be v_1, v_2.

Let π be specified by $V_\pi : v_1, v_2$ and $E_\pi : e_1$.

Remove e_1 from E and v_1 and v_2 from V to create G_1.

Step 2. Suppose that $V_\pi : v_1, v_2, ... v_k$ and $E_\pi : e_1, e_2, ...e_{k-1}$ have been constructed so far, and that all of these edges and vertices have been removed from v and E to form G_{k-1}.

Since v_k has even degree, and e_{k-1} ends there, there must be an edge e_k in G_{k-1} that also has v_k as a vertex.

If there is more than one such edge, select one that is not a bridge for G_{k-1}.

Denote the vertex of e_k other than v_k by v_{k+1}, and extend V_π and E_π to $V_\pi : v_1, v_2, ..., v_k, v_{k+1}$ and $E_\pi : e_1, e_2, ..., e_{k-1}, e_k$.

Step 3. Repeat step 2 until no edges remain in E.

End of algorithm.

Problem 1.95. *Use Fleury's algorithm to construct an Euler circuit for the graph in Figure (1).*

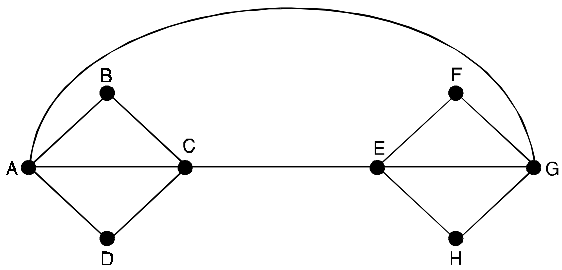


Fig. (1)

Solution. According to step 1, we may begin anywhere.

Arbitrarily choose vertex A. We summarize the results of applying step 2 repeatedly in Table.

Current Path	*Next Edge*	*Reasoning*
π : A	{A, B}	No edge from A is a bridge. Choose any one.
π : A, B	{B, C}	Only one edge from B remains.
π : A, B, C	{C, A}	No edges from C is a bridge. Choose any one.
π : A, B, C, A	{A, D}	No edges from A is a bridge. Choose any one.
π : A, B, C, A, D	{D, C}	Only one edge from D remains.
π : A, B, C, A, D, C	{C, E}	Only one edge from C remains.
π : A, B, C, A, D, C, E	{E, G}	No edge from E is a bridge. Choose any one.
π : A, B, C, A, D, C, E, G	{G, F}	{A, G} is a bridge. Choose {G, F} or {G, H}.
π : A, B, C, A, D, C, E, G, F	{F, E}	Only one edge from F remains.
π : A, B, C, A, D, C, E, G, F, E	{E, H}	Only one edge from E remains.
π : A, B, C, A, D, C, E, G, F, E, H	{H, G}	Only one edge from H remains
π : A, B, C, A, D, C, E, G, F, E, H, G	{G, A}	Only one edge from G remains.
π : A, B, C, A, D, C, E, G, F, E, H, G, A		

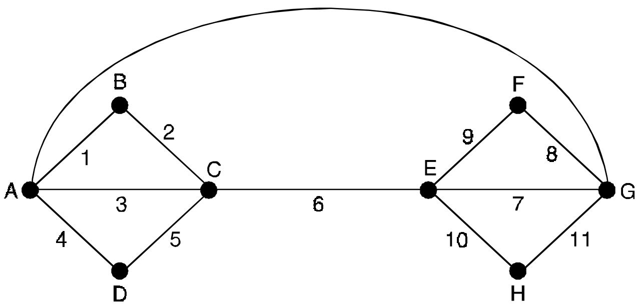


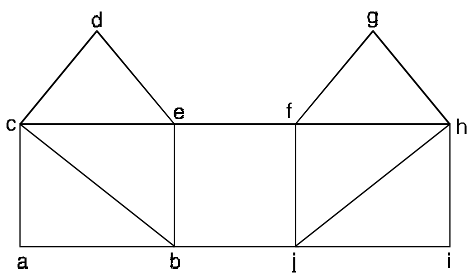
Fig. (2)

The edges in Figure (*b*) have been nembered in the order of their choice in applying step 2.

In several places, other choices could have been made.

In general, if a graph has an Euler circuit, it is likely to have several different Euler circuits.

Problem 1.96. *Using Fleury's algorithm, find Euler circuit in the graph of Figure.*

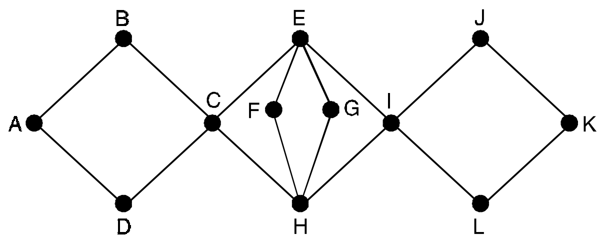


Solution. The degrees of all the vertices are even. There exists an Euler circuit in it.

Current Path	*Next Edge*	*Remark*
$\pi : a$	$\{a, j\}$	No edge from *a* is a bridge choose (*a*, *j*). Add *j* to π and remove (*a*, *j*) from E.
$\pi : aj$	$\{j, f\}$	No edge from *j* is a bridge. Choose (*j*, *f*). Add *f* to π and remove (*j*, *f*) from E.
$\pi : ajf$	$\{f, g\}$	(*f*, *e*) is a bridge and (*f*, *g*) is not a bridge. Other option (*f*, *h*)
$\pi : ajfg$	$\{g, h\}$	(*g*, *h*) is the only edge.
$\pi : ajfgh$	$\{h, i\}$	(*h*, *i*) is the other option
$\pi : ajfghi$	$\{i, j\}$	(*i*, *j*) is the only edge.
$\pi : ajfghij$	$\{j, h\}$	(*j*, *h*) is the only edge.
$\pi : ajfghijh$	$\{h, f\}$	(*h*, *f*) is the only edge
$\pi : ajfghijhf$	$\{f, e\}$	(*f*, *e*) is the only edge
$\pi : ajfghijhfe$	$\{e, d\}$	Other options are (*e*, *c*), (*e*, *a*)
$\pi : ajfghijhfed$	$\{d, c\}$	(*d*, *c*) is the only option.
$\pi : ajfghijhfedc$	$\{c, b\}$	Other options are (*c*, *e*), (*c*, *a*).
$\pi : ajfghijkfedcb$	$\{b, a\}$	(*b*, *a*) is the only option.
$\pi : ajfghijkfedcba$	$\{a, c\}$	Other options are (*a*, *e*)
$\pi : ajfghijkfedcbac$	$\{c, e\}$	(*c*, *e*) is the only option.
$\pi : ajfghijkfedcbace$	$\{e, a\}$	(*e*, *a*) is the only option.
$\pi : ajfghijkfedcbacea$		No edge is remaining in E.

This is the Euler circuit.

Problem 1.97. *Using Fleury's algorithm, find Euler circuit in the graph of Figure.*



Solution. The degree spectrum of the graph is (2, 2, 4, 2, 4, 2, 2, 4, 4, 2, 2, 2) considering the node from A to L in alphabetical order. Since all values are even there exists an Euler circuit in it. The process is summarized in the following table. The start node is A.

S.No.	*Current path*	*Next Edge Considered*	*Remark*
1.	π : A	{A, B}	We select (A, B). Add B to π and remove (A, B) from E.
2.	π : AB	{B, C}	It is the only option. Remove (B, C) from E and B from V. Add C to π.
3.	π : ABC	{C, E}	(C, D) cannot be selected, as it is a bridge. Add E to π and remove (C, E) from E.
4.	π : ABCE	{E, F}	Other options are there.
5.	π : ABCEF	{F, H}	Other option is (H, I). We cannot select
6.	π : ABCEFH	{H, G}	(H, C), as it is a bridge.
7.	π : ABCEFHG	{G, E}	As in Sl. No. 2
8.	π : ABCEFHGE	{E, I}	As in Sl. No. 2
			Other options are also there. Edge (I, H) is a
9.	π : ABCEFHGEI	{I, J}	bridge.
10.	π : AFCEFHGEIJ	{J, K}	As in Sl. No. 2.
11.	π : ABCEFHGEIJK	{K, L}	As in Sl. No. 2
12.	π : ABCEFHGEIJKL	{L, I}	As in Sl. No. 2
13.	π : ABCEFHGEIJKLI	{I, H}	As in Sl. No. 2
14.	π : ABCEFHGEIJKLIH	{H, C}	As in Sl. No. 2
15.	π : ABCEFHGEIJKLIHC	{C, D}	As in Sl. No. 2
16.	π : ABCEFHGEIJKLIHCD	{D, A}	As in Sl. No. 2
17.	π : ABCEFHGEIJKLIHCDA		This is the Euler cycle

1.16 HAMILTONIAN GRAPHS

Hamiltonian graphs are named after Sir William Hamilton, an Irish mathematician who introduced the problems of finding a circuit in which all vertices of a graph appear exactly once.

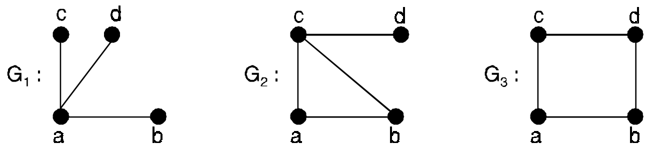
A circuit in a graph G that contains each vertex in G exactly once, except for the starting and ending vertex that **appears twice** is known as **Hamiltonian circuit.**

A graph G is called a **Hamiltonian graph,** if it contains a Hamiltonian circuit.

A Hamiltonian path is a simple path that contains all vertices of G where the end points may be distinct.

Note that an Eulerian circuit traverses every edge exactly once, but may repeat vertices, while a Hamiltonian circuit visists each vertex exactly once but may repeat edges. While there is a criterion for determining whether or not a graph contains an Eulerian circuit, a similar criterion does not exist for Hamiltonian ciruits.

In the following figures, hamiltonian path and cycles are shown :



The graph G_1 has no hamiltonian path (and so hamiltonian cycle), G_2 has hamiltonian path *abcd* but no hamiltonian cycle, while G_3 has hamiltonian cycle *abdca*.

The cycle C_n with n distinct (and n edges) is hamiltonian. Moreover given hamiltonian graph G then if G′ is a subgraph obtained by adding in new edges between vertices of G, G′ will also be hamiltonian. Since any hamiltonian cycle in G will also be hamiltonian cycle in G′. In particular k_n, the complete graph on n vertices, in such a supergraph of a cycle, k_n is hamiltonian.

A simple graph G is called maximal non-hamiltonian if it is not hamiltonian but the addition to it any edge connecting two non-adjacent vertices forms a hamiltonian graph. The graph G_2 is a maximal non-hamiltonian since the addition of an edge *bd* gives hamiltonian graph G_3.

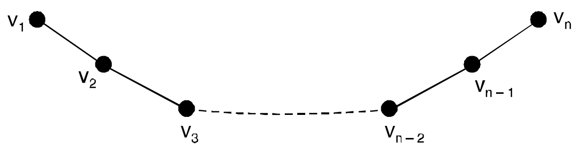
1.17 DIRAC'S THEOREM (1.23)

Let G be a graph of order $p \geq 3$. If $\deg v \geq \dfrac{p}{2}$ for every vertex v of G, then G is hamiltonian.

Proof. If $p = 3$, then the condition on G implies that $G \cong k_3$ and hence G is hamiltonian.

We may assume, therefore, that $p \geq 4$.

Let P : $v_1, v_2, \ldots\ldots\, v_n$ be a longest path in G (see Figure). Then every neighbour of v_1 and every neighbour of v_n is on P.



Otherwise, there would be a longer path than P.

Consequently, $n \geq 1 + \dfrac{p}{2}$.

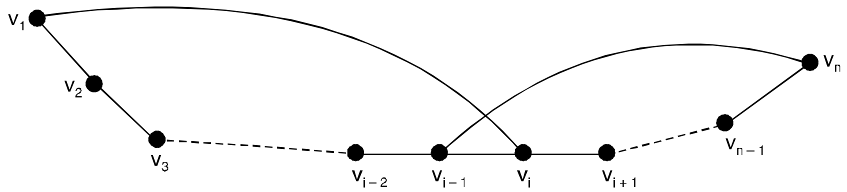
There must be some vertex v_i where $2 \leq i \leq n$, such that v_1 is adjacent to v_i and v_n is adjacent to v_{i-1}.

If this were not the case, then whenever v_1 is adjacent to a vertex v_i, the vertex v_n is not adjacent to v_{i-1}.

Since atleast $\dfrac{p}{2}$ of $p - 1$ vertices different from v_n are not adjacent to v_n.

Hence, $\deg v_n \leq (P - 1) - \dfrac{p}{2} < \dfrac{p}{2}$, which contradicts the fact that $\deg v_n \geq \dfrac{p}{2}$.

Therefore as we claimed, there must be a vertex v_i adjacent to v_1 and v_{i-1} is adjacent to v_n (see Figure).



We now see that G has cycle C : $v_1, v_i, v_{i+1}, \ldots\ldots v_{n-1}, v_n, v_{i-1}, v_{i-2}, \ldots\ldots, v_2, v_1$ that contains all the vertices of P.

If C contains all the vertices of G (if $n = p$) then C is a hamiltonian cycle, and the proof.

Otherwise, there is some vertex u of G that is not on C.

By hypothesis, deg $u \geq \frac{p}{2}$. Since P contains at least $1 + \frac{p}{2}$ vertices, there are fewer than $\frac{p}{2}$ vertices not on C ; so u must be adjacent to a vertex v that lies on C.

However, the edge uv plus the cycle C contain a path whose length is greater than that of P, which is impossible.

Thus C contains all vertices of G and G is hamiltonian.

Hence the proof.

Corollary :

Let G be a graph with p-vertices. If deg $v \geq \frac{p-1}{2}$ for every vertex v of G then G contains a hamiltonian path.

Proof. If $p = 1$ then G $\cong k_1$ and G contains a (trivial) Hamiltonian path.

Suppose then that $p \geq 2$ and define H = G + k_1.

Let v denote the vertex of H that is not in C.

Since H has vertex $p + 1$, it follows that deg $v \geq p$.

Moreover, for every vertex u of G,

$$\deg_H u = \deg_G u + 1 \geq \frac{p-1}{2} + 1 = \frac{p+1}{2} = \frac{|V(H)|}{2}.$$

By Dirac's theorem, H contains a hamiltonian cycle C. By removing the vertex v from C, we obtain a hamiltonian path in G.

Hence the proof.

Theorem 1.25. *If G is a connected graph of order three or more which is not hamiltonian, then the length k of a longest path of G satisfies $k \geq 2\delta(G)$.*

Proof. Let $p : u_0, u_1, \ldots\ldots, u_k$ be a longest path in G.

Since P is longest path, each of u_0 and u_k is adjacent only two vertices of P.

If $u_0u_i \in$ E(G), $1 \leq i \leq k$, then $u_{i-1} u_k \notin$ E(G) for otherwise the cycle C : $u_0, u_1, u_2, \ldots\ldots, u_{i-1}, u_k, u_{k-1}, u_{k-2}, \ldots\ldots, u_i, u_0$ of length $k + 1$ is present in G.

The cycle C cannot contain all vertices of G, since G is not Hamiltonian.

Therefore, there exists a vertex w not on C adjacent with a vertex of C, however this implies G contains a path of length $k + 1$, which is impossible.

Hence for each vertex of $\{u_1, u_2, ..., u_k\}$ adjacent to u_0 there is a vertex of $\{u_0, u_1, ..., u_{k-1}\}$ not adjacent with u_k.

Thus $\deg u_k \leq k - \deg u_0$ so that

$$k \geq \deg u_0 + \deg u_k \geq 2\delta(G).$$

Hence the proof.

Theorem 1.26. *Let G be a simple graph with n vertices and let u and v be an edge. Then G is hamiltonian if and only if G + uv is hamiltonian.*

Proof. Suppose G is hamiltonian. Then the super graph G + uv must also be hamiltonian.

Conversely, suppose taht G + uv is hamiltonian.

Then if G is not hamiltonian.

i.e., if G is a graph with $p \geq 3$ vertices such that for all non adjacent vertices u and v, $\deg u + \deg v \geq p$.

We obtain the inequality $\deg u + \deg v < n$.

However by hypothesis, $\deg u + \deg v \geq n$.

Hence G must be hamiltonian.

This completes the proof.

1.18 ORE'S THEOREM (1.27)

If G is a group with $p \geq 3$ vertices such that for all non adjacent vertices u and v, $\deg u + \deg v \geq p$, then G is hamiltonian.

Proof. Let k denotes the number of vertices of G whose degree does not exceed n,

$$\text{where} \quad 1 \leq n \leq \frac{p}{2}$$

These k vertices induce a subgraph H which is complete, for if any two vertices of H were not adjacent, there would exist two non adjacent vertices, the sum of whose degree is less than p.

This implies that $k \leq n + 1$. However $k \neq n + 1$, for otherwise each vertex of H is adjacent only two vertices of H, and if $u \in V(H)$ and $v \in V(G) - V(H)$, then $\deg u + \deg v \leq n + (p - n - 2) = p - 2$, which is a contradiction.

Further $k \neq n$; otherwise each vertex of H is adjacent to at most one vertex of G not in H.

However, since $k = n < \frac{p}{2}$, there exists a vertex $w \in V(G) - V(H)$ adjacent to no vertex of H. Then if $u \in V(H)$, $\deg u + \deg \omega \leq n + (p - n - 1) = p - 1$, which again a contradiction.

Therefore $k < n$, which implies that G satisfies the condition, so that G is Hamiltonian.

Hence the proof.

Problem 1.98. *Let G be a simple graph with n vertices and m edges where m is at least 3. If* $m \geq \frac{1}{2}(n-1)(n-2) + 2$.

Prove that G is Hamiltonian. Is the converse true ?

Solution. Let u and v be any two non-adjacent vertices in G.

Let x and y be their respective degrees.

If we delete u, v from G, we get a subgraph with $n - 2$ vertices.

If this subgraph has q edges then $q \leq \frac{1}{2}(n-2)(n-3)$.

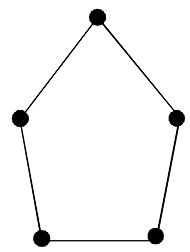
Since u and v are non-adjacent, $m = q + x + y$

Thus, $x + y = m - q \geq \left\{\frac{1}{2}(n-1)(n-2)+2\right\} - \left\{\frac{1}{2}(n-2)(n-3)\right\} = n$.

Therefore, the graph is Hamiltonian.

The converse of the result just proved is not always true.

Because, a 2-regular graph with 5-vertices (see Figure below) is Hamiltonian but the inequality does not hold.



Theorem 1.28. *In a complete graph* K_{2n+1} *there are n edge disjoint Hamiltonian cycles.*

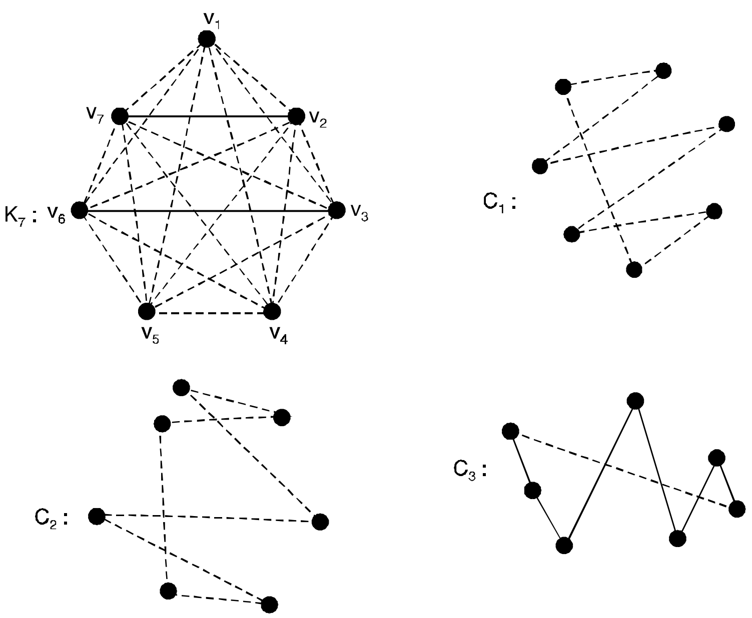
Proof. We first label the vertices of K_{2n+1} as $v_1, v_2, \ldots\ldots v_{2n+1}$ then we construct n paths $P_1, P_2, \ldots\ldots P_n$ on the vertices $v_1, v_2, \ldots\ldots, v_{2n}$ as follows :

$P_i = v_i v_{i-1} v_{i+1} v_{i-2}, \ldots\ldots, v_{i+n-1}, v_{i-n}, \quad 1 \leq i \leq n.$

We note that the jth vertex of P_i is v_k where $k = i + (-1)^{j+1}\left(\frac{j}{2}\right)$, and all subscripts are taken as the integers 1, 2, ..., $2n$ (mod $2n$).

The Hamiltonian cycle C_1 is got by joining v_{2n+1} to the end vertices of P_i.

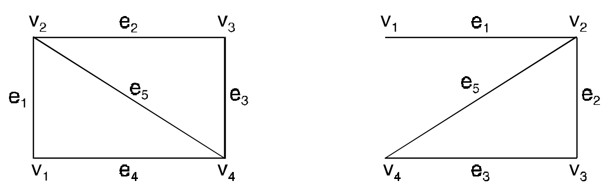
The Figure below illustrates the construction of Hamiltonian cycles in k_7.



It is still an open problem to find a convenient method to determine which graphs are Hamiltonian.

A graph G in which every edge is assigned a real number is called a weighted graph. The real number associated with an edge is called its weight, and the sum of the weights of the edges of G is called the weight of G.

Problem 1.99. *Which of the graphs given in Figure below is Hamiltonian circuit. Give the circuits on the graphs that contain them.*



(a) (b)

Solution. The graph shown in Figure (a) has Hamiltonian circuit given by $v_1e_1v_2e_2v_3e_3v_4e_4v_1$.

Note that all vertices appear in this a circuit but not all edges.

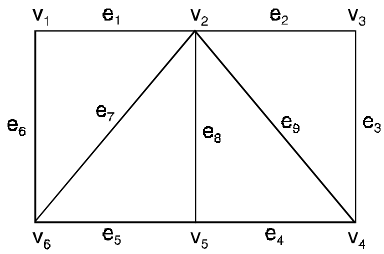
The edge e_5 is not used in the circuit.

The graph shown in Figure (b) does not contain circuit since every circuit containing every vertex must contain the e_1 twice.

But the graph does have a Hamiltonian path $v_1 - v_2 - v_3 - v_4$.

Problem 1.100. *Give an example of a graph which is Hamiltonian but not Eulerian and vice-versa.*

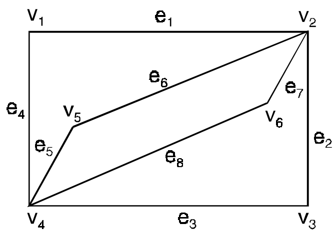
Solution. The following graph shown in Figure below is Hamiltonian but non-Eulerian.



The graph contains a Hamiltonian circuit $v_1e_1v_2e_2v_3e_3v_4e_4v_5e_5v_6e_6v_1$.

Since the degree of each vertex is not n even the graph is non-Eulerian.

The graph shown in Figure below is Eulerian but not Hamiltonian.



The graph is Eulerian since the degree of each vertex is even.

It does not contain Hamiltonian circuit.

This can be seen by noting any circuit containing every vertex must contain a vertex twice except starting vertex and ending vertex.

Problem 1.101. *Show that any k-regular simple graph with 2k – 1 vertices is Hamiltonian.*

Solution. In a k-regular graph, the degree of every vertex is k and $k > k - \frac{1}{2} = \frac{1}{2}(2k-1) = \frac{n}{2}$.

Where $n = 2k - 1$ is the number of vertices. Therefore, the graph considered is Hamiltonian.

Problem 1.102. *Prove that the complete graph k_n, $n \geq 3$ is a Hamiltonian graph.*

Solution. In k_n,the degree of every vertex is $n - 1$. If $n > 2$, we have $n - 2 > 0$ or $2n - 2 > n$ or $n - 1 > \frac{n}{2}$.

Thus, in k_n, where $n > 2$, the degree of every vertex is greater than $\frac{n}{2}$.

Hence k_n is Hamiltonian.

Theorem 1.29. *Let G be a simple graph on n vertices. If the sum of degrees of each pair of vertices in G is at least n – 1, then there exists a Hamiltonian path in G.*

Proof. We first prove that G is connected.

If not, then G contains at least two components say G_1 and G_2.

Let n_1 and n_2 be the number of vertices of G in the components G_1 and G_2.

Then $n_1 + n_2 \leq n$, the degree of a vertex x of G that is in the component G_1 is atmost $n_1 - 1$ and the degree of a vertex y of G that is in the component G_2 is atmost $n_2 - 1$.

Hence the sum of degrees of the vertices x and y of G is at most $(n_1 + n_2) - 2 \leq n - 2 < n - 1$, a contradiction.

Now we show the existence of the Hamiltonian path, by construction. The construction is as follows :

Step 1. Choose a vertex a of G.

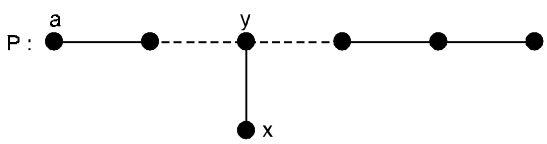
Step 2. Starting from 'a' construct a path P in G.

Step 3. If P is a Hamiltonian path stop, otherwise go to step 4.

Step 4. Extend the path on both the ends to the maximum (make P be a maximal path).

That is if x is a vertex of G adjacent to the end vertex of the path P and not in P, then includes the vertex to P with the corresponding edge and repeats the process. Call the path so obtained as P.

Step 5. If P is a Hamiltonian path then stop. Otherwise, we observe that there exists a vertex x in G that is not in P and adjacent to a vertex y in P (but y is not an end vertex of P).

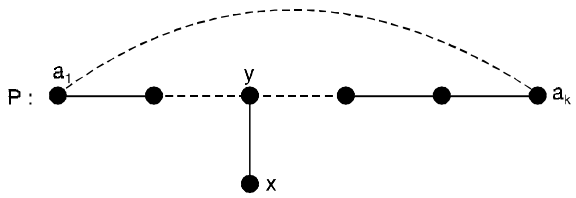


Step 6. Since P is maximal, no vertices of G which are not in P adjacent to the end vertex P.

The end vertices are adjacent to only those vertices in P.

Let P : $a = a_1, a_2, ..., a_k$. Then $k < n$ (otherwise, process stops at step 5).

If a_1 is adjacent to a_k, then obtain a circuit C by join a_1 and a_k, go to step 8. Otherwise, go to step 7 with the following observation.



We observe that there exist i, $1 \leq i \leq k$, such that a_1a_{i+1} and a_ia_k are the edges in G.

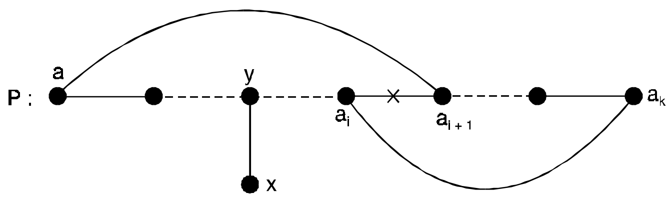
If not, then a_1 is not adjacent to any vertex a_{j+1} in P, which is adjacent to a_k.

But the vertices adjacent to a_k are only the vertices of P (follows by the construction of P), it follows that, if degree of a_k is m, then there are m vertices which are not adjacent to a_1 in P.

Thus, there are at most $k - m - 1$ vertices of P (since a_1 is not adjacent to a_1).

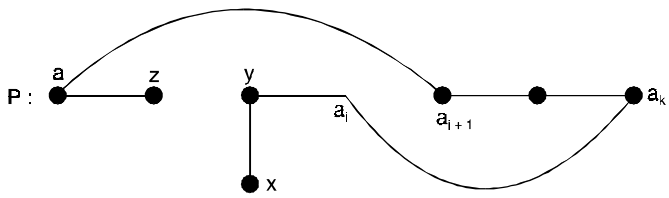
Hence degree of a_1 + degree of $a_k \leq (k - m - 1) + m = k - 1 < n - 1$, a contradiction to the assumption made in the statement of the theorem.

Step 7. Construct a circuit C by deleting an edge $a_i a_{i+1}$ in P and joining the edges $a_1 a_{i+1}$ and $a_i a_k$ to P.



Step 8. Join the edge between the vertex x of G and the vertex y in P (the vertices x and y are those vertices which are observed in step 5) to the circuit C. And delete an edge yz incident with y in C.

Step 9. Step 8 yields a path between the vertex x and the vertex z. This path contains one more vertex than the path P so far we have in our hand (*i.e.*, obtained in step 4) call this path as P and go to step 4.



Finally, we note that the process terminates as in each time we are getting a path on one more vertex (that is not in the earlier path) than the earlier path. Moreover, the final output is the desired Hamiltonian path.

Hence the proof.

Theorem 1.30. *In a complete graph with n-vertices there are* $\frac{n-1}{2}$ *edge-disjoint hamiltonian circuits, if n is an odd number* ≥ 3.

Proof : A complete graph with n vertices has $\frac{n(n-1)}{2}$ edges, and a hamiltonian circuit consists of n edges.

Therefore, the number of edge-disjoint hamiltonian circuits in G cannot exceed $\frac{(n-1)}{2}$.

This implies there are $\frac{n-1}{2}$ edge-disjoint hamiltonian circuits, when n is odd it can be shown as by keeping the vertices fixed on a circle, rotate the polygonal pattern clockwise by

$$\frac{360}{(n-1)}, \frac{2.360}{(n-1)}, \frac{3.360}{(n-1)}, \ldots\ldots, \frac{n-3}{2} \cdot \frac{360}{(n-1)} \text{ degrees.}$$

At each rotation we get a hamiltonian circuit that has no edge in common with any of the previous ones. Thus we have $\frac{n-3}{2}$ new hamiltonian circuits, all edges disjoint from one and also edge disjoint among themselves.

Hence the proof.

Problem 1.103. *Which of the simple graphs in Figure have a Hamilton circuit or, if not, a Hamilton path ?*

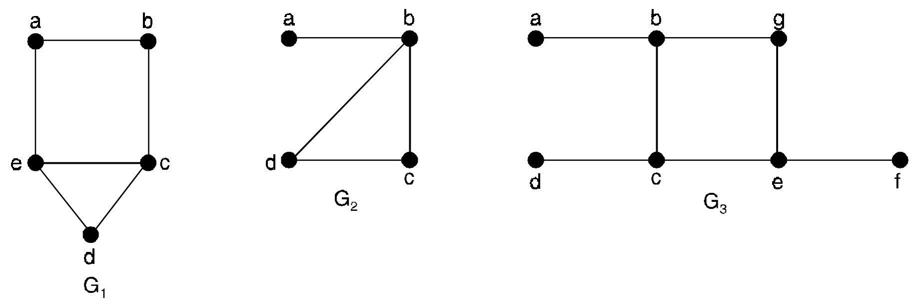


Fig. Three simple graphs.

Solution. G_1 has a Hamilton circuit : *a*, *b*, *c*, *d*, *e*, *a*.

There is no Hamilton circuit in G_2, but G_2 does have a Hamilton path, namely *a*, *b*, *c*, *d*. G_3 has neither a Hamilton circuit nor a Hamilton path, since any path containing all vertices must contain one of the edges {*a*, *b*} {*e*, *f*} and {*c*, *d*} more than once.

Problem 1.104. *Show that neither graph displayed in Figure has a Hamilton circuit.*

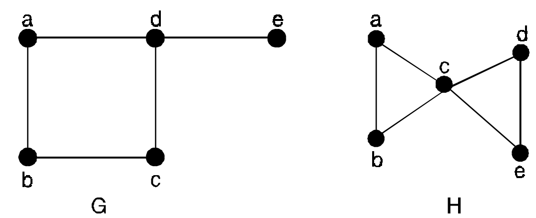
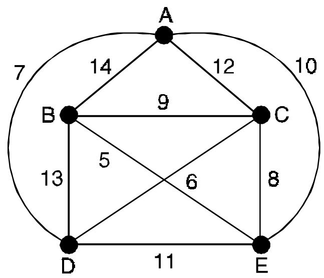


Fig. Two graphs that do not have a Hamilton circuit.

Solution. There is no Hamilton circuit in G since G has a vertex of degree one, namely, *e*. Now consider H. Since the degrees of the vertices *a*, *b*, *d* and *e* are all two, every edge incident with these vertices must be part of any Hamilton circuit. It is now easy to see that no Hamilton circuit can exist in H, for any Hamilton circuit would have to contain four edges incident with C, which is impossible.

Problem 1.105. *Find the minimum Hamiltonian circuit starting from node E in the graph of the Figure.*



Solution. We have to start with the node E. Closest node to E is the node B. Move to B. Now closest node to B is C move to C, extend path up to C and drop node B and all edges from it, from the graph. From C move to D.

From D, move to A and then to E back.

Finally, we have only node E left in the graph.

Thus, we have a Hamiltonian circuit in the graph, which is π : EBCDAE.

The total minimum of this circuit is :

$$EB + BC + CD + DA + EA = 5 + 9 + 6 + 7 + 10 = 37.$$

Problem 1.106. *At a committee meeting of 10 people, every member of the committee has previously sat next to at most four other members. Show that the members may be seated round a circular table in such a way that no one is next to some one they have previously sat beside.*

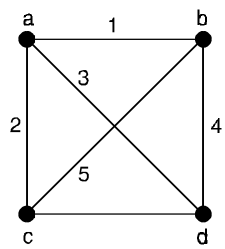
Solution. Consider a graph with 10 vertices representing the 10 members.

Let two vertices be joined by an edge if the corresponding members have not previously sat next to each other.

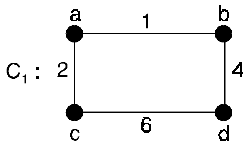
Since any member has not previously sat next to at least five members, the degree of every vertex is at least five.

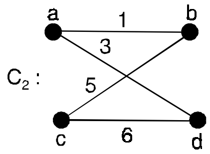
Therefore, the graph has a Hamiltonian circuit. This circuit provides a seating arrangement of the desired type.

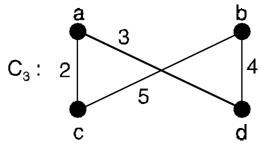
Problem 1.107. *Find three distinct Hamiltonian cycles in the following graph. Also find their weights.*



Solution. The cycles C_1, C_2 and C_3 are three distinct Hamiltonian cycles.







Weight of the cycle $C_1 = 1 + 4 + 6 + 2 = 13$.

Weight of the cycle $C_2 = 1 + 5 + 6 + 3 = 15$

Weight of the third cycle $C_3 = 3 + 4 + 5 + 2 = 14$

Hence the first cycle is of minimum weight.

Theorem 1.31. *A complete graph k_{2n} has a decomposition into n Hamiltonian paths.*

Proof. Consider a complete graph k_{2n}.

Now join a vertex x into K_{2n} and the edges $xv_i \; \forall \; i, \quad 1 \le i \le 2n$.

Then the graph G′ so obtained in K_{2n+1}.

Hence G′ can be decomposed into n Hamiltonian cycles.

Removal of the vertex x from each of these cycles we get n edge disjoint Hamiltonian paths which are the required decomposition of K_{2n}.

Theorem 1.32. *Let G be a connected graph with n vertices, n > 2, and no loops or multiple edges. G has a Hamiltonian circuit if for any two vertices u and v of G that are not adjacent, the degree of u plus the degree of v is greater than or equal to n.*

Corollary : G has a Hamiltonian circuit if each vertex has degree greater than or equal to $\frac{n}{2}$.

Proof. The sum of the degrees of any two vertices is at least $\frac{n}{2} + \frac{n}{2} = n$.

Theorem 1.33. *Let the number of edges of G be m. Then G has a Hamiltonian circuit if $m \ge \frac{1}{2}(n^2 - 3n + 6)$.*

(recall that n is the number of vertices)

Proof. Suppose that u and v are any two vertices of G that are not adjacent. We write deg (u) for the degree of u.

Let H be the graph produced by eliminating u and v from G along with any edges that have u or v as end points.

The H has $n - 2$ vertices and $m - \deg(u) - \deg(v)$ edges (one fewer edge would have been removed if u and v had been adjacent).

The maximum number of edges that H could possibly have is ${}_{n-2}C_2$.

This happens when there is an edge connecting every distinct pair of vertices.

Thus the number of edges of H is at most

$$ {}_{n-2}C_2 = \frac{(n-2)(n-3)}{2} \quad \text{or} \quad \frac{1}{2}(n^2 - 5n + 6) $$

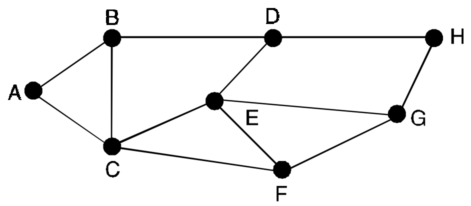
We then have $m - \deg(u) - \deg(v) \leq \frac{1}{2}(n^2 - 5n + 6)$.

Therefore, $\deg(u) + \deg(v) \geq m - \frac{1}{2}(n^2 - 5n + 6)$

By the hypothesis of the theorem,

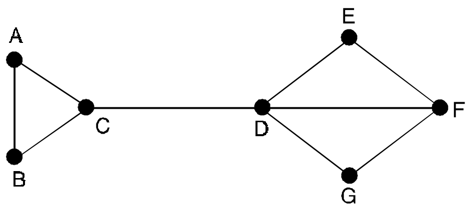
$$\deg(u) + \deg(v) \geq \frac{1}{2}(n^2 - 3n + 6) - \frac{1}{2}(n^2 - 5n + 6) = n.$$

Problem 1.108. *Determine whether a Hamiltonian path or circuit exists in the graph of Figure.*



Solution. Let us take node A to start with. Next, move to either B or C, say B. Extend the path upto B. Next move to D and not to C as a cycle of length 3 could be formed here. Extend the path upto D and drop node B and edges (B, A), (B, C) and (B, D). Then move to H. Drop D and edges from it. Then move to G, then to F (or E) then to E (or F), then to C and finally to A dropping the nodes and edges from them on the way. At the end, only one node A is left with degree zero and π is ABDHGFECA. This is a Hamiltonian cycle.

Problem 1.109. *Determine whether a Hamiltonian path or circuit exists in the graph of Figure.*

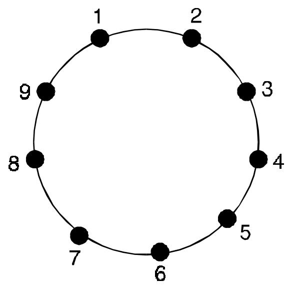


Solution. Let us start with the node A. We can select any one but node C and D. Initialize the path π : A. Next move to the node B we cannot move to C from A. Because any move to D from B and to D from B need node C. Extend the path upto B. Then move to node C, extend the path upto C and drop node B together with edges (B, A) and (B, C). We have now the path π : ABC. Now move to D, extend the path upto D and drop node C together with arcs (C, A) and (C, D). Then move to either node G or E but not to F. Extend the path and do the rest. Finally, proceeding in this way, we get π : ABCDEFG. And two nodes A and G, with degree zero, are left. Thus, this graph has a Hamiltonian path π but no Hamiltonian circuit.

1.19 PROBLEM OF SEATING ARRANGEMENT 1.109

Nine members of a club meet every day for a dinner. They sit in a round table for a dinner, but no two members who sat together will sit together in future. How long is this possible ?

Solution. The seating arrangement can be represented as follows :



Any two numbers can occupy consecutive tables. The neighbouring persons can be represented by an edge. Then each arrangements is a cycle on 9 vertices. These cycles can be chosen from k_9 (since each member can sit with anybody in the beginning).

Thus distinct arrangements as they desired are the edge disjoint (no edges should re-appear, *i.e.,* none of the persons sitting together will sit together in next arrangements) Hamiltonian cycles of K_9, which is possible only for four days (as $9 = 2n + 1 \quad \Rightarrow \quad n = 4$). However this is also possible for 10 members for 4 days only.

If we consider a bench instead of a round table, then for 10 members it is possible for 5 days. (Hamiltonian paths of k_{10}). What can you say about the same situation for nine members.

1.20 TRAVELING-SALESMAN PROBLEM

The traveling-salesman problem, stated as follows :

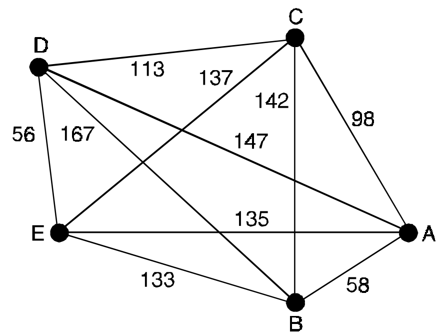
''A salesman is required to visit a number of cities during a trip. Given the distances between the cities, in what order should he travel so as to visit every city precisely once and return home, with the minimum mileage traveled ?''Representing the cities by vertices and the roads between them by edges. We get a graph. In this graph, with every edge e_i there is associated a real number (the distance in miles, say), $w(e_i)$. Such a graph is called a weighted graph ; $w(e_i)$ being the weight of edge e_i.

(*i.e.,* A traveling salesman wants to visit each of n cities exactly once and return to his starting point) if each of the cities has a road to very other city, we have a complete weighted grap.

For example, suppose that the salesman wants to visit five cities, namely, A, B, C, D and E (see Figure). In which order should he visit these cities to travel the minimum total distance ? To solve this problem we can assume the salesman starts in A (since this must be part of the circuit) and examine all possible ways for him to visit the other four cities and then return to A (starting elsewhere will produce the same circuits). There are a total of 24 such circuits, but since we travel the same distance when we travel a circuit in reverse order, we need only consider 12 different circuits to find the minimum total distance he must travel. We list these 12 different circuits and the total distance traveled for each circuit.

As can be seen from the list, the minimum total distance of 458 miles is traveled using the circuit A – B – E – D – C – A (or its reverse).

Route	*Total Distance (miles)*
A – B – D – C – E – A	610
A – B – D – E – C – A	516
A – B – E – C – D – A	588
A – B – E – D – C – A	458
A – B – C – E – D – A	540
A – B – C – D – E – A	504
A – C – B – D – E – A	598
A – C – B – E – D – A	576
A – C – E – B – D – A	682
A – C – D – B – E – A	646
A – D – C – B – E – A	670
A – D – B – C – E – A	728



The graph showing the distance between five cities (A, B, C, D, E)

The traveling salesman problem asks for the circuit of minimum total weight in a weighted, complete, undirected graph that visits each vertex exactly once and returns to its starting point. This is equivalent to asking for a Hamilton circuit with minimum total weight in the complete graph, since each vertex is visited exactly once in the circuit.

The most straight forward way to solve an instance of the traveling salesman problem is to examine all possible Hamilton circuits and select one of minimum total length.

How many circuits do we have to examine to solve the problem if there are n vertices in the graph ? Once a starting point is chosen, there are $(n – 1)$! different Hamilton circuits to examine, since there are $n – 1$ choices for the second vertex, $n – 2$ choices for the third vertex, and so on.

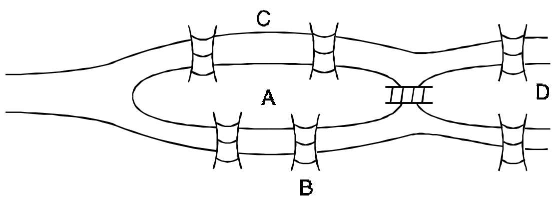
Since a Hamilton circuit can be traveled in reverse order, we need only examine $\frac{(n-1)!}{2}$ circuits to find our answer.

Note that $\frac{(n-1)!}{2}$ grows extremely rapidly. Trying to solve a traveling salesman problem in this way when there are only a few dozen vertices is impractical.

For example, with 25 vertices, a total of $\frac{24!}{2}$ (approximately 3.1×10^{23}) different Hamilton circuits would have to be considered.

If it took just one nanosecond (10^{-9} second) to examine each Hamilton circuit, a total of approximately ten million year would be required to find a minimum-length Hamilton circuit in this graph by exhaustive search techniques.

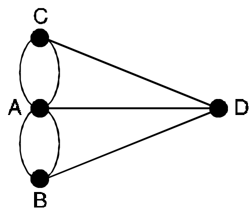
1.21 KÖNIGSBERG'S BRIDGE PROBLEM



There were two islands linked to each other to the bank of the Pregel river by seven bridges as shown above.

The problem was to begin at any of the four land areas, walk across each bridge exactly once and return to the starting point.

One can easily try to solve this problem, but all attempts must be unsuccessful. In proving that, the problem is unsolvable. Euler replaced each land area by a vertex and each bridge by an edge joining the corresponding vertices, there by producing a 'graph' as shown below :



1.22 REPRESENTATION OF GRAPHS

Although a diagrammatic representation of a graph is very convenient for a visual study but this is only possible when the number of nodes and edges is reasonably small.

Two types of representation are given below :

1.22.1. Matrix representation

The matrix are commonly used to represent graphs for computer processing. The advantages of representing the graph in matrix form lies on the fact that many results of matrix algebra can be readily applied to study the structural properties of graphs from an algebraic point of view. There are number of matrices which one can associate witch any graph. We shall discuss adjacency matrix and the incidence matrix.

1.22.2. Adjacency matrix

1.22.2. (*a*) Representation of undirected graph

The adjacency matrix of a graph G with n vertices and no parallel edges is an n by n matrix A = $\{a_{ij}\}$ whose elements are given by a_{ij} = 1, if there is an edge between ith and jth vertices, and

= O, if there is no edge between them.

Note that for a given graph, the adjacency matrix is based on ordering chosen for the vertices.

Hence, there are as many as n ! different adjacency matrices for a graph with n vertices, since there are n ! different ordering of n vertices.

However, any two such adjacency matrices are closely related in that one can be obtained from the other by simply interchanging rows and columns.

There are a number of observations that one can make about the adjacency matrix A of a graph G are :

Observations :

(*i*) A is symmetric *i.e.* $a_{ij} = a_{ji}$ for all i and j

(*ii*) The entries along the principal diagonal of A all zeros if and only if the graph has no self loops. A self loop at the vertex corresponding to a_{ij} = 1.

(*iii*) If the graph is simple (no self loop, no parallel edges), the degree of vertex equals the number of 1's in the corresponding row or column of A.

(*iv*) The (i, j) entry of A^m is the number of paths of length (no. of occurence of edges) m from vertex v_i vertex v_j.

(*v*) If G be a graph with n vertices v_1, v_2, ... v_n and let A denote the adjacency matrix of G with respect to this listing of the vertices. Let B be the matrix.

$$B = A + A^2 + A^3 + ... + A^{n-1}$$

Then G is a connected graph if B has no zero entries of the main diagonal.

This result can be also used to check the connectedness of a graph by using its adjacency matrix.

Adjacency can also be used to represent undirected graphs with loops and multiple edges. A loop at the vertex v_1 is represented by a 1 at the (i, j)th position of the adjacency matrix. When multiple edges are present, the adjacency matrix is no longer a zero-one matrix, since the (i, j)th entry equals the number of edges these are associated to $\{v_i - v_j\}$.

All undirected graphs, including multigraphs and pseudographs, have symmetric adjacency matrices.

1.22.2 (*b*) Representation of directed graph

The adjacency matrix of a diagonal D, with n vertices is the matrix A = $\{a_{ij}\}_{n \times n}$ in which

$$a_{ij} = 1 \text{ if arc } \{v_i - v_j\} \text{ is in D}$$
$$= 0 \text{ otherwise.}$$

One can make a number of observations about the adjacency matrix of a diagonal.

Observations

(*i*) A is not necessary symmetric, since there may not be an edges from v_i to v_j when there is an edge from v_i to v_j.

(*ii*) The sum of any column of j of A is equal to the number of arcs directed towards v_j.

(*iii*) The sum of entries in row i is equal to the number of arcs directed away from vertex v_i (out degree of vertex v_i)

(*iv*) The (i, j) entry of A^m is equal to the number of path of length m from vertex v_i to vertex v_j entries of A^T. A shows the in degree of the vertices.

The adjacency matrices can also be used to represent directed multigraphs. Again such matrices are not zero-one matrices when there are multiple edges in the same direction connecting two vertices.

In the adjacency matrix for a directed multigraph a_{ij} equals the number of edges that are associated to (v_i, v_j).

1.22.3. Incidence matrix

1.22.3. (*a*) Representation of undirected graph

Consider a undirected graph G = (V, E) which has n vertices and m edges all labelled. The incidence matrix $B = \{b_{ij}\}$, is then $n \times m$ matrix,

where $b_{ij} = 1$ when edge e_j is incident with v_i

$= 0$ otherwise

We can make a number of observations about the incidence matrix B of G.

Observations :

(*i*) Each column of B comprises exactly two unit entries.

(*ii*) A row with all O entries corresponds to an isolated vertex.

(*iii*) A row with a single unit entry corresponds to a pendent vertex.

(*iv*) The number of unit entries in row i of B is equal to the degree of the corresponding vertex v_i.

(*v*) The permutation of any two rows (any two columns) of B corresponds to a labelling of the vertices (edges) of G.

(*vi*) Two graphs are isomorphic if and only if their corresponding incidence matrices differ only by a permutation of rows and columns.

(*vii*) If G is connected with n vertices then the rank of B is $n - 1$.

Incidence matrices can also be used to represent multiple edges and loops. Multiple edges are represented in the incidence matrix using columns with identical entries. Since these edges are incident with the same pair of vertices. Loops are represented using a column with exactly one entry equal to 1, corresponding to the vertex that is incident with this loop.

1.22.3. (*b*) Representation of directed graph

The incidence matrix $B = \{b_{ij}\}$ of digraph D with n vertices and m edges is the $n \times m$ matrix in which

$b_{ij} = 1$ if arc j is directed away from a vertex v_i

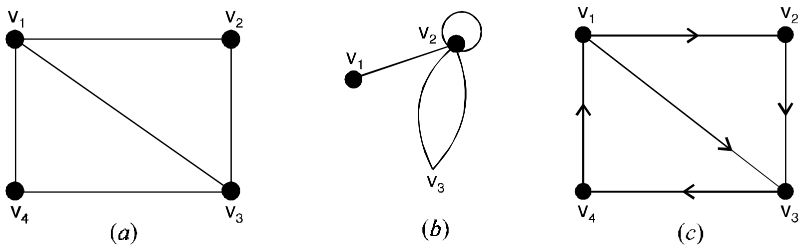
$= -1$ if arc j directed towards vertex v_i

$= 0$ otherwise.

1.22.4. Linked representation

In this representation, a list of vertices adjacent to each vertex is maintained. This representation is also called adjacency structure representation. In case of a directed graph, a case has to be taken, according to the direction of an edge, while placing a vertex in the adjacent structure representation of another vertex.

Problem 1.110. *Use adjacency matrix to represent the graphs shown in Figure below*



Solution. We order the vertices in Figure (1)(*a*) as v_1, v_2, v_3 and v_4.

Since there are four vertices, the adjacency matrix representing the graph will be a square matrix of order four. The required adjacency matrix A is

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

We order the vertices in Figure (1)(*b*) as v_1, v_2 and v_3. The adjacency matrix representing the graph is given by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

Taking the order of the vertices in Figure(1)(*c*) as v_1, v_2, v_3 and v_4. The adjacency matrix representing the graph is given by

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

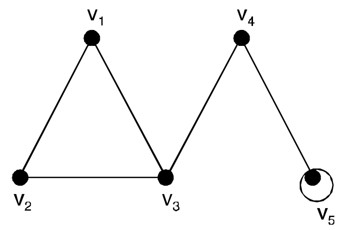
Problem 1.111. *Draw the undirected graph represented by adjacency matrix A given by*

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Solution. Since the given matrix is a square of order 5, the graph G has five vertices v_1, v_2, v_3, v_4 and v_5.

Draw an edge from v_i to v_j where $a_{ij} = 1$.

The required graph is drawn in Figure below.

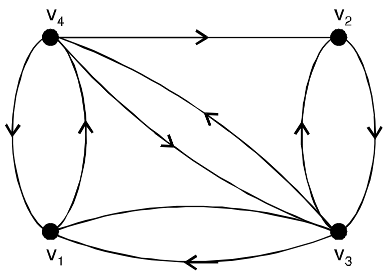


Problem 1.112. *Draw the digraph G corresponding to adjacency matrix*

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Solution. Since the given matrix is square matrix of order four, the graph G has 4 vertices v_1, v_2, v_3 and v_4. Draw an edge from v_i to v_j where $a_{ij} = 1$.

The required graph is shown in Figure below.



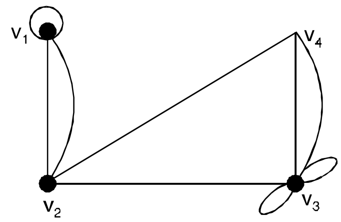
Problem 1.113. *Draw the undirected graph G corresponding to adjacency matrix*

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

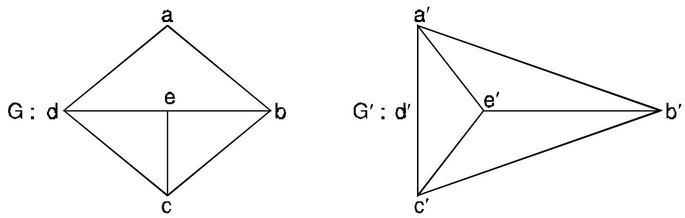
Solution. Since the given adjacency matrix is square matrix of order 4, G has four vertices v_1, v_2, v_3 and v_4. Draw n edges from v_i to v_j where $a_{ij} = n$.

Also draw n loop at v_i where $a_{ij} = n$.

The required graph is shown in Figure below.



Problem 1.114. *Show that the graphs G and G ′ are isomorphic*



Solution. Consider the map $f: G \rightarrow G'$ define as $f(a) = d'$, $f(b) = a'$, $f(c) = b'$, $f(d) = c'$ and $f(e) = e'$.

The adjacency matrix of G for the ordering *a*, *b*, *c*, *d* and *e* is

$$A(G) = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

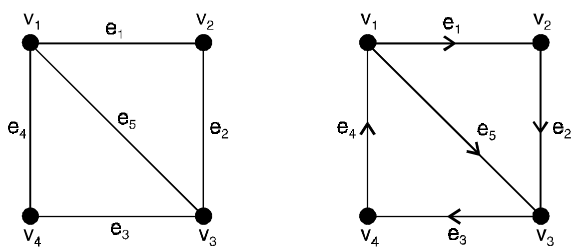
The adjacency matrix of G′ for the ordering *d*′, *a*′, *b*′, *c*′ and *e*′ is

$$A(G') = \begin{array}{c} \\ d' \\ a' \\ b' \\ c' \\ e' \end{array} \begin{array}{c} \begin{array}{ccccc} d' & a' & b' & c' & e' \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

i.e., A(G) = A(G′)

∴ G and G′ are isomorphic.

Problem 1.115. *Find the incidence matrix to represent the graph shown in Figure below :*



(*a*) (*b*)

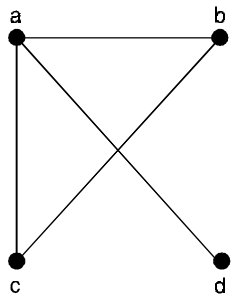
Solution. The incidence matrix of Figure (*a*) is obtained by entering for row *v* and column *e* is 1 if *e* is incident on *v* and 0 otherwise. The incidence matrix is

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}\begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

The incidence matrix of the graph of Figure (*b*) is

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix}$$

Problem 1.116. *Use an adjacency matrix to represent the graph shown in Figure below :*



Solution. We order the vertices as *a*, *b*, *c*, *d*.

The matrix representing this graph is

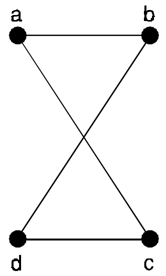
$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Problem 1.117. *Draw a graph with the adjacency matrix*

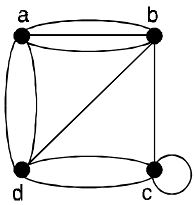
$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

with respect to the ordering of vertices a, b, c, d.

Solution. A graph with this adjacency matrix is shown in Figure below :



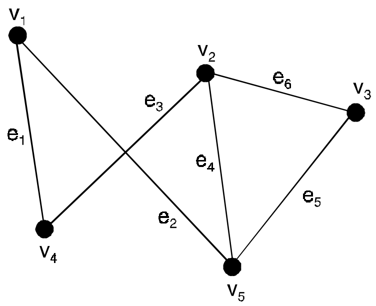
Problem 1.118. *Use an adjacency matrix to represent the pseudograph shown in Figure below :*



Solution. The adjacency matrix using the ordering of vertices *a*, *b*, *c*, *d* is

$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

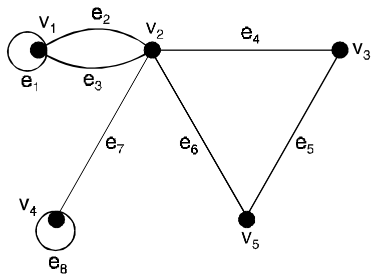
Problem 1.119. *Represent the graph shown in Figure below, with an incidence matrix.*



Solution. The incidence matrix is

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

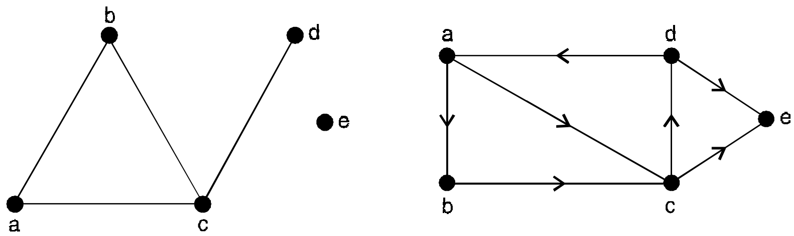
Problem 1.120. *Represent the Pseudograph shown in Figure below, using an incidence matrix.*



Solution. The incidence matrix for this graph is :

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Problem 1.121. *Write adjacency structure for the graphs shown in Figure (1)*



(a) (b)

Solution. The adjacency structure representation is given in the table for Figure (a). Here the symbol ϕ is used to denote the empty list.

Vertex	*Adjacency list*
a	*b*, *c*
b	*a*, *c*
c	*a*, *b*, *d*
d	*e*
e	ϕ

The adjacency structure representation is given in the table for the directed graph shown in Figure (*b*).

Vertex	*Adjacency list*
a	*b*, *c*
b	*c*
c	*d*
d	*a*, *e*
e	*c*

Problem Set 1.1

1. How many vertices do the following graphs have if they contain

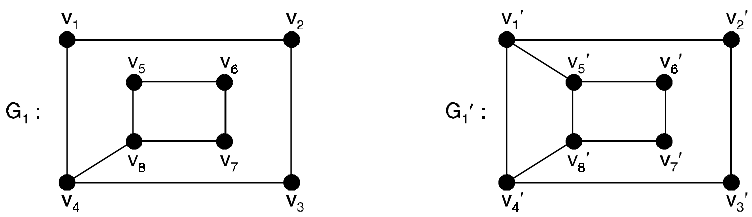
(*i*) 16 edges and all vertices of degree 2

(*ii*) 21 edges, 3 vertices of degree 4 and others each of degree 3.

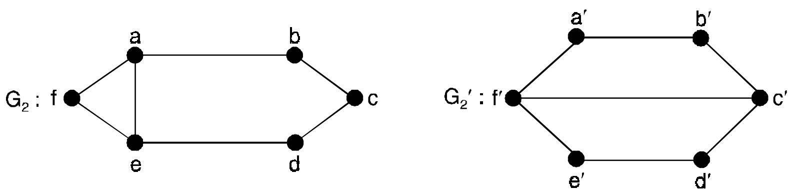
2. Suppose a graph has vertices of degree 0, 2, 2, 3 and 9. How many edges does the graph have ?

3. Determine whether the following graphs are isomorphic

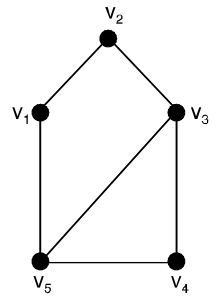
(*i*)

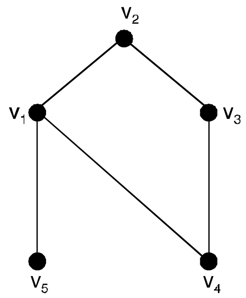


(*ii*)



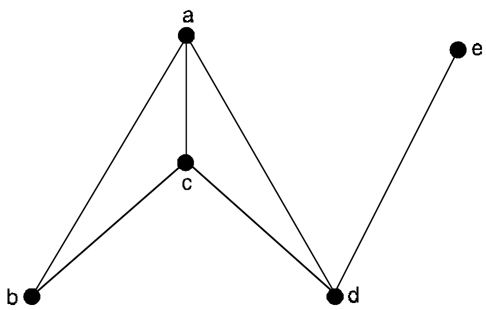
4. Show that graphs are not isomorphic

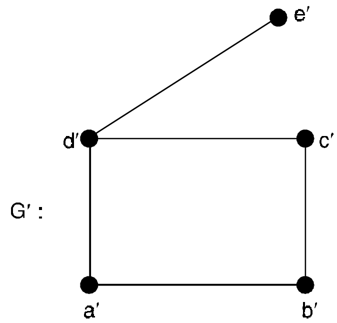




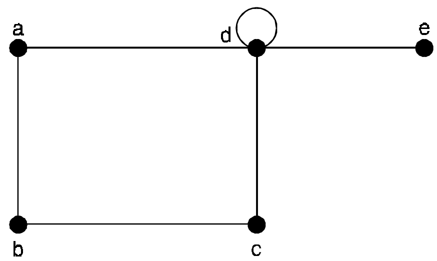
5. Show that the following graphs are isomorphic

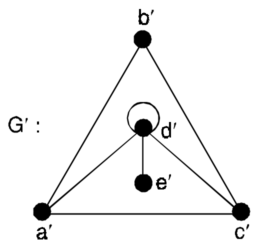
(*i*) G :





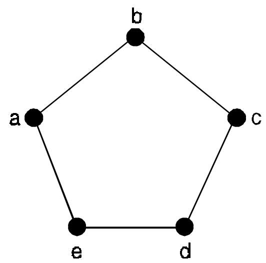
(*ii*) G :

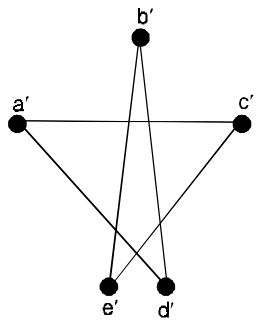




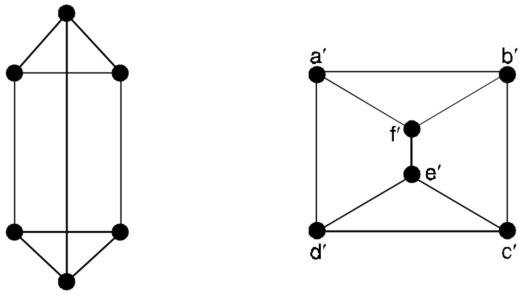
6. Show that the given pairs of graphs are isomorphic

(*i*)



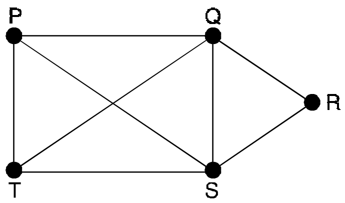


(*ii*)

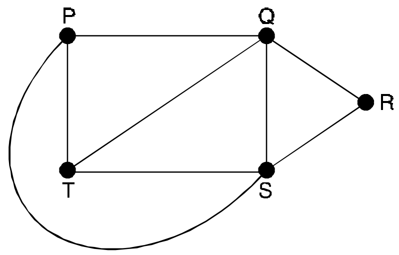


7. Write down the number of vertices, the number of edges, and the degree of each vertex, in

(*i*) the graph in Fig. (*a*) (*ii*) the tree in Fig. (*b*).



(*a*)

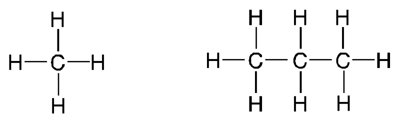


(*b*)

8. Figure below represents the chemical molecules of methane (CH_4) and propane (C_3H_8).

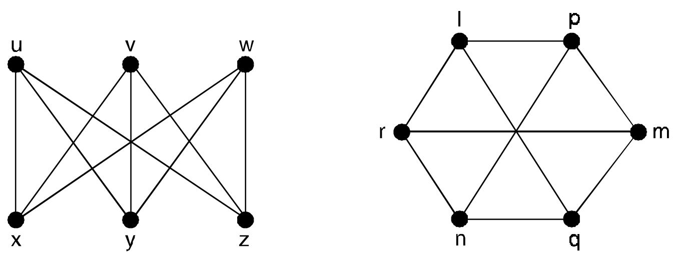
(*i*) Regarding these diagrams as graphs, what can you say about the vertices representing carbon atoms (C) and hydrogen atoms (H) ?

(*ii*) There are two different chemical molecules with formula C_4H_{10}. Draw the graphs corresponding to these molecules.



Methane Propane

9. Write down the vertex set and edge set of each graph in Figure below :



10. Draw (*i*) a simple graph,
(*ii*) a non-simple graph with no loops,
(*iii*) a non-simple graph with no multiple edges, each with five vertices and eight edges.

11. (*i*) Show that there are exactly $2^{n(n-1)/2}$ labelled simple graphs on n vertices
(*ii*) How many of these have exactly m edges ?

12. (*i*) By suitably labelling the vertices, show that the two graphs in Fig. (*a*) are isomorphic
(*ii*) Explain why the two graphs in Fig. (*b*) are not isomorphic.

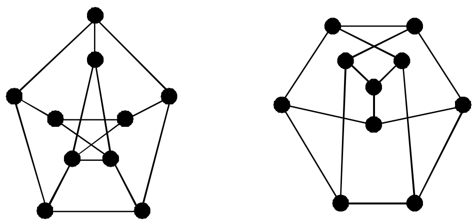


Fig. (*a*)

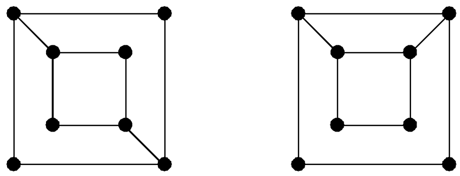
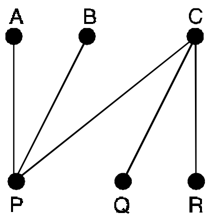
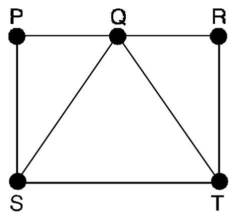


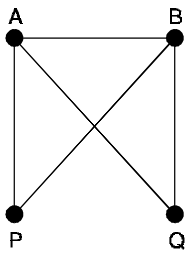
Fig. (*b*)

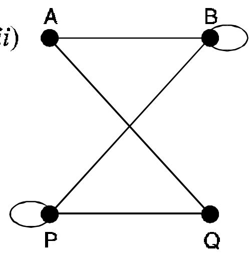
13. For the graphs shown below, indicate the number of vertices, the number of edges and the degrees of vertices.

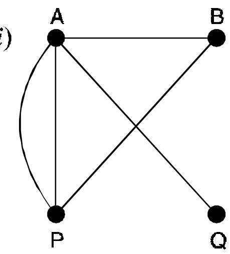
(*i*)


(*ii*)


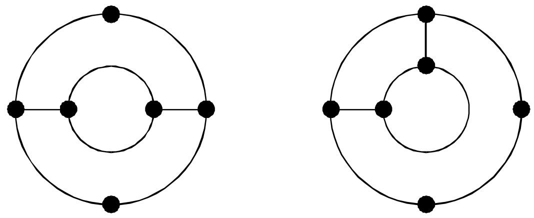
14. Describe the graphs shown below :

(*i*)


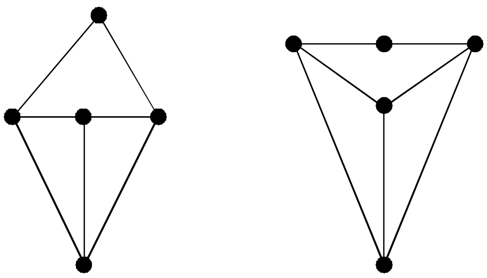
(*ii*)


(*iii*)


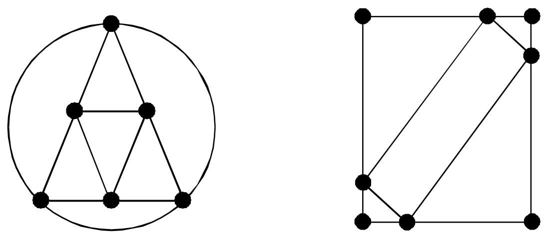
15. Show that the following graphs are not isomorphic :



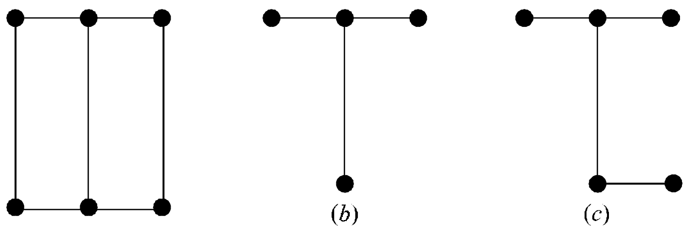
16. Verify that the following graphs are isomorphic :



17. Show that the following graphs are not isomorphic :



18. Three graphs G_1, G_2, G_3 are shown in Figure (*a*), (*b*), (*c*) respectively. Is G_1 a supergraph of G_2 and G_3 ?



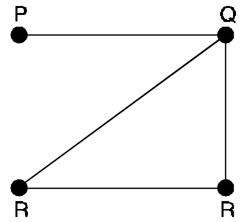
(*a*) (*b*) (*c*)

19. Let G be the graph shown in Figure below. Verify whether H = (V′, E′) is a subgraph of G in the following cases :

(*i*) V′ = {P, Q, S}, E′ = {(P, Q), (P, S)}

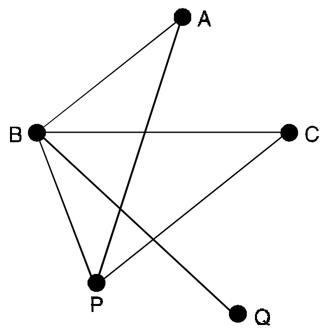
(*ii*) V′ = {Q}, E′ = ϕ, the null set

(*iii*) V′ = {P, Q, R}, E′ = {(P, Q), (Q, R), (Q, S)}



20. For the graph shown in the following Figure, find the nature of the following sequence :

(*i*) BAPCB (*ii*) PABQ (*iii*) CBAPBQ.



21. Prove that the edge set of every closed walk can be partitioned into pairwise edge-disjoint circuits.

22. Show that in a graph with n vertices, the length of a path cannot exceed $n - 1$ and the length of a circuit cannot exceed n.

23. Prove that if u is an odd vertex in a graph G then there must be a path in G from u to another odd vertex v in G.

24. In a graph G, let P_1 and P_2 be two different paths between two given vertices. Prove that G has a circuit in it.

25. Suppose G_1 and G_2 are isomorphic. Prove that if G_1 is connected then G_2 is also connected.

26. Prove that any two simple connected graphs with n vertices, all of degree two, are isomorphic.

27. Show that if G is a connected graph in which every vertex has degree either 1 or 0 then G is either a path or a cycle.

28. Let G be a graph with 15 vertices and 4 components. Prove that atleast one component of G has atleast 4 vertices.

29. If G is a simple graph with n vertices and k components, prove that G has atleast $n - k$ number of edges.

30. Prove that a connected graph of order n contains exactly one circuit if and only if its size is also n.

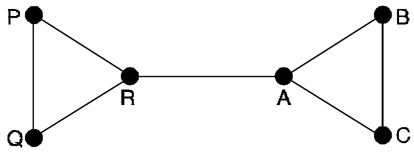
31. Let G be a simple graph. Show that if G is not connected then its complement $\overline{G}$ is connected.

32. Prove that if a connected graph G is decomposed into two subgraphs H_1 and H_2, there must be atleast one vertex common to H_1 and H_2.

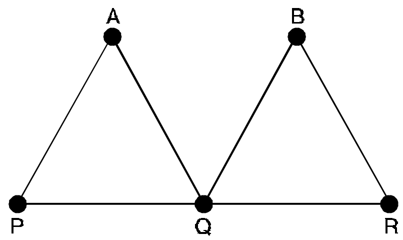
33. Prove that a connected graph is semi-Eulerian if and only if it has exactly zero or two vertices of odd degree.

34. Prove that the Petersen graph is neither Eulerian nor semi-Eulerian.

35. Show that the following graph is not Eulerian :



36. Show that the following graph is Eulerian :

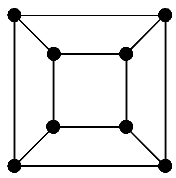


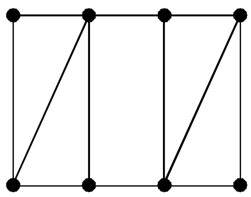
37. Show that the complete graph K_n contains $\frac{1}{2}(n-1)!$ different Hamiltonian circuits.

38. Prove that, if G is a bipartite graph with an odd number of vertices then G is non-Hamiltonian.

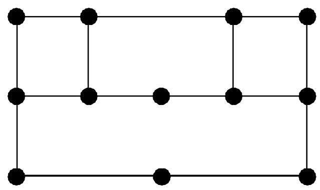
39. If the degree of each vertex of a simple graph is atleast $\frac{(n-1)}{2}$, where n is the number of vertices, show that the graph has a Hamiltonian path.

40. Show that the following graphs are Hamiltonian but not Eulerian.

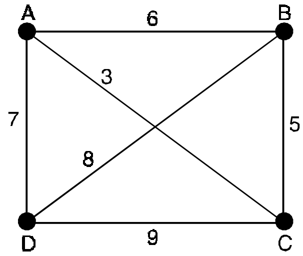




41. Show that the following graph is Hamiltonian.



42. Solve the travelling salesman problem for the weighted graph shown below :



Answers 1.1

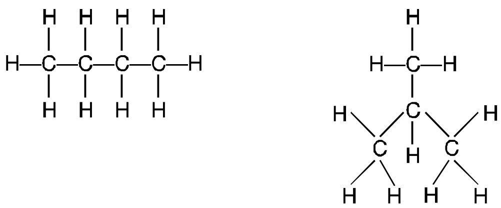
1. (*i*) 16 (*ii*) 13 **2.** 8 **3.** (*i*) Not isomorphic (*ii*) Not isomorphic

7. (*i*) There are 5 vertices and 8 edges ; vertices P and T have degree 3, vertices Q and S have degree 4, and vertex R has degree 2.

(*ii*)There are 6 vertices and 5 edges ; vertices A, B, E and F have degree 1 and vertices C and D have degree 3.

8. (*i*) Each carbon atom vertex has degree 4 and each hydrogen atom vertex has degree 1.

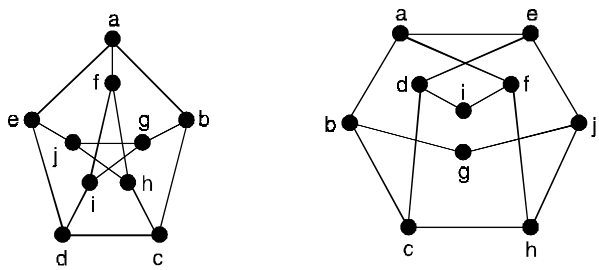
(*ii*) The graphs are as follows :



9. V(G) = {*u, v, w, x, y, z*}, E(G) = {*ux, uy, uz, vx, vy, vz, wx, wy, wz*} ;

V(G) = {*l, m, n, p, q, r*}, E(G) = {*lp, lq, lr, mp, mq, mr, np, nq, nr*}.

12. (*i*) We can label the vertices as follows :



(*ii*) In the first graph, no vertices of degree 2 are adjacent, in the second graph they are adjacent in pairs, since isomorphism preserves adjacency of vertices, the graphs are not isomorphic.

13. (*i*) There are 6 vertices and 5 edges ; vertices A, B, Q, R are pendant vertices and vertices C and P have degree 3.

(*ii*) There are 5 vertices and 7 edges ; vertices P and Q have degree 2, S and T have degree 3 and Q has degree 4.

14. (*i*) This is a simple graph with four vertices and five edges. Vertices A and B are of degree 3 and vertices P, Q are of degree 2.

(*ii*) This is a general graph with four vertices and six edges, of which two are self-loops. The vertices A and Q are of degree 2, and B and P are of degree 4.

(*iii*) This is a multigraph with four vertices and five edges. There are parallel edges joining A and P. The degree of A is 4, degree of P is 3, degree of B is 2 and Q is a pendant vertex.

18. Yes **19.** (*i*) No (*ii*) Yes (*iii*) No.

20. (*i*) Circuit (*ii*) Path (*iii*) Open walk which is not a path.

35. Starting with any vertex, it is not possible to return to that vertex without traversing the edge RA twice.

36. The graph contains an Euler line : PAQBRQP.

42. Circuit of least weight : ADBCA ; least total weight 23.

CHAPTER

2

Planar Graphs

INTRODUCTION

In this section we will study the question of whether a graph can be drawn in the plane without edges crossing. In particular, we will answer the houses-and-utilities problem. There are always many ways to represent a graph. When is it possible to find atleast one way to represent this graph in a plane without any edges crossing. Consider the problem of joining three houses to each of three separate utilities, as shown in figure below. Is it possible to join these houses and utilities so that none of the connections cross ? This problem can be modeled using the complete bipartite graph $K_{3,\,3}$. The original question can be rephrased as : can $K_{3,\,3}$ be drawn in the plane so that no two of its edges cross ?

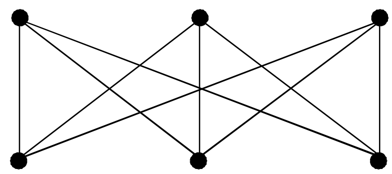


Fig. 2.1. Three houses and three utilities.

2.1 COMBINATORIAL AND GEOMETRIC GRAPHS (REPRESENTATION)

An abstract graph G can be defined as G = (V, E, Ψ) where the set V consists of the five objects named *a, b, c, d* and *e*, that is, V = {*a, b, c, d, e*} and the set E consists of seven objects (none of which is in set V) named 1, 2, 3, 4, 5, 6 and 7, that is,

$$E = \{1, 2, 3, 4, 5, 6, 7\}$$

and the relationship between the two sets is defined by the mapping Ψ, which consists of combinatorial representation of the graph.

$$\Psi = \begin{bmatrix} 1 \longrightarrow (a, c) \\ 2 \longrightarrow (c, d) \\ 3 \longrightarrow (a, d) \\ 4 \longrightarrow (a, b) \\ 5 \longrightarrow (b, d) \\ 6 \longrightarrow (d, e) \\ 7 \longrightarrow (b, e) \end{bmatrix} \longrightarrow \text{Combinatorial representation of a graph}$$

Here, the symbol $1 \longrightarrow (a, c)$ says that object 1 from set E is mapped onto the (unordered) pair (a, c) of objects from set V.

It can be represented by means of geometric figure as shown below. It is true that graph can be represented by means of such configuration.

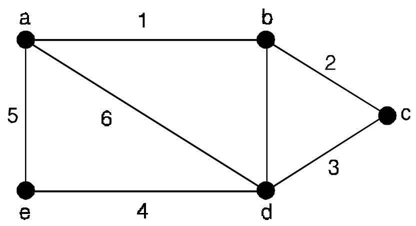


Fig. 2.2. Geometric representation of a graph.

2.2 PLANAR GRAPHS

A graph G is said to be **planar** if there exists some geometric representation of G which can be drawn on a plane such that no two of its edges intersect. The points of intersection are called crossovers.

A graph that cannot be drawn on a plane without a crossover between its edges crossing is called a plane graph. The graphs shown in Figure 2.3(*a*) and are planar graphs.

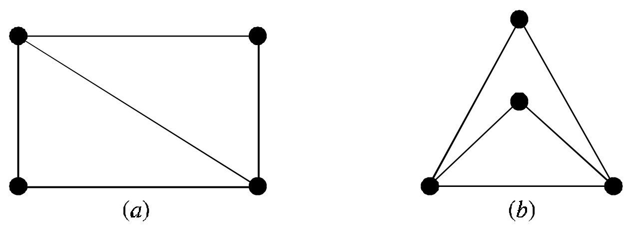


Fig. 2.3.

A drawing of a geometric representation of a graph on any surface such that no edges intersect is called embedding.

Note that if a graph G has been drawn with crossing edges, this does not mean that G is non planar, there may be another way to draw the graph without crossovers. Thus to declare that a graph G is non planar. We have to show that all possible geometric representations of G none can be embedded in a plane.

Equivalently, a graph G is planar is there if there exists a graph isomorphic to G that is embedded in a plane, otherwise G is non planar.

For example, the graph in Figure 2.4(*a*) is apparently non planar. However, the graph can be redrawn as shown in Figure (2.4)(*b*) so that edges don't cross, it is a planar graph, though its appearance is non coplanar.

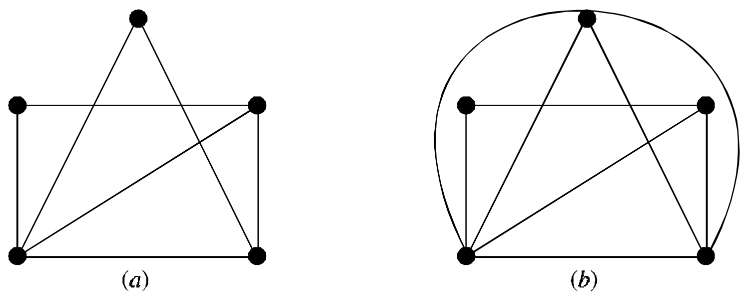


Fig. 2.4.

2.3 KURATOWSKI'S GRAPHS

For this we discuss two specific non-planar graphs, which are of fundamental importance, these are called Kuratowski's graphs. The complete graph with 5 vertices is the first of the two graphs of Kuratowski. The second is a regular, connected graph with 6 vertices and 9 edges.

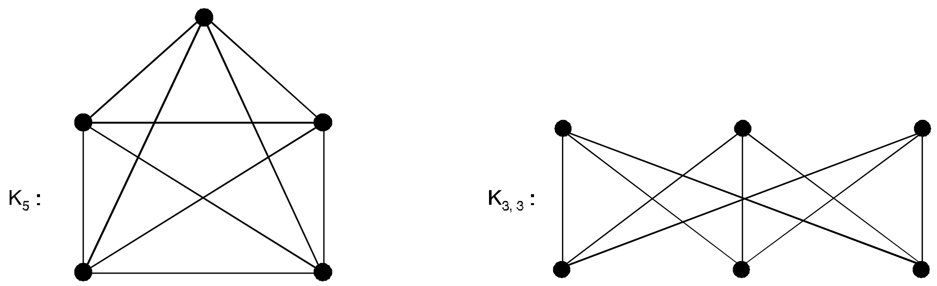


Fig. 2.5.

Observations

(*i*) Both are regular graphs

(*ii*) Both are non-planar graphs

(*iii*) Removal of one vertex or one edge makes the graph planar

(*iv*) (Kuratowski's) first graph is non-planar graph with smallest number of vertices and (Kuratowski's) second graph is non-planar graph with smallest number of edges. Thus both are simplest non-planar graphs.

The first and second graphs of Kuratowski are represented as K_5 and $K_{3,3}$. The letter K being for Kuratowski (a polish mathematician).

2.4 HOMEOMORPHIC GRAPHS

Two graphs are said to be homeomorphic if and only if each can be obtained from the same graph by adding vertices (necessarily of degree 2) to edges.

The graphs G_1 and G_2 in Figure (2.6) are homeomorphic since both are obtainable from the graph G in that figure by adding a vertex to one of its edges.

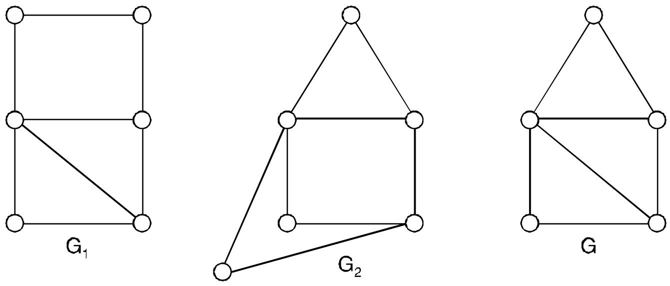


Fig. 2.6. Two homeomorphic graphs obtained from G by adding vertices to edges.

In Figure 2.7, we show two homeomorphic graphs, each obtained from K_5 by adding vertices to edges of K_5 (In each case, the vertices of K_5 are shown with solid dots).

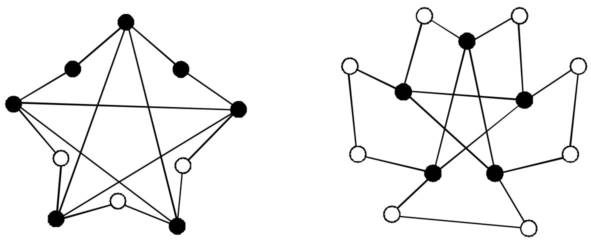


Fig. 2.7. Two homeomorphic graphs obtained from K_5.

2.5 REGION

A plane representation of a graph divides the plane into regions (also called windows, faces, or meshes) as shown in figure below. A region is characterized by the set of edges (or the set of vertices) forming its boundary.

Note that a region is not defined in a non-planar graph or even in a planar graph not embedded in a plane.

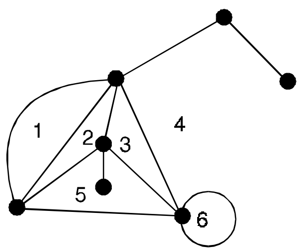


Fig. 2.8. Plane representation (the numbers stand for regions).

For example, the geometric graph in figure below does not have regions.

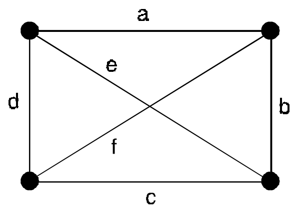


Fig. 2.9.

2.6 MAXIMAL PLANAR GRAPHS

A planar graph is maximal planar if no edge can be added without loosing planarity. Thus in any maximal planar graph with $p \geq 3$ vertices, the boundary of every region of G is a triangle for this maximal planar graphs (or plane graphs) are also refer to as triangulated planar graph (or plane graph).

2.7 SUBDIVISION GRAPHS

A subdivision of a graph is a graph obtained by inserting vertices (of degree 2) into the edges of G. For the graph G of the figure below, the graph H is a subdivision of G, while F is not a subdivision of G.

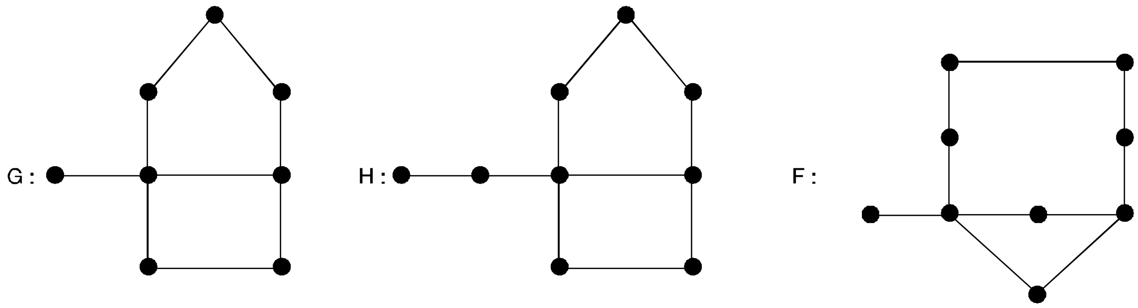


Fig. 2.10.

2.8 INNER VERTEX SET

A set of vertices of a planar graph G is called an inner vertex set I(G) of G. If G can be drawn on the plane in such a way that each vertex of I(G) lies only on the interior region and I(G) contains the minimum possible vertices of G. The number of vertices i(G) of I(G) is said to be the inner vertex number if they lie in interior region of G.

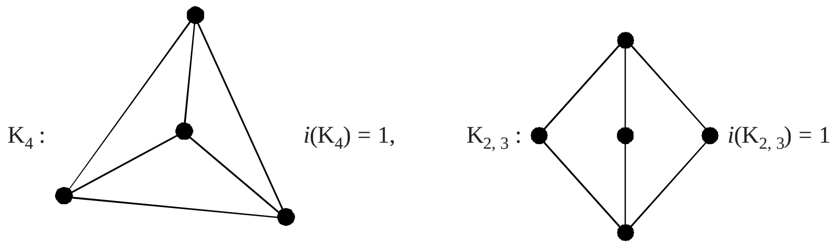


Fig. 2.11.

For any cycle C_p, $i(C_p) = 0$.

2.9 OUTER PLANAR GRAPHS

A planar graph is said to be outer planar if $i(G) = 0$. For example, cycles, trees, $K_4 - x$.

2.9.1. Maximal outer planar graph

An outer planar graph G is maximal outer planar if no edge can be added without losing outer planarity.

For example,

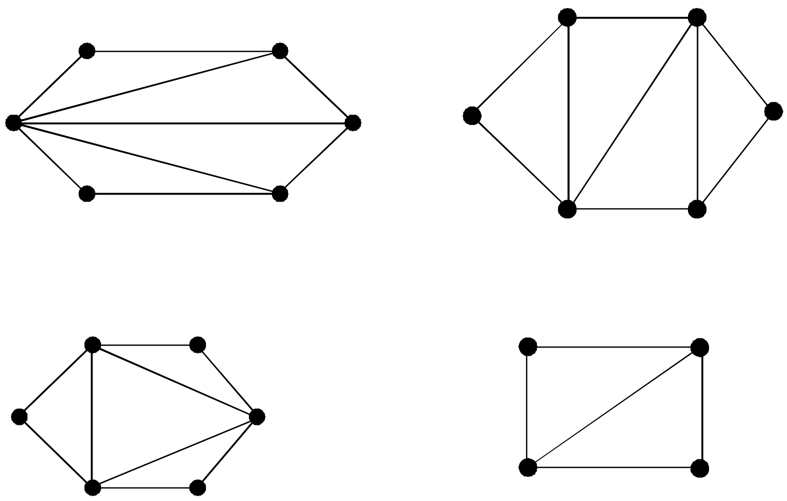
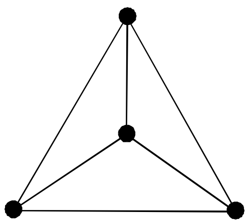


Fig. 2.12. Maximal outer planar graphs.

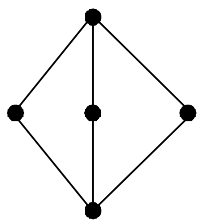
2.9.2. Minimally non-outer planar graphs

A planar graph G is said to be minimally non outer planar if $i(G) = 1$

For example, K_4 :



$K_{2, 3}$:



2.10 CROSSING NUMBER

The crossing number C(G) of a graph G is the minimum number of crossing of its edges among all drawings of G in the plane.

A graph is planar if and only if $C(G) = 0$. Since K_4 is planar $C(K_4) = 0$ for $p \leq 4$. On the other hand $C(K_5) = 1$. Also $K_{3, 3}$ is non planar and can be drawn with one crossing.

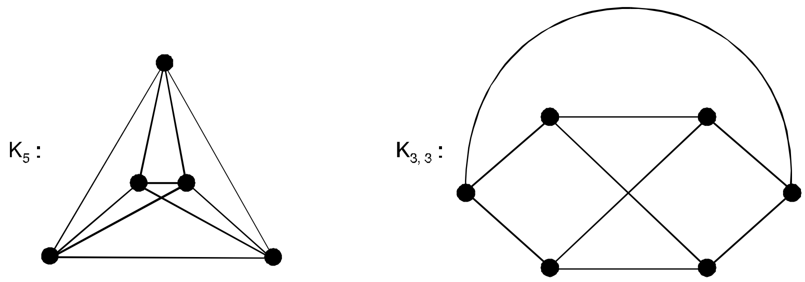


Fig. 2.13. K_5 and $K_{3,3}$ are non planar graphs with one crossing.

2.11 BIPARTITE GRAPH

A graph G = (V, E) is bipartite if the vertex set V can be partitioned into two subsets (disjoint) V_1 and V_2 such that every edge in E connects a vertex in V_1 and a vertex V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2). (V_1, V_2) is called a bipartition of G. Obviously, a bipartite graph can have no loop.

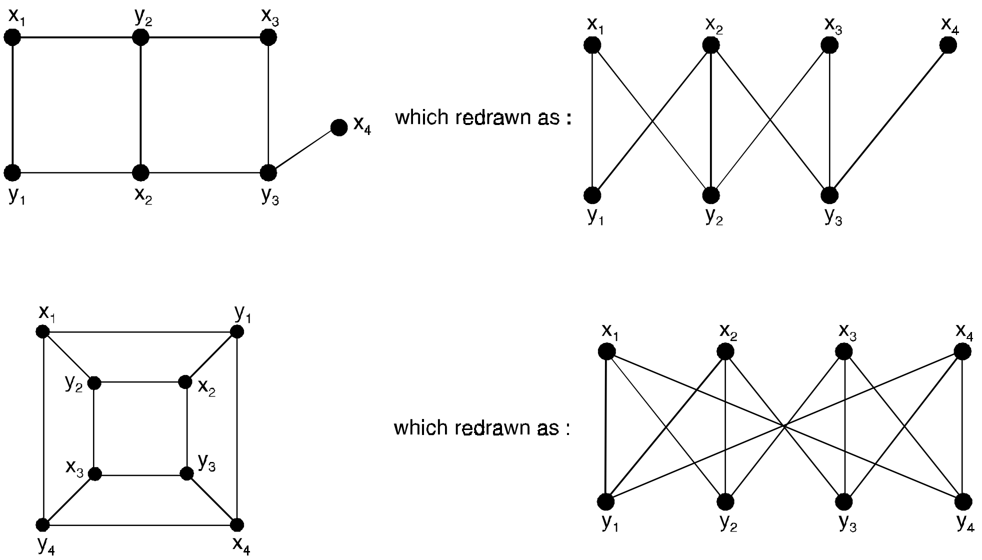


Fig. 2.14. Some bipartite graphs.

2.11.1. Complete bipartite graph

The complete bipartite graph on *m* and *n* vertices, denoted $K_{m,n}$ is the graph, whose vertex set is partitioned into sets V_1 with *m* vertices and V_2 with *n* vertices in which there is an edge between each pair of vertices V_1 and V_6. Where V_1 is in V_1 and V_2 is in V_2. The complete bipartite graphs $K_{2,3}$, $K_{3,3}$, $K_{3,5}$ and $K_{2,6}$ are shown in Figure below. Note that $K_{r,s}$ has $r + s$ vertices and rs edges.

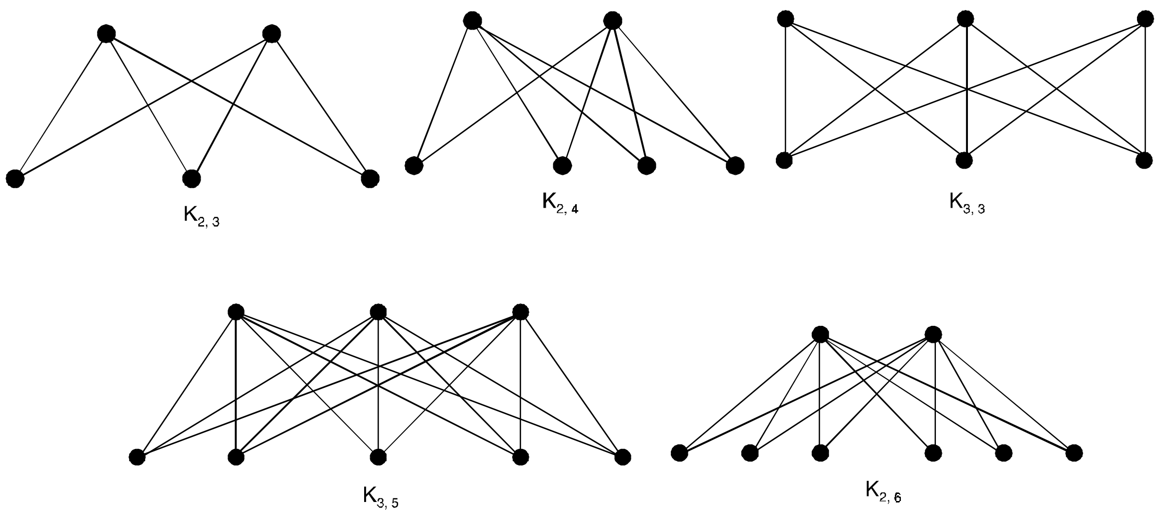


Fig. 2.15. Some complete bipartite graphs.

A complete bipartite graph $K_{m,\,n}$ is not a regular if $m \neq n$.

Problem 2.1. *Show that C_6 is a bipartite graph.*

Solution. C_6 is a bipartite graph as shown in Figure below.

Since its vertex set can be partitioned into two sets $V_1 = \{v_1, v_3, v_5\}$ and $V_2 = \{v_2, v_4, v_6\}$ and every edge of C_6 connects a vertex in V_1 and a vertex in V_2.

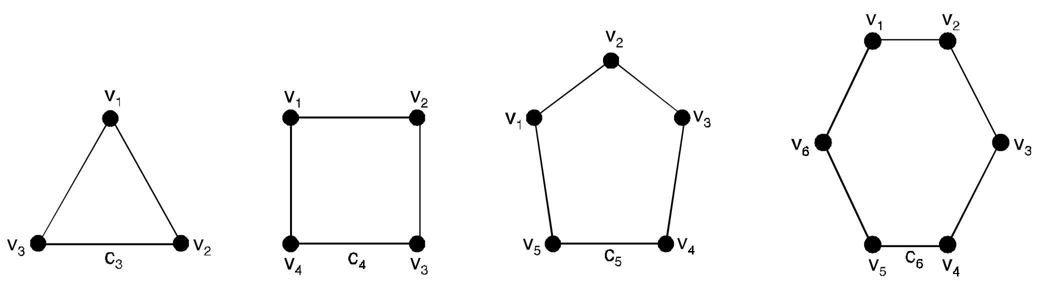
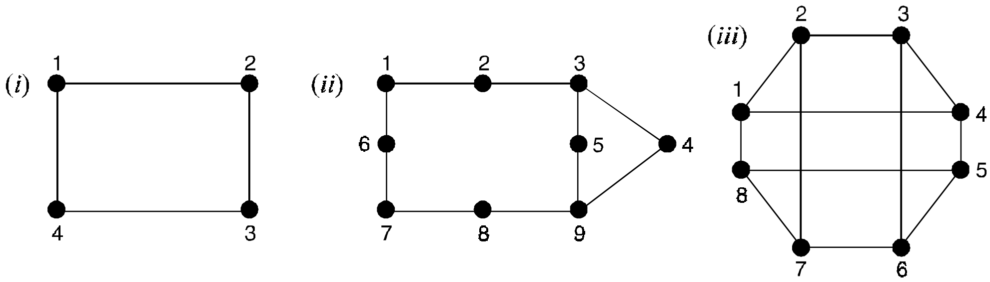


Fig. 2.16.

Problem 2.2. *Prove that a graph which contains a triangle cannot be bipartite.*

Solution. At least two of three vertices must lie in one of the bipartite sets, since these two are joined by two are joined by edge, the graph can not be bipartite.

Problem 2.3. *Determine whether or not each of the graphs is bipartite. In each case, give the bipartition sets or explain why the graph is not bipartite.*



Solution. (*i*) The graph is not bipartite because it contains triangles (in fact two triangles).

(*ii*) This is bipartite and the bipartite sets are {1, 3, 7, 9} and {2, 4, 5, 6, 8}

(*iii*) This is bipartite and the bipartite sets are {1, 3, 5, 7} and {2, 4, 6, 8}.

2.12 EULER'S FORMULA

The basic results about planar graph known as Euler's formula is the basic computational tools for planar graph.

Theorem 2.1. Euler's Formula

If a connected planar graph G has n vertices, e edges and r region, then $n - e + r = 2$.

Proof. We prove the theorem by induction on e, number of edges of G.

Basis of induction : If $e = 0$ then G must have just one vertex.

i.e., $n = 1$ and one infinite region, *i.e.*, $r = 1$

Then $n - e + r = 1 - 0 + 1 = 2$.

If $e = 1$ (though it is not necessary), then the number of vertices of G is either 1 or 2, the first possibility of occurring when the edge is a loop.

These two possibilities give rise to two regions and one region respectively, as shown in Figure (2.17) below.

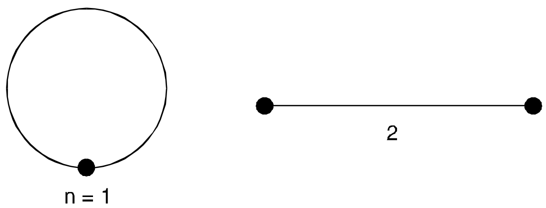


Figure. 2.17. Connected plane graphs with one edge.

In the case of loop, $n - e + r = 1 - 1 + 2 = 2$ and in case of non-loop, $n - e + r = 2 - 1 + 1 = 2$.

Hence the result is true.

Induction hypothesis :

Now, we suppose that the result is true for any connected plane graph G with $e - 1$ edges.

Induction step :

We add one new edge K to G to form a connected supergraph of G which is denoted by G + K.

There are following three possibilities.

(*i*) K is a loop, in which case a new region bounded by the loop is created but the number of vertices remains unchanged.

(*ii*) K joins two distinct vertices of G, in which case one of the region of G is split into two, so that number of regions is increased by 1, but the number of vertices remains unchanged.

(*iii*) K is incident with only one vertex of G on which case another vertex must be added, increasing the number of vertices by one, but leaving the number of regions unchanged.

If let n', e' and r' denote the number of vertices, edges and regions in G and n, e and r denote the same in G + K. Then

In case (*i*) $n - e + r = n' - (e' + 1) + (r' + 1) = n' - e' + r'$.

In case (*ii*) $n - e + r = n' - (e' + 1) + (r' + 1) = n' - e' + r'$

In case (*iii*) $n - e + r = (n' + 1) - (e' + 1) + r' = n' - e' + r'$.

But by our induction hypothesis, $n' - e' + r' = 2$.

Thus in each case $n - e + r = 2$.

Now any plane connected graph with e edges is of the form G + K, for some connected graph G with $e - 1$ edges and a new edge K.

Hence by mathematical induction the formula is true for all plane graphs.

Corollary (1)

If a plane graph has K components then $n - e + r = K + 1$.

The result follows on applying Euler's formula to each component separately, remembering not to count the infinite region more than once.

Corollary (2)

If G is connected simple planar graph with n (≥ 3) vertices and e edges, then $e \leq 3n - 6$.

Proof. Each region is bounded by atleast three edges (since the graphs discussed here are simple graphs, no multiple edges that could produce regions of degree 2 or loops that could produce regions of degree 1, are permitted) and edges belong to exactly two regions.

$$2e \geq 3r$$

If we combine this with Euler's formula, $n - e + r = 2$, we get $3r = 6 - 3n + 3e \leq 2e$ which is equivalent to $e \leq 3n - 6$.

Corollary (3)

If G is connected simple planar graph with n (≥ 3) vertices and e edges and no circuits of length 3, then $e \leq 2n - 4$.

Proof. If the graph is planar, then the degree of each region is atleast 4.

Hence the total number of edges around all the regions is atleast $4r$.

Since every edge borders two regions, the total number of edges around all the regions is $2e$, so we established that $2e \geq 4r$, which is equivalent to $2r \leq e$.

If we combine this with Euler's formula $n - e + r = 2$, we get

$$2r = 4 - 2n + 2e \leq e$$

which is equivalent to $e \leq 2n - 4$.

Problem 2.4. *Show that the graph K_5 is not coplanar.*

Solution. Since K_5 is a simple graph, the smallest possible length for any cycle K_5 is three.

We shall suppose that the graph is planar.

The graph has 5 vertices and 10 edges so that $n = 5$, $e = 10$.

Now $3n - 6 = 3.5 - 6 = 9 < e$.

Thus the graph violates the inequality $e \le 3n - 6$ and hence it is not coplanar.

This may be noted that the inequality $e \le 3n - 6$ is only by a necessary condition but not a sufficient condition for the planarity of a graph.

For example, graph $K_{3,\,3}$ satisfies the inequality because $e = 9 \le 3.6 - 6 = 12$, yet the graph is non planar.

Problem 2.5. *Show that the graph $K_{3,\,3}$ is not coplanar.*

Solution. Since $K_{3,\,3}$ has no circuits of length 3 (it is bipartite) and has 6 vertices and 9 edges.

i.e., $n = 6$ and $e = 9$ so that $2n - 4 = 2.6 - 4 = 8$.

Hence the inequality $e \le 2n - 4$ does not satisfy and the graph is not coplanar.

Problem 2.6. *A connected plane graph has 10 vertices each of degree 3. Into how many regions, does a representation of this planar graph split the plane ?*

Solution. Here $n = 10$ and degree of each vertex is 3

$\Sigma \deg(v) = 3 \times 10 = 30$

But $\Sigma \deg(v) = 2e$ $\Rightarrow$ $30 = 2e$ $\Rightarrow$ $e = 15$

By Euler's formula, we have $n - e + r = 2$ $\Rightarrow$ $10 - 15 + r = 2$ $\Rightarrow$ $r = 7$.

Problem 2.7. *Show that K_n is a planar graph for $n \le 4$ and non-planar for $n \ge 5$.*

Solution. A K_4 graph can be drawn in the way as shown in the Figure (2.18). This does not contain any false crossing of edges.

Thus, it is a planar graph.

Graphs K_1, K_2 and K_3 are by construction a planar graph, since they do not contain a false crossing of edges.

K_5 is shown in the Figure (2.19)

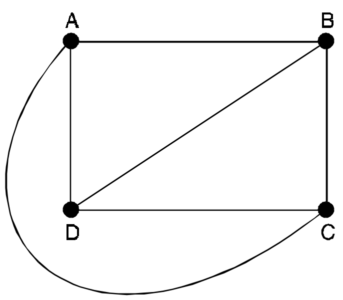


Fig. 2.18.

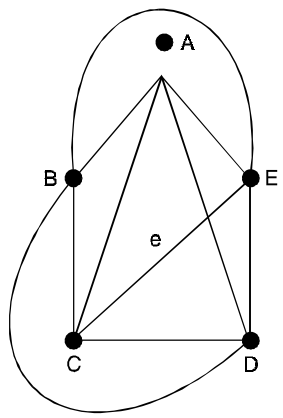


Fig. 2.19.

It is not possible to draw this graph on a 2-dimentional plane without false crossing of edges. Whatever way we adopt, at least one of the edges, say *e*, must cross the other for graph to be completed.

Hence K_5 is not a planar graph.

For any $n > 5$, K_n must contain a subgraph isomorphic to K_5.

Since K_5 is not planar, any graph containing K_5 as its one of the subgraph cannot be planar.

Problem 2.8. *Show that $K_{3,3}$ is a non-planar graph.*

Solution. Graph $K_{3,3}$ is shown in the Figure (2.20) below.

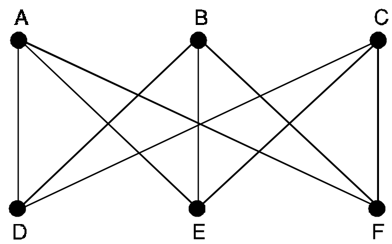


Fig. 2.20.

It is not possible to draw this graph such that there is no false crossing of edges. This is classic problem of designing direct lanes without intersection between any two houses, for three houses on each side of a road.

In this graph there exists an edge, say *e*, that cannot be drawn without crossing another edge.

Hence $K_{3,3}$ is a non-planar graph.

It is easy to determine that the chromatic number of this graph is 2.

Theorem 2.2. *Sum of the degrees of all regions in a map is equal to twice the number of edges in the corresponding graph.*

Proof. As discussed earlier, a map can be drawn as a graph, where regions of the map is denoted by vertices in the graph and adjoining regions are connected by edges.

Degree of a region in a map is defined as the number of adjoining region.

Thus, degree of a region in a map is equal to the degree of the corresponding vertices in the graph.

We know that the sum of the degrees of all vertices in a graph is equal to the twice the number of edges in the graph.

Therefore, we have $2e = \Sigma\text{deg}(R_i)$.

Problem 2.9. *Prove that K_4 and $K_{2,2}$ are planar.*

Solution. In K_4, we have $v = 4$ and $e = 6$

Obviously, $6 \le 3 * 4 - 6 = 6$

Thus this relation is satisfied for K_4.

For $K_{2,2}$, we have $v = 4$ and $e = 4$.

Again in this case, the relation $e \le 3v - 6$

i.e., $4 \le 3 * 4 - 6 = 6$ is satisfied.

Hence both K_4 and $K_{2,2}$ are planar.

Problem 2.10. *Determine the number of vertices, the number of edges, and the number of region in the graphs shown below. Then show that your answer satisfy Euler's theorem for connected planar graphs.*

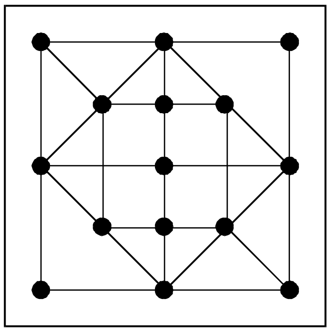


Fig. 2.21.

Solution. There are 17 vertices, 34 edges and 19 regions. So $v - e + r = 17 - 34 + 19 = 2$ which verifies Euler's theorem.

Problem 2.11. *If every region of a simple planar graph with n-vertices and e-edges embeded in a plane is bounded by k-edges then show that* $e = \dfrac{k(n-2)}{k-2}$.

Solution. Since every region is bounded by K-edges, then r-regions are bounded by Kr-edges. Also each edge is counted twice, once for two of its adjacent regions.

Hence we have $2e = \text{K}r \quad \Rightarrow \quad r = \dfrac{2e}{\text{K}}$ ...(1)

i.e., if G is a connected planar graph with n-vertices e-edges and r-regions, then $n - e + r = 2$.

From (1), we have

$$n - e + \frac{2e}{\text{K}} = 2$$

$$\Rightarrow \quad n\text{K} - e\text{K} + 2e = 2\text{K}$$

$$\Rightarrow \quad n\text{K} - 2\text{K} = e\text{K} - 2e$$

$$\Rightarrow \quad \text{K}(n-2) = e(\text{K}-2)$$

$$\Rightarrow \quad e = \frac{\text{K}(n-2)}{\text{K}-2}.$$

Problem 2.12. *Determine whether the graph G shown in Figure (2.22), is planar.*

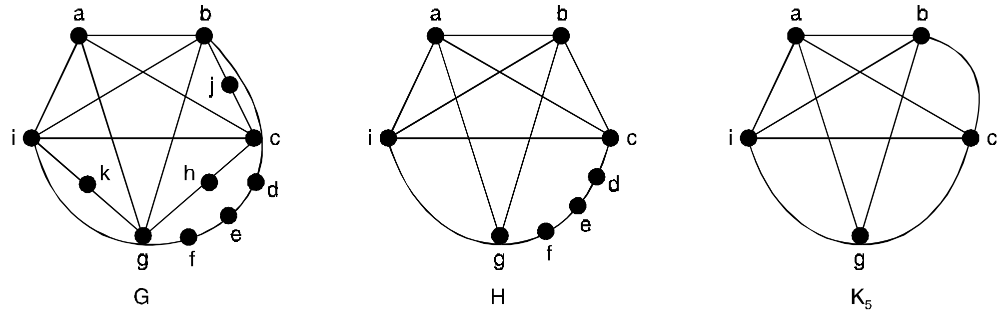


Fig. 2.22. The undirected graph G, a subgraph H homeomorphic to K_5 and K_5.

Solution. G has a subgraph H homeomorphic to K_5, H is obtained by deleting *h, j* and K and all edges incident with these vertices. H is homeomorphic to K_5 since it can be obtained from K_5 (with vertices *a, b, c, g* and *i*) by a sequence of elementary subdivisions, adding the vertices *d*, *e* and *f*.

Hence G is non planar.

Theorem 2.3. KURATOWSKI'S

$K_{3,3}$ and K_5 are non-planar.

Proof. Suppose first that $K_{3,3}$ is planar.

Since $K_{3,3}$ has a cycle $u \to v \to w \to x \to y \to z \to u$ of length 6, any plane drawing must contain this cycle drawn in the form of hexagon, as in Figure (2.23).

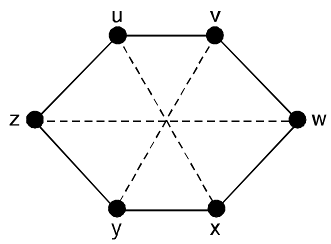


Fig. 2.23.

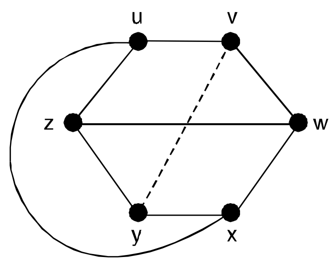


Fig. 2.24.

Now the edge *wz* must lie either wholly inside the hexagon or wholly outside it. We deal with the case in which *wz* lies inside the hexagon, the other case is similar.

Since the edge *ux* must not cross the edge *wz,* it must lie outside the hexagon ; the situation is now as in Figure (2.24).

It is then impossible to draw the edge *vy*, as it would cross either *ux* or *wz*.

This gives the required contradiction.

Now suppose that K_5 is planar.

Since K_5 has a cycle $v \to w \to x \to y \to z \to v$ of length 5, any plane drawing must contain this cycle drawn in the form of a pentagon as in Figure (2.25).

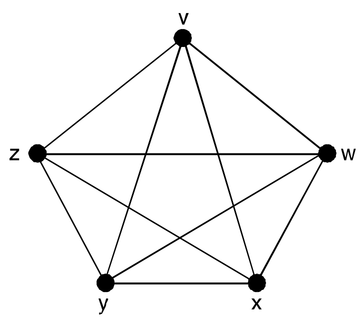


Fig. 2.25.

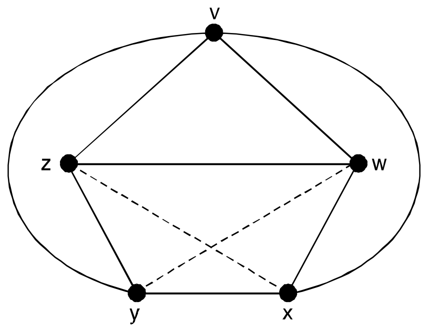


Fig. 2.26.

Now the edge *wz* must lie either wholly inside the pentagon or wholly outside it.

We deal with the case in which *wz* lies inside the pentagon, the other case is similar.

Since the edges *vx* and *vy* do not cross the edge *wz*, they must both lie outside the pentagon, the situation is now as in Figure (2.26)

But the edge *xz* cannot cross the edge *vy* and so must lie inside the pentagon.

Similarly the edge *wy* must lie inside the pentagon, and the edges *wy* and *xz* must then cross.

This gives the required contradiction.

Theorem 2.4. *Let G be a simple connected planar (p, q)-graph having at least K edges in a boundary of each region. Then* $(k-2)q \leq k(p-2)$.

Proof : Every edge on the boundary of G, lies in the boundaries of exactly two regions of G.

Further G may have some pendent edges which do not lie in a boundary of any region of G.

Thus, sum of lengths of all boundaries of G is less than twice the number of edges of G.

i.e., $\quad kr \leq 2q \qquad$ ...(1)

But, G is a connected graph, therefore by Euler's formula

We have $\quad r = 2 + q - p \qquad$ ...(2)

Substituting (2) in (1), we get

$$k(2 + q - p) \leq 2q$$

$$\Rightarrow \quad (k-2)q \leq k(p-2).$$

Problem 2.13. *Suppose G is a graph with 1000 vertices and 3000 edges. Is G planar ?*

Solution. A graph G is said to be planar if it satisfies the inequality. *i.e.,* $q \leq 3p - 6$

Here P = 1000, $q = 3000$ then

$$3000 \leq 3p - 6$$

i.e., $\quad 3000 \leq 3000 - 6$

or $\quad 3000 \leq 2994$ which is impossible.

Hence the given graph is not a planar.

Problem 2.14. *A connected graph has nine vertices having degrees 2, 2, 2, 3, 3, 3, 4, 4 and 5. How many edges are there ? How many faces are there ?*

Solution. By Handshaking lemma,

$$\sum_{i=1}^{n} \deg v_i = 2q$$

i.e., $2q = 2 + 2 + 2 + 3 + 3 + 3 + 4 + 4 + 5 = 28$

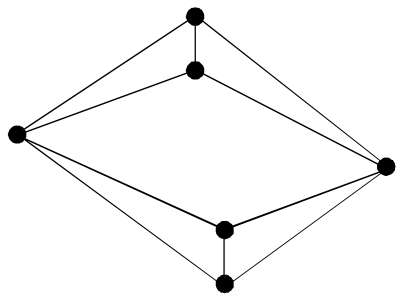
$\Rightarrow$ $q = 24$

Now by Euler's formula $p - q + r = 2$ or $9 - 14 + r = 2$ $\Rightarrow$ $r = 7$

Hence there are 14 edges and 7 regions in the graph.

Problem 2.15. *Find a graph G with degree sequence (4, 4, 3, 3, 3, 3) such that (i) G is planar (ii) G is non planar.*

Solution. For (*i*) we have drawn a planar graph with six vertices with degree sequence 4, 4, 3, 3, 3, 3 as shown below.



For (*ii*) By Handshaking lemma

$$\sum_{i=1}^{n} \deg v_i = 2q$$

i.e., $2q = 4 + 4 + 3 + 3 + 3 + 3$

$2q = 20$

$\Rightarrow$ $q = 10$

Hence the graph with P = 6, is said to be planar if it satisfies the inequality.

i.e., $q = 3p - 6$

i.e., $10 \leq 3 \times 6 - 6$

or $10 \leq 18 - 6$

$10 \leq 12$

Hence it is not possible to draw a non planar graph with given degree sequence 4, 4, 3, 3, 3, 3.

Problem 2.16. *Determine the number of regions defined by a connected planar graph with 6 vertices and 10 edges. Draw a simple and a non-simple graph.*

Solution. Given $p = 6$, $q = 10$

Hence by Euler's formula for a planar graph

$$p - q + r = 2$$

$$6 - 10 + r = 2 \quad \Rightarrow \quad r = 6$$

Hence the graph should have 6 regions.

Simple and non-simple graphs with $p = 6$, $q = 10$ and $r = 6$ are shown below.

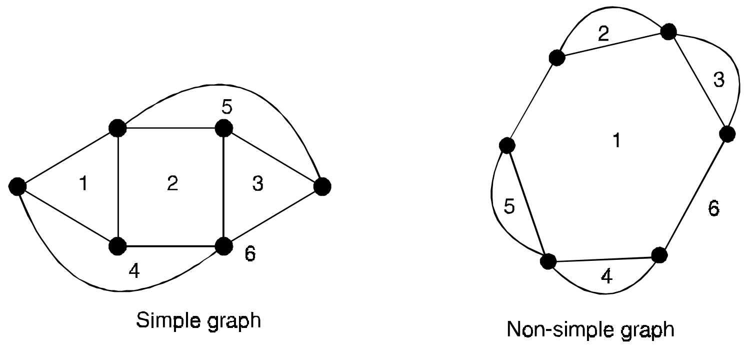
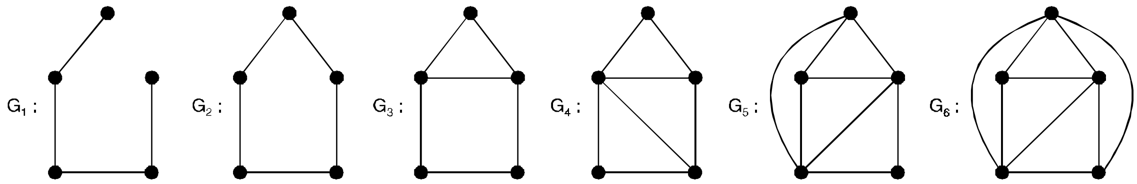


Fig. 2.27.

Problem 2.17. *Draw all planar graphs with five vertices, which are not isomorphic to each other.*

Solution. We have drawn all planar graphs with 5 vertices as shown below.



Problem 2.18. *How many edges must a planar graph have if it has 7 regions and 5 vertices. Draw one such graph.*

Solution. According to Euler's formula, in a planar graph G.

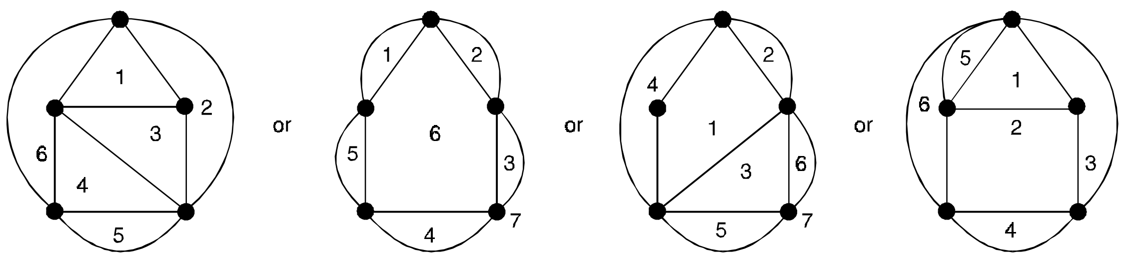
$$p - q + r = 2$$

Here $p = 5$, $r = 7$, $q = ?$

Since the graph is planar, therefore $5 - q + 7 = 2 \quad \Rightarrow \quad q = 10.$

Hence the given graph must have 10-edges.

Here we have drawn more than one graph as shown below.



Problem 2.19. *By drawing the graph, show that the following graphs are planar graphs.*

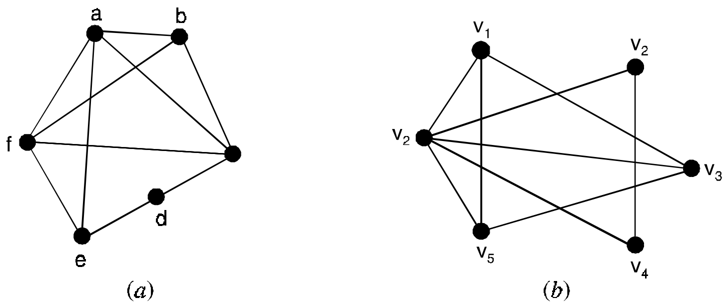


Fig. 2.28.

Solution. The graphs shown in Figure (2.28)(*a*, *b*) can be redrawn as planar graphs as follows see Figure (2.29) (*a*, *b*).

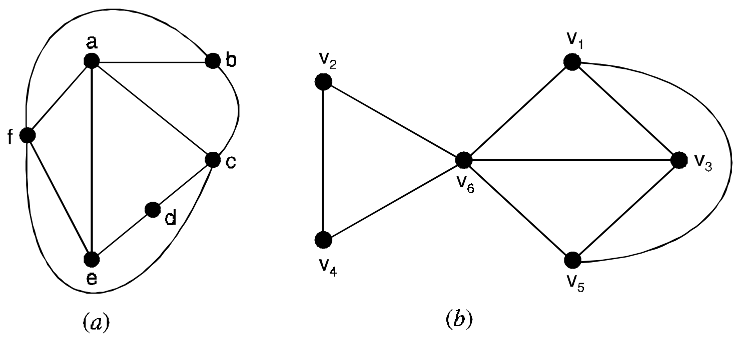


Fig. 2.29.

Problem 2.20. *Show that the Petersen graph is non planar.*

Solution. Petersen graph is well known non planar graph. Since G has some similarity with K_5 because of 5-cycle, ABCDEA. However since K_5 has vertices of degree 4 only subdivision of K_5 will also have such vertices so G can not have only subdivision of K_5.

Since its vertices each have degree 3. So we look for a subgraph of G which is subdivision of the bipartite graph $K_{3,\,3}$.

The Petersen graph shown in Figure (2.30)(*a*) is non planar since it contains a subgraph homeomorphic to $K_{3,\,3}$ as shown in Figure (2.30)(*c*). Note that the Petersen graph does not contain a subgraph homeomorphic to K_5,

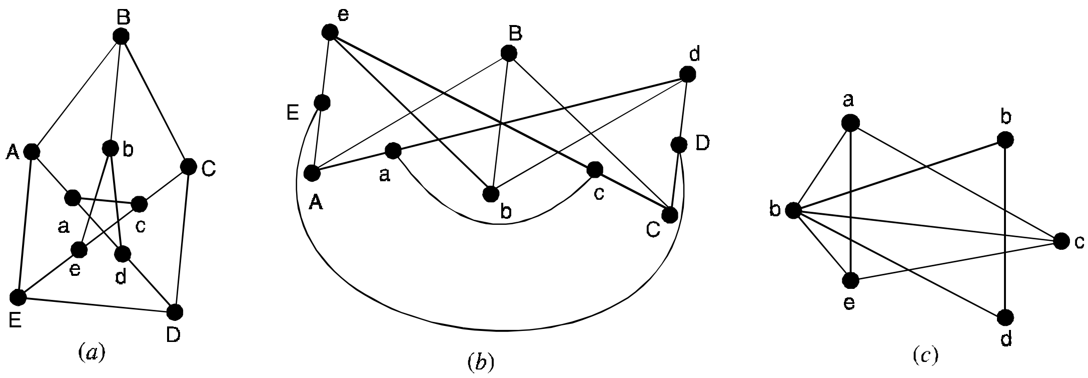


Fig. 2.30.

Problem 2.21. *Find a smallest planar graph that is regular of degree 4.*

Solution. For the graph with two vertices, which is complete, then degree of each vertex is one. For the next smallest graphs are with vertices 3 and 4, if they are complete then degree of each vertex is 2 and 3.

The next graph is with 5 vertices. If degree of each vertex is 4, then it is complete graph with 5 vertices K_5 which is non planar. For the next graph with 6 vertices, if it complete then degree of each vertex is P – 1. *i.e.,* 5. To make this graph 4 regular or regular of degree 4. Remove any 3 non adjacent edges from K_6 we get $K_6 - 3x$ where x is an edge of G, as shown in Figure (2.31), which is regular of degree 4.

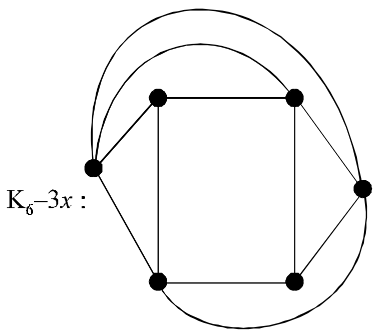
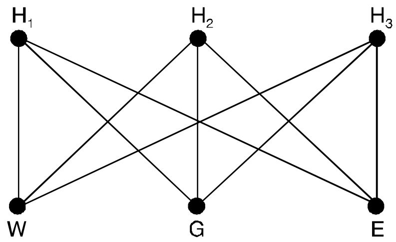


Fig. 2.31.

2.12.1. Three utility problem (2.22)

There are three homes H_1, H_2 and H_3 each to be connected to each of three utilities Water (W), Gas(G) and Electricity(E) by means of conduits. Is it possible to make such connections without any crossovers of the conduits ?

Solution.



The problem can be represented by a graph shown in Figure the conduits are shown as edges while the houses and utility supply centers are vertices.

The above graph is a complete bipartite graph $K_{3, 3}$ which is a non planar graph. Hence it is not possible to draw without crossover. Therefore it is not possible to make the connection without any crossover of the conduits.

Problem 2.23. *Is the Petersen graph, shown in Figure below, planar ?*

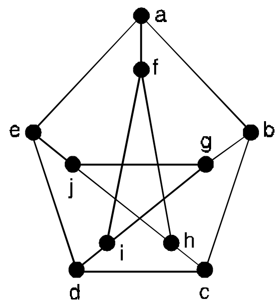


Fig. 2.32. Petersen graph

Solution. The subgraph H of the Petersen graph obtained by deleting *b* and the three edges that have *b* as an end point, shown in Figure (2.33) below, is homeomorphic to $K_{3, 3}$ with vertex sets $\{f, d, j\}$

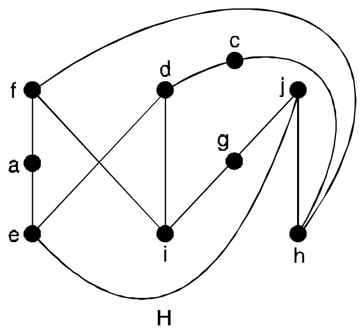


Fig. 2.33.

and $\{e, i, h\}$, since it can be obtained by a sequence of elementary subdivisions, deleting $\{d, h\}$ and adding $\{c, h\}$ and $\{c, d\}$, deleting $\{e, f\}$ and adding $\{a, e\}$ and $\{a, f\}$ and deleting $\{i, j\}$ and adding $\{g, i\}$ and $\{g, j\}$.

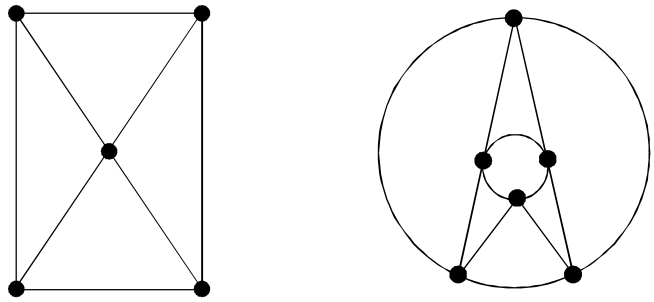
Hence the Petersen graph is not planar.

Problem 2.24. *Show that the following graphs are planar :*

(*i*) *Graph of order 5 and size 8* (*ii*) *Graph of order 6 and size 12.*

Solution. To show that a graph is planar, it is enough if we draw one plane diagram representing the graph in which no two edges cross each other.

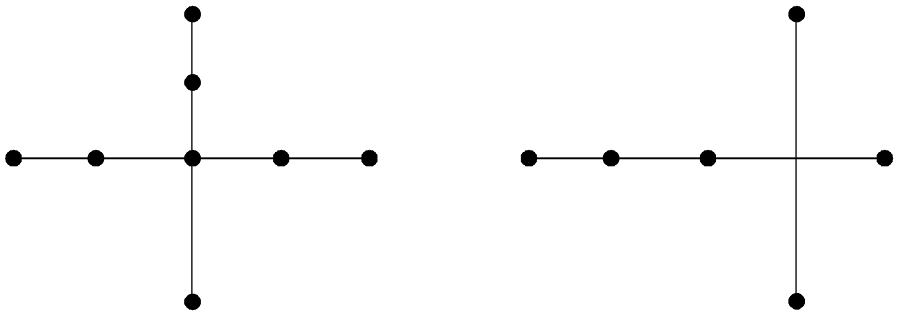
Figure (2.34) (*a*) and (*b*) show that the given graphs are planar.



(*a*) (*b*)

Fig. 2.34.

Problem 2.25. *Verify that the following two graphs are homeomorphic but not isomorphic.*



Solution. Each graph can be obtained from the other by adding or removing appropriate vertices.

Therefore, they are homeomorphic.

That they are not isomorphic is evident if we observe that the incident relationship is not identical.

Problem 2.26. *Show that if a planar graph G of order n and size m has r regions and K components, then $n - m + r = k + 1$.*

Solution. Let H_1, H_2, ... H_k be the K components of G.

Let the number of vertices, the number of edges and the number of non-exterior regions in H_i be n_i, m_i, r_i respectively, $i = 1, 2, ... k$.

The exterior region is the same for all components.

Therefore, $\Sigma n_i = n,$ $\Sigma m_i = m,$ $\Sigma r_i = r - 1$

If the exterior region is not considered, then the Euler's formula applied to H_i yields

$$n_i - m_i + r_i = 1$$

On summation (from $i = 1$ to $i = k$) this yields

$$n - m + (r - 1) = k$$

$$\Rightarrow \qquad n - m + r = k + 1$$

Problem 2.27. *Let G be a connected simple planar (n, m) graph in which every region is bounded by at least k edges. Show that* $m \le \dfrac{k(n-2)}{(k-2)}$.

Solution. Since every region in G is bounded by at least k edges, we have $2m \ge kr$ ...(1)

Where r is the number of regions

Substituting for r from the Euler's formula in (1), we get

$$2m \ge k(m - n + 2)$$

$$\Rightarrow \qquad k(n - 2) \ge km - 2m$$

$$\Rightarrow \qquad m \le \frac{k(n-2)}{(k-2)}$$

Problem 2.28. *Let G be a simple connected planar graph with fewer than 12 regions, in which each vertex has degree at least 3. Prove that G has a region bounded by at most four edges.*

Solution. Suppose every region in G bounded by at least 5 edges.

Then, if G has n vertices and m edges,

we have, $\qquad 2m \ge 5r$ ...(1)

Since each vertex has degree atleast 3, the sum of the degrees of the vertices is greater than or equal to $3n$. By virtue of the handshaking property, this means that

$$2m \ge 3n \qquad \text{...(2)}$$

By Euler's formula, we have

$$r = m - n + 2$$

$$\ge m - \left(\frac{2}{3}\right)m + 2 \qquad (\because \quad (2))$$

$$= \frac{m}{3} + 2 \ge \frac{5}{6}r + 2 \qquad (\because \quad (1))$$

This yields $6r \ge 5r + 12$, $r \ge 12$.

This is a contradiction, because G has fewer than 12 regions.

Hence, some region in G is bounded by atmost four edges.

Problem 2.29. *Show that these does not exist a connected simple planar graph with $m = 7$ edges and with degree $\delta = 3$.*

Solution. Suppose there is a graph G of the desired type.

Then, for this graph, the inequality $\delta \le \left(\dfrac{2m}{n}\right)$ gives $3n \le 14$.

On the other hand, $7 \leq 3n - 6$ or $3n \geq 13$.

Thus, we have $13 \leq 3n \leq 14$ which is not possible (because n has to be a positive integer).

Hence the graph of the desired type does not exist.

Problem 2.30. *Show that every simple connected planar graph G with less than 12 vertices must have a vertex of degree ≤ 4.*

Solution. Suppose every vertex of G has degree greater than or equal to 5.

Then, if $d_1, d_2, d_3, \ldots\ldots d_n$ are the degrees of n vertices of G, we have $d_1 \geq 5, d_2 \geq 5, \ldots\ldots d_n \geq 5$.

So that $d_1 + d_2 + \ldots\ldots + d_n \geq 5n$.

or $2m \geq 5n$, by handshaking property,

or
$$\frac{5n}{2} \leq m \qquad \ldots(1)$$

On the other hand, $m \leq 3n - 6$

Thus, we have, in view of (1)

$$\frac{5n}{2} \leq 3n - 6 \quad \text{or} \quad n \geq 12.$$

Thus, if every vertex of G has degree ≥ 5, then G must have at least 12 vertices.

Hence, if G has less than 12 vertices, it must have a vertex of degree < 5.

Problem 2.31. *Show that the condition $m \leq 3n - 6$ is not a sufficient condition for a connected simple graph with n vertices and m edges to be planar.*

Solution. Consider the graph $K_{3,\,3}$ which is simple and connected and which has $n = 6$ vertices and $m = 9$ edges.

We check that, for this graph, $m \leq 3n - 6$.

But the graph is non-planar.

Problem 2.32. *What is the minimum number of vertices necessary for a simple connected graph with 11 edges to be planar ?*

Solution. For a simple connected planar (n, m) graph,

We have, $m \leq 3n - 6$

or
$$n \geq \frac{1}{3}(m + 6)$$

When $m = 11$, we get $n \geq \dfrac{17}{3}$.

Thus, the required minimum number of vertices is 6.

Problem 2.33. *Verify Euler's formula for the graph shown in Figure (2.35).*

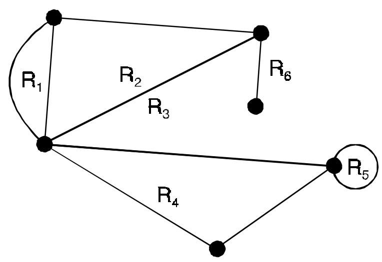


Fig. 2.35.

Solution. The graph has $n = 6$ vertices, $m = 10$ edges and $r = 6$ regions.

Therefore $\quad n - m + r = 6 - 10 + 6 = 2$

Thus, Euler's formula is verified.

Problem 2.34. *What is the maximum number of edges possible in a simple connected planar graph with eight vertices ?*

Solution. When $\quad n = 8,$

$$m \leq 3n - 6 = 18$$

Thus, the maximum number of edges possible is 18.

Theorem 2.5. *A graph is planar if and only if each of its blocks is planar.*

Theorem 2.6. *Every 2-connected plane graph can be embedded in the plane so that any specified face is the exterior.*

Proof. Let f be a non exterior face of a plane block G. Embed G on a sphere and call some point interior to f the North pole.

Consider a plane tangent to the sphere at the South pole and project G onto that plane from the North pole.

The result is a plane gr aph isomorphic to G in which f is the exterior face.

Corollary :

Every planar graph can be embedded in the plane so that a prescribed line is an edge of the exterior region.

Theorem 2.7. *Every maximal planar graph with $P \geq 4$ points is 3-connected.*

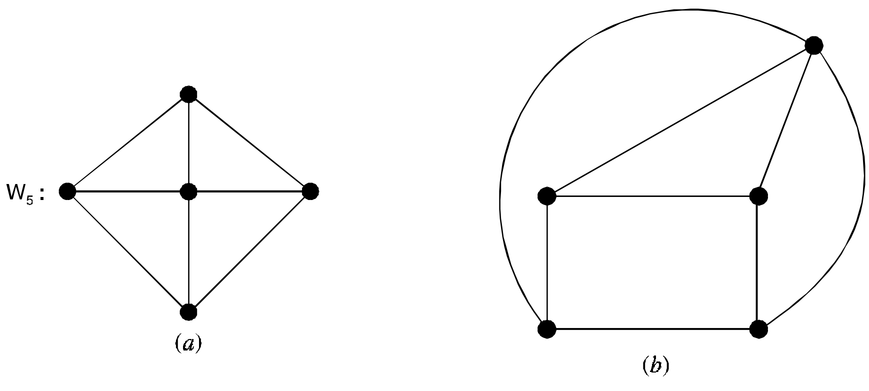


Fig. 2.36. Plane wheels.

There are five ways of embedding the 3-connected wheel W_5 in the plane : one looks like Figure (2.36)(*a*) and the other four look like Figure (2.36)(*b*).

However, there is only one way of embedding W_5 on a sphere, an observation which holds for all 3-connected graphs.

Theorem 2.8. *Every 3-connected planar graph is uniquely embeddable on the sphere.*

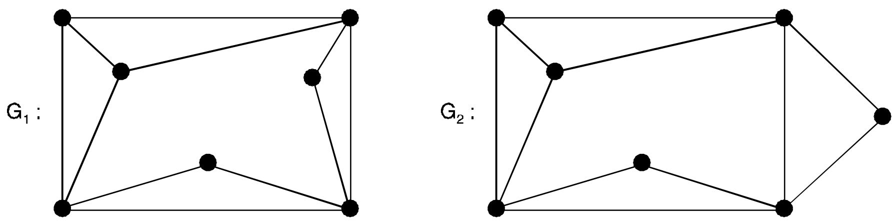


Fig. 2.37. Two plane embeddings of a 2-connected graph.

To show the necessity of 3-connectedness, consider the isomorphic graphs G_1 and G_2 of connectivity 2 shown in Figure above.

The graph G_1 is embedded on the sphere so that none of its regions are bounded by five edges while G_2 has two regions bounded by five edges.

Theorem 2.9. *A graph is the 1-skeleton of a convex 3-dimensional polyhedron if and only if it is planar and 3-connected.*

Theorem 2.10. *Every planar graph is isomorphic with a plane graph in which all edges are straight segments.*

Theorem 2.11. *A graph G is outer planar if and only if each of its blocks is outerplanar.*

Theorem 2.12. *Let G be a maximal outerplane graph with $P \geq 3$ vertices all lying on the exterior face. Then G has $P - 2$ interior faces.*

Proof. Obviously the result holds for $P = 3$.

Suppose it is true for $P = n$ and let G have $P = n + 1$ vertices and m interior faces.

Clearly G must have a vertex v of degree 2 on its exterior face.

In forming $G - v$ we reduce the number of interior faces by 1 so that $m - 1 = n - 2$.

Thus $m = n - 1 = P - 2$, the number of interior faces of G.

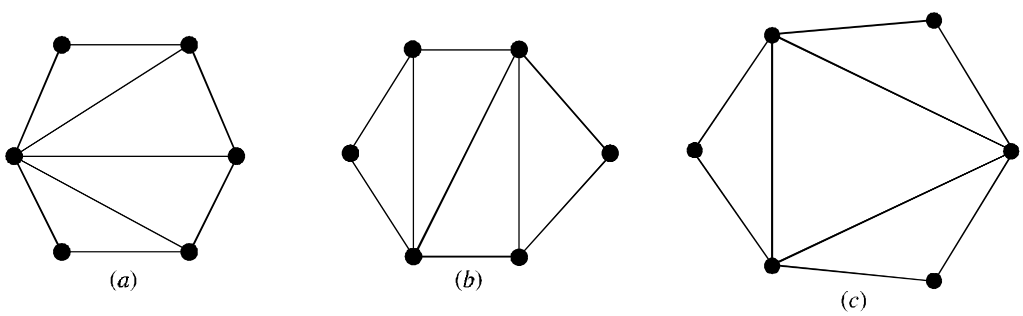


Fig. 2.38. Three maximal outerplanar graphs.

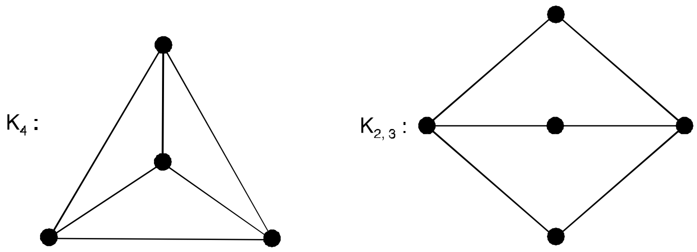


Fig. 2.39. The forbidden graphs for outer planarity.

Corollary :

Every maximal out planar graph G with P points has

(*a*) 2P – 3 lines

(*b*) at least three points of degree not exceeding 3.

(*c*) at least two points of degree 2.

(*d*) K(G) = 2.

All plane embeddings of K_4 and $K_{2,3}$ are of the forms shown in Figure (2.39) above, in which each has a vertex inside the exterior cycle.

Therefore, neither of these graphs is outer planar.

Theorem 2.13. *A graph is outer planar if and only if it has no subgraph homeomorphic to K_4 or $K_{2,3}$ except $K_4 - x$.*

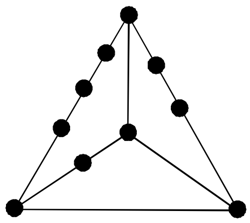


Fig. 2.40. A homeomorph of K_4.

Theorem 2.14. *Every planar graph with atleast nine points has a non planar complement, and nine is the smallest such number.*

Theorem 2.15. *Every outerplanar graph with atleast seven points has a non outer planar complement, and seven is the smallest such number.*

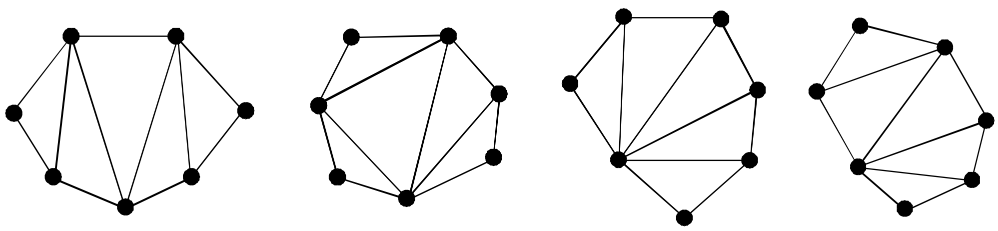


Fig. 2.41. The four maximal outer planar graphs with seven points.

Proof. To prove the first part, it is sufficient to verify that the complement of every maximal outerplanar graph with seven points is not outer planar.

This holds because there are exactly four maximal outer planar graphs with P = 7. (See Figure above) and the complement of each is readily seen to be non outer planar.

The minimality follows from the fact that the (maximal) outer planar graph of Figure below, with six points has an outer planar complement.

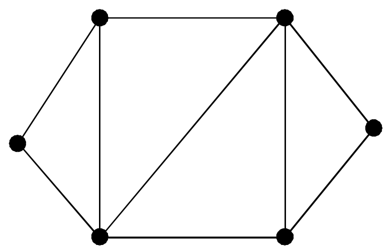


Fig. 2.42.

Lemma 1.

There is a cycle in F containing u_0 and v_0.

Proof. Assume that there is no cycle in F containing u_0 and v_0.

Then u_0 and v_0 lie in different blocks of F.

Hence, there exists a cut point W of F lying on every $u_0 - v_0$ path.

We form the graph F_0 by adding to F the lines wu_0 and wv_0 if they are not already present in F.

In the graph F_0, u_0 and v_0 still lie in different blocks, say B_1 and B_2, which necessarily have the point W in common. Certainly, each of B_1 and B_2 has fewer lives than G, so either B_1 is planar or it contains a subgraph homeomorphic to K_5 or $K_{3,\,3}$.

If, however, the insertion of wu_0 produces a subgraph H of B_1 homeomorphic to K_5 or $K_{3,\,3}$, then the subgraph of G obtained by replacing wu_0 by a path from u_0 to W which begins with x_0 is necessarily homeomorphic to H and so to K_5 or $K_{3,\,3}$, but this is a contradiction.

Hence, B_1 and similarly B_2 is planar. Both B_1 and B_2 can be drawn in the plane so that the lines wu_0 and wv_0 bound the exterior region.

Hence it is possible to embed the graph F_0 in the plane with both wu_0 and wv_0 on the exterior region.

Inserting x_0 cannot then destroy the planarity of F_0. Since G is a subgraph of $F_0 + x_0$, G is planar, this contradiction shows that there is a cycle in F containing u_0 and v_0.

Let F be embedded in the plane in such a way that a cycle Z containing u_0 and v_0 has a maximum number of regions interior to it.

Orient the edges of Z in a cyclic fashion, and let Z[u, v] denote the oriented path from u to v along Z.

If v does not immediately follow u to z, we also write Z(u, v) to indicate the subgraph of Z[u, v] obtained by removing u and v.

By the exterior of cycle Z, we mean the subgraph of F induced by the vertices lying outside Z, and the components of this subgraph are called the exterior components of Z.

By an outer piece of Z, we mean a connected subgraph of F induced by all edges incident with atleast one vertex in some exterior component or by an edge (if any) exterior to Z meeting two vertices of Z. In a like manner, we define the interior of cycle Z, interior component, and inner piece.

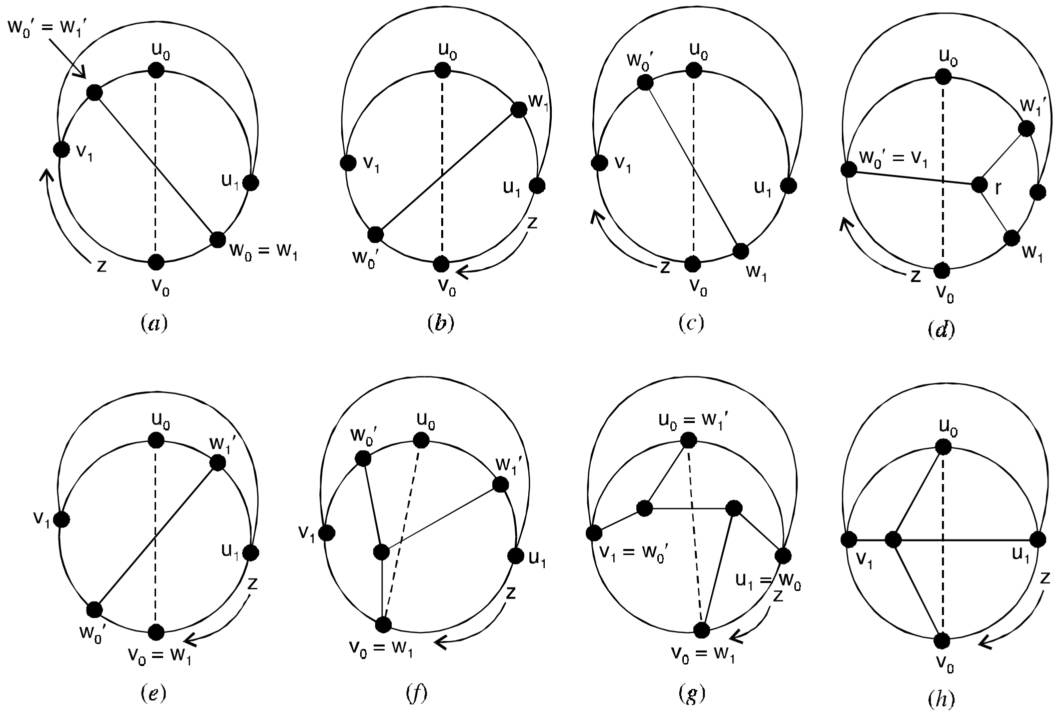


Fig. 2.43. Separating cycle Z illustrating lemma.

An outer or inner piece is called $u - v$ separating if it meets both Z(u, v) and Z(v, u).

Clearly, an outer or inner piece cannot be $u - v$ separating if u and v are adjacent on Z.

Since F is connected, each outer piece must meet Z, and because F has no cut vertices, each outer piece must have atleast two vertices in common with Z.

No outer piece can meet Z(u_0, v_0) or Z(v_0, u_0) in more than one vertex, for otherwise there would exist a cycle containing u_0 and v_0 with more interior regions than Z.

For the same region, no outer piece can meet u_0 or v_0.

Hence every outer piece meets Z in exactly two vertices and is $u_0 - v_0$ separating.

Further more, since x_0 cannot be added to F in planar fashion, there is at least one $u_0 - v_0$ separating inner piece.

Lemma 2.

There exists a $u_0 - v_0$ separating outer piece meeting Z(u_0, v_0), say at u_1, and Z(v_0, u_0), say at v_1, such that there is an inner piece which is both $u_0 - v_0$ separating and $u_1 - v_1$ separating.

Proof. Suppose, to the contrary, that the lemma does not hold. It will be helpful in understanding this proof to refer to Figure (2.43).

We order the $u_0 - v_0$ separating inner pieces for the purpose of relocating them in the plane. Consider any $u_0 - v_0$ separating inner piece I_1 which is nearest to u_0 in the sense of encountering points of this inner piece on moving along Z from u_0. Continuing out from u_0, we can index the $u_0 - v_0$ separating inner pieces I_2, I_3 and so on.

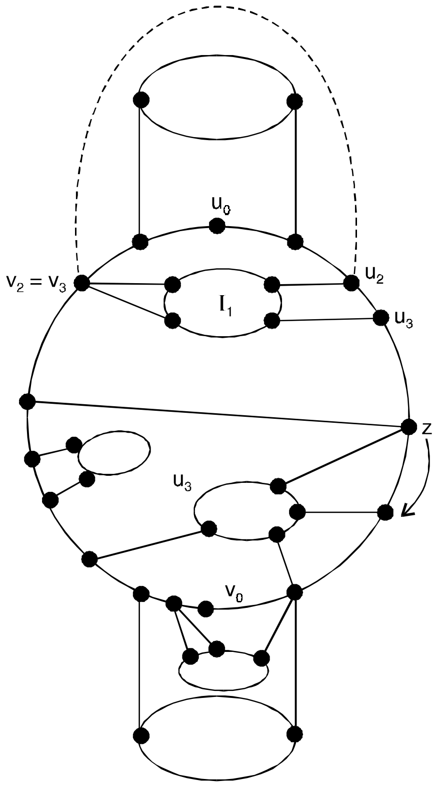


Fig. 2.44. The possibilities for non planar graphs.

Let u_2 and u_3 be the first and last points of I_1 meeting $Z(u_0, v_0)$ and v_2 and v_3 be the first and last vertices of I_1 meeting $Z(v_0, u_0)$.

Every outer piece necessarily has both its common vertices with Z on either $Z[v_3, u_2]$ or $Z[u_2, v_2]$, for otherwise, there would exist an outer piece meeting $Z(u_0, v_0)$ at u_1 and $Z(v_0, u_0)$ at v_1 and an inner piece which is both $u_0 - v_0$ separating and $u_1 - v_1$ separating, contrary to the supposition that the lemma is false.

Therefore, a curve C joining v_3 and u_2 can be drawn in the exterior region so that it meets no edge of F (see Figure (2.43).

Thus, I_1 can be transferred outside of C in a planar manner.

Similarly, the remaining $u_0 - v_0$ separating inner pieces can be transferred outside of Z, in order, so that the resulting graph is plane.

However, the edge x_0 can then be added without destroying the planarity of F, but this is a contradiction, completing the lemma.

2.12.2. Kuratowski's Theorem

A graph is planar if and only if it has no subgraph homeomorphic to K_5 or $K_{3, 3}$.

Proof. Let H be the inner piece guaranteeed by lemma (2) which is both $u_0 - v_0$ separating and $u_1 - v_1$ separating. In addition, let w_0, w_0', w_1 and w_1' be vertices at which H meets $Z(u_0, v_0)$, $Z(v_0, u_0)$, $Z(u_1, v_1)$ and $Z(v_1, u_1)$ respectively.

There are now four cases to consider, depending on the relative position on Z of these four vertices.

Case 1. One of the vertices w_1 and w_1' is on $Z(u_0, v_0)$ and the other is on $Z(v_0, u_0)$.

We can then take, say, $w_0 = w_1$ and $w_0' = w_1'$, in which case G contains a subgraph homeomorphic to $K_{3, 3}$ as indicated in Figure (2.44)(*a*) in which the two sets of vertices are indicated by open and closed dots.

Case 2. Both vertices w_1 and w_1' are on either $Z(u_0, v_0)$ or $Z(v_0, u_0)$.

Without loss of generality we assume the first situation. There are two possibilities : either $v_1 \neq w_0'$ or $v_1 = w_0'$.

If $v_1 \neq w_0'$, then G contains a subgraph homeomorphic to $K_{3, 3}$ as shown in Figure (2.44)(*b* or *c*), dependending on whether w_0' lies on $Z(u_1, v_1)$ or $Z(v_1, u_1)$ respectively.

If $v_1 = w_0'$ (see Figure 2.44), then H contains a vertex *r* from which there exist disjoint paths to w_1, w_1' and v_1, all of whose vertices (except w_1, w_1' and v_1) belong to H.

In this case also, G contains a subgraph homeomorphic to $K_{3, 3}$.

Case 3. $w_1 = v_0$ and $w_1' \neq u_0$.

Without loss of generality, let w_1' be on $Z(u_0, v_0)$. Once again G contains a subgraph homeomorphic to $K_{3, 3}$.

If w_0' is on (v_0, v_1), then G has a subgraph $K_{3, 3}$ as shown in Figure 2.44(*e*).

If, on the other hand, w_0' is on $Z(v_1, u_0)$, there is a $K_{3, 3}$ as indicated in Figure 2.44(*f*).

This Figure is easily modified to show G contains $K_{3, 3}$ if $w_0' = v_1$.

Case 4. $w_1 = v_0$ and $w_1' = u_0$.

Here we assume $w_0 = u_1$ and $w_0' = v_1$, for otherwise we are in a situation covered by one of the first 3 cases.

We distinguish between two subcases.

Let P_0 be a shortest path in H from u_0 to v_0, and let P_1 be such a path from u_1 to v_1,

The paths P_0 and P_1 must intersect.

If P_0 and P_1 have more than one vertex in common, then G contains a subgraph homeomorphic to $K_{3, 3}$ as shown in Figure 2.44(*g*).

Otherwise, G contains a subgraph homeomorphic to K_5 as in Figure 2.44(*h*).

Since these are all possible cases, the theorem has been proved.

Theorem 2.17. *A graph is planar if and only if it does not have a subgraph contractible to K_5 or $K_{3,3}$.*

2.13 DETECTION OF PLANARITY OF A GRAPH :

If a given graph G is planar or non planar is an important problem. We must have some simple and efficient criterion. We take the following simplifying steps :

Elementary Reduction :

Step 1 : Since a disconnected graph is planar if and only if each of its components is planar, we need consider only one component at a time. Also, a separable graph is planar if and only if each of its blocks is planar. Therefore, for the given arbitrary graph G, determine the set.

$$G = \{G_1, G_2, \ldots\ldots G_k\}$$

where each G_i is a non separable block of G.

Then we have to test each G_i for planarity.

Step 2 : Since addition or removal of self-loops does not affect planarity, remove all self-loops.

Step 3 : Since parallel edges also do not affect planarity, eliminate edges in parallel by removing all but one edge between every pair of vertices.

Step 4 : Elimination of a vertex of degree two by merging two edges in series does not affect planarity. Therefore, eliminate all edges in series.

Repeated application of step 3 and 4 will usually reduce a graph drastically.

For example, Figure (2.46) illustrates the series-parallel reduction of the graph of Figure (2.45).

Let the non separable connected graph G_i be reduced to a new graph H_i after the repeated application of step 3 and 4. What will graph H_i look like ?

Graph H_i is

1. A single edge, or
2. A complete graph of four vertices, or
3. A non separable, simple graph with $n \geq 5$ and $e \geq 7$.

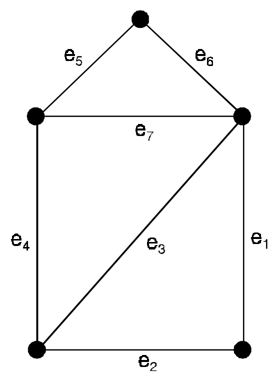


Fig. 2.45.

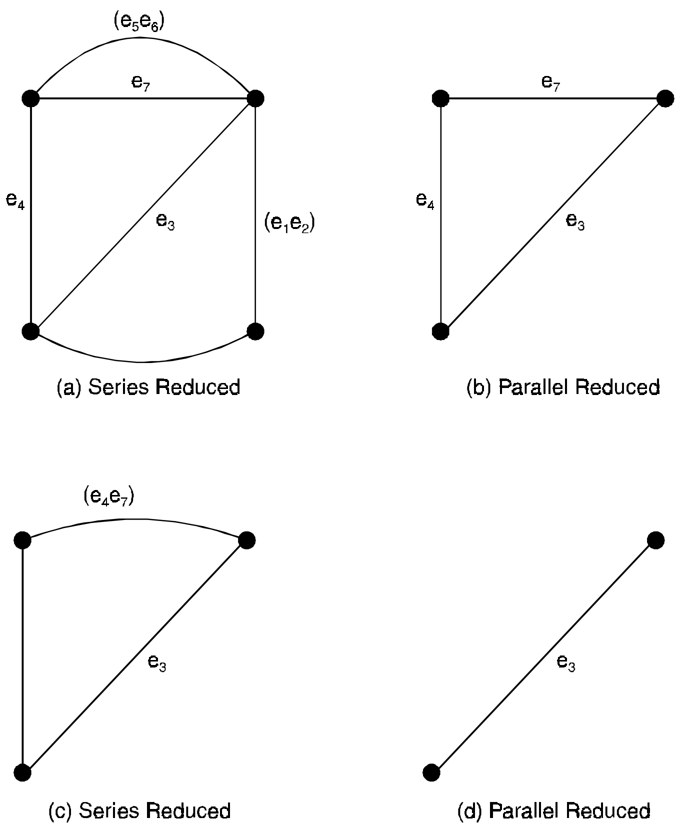


Fig. 2.46. Series-parallel reduction of the graph in Figure 2.45

Problem 2.35. *Check the planarity of the following graph by the method of elementary deduction.*

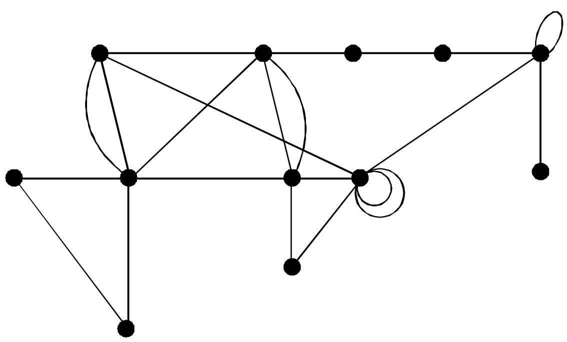


Fig. 2.47.

Solution. Step 1 : Does not apply, because the graph is connected.

Step 2 : Separating blocks of G

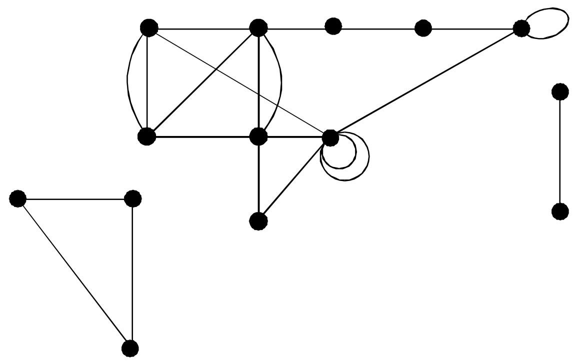


Fig. 2.48.

Step 3 : Removing self-loops and parallel edges

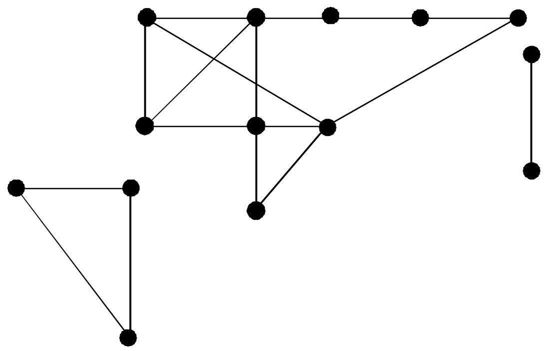


Fig. 2.49.

Step 4 : Merging the series edges.

The final graph contains three components. Largest component contains 7 vertices. Remaining two is triangle and an edge hence they are planar. The largest component contains no subgraph isomorphic to K_5 or $K_{3,\,3}$ and hence it is planar.

Thus the given graph is planar.

Problem 2.36. *Check the planarity of the following graph by the method of elementary reduction.*

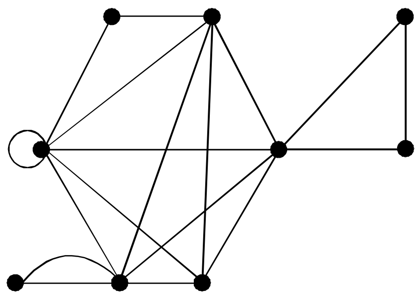


Fig. 2.50.

Solution. The elementary reduction of the given graph G consists of the following stages :

Step 1 : Splitting G into blocks. This splitting is shown below :

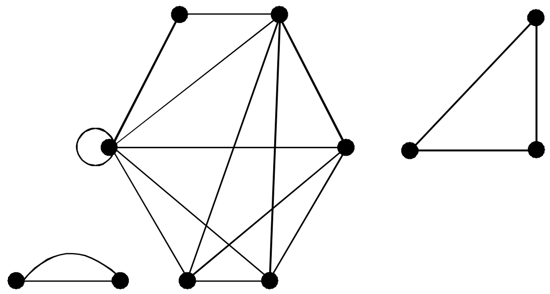


Fig. 2.51.

Step 2 : Removing self-loops and eliminating multiple edges. The resulting graph is as shown below :

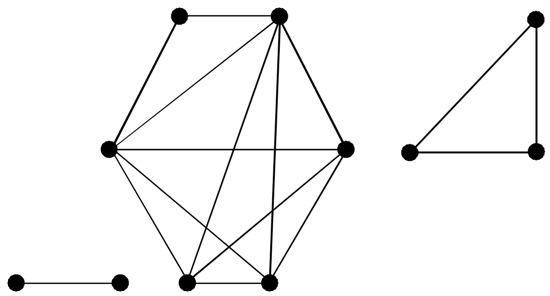


Fig. 2.52.

Step 3 : Merging the edges incident on vertices of degree 2. The resulting graph is as shown below :

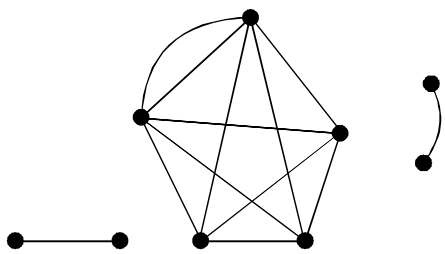


Fig. 2.53.

Step 4 : Eliminating parallel edges. The resulting graph is shown below :

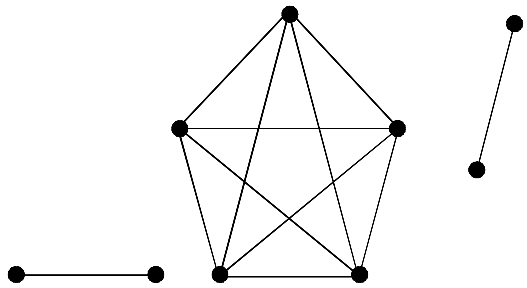


Fig. 2.54.

The reduction is now complete. The final reduced graph (shown in Figure above) has three blocks, of which the first and the third are obviously planar. The second one is evidently the complete graph K_5, which is non planar.

Thus, the given graph contains K_5 as a subgraph and is therefore non planar.

Problem 2.37. *Carryout the elementary reduction process for the following graph :*

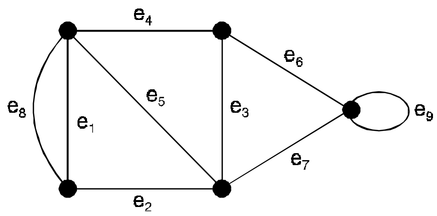


Fig. 2.55.

Solution. The given graph G is a single non separable block. Therefore, the set A of step 1 contains only G. As per step 2, we have to remove the self loops. In the graph, there is one self-loop consisting of the edge e_9. Let us remove it.

As per step 3, we have to remove one of the two parallel edges from each vertex pair having such edges. In the given graph, e_1, e_8 are parallel edges. Let us remove e_8 from the graph.

The graph left-out after the first three steps is as shown below :

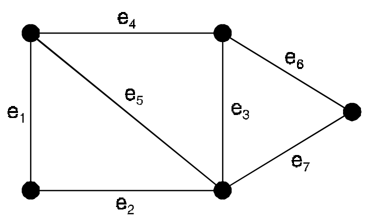


Fig. 2.56.

As per step 4, we have to eliminate the vertices of degree 2 by merging the edges incident on these vertices.

Thus, we merge (*i*) the edges e_1 and e_2 into an edge e_{10} (say) and (*ii*) the edges e_6 and e_7 into an edge e_{11} (say).

The resulting graph will be as shown below :

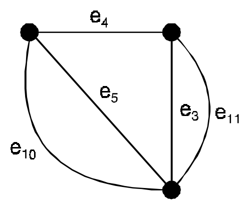


Fig. 2.57.

As per step 3, let us remove one of the parallel edges e_5 and e_{10} and one of the parallel edges e_3 and e_{11}. The graph got by removing e_{10} and e_{11} will be as shown below :

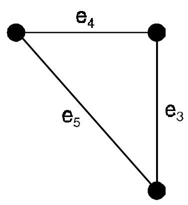


Fig. 2.58.

As per step 4, we merge the edges e_3 and e_4 into an edge e_{12} (say) to get the following graph.

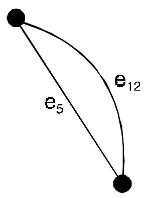
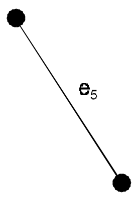


Fig. 2.59.

As per step 3, we remove one of the two parallel edges, say e_{12}. Thus, we get the following graph :



This graph is the final graph obtained by the process of elementary reduction applied to the graph in Figure (1). This final graph which is a single edge is evidently a planar graph.

Therefore, the graph in Figure (1) is also planar.

2.14 DUAL OF A PLANAR GRAPH

Consider the plane representation of a graph in Figure (2.60)(*a*) with six regions of faces F_1, F_2, F_3, F_4, F_5 and F_6.

Let us place six points P_1, P_2, ... P_6, one in each of the regions, as shown in Figure (2.60)(*b*).

Next let us join these six points according to the following procedure :

(*i*) If two regions F_i and F_j are adjacent (*i.e.*, have a common edge), draw a line joining points P_i and P_j that intersects the common edge between F_i and F_j exactly once.

(*ii*) If there is more than one edge common between F_i and F_j, draw one line between points P_i and P_j for each of the common edges.

(*iii*) For an edge *e* lying entirely in one region, say F_k, draw a self-loop at point P_k intersecting *e* exactly once.

By this procedure we obtained a new graph G* (in broken lines in Figure (2.60)(*c*) consisting of six vertices, P_1, P_2, ... P_6 and of edges joining these vertices. Such a graph G* is called **dual** (a geometrical dual) of G

Clearly, there is a one-to-one correspondence between the edges of graph G and its dual G*— one edge of G* intersecting one edge of G. Some simple observations that can be made about the relationship between a planar graph G and its dual G* are :

(*i*) An edge forming a self-loop in G yields a pendant edge in G*.

(*ii*) A pendant edge in G yields a self-loop in G*.

(*iii*) Edges that are in series in G produce parallel edges in G*.

(*iv*) Parallel edges in G produce edges in series in G*.

(*v*) Remarks (*i*)-(*iv*) are the result of the general observation that the number of edges constituting the boundary of a region F_i in G is equal to the degree of the corresponding vertex P_i in G*.

(*vi*) Graph G* is also embedded in the plane and is therefore planar.

(*vii*) Considering the process of drawing a dual G* from G, it is evident that G is a dual of G* (see Fig. (2.60) (*c*)). Therefore, instead of calling G* a dual of G, we usually say that G and G* are dual graphs.

(*viii*) If n, e, f, r and μ denote as usual the numbers of vertices, edges, regions, rank, and nullity of a connected planar graph G, and if n^*, e^*, f^*, r^* and μ^* are the corresponding numbers in dual graph G*, then

$$n^* = f,\ e^* = e,\ f^* = n.$$

Using the above relationship, one can immediately get $r^* = \mu,\ \mu^* = r$.

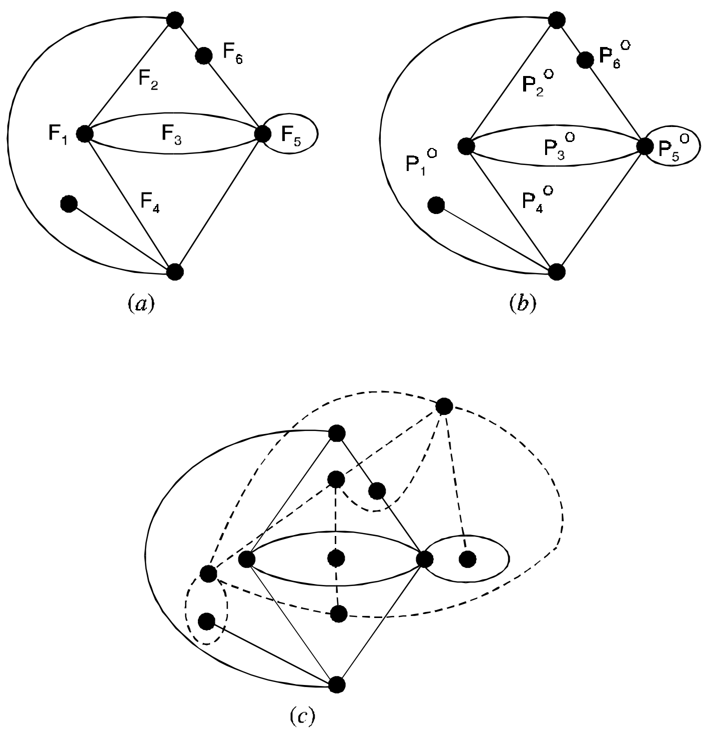


Fig. 2.60. Construction of a dual graph.

2.14.1. Uniqueness of the dual

Given a planar graph G, we can construct more than one geometric dual of G. All the duals so constructed have one important property. This property is stated in the following result :

All geometric duals of a planar graph G are 2-isomorphic, and every graph 2-isomorphic to a geometric dual of G is also a geometric dual of G.

2.14.2. Double dual

Given a planar graph G, suppose we construct its geometric dual G* and the geometric dual G** of G*.

Then G** is called a double geometric dual of G.

If G is a planar graph, then G** and G are 2-isomorphic.

2.14.3. Self-dual graphs

A planar graph G is said to be self-dual if G is isomorphic to its geometric dual G*, *i.e.*, if $G \approx G^*$.

Consider the complete graph K_4 of four vertices show in Figure (2.61)(*a*). Its geometric dual K_4^* can be constructed. This is shown in Figure (2.61)(*b*).

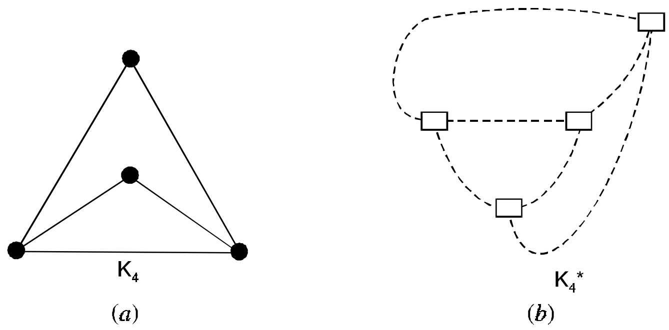


Fig. 2.61.

We observe that K_4* has four vertices and six edges. Also, every two vertices of K_4* are joined by an edge. This means that K_4* also represents the complete graph of four vertices. As such, K_4 and K_4* are isomorphic. In other words, K_4 is a self-dual graph.

2.14.4. Dual of a subgraph

Let G be a planar graph and G* be its geometric dual. Let *e* be an edge in G and *e** be its dual in G*. Consider the subgraph G – *e* got by deleting *e* from G. Then, the geometric dual of G – *e* can be constructed as explained in the two possible cases.

Case (1) :

Suppose *e* is on a boundary common to two regions in G.

Then the removal of *e* from G will merge these two regions into one.

Then the two corresponding vertices in G* get merged into one, and the edge *e** gets deleted from G*.

Thus, in this case, the dual of G – *e* can be obtained from G* by deleting the edge *e** and then fusing the two end vertices of *e** in G* – *e**.

Case (2) :

Suppose *e* is not on a boundary common to two regions in G.

Then *e* is a pendant edge and *e** is a self-loop.

The dual of G – *e* is now the same as G* – *e**.

Thus, the geometric dual of G – *e* can be constructed for all choices of the edge *e* of G.

Since every subgraph H of a graph is of the form G – *s* where *s* is a set edges of G.

2.14.5. Dual of a homeomorphic graph

Let G be a planar graph and G* be its geometric dual.

Let *e* be an edge in G and *e** be its dual in G*.

Suppose we create an additional vertex in G by introducing a vertex of degree 2 in the edge *e*. This will simply add an edge parallel to *e** in G*. If we merge two edges in series in G then one of the corresponding parallel edges in G* will be eliminated. The dual of any graph homeomorphic to G can be obtained from G*.

2.14.6. Abstract dual

Given two graphs G_1 and G_2, we say that G_1 and G_2 are abstract duals of each other if there is a one-to-one correspondence between the edges in G_1 and the edges in G_2, with the property that a set of edges in G_1 forms a circuit in G_1 if and only if the corresponding set of edges in G_2 forms a cut-set in G_2.

Consider the graphs G_1 and G_2 shown in Figure (2.62).

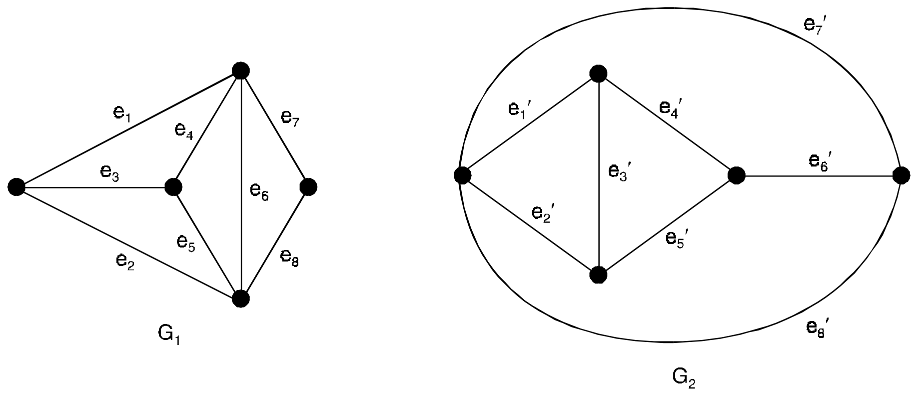


Fig. 2.62.

We observe that there is a one-to-one correspondence between the edges in G_1 and the edges in G_2 with the edge e_i in G_1 corresponding to the edge e_i' in G_2, $i = 1, 2, ... 8$.

Further, note that a set of edges in G_1 which forms a circuit in G_1 corresponds to a set of edges in G_2 which forms a cut sets in G_2.

For example, $\{e_6, e_7, e_8\}$ is a circuit in G_1 and $\{e_6', e_7', e_8')$ is a cut-set in G_2.

Accordingly, G_1 and G_2 are abstract duals of each other.

2.14.7. Combinatorial dual

Given two planar graphs G_1 and G_2, we say that they are combinatorial duals of each other if there is a one-to-one correspondence between the edges of G_1 and G_2 such that if H_1 is any subgraph of G_1 and H_2 is the corresponding subgraph of G_2, then

Rank of $(G_2 - H_2)$ = Rank of G_2 – Nullity of H_1

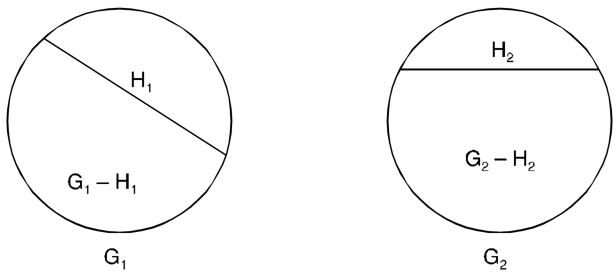
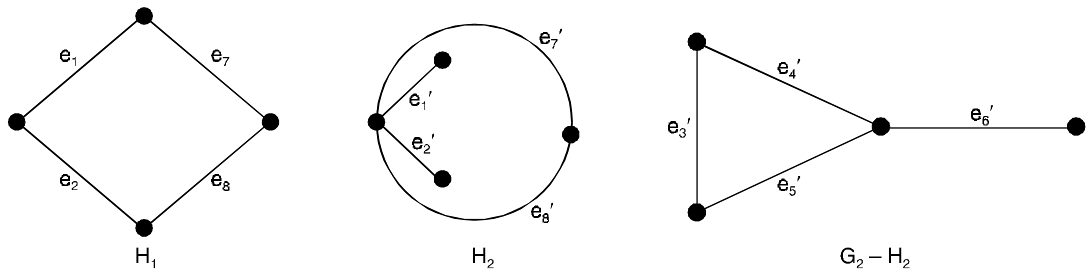


Fig. 2.63.

Consider the graph G_1 and G_2 shown in Figure (2.62) above, and their subgraphs H_1 and H_2 shown in Figure (2.64)(*a*, *b*).



(*a*) (*b*) (*c*)

Fig. 2.64.

Note that there is one-to-one correspondence between the edges of G_1 and G_2 and that the subgraphs H_1 and H_2 correspond to each other.

The graph of $G_2 – H_2$ is shown in Figure (2.64)(*c*).

This graph is disconnected and has two components.

Rank of $G_2 = 5 – 1 = 4$, Rank of $H_1 = 4 – 1 = 3$

Nullity of $H_1 = 4 – 3 = 1$

Rank of $(G_2 – H_2) = 5 – 2 = 3$.

$\Rightarrow$ Rank of $(G_2 – H_2) = 3$ = Rank of G_2 – Nullity of H_1.

Hence, G_1 and G_2 are combinatorial duals of each other.

Theorem 2.18. *If G is a plane connected graph, then G** is isomorphic to G.*

Proof. The result follows immediately, since the construction that gives rise to G* from G can be reversed to give G from G*,

For example, in Figure (2.65), the graph G is the dual of the graph G*

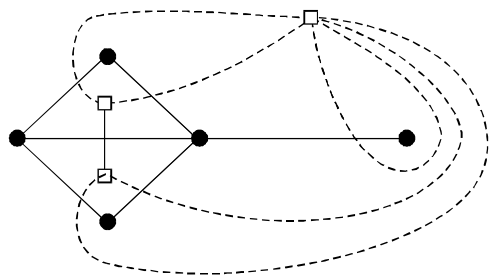


Fig. 2.65.

We need to check only that a face of G* cannot contain more than one vertex of G (it certainly contains at least one) and this follows immediately from the relations $n^{**} = f^* = n$, where n^{**} is the number of vertices of G**.

Theorem 2.19. *Let G be a planar graph and let G* be a geometric dual of G. Then a set of edges in G forms a cycle in G if and only if the corresponding set of edges of G* forms a cutset in G*.*

Proof. We can assume that G is a connected plane graph. If C is a cycle in G, then C encloses one or more finite faces C, and thus contains in its interior a non-empty set S of vertices of G*.

It follows immediately that choose edges of G* that cross the edges of C form a cutset of G* whose removal disconnects G* into two subgraphs, one with vertex set S and the other containing those vertices that do not lie in S (see Figure 2.66).

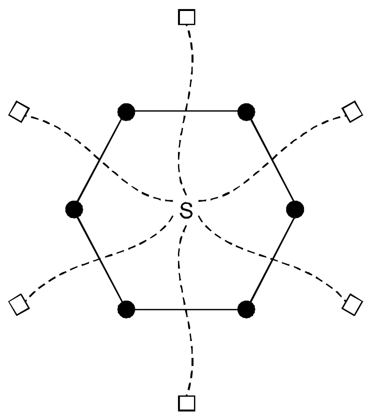


Fig. 2.66.

Corollary : A set of edges of G forms a cutset in G if and only if the corresponding set of edges of G* forms a cycle in G*.

Theorem 2.20. *If G* is an abstract dual of G, then G is an abstract dual of G*.*

Proof. Let C be a cutset of G and let C* denote the corresponding set of edges of G*.

We show that C* is a cycle of C*.

C has an even number of edges in common with any cycle of G, and so C* has an even number of edges in common with any cut set of G*.

C* is either a cycle in G* or an edge-disjoint union of at least two cycles.

But the second possibility cannot occur, since we can show similarly that cycles in C* correspond to edge-disjoint unions of cut sets in G, and so C would be an edge-disjoint union of at least two cutsets, rather than a single cutset.

Theorem 2.21. *A graph is planar if and only if it has an abstract dual.*

Proof. It is sufficient to prove that if G is a graph with an abstract dual G*, then G is planar. The proof is in four steps.

(*i*) We note first that if an edge *e* is removed from G, then the abstract dual of the remaining graph may be obtained from G* by contracting the corresponding edge *e**.

On the repeating this procedure, we deduce that, if G has an abstract dual, then so does any subgraph of G.

(*ii*) We next observe that if G has an abstract dual, and G′ is homeomorphic to G, then G′ also has an abstract dual.

This follows from the fact that the insertion or removal in G of a vertex of degree 2 results in the addition or deletion of a multiple edge in G*.

(*iii*) The third step is to show that neither K_5 nor $K_{3,3}$ has an abstract dual.

If G* is a dual of $K_{3,3}$ then since $K_{3,3}$ contains only cycles of length 4 or 6 and no cutsets with two edges, G* contains no multiple edges and each vertex of G* has degree at least 4.

Hence G* be have at least five vertices, and thus atleast $\frac{(5\times 4)}{2}$ = 10 edges, which is a contradiction.

The argument for K_5 is similar and is omitted.

(*iv*) Suppose, now, that G is a non-planar graph with an abstract dual G*.

Then, by Kuratowski's theorem, G has a subgraph H homeomorphic to K_5 to $K_{3,3}$.

It follows from (*i*) and (*ii*) that H, and hence also K_5 or $K_{3,3}$, must have an abstract dual, contradicting (*iii*).

Theorem 2.22. *Let G be a connected planar graph with n vertices, m edges and r regions, and let its geometric dual G* have n* vertices, m* edges and r* regions. Then n* = r, m* = m, r* = n.*

Further, if ρ and ρ are the ranks and μ and μ* are the nullities of G and G* respectively, then ρ* = μ and μ* = ρ.*

Proof. Every region of G yields exactly one vertex of G* and G* has no other vertex.

Hence the number of regions in G is precisely equal to the number of vertices of G*,

i.e., $r = n^*$. ...(1)

Corresponding to every edge e of G, there is exactly one edge e^* of G* that crosses e exactly once, and G* has no other edge.

Thus G and G* have the same number of edges,

i.e., $m = m^*$ ...(2)

Now, the Euler's formula applied to G* and G yields

$$r^* = m^* - n^* + 2$$
$$= m - r + 2$$
$$= n$$

Since G and G* are connected, we have

$$\rho = n - 1, \quad \mu = m - n + 1$$
$$\rho^* = n^* - 1, \quad \mu^* = m^* - n^* + 1$$

These together with the results (1) and (2) and the Euler's formula yield

$$\rho^* = n^* - 1 = r - 1 = (m - n + 2) - 1$$
$$= m - n + 1 = \mu$$
$$\mu^* = m^* - n^* + 1 = m - r + 1$$
$$= m - (m - n + 2) + 1 = n - 1 = \rho.$$

Theorem 2.23. *A graph has a dual if and only if it is planar.*

Proof. Suppose that a graph G is planar.

Then G has a geometric dual in G*.

Since G* is a geometric dual, it is a dual.

Thus G has a dual.

Conversely, suppose G has a dual.

Assume that G is non planar. Then by Kuratowski's theorem, G contains K_5 and $K_{3, 3}$ or a graph homeomorphic to either of these as a subgraph.

But K_5 and $K_{3, 3}$ have no duals and therefore a graph homeomorphic to either of these also has no dual.

Thus, G contains a subgraph which has no dual.

Hence G has no dual. This is a contradiction.

Hence G is planar if it has a dual.

Problem 2.38. *If G is a 3-connected planar graph, prove that its geometric dual is a simple graph.*

Solution. If G is 3-connected, then G has no vertices of degree 1 or 2.

Therefore, G* has no self-loops or multiple edges. That is, G* is simple.

Problem 2.39. *Show that a connected planar self-dual graph G with n vertices should have 2n – 2 edges.*

Solution. Since the graph G is self-dual, we have $n = n^*$. But $n^* = r$,

Therefore, in G, $n = r$,

The Euler's formula now gives $n = m - n + 2$

or $$m = 2n - 2.$$

Problem 2.40. *Show that a set of edges in a connected planar graph G forms a spanning tree of G if and only if the set of duals of the remaining edges forms a spanning tree of a geometric dual of G.*

Solution. Consider a connected planar graph G with n vertices and m edges.

Let T be a spanning tree of G. This is a set of $n - 1$ edges. The remaining edges are $m - (n - 1)$ in number.

The duals of these edges are also $m - (n - 1)$ in number.

The set T* of these duals belong to G*.

Since G* has $m - n + 2$ vertices, the set T* which consists of $m - n + 1$ vertices is a spanning tree of G*.

This proves the first part of the required result.

By reversing the roles of G and G* in the above argument, we get the second proof.

Problem 2.41. *Show that there is no planar graph with five regions such that there is an edge between every pair of regions.*

Solution. Suppose there is a planar graph G having the desired property.

Then, the geometric dual G* of G will have five vertices such that there is an edge between every pair of vertices.

This means that G* is the graph K_5.

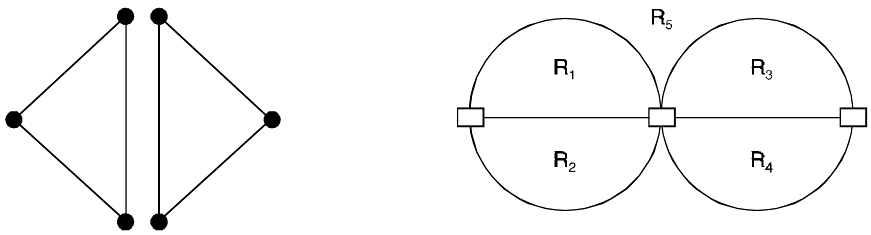
Therefore, G* is non planar.

This is a contradiction because G* has to be planar. (like G).

Hence, a planar graph of the desired type does not exist.

Problem 2.42. *Disprove that the geometric dual of the geometric dual of a planar graph G is the same as the abstract dual of the abstract dual of G.*

Solution. Consider the disconnected graph G with two components, each of which is a triangle as shown in Figure (2.67)(*a*).



(*a*) (*b*)

Fig. 2.67.

The geometric dual G* is shown in Figure (2.67)(*b*), we observe that G* has five regions.

Therefore, the geometric dual G** of G* has five vertices.

On the other hand, if G′ is the abstract dual of G, then G is the abstract dual of G′.

Hence, G is the abstract dual of the abstract dual of G. *i.e.*, G = G″.

Since G has six vertices, it follows that G″ cannot be the same as G** (which has five vertices).

The above counter example disproves that the geometric dual of the geometric dual is the same as the abstract dual of the abstract dual.

Problem 2.43. *Let G be a connected planar graph. Prove that G is bipartite if and only if its dual is on Euler graph.*

Solution. If G is bipartite, then each circuit of G has even length.

Therefore, each cutset of its dual G′ has an even number of edges.

In particular, each vertex of G′ has even degree.

Therefore G′ is an Euler graph.

Theorem 2.24. *Let G be a plane connected graph. Then G is isomorphic to its double dual G**.*

Proof. Let f be any face of the dual G* contains atleast one vertex of G, namely its corresponding vertex *v*.

In fact this is the only vertex of G that *f* contains since by theorem.

i.e., a connected graph G with *n*-vertices, *e*-edges, *f*-faces and n^*, e^*, f^* denotes the vertices, edges and faces of G* then $n^* = f$, $e^* = e$, $f^* = n$, the number of faces of G* is the same as the number of vertices of G.

Hence in the construction of double dual G**, we may choose the vertex *v* to be the vertex in G** corresponding to face *f* of G*.

This choice gives our required result.

Theorem 2.25. *Let G be a connected plane graph with n-vertices e-edges and f-faces. Let n*, e* and f* denote the number of vertices, edges and faces respectively of G*, then n* = f, e* = e and f* = n.*

Proof. The first two relations are direct consequence of the definition of G, the third relation follows immediately on substituting these two relations into Euler's theorem applied to both G and G*.

If G is a plane graph then G* is also a plane graph. We may also construct the dual of G*, called the double dual of G and denoted by G**.

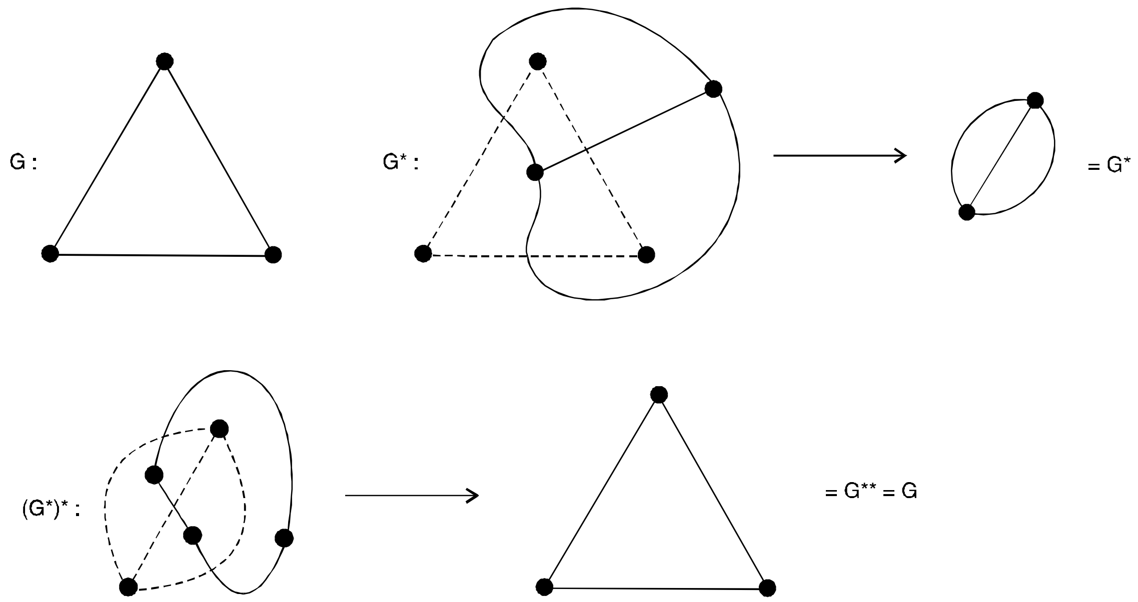


Fig. 2.68.

2.15 GRAPH COLORING

Coloring problem

Suppose that you are given a graph G with *n* vertices and are asked to paint its vertices such that no two adjacent vertices have the same color. What is the minimum number of colors that you would require. This constitutes a coloring problem.

2.15.1. Partitioning problem

Having painted the vertices, you can group them into different sets—one set consisting of all red vertices, another of blue, and so forth. This is a partitioning problem.

For example, finding a spanning tree in a connected graph is equivalent to partitioning the edges into two sets—one set consisting of the edges included in the spanning tree, and the other consisting of the remaining edges. Identification of a Hamiltonian circuit (if it exists) is another partitioning of set of edges in a given graph.

2.15.2. Properly coloring of a graph

Painting all the vertices of a graph with colours such that no two adjacent vertices have the same colour is called the proper colouring (or simply colouring) of a graph.

A graph in which every vertex has been assigned a colour according to a proper colouring is called a properly coloured graph.

Usually a given graph can be properly coloured in many different ways. Figure (2.69)(*a*) shows three different proper colouring of a graph.

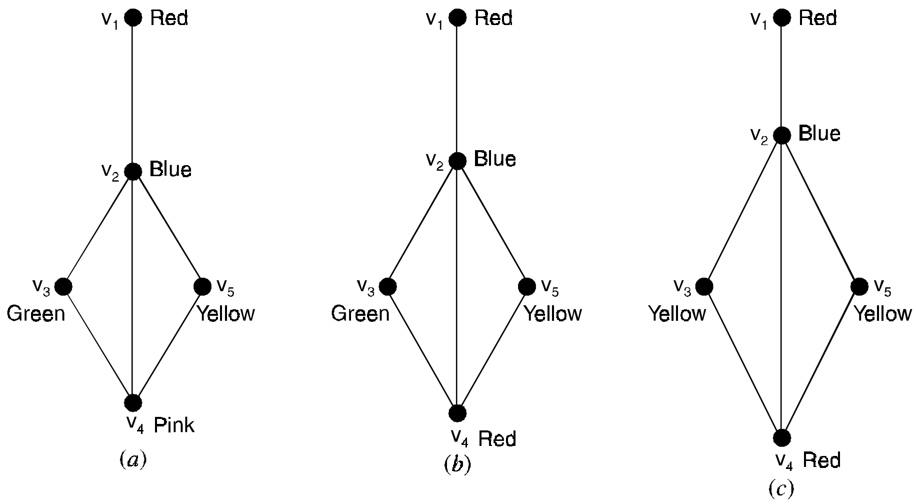


Fig. 2.69. Proper colours of a graph.

The K-colourings of the graph G is a colouring of graph G using K-colours. If the graph G has colouring, then the graph G is said to be K-colourable.

2.15.3. Chromatic number

A graph G is said to be K-colourable if we can properly colour it with K (number of) colours.

A graph G which is *n*-colourable but not (K – 1)-colourable is called a K-chromatic graph.

In other words, a K-chromatic graph is a graph that can be properly coloured with K-colours but not with less than K colours.

If a graph G is K-chromatic, then K is called chromatic number of the graph G. Thus the chromatic number of a graph is the smallest number of colours with which the graph can be properly coloured.The chromatic number of a graph G is usually denoted by $\chi(G)$.

We list a few rules that may be helpful :

1. $\chi(G) \leq |V|$, where $|V|$ is the number of vertices of G.

2. A triangle always requires three colours, that is $\chi(K_3) = 3$; more generally, $\chi(K_n) = n$, where K_n is the complete graph on n vertices.

3. If some subgraph of G requires K colors then $\chi(G) \geq K$.

4. If degree $(v) = d$, then atmost d colours are required to colour the vertices adjacent to v.

5. $\chi(G)$ = maximum $\{\chi(C)/C$ is a connected component of G$\}$

6. Every K-chromatic graph has at least K vertices v such that degree $(v) \geq k - 1$.

7. For any graph G, $\chi(G) \leq 1 + \Delta(G)$, where $\Delta(G)$ is the largest degree of any vertex of G.

8. When building a K-colouring of a graph G, we may delete all vertices of degree less than K (along with their incident edges).

In general, when attempting to build a K-colouring of a graph, it is desirable to start by K-colouring a complete subgraph of K vertices and then successively finding vertics adjacent to K – 1 different colours, thereby forcing the color choice of such vertices.

9. These are equivalent :

(*i*) A graph G is 2-colourable (*ii*) G is bipartite

(*iii*) Every cycle of G has even length.

10. If $\delta(G)$ is the minimum degree of any vertex of G, then $\chi(G) \geq \dfrac{|V|}{|V|} - \delta(G)$ where | V | is the number of vertices of G.

2.15.4. K-Critical graph

If the chromatic number denoted by (G) = K, and (G – v) is less than equal to K – 1 for each vertex v of G, then

2.16 CHROMATIC POLYNOMIAL

A given graph G of n vertices can be properly coloured in many different ways using a sufficiently large number of colours. This property of a graph is expressed elegantly by means of a polynomial. This polynomial is called the chromatic polynomial of G.

The value of the chromatic polynomial $P_n(\lambda)$ of a graph with n vertices gives the number of ways of properly colouring the graph, using λ of fewer colours. Let C_i be the different ways of properly colouring G using exactly i different colours. Since i colours can be chosen out of λ colours in $\binom{\lambda}{i}$ different ways, there are $c_i \binom{\lambda}{i}$ different ways of properly colouring G using exactly i colours out of λ colours.

Since i can be any positive integer from 1 to n (it is not possible to use more than n colours on n vertices), the chromatic polynomial is a sum of these terms, that is,

$$P_n(\lambda) = \sum_{i=1}^{n} C_i \binom{\lambda}{i}$$

$$= C_1 \frac{\lambda}{1!} + C_2 \frac{\lambda(\lambda-1)}{2!} + C_3 \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + \ldots\ldots$$

$$\ldots + C_n \frac{\lambda(\lambda-1)(\lambda-2)\ldots\ldots(\lambda-n+1)}{n!}$$

Each C_i has to be evaluated individually for the given graph.

For example, any graph with even one edge requires at least two colours for proper colouring, and therefore $C_1 = 0$.

A graph with n vertices and using n different colours can be properly coloured in n ! ways.

that is, $C_n = n\,!$.

Problem 2.44. *Find the chromatic polynomial of the graph given in Figure (2.70).*

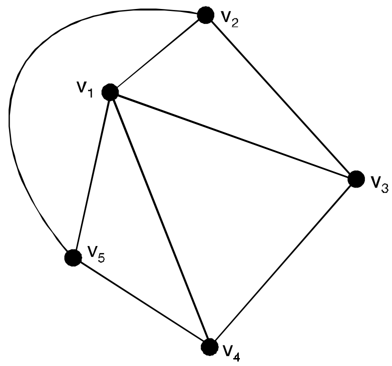


Fig. 2.70. A 3-chromatic graph.

Solution. $P_5(\lambda) = C_1\lambda + C_2 \dfrac{\lambda(\lambda-1)}{2\,!} + C_3 \dfrac{\lambda(\lambda-1)(\lambda-2)}{3\,!}$

$$+ C_4 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4\,!} + C_5 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5\,!}$$

Since the graph in Figure 2.70 has a triangle, it will require at least three different colours for proper colourings.

Therefore, $C_1 = C_2 = 0$ and $C_5 = 5\,!$

Moreover, to evaluate C_3, suppose that we have three colours x, y and z.

These three colours can be assigned properly to vertices v_1, v_2 amd v_3 in 3 ! = 6 different ways.

Having done that, we have no more choices left, because vertex v_5 must have the same colour as v_3 and v_4 must have the same colour as v_2.

Therefore, $C_3 = 6$.

Similarly, with four colours, v_1, v_2 and v_3 can be properly coloured in 4 • 6 = 24 different ways.

The fourth colour can be assigned to v_4 or v_5, thus providing two choices.

The fifth vertex provides no additional choice.

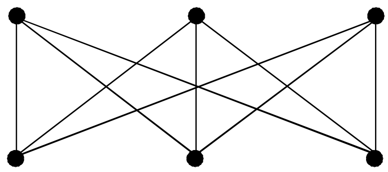
Therefore, $C_4 = 24 \bullet 2 = 48$.

Substituting these coefficients in $P_5(\lambda)$, we get, for the graph in Figure (2.70).

$$P_5(\lambda) = \lambda(\lambda - 1)(\lambda - 2) + 2\lambda(\lambda - 1)(\lambda - 2)(\lambda - 3) + \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)(\lambda - 4)$$
$$= \lambda(\lambda - 1)(\lambda - 2)(\lambda^2 - 5\lambda + 7)$$

The presence of factors $\lambda - 1$ and $\lambda - 2$ indicates that G is at least 3-chromatic.

Problem 2.45. *Find the chromatic polynomial and chromatic number for the graph* $K_{3,\,3}$.



Solution. Chromatic polynomial for $K_{3,\,3}$ is given by $\lambda(\lambda - 1)^5$.

Thus chromatic number of this graph is 2.

Since $\lambda(\lambda - 1)^5 > 0$ first when $\lambda = 2$.

Here, only two distinct colours are required to colour $K_{3,\,3}$.

The vertices *a*, *b* and *c* may have one colours, as they are not adjacent.

Similarly, vertices *d*, *e* and *f* can be coloured in proper way using one colour.

But a vertex from {*a*, *b*, *c*} and a vertex from {*d*, *e*, *f*} both cannot have the same colour.

In fact every chromatic number of any bipartite graph is always 2.

Problem 2.46. *Find the chromatic polynomial and hence the chromatic number for the graph shown below.*

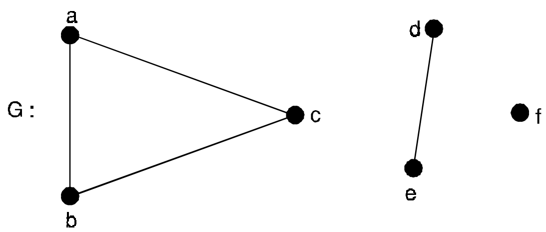


Fig. 2.71.

Solution. Since G is made up of components of G_1, G_2 and G_3 where $G_1 = K_3$, G_2 is a linear graph and G_3 is an isolated vertex.

Now G_1 can be coloured in $\lambda(\lambda - 1)(\lambda - 2)$ ways, G_2 can be coloured in $\lambda(\lambda - 1)$ ways and G_3 is λ ways.

Therefore, by the rule of product G can be coloured be

$$\lambda(\lambda - 1)(\lambda - 2)\lambda(\lambda - 1)\lambda = \lambda^3(\lambda - 1)^2(\lambda - 2).$$

2.16.1. Decomposition theorem (2.26)

If $G = (V, E)$ is a connected graph and $e = \{a, b\} \in E$, then $P(G_e, \lambda) = P(G, \lambda) + P(G_e', \lambda)$.

Where G_e denotes the subgraph of G obtained by deleting *e* from G without removing vertices *a* and *b*.

i.e., $G_e = G - e$ and G_e' is a second subgraph of G obtained from G_e' by colouring the vertices *a* and *b*.

Proof. Let $e = \{a, b\}$. The number of ways to properly color the vertices in $G_e = G - e$ with (atmost) λ colours in $P(G_e, \lambda)$.

Those colourings where end points *a* and *b* of *e* have different colours are proper colourings of G.

The colourings of G_e that are not proper colourigns of G occur when *a* and *b* have the same color.

But each of these colourings corresponds with a proper colouring for G_e'.

This partition of the $P(G_e, \lambda)$ proper colourings of G_e into two disjoint subsets results in the equation

$$P(G_e, \lambda) = P(G, \lambda) + P(G_e', \lambda)$$

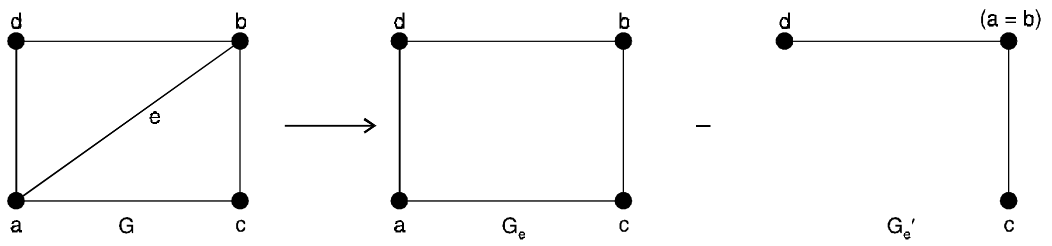
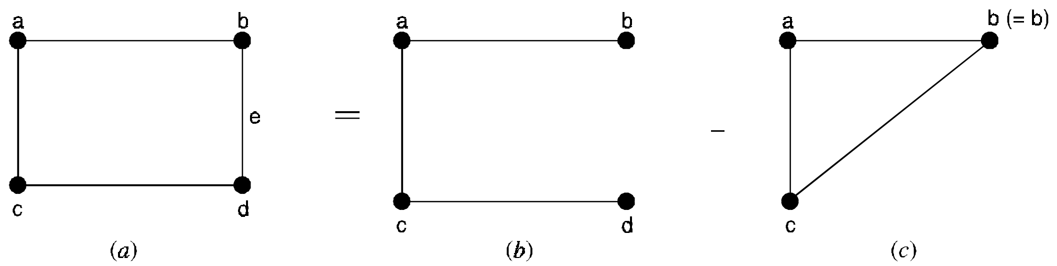


Fig. 2.72.

Problem 2.47. *Using decomposition theorem find the chromatic polynomial and hence the chromatic number for the graph given below in Figure (2.73).*



(*a*) (*b*) (*c*)

Fig. 2.73.

Solution. Deleting the edge *e* from G, we get G_2 as shown in Figure (*b*). Then the chromatic polynomial of G_e is

$$P(G_e, \lambda) = \lambda(\lambda - 1)(\lambda - 2)$$

By colouring the endpoints of *e*, *i.e.*, *a* and *b*, we get G_e' as shown in Figure (*c*). Then the chromatic polynomial of G_e is

$$P(G_e', \lambda) = \lambda(\lambda - 1)^3.$$

Hence, by decomposition theorem, the chromatic polynomial of G is

$$\begin{aligned} P(G, \lambda) &= \lambda(\lambda - 1)^3 - \lambda(\lambda - 1)(\lambda - 2) \\ &= \lambda(\lambda - 1)\,[(\lambda - 1)^2(\lambda - 2)] \\ &= \lambda(\lambda - 1) - (\lambda^3 - 3\lambda + 3)\,\lambda^4 \\ &= 4\lambda^3 + 6\lambda - 3\lambda. \end{aligned}$$

Theorem 2.27. *For each graph G, the constant term in $P(G, \lambda)$ is 0.*

Proof. For each graph G, $\lambda(G) > 0$ because $V \neq \phi$.

If $P(G, \lambda)$ has constant term a, then $P(G, 0) = a \neq 0$.

This implies that there are a ways to colour G properly with 0 colours, a contradiction.

Theorem 2.28. *Let $G = (V, E)$ with $| E | > 0$. Then the sum of the coefficients in $P(G, \lambda)$ is 0.*

Proof. Since $| E | \geq 1$, we have $\lambda(G) \geq 2$, so we cannot properly colour G with only one colour. Consequently, $P(G, 1) = 0$ = the sum of the coefficients in $P(G, \lambda)$.

Problem 2.48. *Explain why each of the following polynomials cannot be a chromatic polynomial*

(*i*) $\lambda^3 + 5\lambda^2 - 3\lambda + 5 = 0$

(*ii*) $\lambda^4 + 3\lambda^3 - 3\lambda^2 = 0.$

Solution. (*i*) The polynomial cannot be a chromatic polynomial since the constant term is 5, not 0.

(*ii*) The polynomial cannot be a chromatic polynomial since the sum of the coefficient is 1, not 0.

Theorem 2.29. *(Vizing) If G is a simple graph with maximum vertex degree Δ then $\Delta \leq \chi'(G) \leq \Delta + 1$.*

Theorem 2.30. *Let $\Delta(G)$ be the maximum of the degrees of the vertices of a graph G. Then $\chi(G) \leq 1 + \Delta(G)$.*

Proof. The proof is by induction on V, the number of vertices of the graph.

When $V = 1$, $\Delta(G) = 0$ and $\chi(G) = 1$, so the result clearly holds.

Now let K be an integer $K \geq 1$, and assume that the result holds for all graphs with $V = K$ vertices.

Suppose G is a graph with $K + 1$ vertices.

Let v be any vertex of G and let $G_0 = \dfrac{G}{\{v\}}$ be the subgraph with v (and all edges incident with it) deleted.

Note that $\Delta(G_0) \leq \Delta(G)$. Now G_0 can be be coloured with $\chi(G_0)$ colours.

Since G_0 has K vertices, we can use the induction hypothesis to conclude that $\chi(G_0) \leq 1 + \Delta(G_0)$.

Thus, $\chi(G_0) \leq 1 + \Delta(G)$, so go can be coloured with atmost $1 + \Delta(G)$ colours.

Since there are atmost $\Delta(G)$ vertices adjacent to v, one of the variable $1 + \Delta(G)$ colours remains for v.

Thus, G can be coloured with atmost $1 + \Delta(G)$ colours.

Theorem 2.31. (Kempe, Heawood). *If G is a planar graph, then $\chi(G) \leq 5$.*

Proof. We must prove that any planar graph with V vertices has a 5-colouring.

Again we use induction on V and note that if V = 1, the result is clear.

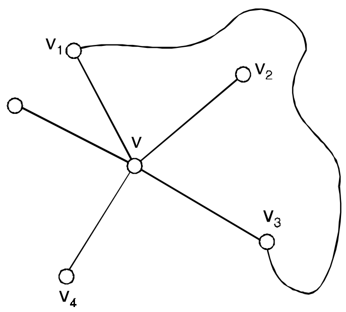


Fig. 2.74.

Let K ≥ 1 be an integer and suppose that any planar graph with K vertices has a 5-colouring.

Let G be a planar graph with K + 1 vertices and assume that G has been drawn as a plane graph with straight edges. We describe how to obtain a 5-colouring of G.

First, G contains a vertex v of degree atmost 5.

Let $G_0 = \dfrac{G}{\{v\}}$ be the subgraph obtained by deleting v (and all edges with which it is incident).

By the induction hypothesis, G_0 has a 5-colouring.

For convenience, label the five colours 1, 2, 3, 4 and 5.

If one of these colours was not used to colour the vertices adjacent to v, then it can be used for v and G has been 5-coloured.

Thus, we assume that v has degree 5 and that each of the colour 1 through 5 appears on the vertices adjacent to v.

In clockwise order, label these vertices v_1, v_2, ... v_5 and assume that v_i is coloured with colour i (see Figure 2.74).

We show how to recolour certain vertices of G_0 so that a colour becomes available for v.

There are two possibilities :

Case 1 : There is no path in G_0 from v_1 to v_3 through vertices all of which are coloured 1 or 3.

In this situation, let H be the subgraph of G consisting of the vertices and edges of all paths through vertices coloured 1 or 3 which start at v_1.

By assumption, v_3 is not in H. Also, any vertex which is not in H but which is adjacent to a vertex of H is coloured neither 1 nor 3.

Therefore, interchanging colours 1 and 3 throughout H produces another 5-colouring of G_0.

In this new 5-colouring both v_1 and v_3 acquire colour 3, so we are now free to give color 1 to v, thus obtaining a 5-colouring of G.

Case 2 : There is a path P in G_0 from v_1 to v_3 through vertices all of which are coloured 1 or 3.

In this case, the path P, followed by v and v_1, gives a circuit in G which does not enclose both v_2 and v_4. Thus, any path from v_2 to v_4 must cross P and, since G is a plane graph, such a crossing can occur only at a vertex of P.

It follows that there is no path in G_0 from v_2 to v_4 which uses just colours 2 and 4.

Now we are in the situation described in case (1), where we have already shown that a 5-colouring for G exists.

Problem 2.49. *$\chi(K_n) = n$, $\chi(K_{m,n}) = 2$, why ?*

Solution. It takes n colours to colour K_n because any two vertices of K_n are adjacent. $\chi(K_n) = n$.

On the otherhand, $\chi(K_{m,n}) = 2$, colouring the vertices of each bipartition set the same colour produces a 2-colouring of $K_{m,n}$.

Problem 2.50. *What is the chromatic number of the graph in Figure (2.75).*

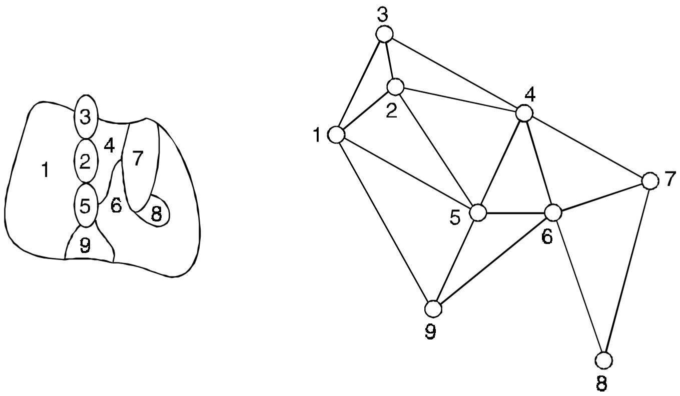


Fig. 2.75. A map and an associated planar graph.

Solution. A way to 4-colour the associated graph, was given in the text. From this, we deduce that $\chi(G) \leq 4$.

To see that $\chi(G) = 4$, we investigate the consequences of using fewer than four colours.

Vertices 1, 2, 3 from a triangle, so three different colours are needed for these.

Suppose we use red, blue and green, respectively, as before.

To avoid a fourth colour, vertex 4 has to be coloured red and vertex 5 green.

Thus, vertex 6 has to be blue.

Since vertex 9 is adjacent to vertices 1, 5 and 6 of colours red, green and blue, respectively. Vertex 9 requires a fourth colour.

Problem 2.51. *Show that $\chi(G) = 4$ for the graph of G of Figure (2.76).*

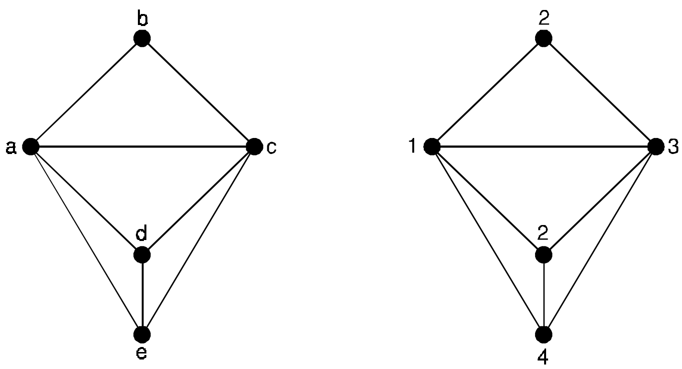


Fig. 2.76.

Solution. Clearly the triangle *abc* requires three colours, assign the colours 1, 2 and 3 to *a*, *b* and *c* respectively.

Then since *d* is adjacent to *a* and *c*, *d* must be assigned a colour different from the colours for *a* and *c*, colour *d* is colour 2.

But then *e* must be assigned a colour different from 2 since *e* is adjacent to *d*.

Likewise *e* must be assigned a colour different from 1 or 3 because *e* is adjacent to *a* and to *c*.

Hence a fourth colour must be assigned to *e*.

Thus, the 4-colouring exhibited incidates $\chi(G) \le 4$.

But, at the same time, we have argued that $\chi(G)$ cannot be less than 4.

Hence $\chi(G) = 4$.

Theorem 2.32. *The minimum number of hours for the schedule of committee meetings in our scheduling problem is* $\chi(G_0)$.

Proof. Suppose $\chi(G_0) = K$ and suppose that the colours used in colouring G_0 are 1, 2, ... K.

First we assert that all committees can be scheduled in K one-hour time periods.

In order to see this, consider all those vertices coloured 1, say, and the committees corresponding to these vertices.

Since no two vertices coloured 1 are adjacent, no two such committees contain the same member.

Hence, all these committees can be scheduled to meet at the same time.

Thus, all committees corresponding to same-coloured vertices can meet at the same time.

Therefore, all committees can be scheduled to meet during K time periods.

Next, we show that all committees cannot be scheduled in less than K hours. We prove this by contradiction.

Suppose that we can schedule the committees in *m* one-hour time periods, where $m < K$.

We can then give G_0 an *m*-colouring by colouring with the same colour all vertices which correspond to committees meeting at the same time.

To see that this is, infact, a legitimate *m*-colouring of G_0, consider two adjacent vertices.

These vertices correspond to two committees containing one or more common members.

Hence, these committees meet at different times, and thus the vertices are coloured differently.

However, an *m*-colouring of G_0 gives a contradiction since we have $\chi(G_0) = K$.

Problem 2.52. *Suppose* $\chi(G) = 1$ *for some graph G. What do you know about G ?*

Solution. If G has an edge, its end vertices must be coloured differently, so $\chi(G) \ge 2$.

Thus $\chi(G) = 1$ if and only if G has no edges.

Problem 2.53. *Any two cycles are homeomorphic. Why ?*

Solution. Any cycle can be obtained from a 3-cycle by adding vertices to edges.

Problem 2.54. *Find the number N defined in this proof for the graph of Figure (2.77). Verify that* $N \le 2E$. *Give an example of an edge which is counted just once.*

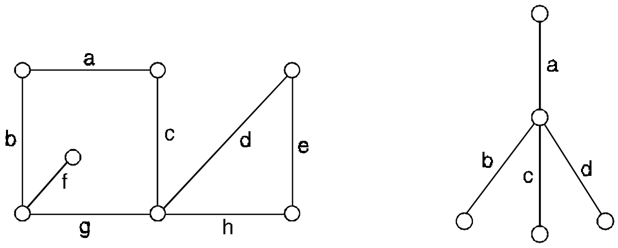


Fig. 2.77.

Solution. The boundaries of the regions are gievn :

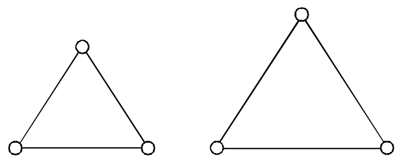
{*d*, *e*, *h*}, {*a*, *b*, *f*, *g*, *c*} and {*a*, *b*, *g*, *c*, *d*, *e*, *h*}

N = 3 + 5 + 7 = 15 ≤ 16 = 2 E.

Edge *f* is counted only once.

Problem 2.55. *Show that, Euler's theorem is not necessarily true if "connected" is omitted from its statement.*

Solution.



In the graph shown, V – E + R = 6 – 6 + 3 = 3.

Problem 2.56. *Consider the plane graph shown on the left of Figure 2.78, below :*

(*a*) *How many regions are there ?*

(*b*) *List the edges which form the boundary of each region.*

(*c*) *Which region is exterior ?*

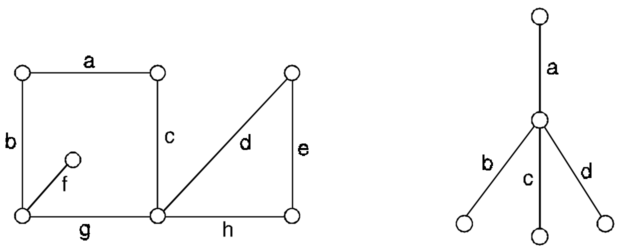


Figure 2.78.

Solution. The graph on the left of Figure 2.78 has three regions whose boundaries are {*d*, *e*, *h*}, {*a*, *b*, *f*, *g*, *c*} and {*a*, *b*, *g*, *c*, *d*, *e*, *h*}, the last region is exterior.

The graph on the right is a tree, it determines only one region, the exterior one, with boundary {*a*, *b*, *c*, *d*}.

2.16.2. Scheduling Final Exams (2.57)

How can the final exams at a university be scheduled so that no student has two exams at the same time ?

Solution. This scheduling problem can be solved using a graph model, with vertices representing courses and with an edge between two vertices if there is a common student in the courses they represent. Each time slot for a final exam is represented by a different colour. A scheduling of the exams corresponds to a colouring of the associated graph.

For instance, suppose there are seven finals to be scheduled. Suppose the courses are numbered 1 through 7. Suppose that the following pairs of courses have common students : 1 and 2, 1 and 3, 1 and 4, 1 and 7, 2 and 3, 2 and 4, 2 and 5, 2 and 7, 3 and 4, 3 and 6, 3 and 7, 4 and 5, 4 and 6, 5 and 6, 5 and 7, and 6 and 7.

In Figure 2.79, the graph associated with this set of classes is shown.

A scheduling consists of a colouring of this graph.

Since the chromatic number of this graph is 4, four times slots are needed.

A colouring of the graph using four colours and the associated schedule are shown in Figure 2.80.

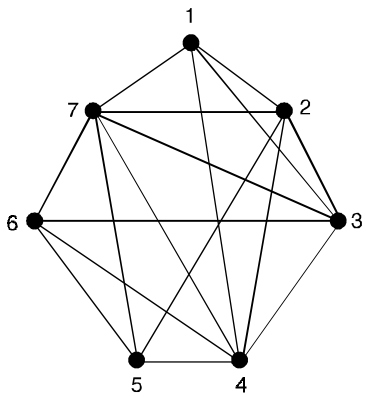


Fig. 2.79.

The graph representing the scheduling of final exams

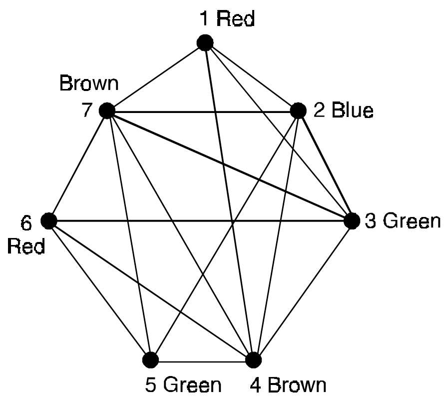


Fig. 2.80.

Using a colouring to schedule final exams.

Time period	Courses
I	1, 6
II	2
III	3, 5
IV	4, 7

2.16.3. Frequency assignments (2.58)

Television channels 2 through 13 are assigned to stations in New Delhi so that no two stations within 150 miles can operate on the same channel. How can the assignment of channels be modeled by graph colouring ?

Solution. Construct a graph by assigning a vertex to each station.

Two vertices are connected by an edge if they are located within 150 miles of each other.

An assignment of channels corresponds to a colouring of the graph. Where each colour represents a different channel.

2.16.4. Index registers (2.59)

In efficient compilers the execution of loops is speeded up when frequently used variables are stored temporarily in index registers in the central processing unit, instead of in regular memory. For a given loop, how many index registers are needed ?

Solution. This problem can be addressed using a graph colouring model.

To set up the model, let each vertex of a graph represent a variable in the loop.

There is an edge between two vertices if the variables they represent must be stored in index registers at the same time during the execution of the loop.

Thus, the chromatic number of the graph gives the number of index registers needed, since different registers must be assigned to variables when the vertices respresenting these variables are adjacent in the graph.

Problem 2.60. *What is the chromatic number of the graph C_n ?*

Solution. We will first consider some individual cases.

To begin, let $n = 6$. Pick a vertex and colour it red.

Proceed clockwise in the planar depiction of C_6 shown in Figure (2.81).

It is necessary to assign a second colour, say blue, to the next vertex reached.

Continue in the clockwise direction, the third vertex can be coloured red, the fourth vertex blue, and the fifth vertex red.

Finally, the sixth vertex, which is adjacent to the first, can be coloured blue.

Hence, the chromatic number of C_6 is 2. Figure (2.81) displays the colouring constructed here.

Next, let $n = 5$ and consider C_5. Pick a vertex and colour it red.

Proceeding clockwise, it is necessary to assign a second colour, say blue, to the next vertex reached.

Continuing in the clockwise direction, the third vertex can be coloured red, and the fourth vertex can be coloured blue.

The fifth vertex cannot be coloured either red or blue, since it is adjacent to the fourth vertex and the first vertex.

Consequently, a third colour is required for this vertex.

Note that we would have also needed three colours if we had coloured vertices in the counter clockwise direction.

Thus, the chromatic number of C_5 is 3. A colouring of C_5 using three colours is displayed in Figure (2.81).

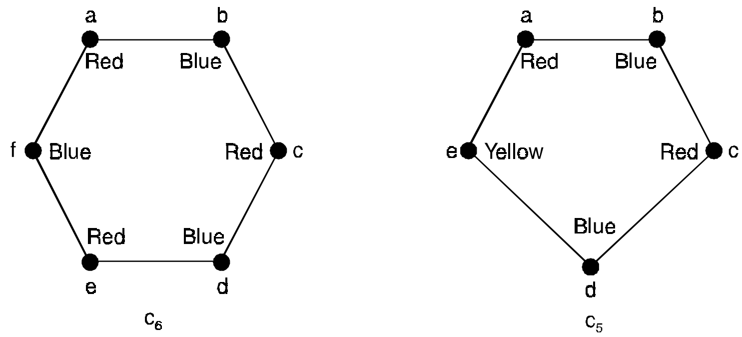


Fig. 2.81. Colourings of C_5 and C_6.

In general, two colours are needed to colours C_n when n is even. To construct such a colouring, simply pick a vertex and colour it red.

Proceeding around the graph in a clockwise direction (using a planar representation of the graph) colouring the second vertex blue, the third vertex red, and so on.

The nth vertex can be colored blue, since the two vertices adjacent to it, namely the $(n - 1)$st and the first vertices, are both coloured red.

When n is odd and $n > 1$, the chromatic number of C_n is 3.

To see this, pick an initial vertex. To use only two colours, it is necessary to alternate colours as the graph is traversed in a clockwise direction.

However, the nth vertex reached is adjacent to two vertices of different colours, namely, the first and $(n - 1)$st.

Hence, a third colour must be used.

Problem 2.61. *What is the chromatic number of the complete bipartite graph $K_{m, n}$, where m and n are positive integers ?*

Solution. The number of colours needed may seem to depend on m and n.

However, only two colours are needed. Colour the set of m vertices with one colour and the set of n vertices with a second colour.

Since edges connect only a vertex from the set of m vertices and a vertex from the set of n vertices, no two adjacent vertices have the same colour.

A colouring of $K_{3, 4}$ with two colours is displayed in Figure (2.82).

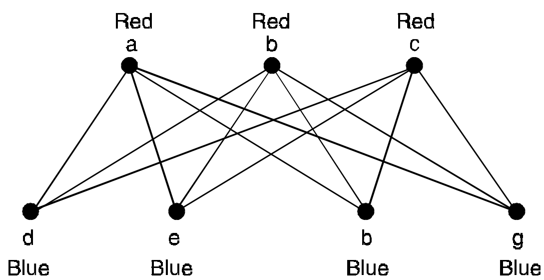


Fig. 2.82. A colouring of $K_{3, 4}$.

Problem 2.62. *What is the chromatic number of K_n ?*

Solution. A colouring of K_n can be constructed using n colours by assigning a different color to each vertex. Is there a colouring using fewer colours ? The answer is no. No two vertices can be assigned the same colour, since every two vertices of this graph are adjacent.

Hence, the chromatic number of $K_n = n$.

A colouring of K_5 using five colours is shown in Figure (2.83).

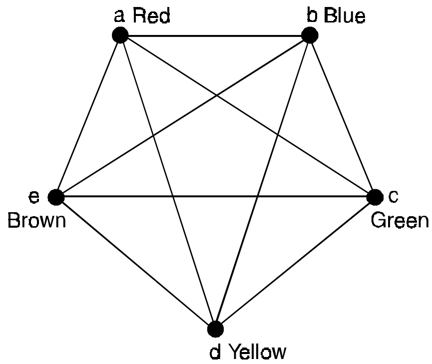


Fig. 2.83. A colouring of K_5.

Problem 2.63. *What is the chromatic numbers of the graphs G and H shown in Figure (2.84).*

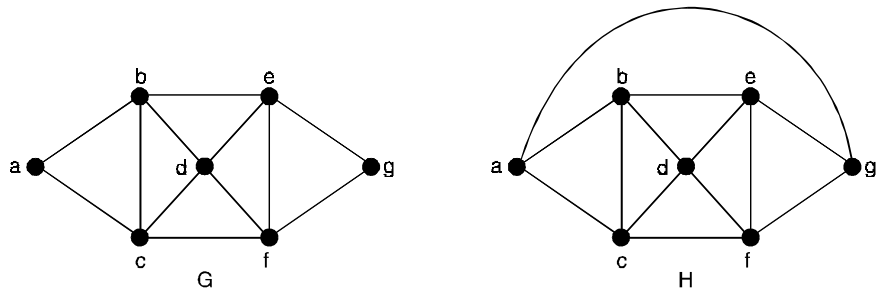


Fig. 2.84. The simple graphs G and H.

Solution. The chromatic number of G is at least three, since the vertices a, b and c must be assigned different colorus.

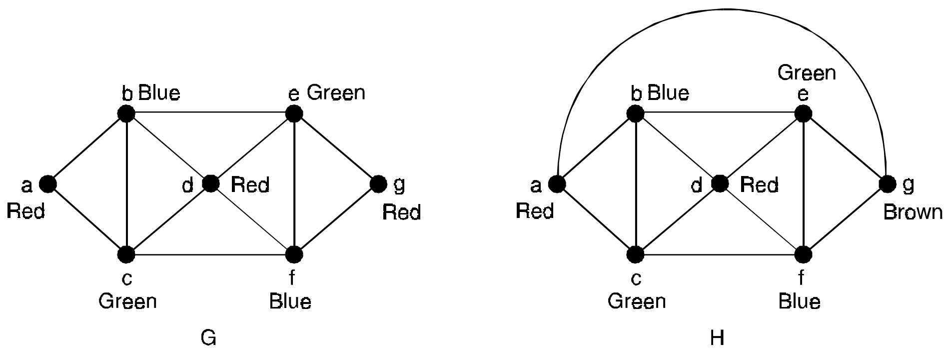


Fig. 2.85. Colourings of the graphs G and H.

To see if G can be colourd with three colours, assign red to *a*, blue to *b*, and green to *c*. Then, *d* can (and must) be coloured red since it is adjacent to *b* and *c*.

Furthermore, *e* can (and must) be coloured green since it is adjacent only to vertices coloured red and blue, and *f* can (and must) be coloured blue since it is adjacent only to vertices coloured red and green.

Finally, *g* can (and must) be coloured red since it is adjacent only to vertices coloured blue and green.

This produces a colouring of G using exactly three colours. Figure (2.85) displays such a colouring.

The graph H is made up of the graph G with an edge connecting *a* and *g*.

Any attempt to colour H using three colours must follow the same reasoning as that used to colour G, except at the last stage, when all vertices other than *g* have been coloured.

Then, since *g* is adjacent (in H) to vertices coloured red, blue, and green, a fourth colour, say brown, needs to be used.

Hence, H has a chromatic number equal to 4.

A colouring of H is shown in Figure (2.85).

Problem 2.64. *Suppose that in one particular semester, there are students taking each of the following combinations of courses.*

** Mathematics, English, Biology, Chemistry*

** Mathematics, English, Computer Science, Geography*

** Biology, Psychology, Geography, Spanish*

** Biology, Computer Science, History, French*

** English, Psychology, History, Computer Science*

** Psychology, Chemistry, Computer Science, French*

** Psychology, Geography, History, Spanish.*

What is the minimum number of examination periods required for exams in the ten courses specified so that students taking any of the given combinations of courses have no conflicts ?

Find a possible schedule which uses this minimum number of periods.

Solution. In order to picture the situation, we draw a graph with ten vertices labeled M, E, B, ... corresponding to Mathematics, English, Biology and so on, and join two vertices with an edge if exams in the corresponding subjects must not be scheduled together.

The minimum number of examination periods is evidently the chromatic number of this graph. What is this ? Since the graph contains K_5 (with vertices M, E, B, G, CS), at least five different colours are needed. (The exams in the subjects which these vertices represent must be scheduled at different times). Five colours are not enough, however, since P and H are adjacent to each other and to each of E, B, G and CS.

The chromatic number of the graph is, infact 6.

In Figure (2.86), we show a 6-colouring and the corresponding exam schedule.

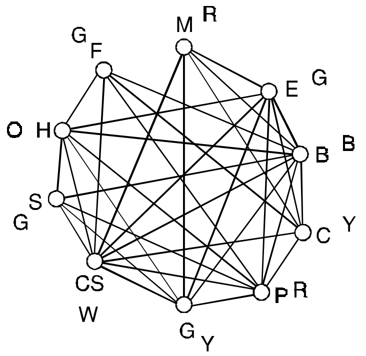


Fig. 2.86.

Period 1	Mathematics, Psychology
Period 2	English, Spanish, French
Period 3	Biology
Period 4	Chemistry, Geography
Perido 5	Computer Science
Period 6	History

Theorem 2.33. *A graph G is bipartite if and only if it does not contain a odd cycle.*

Proof. Let G be bipartite. Then the vertex set G can be partitioned into two subsets V_1 and V_2 such that every edge in G joins a vertex in V_1 with a vertex in V_2.

Suppose G contains a cycle. Let v be a vertex of this cycle. Then to trace the cycle starting from v we have to travel on the edges of G.

The edges of G are the only edges between V_1 and V_2.

Thus starting from v to come back to v along the cycle of G we have to travel exactly even number of times between V_1 and V_2.

That is, the number of edges in C is even, that is, the length of C is even.

Conversely, without loss of generality we assume G is connected.

Let G does not contain a odd cycle. Choose a vertex x of G. Colour the vertex by the Colour Black. Colour all the vertices that are at odd distances from x with the colour Red. Color all the vertices that are at even distances from x with colour Black. Since every distance is either a odd or even (but not both), every vertex of G is now coloured.

We now show that the graph G is now properly coloured. Suppose G is not properly coloured, the G contains two adjacent vertices say u and v, colored with the same colour. Then distance from the vertex x to both the vertices u and v is odd.

Let P_1 and P_2 be shortest paths from x to u and x to v respectively.

Let y be the last vertex common to P_1 and P_2 (*i.e.,* the path from y to u and path from y to v along P_1 and P_2 are disjoint). Then $d(x, y)$ along P_1 is same along P_2 (since both P_1 and P_2 are shortest paths).

Otherwise, if the $d(x, y)$ along P_1 is smaller than that on P_2, then the path from x to y along P_1 with the path from y to v along P_2 is shorter than P_2, which is a contradiction to the fact that P_2 is shortest.

Let $d(x, u) = m$ and $d(x, v) = n$, then both m and n are odd numbers or both are even numbers (since u and v are coloured with same colour).

Then $d(y, u)$ and $d(y, v)$ are both either odd or even and hence the sum is even.

Hence, the circuit formed due to these paths together with the edge uv is of odd length, which is a contradiction.

Thus we conclude that the colouring is proper.

Now consider the set V_1 of all vertices of G coloured by Black and the set V_2 of all the vertices of G coloured by the colour Red.

These sets are the partition of G such that no two vertices in the same set are adjacent.

Hence G is bipartite.

Theorem 2.34. *A graph of n vertices is a complete graph if and only if its chromatic polynomial is*

$$P_n(\lambda) = \lambda(\lambda - 1)(\lambda - 2) \ldots\ldots (\lambda - n + 1).$$

Proof. With λ colours, there are λ different ways of colouring any selected vertex of a graph.

A second vertex can be coloured properly in exactly $\lambda - 1$ ways, the third in $\lambda - 2$ ways, the fourth in $\lambda - 3$ ways, ..., and the nth in $\lambda - n + 1$ ways if and only if every vertex is adjacent to every other.

That is, if and only if the graph is complete.

Theorem 2.35. *Let a and b be two non adjacent vertices in a graph G. Let G′ be a graph obtained by adding an edge between a and b. Let G″ be a simple graph obtained from G by fusing the vertices a and b together and replacing sets of parallel edges with single edges. Then*

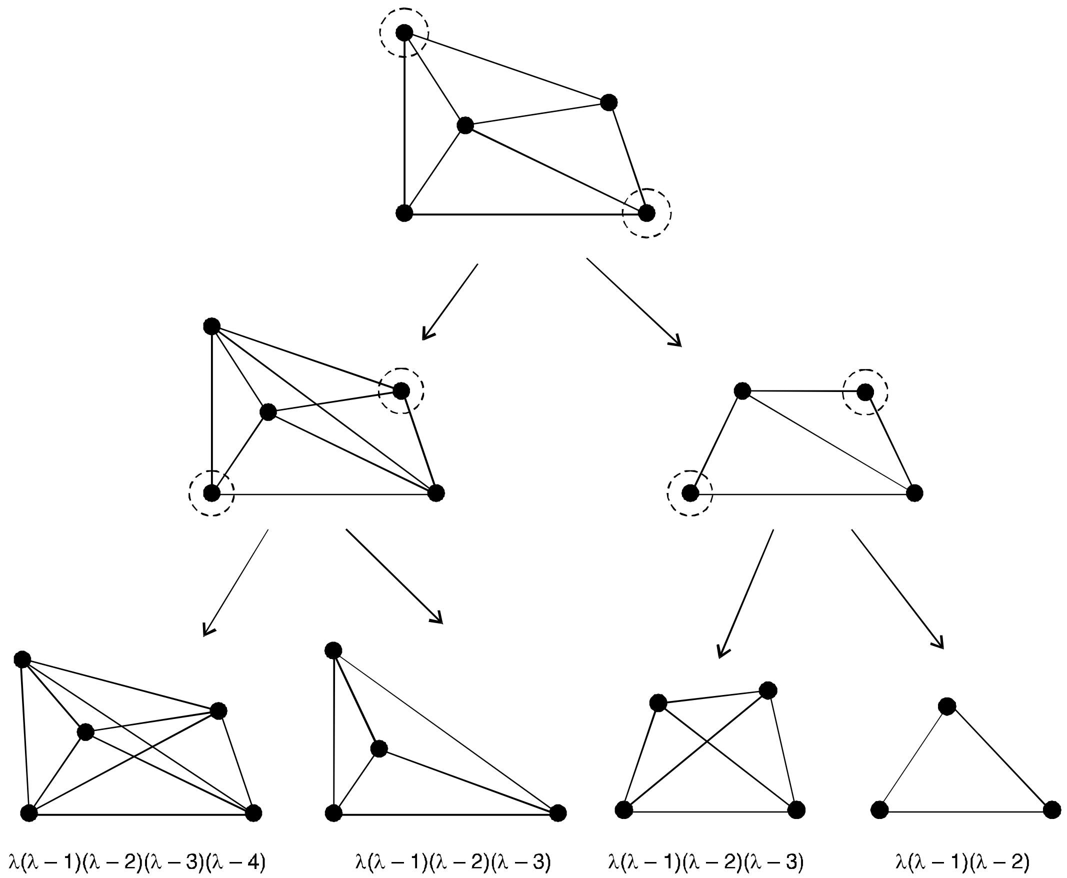
$$P_n(\lambda) \text{ of } G = P_n(\lambda) \text{ of } G' + P_{n-1}(\lambda) \text{ of } G''.$$

Proof. The number of ways of properly colouring G can be grouped into two cases, one such that vertices *a* and *b* are of the same colour and the other such that *a* and *b* are of different colours.

Since the number of ways of properly colouring G such that *a* and *b* have different colours = number of ways of properly colouring G′, and

Number of ways of properly colouring G such that *a* and *b* have the same colour = number of ways of properly colouring G″.

$$P_n(\lambda) \text{ of } G = P_n(\lambda) \text{ of } G' + P_{n-1}(\lambda) \text{ of } G''$$



$$P_5(\lambda) \text{ of } G = \lambda(\lambda-1)(\lambda-2) + 2\lambda(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)$$
$$= \lambda(\lambda-1)(\lambda-2)(\lambda^2 - 5\lambda + 7)$$

Fig. 2.87. Evaluation of a chromatic polynomial.

Theorem 2.36. *A graph is bicolourable if and only if it has no odd cycles.*

Theorem 2.37. *For any graph G, $\chi(G) \le 1 + \max \delta(G')$,*

Where the maximum is taken over all induced subgraphs G′ of G.

Proof. The result is obvious for totally disconnected graphs.

Let G be an arbitrary n-chromatic graph, $n \ge 2$.

Let H be any smallest induced subgraph such that $\chi(H) = n$

The graph H therefore has the property that

$$\chi(H - v) = n - 1 \text{ for all its points } v.$$

It follows that deg $v \ge n - 1$ so that $\delta(H) \ge n - 1$ and hence

$$n - 1 \le \delta(H) \le \max \delta(H') \le \max \delta(G')$$

The first maximum taken over all induced subgraphs H′ of H and the second over all induced subgraphs G′ of G.

This implies that

$$\chi(G) = n < 1 + \max \delta(G')$$

Corollary : For any graph G, the chromatic number is atmost one greater than the maximum degree $\chi \le 1 + \Delta$.

Theorem 2.38. If $\Delta(G) = n \ge 2$, then G is n-colourable unless, or

(i) $n = 2$ and G has a component which is an odd cycle, or

(ii) $n > 2$ and K_{n+1} is a component of G.

Theorem 2.39. *For any graph G,* $\dfrac{P}{\beta_0} \le \chi \le P - \beta_0 + 1.$

Proof. If $\chi(G) = n$, then V can be partitioned into n colour classes $V_1, V_2, \ldots\ldots V_n$, each of which, as noted above, is an independent set of points.

If $| V_i | = P_i$, then every $P_i \le \beta_0$ so that

$$P = \Sigma P_i \le n\, \beta_0$$

To verify the upper bound, let S be a maximal independent set containing β_0 points.

It is clear that $\chi(G - S) \ge \chi(G) - 1$.

Sicne G – S has $P - \beta_0$ points, $\chi(G - S) \le P - \beta_0$

Therefore, $\chi(G) \le \chi(G - S) + 1 \le P - \beta_0 + 1$.

Theorem 2.40. *For every two positive integers m and n, there exists an n-chromatic graph whose girth exceeds m.*

Theorem 2.41. *For any graph G, the sum and product of χ and $\bar{\chi}$ satisfy the inequalities :*

$$2\sqrt{P} \le \chi + \bar{\chi} \le P + 1,$$

$$P \le \chi\bar{\chi} \le \left(\frac{P+1}{2}\right)^2$$

Proof. Let G be n-chromatic and let $V_1, V_2, \ldots V_n$, be the colour classes of G, where $|V_i| = P_i$

Then of course $\Sigma P_i = P$ and $\max P_i \geq \dfrac{P}{n}$.

Since each V_i induces a complete subgraph of $\overline{G}$

$\bar{\chi} \geq \max P_i \geq \dfrac{P}{n}$ so that $\chi\bar{\chi} \geq P$.

Since the geometric mean, it follows that $\chi + \bar{\chi} \geq 2\sqrt{P}$.
This establishes both lower bounds.

To show that $\chi + \bar{\chi} \leq P + 1$, we use induction on P, noting that equality holds when P = 1.

We thus assume that $\chi(G) + \bar{\chi}(G) \leq P$ for all graphs G having P – 1 points.

Let H and $\overline{H}$ be complementary graphs with P points, and let v be a point of H.

Then $G = H - v$ and $\overline{G} + \overline{H} - v$ are complementary graphs with P – 1 points.

Let the degree of v in H be d so that the degree of v in $\overline{H}$ is $P - d - 1$.
It is obvious that

$$\chi(H) \leq \chi(G) + 1 \text{ and } \bar{\chi}(H) \leq \bar{\chi}(G) + 1$$

If either

$$\chi(H) < \chi(G) + 1 \text{ or } \bar{\chi}(H) < \bar{\chi}(G) + 1.$$

then $\chi(H) + \bar{\chi}(H) \leq P + 1$.

Suppose then that $\chi(H) = \chi(G) + 1$ and $\bar{\chi}(H) = \bar{\chi}(G) + 1$.
This implies that the removal of v from H, producing G, decreases the chromatic number so that $d \geq \chi(G)$.

Similarly $P - d - 1 \geq \bar{\chi}(G)$,

thus $\chi(G) + \bar{\chi}(G) \leq P - 1$

Therefore, we always have $\chi(H) + \bar{\chi}(H) \leq P + 1$
Finally, applying the inequality

$$4\chi\bar{\chi} \leq (\chi + \bar{\chi})^2 \quad \text{we see that} \quad \chi\bar{\chi} \leq \left[\frac{(P+1)}{2}\right]^2.$$

Theorem 2.42. *Every tree T with two or more vertices is 2-chromatic.*

Proof. Since Tree T is a bipartite graph.

The vertex set V of G can be partitioned into two subsets V_1 and V_2 such that no two vertices of the set V_1 are adjacent and two vertices of the set V_2 are adjacent.

Now colour the vertices of the set V_1 by the colour 1 and the vertices of the set V_2 by the colour 2.

This colouring is a proper colouring.

Hence, chromatic number of G ≤ 2, and since T contains atleast one edge chromatic number of G ≥ 2.

Thus, chromatic number of G is 2.

Theorem 2.43. *A graph G is 2-chroamtic if and only if G is bipartite.*

Proof. Let chromatic index of a graph G be two.

Let G be properly coloured with two colours 1 and 2. Consider the set of vertices coloured with the colour 1 and the set of all vertices coloured with the colour 2.

These sets are precisely partition of the vertex set such that no two of the vertices of the same set are adjacent.

Hence G is bipartite.

Conversely, G is not bipartite then G contains a odd cycle.

The chromatic number of a odd cycle is three.

Hence G contains a subgraph whose chromatic number is three.

Therefore, $K(G) \geq 3$.

Theorem 2.44. *The chromatic number of a graph cannnot exceed one more than the maximum degree of a vertex of G.*

Proof. Since maximum degree of the graph is m, the graph cannot have a subgraph K_n, $n > m + 1$.

Thus $K(G) \leq m + 1$.

Corollary. The chromatic number of a graph cannot exceed maximum degree m of a vertex of G if and only if G does not have a subgraph isomorphic to K_{m+1}.

Theorem 2.45. *If d_{max} is the maximum degree of the vertices in a graph G, chromatic number of G $\leq 1 + d_{max}$.*

Theorem 2.46. (König's theorem)

A graph with atleast one edge is 2-chromatic if and only if it has no circuits of odd length.

Proof. Let G be a connected graph with circuits of only even lengths.

Consider a spanning tree T in G, let us properly color T with two colors. Now add the chords to T one by one.

Since G had no circuits of odd length, the end vertices of every chord being replaced are differently coloured in T.

Thus G is coloured with two colours, with no adjacent vertices having the same colour.

That is, G is 2-chromatic.

Conversely, if G has a circuit of odd length, we would need at least three colours just for that circuit.

Thus the theorem.

Theorem 2.47. *A graph G is 2-chromatic if and only if it is a non-null bipartite graph.*

Proof. Suppose a graph G is 2-chromatic. Then it is non-null, and some vertices of G have one colour, say α, and the rest of the vertices have another colour, say β.

Let V_1 be the set of vertices having colour α and V_2 be the set of vertices having colour β.

Then $V_1 \cup V_2 = V$, the vertex set of G, and $V_1 \cap V_2 = \phi$.

Also, no two vertices of V_1 can be adjacent and no two vertices of V_2 can be adjacent.

As such, every edge in G has one end in V_1 and the other end in V_2.

Hence G is a bipartite graph.

Conversely, suppose G is a non-null bipartite graph. Then the vertex set of G has two partitions V_1 and V_2 such that every edge in G has one end in V_1 and another end in V_2.

Consequently, G cannot be properly coloured with one colour, because then vertices in V_1 and V_2 will have the same colour and every edge has both of its ends of the same colour.

Suppose we assign a colour α to all vertices in V_1 and a different colour β to all vertices in V_2.

This will make a proper colouring of V.

Hence G is 2-chromatic.

Corollary. *Every three with two or more vertices is a bipartite graph.*

Proof. Every tree with two or more vertices is 2-chromatic. Therefore, it is bipartite, by the theorem.

Theorem 2.48. For a graph G, the following statements are equivalent :

(*i*) G is 2-chromatic

(*ii*) G is non-null and bipartite

(*iii*) G has no circuits of odd length.

Corollary. A graph G is a non-null bipartite graph if and only if it has no circuits of odd length.

Theorem 2.49. *If G is a graph with n vertices and degree δ, then* $\chi(G) \geq \dfrac{n}{n-\delta}$.

Proof. Recall that δ is the minimum of the degrees of vertices.

Therefore, every vertex v of G has atleast δ number of vertices adjacent to it.

Hence there are at most $n - \delta$ vertices can have the same colour.

Let K be the least number of colours with which G can be properly coloured.

Then $K = \chi(G)$.

Let $\alpha_1, \alpha_2, \ldots\ldots \alpha_K$ be these colours and let n_1 be the number of vertices having colour α_1, n_2 be the number of vertices having colour α_2 and so on, and finally n_K be the number of vertices having colour α_K.

Then $$n_1 + n_2 + n_3 + \ldots\ldots n_k = n \qquad \ldots(1)$$

and $$n_1 \leq n - \delta,\ n_2 \leq n - \delta,\ \ldots\ldots\ n_k \leq n - \delta \qquad \ldots(2)$$

Adding the K in equalities in (2), we obtain

$$n_1 + n_2 + \ldots\ldots + n_k \leq K(n - \delta)$$

or $$n \leq K(n - \delta), \text{ using (1)}$$

Since $K = \chi(G)$, this becomes

$$\chi(G) \geq \frac{n}{n-\delta}$$

This is the required result.

Problem 2.65. *Write down chromatic polynomial of a given graph on n vertices.*

Solution. Let G be a graph on n vertices.

Let C_i denote the different ways of properly coloring of G using exactly i distinct colors.

These i colors can be chosen out of λ colors in $\binom{\lambda}{i}$ distinct ways.

Thus total number of distinct ways a proper coloring to a graph with i colors out of λ colors is possible in $\binom{\lambda}{i}C_i$ ways.

Hence $\sum_{i=1}^{n} \binom{\lambda}{i}C_i$. Each C_i has to be evaluated individually for the given graph.

Problem 2.66. *Find all maximal independent sets of the following graph.*

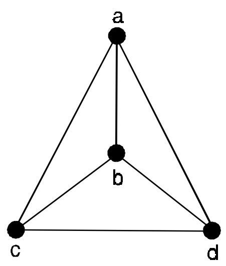


Fig. 2.88.

Solution. The maximal independent sets of G are $\{a\}$, $\{b\}$, $\{c\}$ and $\{d\}$.

Problem 2.67. *Find all maximal independent sets of the following graph.*

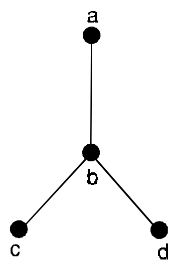


Fig. 2.89.

Solution. Maximal independent sets are $\{a, c, d\}$ and $\{b\}$.

Problem 2.68. *Find all possible maximal independent sets of the following graph using Boolean expression.*

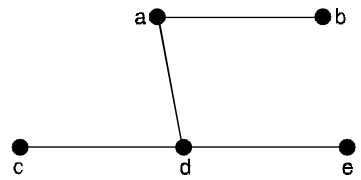


Fig. 2.90.

Solution. The Boolean expression for this graph

$$\phi = \Sigma xy = ab + ad + cd + de \text{ and}$$

$$\begin{aligned}
\phi' &= (a' + b')(a' + d')(c' + d')(d' + e') \\
&= \{a'(a' + d') + b'(a' + d')\}\ \{c'(d' + e') + d'(d' + e')\} \\
&= \{a' + b'a' + b'd'\}\ \{c'd' + c'e' + d'\} \\
&= \{a'(1 + b') + b'd'\}\ \{d'(c' + 1) + c'e'\} \\
&= \{a' + b'd'\}\ \{d' + c'e'\} \\
&= a'd' + a'c'e' + b'd' + b'c'd'e' \\
&= a'd' + a'c'e' + b'd'(1 + c'e') \\
&= a'd' + a'c'e' + b'd'
\end{aligned}$$

Thus $f_1 = a'd'$, $f_2 = a'c'e'$ and $f_3 = b'd'$.

Hence maximal independent sets are $V - \{a, b\} = \{b, c, e\}$

$V - \{a, c, e\} = \{b, d\}$ and $V - \{b, d\} = \{a, c, e\}$.

Problem 2.69. *Find the chromatic polynomial of a connected graph on three vertices.*

Solution. Since the graph is connected it contains an edge, hence minimum two colours are required for any proper colouring of G.

Thus $C_1 = 0$.

Further the number of ways a graph on n vertices with n distinct colours can be properly assigned in n ! ways.

Hence for the graph on 3 vertices $C_3 = 3! = 6$.

If G is a triangle, then G cannot be labeled with two colours.

Hence $C_2 = 0$, thus

$$P_3(\lambda) = \sum_{i=1}^{3} \binom{\lambda}{i} C_i = 0 + 0 + \binom{\lambda}{3} 6$$

$$= \frac{\lambda(\lambda-1)(\lambda-2)}{3!}\, 6 = \lambda(\lambda - 1)(\lambda - 2)$$

If G is a path, then end vertices can be coloured with only two ways with two colours and for each choice of end vertex only one choice of another colour is possible for the middle vertex. Thus $C_2 = 2$ and similar to above argument $C_3 = 3!$.

Therefore, $P_3(\lambda) = \sum_{i=1}^{3} \binom{\lambda}{i} C_i = 0 + \binom{\lambda}{2} 2 + \binom{\lambda}{3} 6$

$$= \frac{\lambda(\lambda-1)}{2!}2 + \frac{\lambda(\lambda-1)(\lambda-2)}{3!}6$$

$$= \lambda(\lambda - 1) + \lambda(\lambda - 1)(\lambda - 2)$$

$$= \lambda(\lambda - 1)(1 + (\lambda - 2))$$

$$= \lambda(\lambda - 1)^2.$$

Theorem 2.50. *An n-vertex graph is a tree if and only if its chromatic polynomial* $P_n(\lambda) = \lambda(\lambda - 1)^{n-1}$.

Proof. Let G be a tree on n vertices.

We prove the result by induction on n.

If $n = 1$, then G contains only one vertex which can be coloured in λ distinct ways only.

Hence the result holds in this case.

If $n = 2$, then G contains one edge, so that exactly two colours are required for the proper colouring of the graph.

Hence $C_1 = 0$ and two colours can be assigned in two different ways for the vertices of the graph.

Therefore, $C_2 = 2$.

Thus $P_n(\lambda) = 0 + \left[\frac{\lambda(\lambda-1)}{2!}\right]2 = \lambda(\lambda - 1)$

Hence the result holds with $n = 2$.

Now assume the result for lesser values of n, $n \geq 2$.

Since the graph G is a tree, it contains a pendent vertex. Let v be a pendent vertex of the graph. Let G′ be the graph obtained by deleting the vertex v. Then by inductive hypothesis the chromatic polynomial of G′ is $\lambda(\lambda - 1)^{n-2}$.

Now for each proper coloring of G′ the given graph can be properly colored by painting the vertex v with the colour other than vertex adjacent to the vertex v.

Thus we can choose $(\lambda - 1)$ colors to v for each proper colouring of G′.

Hence total $\lambda(\lambda - 1)^{n-2} (\lambda - 1) = \lambda(\lambda - 1)^{n-1}$ ways we can properly colour the given tree.

Thus the result hold by induction.

Problem 2.70. *How many ways a tree on 5 vertices can be properly coloured with at most 4 colors.*

Solution. We have a tree with n vertices can be coloured with at most λ colours in $\lambda(\lambda - 1)^{n-1}$ ways.

Therefore a tree on $n = 5$ vertices can be properly coloured with at most $\lambda = 4$ colours in $\lambda(\lambda - 1)^{n-1} = 4 \cdot 3^4 = 4 \cdot 81 = 324$ ways.

Problem 2.71. *Write down the chromatic polynomial of the graph $K_4 - e$.*

Solution.

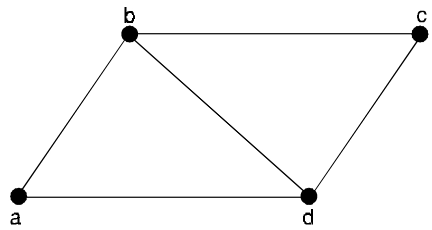


Fig. 2.91.

The graph $K_4 - e$ is shown below. It contains exactly two non-adjacent vertices.

Let G′ be a graph obtained by adding the edge between these non adjacent vertices.

Then G′ is a complete graph K_4.

Hence $P_4(\lambda)$ of $G' = \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)$

Let G″ be the graph obtained by fusing these vertices and replacing the parallel edges.

Then G″ is a complete graph K_3.

Hence $P_3(\lambda)$ of $G'' = \lambda(\lambda - 1)(\lambda - 2)$

Now,
$$\begin{aligned} P_4(\lambda) \text{ of } G &= P_4(\lambda) \text{ of } G' + P_{4-1}(\lambda) \text{ of } G'' \\ &= \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3) + \lambda(\lambda - 1)(\lambda - 2) \\ &= \lambda(\lambda - 1)(\lambda - 2)(1 + \lambda - 3) \\ &= \lambda(\lambda - 1)(\lambda - 2)^2. \end{aligned}$$

Problem 2.72. *Find the chromatic number of the following graphs*

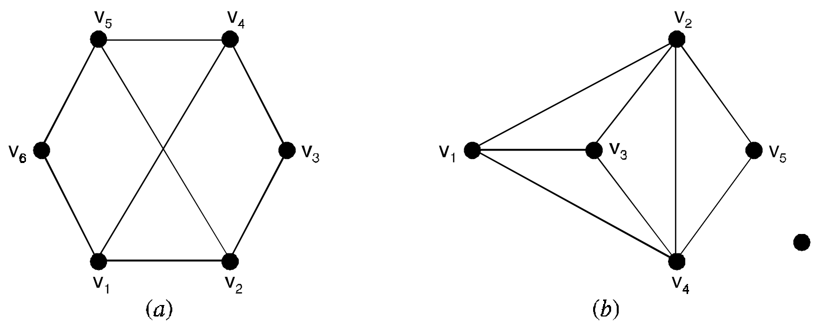


Fig. 2.92.

Solution. (*i*) For the graph in Figure 2.92(*a*), let us assign a colour α to the vertex v_1.

Then, for a proper colouring, we have to assign a different colour to its neighbours v_2, v_4, v_6.

Since v_2, v_4, v_6 are non adjacent vertices, they can have the same color, say β (which is different from α).

Since v_3, v_5 are not adjacent to v_1, these can have the same colours as v_1, namely α.

Thus, the graph can be properly coloured with at least two colours, with the vertices v_1, v_3, v_5 having one colour α and v_2, v_4, v_6 having a different colour β.

Hence the graph is 2-chromatic

(*i.e.,* the chromatic number of the graph is 2).

(*ii*) For the graph in Figure 2.92(*b*), let us again the colour α to the vertex v_1.

Then, for a proper colouring, its neighbours v_2, v_3 and v_4 cannot have the colour α, but v_5 can have the colour α.

Further more, v_2, v_3, v_4 must have different colours, say β, γ, δ.

Thus, at least four colours are required for a proper colouring of the graph.

Hence, the graph is 4-chromatic (*i.e.,* the chroamtic number of the graph is 4).

Problem 2.73. *Prove that a simple planar graph G with less than 30 edges in 4-colorable.*

Solution. If G has 4 or less number of vertices, the required result is true.

Assume that the result is true for any graph with n = K vertices.

Consider a graph G′ with K + 1 vertices and less than 30 edges.

Then, G′ has at least one vertex v of degree at most 4.

Now, considering the graph G′ – v we find that G′ is 4-colorable.

Problem 2.74. *Prove that a graph of order n (≥ 2) consisting of a single circuit is 2-chromatic if n is even, and 3-chromatic if n is odd.*

Solution. The given graph is the cycle graph C_n, $n \geq 2$ as shown in figure below.

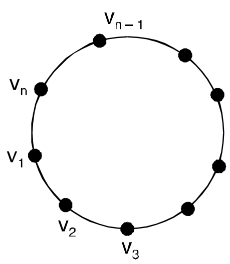


Fig. 2.93.

Obviously, the graph cannot be properly colored with a single colour. Assign two colours alternatively to the vertices, starting with v_1.

That is, the odd vertices v_1, v_3, v_5 etc, will have a colour α and the even vertices v_2, v_4, v_6 etc., will have a different colour β.

Suppose n is even. Then the vertex v_n is an even vertex and therefore will have the colour β, and the graph gets properly coloured.

Therefore, the graph is 2-chormatic.

Suppose n is odd. Then the vertex v_n is an odd vertex and therefore will have the colour α, and the graph is not properly coloured. To make it properly coloured. v_n should be assigned a third colour γ. Thus, in this case, the graph is 3-chromatic.

Problem 2.75. *Prove that every tree with two or more vertices is 2-chromatic.*

Solution. Consider a tree T rooted at a vertex v as shown in figure below. Assign a colour α to v and a different colour β to all vertices adjacent to v. Then the vertices adjacent to those which have the color β are not adjacent to v (because a tree has no circuits) and are at a distance 2 from v. Assign the colour α to these vertices. Repeat the process until all vertices are coloured.

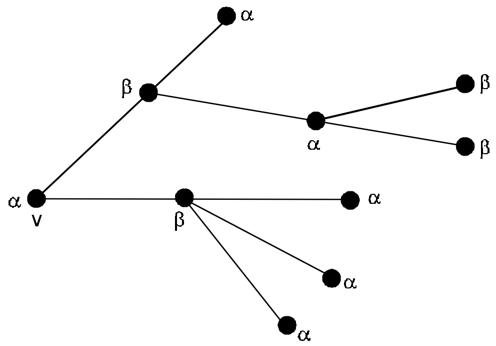


Fig. 2.94.

Thus, v and all vertices which are at distances 2, 4, 6, ... from v have α as their color and all vertices which are at distances 1, 3, 5, ... from v have β as their colour.

Accordingly, along any path of T the vertices are of alternating colours.

Since there is one and only one path between any two vertices in a tree, no two adjacent vertices will have the same colour.

Thus, T has been properly coloured with 2 colours.

If T has two or more vertices, it has one or more edges. As such, it cannot be coloured with 1 colour. This proves that the chromatic number of T is 2, that is 2-chromatic.

Problem 2.76. *Find the chromatic number of a cubic graph with $p \geq 6$ vertices.*

Solution. Every cubic graph contains of odd degree and in which there exists at least one triangle.

Hence $\chi(G) = 3$, where G is a cubic graph.

The following Figure (2.95) gives the result :

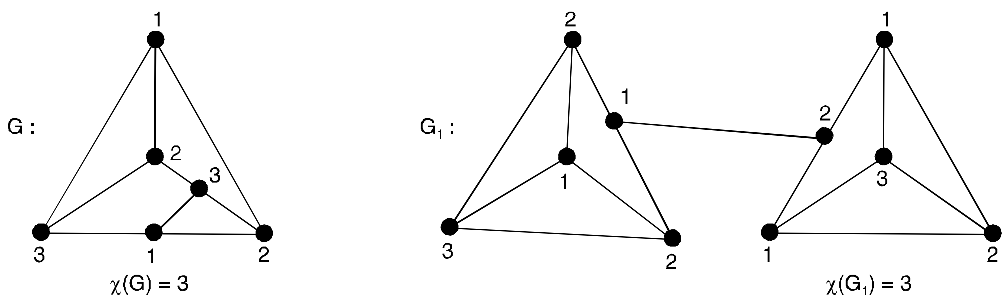


Fig. 2.95

Problem 2.77. *Find the chromatic polynomial of a complete graph on n vertices.*

Solution. Since minimum n colours required for the proper colouring of complete graph K_n on n vertices.

We have $\quad C_i = 0$ for all $i = 1, 2, \ldots\ldots n - 1$.

Further since the graph contains n vertices, n distinct colours can be assigned in n ! ways.

Thus $C_n = n$!.

$$\text{Therefore,} \quad P_n(\lambda) = \sum_{i=1}^{n} \binom{\lambda}{i} C_i = \binom{\lambda}{n} C_n$$

$$= \frac{\lambda(\lambda-1)(\lambda-2) \ldots\ldots \lambda-(n+1)}{n\,!}\, n\,!$$

$$= \lambda(\lambda - 1)(\lambda - 2) \ldots\ldots (\lambda - n + 1).$$

Problem 2.78. *Show that the chromatic number of a graph G is $\lambda(\lambda - 1)(\lambda - 2) \ldots\ldots (\lambda - n + 1)$ if and only if G is a complete graph on n vertices.*

Solution. For a given λ, the first vertex of a graph can be colored in λ ways.

A second vertex can be coloured properly with $\lambda - 1$ ways, the third vertex in only $\lambda - 2$ ways if and only if this vertex is adjacent to first two vertices. Continuing like this we have, the last vertex can be coloured with $(\lambda - n + 1)$ ways if and only if the graph is complete.

Problem 2.79. *Prove that, for a graph G with n vertices*

$$\beta(G) \geq \frac{n}{\chi(G)}.$$

Solution. Let K be the minimum number of colours with which G can be properly colored.

Then $K = \chi(G)$. Let $\alpha_1, \alpha_2, \ldots\ldots \alpha_K$ be these colours and let $n_1, n_2, \ldots\ldots n_K$ be the number of vertices having colours $\alpha_1, \alpha_2, \ldots\ldots \alpha_K$ respectively.

Then $n_1, n_2, \ldots\ldots n_K$ are the orders of the maximal independent sets, because a set of all vertices having the same colour contain all vertices which are mutually non-adjacent.

Sicne $\beta(G)$ is the order of a maximal independent set with largest number of vertices, none of $n_1, n_2, \ldots\ldots n_K$ can exceed $\beta(G)$.

i.e., $\quad n_1 \leq \beta(G), \quad n_2 \leq \beta(G), \ldots\ldots n_K \leq \beta(G)$

Adding these inequalities, we get

$$n_1 + n_2 + \ldots\ldots + n_K \leq K\beta(G)$$

Since $n_1 + n_2 + \ldots\ldots + n_K = n$ and $K = \chi(G)$, this becomes

$$n \leq \chi(G) \,.\, \beta(G)$$

or

$$\beta(G) \geq \frac{n}{\chi(G)}.$$

Problem 2.80. *Show that the following graph is uniquely colourable.*

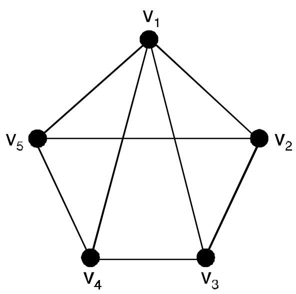


Fig. 2.96.

Solution. We check that the given graph G has only the following independent sets both of which are maximal.

$$W_1 = \{v_2, v_4\}, W_2 = \{v_3, v_5\}$$

Both of these have 2 vertices, and as such $\beta(G) = 2$.

The sets W_1 and W_2 are mutually disjoint and yield only one chromatic partition given below :

$$P = \{W_1, W_2, \{v_1\}\}$$

In view of this single possible chromatic partitioning of G, we infer that G is uniquely colourable.

2.17 COLOUR PROBLEM

The most famous unsolved problem in graph theory and perhaps in all of Mathematics is the celebrated four colour conjecture. This remarkable problem can be explained in five minutes by any mathematician to the socalled man in the steet. At the end of the explanation, both will understand the problem, but neither will be able to solve it.

The conjecture states that, any map on a plane or the surface of a sphere can be coloured with only four colours so that no two adjacent countries have the same colour. Each country must consist of a single connected region, and ajdacent countries are those having a boundary line in common. The conjecture has acted as a catalyst in the branch of mathematics known as combinatorial topology and is closely related to the currently fashionable field of graph theory. More than half a century of work by many mathematicians has yielded proofs for special cases ..... . The consensus is that the conjecture is correct but unlikely to be proved in general.

It seems destined to retain for some time the distinction of being both the simplest and most fascinating unsolved problem of mathematics.

The four colour conjecture has an interesting history, but its origin remains some what vague. There have been reports that Möbius was familiar with this problem in 1840, but it is only definite that the problem was communicated to De Morgan by Guthrie about 1850.

The first of many erroneous proofs of the conjecture was given in 1879 by Kempe. An error was found in 1890 by Heawood who showed, however, that the conjecture becomes true when ‘four’ is replaced by ‘five’.

A counter example, if ever found, will necessarily be extremely large and complicated, for the conjecture was proved most recently by Ore and Stemple for all maps with fewer than 40 countries.

The four colour conjecture is a problem in graph theory because every map yields a graph is which the countries are the points, and two points are joined by a line whenever the corresponding countries are adjacent. Such a graph obviously can be drawn in the plane without intersecting lines.

Thus, if it is possible to colour the points of every **planar graph** with four or fewer colours so that adjacent points have different colours, then the four colour conjecture will have been proved.

2.17.1. The Four colour theorem : 2.51

Every planar graph is 4-colorable.

Assume the four colour conjecture holds and let G be any plane map.

Let G* be the underlying graph of the geometric dual of G.

Since two regions of G are adjacent if and only if the corresponding vertices of G* are adjacent, map G is 4-colorable because graph G* is 4-colorable.

Conversely, assume that every plane map is 4-colorable and let H be any planar graph.

Without loss of generality, we suppose H is a connected plane graph.

Let H* be the dual of H, so drawn that each region of H* encloses precisely one vertex of H. The connected plane pseudograph H* can be converted into a plane graph H′ by introducing two vertices into each loop of H* and adding a new vertex into each edge in a set of multiple edges.

The 4-colorability of H′ now implies that H is 4-colorable, completing the verification of the equivalence.

If the four color conjecture is ever proved, the result will be best possible, for it is easy to give examples of planar graphs which are 4-chromatic, such as K_4 and W_6 (see Figure 2.97 below).

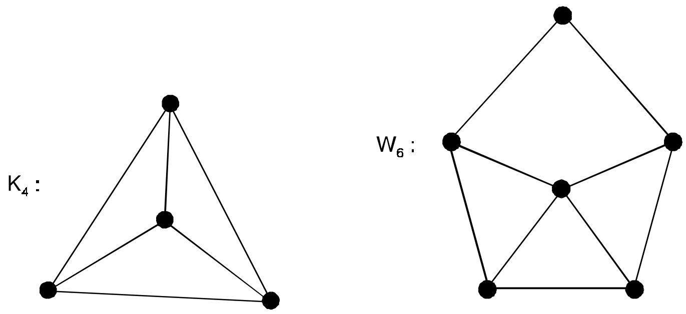


Fig. 2.97. Two 4-chromatic planar graphs.

Theorem 2.52. *Every planar graph with fewer than 4 triangles is 3-colourable.*

Corollary. Every planar graph without triangle is 3-colourable.

Theorem 2.53. *The four colour conjecture holds if and only if every cubic bridgeless plane map is 4-colourable.*

Proof. We have already seen that every plane map is 4-colourable if and only if the four colour conjecture holds.

This is also equivalent to the statement that every bridgeless plane map is 4-colourable since the elementary contraction of identifying the end vertices of a bridge affects neither the number of regions in the map nor the adjacency of any of the regions.

Certainly, if every bridgeless plane map is 4-colorable, then every cubic bridgeless plane map is 4-colorable.

In order to verify the converse, let G be a bridgeless plane map and assume all cubic bridgeless plane maps are 4-colourable.

Since G is bridgeless, it has no end vertices.

If G contains a vertex v of degree 2 incident with edges y and z, we subdivide y and z, denoting the subdivision vertices by u and w respectively.

We now remove v, identify u with one of the vertices of degree 2 in a copy of the graph $K_4 - x$ and identify w with the other vertex of degree 2 in $K_4 - x$.

Observe that each new vertex added has degree 3 (see Figure 2.98).

If G contains a vertex v_0 of degree $n \geq 4$ incident with edges x_1, x_2, ... x_n, arranged cyclically about v_0, we subdivide each x_i producing a new vertex v_i.

We then remove v_0 and add the new edges v_1v_2, v_2v_3, ..., $v_{n-1}v_n$, v_nv_1.

Again each of the vertices so added has degree 3.

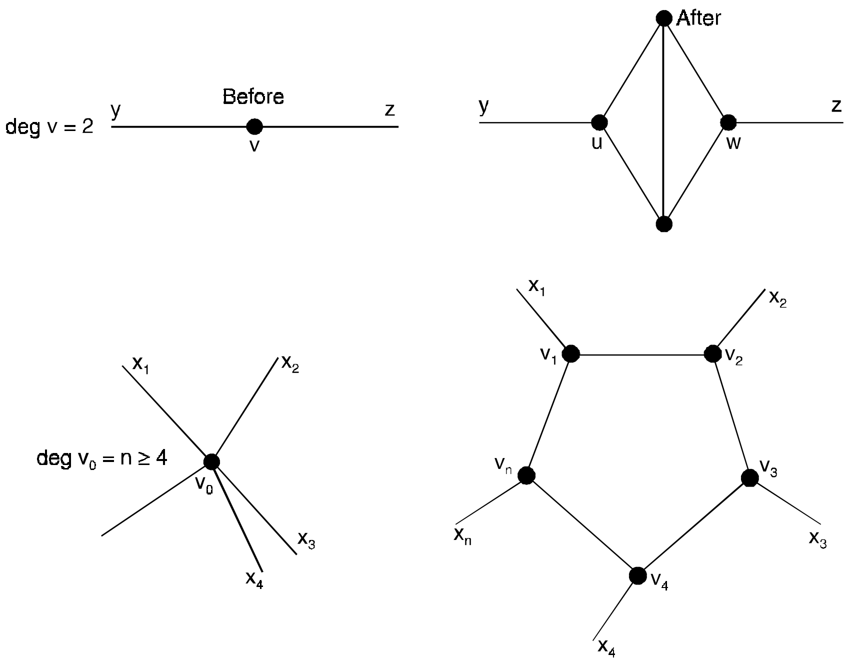


Fig. 2.98. Conversion of a graph into a cubic graph.

Denote the resulting bridgeless cubic plane map by G′, which, by hypothesis, is 4-colourable.

If for each vertex v of G with deg $v \neq 3$, we identify all the newly added vertices associated with v in the formation of G′, we arrive at G once again. Thus, let there be given a 4-colouring of G′. The above mentioned contradiction of G′ into G induces an m-colouring of G, $m \leq 4$, which completes the proof.

Theorem 2.54. *The four color conjecture holds if and only if every hamiltonian planar graph is 4-colorable.*

Theorem 2.55. *For any graph G, the line chromatic number satisfies the inequalties*

$$\Delta \leq \chi' \leq \Delta + 1.$$

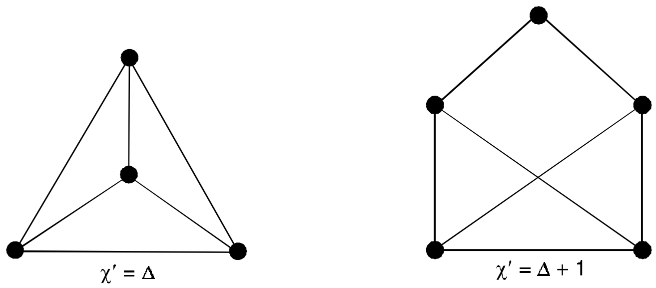


Fig. 2.99. The two values for the line-chromatic number.

2.17.2. The Five colour theorem 2.56

Every planar graph is 5-colorable.

Proof. We proceed by induction on the number P of points. For any planar graph having P ≤ 5 points, the result follows trivially since the graph is P-colorable.

As the inductive hypothesis we assume that all planar graphs with P points, P ≥ 5, are 5-colourable.

Let G be a plane graph with P + 1 vertices, G contains a vertex v of degree 5 or less.

By hypothesis, the plane graph G – v is 5-colourable.

Consider an assignment of colours to the vertices of G – v so that a 5-colouring results, when the colours are denoted by C_i, $1 \leq i \leq 5$.

Certainly, if some colour, say C_j, is not used in the colouring of the vertices adjacent with v, then by assigning the colour C_j to v, a 5-colouring of G results.

This leaves only the case to consider in which deg $v = 5$ and five colours are used for the vertices of G adjacent with v.

Permute the colours, if necessary, so that the vertices coloured C_1, C_2, C_3, C_4 and C_5 are arranged cyclically about v,

Now label the vertex adjacent with v and coloured C_i by v_i, $1 \leq i \leq 5$ (see Figure 2.100)

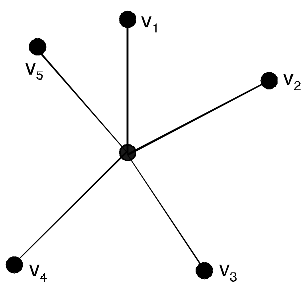


Fig. 2.100. A step in the proof of the five colour theorem.

Let G_{13} denote the subgraph of $G - v$ induced by those vertices coloured C_1 or C_3.

If v_1 and v_3 belong to different components of G_{13}, then a 5-coloring of $G - v$ may be accomplished by interchanging the colors of the vertices in the component of G_{13} containing v_1.

In this 5-coloring however, no vertex adjacent with v is colored C_1, so by coloring v with the color C_1, a 5-coloring of G results.

If, on the other hand, v_1 and v_3 belong to the same component of G_{13}, then there exists in G a path between v_1 and v_3 all of whose vertices are colored C_1 or C_3.

This path together with the path $v_1\ vv_3$ produces a cycle which necessarily encloses the vertex v_2 or both the vertices v_4 and v_5.

In any case, there exists no path joining v_2 and v_4, all of whose vertices are coloured C_2 or C_4.

Hence, if we let G_{24} denote the subgraph of $G - v$ induced by the vertices coloured C_2 or C_4, then v_2 and v_4 belong to different components of G_{24}.

Thus if we interchange colors of the vertices in the component of G_{24} containing v_2, a 5-colouring of $G - v$ is produced in which no vertex adjacent with v is coloured C_2.

We may then obtain a 5-coloring of G by assigning to v the colour C_2.

Problem Set 2.1

1. (*a*) Show that the graph is planar by drawing an isomorphic plane graph with straight edges.

(*b*) Label the regions defined by your plane graph and list the edges which form the boundary of each region.

(*c*) Verify that $V - E + R = 2$, $N \leq 2E$, and $E \leq 3V - 6$.

2. Verify Euler's formula $V - E + F = 2$ for each of the five platanoic solids.

3. If G a connected plane graph with $V \geq 3$ vertices and R regions, show that $R \leq 2V - 4$.

4. (*a*) Give an example of a connected planar graph for which $E = 3V - 6$.

(*b*) Let G be a connected plane graph for which $E = 3V - 6$ show that every region of G is a triangle.

5. (*a*) If G is a connected plane graph with at least three vertices such that no boundary of a region is a triangle, prove that $E \leq 2V - 4$.

(*b*) Let G be a connected planar bipartite graph with E edges and $V \geq 3$ vertices. Prove that $E \leq 2V - 4$.

6. (*a*) For which n is K_n planar ?

(*b*) For which m and n is $K_{m,\,n}$ planar ?

7. Show that $K_{2,\,2}$ is homeomorphic to K_3.

8. Suppose a graph G_1 with V_1 vertices and E_1 edges is homeomorphic to a graph G_2 with V_2 vertices and E_2 edges prove that $E_2 - V_2 = E_1 - V_1$.

9. Show that any graph homeomorphic to K_5 or $K_{3,\,3}$ is obtainable from K_5 or $K_{3,\,3}$ respectively, by addition of vertices to edges.

10. (*a*) Let G be a connected graph with V_1 vertices and E_1 edges and let H be a subgraph with V_2 vertices and E_2 edges. Show that $E_2 - V_2 \leq E_1 - V_1$.

(*b*) Let G be a connected graph with V vertices ; E edges, and $E \leq V + 2$. Show that G is planar.

11. Let G be a graph and let H be obtained from G by adjoining a new vertex of degree 1 to some vertex of G. Is it possible for G and H to be homeomorphic ? Explain.

12. (*a*) Show that any planar graph all of whose vertices have degree at least 5 must have at least 12 vertices.

(*b*) Find a planar graph each of whose vertices has degree at least 5.

13. (*a*) Prove that if G is a planar graph with *n* connected components, each components having at least three vertices then $E \leq 3V - 6n$.

(*b*) Prove that if G is a planar graph with *n* connected components, then it is always true that $E \leq 3V - 3n$.

14. (*a*) Prove that every planar graph with $V \geq 2$ vertices has at least two vertices of degree $d \leq 5$.

(*b*) Prove that every planar graph with $V \geq 3$ vertices has at least three vertices of degree $d \leq 5$.

(*c*) Prove that every planar graph with $V \geq 4$ vertices has at least four vertices of degree of $d \leq 5$.

15. (*a*) A connected planar graph G has 20 vertices. Prove that G has at most 54 edges.

(*b*) A connected planar graph G has 20 vertices, seven of which have degree 1. Prove that G has at most 40 edges.

16. Suppose G is a connected planar graph in which every vertex has degree at least 3. Prove that at least two regions of G have at most five edges on their boundaries.

17. Draw a graph corresponding to the map shown at the right and find a coloring which requires the least number of colors. What is the chromatic number of the graph ?

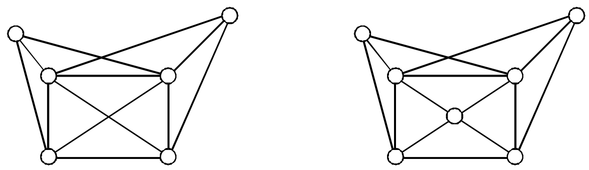
18. (*a*) What is $\chi(K_{14})$? What is $\chi(K_5, 14)$? Why ?

(*b*) Let G_1 and G_2 by cycles with 38 and 107 edges, respectively. What is $\chi(G_1)$? What is $\chi(G_2)$? Explain.

19. Let $n \geq 4$ be a natural number. Let G be the graph which consists of the union of K_{n-3} and a 5-cycle C together with all possible edges between the vertices of these graphs. Show that $\chi(G) = n$, yet G does not have K_n as a subgraph.

20. Find a formula for V – E + R which applies to planar graphs which are not necessarily connected.

21. Find its chromatic number and explain why this piece of information is consistent with the four color problem.



22. Prove that every subgraph of a planar graph is planar.

23. Prove that :

(*i*) K_5 is the non planar graph with the smallest number of vertices.

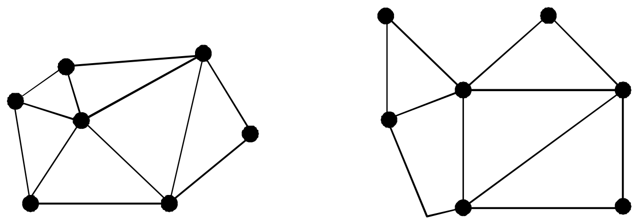
(*ii*) $K_{3, 3}$ is the non planar graph with the smallest number of edges.

24. Show that every graph with four or fewer vertices is planar.

25. Show that the graphs $K_{1,S}$ for $S \geq 1$ and $K_{2,S}$ for $S \geq 2$ are planar.

26. Let G be a simple connected graph with at least 11 vertices. Prove that either G or its complement $\overline{G}$ must be non planar.

27. Verify the Euler's formula for the graphs shown below :



28. Verify the Euler's formula for the graphs W_8, $K_{1,5}$ and $K_{2,7}$.

29. Prove that every simple connected planar graph with $n \geq 4$ vertices has at least four vertices of degree five or less.

30. Let G be a connected planar graph with more than two vertices. If G has exactly n_K vertices of degree K and $\Delta(G) = P$, show that

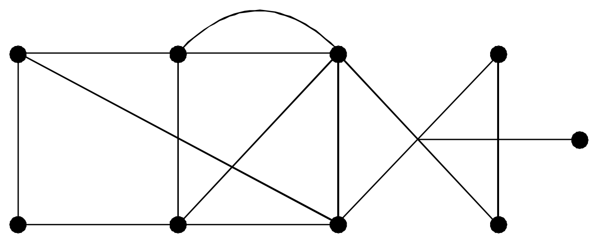
$$5n_1 + 4n_2 + 3n_3 + 2n_4 + n_5 \geq n_7 + 2n_8 + \ldots\ldots + (P - 6)\, n_P + 12.$$

31. Prove that the sum of the degree of the regions of a planar graph is equal to twice the number of edges in the graph.

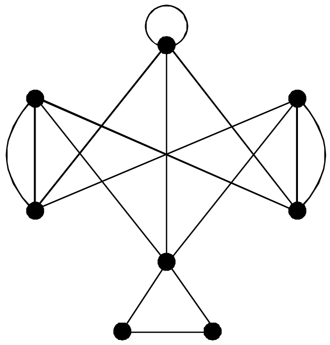
32. Show that a simple planar connected graph with less than 30 edges must have a vertex of degree ≤ 4.

33. What is the minimum number of vertices necessary for a simple connected graph with 7 edges to be planar ?

34. By using the method of elementary reduction, show that the following graph is planar.



35. By the method of elementary reduction, show that the following graph is non planar.



36. Prove that a planar graph G is isomorphic to G** if and only if G is connected.

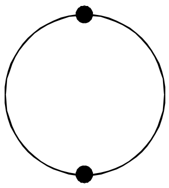
37. Let G be a planar connected graph. Prove that G is bipartite if and only if G* is an Euler graph.

38. Prove that a self loop free planar graph G is 2-connected if and only if G* is 2-connected.

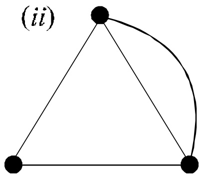
39. Prove that 5-connected planar graph has at least 12 vertices.

40. Show that the following graphs are self dual.

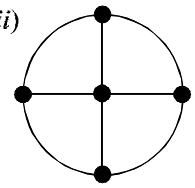
(*i*)



(*ii*)



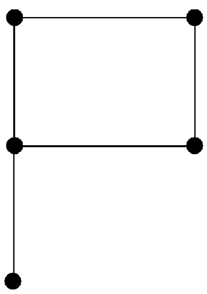
(*iii*)



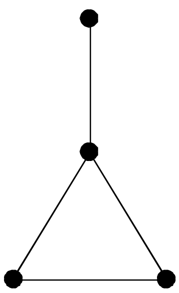
41. Show that a simple graph with n vertices and more than $\left[\dfrac{n^2}{4}\right]$ edges cannot be 2-chromatic.

42. Prove or disprove that in a graph of order n and size m, $\chi(G) \le 1 + \dfrac{2m}{n}$.

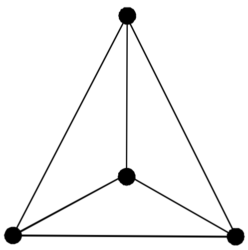
43. Find the chromatic numbers of the following



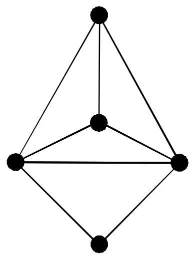
(*i*)



(*ii*)



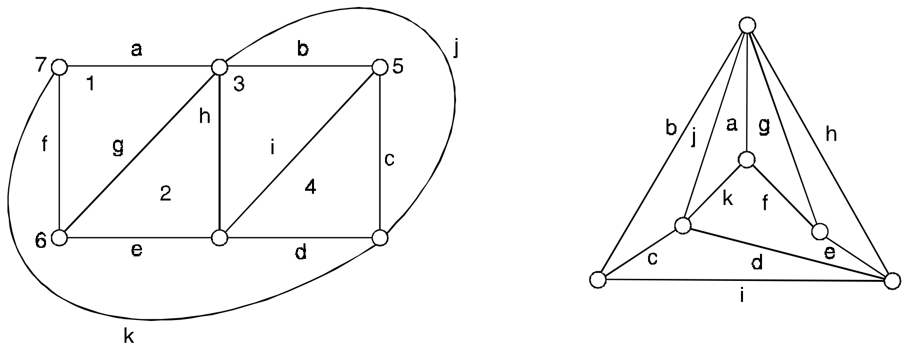
(*iii*)



(*iv*)

Answers 2.1

1. (*a*) We draw the graph quickly as a planar and then, after some thinking, as a planar graph with straight edges.



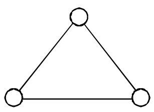
(*b*) There are seven regions, numbered 1, 2, ... 7, with boundaries *afg, ghe, hbi, icd, bjc, fedk,* and *ajk*, respectively.

(*c*) E = 11, V = 6, R = 7, N = 22, so V – E + R = 6 – 11 + 7 = 2 ; N = 22 ≤ 22 = 2E and E = 11 ≤ 12 ≤ 3V – 6.

2.

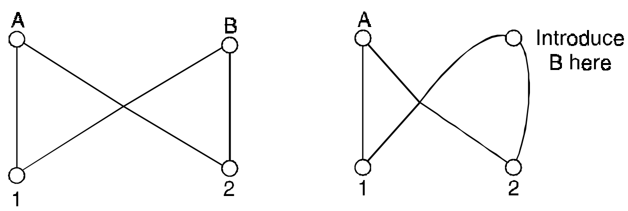
Solid	*V*	*E*	*F*	*V – E + F*
tetrahedron	4	6	4	2
cube	8	12	6	2
octahedron	6	12	8	2
dodecahedron	201	30	12	2
icosahedron	12	30	20	2

3. We know that E ≤ 3V – 6, substituting E = V + R – 2, we obtain V + R – 2 ≤ 3V – 6 or R ≤ 2V – 4, as required.

4. (*a*)  E = 3 = 3(3) – 6 = 3V – 6.

6. (*a*) K_n is planar if and only if $n \leq 4$.

7. (*a*)



10. (*a*) Assume the result is not true. Then there is some counter example G and subgraph H ; that is, for these graphs $E_1 - V_1 < E_2 - V_2$. (In particular, $V_1 \neq V_2$), choose H such that $V_1 - V_2$ is as small as possible. Since G is connected, we can find a vertex v which is in G, but not H which is joined to some vertex in H. Let K be that subgraph of G consisting of H, v and all edges joining v to vertices in H. Letting v_3 and E_3 denote the number of vertices and edges, respectively, in K, we have $V_3 = V_2 + 1$, while $E_3 \geq E_2 + 1$. Hence $E_2 - V_2 \leq E_3 - V_3$ and so $E_1 - V_1 < E_3 - V_3$. Thus, G and its subgraph K provide another counter example, but this contradicts the minimality of $V_1 - V_2$ since $V_1 - V_3 < V_1 - V_2$.

11. Yes. An example is shown to the right. The graphs are homeomorphic since the one on the right is obtainable from the other by adding a vertex of degree 2.

13. (*a*) Let $G_1, G_2, ..., G_n$ be the connected components of G. Since G_i has atleast three vertices, we have $E_{G_i} \leq 3\, V_{G_i} - 6$. Hence, $\Sigma\, E_{G_i} \leq 3\, \Sigma\, V_{G_i} - 6n$, so $E \leq 3V - 6n$ is required.

14. (*a*) We may assume that G is connected. Say there is only one vertex of degree atmost 5. Then

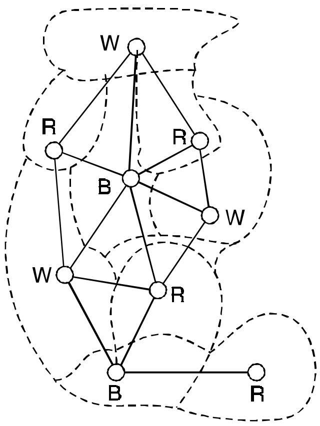
$\Sigma \deg v_i \geq 6(V - 1) = 6V - 6$, contradicting

$\Sigma \deg v_i = 2E \leq 6V - 12$.

15. (*a*) $E \leq 3V - 6$, so $E \leq 3(20) - 6 = 54$.

16. Say at most one region has atmost five edges on its boundary. Then, $N \geq 6(R - 1)$. But $N \leq 2E$, so $2E \geq 6R - 6$, $3R \leq E + 3$. Sinve $V - E + R = 2$, $6 = 3V - 3E + 3R \leq 3V - 2E + 3$ that is, $2E \leq 3V - 3$. But $2E = \Sigma\deg v_i \geq 3V$ by assumption, and this is a contradiction.

17. We show the graph superimposed over the given map. Since this graph contains triangles, atleast three colours are necessary. A 3-colouring is shown, so the chromatic number is 3.



18. (*a*) For any n, $\chi(K_n) = n$ and for any m, n, $\chi(K_{m,n}) = 2$. Thus, $\chi(K_{14}) = 14$ and $\chi(K_{5,14}) = 2$.

20. Letting x denote the number of connected components of G, we have $V - E + R = 1 + x$.

For each component C, $V_C - E_C + R_C = 2$. Adding, we get

$\Sigma V_C - \Sigma E_C + \Sigma R_C = 2x$. We have $\Sigma V_C = V$ and $\Sigma E_C = E$, but $\Sigma R_C = R + (x - 1)$ since the exterior region is common to all components.

Thus, $V - E + R + x - 1 = 2x$, $V - E' + R' = x + 1$.

42. (*i*) 2, (*ii*) 3, (*iii*) 4 (*iv*) 4.

CHAPTER

3 Trees

INTRODUCTION

Kirchhoff developed the theory of trees in 1847, in order to solve the system of simultaneous linear equations which give the current in each branch and arround each circuit of an electric network.

In 1857, Cayley discovered the important class of graphs called **trees** by considering the changes of variables in the differential calculus. Later, he was engaged in enumerating the isomers of saturated **hydro carbons** $\mathbf{C}_n \mathbf{H}_{2n+2}$ with a given number of n of carbon atoms as

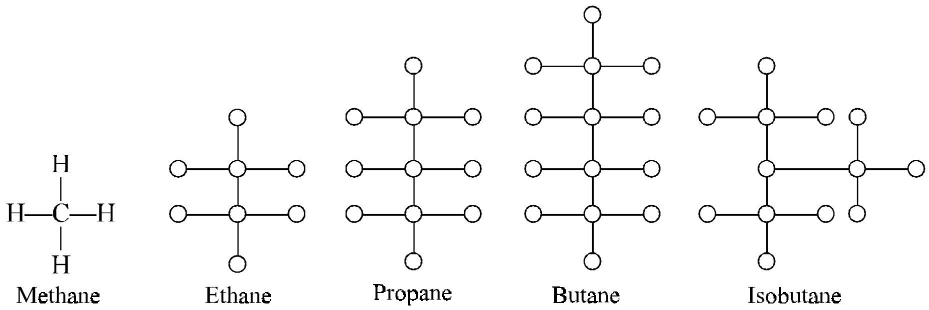


Fig. 3.1.

3.1 TREE

3.1.1. Acyclic graph

A graph is acyclic if it has no cycles.

3.1.2. Tree

A tree is a connected acyclic graph.

3.1.3. Forest

Any graph without cycles is a forest, thus the components of a forest are trees.

The tree with 2 points, 3 points and 4-points are shown below :

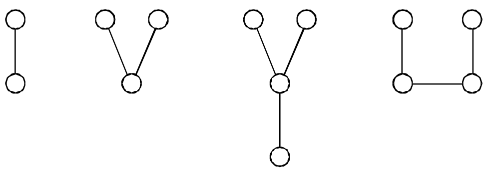


Fig. 3.2.

Note :

(1) Every edge of a tree is a bridge.

i.e., every block of G is acyclic.

Conversely, every edge of a connected graph G is a bridge, then G is a tree.

(2) Every vertex of G (tree) which is not an end vertex is neccessarily a cut-vertex.

(3) Every nontrivial tree G has at least two end vertices.

3.2 SPANNING TREE

A spanning tree is a spanning subgraph, that is a tree.

3.2.1. Branch of tree

An edge in a spanning tree T is called a branch of T.

3.2.2. Chord

An edge of G that is not in a given spanning tree is called a chord.

Note :

(1) The branches and chords are defined only with respect to a given spanning tree.

(2) An edge that is a branch of one spanning tree T_1 (in a graph G) may be chord, with respect to another spanning tree T_2.

3.3 ROOTED TREE

A rooted tree T with the vertex set V is the tree that can be defined recursively as follows :

T has a specially designated vertex $v_1 \in$ V, called the **root** of T. The subgraph of T_1 consisting of the vertices V – {v} is partitionable into subgraphs.

T_1, T_2, ..., T_r each of which is itself a rooted tree. Each one of these r-rooted tree is called a **subtree of v_1.**

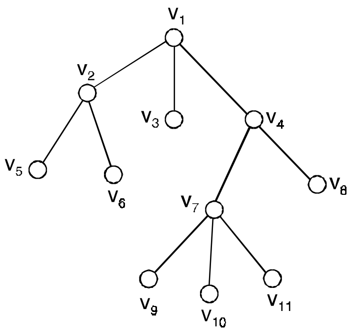


Fig. 3.3. A rooted tree.

3.3.1. Co tree

The cotree T* of a spanning tree T in a connected graph G is the spanning subgraph of G containing exactly those edges of G which are not in T. The edges of G which are not in T* are called its twigs.

For example :

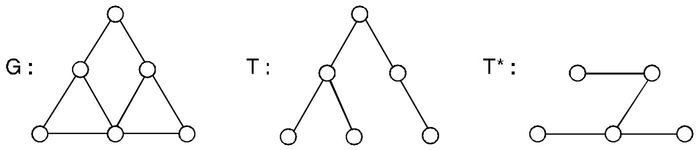


Fig. 3.4.

3.4 BINARY TREES

A binary tree is a rooted tree where each vertex v has atmost two subtrees ; if both subtrees are present, one is called a left subtree of v and the other right-subtree of v. If only one subtree is present, it can be designated either as the left subtree or right subtree of v.

In other words, a binary tree is a 2-ary tree in which each child is designated as a left child or right child.

In a binary tree *e* very vertex has two children or no children.

Properties : (Binary trees) :

(1) The number of vertices n in a complete binary tree is always odd. This is because there is exactly one vertex of even degree, and remaining $n - 1$ vertices are of odd degree. Since from theorem (*i.e.*, the number of vertices of odd degree is even), $n - 1$ is even. Hence n is odd.

(2) Let P be the number of end vertices in a binary tree T. Then $n - p - 1$ is the number of vertices of degree 3. The number of edges in T is

$$\frac{1}{2}[p + 3(n - p - 1) + 2] = n - 1 \qquad \text{or} \qquad p = \frac{n+1}{2} \qquad ...(1)$$

(3) A non end vertex in a binary tree is called an **internal vertex.** It follows from equation (1) that the number of internal vertices in a binary is one less than the number of end vertices.

(4) In a binary tree, a vertex v_i is said to be at **level l_i** if v_i is at a distance l_i from the root. Thus the root is at level O.

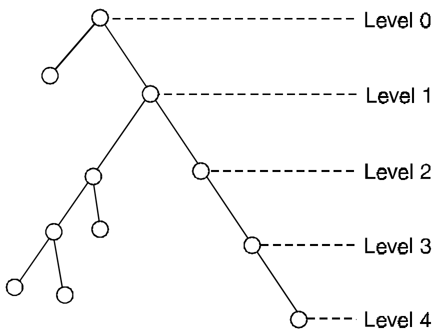


Fig. 3.5. 13-vertices, 4-level binary tree.

The maximum numbers of vertices possible in a k-level binary tree is $2^0 + 2^1 + 2^2 + ... + 2^k \geq n$,

The maximum level, l_{max} of any vertex in a binary tree is called the **height** of the tree.

On the other hand, to construct a binary tree for a given n such that the farthest vertex is as for as possible from the root, we must have exactly two vertices at each level, except at the O level.

Hence max $l_{max} = \dfrac{n-1}{2}$.

For example,

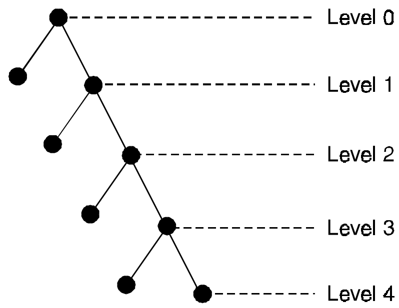


Fig. 3.6.

Max $l_{max} = \dfrac{9-1}{2} = 4$

The minimum possible height of n-vertex binary tree is min $l_{max} = [\log_2(n + 1) - 1]$

In analysis of algorithm, we are generally interested in computing the sum of the levels of all end vertices. This quantity, known as the **path length** (or external path length) of a tree.

3.4.1. Path length of a binary tree

It can be defined as the sum of the path lengths from the root to all end vertices.

For example,

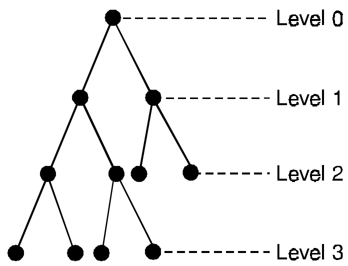


Fig. 3.7.

Here the sum is 2 + 2 + 3 + 3 + 3 + 3 = 16 is the path length of a given above binary tree.

The path length of the binary tree is often directly related to the executive time of an algorithm.

3.4.2. Binary tree representation of general trees

There is a straight forward technique for converting a general tree to a binary tree form. The algorithm has two easy steps :

Step 1 :

Insert edges connecting siblings and delete all of a parents edges to its children except to its left most off spring.

Step 2 :

Rotate the resulting diagram 45° to distinguish between left and right subtrees.

For example,

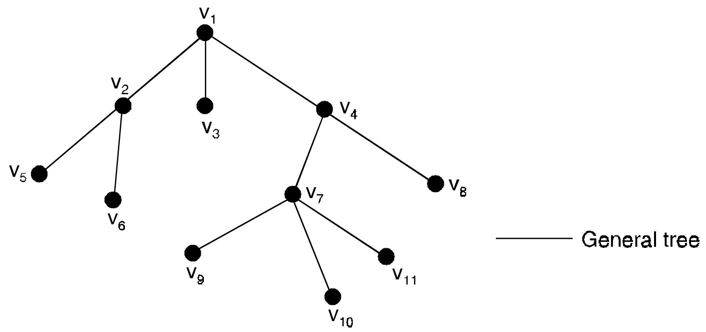


Fig. 3.8.

Here v_2, v_3 and v_4 are siblings to the parent v_1, now apply the steps given above we have a binary tree as shown here.

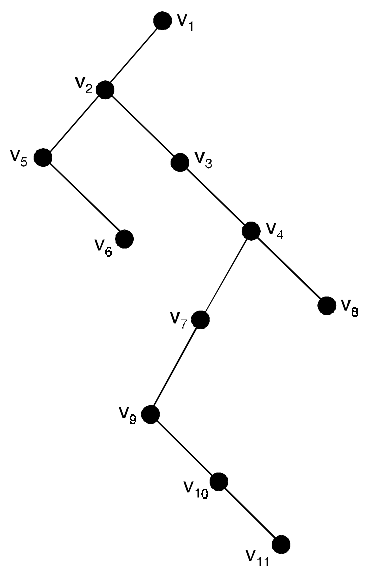


Fig. 3.9.

Theorem 3.1. *A(p, q) graph is a tree if and only if it is acyclic and $p = q + 1$ or $q = p - 1$.*

Proof. If G is a tree, then it is acyclic.

By definition to verify the equality $p = q + 1$.

We employ induction on p.

For $p = 1$, the result is trivial.

Assume, then that the equality $p = q + 1$ holds for all (p, q) trees with $p \geq 1$ vertices.

Let G_1 be a tree with $p + 1$ vertices.

Let v be an end-vertex of G_1.

The graph $G_2 = G_1 - v$ is a tree of order p, and so $p = |E(G_2)| + 1$.

Since G_1 has one more vertex and one more edge than that of G_2.

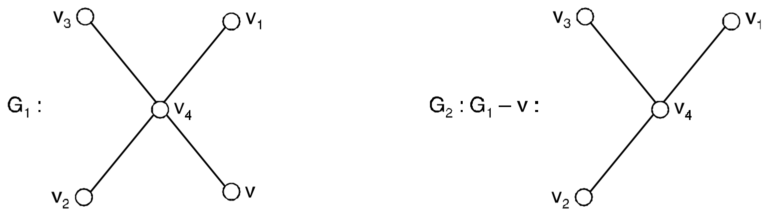


Fig. 3.10.

$$|V(G_1)| = p + 1 = (|E(G_2)| + 1) + 1$$
$$= |E(G_1)| + 1$$
$$\therefore \quad |V(G_1)| = |E(G_1)| + 1.$$

Conversely : Let G be an acyclic (p, q) graph with $p = q + 1$.

To show G is a tree, we need only verify that G is connected. Denote by $G_1, G_2, ..., G_k$, the components of G, where $k \geq 1$.

Furthermore, let G_i be a (p_i, q_i) graph.

Since each G_i is a tree, $p_i = q_i + 1$.

Hence $\quad p - 1 = q = \sum_{i=1}^{k} q_i$

$$= \sum_{i=1}^{k} (p_i - 1) = p - k$$

$\Rightarrow \quad p - 1 = p - k \quad \Rightarrow \quad k = 1$ and G is connected.

Hence, (p, q) graph is a tree.

Hence the proof.

Corollary : A forest G of vertices p has $p - k$ edges where k is the number of components.

Theorem 3.2. *A (p, q) graph G is a tree if and only if G is connected and $p = q + 1$.*

Proof. Let G be a (p, q) tree.

By definition of G, it is connected and by theorem : *i.e.,* A(p, q) graph is a tree if and only if it is acyclic and $p = q + 1$), $p = q + 1$.

Conversely : We assume G is connected (p, q) graph with $p = q + 1$.

It is sufficient to show that G is acyclic.

If G contains a cycle C and e is an edge of C, then G – e is a connected graph with p vertices having $p - 2$ edges.

This is impossible by the definition (*i.e.*, A(p, q) graph has $q < p - 1$ then G is disconnected).

This contradicts our assumption.

Hence G is connected.

Theorem 3.3. *A complete* ***n-ary*** *tree with m internal nodes contains n × m + 1 nodes.*

Proof. Since there are m internal nodes, and each internal node has n descendents, there are $n \times m$ nodes in three other than root node.

Since there is one and only one root node in a tree, the total number of nodes in the tree will $n \times m + 1$.

Problem 3.1. *A tree has five vertices of degree 2, three vertices of degree 3 and four vertices of degree 4. How many vertices of degree 1 does it have ?*

Solution. Let x be the number of nodes of degree one.

Thus, total number of vertices

$$= 5 + 3 + 4 + x = 12 + x.$$

The total degree of the tree $= 5 \times 2 + 3 \times 3 + 4 \times 4 + x = 35 + x$

Therefore number of edges in the three is half of the total degree of the tree.

If G = (V, E) be the tree, then, we have

$$|V| = 12 + x \text{ and } |E| = \frac{35 + x}{2}$$

In any tree, $|E| = |V| - 1$.

Therefore, we have $\frac{35 + x}{2} = 12 + x - 1$

$\Rightarrow \quad 35 + x = 24 + 2x - 2$

$\Rightarrow \quad x = 13$

Thus, there are 13 nodes of degree one in the tree.

Problem 3.2. *A tree has 2n vertices of degree 1, 3n vertices of degree 2 and n vertices of degree 3. Determine the number of vertices and edges in the tree.*

Solution. It is given that total number of vertices in the tree is $2n + 3n + n = 6n$.

The total degree of the tree is $2n \times 1 + 3n \times 2 + n \times 3 = 11n$.

The number of edges in the tree will be half of $11n$.

If G = (V, E) be the tree then, we have

$$|V| = 6n \quad \text{and} \quad |E| = \frac{11n}{2}$$

In any tree, $|E| = |V| - 1$.

Therefore, we have

$$\frac{11n}{2} = 6n - 1$$

$\Rightarrow \quad 11n = 12n - 2$

$\Rightarrow \quad n = 2$

Thus, there are 6 × 2 = 12 nodes and 11 edges in the tree.

Theorem 3.4. *There are at the most* ***n^h leaves in an n-ary tree*** *of height h.*

Proof. Let us prove this theorem by mathematical induction on the height of the tree.

As basis step take $h = 0$, *i.e.*, tree consists of root node only.

Since $n° = 1$, the basis step is true.

Now let us assume that the above statement is true for $h = k$.

i.e., an n-ary tree of height k has at the most n^k leaves.

If we add n nodes to each of the leaf node of n-ary tree of height k, the total number of leaf nodes will be at the most $n^h \times n = n^{h+1}$.

Hence inductive step is also true.

This proves that above statement is true for all $h \geq 0$.

Theorem 3.5. *In a complete n-ary tree with m internal nodes, the number of leaf node l is given by the formula*

$$l = \frac{(n-1)(x-1)}{n}.$$

where, x is the total number of nodes in the tree.

Proof. It is given that the tree has m internal nodes and it is complete n–ary, so total number of nodes

$$x = n \times m + 1.$$

Thus, we have $\quad m = \dfrac{(x-1)}{n}$

It is also given that l is the number of leaf nodes in the tree.

Thus, we have $\quad x = m + l + 1$

Substituting the value of m in this equation, we get

$$x = \left(\frac{x-1}{n}\right) + l + 1$$

or

$$l = \frac{(n-1)(x-1)}{n}$$

Theorem 3.6. *If T = (V, E) be a rooted tree with v_0 as its root then*

(*i*) *T is a acyclic*

(*ii*) *v_0 is the only root in T*

(*iii*) *Each node other than root in T has in degree 1 and v_0 has indegree zero.*

Proof. We prove the theorem by the method of contradiction.

(*i*) Let there is a cycle π in T that begins and end at a node v.

Since the in degree of root is zero, $v \neq v_0$.

Also by the definition of tree, there must be a path from v_0 to v, let it be p.

Then πp is also a path, distinct from p, from v_0 to v.

This contradicts the definition of a tree that there is unique path from root to every other node.

Hence T cannot have a cycle in it.

i.e., a tree is always acyclic.

(*ii*) Let v_1 is another root in T.

By the definition of a tree, every node is reachable from root.

This v_0 is reachable from v_1 and v_1 is reachable from v_0 and the paths are π_1 and π_2 respectively.

Then $\pi_1\pi_2$ combination of these two paths is a cycle from v_0 and v_0.

Since a tree is always acyclic, v_0 and v_1 cannot be different.

Thus, v_0 is a unique root.

(*iii*) Let w be any non-root node in T.

Thus, $\exists$ a path $\pi : v_0, v_1, ..., v_k w$ from v_0 to w in T.

Now let us suppose that indegree of w is two.

Then $\exists$ two nodes w_1 and w_2 in T such that edges (w_1, v_0) and (w_2, v_0) are in E.

Let π_1 and π_2 be paths from v_0 to w_1 and w_2 respectively.

Then $\pi_1 : v_0 v_1 ...... v_k w_1 w$ and $\pi_2 : v_0 v_1 ...... v_k w_2 w$ are two possible paths from v_0 to w.

This is in contradiction with the fact that there is unique path from root to every other nodes in a tree.

Thus indegree of w cannot be greater than 1.

Next, let indegree of $v_0 > 0$. Then $\exists$ a node v in T such that $(v, v_0) \in$ E.

Let π be a path from v_0 to v, thus $\pi(v, v_0)$ is a path from v_0 to v_0 that is a cycle.

This is again a contradiction with the fact that any tree is acyclic.

Thus indegree of root node v_0 cannot be greater than zero.

Problem 3.3. *Let T = (V, E) be a rooted tree. Obviously E is a relation on set V. Show that*

(*i*) *E is irreflexive*

(*ii*) *E is asymmetric*

(*iii*) *If* $(a, b) \in E$ *and* $(b, c) \in E$ *then* $(a, c) \notin E, \forall a, b, c \in V$.

Solution. Since a tree is acyclic, there is no cycle of any length in a tree.

This implies that there is no loop in T.

Thus, $(v, v) \notin \text{E} \; \forall a \in \text{V}$.

Thus E is an irreflexive relation on V.

Let $(x, y) \in$ E. If $(y, x) \in$ E, then there will be cycle at node x as well as on node y.

Since no cycle is permissible in a tree, either pair (x, y) or (y, x) can be in E but never both.

This implies that presence of (x, y) excludes the presence of (y, x) in E and *vice versa*.

Thus E is a asymmetric relation on V.

Let $(a, c) \in$ E.

Thus presence of pairs (b, c) and (a, c) in E implies that c has indegree > 1.

Hence $(a, c) \notin$ E.

Problem 3.4. *Prove that a tree T is always separable.*

Solution. Let w be any internal node in T and node v is the parent of w.

By the definition of a tree, in degree of w is one.

If w is dropped from the tree T, the incoming edge from v to w is also removed.

Therefore all children of w will be unreachable from root and tree T will become disconnected.

See the forest of the Figure (3.11), which has been obtained after removal of node F from the tree of Figure (3.12).

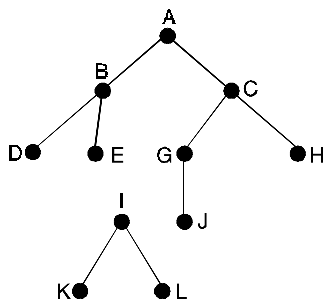


Fig. 3.11

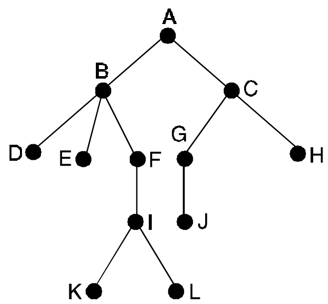


Fig. 3.12

Problem 3.5. *Let A = $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}$ and let*

$$T = \{(v_2, v_3), (v_2, v_1), (v_4, v_5), (v_4, v_6), (v_5, v_8), (v_6, v_7), (v_4, v_2), (v_7, v_9), (v_7, v_{10})\}.$$

Show that T is a rooted tree and identify the root.

Solution. Since no paths begin at vertices v_1, v_3, v_8, v_9 and v_{10}, these vertices cannot be roots of a tree.

There are no paths from vertices v_6, v_7, v_2 and v_5 to vertex v_4, so we must eliminate these vertices as possible roots.

Thus, if T is a rooted tree, its root must be vertex v_4.

It is easy to show that there is a path from v_4 to every other vertex.

For example, the path v_4, v_6, v_7, v_9 leads from v_4 and v_9, since (v_4, v_6), (v_6, v_7) and (v_7, v_9) are all in T.

We draw the digraph of T, beginning with vertex v_4, and with edges shown downward.

The result is shown in Fig. (3.13). A quick inspection of this digraph shows that paths from vertex v_4 to every other vertex are unique, and there are no paths from v_4 and v_4.

Thus T is a tree with root v_4.

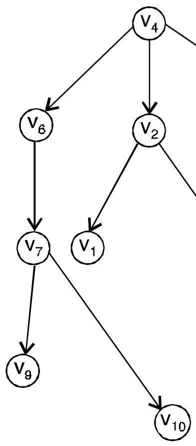


Fig. 3.13

Theorem 3.7. *There is one and only one path between every pair of vertices in a tree T.*

Proof. Since T is a connected graph, there must exist atleast one path between every pair of vertices in T.

Let there are two distinct paths between two vertices u and v of T.

But union of these two paths will contain a cycle and then T cannot be a tree.

Theorem 3.8. *If in a graph G there is one and only one path between every pair of vertices, G is a tree.*

Proof. Since there exists a path between every pair of vertices then G is connected.

A cycle in a graph (with two or more vertices) implies that there is atleast one pair of vertices u, v such that there are two distinct paths between u and v.

Since G has one and only one path between every pair of vertices, G can have no cycle.

Therefore, G is a tree.

Theorem 3.9. *A tree T with n vertices has n – 1 edges.*

Proof. The theorem is proved by induction on n, the number of vertices of T.

Basis of Inductive : When $n = 1$ then T has only one vertex. Since it has no cycles, T can not have any edge.

i.e., it has $e = 0 = n - 1$

Induction step : Suppose the theorem is true for $n = k \geq 2$ where k is some positive integer.

We use this to show that the result is true for $n = k + 1$.

Let T be a tree with $k + 1$ vertices and let uv be edge of T. Let uv be an edge of T. Then if we remove the edge uv from T we obtain the graph T – uv. Then the graph is disconnected since T – uv contains no (u, v) path.

If there were a path, say u, v_1, v_2 ...... v from u to v then when we added back the edge uv there would be a cycle u, v_1, v_2, ... v, u in T.

Thus, T – uv is disconnected. The removal of an edge from a graph can disconnected the graph into at most two components. So T – uv has two components, say, T_1 and T_2.

Since there were no cycles in T to begin with, both components are connected and are without cycles.

Thus, T_1 and T_2 are trees and each has fewer than n vertices.

This means that we can apply the induction hypothesis to T_1 and T_2 to give

$$e(T_1) = v(T_1) - 1$$
$$e(T_2) = v(T_2) - 1$$

But the construction of T_1 and T_2 by removal of a single edge from T gives that

$$e(T) = e(T_1) + e(T_2) + 1$$

and that $\quad v(T) = v(T_1) + v(T_2)$

it follows that

$$\begin{aligned} e(T) &= v(T_1) - 1 + v(T_2) - 1 + 1 \\ &= v(T) - 1 \\ &= k + 1 - 1 = k. \end{aligned}$$

Thus T has k edges, as required.

Hence by principle of mathematical induction the theorem is proved.

Theorem 3.10. *For any positive integer n, if G is a connected graph with n vertices and n – 1 edges, then G is a tree.*

Proof. Let n be a positive integer and suppose G is a particular but arbitrarily chosen graph that is connected and has n vertices and $n - 1$ edges.

We know that a tree is a connected graph without cycles. (We have proved in previous theorem that a tree has $n - 1$ edges).

We have to prove the converse that if G has no cycles and $n - 1$ edges, then G is connected.

We decompose G into k components, $c_1, c_2, \ldots\ldots c_k$.

Each component is connected and it has no cycles since G has no cycles.

Hence, each C_k is a tree.

Now $e_1 = n_1 - 1$ and $\sum_{i=1}^{k} e_i = \sum_{i=1}^{k} (n_i - 1) = n - k$

$\Rightarrow \quad e = n - k$

Then it follows that $k = 1$ or G has only one component.

Hence G is a tree.

Problem 3.6. *Consider the rooted tree in Figure (3.14).*

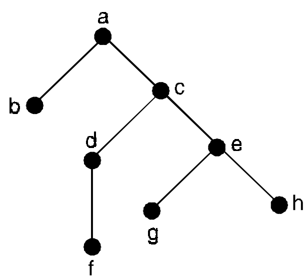


Fig. 3.14.

(*a*) *What is the root of T ?*

(*b*) *Find the leaves and the internal vertices of T.*

(*c*) *What are the levels of c and e.*

(*d*) *Find the children of c and e.*

(*e*) *Find the descendants of the vertices a and c.*

Solution. (*a*) Vertex a is distinguished as the only vertex located at the top of the tree.

Therefore a is the root.

(*b*) The leaves are those vertices that have no children. These b, f, g and h. The internal vertices are c, d and e.

(*c*) The levels of c and e are 1 and 2 respectively.

(*d*) The children of c are d and e and of e are g and h.

(*e*) The descendants of a are b, c, d, e, f, g, h.

The descendants of c are d, e, f, g, h.

Theorem 3.11. *A full m-ary tree with i internal vertex has n = mi + 1 vertices.*

Proof. Since the tree is a full m-ary, each internal vertex has m children and the number of internal vertex is i, the total number of vertex except the root is mi.

Therefore, the tree has $n = mi + 1$ vertices.

Since 1 is the number of leaves, we have $n = l + i$ using the two equalities $n = mi + 1$ and $n = 1 + i$, the following results can easily be deduced.

A full m-ary tree with

(*i*) n vertices has $i = \dfrac{(n-1)}{m}$ internal vertices and $l = \dfrac{[(m-1)(n+1)]}{m}$ leaves.

(*ii*) i internal vertices has $n = mi + 1$ vertices and $l = (m - 1)i + 1$ leaves.

(*iii*) l leaves has $n = \dfrac{(ml-1)}{(m-1)}$ vertices and $i = \dfrac{(l-1)}{(m-1)}$ internal vertices.

Theorem 3.12. *There are at most m^h leaves in an m-ary tree of height h.*

Proof. We prove the theorem by mathematical induction.

Basis of Induction :

For $h = 1$, the tree consists of a root with no more than m children, each of which is a leaf.

Hence there are no more than $m^1 = m$ leaves in an m-ary of height 1.

Induction hypothesis :

We assume that the result is true for all m-ary trees of heights less than h.

Induction step :

Let T be an m-ary tree of height h. The leaves of T are the leaves of subtrees of T obtained by deleting the edges from the roots to each of the vertices of level 1.

Each of these subtrees has at most m^{h-1} leaves. Since there are at most m such subtrees, each with a maximum of m^{h-1} leaves, there are at most $m \cdot m^{h-1} = m^h$.

Problem 3.7. *Find all spanning trees of the graph G shown in Figure 3.15.*

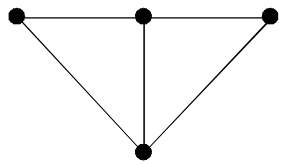


Fig. 3.15.

Solution. The graph G has four vertices and hence each spanning tree must have 4 – 1 = 3 edges.

Thus each tree can be obtained by deleting two of the five edges of G.

This can be done in 10 ways, except that two of the ways lead to disconnected graphs.

Thus there are eight spanning trees as shown in Figure (3.16).

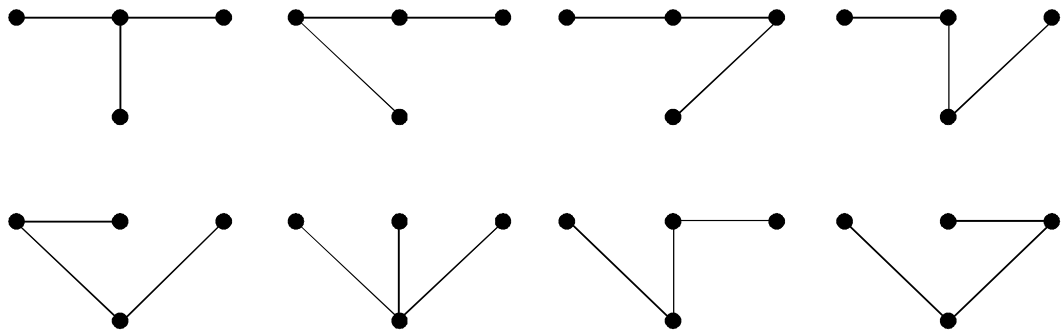


Fig. 3.16.

Problem 3.8. *Find all spanning trees for the graph G shown in Figure 3.17, by removing the edges in simple circuits.*

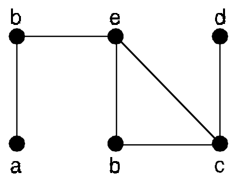


Fig. 3.17.

Solution. The graph G has one cycle *cbec* and removal of any edge of the cycle gives a tree.

There are three trees which contain all the vertices of G and hence spanning trees.

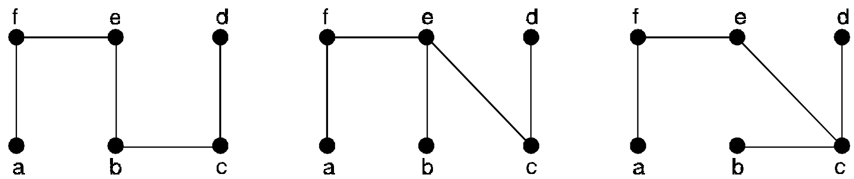


Fig. 3.18.

Theorem 3.13. *A simple graph G has a spanning tree if and only if G is connected.*

Proof. First, suppose that a simple graph G has a spanning tree T. T contains every vertex of G. Let a and b be vertices of G. Since a and b are also vertices of T and T is a tree, there is a path P between a and b.

Since T is subgraph, P also serves as path between a and b in G.

Hence G is connected.

Conversely, suppose that G is connected.

If G is not a tree, it must contain a simple circuit. Remove an edge from one of these simple circuits. The resulting subgraph has one fewer edge but still contains all the vertices of G and is connected.

If this subgraph is not a tree, it has a simple circuit, so as before, remove an edge that is in a simple circuit.

Repeat this process until no simple circuit remain.

This is possible because there are only a finite number of edges in the graph, the process terminates when no simple circuits remain.

Thus we eventually produce an acyclic subgraph T which is a tree.

The tree is a spanning tree since it contains every vertex of G.

Theorem 3.14. *There is one and only path between every pair of vertices in a tree.*

(OR)

A graph G is a tree if and only if every two distinct vertices of G are joined by a unique path of G.

Proof. Since T is a connected graph, there must exist atleast one path between pair of vertices in T.

Now suppose that between two vertices a and b of T there are two distinct paths.

The union of these two paths will contain a cycle, and T cannot be a tree.

Conversely, suppose in a graph G there is one and only one path between every pair of vertices, then G is a tree.

If there exists a path between every pair of vertices, then G is connected.

A cycle in a graph implies that there is atleast one pair of vertices a and b such that there are two distinct paths between a and b.

Sicne G has one and only one path between every pair of vertices, G can have no cycle.

Therefore, G is a tree.

Theorem 3.15. *Every non trivial tree contains atleast two end vertices.*

Proof. Suppose that T is a tree with p-vertices and q-edges and let $d_1, d_2, \ldots\ldots d_p$ denotes the degrees of its vertices, ordered so that $d_1 \le d_2 \le \ldots\ldots \le d_p$.

Since T is connected and non trivial, $d_i \ge 1$ for each $i(1 \le i \le p)$.

If T does not contain two end vertices, then $d_i \ge 1$ and $d_i \ge 2$ for $2 \le i \le p$,

So $$\sum_{i=1}^{p} d_i \ge 1 + 2\,(p-1) = 2p - 1 \qquad \ldots(1)$$

However from the results *i.e.*, $\sum_{i=1}^{p} \deg v_i = 2q$ and a tree with p-vertices has $p - 1$ edges.

$$\sum_{i=1}^{p} d_i = 2q = 2(p-1) = 2p-2 \text{ which contradicts in equality (1).}$$

Hence T contains atleast two end vertices.

Theorem 3.16. *If G is a tree and if any two non adjacent vertices of G are joined by an edge e, then G + e has exactly one cycle.*

Proof. Suppose G is a tree. Then there is exactly one path joining any two vertices of G.

If we add an edge of G, that edge together with unique path joining u and v forms a cycle.

Theorem 3.17. *A graph G is connected if and only if it contains a spanning tree.*

Proof. It is immediate that, if a graph contains a spanning tree, then it must be connected.

Conversely, if a connected graph does not contain any cycle then it is a tree.

For a connected graph containing one or more cycles, we can remove an edge from one of the cycles and still have a connected subgraph. Such removal of edges from cycles can be repeated until we have a spanning tree.

Theorem 3.18. *If u and v are distinct vertices of a tree T contains exactly one u – v path.*

Proof. Suppose, to the contrary that T contains two $u – v$ paths say P and Q are different $u – v$, paths there must be a vertex x (*i.e.*, $x = u$) belonging to both P and Q such that the vertex immediately following x on Q. See Figure 3.19.

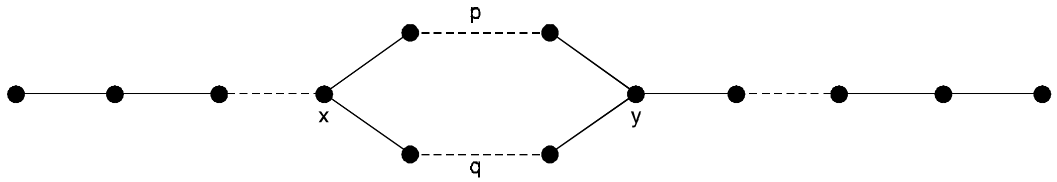


Fig. 3.19.

Let y be the first vertex of P following x that also belongs to Q (y could be v).

Then this produces to $x – y$ paths that have only x and y in common.

These two paths produces a cycle in T, which contradicts the fact that T is a tree.

Therefore, T has only one $u – v$ path.

Problem 3.9. *Construct two non-isomorphic trees having exactly 4 pendant vertices on 6 vertices.*

Solution.

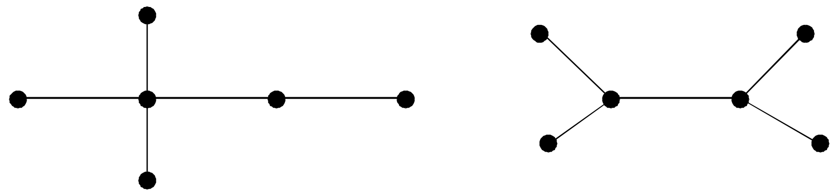


Fig. 3.20.

Problem 3.10. *Construct three distinct trees with exactly*

(i) *one central vertex* (ii) *two central vertices.*

Solution. (i) The following trees contain only one central vertex.

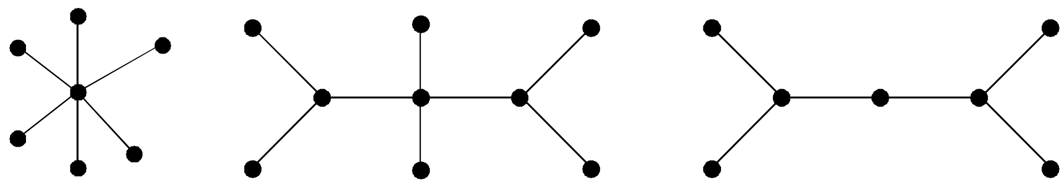


Fig. 3.21.

(*ii*) The following trees contain exactly two central vertices.

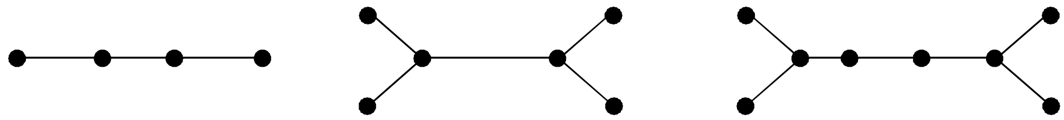


Fig. 3.22.

Problem 3.11. *Count the number of vertices of degree three in a binary tree on n vertices having k number of pendant vertices.*

Solution. Since the binary tree contains k number of pendant vertices and one vertex of degree two, we have total number of remaining vertices which are of degree three is $n - k - 1$.

Problem 3.12. *Let T be a tree with 50 edges. The removal of certain edge from T yields two disjoint trees T_1 and T_2. Given that the number of vertices in T_1 equals the number of edges in T_2, determine the number of vertices and the number of edges in T_1 and T_2.*

Solution. We have removal of an edge from a graph will not remove any vertex from the graph.

Thus $|V(T_1)| + |V(T_2)| = |V(T)|$

Since T_1 and T_2 are trees and number of vertices of T_1 is equal to the number of edges in T_2, we get

$$\begin{aligned} |V(T)| &= |V(T_1)| + |V(T_2)| \\ &= (|V(T_2)| - 1) + |V(T_2)| \\ &= 2|V(T_2)| - 1 \end{aligned}$$

but $|V(T)| = |E(T) + 1| = 50 + 1 = 51$

Hence $2|V(T_2)| - 1 = 51$

$\Rightarrow \quad |V(T_2)| = 26$ and $|V(T_1)| = 25$

Therefore, there are 26 vertices and hence 25 edges in T_2 and there are 25 vertices hence 24 edges in T_1.

Thus $50 - (25 + 24) = 1$ edge is removed from the tree T.

Problem 3.13. *What is the maximum number of end vertices a tree on n vertices may have ?*

Solution. The graph $K_{1,n}$ contains maximum number of end vertices.

Thus a tree on n vertices may contain a maximum of $n - 1$ end vertices.

Problem 3.14. *Prove that a pendant edge in a connected graph G is contained in every spanning tree of G.*

Solution. By a pendant edge, we mean an edge whose one end vertex is a pendant vertex.

Let e be a pendant edge of a connected graph G and let v be the corresponding pendant vertex.

Then e is the only edge that is incident on v.

Suppose there is a spanning tree of T for which e is not a branch.

Then, T cannot contain the vertex v.

This is not possible, because T must contain every vertex of G.

Hence there is no spanning tree of G for which e is not a branch.

Problem 3.15. *Show that a Hamiltonian path is a spanning tree.*

Solution. Recall that a Hamiltonian path P in a connected graph G, if there is a path which contains every vertex of G and that if G has n vertices then P has $n - 1$ edges.

Thus, P is a connected subgraph of G with n vertices and $n - 1$ edges.

Therefore, P is a tree. Since P contains all vertices of G, it is a spanning tree of G.

Problem 3.16. *Prove that the number of branches of a spanning tree T of a connected graph G is equal to the rank of G and the number of the corresponding chords is equal to the nullity of G.*

Solution. Let n be the number of vertices and m be the number of edges in a connected graph G. Then

Rank of G = $\rho(G) = n - 1$

= no. of branches of a spanning tree T of G.

Nullity of G = $\mu(G) = m - (n - 1)$

= no. of chords relative to T.

Problem 3.17. *Prove that every circuit in a graph G must have atleast one edge in common with a chord set.*

Solution. Recall that a chord set is the complement of a spanning tree.

If there is a circuit that has no common edge with this set, the circuit must be containined in a spanning tree.

This is impossible, because a tree does not contain a circuit.

Problem 3.18. *Let G be a graph with k components, where each component is a tree. If n is the number of vertices and m is the number of edges in G, prove that n = m + k.*

Solution. Let $H_1, H_2, \ldots\ldots H_k$ be the components of G.

Since each of these is a tree, if n_i is the number of vertices in H_i and m_i is the number of edges in H_i

We have $\quad m_i = n_i - 1, \quad i = 1, 2, \ldots\ldots k$

this gives $\quad m_1 + m_2 + \ldots\ldots + m_k = (n_1 - 1) + (n_2 - 1) + \ldots\ldots (n_k - 1)$

$$= n_1 + n_2 + \ldots\ldots + n_k - k$$

But $\quad m_1 + m_2 + \ldots\ldots + m_k = m \quad$ and

$$n_1 + n_2 + \ldots\ldots + n_k = n$$

Therefore $\quad m = n - k$

$\Rightarrow \quad n = m + k.$

Problem 3.19. *Show that, in a tree ; if the degree of every non-pendant vertex is 3, the number of vertices in the tree is even.*

Solution. Let n be the number of vertices in a tree T.

Let k be the number of pendant vertices.

Then, if each non-pendant vertex is of degree 3, the sum of the degrees of vertices is $k + 3(n - k)$.

This must be equal to $2(n - 1)$

Thus, $k + 3(n - k) = 2(n - 1)$

$\Rightarrow \quad n = 2(k - 1)$

Therefore, n is even.

Problem 3.20. *Suppose that a tree T has N_1 vertices of degree 1, N_2 vertices of degree 2, N_3 vertices of degree 3, ... N_k vertices of degree k. Prove that*

$$N_1 = 2 + N_3 + 2N_4 + 3N_5 + ... + (K - 2)\, N_k.$$

Solution. Note that a tree T,

The total number of vertices $= N_1 + N_2 + ... + N_k$

Sum of the degrees of vertices $= N_1 + 2N_2 + 3N_3 + ... kN_k$

Therefore, the total number of edges in T is

$$N_1 + N_2 + ... + N_k - 1, \text{ and}$$

the handshaking property, gives

$$N_1 + 2N_2 + 3N_3 + 4N_4 + 5N_5 + ... kN_k$$
$$= 2(N_1 + N_2 + ... + N_k - 1)$$

Rearranging terms, which gives

$$N_3 + 2N_4 + 3N_5 + ... + (k - 2)\, N_k = N_1 - 2$$

$\Rightarrow \quad N_1 = 2 + N_3 + 2N_4 + 3N_5 + ... + (k - 2)\, N_k.$

Problem 3.21. *Show that if a tree has exactly two pendant vertices, the degree of every other vertex is two.*

Solution. Let n be the number of vertices in a tree T.

Suppose, it has exactly two pendant vertices, and let $d_1, d_2, ... d_{n-2}$ be the degrees of the other vertices.

Then, since T has exactly $n - 1$ edges.

We have $\quad 1 + 1 + d_1 + d_2 + ... + d_{n-2} = 2(n - 1)$

$\Rightarrow \quad d_1 + d_2 + ... + d_{n-2} = 2n - 4 = 2(n - 2)$

The left hand side of the above condition has $n - 2$ terms d's, and none of these is one or zero.

Therefore, this condition holds only if each of the $d_i s$ is equal to two.

Problem 3.22. *Show that the complete graph K_n is not a tree, when $n > 2$.*

Solution. If v_1, v_2, v_3 are any three vertices of K_n, $n > 2$ then the closed walk $v_1v_2v_3v_1$ is a circuit in K_n.

Since K_n has a circuit, it cannot be a tree.

Problem 3.23. *Suppose that a tree T has two vertices of degree 2, four vertices of degree 3 and three vertices of degree 4. Find the number of pendant vertices in T.*

Solution. Let N be the number of pendant vertices in T.

It is given that T has two vertices of degree 2, four vertices of degree 3 and three vertices of degree 4.

Therefore, the total number of vertices

$$= N + 2 + 4 + 3$$
$$= N + 9.$$

Sum of the degrees of vertices $= N + (2 \times 2) + (4 \times 3) + (3 \times 4)$

$$= N + 28.$$

Since T has N + 9 vertices, it has N + 9 – 1 = N + 8 edges.

Therefore, by handshaking property, we have

$$N + 28 = 2(N + 8)$$

$$\Rightarrow \qquad N = 12$$

Thus, the given tree has 12 pendant vertices.

Problem 3.24. *Show that the complete bipartite graph $K_{r,\, s}$ is not a tree if $r \geq 2$.*

Solution. Let v_1 and v_2 be any two vertices in the first partition and v_1', v_2' be any two vertices in the second partition of $K_{r,\, s}$ $s \geq r > 1$.

Then the closed walk $v_1 v_1' v_2 v_2' v_1$ is a circuit in $K_{r,\, s}$.

Since $K_{r,\, s}$ has a circuit, it cannot be a tree.

Problem 3.25. *Prove that, in a tree with two or more vertices, there are atleast two leaves (pendant vertices).*

Solution. Consider a tree T with n vertices, $n \geq 2$. Then, it has $n - 1$ edges.

Therefore, the sum of the degrees of the n vertices must be equal to $2(n - 1)$.

Thus, if $d_1, d_2, \ldots\ldots d_n$ are the degrees of vertices.

We have $\quad d_1 + d_2 + \ldots\ldots + d_n = 2(n - 1) = 2n - 2.$

If each of $d_1, d_2, \ldots\ldots d_n$ is ≥ 2, then their sum must be at least $2n$.

Since this is not true, atleast one of the d's is less than 2.

Thus, there is a d which is equal to 1.

Without loss of generality, let us take this to be d_1. Then

$$d_2 + d_3 + \ldots\ldots + d_n = (2n - 2) - 1 = 2n - 3.$$

This is possible only if atleast one of $d_2, d_3 \ldots\ldots d_n$ is equal to 1.

So, there is atleast one more d which is equal to 1.

Thus, there are atleast two vertices with degree 1.

Problem 3.26. *Prove that a graph with n vertices, n – 1 edges, and no circuits is connected.*

Solution. Consider a graph G which has n vertices, $n - 1$ edges and no circuits.

Suppose G is not connected.

Let the components of G be H_i, $i = 1, 2, \ldots\ldots k$.

If H_i has n_i vertices, we have

$$n_1 + n_2 + \ldots\ldots + n_k = n.$$

Since G has no circuits, $H_i s$ is also do not have circuits.

Further, they are all connected graphs.

Therefore, they are trees.

Consequently, each H_i must have $n_i - 1$ edges.

Therefore, the total number of edges in these $H_i s$ is $(n_1 - 1) + (n_2 - 1) + \ldots\ldots (n_k - 1) = n - k$.

This must be equal to the total number of edges in G, that is $n - k = n - 1$.

This is not possible, since $k > 1$.

Therefore, G must be connected.

Problem 3.27. *Construct three distinct binary trees on 11 vertices.*

Solution. (*i*) (*ii*) (*iii*)

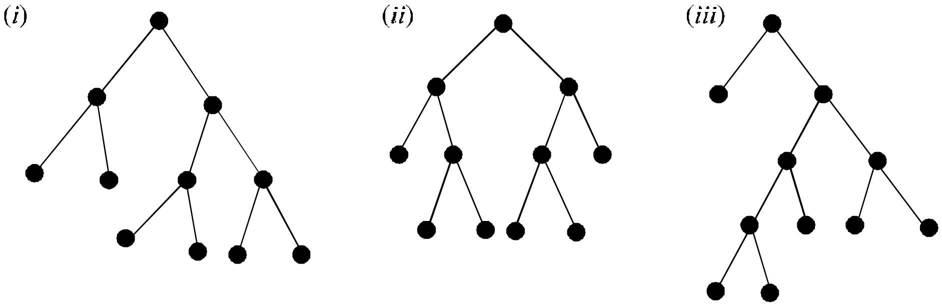


Fig. 3.23.

Problem 3.28. *What is the minimum possible height of a binary tree on 2n – 1 (n ≥ 1) vertices ?*

Solution. Let k be the minimum height of a binary tree on $2n - 1$ vertices.

For minimum height we have to keep maximum number of vertices in the previous level before placing any vertex in the next level.

Thus, k should satisfy the inequality

$$2n - 1 \le 2^0 + 2^1 + 2^2 + \ldots\ldots + 2^k$$

$$= \frac{1(1-2^{k+1})}{1-2}$$

Since right hand side is a G.P. series with first term is 1 and common ratio having $k + 1$ terms.

i.e., $2n - 1 \le 2^{k+1} - 1 \quad \Rightarrow \quad 2n \le 2^{k+1}$

$$\Rightarrow \quad n \le 2^k.$$

Now taking natural log on both sides we get

$$\log_2 n \le k \quad \Rightarrow \quad k \ge \log_2 n.$$

Since k is an integer, this implies that the minimum value of $k = [\log_2 n]$.

Problem 3.29. *What is the maximum possible number of vertices in any k-level tree ?*

Solution. The level of a root is zero and it is the only one vertex at level zero.

There are two vertices that are adjacent to the root, at which are at levels one.

From these vertices we can find maximum four vertices at level 2 so on ...... to get a minimum heighten tree we have to keep the vertex at higher level only after filling all the vertices in its lower level.

Trees maximum number of vertices possible for such a k-level tree is therefore

$$n \leq 2^0 + 2^1 + 2^2 + \ldots\ldots\ 2^k = \frac{1(1-2^{k+1})}{1-2} = 2^{k+1} - 1.$$

Problem 3.30. *What is the maximum possible level (height) of a binary tree on 2n + 1 (n ≥ 0) vertices.*

Solution. Let k be the height of a binary tree on $2n + 1$ vertices.

To get a vertex in maximum level we must keep exactly two (minimum) vertices in each level except the root vertex.

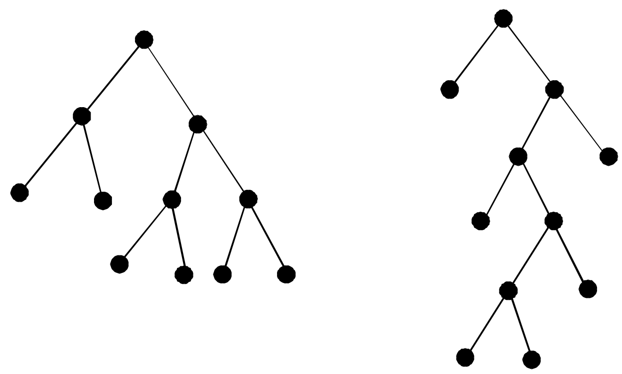
That is out of $2n + 1$ vertices one is a root and the remaining $2n$ vertices can keep in exactly n levels.

Thus the maximum height of a tree is n.

Hence maximum possible value of k is n.

Problem 3.31. *Sketch two different binary trees on 11 vertices with one having maximum height and the other with minimum height.*

Solution. Required binary trees on 11 vertices are



(*i*) **With minimum height 3**

(*ii*) **With maximum height 5.**

Fig. 3.24.

Problem 3.32. *Show that the number of vertices in a binary tree is always odd.*

Solution. Consider a binary tree on n vertices. Since it contains exactly one vertex of degree two and other vertices are of degree one or three, it follows that there are $n - 1$ odd degree vertices in the graph.

But if the number of odd degree vertices of a graph is even, it follows that $n - 1$ is even and hence n is odd.

Problem 3.33. *In any binary tree T on n vertices, show that the number of pendant vertices (edges) is equal to* $\frac{(n+1)}{2}$.

Solution. Let the number of pendant edges in a binary tree on n vertices be k.

Then we have there are $n - k - 1$ vertices of degree three, one vertex of degree two, k vertices of degree one and $n - 1$ edges.

Therefore, sum of degrees of vertices = 2 × number of edges.

$$(n - k - 1) \times 3 + 2 + k \times 1 = 2(n - 1)$$

$\Rightarrow \quad 3n - 3k - 3 + 2 + k = 2n - 2$

$\Rightarrow \quad 2k = 3n - 2n + 1 = n + 1$

$\Rightarrow \quad k = \frac{(n+1)}{2}.$

Problem 3.34. *Draw a tree with 6 vertices, exactly 3 of which have degree 1.*

Solution. A tree with 6 vertices which contains 3 pendant vertices is given in Figure (3.25).

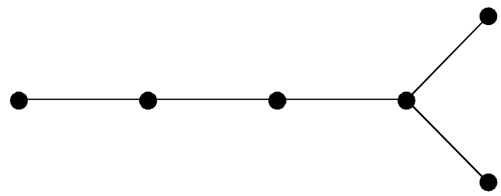


Fig. 3.25.

Problem 3.35. *Which trees are complete bipartite graphs ?*

Solution. Suppose T is a tree which is a complete bipartite graph.

Let T = $K_{m,n}$ then the number of vertices in T is $(m + n)$.

Hence the tree T contains $(m + n - 1)$ number of edges.

But the graph $K_{m,n}$ has (m, n) number of edges.

Therefore $\quad m + n - 1 = mn$

$\Rightarrow \quad mn - m - n + 1 = 0$

$\Rightarrow \quad m(n - 1) - 1(n - 1) = 0$

$\Rightarrow \quad (m - 1)(n - 1) = 0$

$\Rightarrow \quad m = 1$ or $n = 1$

This means T is either $K_{1,n}$ or $K_{m,1}$ that is T is a star.

Problem 3.36. *Draw all non-isomorphic trees with 6 vertices.*

Solution. All non isomorphic trees with 6 vertices are shown below :

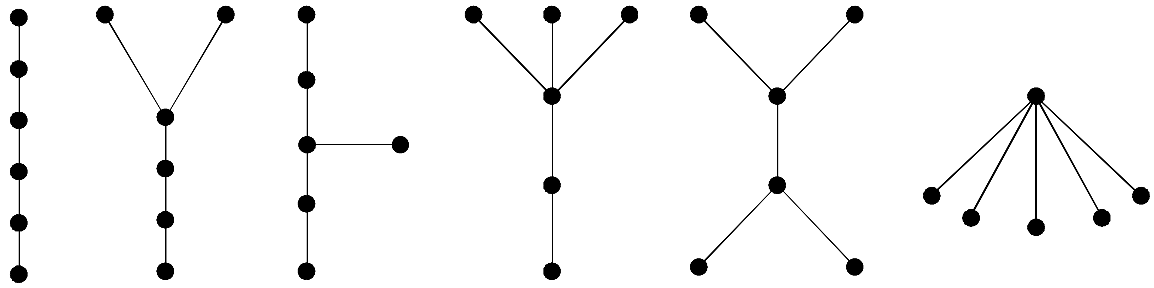


Fig. 3.26.

Problem 3.37. *Is it possible to draw a tree with five vertices having degrees 1, 1, 2, 2, 4.*

Solution. Since the tree has 5 vertices hence it has 4 edges.

Now given the vertices of tree are having degrees

1, 1, 2, 2, 4.

i.e., the sum of the degrees of the tree = 10

By handshaking lemma, $2q = \sum_{i=1}^{5} d(v_i)$

Where q is the number of edges in the graph

$2q = 10 \quad \Rightarrow \quad q = 5$

Which is contradiction to the statement that the tree has 4 edges with 5 vertices.

Hence the tree with given degrees of vertices does not exist.

3.5. COUNTING TREES

The subject of graph enumeration is concerned with the problem of finding out how many non-isomorphic graphs possess a given property. The subject was initiated in the 1850's by Arthur Cayley, who later applied it to the problem of enumerating alkanes $C_n H_{2n+2}$ with a given number of carbon atoms. This problem is that of counting the number of trees in which the degree of each vertex is either 4 or 1. Many standard problems of graph enumeration have been solved.

For example, it is possible to calculate the number of graphs, connected graphs, trees and Eulerian graphs with a given number of vertices and edges, corresponding general results for planar graphs and Hamiltonian graphs have, however, not yet been obtained. Most of the known results can be obtained by using a fundamental enumeration theorem due to Polya, a good account of which may be found in Harary and Palmer.

Unfortunately, in almost every case it is impossible to express these results by means of simple formulas.

Consider Fig. (3.27), which shows three ways of labelling a tree with four vertices. Since the second labelled tree is the reverse of the first one, these two labelled trees are the same. On the other hand, neither is isomorphic to the third labelled tree, as you can see by comparing the degrees of vertex 3.

Thus, the reverse of any labelling does not result in a new one, and so the number of ways of labelling this tree is $\frac{(4!)}{2}$ = 12.

Similarly, the number of ways of labelling the tree in Fig. (3.28) is 4, since the central vertex can be labelled in four different ways, and each one determines the labelling.

Thus, the total number of non-isomorphic labelled trees on four vertices is 12 + 4 = 16.

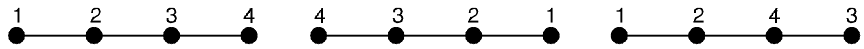


Fig. 3.27.

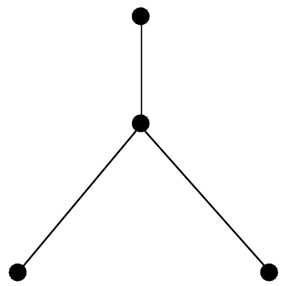


Fig. 3.28.

Theorem 3.19. *Let T be a graph with n vertices. Then the following statements are equivalent :*

(*i*) *T is a tree*

(*ii*) *T contains no cycles, and has n – 1 edges*

(*iii*) *T is connected and has n – 1 edges*

(*iv*) *T is connected and each edge is a bridge*

(*v*) *Any two vertices of T are connected by exactly one path*

(*vi*) *T contains no cycles, but the addition of any new edge creates exactly one cycle.*

Proof. If $n = 1$, all six results are trivial, we therefore assume that $n \geq 2$.

(*i*) $\Rightarrow$ (*iii*)

Since T contains no cycles, the removal of any edge must disconnect T into two graphs, each of which is a tree.

It follows by induction that the number of edges in each of these two trees is one fewer than the number of vertices. We deduce that the total number of edges of T is $n - 1$.

(*ii*) $\Rightarrow$ (*iii*)

If T is disconnected, then each component of T is a connected graph with no cycles and hence, by the previous part, the number of vertices in each component exceeds the number of edges by 1.

It follows that the total number of vertices of T exceeds the total number of edges by atleast 2, contradicting the fact that T has $n - 1$ edges.

(iii) $\Rightarrow$ (iv)

The removal of any edge results in a graph with n vertices and $n - 2$ edges, which must be disconnected.

(iv) $\Rightarrow$ (v)

Since T is connected, each pair of vertices is connected by atleast one path.

If a given pair of vertices is connected by two paths, then they enclose a cycle, contradicting the fact that each edge is a bridge.

(v) $\Rightarrow$ (vi)

If T contained a cycle, then any two vertices in the cycle would be connected by atleast two paths, contradicting statement (v).

If an edge e is added to T, then, since the vertices incident with e are already connected in T, a cycle is created.

The fact that only one cycle is obtained.

(vi) $\Rightarrow$ (i)

Suppose that T is disconnected.

If we add to T any edge joining a vertex of one component to a vertex in another, then no cycle is created.

Corollary :

If G is a forest with n vertices and k components, then G has $n - k$ edges.

Theorem 3.20. *If T is any spanning forest of a graph G, then*

(i) *each cutset of G has an edge in common with T*

(ii) *each cycle of G has an edge in common with the complement of T.*

Proof. (i) Let C* be a cutset of G, the removal of which splits a component of G into two subgraphs H and K.

Since T is a spanning forest, T must contain an edge joining a vertex of H to a vertex of K, and this edge is the required edges.

(ii) Let C be a cycle of G having no edge is common with the complement of T.

Then C must be contained in T, which is a contradiction.

3.5.1. Cayley theorem (3.21)

There are n^{n-2} distinct labelled trees on n vertices.

Remark. The following proofs are due to Prüfer and Clarke.

Proof. First proof :

We establish a one-one correspondence between the set of labelled trees of order n and set of sequences (a_1, a_2, ... a_{n-2}), where each a_i is an integer satisfying $1 \le a_i \le n$.

Since there are precisely n^{n-2} such sequence, the result follows immediately.

We assume that $n \ge 3$, since the result is trivial if $n = 1$ or 2.

In order to establish the required correspondence, we first let T be a labelled tree of order n, and show how the sequence can be determined.

If b_1 is the smallest label assigned to an end-vertex, we let a_1 be the label of the vertex adjacent to the vertex b_1.

We then remove the vertex b_1 and its incident edge, leaving a labelled tree of order $n - 1$.

We next let b_2 be the smallest label assigned to an end-vertex of our new tree, and let a_2 be the label of the vertex adjacent to the vertex b_2.

We then remove the vertex b_2 and its incident edge, as before.

We proceed in this way until there are only two vertices left, and the required sequence is (a_1, a_2, ... a_{n-2}).

For example, if T is the labelled tree in Figure (3.29),

then $b_1 = 2$, $a_1 = 6$, $b_2 = 3$, $a_2 = 5$, $b_3 = 4$, $a_3 = 6$

$b_4 = 6$, $a_4 = 5$, $b_5 = 5$, $a_5 = 1$

The required sequence is therefore (6, 5, 6, 5, 1)

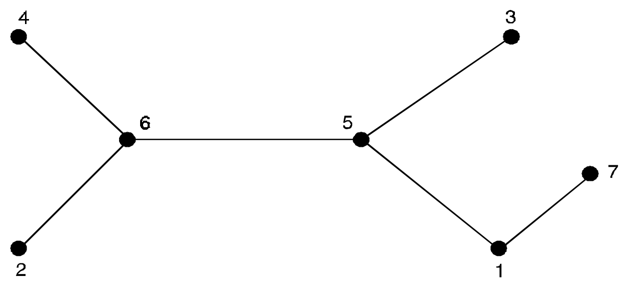


Fig. 3.29.

To obtain the reverse correspondence, we take a sequence (a_1, ... a_{n-2}).

Let b_1 be the smallest number that does not appear in it, and join the vertices a_1 and b_1.

We then remove a_1 from the sequence, remove the number b_1 from consideration, and proceed as before.

In this way we build up the tree, edge by edge,

For example, if we start with the sequence (6, 5, 6, 5, 1), then $b_1 = 2$, $b_2 = 3$, $b_3 = 4$, $b_4 = 6$, $b_5 = 5$, and the corresponding edges are 62, 53, 64, 56, 15.

We conclude by joining the last two vertices not yet crossed out-in this case, 1 and 7.

It is simple to check that if we start with any labelled tree, find the corresponding sequence, and then find the labelled tree corresponding to that sequence, then we obtain the tree we started from.

We have therefore established the required correspondence and the result follows.

Second Proof :

Let T(n, k) be the number of labelled trees on n vertices in which a given vertex v has degree k.

We shall derive an expression for T(n, k), and the result follows on summing from $k = 1$ to $k = n - 1$.

Let A be any labelled tree in which deg (v) = $k - 1$.

The removal from A of any edge wz that is not incident with v leaves two subtrees, one containing v and either w or z (w, say), and the other containing z.

If we now join the vertices v and z, we obtain a labelled tree B in which deg (v) = K see Fig. (3.30).

We call a pair (A, B) of labelled trees of linkage if B can be obtained from A by the above construction.

Our aim is to count the possible linkages (A, B).

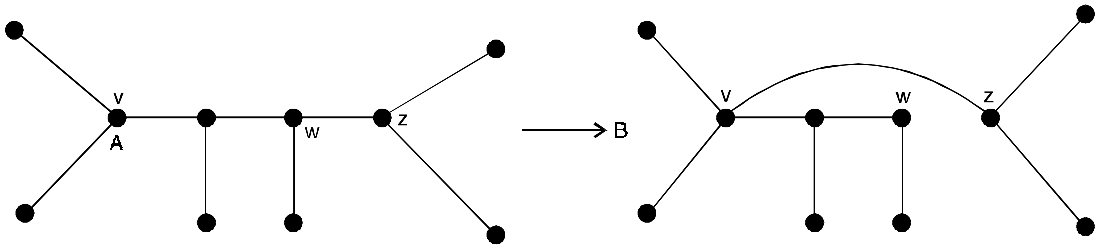


Fig. 3.30.

Since A may be chosen in T(n, k – 1) ways, and since B is uniquely defined by the edge wz which may be chosen in $(n – 1) – (k – 1) = (n – k)$ ways, the total number of linkages (A, B) is $(n – k)$ T(n, k – 1).

On the other hand, let B be a labelled tree in which deg (v) = k, and let T_1, ... T_k be the subtrees obtained from B by removing the vertex v and each edge incident with v.

Then we obtain a labelled tree A with deg (v) = k – 1 by removing from B just one of these edges (vw_i, say, where w_i lies in T_i), and joining w_i to any vertex u of any other subtree T (see Fig. 3.31).

Note that the corresponding pair (A, B) of labelled trees is a linkage, and that all linkages may be obtained in this way.

Since B can be chosen in T(n, k) ways, and the number of ways of joining w_i to vertices in any other T_j is $(n – 1) – n_i$, where n_i is the number of vertices of T_i, the total number of linkages (A, B) is

T(n, k) $\{(n – 1 – n_1) + ... + (n – 1 – n_k)\} = (n – 1)(k – 1)$ T(n, k), since $n_1 + ... + n_k = n – 1$

We have thus shown that

$$(n – k)\, T(n, k – 1) = (n – 1)(k – 1)\, T(n, k).$$

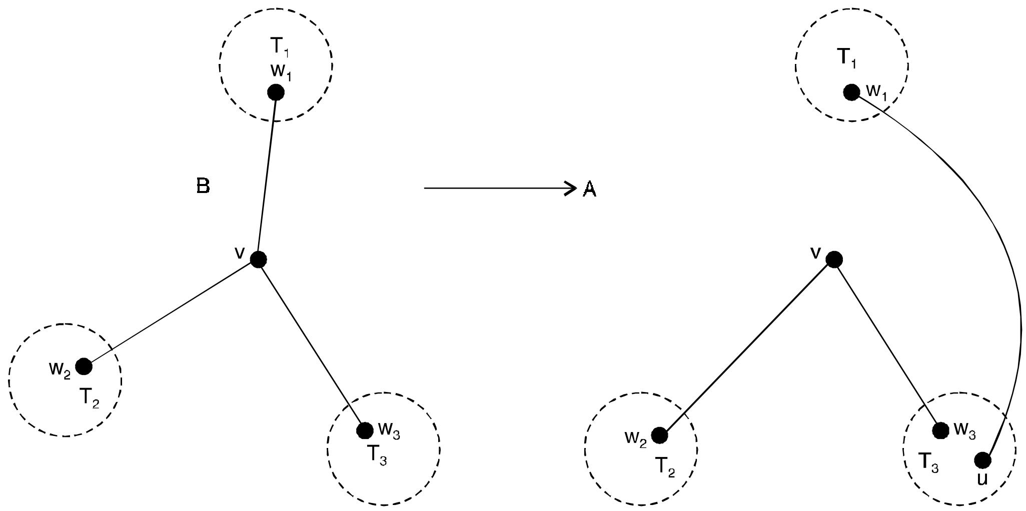


Fig. 3.31.

On iterating this result, and using the obvious fact that T(n, n – 1) = 1, we deduce immediately that

$$T(n, k) = \binom{n-2}{n-1} (n-1)_{n-k-1}$$

On summing over all possible values of k, we deduce that the number T(n) of labelled trees on n vertices is given by

$$T(n) = \sum_{k=1}^{n-1} T(n, k) = \sum_{k=1}^{n-1} \binom{n-2}{k-1}^{n-k-1}$$

$$= \{(n-1) + 1\}^{n-2} = n^{n-2}.$$

Corollary :

The number of spanning trees of K_n is n^{n-2}.

Proof. To each labelled tree on n vertices there corresponds a unique spanning tree of K_n.

Conversely, each spanning tree of K_n gives rise to a unique labelled tree on n vertices.

Theorem 3.22. *Prove that the maximum number of vertices in a binary tree of depth d is* $2^d - 1$, *where* $d \geq 1$.

Proof. We shall prove the theorem by induction.

Basis of induction :

The only vertex at depth d = 1 is the root vertex.

Thus the maximum number of vertices on depth

$d = 1$ is $2^1 - 1 = 1$.

Induction hypothesis :

We assume that the theorem is true for depth k,

$d > k \geq 1$

Therefore, the maximum number of vertices on depth k is $2^k - 1$.

Induction step :

By induction hypothesis, the maximum number of vertices on depth k – 1 is $2^{k-1} - 1$.

Since, we know that each vertex in a binary tree has maximum degree 2, therefore, the maximum number of vertices on depth $d = k$ is twice the maximum number of vertices on depth k – 1.

So, at depth k, the maximum number of vertices is $2.2^{k-1} - 1 = 2^k - 1$.

Hence proved.

Problem 3.38. *What are the left and right children of b shown in Fig. 3.32 ? What are the left and right subtrees of a ?*

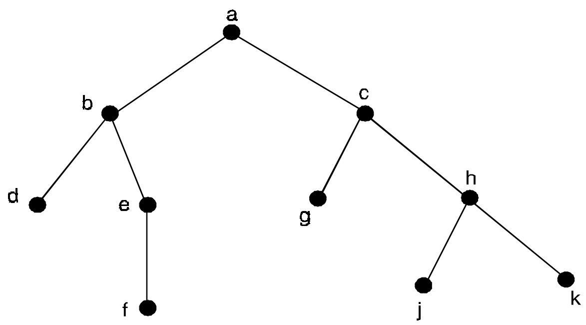
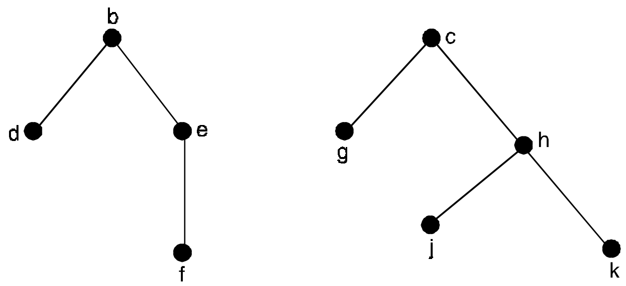


Fig. 3.32.

Solution. The left child of *b* is *d* and the right child is *e*. The left subtree of the vertex *a* consists of the vertices *b*, *d*, *e* and *f* and the right subtree of a consists of the vertices *c*, *g*, *h*, *j* and *k* whose figures are shown in Fig. 3.33. (*a*) and (*b*) respectively.



(*a*) (*b*)

Fig. 3.33.

Theorem 3.23. *Prove that the maximum number of vertices on level n of a binary tree is* 2^n*, where* $n \geq 0$*.*

Proof. We prove the theorem by mathematical induction.

Basis of induction :

When $n = 0$, the only vertex is the root.

Thus the maximum number of vertices on level $n = 0$ is $2^0 = 1$.

Induction hypothesis :

We assume that the theorem is true for level K, where $n \geq k \geq 0$.

So the maximum number of vertices on level k is 2^k.

Induction step :

By induction hypothesis, maximum number of vertices on level $k - 1$ is 2^{k-1}.

Since each vertex in binary tree has maximum degree 2, then the maximum number of vertices on level k is twice the maximum number of level $k - 1$.

Hence, the maxmum number of vertices at level k is $= 2.2^{k-1} = 2^k$.

Hence, the theorem is proved.

Theorem 3.24. *If T is full binary tree with i internal vertices, then T has i + 1 terminal vertices and 2i + 1 total vertices.*

Proof. The vertices of T consists of the vertices that are children (of some parent O) and the vertices that are not children (of any parent).

There is one non child-the root. Since there are i internal vertices, each parent having two children, there are $2i$ children.

Thus, there are total $2i + 1$ vertices and the number of terminal vertices is $(2i + 1) – i = i + 1$.

3.6. TREE TRAVERSAL

A traversal of a tree is a process to traverse a tree in a systematic way so that each vertex is visited exactly once. Three commonly used traversals are preorder, postorder and inorder. We describe here these three process that may be used to traverse a binary tree.

3.6.1. Preorder traversal

The preorder traversal of a binary tree is defined recursively as follows

(*i*) Visit the root

(*ii*) Traverse the left subtree in preorder.

(*iii*) Traverse the right subtree in preorder

3.6.2. Postorder traversal

The postorder traversal of a binary tree is defined recursively as follows

(*i*) Traverse the left subtree in postorder

(*ii*) Traverse the right subtree in postorder

(*iii*) Visit the root.

3.6.3. Inorder traversal

The inorder traversal of a binary tree is defined recursively as follows

(*i*) Traverse in inorder the left subtree

(*ii*) Vist the root

(*iii*) Traverse in inorder the right subtree

Given an order of traversal of a tree it is possible to construct a tree.

For example,

Consider the following order :

Inorder = *d b e a c*

We can construct the binary trees shown below in Fig. (3.36) using this order of traversal.

3.7. COMPLETE BINARY TREE

If all the leaves of a full binary tree are at level d, then we call a tree as a complete binary tree of depth d. A complete binary tree of depth of 3 is shown in Fig. (3.34).