Q5.

$$G_f(x) = \frac{3x}{(1-x)^5} + \frac{1}{(1-x)^3}$$

$$G_f(x) = 3x(1-x)^{-5} + (1-x)^{-3}$$

$$G_f(x) = 3x\sum_{k\geq 0}\binom{-5}{k}(-x)^k + \sum_{k\geq 0}\binom{-3}{k}(-x)^k$$

$f(n) =$ Coeff of x^n in $G_f(x)$

$$= 3\binom{-5}{n}(-1)^n + \binom{-3}{n}(-1)^n$$

$$= \cancel{(-1)^n\left(3\binom{-5}{n}\right)}$$

$$= 3(-1)^n\binom{-5}{n} + (-1)^n\binom{-3}{n}$$

$$a = (-1)^n\binom{-5}{n}, \quad b = \binom{-3}{n}(-1)^n$$

where $\binom{a}{k} = \frac{a(a-1)\cdots(a-k+1)}{k!}$ & $\binom{a}{0} = 1$

Q 57

$$\boxed{\text{~~length~~ size of maximal chain} = 1 + \sum_{i=1}^{t} d_i}$$

Consider following sequence of size $\sum_{i=1}^{t} d_i = S$

$$= (\underbrace{P_1, \cdots P_1}_{d_1 \text{ times}}, \underbrace{P_2 \cdots P_2}_{d_2 \text{ times}}, \cdots \underbrace{P_t \cdots P_t}_{d_t \text{ times}})$$

~~Claim~~. Number of maximal chains = No of permutations of the above sequence.

~~clearly the si~~ chain starts from 1 (so the size of chain is $1 + \sum_{i=1}^{t} d_i$)

Let a permutation of the sequence be $(x_1, x_2 \cdots x_S)$

you can construct a chain as follows

$$1 \to x_1 \to x_1 x_2 \to x_1 x_2 x_3 \cdots \to \left(\prod_{i=1}^{S} x_i = a\right)$$

clearly a is the maximum element.

Here we are just multiplying $(i-1)^{th}$ position in the chain with x_i to get element at i^{th} position in the chain.

$$\text{Ans} = \frac{S!}{d_1! d_2! \cdots d_t!} \quad \text{where } S = \sum_{i=1}^{t} d_i$$

Q8. Let $Q(k)$ be the required answer.

If $k < 2n+1$

$Q(k) = 0$

as all $x_i \geq 2$ & $x_n > x_{n-1}$

If $k \geq 2n+1$

Consider

~~Consid~~ $x_1 + \cdots + x_n = k$

where $2 \leq x_1 \leq \cdots \leq x_{n-1} < x_n$

Let $x_i = x_i' + 1$

$\Rightarrow x_1' + x_2' - - - + x_n' = k - n$

where $1 \leq x_1' \leq x_2' \leq \cdots \leq x_{n-1}' < x_n'$

all possible values of $x_n' = \{\cancel{1, 2}\} \{1, \ldots \underbrace{k-2n+1}_{+ve (>1)}\}$

$x_1' + x_2' + \cdots + x_{n-1}' = k - n - x_n'$ → Let this be some number from the above set.

where $1 \leq x_1' \cdots \leq x_{n-1}'$

$$\Rightarrow \cancel{Q(} \boxed{Q(k) = \sum_{k' = k-n-1}^{n-1} P_{n-1}(k')}$$

Q9.

claim: $\gcd(a,b,c)=1 \iff \exists (x,y,z)\in \mathbb{Z}^3$ such that

$ax+by+cz=1$

Proof:

We know that, $\forall\, x,y \in \mathbb{Z}$

$ax+by=\gcd(a,b)k$

where k is some integer.

Let $g=\gcd(a,b)$

for any given k you can find $(x,y)\in \mathbb{Z}^2$

such that $ax+by=gk$.

Because, by extended euclidian algorithm you can

find $(u,v)\in\mathbb{Z}^2$ such that $au+bv=g$.

~~Now~~ $\Rightarrow a(uk)+b(vk)=gk$

Hence $x=uk$, $y=vk$ are solutions.

So k is arbitrary here.

We know that

$\rightarrow \boxed{gk+cz=1} \Rightarrow (\gcd(g,c)=\gcd(a,b,c)=1)$

~~And also $\exists (k,c)\in\mathbb{Z}^2$~~

And also;

$\boxed{\gcd(g,c) \iff \exists (k,z)\in\mathbb{Z}^2 \text{ such that } gk+cz=1}$

So find (k, z) by extended euclian algorithm &

find x, y using that k.

QED.

$a = 18$, $b = 26$, $c = 35$.

$\gcd(18, 26, 35) = 1$ $\therefore$ solution exists.

$\gcd(18, 26) = 2$

$2k + 35z = 1$

$2(-17) + 35(1) = 1$

$\Rightarrow \boxed{k = -17, z = 1}$

$18u + 26v = 2$

$\Rightarrow 18(3) + 26(-2) = 2$

$\Rightarrow u = 3, v = -2$

$x = uk$, $y = vk$

$\Rightarrow (x, y, z) = (-51, 34, 1)$

Q10.

~~We need last two least sign~~

We need $(103)^{(104)^{112}} \% 100$

Let $(104)^{112} = x$

$\Rightarrow$ ~~$(103)^{x} \% 100 = 3^{x} \% 100$~~

$(103)^{x} \equiv 3^{x} \pmod{100}$

$3 \in \mathbb{Z}_{100}^{*}$ so,

$3^{x} \equiv 3^{x \% \phi(100)} \pmod{100}$

where $\phi(100) = 100(1-\frac{1}{2})(1-\frac{1}{5})$ (totient function)

$= 100(\frac{1}{2})(\frac{4}{5})$

$= 40$

we need.
$\boxed{x \% 40}$

$(104)^{112} \equiv (24)^{112} \pmod{40}$

$40 = 8 \times 5$ & $\gcd(8,5) = 1$

So By using CRT, we can represent

$(24)^{112} \pmod{40}$ as (x, y) uniquely

where $(24)^{112} \equiv x \pmod{8}$

& $(24)^{112} \equiv y \pmod{5}$

clearly $x = 0$ as $8 | 24$

$(24)^{112} \equiv y \pmod 5$

~~gcd~~ $\equiv 4^{112} \pmod 5$

$4 \in Z_5^*$ so,

$4^{112} \equiv 4^{112 \% \phi(5)} \pmod 5$ $\quad (\phi(5) = 4)$

$\equiv 4^0 \pmod 5$

$\equiv 1 \pmod 5$

$\therefore \boxed{y = 1}$

$(x, y) = (0, 1)$

$8u + 5v = 1 \quad (\because \gcd(8,5) = 1)$

$8(2) + 5(-3) = 1$

$u = 2, v = -3$

$(24)^{112} \pmod{40} \equiv (5vx + 8uy) \pmod{40}$

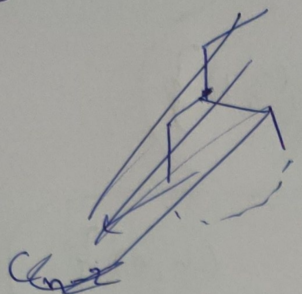
$\equiv 16 \pmod{40}$

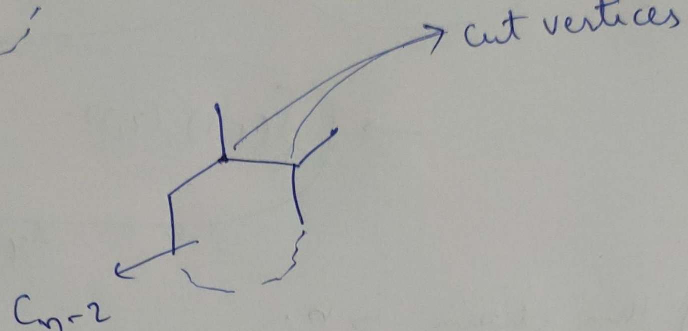
$\Rightarrow 3^{(104)^{112}} \equiv 3^{16} \pmod{100}$

$\equiv (43046721) \pmod{100} \equiv 21 \pmod{100}$

$\therefore (103)^{(104)^{112}} \% 100 = 21$

Q11a





Q11b

$$\underset{x_1}{\{(1,1)\}} \to \underset{x_2}{\{(1,1),(2,2)\}} \to \underset{x_3}{\{(1,1),(2,2),(3,3)\}}$$

$$\cdots \to \underset{x_4}{\{(1,1),(2,2)\cdots(n,n)\}}$$

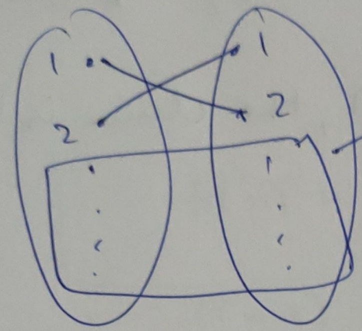
Size of this chain is n.

clearly $\boxed{x_i \subseteq x_{i+1}}$

~~Q12b~~ $X = [n]$

Q13a Let $f(1) = 2$

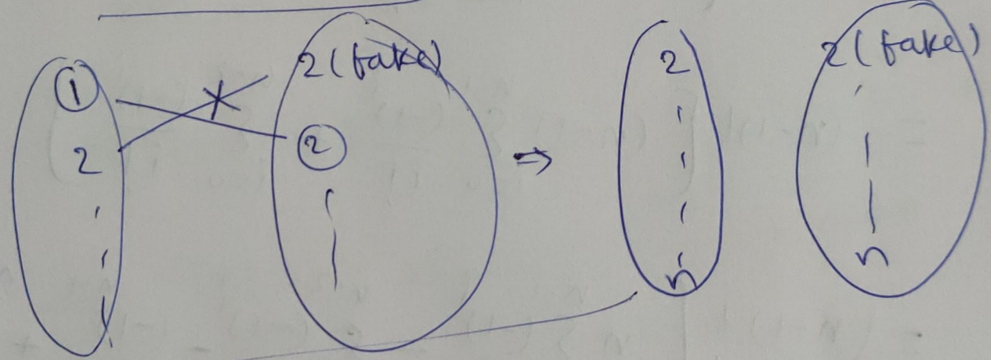
Case 1: $f(2) = 1$



~~No~~ Now this ~~is~~ derangements of size $n-2$. ($\equiv$ over $[n-2]$)

$= D(n-2)$

Case 2: $f(2) \neq 1$. Give 1 in co-domain fake value 2. as anyways 2 won't be mapped to it.



Now this is $D(n-1)$

total $= D(n-2) + D(n-1)$

this is for $f(1) = 2$. But ~~f~~ $f(1)$ can be equal to any number $\in \{2, \cdots n\}$

$\therefore \; D(n) = (n-1)\left(D(n-1) + D(n-2)\right)$

$a = n-1$, $b = n-2$

Q13b. $D(n) = (n-1)\left(D(n-1) + D(n-2)\right)$

claim: $\forall n \geq 1 \quad D(n) = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$

Base case: $D(1) = 0$, $D(2) = 2!\left(1 - 1 + \frac{1}{2}\right) = 1$ → So claim is true for $n = 1, 2$

Induction hypothesis: Assume, for all $1 \leq k < n$, the claim is true

where $n \geq 3$

$$D(n) = (n-1)\left(D(n-1) + D(n-2)\right)$$

$$= (n-1)\left[(n-1)! \sum_{i=0}^{n-1} \frac{(-1)^i}{i!} + (n-2)! \sum_{i=0}^{n-2} \frac{(-1)^i}{i!}\right]$$

$$= (n-1)!\left[(n-1) \sum_{i=0}^{n-1} \frac{(-1)^i}{i!} + \sum_{i=0}^{n-2} \frac{(-1)^i}{i!}\right]$$

$$= (n-1)!\left[n \sum_{i=0}^{n-1} \frac{(-1)^i}{i!} - \cancel{\sum_{i=0}^{n-2} \frac{(-1)^i}{i!}} - \frac{(-1)^{n-1}}{(n-1)!} + \cancel{\sum_{i=0}^{n-2} \frac{(-1)^i}{i!}}\right]$$

$$= (n-1)!\left[n \sum_{i=0}^{n-1} \frac{(-1)^i}{i!} + \frac{n(-1)^n}{n!}\right]$$

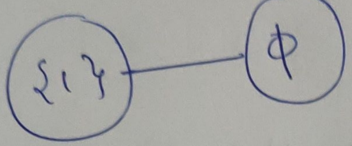
$$\boxed{D(n) = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}}$$

Q.E.D

Q14a

X_1 : $V = \{\{1\}, \phi\}$

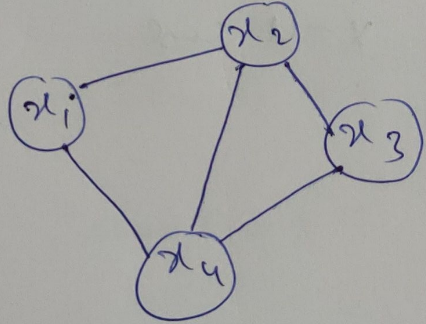
$E = \{\{\{1\}, \phi\}\}$



X_2: $V = \{\overset{x_1}{\{1\}}, \overset{x_2}{\{1,2\}}, \overset{x_3}{\{2\}}, \overset{x_4}{\phi}\}$

$E = \{\{\{1\}, \{1,2\}\}, \{\{2\}, \{1,2\}\}, \{\{1\}, \phi\}, \{\{2\}, \phi\},$

$\{\{1,2\}, \phi\},$



Q14b

WLOG,

let $u = \{1, \cdots k\}$

Count of X such that $X \subsetneq u$

$$= 2^{k} - 1$$

2^k → total subsets

1 → subtracting u itself. ($X = u$ case)

Count of X such that $u \subsetneq X$

$$= 2^{n-k} - 1$$

2^{n-k} → ~~pick~~ no of subsets of $\{k+1, \cdots n\}$ so that these can be appended to u to form X

1 → subtracting $X = u$ case.

~~To~~ degree $= \cancel{\Sigma}\,(2^{k}-1) + (2^{n-k}-1)$

$$\boxed{\text{deg} = 2^{k} + 2^{n-k} - 2}$$

Q14c Let $S_1 = P([n-1])$ (size 2^{n-1})

No of edges in X_{n-1} (vertex set $= S_1$) $= f(n-1)$

& $S_2 = P([n-1]) + \{n\}$ (size 2^{n-1})

i.e n appended to all the elements in $P([n-1])$

~~This gives~~ ~~$P([$~~

$S_1 + S_2 = P([n])$ (size 2^n)

Consider an edge in X_{n-1} with vertices x_1, x_2

$x_1 \subsetneq x_2$. We can map this edge to $\{x_1, x_2 + \{n\}\}$ edge. Like this we can map all the edges in X_{n-1} to an extra edge (relative to X_{n-1}) in X_n.

No of edges becomes $2f(n-1)$.

Now add all edges $\{x_1, x_2\}$ such that $x_1 \in S_1$ & $x_2 \in S_2$ & $x_2 - x_1 = \{n\}$. One edge per vertex in X_1. So no of edges $= 2^{n-1}$

clearly all the edges from S_1 to S_2 are finished.

~~Graph corresponding to~~

~~S_2 is nothing but isomorphic to X~~

~~Graph corresponding to S_2 in X_n~~

Graph induced by S_2 in X_n is isomorphic to X_{n-1}

as $S_2 = P([n-1]) + \{n\}$.

$\Rightarrow$ No of edges among the vertices in $S_2 = f(n-1)$

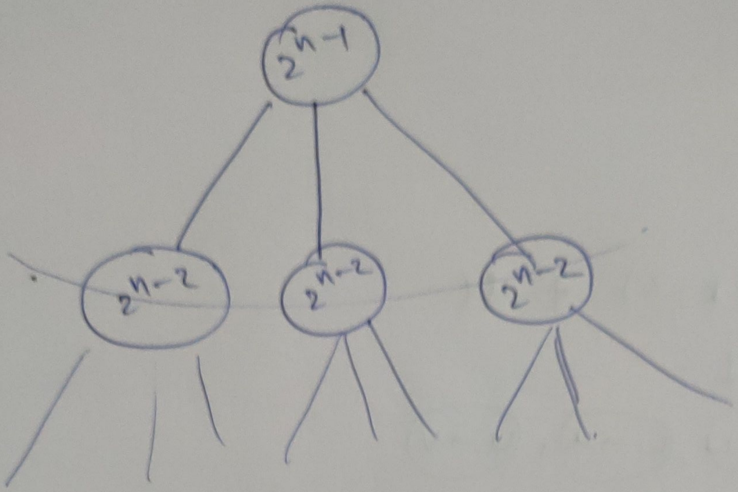
$$\therefore \boxed{f(n) = 3f(n-1) + 2^{n-1}}$$

$f(1) = 3f(0) + 1 = 1$ $\quad (\because f(0) = 0$

$f(2) = 3f(1) + 2 = 5$

$f(3) = 3f(2) + 4 = 19$

Q14d.



$$f(n) = 3f(n-1) + 2^{n-1}$$

$$f(n-1) = 3f(n-2) + 2^{n-2}$$

$$\Rightarrow f(n) = 3^2 f(n-2) + 3 \cdot 2^{n-2} + 2^{n-1}$$

$$f(n-2) = 3f(n-3) + 2^{n-3}$$

$$\rightarrow f(n) = 3^3 f(n-3) + 3^2 \cdot 2^{n-3} + 3 \cdot 2^{n-2} + 2^{n-1}$$

$$\rightarrow f(n) = 3^r f(n-r) + \sum_{k=0}^{r-1} 3^k 2^{n-1-k}$$

put $r = n$ $\quad (\because f(0) = 0)$

$$f(n) = \sum_{k=0}^{n-1} 3^k 2^{n-1-k}$$

$$= 2^{n-1} \sum_{k=0}^{n-1} \left(\frac{3}{2}\right)^k$$

$$= 2^{n-1} \left(\frac{\left(\frac{3}{2}\right)^n - 1}{\frac{3}{2} - 1} \right) = 2^n \left(\left(\frac{3}{2}\right)^n - 1\right)$$

$$\boxed{f(n) = 3^n - 2^n}$$