Let L be a list of integers > 1 which may appear multiple times but the order of which is not important. One can perform the following mutually inverse operations on elements of L.

1. Replace two elements a, b in L with a^b, e. g. $L = [7, 2, 3, 3]$ could become $[7, 2, 81]$ or $[7, 8, 3]$.
2. Conversely, decompose an element of the form a^b in L to a and b. However, notice that b must be > 1. For example, one could revert $L = [7, 2, 81]$ back to $[7, 2, 3, 3]$ but not to $[7, 2, 81, 1]$ since $81 = 81^1$ would not yield a valid decomposition by rule.

Definition. A number n is said to be *creative*, if for any $m > 1$ the list $L = [n]$ can be transformed by the above rules to a list that contains m.

Problem. Find the sum of all creative numbers $\leq 10^{12}$.

Let's play a little bit around with what we have so far.

How about the number $n = 7$. Is it creative? Well, of course not, since 7 is not a power and we thus cannot obtain another list than $[7]$ by the above rules. By the same reasoning, any other non-power cannot be creative.

What about a power then? Let's say $n = 49 = 7^2$. Well, we may transform

$$[49] \implies [7, 2] \implies [49] \implies [7, 2] \implies \dots$$

so here too, we are stuck between the two lists $[49]$ and $[7, 2]$.

What about another power, say $n = 81 = 3^4$? Lets turn on some mental arithmetic here

$$[81] \implies [3, 4] \implies [3, 2, 2] \implies [9, 2] \implies [512] \implies [8, 3] \implies [2, 3, 3] \implies [2, 81] \implies \dots$$

Halt for a moment and look, what we've reached here. We started off with a list $[81]$ and obtained a list $[2, 81]$. It somehow looks like as if we could magically add another 2 to our list. Of course, we can keep applying the same sequence of operations to that 81 again and again to get us as many additional 2s as we want. Now, let m be any number, say $m = 198$. If 81 was creative, we must be able to hit this 198 by the above transformation rules. Here we go. We first create some additional 2s

$$\dots \implies [2, 2, 2, 2, 2, 81] \implies [2, 2, 2, 4, 81] \implies [2, 2, 16, 81] \implies [2, 256, 81] \implies \dots$$

Now, I would like to perform the transform

$$2^{256} = 115792089237316195423570985008687907853269984665640564039457584007913129639936$$

Puh, quite a long number ... For notational sake I'd better leave it as 2^{256}.

$$... \implies [2^{256}, 81] \implies [81^{(2^{256})}] \implies ...$$

Recall exponents rules from high school

$$81^{(2^{256})} = 81^{(2^{198+60})} = 81^{(2^{198} \cdot 2^{60})} = \left(81^{(2^{198})}\right)^{(2^{60})}$$

Thus, we can split up further into

$$... \implies [2^{60}, 81^{(2^{198})}] \implies [2^{60}, 81, 2^{198}] \implies [2, 60, 81, 2, 198]$$

And there we are! $m = 198$ is contained in our list. But what about other numbers m? In order for $n = 81$ to be creative, we must be able to obtain *all* numbers m from the starting list $[81]$, not just $m = 198$. Well, it's actually easy to adapt the above reasoning to any other number m. More generally, it holts

Lemma. *Let L contain at least three elements a, b, c with at least one of being ≥ 3. Then, given $m > 1$ arbitrarily, we can transform L by a sequence of the above operations such that L will eventually contain m.*

Proof. Before we get to the actual proof, let us set up some conveniences. □

1. x, y_1, y_2 can be transformed to $x, y_1 \cdot y_2$, and vice versa

$$x, y_1, y_2 \iff x^{y_1}, y_2 \iff (x^{y_1})^{y_2} = x^{y_1 \cdot y_2} \iff x, y_1 \cdot y_2$$

2. x, y, y, s, t can be transformed to $x, y, s + t$, and vice versa

$$x, y, y, s, t \iff x, y^s, y^t \iff \ldots \iff x, y^s \cdot y^t = y^{s+t} \iff x, y, s + t$$

The careful reader will recognize these steps in the last example above.

Now, let $m > 1$ be arbitrary. By assumption at least one, say a, is ≥ 3. By convenience 1 we get

$$a, b, c \implies a^b, c \implies \underbrace{a, \ldots, a}_{b \geq 2\text{-times}}, c$$

with at least two elements a in the list, since $b \geq 2$. If we remove elements from the list and still be able to eventually generate m, we could have generated m with those elements more than ever. So I will prove to you that we will be able to generate m even with the guaranteed minimum number of two as in our new list. This time however, we have at least two elements being

$$a, a, c \implies a^a, c \implies \underbrace{a, \ldots, a}_{a \geq 3\text{-times}}, c$$

and this time we have at least three as in our list.

$$\begin{aligned} a, a, a, c &\implies a, a^a, c \\ &\implies a, \underbrace{a, \ldots, a}_{a \geq 3\text{-times}}, c \\ &\implies a, \ a, a, a \ , c \\ &\implies a, \ a, a^a \ , c \\ &\implies a, a, \underbrace{a, \ldots, a}_{a \geq 3\text{-times}}, c \\ &\ \vdots \end{aligned}$$

This shows that we can generate as many as as we like! If by the way you wonder why we needed c at all, take a close look at convenience 1.

Now, lets add enough as to our list such that

$$\left((a^a)^{\cdot^{\cdot^{\cdot}}} \right)^a = m + s$$

with some $s > 1$ such that we eventually get a list containing a, c and $m + s$. Now convenience 2 comes into play, but applied reversely, such that we get

$$a, c, c, m, s$$

as desired.

Corollary. *Let $n = a^b$ be a power with $a, b > 1$. Then n is uncreative if and only if one of the following hold*

1. a and b are not both prime.
2. $n \neq 16$

Proof. If a and b are both prime $[a^b]$ and $[a, b]$ are the only lists that can be generated. If $n = 16 = 2^4 = 4^2$ only the lists $[16], [2, 4]$ and $[2, 2, 2]$ can be generated. Otherwise, let a be composite[1], say $a = k \cdot l$. Then k, l and b cannot be all 2 since then n would have been 16. Hence, the condition of the lemma is satisfied. $\square$

Remark. To compute the sum of all creative numbers below some threshold T efficiently, we only check the powers

[1] The case b composite is similar.

$a^b \leq T$, i. e. $b\log(a) \leq \log(T)$. In our case, $T = 10^{12}$ and thus

$$b \leq \frac{12}{\log_{10}(a)} \leq \frac{12}{\log_{10}(2)} \leq 39.9$$

So, we only need to consider exponents $2 \leq b \leq 39$. However, if an exponent is composite, say $b = k \cdot l$, we have $n = a^b = \left(a^k\right)^l$. Hence, for further performance boost, we may only loop through prime exponents. For fixed b we may increase a from $a = 2$ until $a \leq \sqrt[b]{T}$ and then check the criterion in the corollary, that is essentially whether a is prime[2]. Should a fail a primality test we must be careful though to not simply add it to the final creative sum. Why? Here is an example: The number $1024 = 32^2 = 4^5$ is creative. In our algorithm, this number will appear twice! The first time, when $b = 2$ and $a = 32$. The second time when $b = 5$ and $a = 4$. Therefore, each time, we must add $1024/2$ to the creative sum. The interested reader is invited to ponder on the rest of the algorithm.

[2] Notice that b is prime anyway