Bridge Network Maintenance Optimization Using Stochastic Dynamic Programming

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Abstract: This paper presents a stochastic dynamic programming (DP) procedure for multiobjective optimization of bridge network maintenance planning that involves a group of existing highway bridges with various remaining service lifetimes and different reliability importance factors to the bridge network. The complex multiobjective optimization problem is solved by using a two-phase DP approach. The Phase I problem consists of identifying the optimal maintenance plans for individual bridges that have minimum life-cycle maintenance costs, while satisfying both condition and safety requirements for a targeted service lifetime period. This problem is solved by using a specific DP optimization algorithm along with Monte Carlo simulation. The Phase II problem is to rationally allocate the limited annual maintenance budgets in such a way that the identified optimal maintenance plans for individual bridges can be satisfied for as many bridges as possible. A single-objective formulation derived from multiple attribute utility theory with weight assignment from reliability importance factors is developed. This is solved by a binary integer programming algorithm. The ultimate goal of this study is achieved in terms of finding the most efficient combinations of available maintenance actions applied to all bridges in a highway network.

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CE Database subject headings: Bridge maintenance; Computer programming; Monte Carlo method; Life cycles; Safety.

Introduction

Bridge maintenance planning (BMP) as a part of public policy management currently faces a great challenge to balance limited available funds and increasing needs for bridge maintenance, repair, and rehabilitation activities (Heineman 2002; Das 1999; Shepard 2005; Bruehwiler and Adey 2005). This provides an excellent opportunity for applying advanced mathematical programming techniques in this field (Jiang and Sinha 1989: Mayet and Madanat 2002). As a matter of fact, BMP can be considered as a temporal optimization problem, where maintenance decisions and actions are made and taken moment by moment during the entire service lifetime of a bridge, based on the possible future consequences of safety, economic, and political impacts (Whittle 1982). The characteristics of BMP include: (1) present actions can affect future decisions only, that is, any actions taken in the past are not reversible; (2) a maintenance policy that comprises current decision and sequential maintenance decisions in the future are made through current knowledge and information only, although the knowledge and information on BMP generally increase with the passage of time; (3) optimality equations are recursive, that is, the optimal maintenance policy that consists of sequential maintenance decisions can be implemented at several stages along with time, and the selection of the maintenance action at each stage is dependent on the bridge condition and safety at the end of the previous stage, and is also related to the maintenance action taken at the beginning of the previous stage. The cost of the maintenance decision at each stage is a function of the selected maintenance action, application time, and any operating costs during this stage; (4) consequences of the maintenance decisions and actions are uncertain, even if reasonable predictions can be achieved. Either mini-max or a stochastic approach must be employed to find the solutions of the optimality equations; and (5) in general, BMP has multiple objectives that need to be optimized in the views of multiple decision makers. The multiple objectives of BMP could, for example, be: (1) maintaining the allowable condition and safety requirements; (2) minimizing the life-cycle maintenance costs and/or user costs; and (3) maximizing the benefit-cost ratios. The multiple decision makers may involve highway agents of local, state, or federal government, general contractors, maintenance material suppliers, even politicians in some cases. The ultimate goal of BMP is to find the "best" strategies and/or operational plans that are not only technically feasible, but also are considered optimal by all parties of involved decision makers. This can be achieved through decision support systems (DSSs) that provide better understanding of the real-world situations, identifying all possible objectives and conflicts, evaluating as many alternatives as possible, and finally reaching rational plans (Ang and De Leon 2005; Ellingwood 2005). Consequently, stochastic dynamic programming for a multiobjective optimization problem may be applied to find the optimal bridge maintenance plans for both individual bridges and bridge networks.

This paper presents a stochastic dynamic programming (DP)

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procedure for multiobjective optimization of bridge network maintenance planning that involves a group of existing highway bridges with various remaining service lifetimes and different reliability importance factors (RIFs) to the bridge network. The multiple objectives formulation requirements are: (1) the condition indices of all bridges in a highway network must be, at any time during a targeted service lifetime, lower than the maximum allowable condition index; (2) the safety indices of all bridges in a highway network must be, at any time during a targeted service lifetime, higher than the minimum allowable safety index; (3) the identified optimal bridge maintenance plans should have the minimum life-cycle maintenance costs; and (4) the total costs of the maintenance actions performed at each year must be under a predefined annual maintenance budget. This complex multiobjective optimization problem is solved in this study by using a two-phase DP approach. The Phase I problem consists of identifying the optimal maintenance plans for all individual bridges, by using a specific DP optimization algorithm, that has the minimum life-cycle maintenance costs, meanwhile satisfying both condition and safety requirements for a targeted service lifetime period. The Phase II problem is to rationally allocate the limited annual maintenance budgets in such a way that the identified optimal maintenance plans for individual bridges in a highway network can be satisfied for as many bridges as possible. A single-objective function of the probabilities that each of the identified optimal maintenance plans for individual bridges can be satisfied is established from the additive form of multiple attribute utility functions with weight assignments from the reliability importance factor (RIF) of each bridge (Liu and Frangopol 2005). The single-objective optimization problem in Phase II is actually a binary integer programming problem that maximizes the sum of the weighted probabilities obtained from the Phase I problem with the constraint on a predefined annual maintenance budget. This may be solved by either traditional mathematical programming for combinatorial optimization or the advanced heuristic search methods such as genetic algorithms (GAs).

In this paper, bridge condition and safety profiles are briefly reviewed. The effects of four different maintenance actions (i.e., minor concrete repair, silane treatment, cathodic protection, and rebuild) on bridge condition and safety profiles are discussed. These four maintenance actions include both preventive and essential maintenance methods with actual cost data. Then, the computer program for bridge management system (BMS) developed at the University of Colorado, called BMS-DP, that employs a specific DP optimization algorithm along with Monte Carlo simulations is introduced. BMS-DP can be used to identify all possible feasible maintenance plans for individual bridges in a highway network, based on both condition and safety requirements during a targeted lifetime period. In this study, the optimal maintenance plans for individual bridges in a highway network are defined as the feasible maintenance plans that have the minimum total maintenance costs in terms of the net present value (NPV). Monte Carlo simulations are integrated into the proposed DP procedure to generate various optimal maintenance plans for individual bridges. The random variables involved in Monte Carlo simulations include discount rates, maintenance costs for each of the four maintenance actions, and effects of the four maintenance actions on bridge condition and safety profiles, among others. At the end of Phase I, the probability distributions of the application times for each of the four maintenance actions are obtained for each of the individual bridges in a highway network. Phase II starts with the formulation of a binary integer programming problem, which is relevant to the probability distributions of the application times obtained in Phase I, the weight assignment from RIF, and the annual bridge maintenance budgets. The optimization problem in Phase II is to maximize the sum of the weighted probabilities that each of the four maintenance actions may be applied to each of the individual bridges in a highway network at a certain year under the constraint of a predefined maintenance budget at that year. As a result, the efficiency of allocation of the limited maintenance budgets at a certain year may be evaluated by using the results of the Phase II optimization problem. Consequently, the ultimate goal of bridge network maintenance planning has been achieved in terms of finding the most efficient combination of the four maintenance actions applied to all individual bridges in a highway network at a certain year. Finally, a numerical example is provided for illustration purposes.

Bridge Condition and Safety Profiles

The performance of a highway bridge at any time during its effective service lifetime period can be evaluated by using the bridge performance indicators. Bridge condition index (CI) and safety index (SI) are the most widely used performance indicators in modern BMSs. CI evaluates the performance of a bridge based on the observations on the bridge and its components only, without any consideration of traffic loading conditions. The observations used in CI may be obtained from visual inspection and/or special in situ tests such as nondestructive testing (NDT) and structural health monitoring (SHM). On the other hand, SI considers both bridge performance and traffic loading situations, resulting in a true measure of bridge safety. Since the system reliability of a bridge is a function of bridge load capacity, traffic load effect, and failure mode, the system reliability index of the bridge is regarded as SI in this paper. For detailed computations of the bridge system reliability index the interested readers are referred to Estes and Frangopol (1999, 2001). The computation of SI and its evolution with time require much more detailed information than the evaluation of CI. Therefore, the computation of CI is much simpler than SI in practical BMS at the present time. In this study, both CI and SI are used in optimal bridge maintenance planning.

The changes of CI and SI over time produce bridge condition index and safety index profiles, respectively. National bridge inventory (NBI) adopts a condition rating system with the highest score of 9, indicating an excellent bridge condition, and the lowest score of 0, implying structural failure of a bridge (FHWA 1988). Thus, the bridge condition index profile without maintenance decreases with time in NBI. Meanwhile, PONTIS, the most popular BMS in the Unites States, has a condition rating system, which produces a bridge condition profile that increases from 1 to 5 due to aging and deterioration (Thompson 1994). In this study, a similar condition rating system to that used in PONTIS is employed, which produces a typical linear condition index profile as shown in Fig. 1, where the initial condition index CI_{INI}, the rate of deterioration α_c, and consequently the time of reaching the worst condition, $T_{0,c}$, may all be treated as random variables. Furthermore, the bridge safety index profile under no maintenance is considered as a bilinear function of system reliability index (Frangopol et al. 2001), as follows:

$$SI(t) = \begin{cases} \beta_0 & \text{for } 0 \leqslant t < t_0 \\ \beta_0 - \alpha_s(t - t_0) & \text{for } t \geqslant t_0 \end{cases} \quad (1)$$

where β_0=initial system reliability index of a bridge; α_s=deterioration rate of the bridge system reliability index; and

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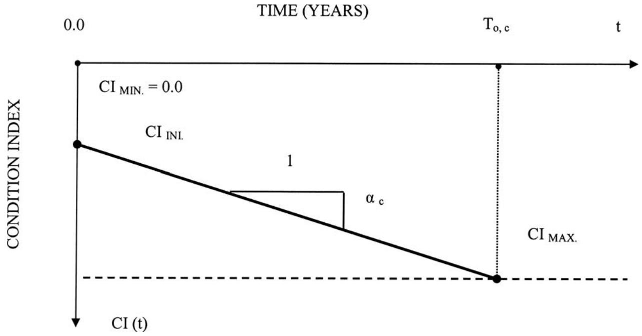

Fig. 1. Bridge linear condition index profile without maintenance

t_0=time of initiation of deterioration of the system reliability index. It is worth noting that β_0, t_0, α_s, and consequently the time of reaching the minimum allowable safety, $T_{0,s}$, may all be treated as random variables. Fig. 2 shows a bilinear bridge safety index profile. The probability density functions (PDFs) of random variables β_0, t_0, and α_s may also be assigned (Frangopol et al. 2001, 2004; van Noortwijk and Frangopol 2004).

Effects of Bridge Maintenance Actions

Bridge maintenance actions can be categorized as preventive and essential. The preventive maintenance actions are carried out on functional bridges and/or their components in order to reduce the probability of unsatisfactory performance and to delay the application times of essential maintenance actions that usually are associated with much higher costs and need much longer application times than the preventive maintenance actions. For example, cleaning, minor concrete repairing, and repainting on metal (e.g., steel or aluminum) components are typical of preventive maintenance. On the other hand, the essential maintenance actions are corrective for malfunctioned bridge components such as eroded bridge foundations, corroded steel girders, and deteriorated concrete decks with too many potholes. The essential maintenance actions may include strengthening structural members, replacement of major bridge components, and even rebuilding entire bridges.

The effects of maintenance actions on bridge condition and safety profiles can be classified as: (1) improvement of current condition and/or safety indices; (2) delay in deterioration occurrence; (3) reduction of deterioration rates; and (4) combinations of the above three effects during the effective period of maintenance action (Frangopol et al., 2001; Neves and Frangopol 2004). Thus, after applying a single maintenance action, the effective service lifetime of a bridge will be extended for condition by ΔT_c, and for safety by ΔT_s, as follows:

$$\Delta T_c = \Delta T_{\text{improvement},c} + \Delta T_{\text{delay},c} + \Delta T_{\text{rate},c} \tag{2}$$

$$\Delta T_s = \Delta T_{\text{improvement},s} + \Delta T_{\text{delay},s} + \Delta T_{\text{rate},s} \tag{3}$$

where $\Delta T_{\text{improvement},c}$ and $\Delta T_{\text{improvement},s}$=extended time due to the improvement of current condition and safety indices, respectively; $\Delta T_{\text{delay},c}$ and $\Delta T_{\text{delay},s}$=extended time due to the delay in deterioration occurrences of bridge condition and safety, respectively; and $\Delta T_{\text{rate},c}$ and $\Delta T_{\text{rate},s}$=extended time due to the reduction of deterioration rates of condition and safety, respectively. Moreover, $\Delta T_{\text{improvement}}$ and ΔT_{rate} can be computed using the model proposed in Frangopol et al. (2001), respectively, as

$$\Delta T_{\text{improvement}} = \frac{\Delta\gamma}{\alpha} \tag{4}$$

$$\Delta T_{\text{rate}} = \frac{\delta}{\alpha} \times t_d \tag{5}$$

where $\Delta\gamma$=improvement of performance (i.e., condition or safety) index; α=performance deterioration rate; δ=reduction of deterioration rate; and t_d=effective period of reduced deterioration rate. Consequently, the effective service lifetime of a bridge is extended by ΔT as follows:

$$\Delta T = \min(\Delta T_c, \Delta T_s) \tag{6}$$

As previously indicated, bridge maintenance actions considered in this study include both preventive and essential maintenance. Tables 1 and 2 present the effects of the four maintenance actions in this study, namely, “minor concrete repair,” “silane treatment,” “cathodic protection,” and “rebuild,” on bridge condition and safety indices over time, respectively (S. Denton, personal communication, 2002). For example, the application of “minor concrete repair” results in a decrease of bridge CI between 2 and 3 with a triangular PDF. The mode of the triangular PDF is 2.5, indicating that the most likely decrease of the bridge condition index is 2.5 (see Table 1). Meanwhile, the “minor concrete repair” causes a delay in deterioration of bridge SI when the bridge condition index is less than 1.0. Therefore, there is no change of SI after the “minor concrete repair” is applied until CI reaches 1.0. The “silane treatment” produces the reduced deterioration rates on both condition and safety indices during the maintenance effective duration that has a triangular PDF between 7.5 and 12.5 years with a mode of 10 years. The bridge condition and safety indices will not change in the first 12.5 years after the “cathodic protection” maintenance action is applied. The “rebuild” is the only essential maintenance action considered in this study. If this action is applied, the bridge condition index will be set to zero (i.e., best possible condition) and the bridge safety index will be assigned to the safety index of the rebuilt bridge, SI_{new}. Meanwhile, the deterioration of the bridge condition index will restart between 10 and 30 years after “rebuild” with a triangular PDF mode of 15 years. The deterioration of the bridge safety index will resume when the bridge condition index reaches 1.0.

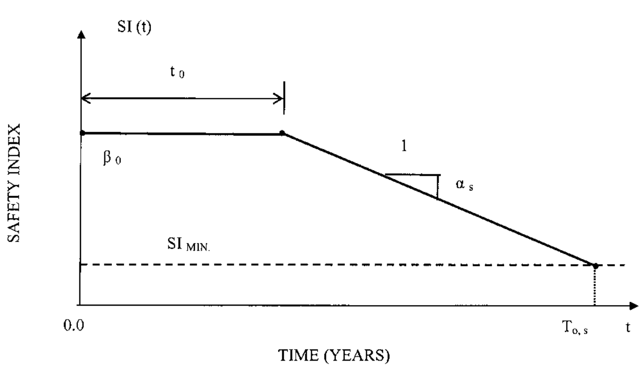

Fig. 2. Bridge bilinear safety index profile without maintenance

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Table 1. Effects of Bridge Maintenance Actions on Mean Condition Index (CI) (Data Provided by Denton, Personal Communication, 2002)

Maintenance action	Decrease in CI $\Delta\gamma_c$	Delay in CI $\Delta T_{\text{delay},c}$ (years)	Reduced deterioration rate δ_c (year^{-1})	Effective duration $t_{d,c}$ (years)
Minor concrete repair	T (2.0, 2.5, 3.0)	0.0	0.0	0.0
Silane treatment	0.0	0.0	T (0.00, 0.01, 0.03)	T (7.5, 10.0, 12.5)
Cathodic protection	0.0	12.5	0.0	12.5
Rebuild	Set to zero	T (10, 15, 30)	0.0	T (10, 15, 30)

Note: T (minimum value, mode, maximum value) represents triangular probability density distribution.

Life-Cycle Bridge Maintenance Cost

The effective service lifetime of a bridge without maintenance, $T_0=\min(T_{0c},T_{0s})$, may not be long enough to reach a targeted level due to aging and deterioration, but it can be extended by applying sequential maintenance actions. Thus, the total service lifetime of a bridge with N maintenance actions, T_m, can be expressed as

$$T_m = T_o + \sum_{i=1}^{N} \Delta T_i = T_o + \sum_{i=1}^{N} \min(\Delta T_{i,c}, \Delta T_{i,s}) \tag{7}$$

where ΔT_i=extended service lifetime due to applied maintenance action i, as indicated in Eq. (6). It can be proven that ΔT_i for the four maintenance actions in this study are generally independent of the application times, that is, the combinations involving the same maintenance actions, but different sequences, result in the same T_m.

A combination of any of the above four maintenance actions, which can extend the bridge effective service lifetime to a targeted level, can be regarded as a feasible bridge maintenance plan. These feasible maintenance plans may require performing different combinations of the four maintenance actions at different application times, resulting in different life-cycle maintenance costs as previously indicated. In order to find an optimal plan, the life-cycle maintenance cost for each feasible maintenance plan must be converted to the NPV, using a reasonable discount rate. An optimal maintenance plan in this study is the feasible plan that has a minimum life-cycle maintenance cost, C_{lm}, as follows:

$$C_{\text{lm}} = \sum_{i=1}^{N} \frac{C_i}{(1+D_r)^{T_i}} \tag{8}$$

where C_i=cost associated with maintenance action i; D_r=discount rate; and T_i=application time of maintenance action i. Table 3 presents the assumed costs for the four maintenance actions in this study, which are approximately based on the reported cost data in S. Denton (personal communication, 2002).

Dynamic Programming for Optimization

Theoretically, an optimal bridge maintenance plan can be obtained by comparing the life-cycle maintenance cost for all feasible plans, using enumeration methods. However, the advanced mathematical optimization techniques such as DP and GAs can provide very efficient approaches, particularly for stochastic systems that involve random variables in computations. In this study, DP is adopted along with Monte Carlo simulations.

DP is an effective tool for finding an optimal sequence of decisions in a multistage decision making process (Bertsekas 1976). Generally, a complex decision optimization problem can be broken down using DP into a sequence of several smaller and simpler subproblems. Each of the subproblems is defined as a stage. There are usually several decision candidates at each stage, and each of the decision candidates can produce a consequence (i.e., state). The optimal decision at each stage can be selected from the decision candidates, based on the state and decision at the previous stage. According to the "principle of optimality" (Bellman 1957; Bellman and Dreyfus 1962), the optimal solution of the original optimization problem is made up of the optimal sequence of solutions at each stage. Since DP is essentially dependent on the recurrence relationships in the optimality equations of a specific problem under consideration, there is no standard mathematical formulation for DP. Therefore, DP is a general strategy for optimization rather than a specific set of rules to find an optimal series of sequential solutions. For this reason, almost all DP problems need individual specification (Smith 1991). DP usually begins with the search of the optimal decision at the last stage, and works backward to the first stage of the original decision optimization problem. In short, "DP starts with a small portion of the problem and finds the optimal solution for this smaller problem, then gradually enlarges the problem, finding the current optimal solution from the previous one until the original problem is solved in its entirety" (Hillier and Lieberman 1967).

The computer program BMS-DP, using DP optimization algorithms along with Monte Carlo simulations, has been developed at the University of Colorado at Boulder for optimal bridge main-

Table 2. Effects of Bridge Maintenance Actions on Mean Safety Index (SI) (Data Provided by Denton, Personal Communication, 2002)

Maintenance action	Increase in SI $\Delta\gamma_s$	Delay in SI $\Delta T_{\text{delay},s}$ (years)	Reduced deterioration rate δ_s (year^{-1})	Effective duration $t_{d,s}$ (years)
Minor concrete repair	0.0	While CI < 1.0	0.0	During CI < 1.0
Silane treatment	0.0	0.0	T (0, 0.007, 0.018)	T (7.5, 10.0, 12.5)
Cathodic protection	0.0	12.5	0.0	12.5
Rebuild	Set to SI_{new}	While CI < 1.0	0.0	During CI < 1.0

Note: T (minimum value, mode, maximum value) represents triangular probability density distribution.

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Table 3. Assumed Cost for Bridge Maintenance Actions

Maintenance action	Cost C_i (relative units)
Minor concrete repair	T (200, 520, 880)
Silane treatment	T (35, 80, 120)
Cathodic protection	T (350, 600, 850)
Rebuild	T (1,000, 1,750, 2,500)

Note: T (minimum value, mode, maximum value) represents triangular probability density distribution.

tenance planning. The flow chart of BMS-DP is presented in Fig. 3. BMS-DP consists of a main program and three subroutines: Monte Carlo simulation (MCS), feasible maintenance plans (FMP), and optimal feasible plans (OFP). MCS is used to generate random numbers in accordance with the predefined probability density function for each of the random variables in BMS-DP. The function of FMP is to identify all feasible plans by using a DP procedure. First of all, the total number of the applied maintenance actions N is set to 1, and T_m in Eq. (7) is computed for each maintenance action. The maintenance plans with T_m equal to or greater than the targeted service lifetime T_g=feasible plans. Otherwise, the maintenance plans are infeasible. Then, N is set to be $N+1$ (i.e., $N=2$), and T_m in Eq. (7) is computed again for each of the infeasible maintenance plans plus one of the maintenance actions. Additional feasible and infeasible plans are identified. This procedure continues until the pool of the infeasible plans is empty. Finally, OFP is developed to search the optimal application time T_i for each maintenance action in a feasible plan in order to minimize the life-cycle maintenance cost C_{lm} of the feasible plan. As shown in Eq. (8), T_i should be as large as possible in order to minimize C_{lm}, provided the discount rate D_r is positive. On the other hand, T_i should comply with both maximum condition index and minimum safety index criteria. This is because the bridge CI increases as the bridge deteriorates with time, but the bridge SI decreases with time due to aging and deterioration. For example, "rebuild" must be conducted before SI reaches 0.91, while the "minor concrete repair" must be performed before CI increases above 3.0. Meanwhile, T_i should be assigned in such a way that all sequential maintenance actions in a feasible plan must be accomplished before T_g, and after T_{i-1}, of which the first application time, T_1, must be either zero (at present time) or positive (in the future). In any case, the bridge condition and safety indices at any time during its entire life must be no greater than

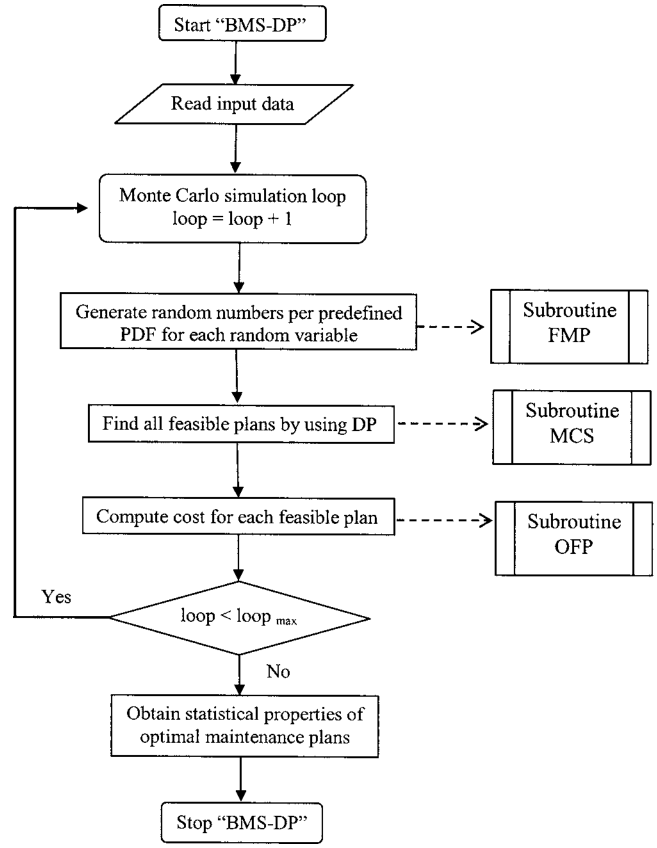

Fig. 3. Flow chart of computer program BMS-DP

Table 4. Parameters in Computer Program BMS-DP

Parameter	Notation	Distribution type	Remarks
Cost for maintenance action i	C_i	Random	See Table 3
Discount rate	D_r	Random	Uniform distribution (2%, 8%)
Extended time due to deterioration delay	$\Delta T_{\mathrm{delay}}$	Random	See Tables 1 and 2
Improvement in performance indices	$\Delta\gamma$	Random	See Tables 1 and 2
Reduction in deterioration rate	δ	Random	See Tables 1 and 2
Effective duration of maintenance action	t_d	Random	See Tables 1 and 2
Application time for maintenance action i	T_i	Deterministic	Assigned by subroutine OFP
Total number of maintenance actions	N	Deterministic	Assigned by subroutine FMP
Lifetime without maintenance actions	T_o	Deterministic	Problem specified
Original deterioration rate	a	Deterministic	Problem specified
Current safety index	$\mathrm{SI}_{\mathrm{ini}}$	Deterministic	Problem specified
Current condition index	$\mathrm{CI}_{\mathrm{ini}}$	Deterministic	Problem specified
New safety index for "rebuild"	$\mathrm{SI}_{\mathrm{new}}$	Deterministic	Problem specified
Targeted service lifetime	T_g	Deterministic	75 years
Minimal safety index (SI)	$\mathrm{SI}_{\mathrm{min}}$	Deterministic	0.91 in this study
Maximum condition index (CI)	$\mathrm{CI}_{\mathrm{max}}$	Deterministic	3.00 in this study
Minimal SI for maintenance action i	SI_i	Deterministic	0.91 for "rebuild"
Maximum CI for maintenance action i	CI_i	Deterministic	2.0 for "cathodic protection" 3.0 for "minor concrete repair"

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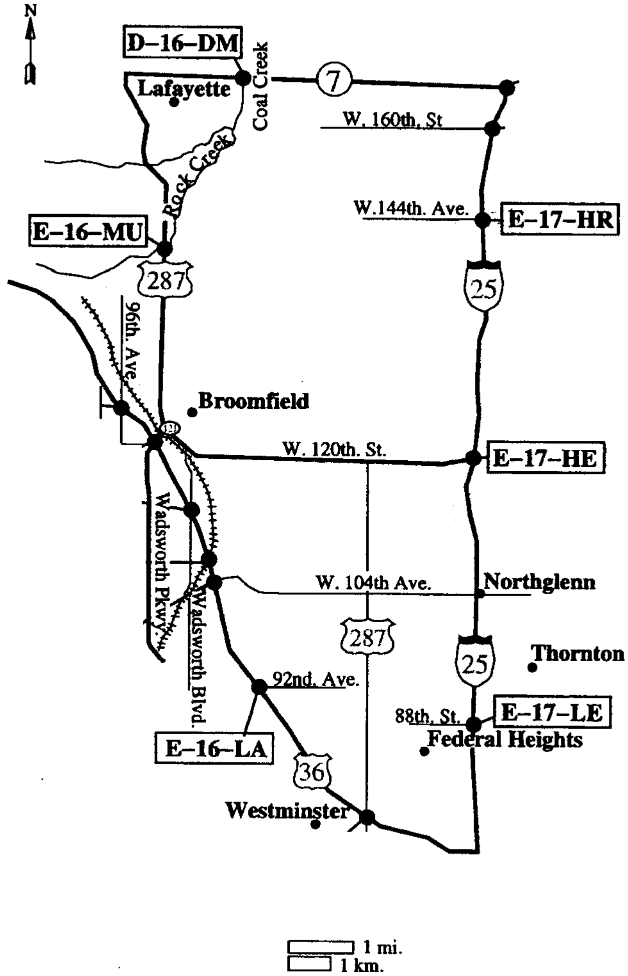

Fig. 4. Regional bridge network in Colorado (adapted by Akgul and Frangopol 2003)

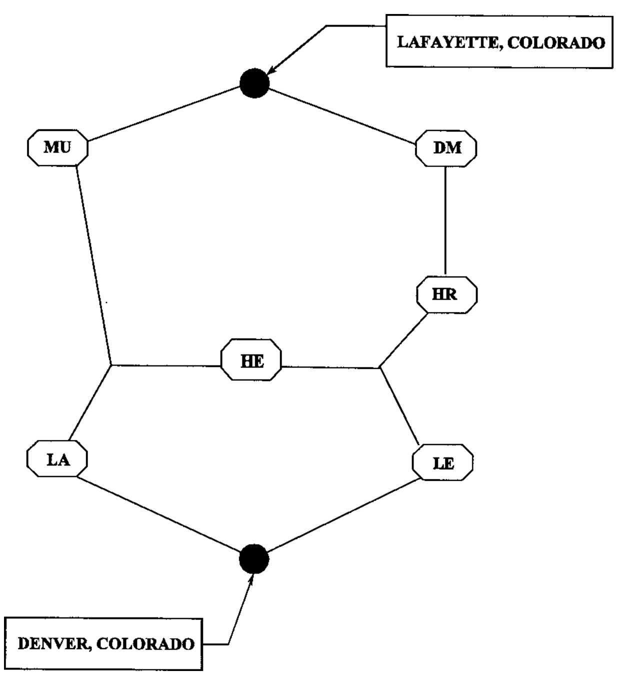

Fig. 5. Model of Colorado bridge network in Fig. 4

Table 5. Bridge Characteristics in Regional Highway Network in Colorado

Bridge name	Abbreviated name	Year built	Computed system reliability at 21st year	RIF at 21st year	Normalized RIF
Steel plate girder bridges					
E-17-HE	HE	1962	2.03	0.489	0.252
E-17-HR	HR	1962	0.91	0.523	0.270
E-17-LE	LE	1972	2.68	0.109	0.056
Prestressed concrete					
E-16-MU	MU	1994	2.67	0.265	0.137
E-16-LA	LA	1983	2.24	0.481	0.248
E-16-DM	DM	1990	2.06	0.073	0.037

the required maximum condition index CI_{max} and no less than the required minimum safety index SI_{min}, respectively. This complex task can be accomplished by BMS-DP.

Table 4 summarizes all parameters in BMS-DP, including random variables and deterministic parameters, where N and T_i must be obtained from Subroutines FMP and OFP, respectively. In this study, C_i is taken from Table 3, while ΔT_{delay}, $\Delta\gamma$, δ, and t_d are based on Tables 1 and 2. D_r and T_g herein are assumed random with a uniform probability density distribution (2%, 8%) and a deterministic value of 75 years, respectively. According to S. Denton (personal communication, 2002), SI_{min} and CI_{max} are 0.91 and 3.0, respectively. Meanwhile, the minimum value $SI_i=0.91$ for applying "rebuild," the maximum value $CI_i=2.0$ for applying "cathodic protection," and the maximum value $CI_i=3.0$ for applying "minor concrete repair" are also suggested by S. Denton (personal communication, 2002).

Optimization for Bridge Network Maintenance Planning

Optimal bridge maintenance planning has to provide answers to several questions such as what kinds of bridge maintenance actions are available at the present time, what sequence of the available maintenance actions should be chosen, and when should the selected maintenance actions take place in order to minimize the life-cycle maintenance cost during an entire targeted lifetime period of a bridge. The life-cycle cost may include the construction costs that bridge owners have to pay for or user's cost that includes the time delays and fuel consumption due to detour and/or congestion caused by the maintenance actions or combination of both construction cost and user's cost. Moreover, bridge network maintenance planning has to deal with multiple bridges in a highway network, and must consider annual maintenance funding limitation. Consequently, multicriteria decision making (MCDM)

Table 6. Problem-Specified Parameters for Bridge HR in Numerical Example

Parameter	Notation	Safety index	Condition index
Original deterioration rate (year^{-1})	α	0.02	0.07
Current safety index	SI_{ini}	1.61	—
Current condition index	CI_{ini}	—	1.50
Safety index after "rebuild"	SI_{new}	1.91	—

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Table 7. One Realization of Monte Carlo Simulations on Bridge HR from BMS-DP with $D_r = 3\%$

Maintenance action	ΔT (max.) SI (years)	ΔT (max.) CI (years)	Cost C_i (units)	$\Delta\gamma$ (max.) SI	$\Delta\gamma$ (max.) CI	ΔT_{delay} (max.) SI (years)	ΔT_{delay} (max.) CI (years)	δ SI (year^{-1})	δ CI (year^{-1})	$C_i/\Delta T$ (max.) SI	$C_i/\Delta T$ (max.) CI
Minor concrete repair	14.3	35.7	550	0.0	2.5	14.3	0.0	0.0	0.0	38.5	15.4
Silane treatment	2.00	1.43	86	0.0	0.0	0.0	0.0	0.004	0.01	43.0	60.1
Cathodic protection	12.5	12.5	450	0.0	0.0	12.5	12.5	0.0	0.0	36.0	36.0
Rebuild	76.3	54.9	1,350	1.0	3.0	26.3	12.0	0.0	0.0	17.7	24.6

Note: SI=based on safety index criterion; CI=based on condition index criterion.

procedures can be used to help bridge owners and/or maintenance managers to develop the optimal bridge network maintenance plans in terms of making rational decisions on allocating the limited maintenance funds every year.

As an important methodology in MCDM, the multiple attribute utility theory (MAUT), is adopted in this study. As a matter of fact, MAUT focuses on the development of the multiple attribute utility functions to model and represent the decision maker's preferential structures (Von Neumann and Morgenstern 1944; Pardalos et al., 1995) when multiple objectives and alternative decisions exist. The multiple attribute utility functions combine all of the marginal utility functions associated with individual attributes of each alternative decision, where the marginal utility functions may be built up by either direct interrogation with decision makers or by indirect methods, as well as by using the analytic hierarchy process (Saaty 1980) that has been mainly used in the United States. The decomposition forms of the multiple attribute utility functions may be: (1) additive; (2) multiplicative; and (3) multilinear (Keeney and Raiffa 1993). The additive form requires mutual preferential independence, that is, every subset of criteria is preferentially independent from the remaining criteria. A subset of criteria is considered to be preferentially independent from the remaining criteria if and only if the decision maker's preferences on the alternative decisions differ only with respect to the criteria, and are independent on the other remaining criteria (Vincke 1992). It should be noted that the very complex decomposition forms are not of interest from a practical point of view. MAUT also employs an interactive and iterative procedure involving policy analyst and decision makers to specify the weight and marginal utility function corresponding to individual attributes of each alternative decision. Finally, the combined multiple attribute utility function associated with each alternative decision can be used as a single-objective function in the traditional mathematical programming in order to identify the final optimal decisions (Doumpos and Zopounidis 2002).

Since multiple bridges in a highway network are taken into account when making an optimal bridge network maintenance plan, the multicriteria are considered as satisfying the optimal bridge maintenance plans for individual bridges, which in this study are identified in the Phase I problem. Therefore, satisfying the optimal bridge maintenance plan for each bridge can be treated as a subset of criteria. The marginal utility function for each alternative decision (i.e., satisfying the optimal bridge maintenance plan for a certain bridge) is assigned to be the probability that each of the four maintenance actions may be applied to a certain bridge at a certain year. Because the mutual preferential independence requirement can be easily satisfied in this case, the additive form of the multiple attribute utility function may be used to form a single-objective function for the Phase II optimization problem. Moreover, as each bridge has its unique role in a highway network, the importance of each bridge to the bridge network should be reflected in an optimal bridge network maintenance plan. Thus, the objective function in terms of the additive form of the multiple attribute utility functions in the Phase II problem is weighted by an importance factor that is dependent on bridge locations, traffic volumes, maintenance needs, and so on. In this study, the importance factor is taken as the RIF of each bridge, where RIF is defined as the sensitivity of the bridge network reliability in terms of connectivity to the change in the individual bridge system reliability (Liu and Frangopol 2005), and is a function of bridge system reliability profiles, network reliability, and network topology. The importance factor can be further assigned to reflect impacts of bridge maintenance actions on traffic capacity, economy, and environment, when considering additional criteria such as user's satisfaction, critical bridge performance in a highway network, and so on (Liu and Frangopol

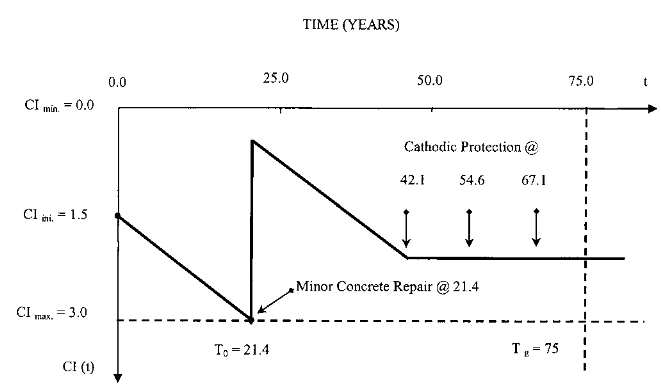

Fig. 6. Bridge condition index profile from one realization of Monte Carlo simulations

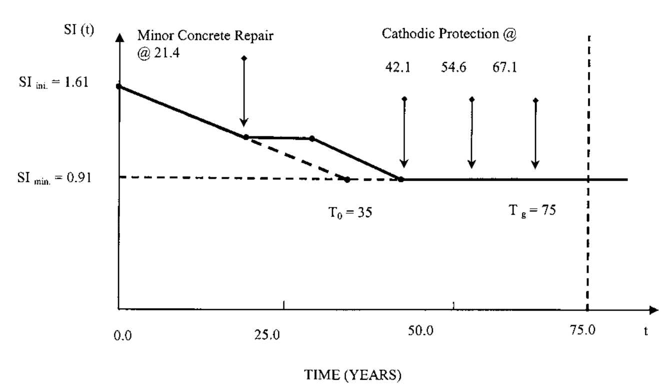

Fig. 7. Bridge safety index profile from one realization of Monte Carlo simulations

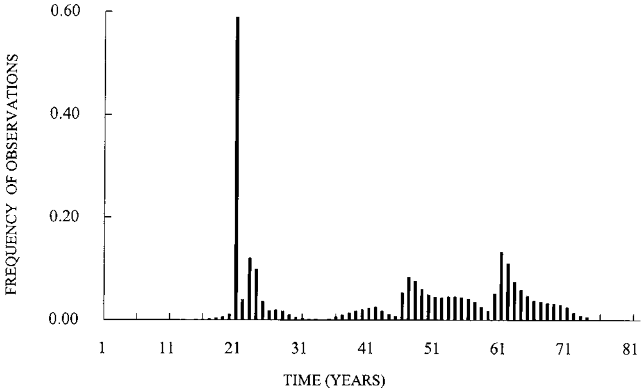

Fig. 8. Bridge HR: Frequency of observations per year for "minor concrete repair"

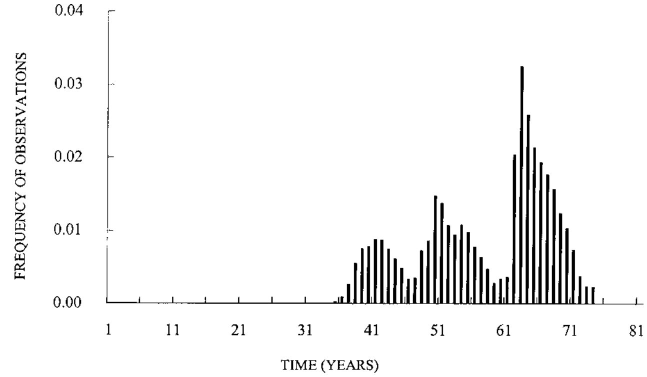

Fig. 10. Bridge HR: Frequency of observations per year for "cathodic protection"

2006b). Consequently, the Phase II optimization problem can be formulated for a certain year k as follows:

Maximize

$$\sum_i \sum_j D_{ij} \times \mathrm{RIF}_i \times P_{ij} \tag{9}$$

subject to

$$\sum_i \sum_j D_{ij} \times C_j \leq C_{\mathrm{budget}} \tag{10}$$

where D_{ij}=binary design variable (i.e., the value of D_{ij} can be either 0 or 1); RIF_i=reliability importance factor of bridge i at year k; P_{ij}=probability that the maintenance action j is applied to bridge i at year k; C_j=cost of maintenance action j; and C_{budget}=annual maintenance budget at year k.

The binary design variable D_{ij} represents the decision on selecting maintenance action j applied to bridge i. Therefore, $D_{ij}=0$ means maintenance action j will not be applied to bridge i, and $D_{ij}=1$ means maintenance action j is selected to be applied to bridge i. In addition, it should be noted that the values of RIF_i and P_{ij} usually vary with time. This is because the time-dependent RIF_i is normalized for all bridges in a highway network, where each bridge may experience different ages and deterioration with time (Liu and Frangopol 2005). Similarly, P_{ij} changes with time, depending on the results from Monte Carlo simulations in the Phase I problem. Furthermore, although C_i is taken as the costs in Table 3, more accurate values of C_i will result in more realistic optimal bridge network maintenance plans. Finally, this combinatorial optimization problem in Phase II can be easily solved by a traditional binary integer programming algorithm.

Numerical Example

As a numerical example for demonstration purposes, a regional highway network that connects the cities of Denver and Lafayette, Colo. is simplified to contain six bridges only, as shown in Figs. 4 and 5. These bridges are either prestressed concrete bridges or steel plate girder bridges built at various years, and their basic characteristics are presented in Table 5. An optimal bridge network maintenance plan needs to be developed for a targeted effective service lifetime of 75 years with the four maintenance actions considered in Tables 2 and 3. Since the deterioration rates and initial conditions of these bridges including condition ratings and safety indices are different at the present time, even if the bridges are located in the same area and are under similar vehicular and environmental conditions, Table 6 presents the values of the problem-specified parameters in Table 4 for Bridge E-17-HR (i.e., Bridge HR). Thus, the effective service lifetime of the bridge without maintenance is based on the CI criterion, i.e., $(3.0-1.5)/0.07=21.4$ years, which is less than the one based on the SI criterion, i.e., $(1.61-0.91)/0.02=35.0$ years. BMS-DP along with Monte Carlo simulations was performed with 50,000 samples, based on the probability distributions and values in Tables 1–3. One realization of Monte Carlo simulations from BMS-DP is presented in Table 7 for Bridge HR. From

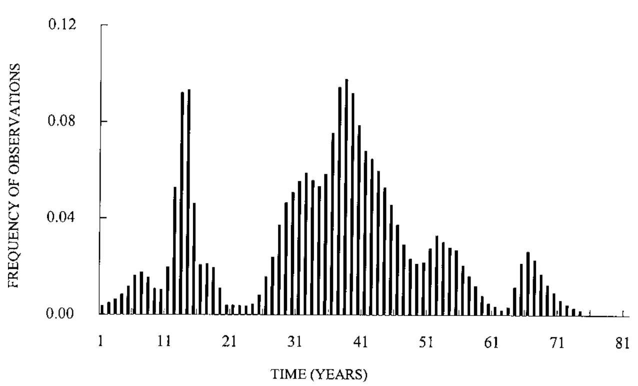

Fig. 9. Bridge HR: Frequency of observations per year for "silane treatment"

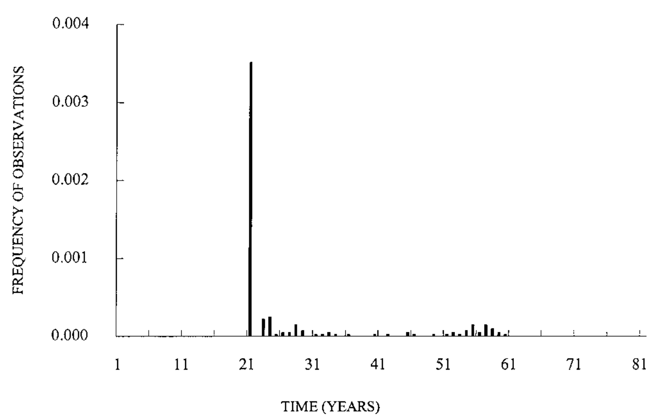

Fig. 11. Bridge HR: Frequency of observations per year for "rebuild"

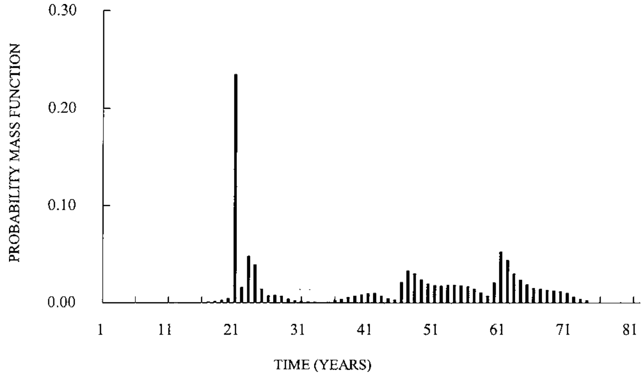

Fig. 12. PMF of "minor concrete repair" applied on bridge HR

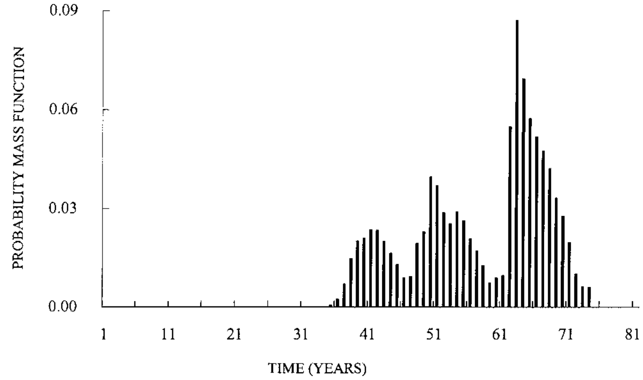

Fig. 14. PMF of "cathodic protection" applied on bridge HR

Tables 6 and 7 as well as the effective duration of $t_d=10$ years for "silane treatment," this optimal maintenance plan for Bridge HR was identified to include one "minor concrete repair" application at 21.4 years from the present time, and three "cathodic protection" applications at 42.1, 54.6, and 67.1 years from the present time, which resulted in the life-cycle bridge maintenance cost of 572.7 units in terms of NPV. The bridge condition index and safety index profiles for this optimal maintenance plan are presented in Figs. 6 and 7, respectively. Furthermore, based on a total of 50,000 Monte Carlo simulations, the frequencies of observations per year of each of the four maintenance actions that may be carried out on Bridge HR are plotted in Figs. 8–11. The expected life-cycle bridge maintenance cost for Bridge HR is 296.9 units in terms of NPV with the coefficient of variation (COV) of 44.6%. Since an optimal bridge maintenance plan may contain the same maintenance action applied at different times (e.g., the "cathodic protection" was selected three times as shown in Figs. 6 and 7), the total number of each of the four maintenance actions that is selected in the optimal bridge maintenance plans may be different from the total number of 50,000 Monte Carlo simulations. For example, "minor concrete repair" was selected 101,641 times, "silane treatment" was selected 88,357 times, "cathodic protection" was selected 15,060 times, and "rebuild" was selected 215 times. The corresponding probability mass functions (PMFs) are presented in Figs. 12–15 for each of the four maintenance actions. It is interesting to observe that most of the maintenance actions identified in the optimal maintenance plans from the Monte Carlo simulations are applied after the effective service life of the bridge without maintenance, i.e., 21.4 years, except for "silane treatment." The relatively low-cost "silane treatment" may be applied before the bridge exhausts its effective service life without maintenance in order to delay the application of more expensive maintenance actions. As shown in Figs. 8 and 11, almost 60% of the identified optimal maintenance plans do nothing before the bridge reaches an effective service life of 21.4 years, but increase the CI at the 21st year by either "minor concrete repair" or "rebuild." More importantly, the results from the Monte Carlo simulations in this numerical example reveal that the relative instead of absolute costs among the maintenance actions dominate the selections of the maintenance actions in the optimal maintenance plans, and the simple benefit-cost analysis may mislead optimal maintenance planning. As shown in Table 7, "rebuild" has the lowest cost to benefit ratio ($C_i/\Delta T(\text{max.})$), but it is seldom chosen in the optimal maintenance plans from the Monte Carlo simulations. The same approach is applied to the other five bridges in the highway network, and similar results are obtained.

According to Akgul and Frangopol (2003), the computed system reliabilities of these six bridges at the 21st year from the time that CI_{INI} and SI_{INI} are attained are listed in Table 5. The corresponding RIFs and their normalized values, based on the bridge network connectivity analysis (Liu and Frangopol 2005), are presented in Table 5 as well. Table 8 summarizes the example values of RIF_i, P_{ij}, and C_i for the Phase II optimization problem, which is subject to a predefined annual maintenance budget constraint of $C_{budget}=2{,}000$. It should be noted that the probability of "do nothing" is considered in Table 8. Therefore, a complete set of choices at a certain year is provided. In other words, the sum of the probabilities of the four maintenance actions and "do nothing" at a certain year should be equal to 1.0. In addition, the costs of

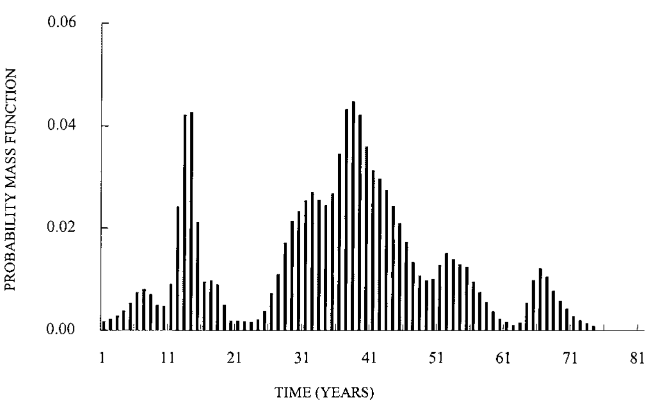

Fig. 13. PMF of "silane treatment" applied on bridge HR

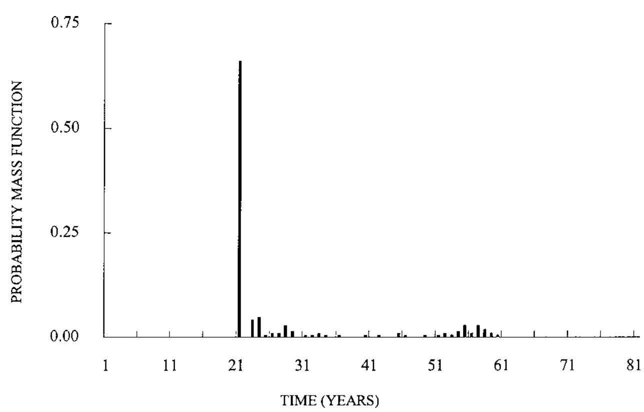

Fig. 15. PMF of "rebuild" applied on bridge HR

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Table 8. Example Values of RIF_i, P_{ij}, and C_i at 21st Year for Six Bridges in Highway Network

	Bridge HR		Bridge HE		Bridge MU		Bridge LA		Bridge DM		Bridge LE	
Maintenance action	Cost	P_{1j}	Cost	P_{2j}	Cost	P_{3j}	Cost	P_{4j}	Cost	P_{5j}	Cost	P_{6j}
Minor concrete repair	780	0.60	675	0.53	890	0.15	653	0.24	758	0.12	560	0.47
Silane treatment	98	0.02	89	0.03	76	0.32	109	0.10	67	0.38	88	0.08
Cathodic protection	550	0.04	324	0.07	320	0.28	378	0.36	260	0.20	450	0.04
Rebuild	1,650	0.02	2,300	0.05	1,890	0.04	1,432	0.02	1,260	0.05	1,900	0.05
Do nothing	0	0.32	0	0.32	0	0.21	0	0.28	0	0.25	0	0.36
Reliability importance Factor (RIF_i)	0.270		0.252		0.137		0.248		0.037		0.056	

the four maintenance actions are treated as deterministic in the Phase II optimization problem. This is because the allocations of the annual maintenance budgets usually are conducted for the near future when the variances in the maintenance costs may be neglected. As a result, the most efficient allocation of $C_{\text{budget}}=2{,}000$ is that: (1) "minor concrete repair" is applied to Bridges HR and HE; (2) "silane treatment" is applied to Bridges MU and DM; (3) "cathodic protection" is applied to Bridge LA; and (4) there are no budgets that should be spent on Bridges LE. This results in a total maintenance cost of 1976 at the 21st year.

Conclusions

This paper presented a dynamic programming procedure integrated with Monte Carlo simulations for optimal bridge network maintenance planning. Based on information provided on maintenance actions (i.e., costs and effects on safety and condition indices), an optimal bridge maintenance plan that had the minimal life-cycle maintenance costs while satisfying both minimum safety index and maximum condition index requirements was developed for each bridge in a highway network (Phase I problem). For this purpose, the computer program BMS-DP had been developed. Furthermore, a combinational optimization problem was formulated and solved by a traditional mathematical programming (Phase II problem) with a single-objective function that was related to the probability distributions of the application times of the maintenance actions obtained in the Phase I problem. The single-objective function in the Phase II problem was weighted by RIFs, which were functions of bridge system reliability profiles, bridge network reliability, and network topology (Liu and Frangopol 2005). The constraint of the optimization problem in Phase II was the limited annual maintenance budget. A numerical example was provided to demonstrate the application of the proposed stochastic DP procedure in bridge network maintenance planning. As a result, the following conclusions can be drawn from this study:

1. Bridge maintenance planning (BMP) is a time-dependent optimization problem involving sequential maintenance decisions that can be implemented at several stages during time. BMP generally has multiple objectives that need to be optimized simultaneously and balanced by decision makers;
2. The optimal maintenance policy in bridge network maintenance planning can be found by the proposed two-phase DP approach. The corresponding computer program BMS-DP has been developed;
3. The proposed stochastic DP procedure considers uncertainties in maintenance decisions and actions by using Monte Carlo simulations. Meanwhile, the allocation of the limited annual maintenance budget in the proposed approach is related to the importance of each bridge to a highway network;
4. The bridge condition and safety index profiles should be updated whenever the allocations of the maintenance budgets are required by using the proposed stochastic DP procedure. The new information on the bridge condition and safety indices may be due to the maintenance actions and/or comes from field inspection and structural health monitoring systems. Consequently, the optimal maintenance policy with the updated bridge condition and safety index profiles may be different from the one with the old information. This reflects the dynamic nature of bridge maintenance planning. It is recognized that the optimal maintenance policy must be based on the best knowledge and latest information at the time the maintenance decisions are made, with the considerations of the targeted effective service lifetime and the minimum life-cycle maintenance costs. As a result, the maintenance decisions on individual bridges in a highway network may change with time. If the decision makers consider bridge network connectivity, correlations among individual bridges have to be taken into account;
5. Based on the numerical example presented, the relative instead of absolute costs of the maintenance actions dominate the selections of the maintenance actions in the optimal maintenance plans. The relatively low-cost preventive maintenance actions such as "silane treatment" may be applied before the bridge exhausts its effective service lifetime without maintenance. In this manner, it is possible to delay the application of more expensive maintenance actions. Otherwise, expensive maintenance actions should be performed as late as possible in order to take advantage of the effects of the discount rate. Therefore, it is crucial to develop a national database system for maintenance actions on highway bridges that will provide more accurate and real-time cost information on the maintenance actions. Study work is in progress at the University of Colorado (NCHRP 14–15, 2005); and
6. Further research is needed on probabilistic bridge maintenance optimization in connection with: (1) incorporation of seismic considerations; and (2) combination of maintenance actions in both space and time. Preliminary developments may be found in Frangopol and Liu (2007), Liu and Frangopol (2006a), Neves et al. (2006), and Marsh and Frangopol (2007).

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