J. Soc. Indust. Appl. Math.
Vol. 3, No. 3, September, 1955
Printed in U.S.A.

EQUIPMENT REPLACEMENT POLICY*

RICHARD BELLMAN

1. Introduction. A problem of continuing interest in industrial economics is that of determining the optimal procedure for the replacement of old equipment by new. One approach to this problem using the conventional tools of variational analysis is contained in the interesting monograph of Alchian [1]. In this paper we wish to show how the problem may be approached by the functional equation technique of the theory of dynamic programming. An exposition of this theory and an extensive bibliography may be found in [2].

We shall first consider the simple case in which there is no technological improvement in equipment or practice, and show that this problem may be resolved quite easily. Then we shall consider the more difficult and important case where we take account of technological improvement. We shall derive the functional equation for the solution of this more realistic problem, and reserve for a further more detailed communication the analytic and computational aspects of the problem which depend, to a great extent, upon the the particular application made.

2. Description of the decision process. Let us assume that we have a piece of equipment with the operating characteristics as shown in Figure 1. Here t is the age of the machine measured in appropriate units. Furthermore, let us assume that the trade-in value of the machine as a function of time has the form shown in Figure 2, where p is the purchase price of a new machine.

Given this information, we wish to determine the age at which the old machine should be replaced by a new machine. In order to make the problem precise, we must furnish some further details. In the first place, let us assume that there is no time lag in delivery, so that a new machine is immediately available. Secondly, we must decide upon the duration of the process in advance. Do we wish to consider a five-year period, a ten-year period, or periods of even greater length?

It simplifies the analysis to consider an unending process. However, if we do this in a crude fashion, we end up with infinite returns and infinite costs. Furthermore, it makes no sense to have the far-off future play any essential role since we have too little knowledge of what is to come. One way of nullifying this ignorance is to introduce a discount factor a, where $0 < a < 1$, which measures the relative value to use of a dollar now to a dollar one time-unit hence. Economically this can be defended, and mathematically it introduces convergence.

* Received by the editors March 8, 1955.

3. Mathematical formulation. Let us now formulate in mathematical terms the problem of determining the replacement policy which maximizes the over-all discounted return.

We begin by defining

(1) $f(t)$ = overall return from a machine of age t employing an optimal replacement policy.

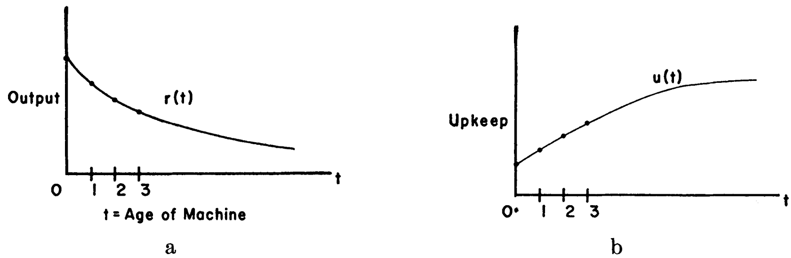

FIG. 1

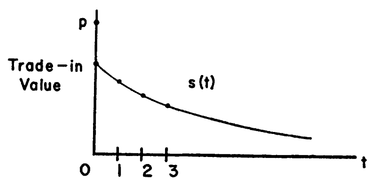

FIG. 2

At each time t, we have one of two courses of action open. We may either keep the machine for another time period, or we may purchase a new machine. In the first case, we see that $f(t)$ satisfies the equation

$$f_K(t) = r(t) - u(t) + af(t+1), \tag{2}$$

which is to say the return over the next time period plus the future discounted return, and in the second case

$$f_P(t) = s(t) - p + r(0) - u(0) + af(1), \tag{3}$$

an expression of precisely the same form.

Hence the functional equation for $f(t)$ is

$$f(t) = \text{Max}\begin{bmatrix} K\colon & r(t) - u(t) + af(t+1) \\ P\colon & s(t) - p + r(0) - u(0) + af(1) \end{bmatrix}, \tag{4}$$

since we wish to maximize our return.

4. Solution. An optimal policy will have the form: keep a new machine until it is T years old and then purchase a new one. This observation yields the following system of equations

$$(1)\qquad \begin{aligned} f(0) &= n(0) + af(1) \\ f(1) &= n(1) + af(2), \\ &\vdots \\ f(T-1) &= n(T-1) + af(T), \\ f(T) &= s(T) - p + n(0) + af(1), \end{aligned}$$

where we have set $n(t) = r(t) - u(t)$. It is clear from (3.4) that $T \neq 0$, since $p > s(0)$. Solving for $f(1)$ recurrently, we obtain

$$(2)\qquad \begin{aligned} f(1) &= n(1) + a(n(2) + af(3)) = n(1) + an(2) + a^2f(3) \\ &= n(1) + an(2) + \cdots + a^{T-2}n(T-1) + a^{T-1}f(T) \\ &= [n(1) + an(2) + \cdots + a^{T-2}n(T-1)] \\ &\quad + a^{T-1}[s(T) - p + n(0) + af(1)]. \end{aligned}$$

Hence

$$(3)\qquad f(1) = \frac{[n(1) + an(2) + \cdots + a^{T-2}n(T-1) + n(0)a^{T-1}] + a^{T-1}[s(T) - p]}{1 - a^T}.$$

Since T is to be chosen to maximize $f(0)$, and hence $f(1)$, we see that the optimal value of T is furnished by the value of T which yields the absolute maximum of the right side of (3) above for $T = 1, 2, \cdots$. Let us call this maximum $\overline{T}$.

This is a very simple computation to perform, given the preceding curves.

5. Over-age machines. Actually we do not have the complete solution to the problem, since we do not know what to do with an over-age machine, one whose age is greater than $\overline{T}$. It is not clear that in this case the optimal policy involves purchasing a new machine immediately, and, as a matter of fact, it is not unrestrictedly true. To determine the optimal procedure, we use the relation in (3.4) for $t \geqq \overline{T} + 1$. There will then be a second critical age T_2, determined very much the same way as above, at which we purchase a new machine starting with this over-age machine. It is clear that by suitably altering $n(t)$ and $s(t)$ for large values of t, we can have as many switchover points as we wish.

6. Technological improvement. Let us now consider the more realistic situation where improved operating techniques will increase the future return from the same machine, and new machines will be developed. There are now two state variables, t, the age of the machine, and τ, absolute time.

Let us define as above

(1) $f(\tau, t) =$ overall return obtained from a machine of age t at time τ, using an optimal replacement policy.

The functional equation is

$$(2) \quad f(\tau, t) = \text{Max} \begin{bmatrix} K: & r(\tau, t) - u(\tau, t) + af(\tau + 1, t + 1) \\ P: & s(\tau, t) - p + r(\tau, 0) - u(\tau, 0) + af(\tau + 1, 1) \end{bmatrix}$$

where the graphs given in §1 typify families of such graphs dependent upon the absolute time. This equation is more difficult to treat than the one above. Here the method of successive approximations based upon an initial policy space approximation will work well.

REFERENCES

1. A. Alchian, *Economic replacement policy*, RAND Report No. R-224.
2. R. Bellman, *The theory of dynamic programming*, Bull. Amer. Math. Soc., vol. 60 (1954), pp. 503-516.

Rand Corporation