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Optimal Inspection Policies: A Review and Comparison

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This paper discusses two models of inspection policies. First, the nearly optimal inspection policies are discussed by introducing the inspection density and are compared to the existing inspection policies with numerical examples. Second, the inspection model with checking time and system deterioration is discussed. The algorithms are given by using the principle of optimality to seek the optimal inspection policies minimizing the total expected discounted costs in two cases without and with renewal. The numerical examples are finally presented for illustration.

1. INTRODUCTION

In a system whose failure can be detected only by inspection, an efficient procedure for detecting its failure is desired. If frequent inspections are executed to detect the failure earlier, then the expenses for inspection increases too much. Conversely, if frequent inspections are done to decrease the expenses for inspection, the interval between the failure and its detection increases, which implies the more expenses for system down (shortage costs). Thus, the efficient method for detecting the system failure must be obtained by balancing the trade-off between the expenses for inspection and for system down. That is, we wish to obtain the optimal inspection policy which minimizes the total expected cost composed of costs for

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inspection and for system down. From this point of view, many contributions have been made to the inspection policies [1–26].

The typical inspection policy among them is discussed by Barlow *et al.* [1, 2]. Their model is the following: The system obeys an arbitrary lifetime distributed $F(t)$ with a *pdf* (probability density functions) $f(t)$, and it is inspected at a prespecified time sequence $\{t_1, t_2, t_3, \ldots\}$, where inspection is perfect and inspection time is instantaneous. The policy terminates when the inspection detects the system failure. Costs considered are one per each inspection c_c and one per unit time suffered for system down k_f. Then, the total expected cost is obtained as

$$C = \sum_{k=0}^{\infty} \int_{t_k}^{t_{k+1}} [c_c(k+1) + k_f(t_{k+1} - t)]\, dF(t). \tag{1.1}$$

They obtained an algerithm for seeking the optimal inspection time sequence which minimizes the total expected cost in (1.1) by using the recurrence formula

$$t_{k+1} - t_k = \frac{F(t_k) - F(t_{k-1})}{f(t_k)} - \frac{c_c}{k_f}, \qquad k = 1, 2, 3, \ldots, \tag{1.2}$$

where $f(t)$ is a PF_2 (Pólya frequency function of order 2) with $f(t+\Delta)/f(t)$ strictly decreasing for $t \geqslant 0$, $\Delta > 0$ and with $f(t) > 0$ for $t > 0$, and $t_0 = 0$.

However, the algorithm obtained by Barlow *et al.* [1, 2] is difficult to execute, because one must calculate the optimal inspection time sequence by applying trial and error in specifying the first inspection time, and the assumption of $f(t)$ is really restricted as described above. Several improved methods for obtaining the nearly optimal inspection time sequence were proposed. For instance, Keller [3] proposed the nearly optimal inspection policy introducing a smooth density which denotes the number of inspections per unit time, and applying the calculus of variations. Further, Kaio and Osaki [4] developed Keller's method using the smooth density (which is called *inspection density*) and obtained the more exact inspection policy. Munford and Shahani [5] presented the nearly optimal inspection policy by assuming that the conditional probability, which is the probability of the failure occurences between the successive inspections, is constant, and they applied this method to a case with Weibull distribution [6]. Further, Tadikamalla [7] discussed a case with gamma distribution by using this method. Nakagawa and Yasui [8] considered an improved method based on Barlow *et al.* [1, 2], and obtained the nearly optimal inspection policy in which the successive inspection times are computed backward assuming that an appropriate inspection time is previously given after a large number of inspections.

On the other hand, several modified models from the Barlow *et al.* one [1, 2] were considered. For instance, Wattanapanom and Shaw [9] considered the inspection model in which the system deterioration is caused by each inspection. Kaio and Osaki [4] treated the inspection models with checking time and with imperfect inspection probability.

This paper discusses two models of inspection policies: First, the nearly optimal inspection policy is discussed for the typical inspection model, and, second, the modified inspection model is discussed. Section 2 is devoted to the inspection policy using the inspection density developed by Kaio and Osaki [4]. This nearly optimal inspection policy is discussed more precisely, and compared to the Barlow *et al.* algorithm [1, 2] with numerical examples. Section 3 is devoted to the modified inspection model by Wattanapanom and Shaw [9], taking account of the time for the inspection (checking time), and obtains the optimal inspection policy with a conditional exponential lifetime distribution. Supposing the conditional exponential lifetime distribution implies that the residual lifetime of the system decreases after each inspection, i.e., the system deteriorates by each inspection. In practice, there is any checking time to inspect the system, and our model is more realistic than the model by Wattanapanom and Shaw [9]. A criterion of optimality is the total expected discounted cost of introducing an exponential type discount rate. The algorithms for seeking the optimal policies minimizing the expected costs are given by applying the *principle of optimality*. Subsection 3.1 discusses a case that when a system fails, the system does not renew, i.e., the policy terminates when the system failure is detected. Subsection 3.2 discusses a case that a system renews when the system fails, i.e., the replacement or the repair is done and its operation is taken over, when the system failure is detected. The numerical examples are presented for each case.

2. Nearly Optimal Inspection Policies Using Inspection Density

Nearly Optimal Inspection Procedure

The inspection model and the notation mentioned above follow Barlow *et al.* [1, 2]. Further, introduce the inspection density at time t, $n(t)$, which is a smooth function and denotes the approximate number of inspections per unit time at time t. Then, the approximately total expected cost to the detection of the system failure is

$$C(n(t)) = c_c \int_0^\infty n(t)\, \bar{F}(t)\, dt + k_f \int_0^\infty 1/[2n(t)]\, dF(t), \tag{2.1}$$

where $\bar{\psi} = 1 - \psi$, in general. The inspection density $n(t)$, minimizing the functional $C(n(t))$ in (2.1), is obtained as

$$n(t) = [k_c r(t)]^{1/2}, \tag{2.2}$$

where $k_c = k_f/(2c_c)$, and $r(t) = f(t)/\bar{F}(t)$, a failure rate.

On the other hand, if the inspection density $n(t)$ is introduced, the inspection time sequence $\{t_1, t_2, t_3, \ldots\}$ satisfies the following equation, in general:

$$i = \int_0^{t_i} n(t)\,dt; \qquad i = 1, 2, 3, \ldots. \tag{2.3}$$

Substituting $n(t)$ in (2.2) into Eq. (2.3) yields the nearly optimal inspection time sequence. For details, see Kaio and Osaki [4].

The nearly optimal inspection time sequences are obtained for a Weibull or a gamma lifetime distribution, in the following. For the numerical examples, when $F(t_N) \geqslant 99.99\%$ for the first time, the inspection time t_N is the final one.

Numerical Examples with Weibull Distribution

Discuss a case that the lifetime obeys a Weibull distribution, i.e.,

$$F(t) = 1 - \exp\left[-\left(\frac{t}{\eta}\right)^m\right]; \qquad \eta, m > 0. \tag{2.4}$$

Then,

$$t_i = \left(\frac{i(m+1)}{2K_a}\right)^{2/m+1}; \qquad i = 1, 2, 3, \ldots, \tag{2.5}$$

where

$$K_a = \left(\frac{k_c m}{\eta^m}\right)^{1/2}. \tag{2.6}$$

Table I shows the numerical results with $c_c = 20$, $k_f = 1$, $\eta = 400$, and $m = 2$, including the optimal inspection time sequences obtained from the Barlow *et al.* algorithm [1, 2]. Several optimal results from the Barlow *et al.* algorithm can be obtained since t_N is the final inspection time when $F(t_N) \geqslant 99.99\%$ for the first time. The two results are presented with the smallest t_1 and the largest one, in which the former policy is regarded as the optimal one since its total expected cost is the smallest among several numerical results. On the other hand, the nearly optimal inspection time sequence is

TABLE I

Optimal and Nearly Optimal Inspection Time Sequences, Their Total Expected Costs and the Sum of Relative Errors ($F(t) = 1 - \exp[-(t/\eta)^m]$, $c_i = 20$, $k_f = 1$, $\eta = 400$, and $m = 2$)

t_i	Barlow *et al.* optimal policy		Nearly optimal policy using inspection density
t_1	220.1561	220.1649	193.0979
t_2	328.7263	328.7419	306.5238
t_3	418.5534	418.5779	401.6598
t_4	498.1838	498.2209	486.5762
t_5	571.0243	571.0809	564.6216
t_6	638.8717	638.9587	637.5951
t_7	702.8173	702.9539	706.6042
t_8	763.5815	763.8007	772.3915
t_9	821.6620	822.0220	835.4860
t_{10}	877.4039	878.0087	896.2810
t_{11}	931.0281	932.0666	955.0790
t_{12}	982.6276	984.4475	1012.1192
t_{13}	1032.1257	1035.3739	1067.5947
t_{14}	1079.1761	1085.0658	1121.6642
t_{15}	1122.9674	1133.7805	1174.4603
t_{16}	1161.8882	1181.8814	1226.0951
t_{17}	1193.0697	1229.9783	—
t_{18}	1212.1220	—	—
t_{19}	1214.0096	—	—
Total expected cost	115.6053	115.6146	116.3844
Sum of relative errors			
For total expected cost	0	0.0001	0.0067
For inspection time sequence	0	0.0714[a]	0.5533[b]

[a] Sum of the relative errors from t_1 to t_{17}.
[b] Sum of the relative errors from t_1 to t_{16}.

specified uniquely, when the inspection procedure using the inspection density is applied. From the results of Table I, is is recognized that the nearly optimal inspection time sequence by the inspection procedure using the inspection density approximates sufficiently to the optimal one from the Barlow *et al.* algorithm, since the sum of relative errors for the inspection time sequence is sufficiently small (0.5533) and the difference between the total expected costs is also sufficiently small (0.0067).

For $m=1$, $\eta=100$, $c_c=20$, and $k_f=1$, the nearly optimal inspection time sequences are $t_i=63.2456 \cdot i$ ($i=1, 2, 3, \ldots, 15$) while the total expected cost is 77.5756, where the inspection procedure using the inspection density is applied.

Further, Table II shows the nearly optimal inspection time sequence by applying the inspection density for $m=0.5$, $\eta=10$, $c_c=20$, and $k_f=1$.

Numerical Examples with Gamma Distribution

A case that the lifetime obeys a gamma distribution is discussed, i.e.,

$$F(t)=\int_0^t \exp(-\gamma\tau)\,\gamma(\gamma\tau)^{m-1}/(m-1)!\,d\tau; \qquad \gamma>0, \tag{2.7}$$

where m is a positive integer.

TABLE II

Nearly Optimal Inspection Time Sequence and Total Expected Cost
($F(t)=1-\exp[-(t/\eta)^m]$, $c_c=20$, $k_f=1$, $\eta=20$, and $m=0.5$)

t_i	Nearly optimal policy usng inspection density
t_1	27.2568
t_2	68.6829
t_3	117.9334
t_4	173.0700
t_5	233.0424
t_6	297.1735
t_7	364.9828
t_8	436.1089
t_9	510.2680
t_{10}	587.2302
t_{11}	666.8046
t_{12}	748.8302
t_{13}	833.1684
t_{14}	919.6990
Total expected cost	51.9545

TABLE III

Optimal and Nearly Optimal Inspection Time Sequences, Their Total Expected Costs, and the Sum of Relative Errors ($F(t) = \int_0^t \exp(-\gamma\tau)\,\gamma(\gamma\tau)^{m-1}/(m-1)!\,d\tau$, $c_c = 20$, $k_f = 1$, $\gamma = 0.01$, and $m = 2$)

t_i	Barlow *et al.* optimal policy		Nearly optimal policy using inspection density
t_1	122.9348	122.9400	113.9234
t_2	199.7056	199.7171	195.3928
t_3	270.1785	270.1996	271.1011
t_4	337.6078	337.6446	343.9661
t_5	403.1867	403.2492	415.0951
t_6	467.4990	467.6043	485.0500
t_7	530.8723	531.0494	554.1427
t_8	593.5015	593.7996	622.5764
t_9	655.4973	656.0008	690.4889
t_{10}	716.9039	717.7567	757.9780
t_{11}	777.6961	799.1447	825.1161
t_{12}	837.7579	840.2249	891.9581
t_{13}	896.8379	901.0459	958.546
t_{14}	954.4683	961.6498	1024.9164
t_{15}	1009.8355	1022.0767	1091.0943
t_{16}	1061.5845	1082.3691	1157.1030
t_{17}	1107.5717	1142.5798	1222.9615
t_{18}	1144.6542	1202.7826	—
t_{19}	1168.7750	—	—
t_{20}	1175.7609	—	—
Total expected cost	95.4186	95.4287	95.7588
Sum of relative errors			
For total expected cost	0	0.0001	0.0036
For inspection time sequence	0	0.1346[a]	0.9306[b]

[a] Sum of the relative errors from t_1 to t_{18}.
[b] Sum of the relative errors from t_1 to t_{17}.

Table III shows the nearly optimal inspection time sequence using the inspection density and the optimal inspection time sequences from the Barlow *et al.* algorithm which are for the smallest t_1 and the largest one, for $c_c = 20$, $k_f = 1$, $\gamma = 0.01$, and $m = 2$. From the results of Table III, it is concluded that the nearly optimal inspection time sequence using the inspection density approximates sufficiently to the optimal one from the Barlow *et al.* algorithm, in a similar fashion as a case with Weibull distribution above.

Remarks

The following are the merits of the nearly optimal inspection policy using the inspection density, developed by Kaio and Osaki [4]:

(1) Once the failure rate $r(t)$ is obtained, the nearly optimal inspection time sequence is obtained uniquely, immediately, and easily, from the formulae (2.2) and (2.3), as shown in the numerical examples. Especially, this procedure can obtain the nearly optimal inspection time sequences for any distributions, while the Barlow *et al.* algorithm cannot treat without the PF_2 distribution, i.e., the examples of the *non*-PF_2 distribution are the Weibull distributions with $1 \geqslant m > 0$ as shown in Table II.

(2) The nearly optimal inspection time sequence, obtained easily by this procedure using the inspection density, approximates sufficiently to the optimal one, as shown in the numerical examples.

(3) The more complicated models can be analyzed and their nearly optimal inspection time sequences can be easily obtained, e.g., the inspection model with imperfect inspection probability [4], if the inspection density is applied.

3. Inspection Model with Checking Time and System Deterioration

3.1. *Inspection Model without Renewal*

Model and Assumptions

Consider a one-unit system, where the system failure can be detected only by an inspection with a checking time and this policy terminates with the detection of the failure. The system begins operation at age t_0 for the first time, and is inspected at age t_k ($k = 1, 2, 3, \ldots$). The system is stopped when it is inspected, and takes over operation again if it does not fail. The system is deteriorated by each inspection. To describe the deterioration of the system, assume the conditional exponential lifetime distribution, which is given by densities as follows (see Wattanapanom and Shaw [9]): for $k = 0, 1, 2, \ldots$,

$$\text{(i)}\quad f(t \mid t_k < t) = \lambda_k \exp[-\lambda_k(t - t_k)]; \qquad t_k < t, \tag{3.1}$$

$$\text{(ii)}\quad f(t \mid t_k < t \leqslant t_{k+1}) = \lambda_k \exp[-\lambda_k(t - t_k)]/[1 - \exp(-\lambda_k d_k)]; \qquad t_k < t \leqslant t_{k+1}, \tag{3.2}$$

where $\psi(\cdot \mid \cdot)$ is the conditional characteristic and condition is given on the right, in general. Since each inspection causes the system deterioration, the residual lifetime decreases after each inspection, and tends to zero as the number of inspections tends to infinity. Thus,

$$\lim_{k \to \infty} \lambda_k = \infty. \tag{3.3}$$

Further, introduce $H(t)$ as a cumulative distribution function of checking time.

The costs considered here are the following: a cost k_c per unit time is suffered for inspection, a cost c_c is associated with each inspection and is suffered when each inspection begins, a return k_r per unit time is earned by the system and defends too frequent inspections. A cost k_f is the same as in the preceding sections. Introduce a continuous (exponential) type discount rate $\alpha(>0)$, where a unit of cost is discounted $\exp(-\alpha t)$ after a time interval t. The planning horizon is infinite.

Optimal Inspection Policy

From the principle of optimality (see Bellman and Dreyfus [27]), the minimum future total expected discounted cost when the system begins operation at age t_k ($k = 0, 1, 2, \ldots$) is as follows:

$$C_r^0(t_k) = \min_{d_k} \left[e^{-(\alpha + \lambda_k) d_k} \left\{ C_r^0(t_{k+1})\, H^*(\alpha) + \frac{k_f + k_r}{\alpha + \lambda_k} \right\} + e^{-\alpha d_k} \left(k_c \frac{\bar{H}^*(\alpha)}{\alpha} + c_c - \frac{k_f}{\alpha} \right) - \frac{k_r}{\alpha} + (k_f + k_r) \frac{\lambda_k}{\alpha(\alpha + \lambda_k)} \right];$$

$$k = 0, 1, 2, \ldots,$$

where $d_k = t_{k+1} - t_k$ and is the time interval between the end of kth inspection and the beginning of $(k+1)$st one. Assume

$$(\alpha + \lambda_k)\, C_r^0(t_{k+1})\, H^*(\alpha) + k_f + k_r \geqslant k_f - \alpha c_c - k_c \bar{H}^*(\alpha) > 0, \tag{3.5}$$

where $\psi^*(\alpha)=\int_0^\infty \exp(-\alpha t)\, d\psi(t)$, in general. Thus, optimal d_k $(k=0, 1, 2,\ldots)$, which minimizes the right-hand side in the formula (3.4), and $C_r^0(t_k)$, which is substituted d_k^0 (optimal d_k), are as follows:

$$d_k^0=\frac{1}{\lambda_k}\ln\frac{(\alpha+\lambda_k)\,C_r^0(t_{k+1})\,H^*(\alpha)+k_f+k_r}{k_f-\alpha c_c-k_c\bar{H}^*(\alpha)};\qquad k=0, 1, 2,\ldots, \tag{3.6}$$

$$C_r^0(t_k)=\frac{\lambda_k}{\alpha+\lambda_k}\left[e^{-\alpha d_k^0}\left(k_c\frac{\bar{H}^*(\alpha)}{\alpha}+c_c-\frac{k_f}{\alpha}\right)-\frac{k_r}{\lambda_k}+\frac{k_f}{\alpha}\right];\qquad k=0, 1, 2,\ldots. \tag{3.7}$$

The following theorem is given when $k\to\infty$.

THEOREM 3.1. *When $k\to\infty$, d_k^0 and $C_r^0(t_k)$ are as follows*:

(1) $$\lim_{k\to\infty} k_d^0=0, \tag{3.8}$$

(2) $$\lim_{k\to\infty} C_r^0(t_k)=k_c\frac{\bar{H}^*(\alpha)}{\alpha}+c_c. \tag{3.9}$$

Now, give the algorithm for seeking d_k^0 $(k=0, 1, 2,\ldots)$ and $C_r^0(t_k)$ from the formulae (3.6) and (3.7) and Theorem 3.1 as follows: In advance, give the maximum inspection number sequence $\{N_0, N_1, N_2,\ldots\}$, whose elements are increasing and positive integers, and where N_0 is the number for the initiation. Assume

$$C_r^0(t_{N_m})=k_c\frac{\bar{H}^*(\alpha)}{\alpha}+c_c;\qquad m=1, 2, 3,\ldots. \tag{3.10}$$

There is no general procedure in presenting the sequence $\{N_0, N_1, N_2,\ldots\}$, because the appropriate procedures are dependent on parameters. The following examples are calculated as $N_0=1$, $N_1=11$, $N_2=21$, $N_3=31,\ldots$. For simplicity of the notation, put d_k^0 and $C_r^0(t_k)$ for the maximum inspection number N_m $(m=0, 1, 2,\ldots)$ to $d_m[k]$ and $C_m[k]$, respectively. The algorithm ends if $|d_m[k]-d_{m-1}[k]|<\delta$ for all $k=0, 1, 2,\ldots, I-1<N_{m-1}$ satisfying

$$1-\Pr\{t\leqslant t_I\}=\exp\left(-\sum_{k=0}^{I-1} d_{m-1}[k]\,\lambda_k\right)<\varepsilon, \tag{3.11}$$

where ε and δ are preassigned as the sufficiently small and positive real numbers, and we obtain the optimal $d_{m-1}[k]$ $(k=0, 1, 2,\ldots, N_{m-1}-1)$ and $C_{m-1}[k]$.

ALGORITHM 3.1.

begin
  $m \leftarrow 0$; for $k \leftarrow 0$ to $N_m - 1$ $d_m[k] \leftarrow 0$;
  repeat
    $m \leftarrow m + 1$
    compute $C_m[N_m]$ using formula (3.10);
    for $k \leftarrow N_m - 1$ to 0 step -1
      compute $d_m[k]$ using formula (3.6)
        and $C_m[k]$ using formula (3.7);
    choose I such that $I \leqslant N_{m-1}$ and

$$\exp\left(-\sum_{k=0}^{I-1} d_{m-1}[k]\,\lambda_k\right) < \varepsilon;$$

  until $(\forall k[0 \leqslant k < I \Rightarrow |d_m[k] - d_{m-1}[k]| < \delta])$
end;

Numerical Examples

Obtain the optimal inspection policies using Algorithm 3.1. Assume that $H(t)$ is a gamma distribution with a shape parameter 2; i.e.,

$$H(t) = 1 - (1 + \gamma t)\exp(-\gamma t); \qquad \gamma > 0, \tag{3.12}$$

and

$$\lambda_k = \lambda_0(1 + k); \qquad k = 0, 1, 2, \ldots. \tag{3.13}$$

Further, put $k_c = 1$, $c_c = 1$, $k_f = 20$, $k_r = 5$, $\lambda_0 = 1$, $\alpha = 0.1$, $\gamma = 20$, $N_0 = 1$, and $N_1 = 11$. When $N_2 = 21$ and $N_3 = 31$, d_i^0's $(0 \leqslant i < 15)$ for N_2 and N_3 are sufficiently near and $\Pr\{t \leqslant t_{15}\} = 0.9999+$ is sufficiently high. Thus, the number of inspections is regarded as 21 times. The results of d_k^0 $(k = 0, 1, 2, \ldots, 20)$ and $C_r^0(t_k)$ are presented in Table IV. Also, the optimal inspection policy in the case that

$$\lambda_k = \lambda_0/\rho^k; \qquad k = 0, 1, 2, \ldots \tag{3.14}$$

is presented in Table V.

3.2. *Inspection Model with Renewal*

Model and Assumptions

Treat the inspection model with renewal, i.e., when the system failure is detected, the replacement or the repair of the system is executed with any

TABLE IV

k	λ_k	d_k^0	$C_r^0(t_k)$
20	21	0.0419	1.6851
19	20	0.0540	1.9103
18	19	0.0591	1.9974
17	18	0.0621	2.0409
16	17	0.0645	2.0719
15	16	0.0669	2.1002
14	15	0.0695	2.1289
13	14	0.0723	2.1593
12	13	0.0754	2.1918
11	12	0.0789	2.2269
10	11	0.0828	2.2650
9	10	0.0874	2.3063
8	9	0.0926	2.3513
7	8	0.0988	2.4005
6	7	0.1063	2.4540
5	6	0.1157	2.5120
4	5	0.1278	2.5736
3	4	0.1445	2.6359
2	3	0.1697	2.6893
1	2	0.2152	2.6998
0	1	0.3399	2.4958

Note. d_k^0 ($k=0, 1, 2,..., 20$) and $C_r^0(t_k)$ ($\lambda_k = \lambda_0(1+k)$, $H(t) = 1-(1+\gamma t)\exp(-\gamma t)$, $k_\iota = 1$, $c_\iota = 1$, $k_f = 20$, $k_r = 5$, $\lambda_0 = 1$, $\alpha = 0.1$, and $\gamma = 20$).

time and the system takes over its operation. $G(t)$ is introduced as the cumulative distribution function of replacement or repair time, and a cost k_m per unit time is considered to be suffered for replacement or repair. The system is renewed as before. An interval from a renewal to the following renewal is defined as one cycle, and $\mathbf{d} = (d_0, d_1, d_2,...)$ is repeated every time the system renews, which is a vector with d_k ($k = 0, 1, 2,...$) in one cycle as elements. For the others, see Model and Assumptions in Subsection 3.1.

Optimal Inspection Policy

When $K(\mathbf{d})$ is defined as the total expected discounted cost per one cycle and $u(\mathbf{d})$ as the discounted unit cost just after one cycle, the the total expected discounted cost when the system begins operation at time 0 is obtained as follows (see Fox [28]):

$$C_T(\mathbf{d}) = K(\mathbf{d}) + \sum_{i=1}^{\infty} [u(\mathbf{d})]^i K(\mathbf{d})$$
$$= K(\mathbf{d})/\bar{u}(\mathbf{d}). \tag{3.15}$$

TABLE V

k	λ_k	d_k^0	$C_r^0(t_k)$
20	8.2253	0.0654	1.7664
19	7.4027	0.0879	2.1353
18	6.6625	0.1022	2.3362
17	5.9962	0.1127	2.4540
16	5.3966	0.1217	2.5314
15	4.8569	0.1301	2.5882
14	4.3712	0.1386	2.6333
13	3.9341	0.1474	2.6703
12	3.5407	0.1566	2.7007
11	3.1866	0.1663	2.7248
10	2.8680	0.1766	2.7421
9	2.5812	0.1876	2.7519
8	2.3231	0.1993	2.7533
7	2.0908	0.2118	2.7453
6	1.8817	0.2252	2.7265
5	1.6935	0.2395	2.6957
4	1.5242	0.2549	2.6513
3	1.3717	0.2714	2.5917
2	1.2346	0.2892	2.5152
1	1.1111	0.3083	2.4200
0	1.0000	0.3289	2.3040

Note. d_k^0 ($k = 0, 1, 2, \ldots, 20$) and $C_r^0(t_k)$ ($\lambda_k = \lambda_0 \cdot \rho^k$, $H(t) = 1 - (1 + \gamma t)\exp(-\gamma t)$, $k_c = 1$, $c = 1$, $k_f = 20$, $k_r = 5$, $\lambda_0 = 1$, $\rho 0.9$, $\alpha = 0.1$, and $\gamma = 20$).

Put that $J(\mu, \mathbf{d}) = K(\mathbf{d}) - \mu\bar{u}(\mathbf{d})$ and $\mathbf{d}(\mu)$ is the optimal $\mathbf{d}$ which minimizes $J(\mu, \mathbf{d})$ for any μ, i.e., $\min_{\mathbf{d}} J(\mu, \mathbf{d}) = J(\mu, \mathbf{d}(\mu))$. Then, there exists the following relationship between $J(\mu, \mathbf{d})$ and $C_T(\mathbf{d})$.

THEOREM 3.2. *When $J(\mu^*, \mathbf{d}(\mu^*)) = 0$ for any μ^*, $\mathbf{d}(\mu^*)$ minimizes $C_T(\mathbf{d})$ also. Then,*

$$C_T(\mathbf{d}(\mu^*)) = \mu^*. \tag{3.16}$$

The next theorem is also given, where $K^0(\mathbf{d}) = \min_{\mathbf{d}} K(\mathbf{d})$.

THEOREM 3.3. *When $K^0(\mathbf{d}) \geqslant 0$, there exists a finite μ^* ($0 \leqslant \mu^* < \infty$) satisfying $J(\mu^*, \mathbf{d}(\mu^*)) = 0$.*

Thus, the next problem is that $\mathbf{d}(\mu)$ is obtained. From the principle of optimality, the function corresponding to $J(\mu, \mathbf{d}(\mu))$ when the system begins operation at age t_k ($k = 0, 1, 2, \ldots$) in the first cycle is as follows:

$$J^0(t_k)=\min_{d_k}\left[e^{-(\alpha+\lambda_k)d_k}\left\{(J^0(t_{k+1})-k_m\frac{\bar{G}^*(\alpha)}{\alpha}+\mu\bar{G}^*(\alpha))\,H^*(\alpha)\right.\right.$$
$$\left.+\frac{k_f+k_r}{\alpha+\lambda_k}\right\}+e^{-\alpha d_k}\left\{k_c\frac{\bar{H}^*(\alpha)}{\alpha}+c_c-\frac{k_f}{\alpha}+k_m\frac{H^*(\alpha)\,\bar{G}^*(\alpha)}{\alpha}\right.$$
$$\left.\left.+\mu H^*(\alpha)\,G^*(\alpha)\right\}-\frac{k_r}{\alpha}+(k_f+k_r)\frac{\lambda_k}{\alpha(\alpha+\lambda_k)}-\mu\right];\qquad k=0,1,2,..., \tag{3.17}$$

where $J^0(t_0)=J(\mu,\mathbf{d}(\mu))$, and this is the function corresponding to $C_r^0(t_k)$ in Subsection 3.1. Assume the following just corresponding to the formula (3.5):

$$(\alpha+\lambda_k)(J^0(t_{k+1})-k_m\bar{G}^*(\alpha)/\alpha+\mu\bar{G}^*(\alpha))\,H^*(\alpha)+k_f+k_r$$
$$\geqslant k_f-(k_c\bar{H}^*(\alpha)+k_mH^*(\alpha)\,\bar{G}^*(\alpha))-\alpha(c_c+\mu H^*(\alpha)\,G^*(\alpha))>0. \tag{3.18}$$

Thus, optimal d_k ($k=0,1,2,...$), which minimizes the right-hand side in the formula (3.17), and $J^0(t_k)$ corresponding to it are as follows:

$$d_k^0=\frac{1}{\lambda_k}$$
$$\times\ln\frac{(\alpha+\lambda_k)(J^0(t_{k+1})-k_m\bar{G}^*(\alpha)/\alpha+\mu\bar{G}^*(\alpha))\,H^*(\alpha)+k_f+k_r}{k_f-(k_c\bar{H}^*(\alpha)+k_mH^*(\alpha)\,\bar{G}^*(\alpha))-\alpha(c_c+\mu H^*(\alpha)\,G^*(\alpha))};$$
$$k=0,1,2,..., \tag{3.19}$$

$$J^0(t_k)=\frac{\lambda_k}{\alpha+\lambda_k}\left[e^{-\alpha d_k^0}\left\{k_c\frac{\bar{H}^*(\alpha)}{\alpha}+c_c-\frac{k_f}{\alpha}+k_m\frac{H^*(\alpha)\,\bar{G}^*(\alpha)}{\alpha}\right.\right.$$
$$\left.\left.+\mu H^*(\alpha)\,G^*(\alpha)\right\}-\frac{k_r}{\lambda_k}+\frac{k_f}{\alpha}-\mu\frac{\alpha+\lambda_k}{\lambda_k}\right];\qquad k=0,1,2,.... \tag{3.20}$$

Thus, obtain the theorem corresponding to Theorem 3.1, when $k\to\infty$.

THEOREM 3.4. *When $k\to\infty$, d_k^0 and $J^0(t_k)$ are as follows*:

(1) $$\lim_{k\to\infty} d_k^0=0, \tag{3.21}$$

(2) $$\lim_{k\to\infty} J^0(t_k)$$
$$=k_c\frac{\bar{H}^*(\alpha)}{\alpha}+c_c+k_m\frac{H^*(\alpha)\,\bar{G}^*(\alpha)}{\alpha}+\mu H^*(\alpha)\,G^*(\alpha)-\mu. \tag{3.22}$$

Now, give the algorithm for seeking d_k^0 ($k=0, 1, 2,\ldots$) and $J^0(t_k)$ from the formulae (3.19) and (3.20) and Theorem 3.4 as follows: Prespecify the maximum inspection number sequence $\{N_0, N_1, N_2,\ldots\}$, whose elements are increasing and positive integers, and $\mu_{\max}$. See Algorithm 3.1. for the sequence $\{N_0, N_1, N_2,\ldots\}$, and $\mu_{\max}$ is a sufficiently large and positive value (e.g., $\mu_{\max}=100$). Assume

$$J^0(t_{N_m}) = k_c\frac{\bar{H}^*(\alpha)}{\alpha} + c_c + k_m\frac{H^*(\alpha)\,\bar{G}^*(\alpha)}{\alpha} + \mu H^*(\alpha)\,G^*(\alpha) - \mu;$$
$$m = 1, 2, 3,\ldots. \tag{3.23}$$

Put d_k^0 and $J^0(t_k)$ for the number N_m ($m=0, 1, 2,\ldots$) to $d_m[k]$ and $J_m[k, \mu]$, respectively, in a similar fashion to Algorithm 3.1, where $J_m[k, \mu]$ is the function of μ also. Prespecify ε and δ as the sufficiently small and positive real numbers.

ALGORITHM 3.2.

```
begin
  m ← 0; for k ← 0 to N_m − 1 d_m[k] ← 0;
  repeat
    m ← m + 1;
    μ_min ← 0;
    μ ← ½μ_max;
    i ← ½μ_max;
    while J_m[0, μ] ≠ 0 do
      begin i ← ½i;
      if J_m[0, μ] J_m[0, μ_min] > 0
      then begin μ_min ← μ;
                 μ ← μ + i
           end
      else       μ ← μ − i
    end;
    choose I such that I ≤ N_{m−1} and

              exp( − Σ_{k=0}^{I−1} d_{m−1}[k] λ_k ) < ε;

  until (∀k[0 ≤ k < I ⇒ |d_m[k] − d_{m−1}[k]| < δ])
end;
function J_m[0, μ];
  begin compute J_m[N_m, μ] using formula (3.23);
    for k ← N_m − 1 to 0 step −1
      compute d_m[k] using formula (3.19) and
              J_m[k, μ] using formula (3.20);
  end;
```

Numerical Examples

The optimal inspection policies using Algorithm 3.2 are obtained. Assume that $H(t)$ and λ_k $(k=0, 1, 2,...)$ are given by the formulae (3.12) and (3.13), respectively, and $G(t)$ is a gamma distribution with a shape parameter 3, i.e.,

$$G(t)=\int_0^t \exp(-\beta\tau)\,\beta(\beta\tau)^2/2d\tau; \qquad \beta>0. \tag{3.24}$$

Further, put $k_c=1$, $c_c=1$, $k_f=20$, $k_r=5$, $k_m=1$, $\lambda_0=1$, $\alpha=0.1$, $\gamma=20$, and $\beta=10$. Then, $\mu=23.8564$ and the number of inspections are obtained as 21 times, in which the optimal inspection policy is presented in Table VI. Also, when λ_k $(k=0, 1, 2,...)$ is given by the formula (3.14) with $\rho=0.9$, then $\mu=17.8100$.

Remarks

For the costs and the parameter of distribution, denote the following: As the shortage cost k_f increases, the interval between the inspections

TABLE VI

k	λ_k	d_k^0	$J^0(t_k)$
20	21	0.0427	0.8525
19	20	0.0524	1.0024
18	19	0.0568	1.0592
17	18	0.0595	1.0851
16	17	0.0618	1.1012
15	16	0.0642	1.1142
14	15	0.0667	1.1263
13	14	0.0694	1.1382
12	13	0.0724	1.1499
11	12	0.0759	1.1615
10	11	0.0798	1.1726
9	10	0.0843	1.1827
8	9	0.0895	1.1911
7	8	0.0958	1.1966
6	7	0.1036	1.1971
5	6	0.1134	1.1892
4	5	0.1265	1.1662
3	4	0.1451	1.1152
2	3	0.1747	1.0064
1	2	0.2320	0.7568
0	1	0.4022	0.0000

Note. d_k^0 $(k=0, 1, 2,..., 20)$ and $J^0(t_k)$ ($\lambda_k=\lambda_0(1+k)$, $H(t)=1-(1+\gamma t)\exp(-\gamma t)$, $G(t)=\int_0^t \exp(-\beta\tau)\,\beta(\beta\tau)^2/2d\tau$, $k_c=1$, $c_c=1$, $k_f=20$, $k_r=5$, $k_m=1$, $\lambda_0=1$, $\alpha=0.1$, $\gamma=20$, $\beta=10$, and $\mu=23.8564$).

decreases, since if the cost k_f is expensive and the detection of the system failure is late, then the cost for the system down is more expensive. Conversely, as the return k_r increases, the interval between the inspections increases, since fewer inspections cause many returns. On the other hand, as λ_k $(k = 0, 1, 2, \ldots)$ increases, the interval between the inspections decreases, since the opportunity of the system failure increases.

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