Comparison of Age, Block, and Failure Replacement Policies

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Key Words—Age, Block, Failure replacement policies.

Reader Aids—
Purpose: Widen state of the art, tutorial.
Special math needed: Renewal theory.
Results useful to: Reliability analysts and engineers.

Abstract—Two widely used preventive replacement policies are the age replacement policy (ARP) and the block replacement policy (BRP). Another replacement policy is the failure replacement policy (FRP) in which no preventive replacements are made at all. In this paper we give a rule for choosing the least costly of the above three policies under conditions specified in the paper. The implementation of this rule is illustrated for two special cases, where the distribution of item life times is uniform, or 2-stage Erlang.

1. INTRODUCTION AND DESCRIPTION OF SOME REPLACEMENT POLICIES

For stochastically failing items, a policy of preventive replacement, i.e., planned replacement of an item while still serviceable, can forestall the expense incurred by its failure. Such a policy obviously entails the loss of the unexpended life of the item being replaced. The problem is to strike a balance between this loss and the saving effected by preventing failures, with a view to minimizing the overall cost.

The three principal types of replacement policies discussed in the literature are age replacement (ARP), block replacement (BRP), and the non-preventive approach known as failure replacement (FRP) under which action is taken only after failure has actually occurred. All three policies contain an element of failure replacement, whose cost can be standardized to be one unit. As for the planned replacement cost it may well be different for the two preventive replacement policies. The aim of this paper is to provide a simple rule for choosing the least costly of the three policies for any possible values of the planned replacement costs.

NOTATION

$G(\cdot)$, $\overline{G}(\cdot)$, $g(\cdot)$, $r(\cdot)$ — Cdf, Sf, pdf, and failure rate for time to failure distribution. For convenience we assume that g exists.

$M(t)$ — Renewal function associated with $G(\cdot)$

$m(t)$ — $dM(t)/dt$, the renewal density associated with $G(\cdot)$

ARP — Age Replacement Policy

BRP — Block Replacement Policy

FRP — Failure Replacement Policy

$A(a,x)$, $(B(b,t))$ — the *s*-expected cost per unit time in the long run for ARP (BRP) with age limit x (block interval t) and planned replacement cost $a(b)$

F — the *s*-expected cost per unit time in the long run for FRP. The cost of a failure replacement is standardized to 1, so that $a<1$ and $b<1$

$*$ — symbol used to denote an optimal value or evaluation of a function at an optimal value

1.1 FRP

Under FRP, the average cost per item per unit time in the long run is according to the elementary renewal theorem [1, p 55]

$$F = 1/\mu \tag{1}$$

where μ is the *s*-expected life of the item.

1.2 ARP

Under ARP an age limit x is set in advance, with the item being replaced on failure or on reaching this limit, whichever comes first. The same procedure is repeated for all succeeding replacement units. From [1, p 87] we have

$$A(a,x) = [G(x) + a\overline{G}(x)]/\int_0^x \overline{G}(u)du \tag{2}$$

Differentiating (2) yields the result that the optimal age $x^*(a)$, at which $A(a,x)$ attains its global minimum, must satisfy

$$r(x^*) \int_0^{x^*} \overline{G}(u)du - G(x^*) = a/(1-a). \tag{3}$$

A sufficient condition for x^* to be unique is that $r(x)$ is an increasing function. If no finite solution of (3) exists then $x^*=x^*(a)\to\infty$, i.e., the optimal ARP becomes FRP. Equations (2) and (3) yield the minimum cost for an ARP.

$$A^*(a) \equiv A(a,x^*) = (1-a)r(x^*). \tag{4}$$

We now observe that $A^*(a)$ is an increasing function of a. To see this let $a_1 < a_2$ be two planned replacement costs and let x_i^* be the optimal age limit associated with a_i $(i=1,2)$. From the definition of x_i^* it follows that

$$A^*(a_1) = A(a_1,x_1^*) \leq A(a_1,x_2^*). \tag{5}$$

On the other hand, assuming $x_2^* < \infty$, we also have

$$A(a_1,x_2^*) < A(a_2,x_2^*) = A^*(a_2)$$

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since $A(a,x)$ is increasing in a for any fixed $x<\infty$. Thus $A^*(a_1) < A^*(a_2)$ if $a_1<a_2$, which completes the proof.

If $G(x)$ possesses a second derivative, one obtains

$$dA^*(a)/da = r(x^*)/[((1-a)\bar{G}(x^*))^{-1}-1] > 0. \tag{6}$$

A consequence of the above is that a one to one correspondence exists between a and $A^*(a)$, but not necessarily between a and $x^*(a)$.

1.3 BRP

Under BRP, items are replaced on failure and preventive replacements are carried out at prescribed times t, $2t$, $3t$,..., irrespective of the failure history. Here the average cost per item per unit time over an infinite time horizon [1, p 95] is

$$B(b,t) = [M(t) + b]/t. \tag{7}$$

Differentiating (7) yields the condition to be satisfied by an optimal time interval $t^*(b)$, at which $B(b,t)$ attains its global minimum

$$t^*m(t^*) - M(t^*) = b. \tag{8}$$

A sufficient condition for t^* to be unique is that $m(t)$ is increasing with t. Again, if no finite solution of (8) exists then $t^*=t^*(b)\rightarrow\infty$, and the optimal BRP becomes FRP. Equations (7) and (8) yield the minimum cost for a BRP

$$B^*(b) \equiv B(b,t^*) = m(t^*). \tag{9}$$

Analogously to what we showed for ARP it can be proved that for BRP, $B^*(b)$ is an increasing function of b.

If $G(x)$ possesses a second derivative, it can be shown that

$$dB^*(b)/db = 1/t^* > 0. \tag{10}$$

From this fact one can again conclude that there is a one to one correspondence between b and $B^*(b)$, but not necessarily between b and $t^*(b)$.

2. COMPARISON OF ARP, BRP, AND FRP

For a given planned replacement cost the optimal ARP was shown to be the globally optimal replacement procedure [2]. In particular this means that ARP is preferable to BRP if $a=b$. However, because of the nature of the two policies it often happens, particularly with multi-unit systems, that $b<a$ and then the question arises which policy should be chosen.

To provide a simple rule for making this decision, we shall partition the unit square in (a,b) space into three regions for which one of the three policies ARP, BRP, FRP is least costly. It can happen, as in Example 1 (Sec. 3), that the FRP region is an empty set. But there are cases, such as Example 2, in which for some (a,b) the optimal procedure is FRP. In general, the monotonicity of $A^*(a)$ and $B^*(b)$ imply that—

1. FRP will be less costly than the optimal ARP i.f.f. $a>a_0$;
2. FRP will be less costly then BRP i.f.f. $b>b_0$;

where

$$a_0 \equiv \inf\{a;\ A^*(a) = F\};\ b_0 \equiv \inf\{b;\ B^*(b) = F\}. \tag{11}$$

Thus a meaningful comparison between the optimal ARP and the optimal BRP is confined to the rectangle

$(0\leqslant a\leqslant a_0,\ 0\leqslant b\leqslant b_0)$.

In [1, p 67] some properties of ARPs and BRPs are compared when the age limit x for an ARP is equal to the block interval t in a BRP. This comparison is not suitable for our purpose because in order to make an optimal choice between the policies, one has to compare their costs when each one of them is carried out optimally, i.e., the ARP with an age limit x^*, and the BRP with a block interval t^*; x^* and t^* need not of course be equal.

Now fix a $(a<a_0)$ and find the b for which

$$B^*(b) = A^*(a). \tag{12}$$

Because of the monotonicity of $A^*(a)$ and $B^*(b)$ the resulting b, which we denote by $b^*(a)$, is

1. unique,
2. less than b_0, since only at the upper right hand corner of the rectangle $(0\leqslant a\leqslant a_0, 0\leqslant b\leqslant b_0)$ is $A^*(a) = B^*(b) = F$,
3. satisfies the equation

$$B^*(b) \lesseqgtr A^*(a) \quad \text{for } b \lesseqgtr b^*(a), \tag{13}$$

4. $b^*(a)$ is increasing with a.

If $G(x)$ possesses a second derivative we can conclude from (12) that

$$\frac{db^*(a)}{da} = \frac{dA^*(a)}{da} \bigg/ \frac{dB^*(b)}{db} \tag{14}$$

and hence by (6) and (10)

$$db^*(a)/da = t^*r(x^*)/[((1-a)\bar{G}^*)^{-1} - 1] > 0, \tag{15}$$

where $x^* \equiv x^*(a)$, $t^* \equiv t^*[b^*(a)]$, $G^* \equiv G(x^*)$.

Therefore the rule for making an optimal choice between ARP, BRP, and FRP is the following:

For $0\leqslant a\leqslant a_0$, choose the optimal BRP if $b<b^*(a)$ and the optimal ARP otherwise. ($b=b^*(a)$ is a curve of indifference separating the ARP and BRP regions).

For $a>a_0$ choose the optimal BRP if $b\leqslant b_0$ and FRP otherwise. A geometrical representation of this rule is given in Fig. 1.

Before closing this section let us say a few words about how we can, in practice, find $b^*(a)$ for a given $a<a_0$. First we use (3) to compute $x^*=x^*(a)$ and then by (4) we immediately obtain $A^*(a)$. Now the unique $b^*(a)$ which satisfies (12) is, by (9), the solution of (16).

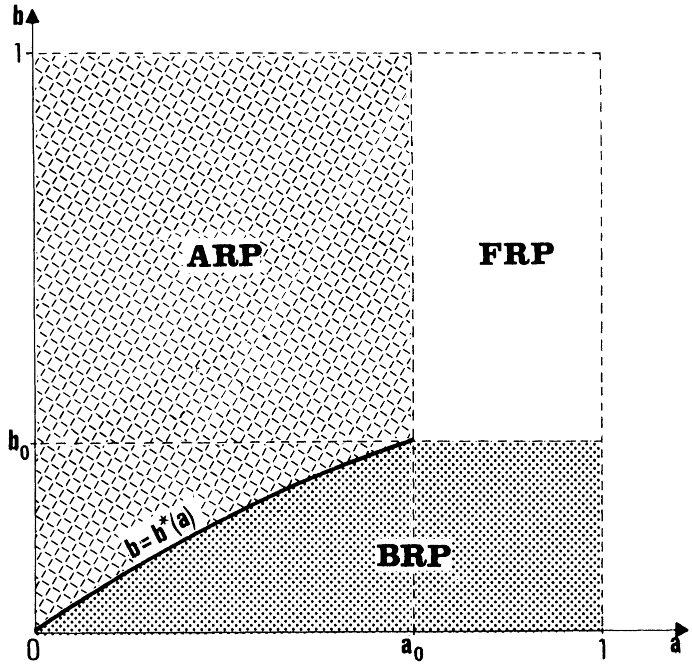

Fig. 1. Possible partitioning of the unit square, $0 \leqslant a \leqslant 1$, $0 \leqslant b \leqslant 1$, into regions where ARP, BRP, FRP are optimal.

$$m(t^*) = A^*(a) \tag{16}$$

where $t^* \equiv t^*(b)$. If $m(t)$ is an increasing function of t then t^* is unique and is—

$$t^* = m^{-1}[A^*(a)]. \tag{17}$$

Inserting (17) into (8) yields $b^*(a)$ immediately. If, however, $m(t)$ is not an increasing function, (16) may have several solutions which we denote by $(t_1, \ldots, t_n)$. Inserting these values into (8) yields a vector of costs $(b_1, \ldots, b_n)$ whose components need not be different.

The next step is obvious. Choose the pairs (b_i, t_i) for which $B(b,t)$ is minimum. The t_i's could be different but the corresponding b_i is always the same. It is this unique $b = b^*(a)$ which we are seeking.

To see that all the b_i's have to be equal assume that there exist $b_{i_1} < b_{i_2}$ such that

$$B(b_{i_1}, t_{i_1}) = B(b_{i_2}, t_{i_2}) \leqslant B(b_j, t_j) \quad \text{for all } j = 1, \ldots, n$$

$$\text{then} \quad B^*(b_{i_1}) = B^*(b_{i_2})$$

which contradicts the previously established result that $B^*(b)$ is a monotonically increasing function of b.

3. EXAMPLES

In this section we compare the optimal ARP, optimal BRP and FRP for two special life-time distributions, using the proposed method. For convenience we standardize the scale parameter, in both examples, to 1.

1: *The Uniform Distribution*

$$g(w) = \begin{cases} 1, & 0 \leqslant w \leqslant 1 \\ 0, & \text{otherwise.} \end{cases}$$

For $0 \leqslant w < 1$ we have

$$r(w) = 1/(1-w), M(w) = \exp(w) - 1, m(w) = \exp(w). \tag{18}$$

The s-expectation of this distribution is 1/2 and hence the cost of a FRP is, by (1),

$$F = 2. \tag{19}$$

Solving (3) for this distribution yields

$$x^* = [-a - 1 + \alpha]/1 - a, \quad \alpha \equiv 1 + \sqrt{2a - a^2}. \tag{20}$$

x^* is unique since $r(w)$ is increasing with w. Inserting (20) into (4) yields the cost of the optimal ARP

$$A^*(a) = \alpha. \tag{21}$$

Comparing (19) and (21) shows that

$$A^*(a) < F, \quad \text{for all } 0 \leqslant a \leqslant 1. \tag{22}$$

In other words, a FRP is never less costly than the optimal ARP which means that $a_0 = 1$ for this distribution.

The cost of the optimal BRP for any particular b is, by (9) and (18),

$$B^*(b) = e^{t^*} \tag{23}$$

where $t^* \equiv t^*(b)$.

t^* is unique since $m(w)$ is increasing with w for this distribution. Comparing (19) and (23) indicates that

$$B^*(b) \leqslant F \quad \text{i.f.f.} \quad t^* \leqslant \ln 2. \tag{24}$$

The l.h.s. of (8) is increasing with t^* for this distribution, and hence

$$t^* \leqslant \ln 2 \quad \text{i.f.f.} \quad b \leqslant 2 \ln 2 - 1 = 0.386. \tag{25}$$

Combining (24) with (25) yields

$$B^*(b) \leqslant F \quad \text{i.f.f.} \quad b \leqslant 0.386, \tag{26}$$

which in turn implies that $b_0 = 0.386$. In this connection see [3] where the optimal BRP and FRP have been compared for the uniform distribution.

Thus the relevant region for a meaningful comparison of the optimal ARP and the optimal BRP is the rectangle $(0 \leqslant a \leqslant 1, 0 \leqslant b \leqslant 0.386)$. Proceeding with our method we now fix a and use (17), (18) and (21) to obtain t^*,

$$t^* = \ln \alpha. \tag{27}$$

Finally, according to the last step in our procedure we insert (27) into (8) to obtain $b^*(a)$

$$b^*(a) = 1 - \alpha(1 - \ln \alpha). \tag{28}$$

$b^*(a)$ is increasing with a, and $b^*(a_0) = b^*(1) = 0.386 = b_0$.

It is easily verified that $b^*(a) \leqslant a$, which means that the optimal ARP is less costly than the optimal BRP when both have the same planned replacement cost. Global optimality of ARP, for a given planned replacement cost, is proved in a more general setting in [2].

We are now in a position to state the rule for the optimal choice among the replacement policies we are considering. For $0 \leqslant a \leqslant 1$ choose the optimal BRP if

$$b < 1 - \alpha(1 - \ln \alpha)$$

and the optimal ARP otherwise.

The appropriate partitioning of the unit square is given in Fig. 2.

2: *The 2-stage Erlang distribution*

$$g(w) = \begin{cases} we^{-w}, & w \geqslant 0 \\ 0, & \text{otherwise.} \end{cases}$$

For $w \geqslant 0$ we have,

$$r(w) = w/1+w, \quad M(w) = (2w-1+e^{-2w})/4$$
$$\text{and } m(w) = (1-e^{-2w})/2. \tag{29}$$

Applying our procedure to this distribution we first find that

$$a_0 = 1/2, \ b_0 = 1/4.$$

The boundary separating the regions in the rectangle ($0 \leqslant a \leqslant 1/2$, $0 \leqslant b \leqslant 1/4$) where the ARP and BRP are optimal, is

$$b^*(a) = [1 - \xi^* (1 - \ln \xi^*)]/4$$
$$\xi^* \equiv [1 + 2x^* \exp(x^*)]^{-1} \tag{30}$$

with x^* expressed as the following implicit function of a

$$a = 1 - (x^*+1)/(2x^*+e^{-x^*}) \tag{31}$$

The r.h.s. of (31) is an increasing function of x^* from 0 (when $x^*=0$) to 1/2 (when $x^* \to \infty$). Hence a unique solution of x^* exists for any a in the relevant rectangle.

For the 2-stage Erlang distribution, unlike the uniform distribution, $b^*(a)$ is an implicit function of a, i.e., has the form $b^*(a,x^*(a))$.

The optimal rule is as follows:

For $0 \leqslant a \leqslant 1/2$, choose the optimal BRP if $b \leqslant b^*(a)$. Otherwise choose the optimal ARP.

For $a > 1/2$, choose the optimal BRP if $b < 1/4$. Otherwise choose FRP.

The appropriate partitioning of the unit square is given in Fig. 3.

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[3] P. Schweitzer, "Optimal replacement policies for hyper-exponentially and uniformly distributed lifetimes", *Operations Research*, vol 15, 1967 Mar, pp 360-362.

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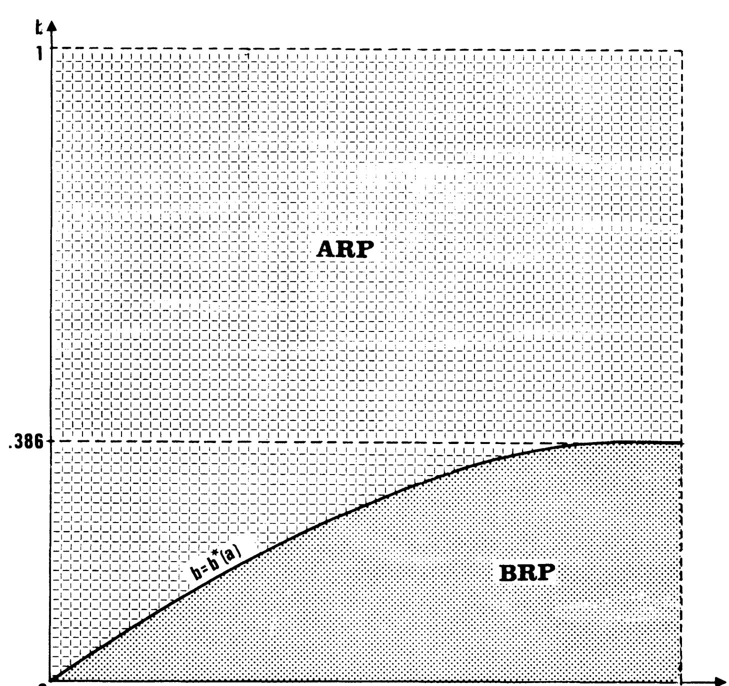

Fig. 2. Partition of the unit square, $0 \leqslant a \leqslant 1$, $0 \leqslant b \leqslant 1$, into regions of optimality for ARP, BRP, for uniformly distributed lifetimes.

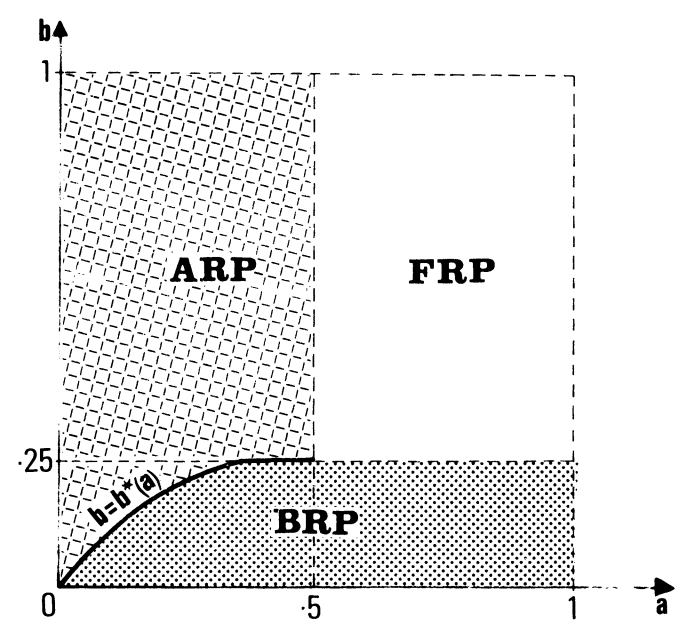

Fig. 3. Partition of the unit square, $0 \leqslant a < 1$, $0 \leqslant b < 1$, into regions where ARP, BRP, and FRP are optimal for a two stage Gamma life distribution.

Menachem Berg was born in Budapest, Hungary in 1944. He has been in Israel since 1950. He received a BA in Mathematics and Statistics from the Hebrew University, Jerusalem in 1967 and the MSc and the DSc degrees in Operations Research at the Technion, Israel Institute of Technology, 1970 and 1975, respectively. His research has been focussed on various aspects of the theory of preventive maintenance. He is a Lecturer, Dept. of Statistics, Haifa University.

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Benjamin Epstein was born in Boston, Mass. in 1918. He received the SB and MSc in Mathematics from MIT in 1937 and 1938 respectively and a PhD in Mathematics from the University of Illinois in 1941. At the Technion he has been instrumental in setting up a program of teaching and research in reliability theory and in establishing a graduate program in Statistics. Dr. Epstein is a Fellow of the Institute of Mathematical Statistics, the American Statistical Association, and the American Society for Quality Control, and was in 1974 elected a member of the International Statistical Institute. He is a founding member of the Israel Society for Quality Assurance (ISQA) and the Israel Statistical Association (ISA) and is a member of the Council of ISQA and Vice President of ISA. He received the 1965 award of the Electronics Division of ASQC and the 1974 Shewhart Medal of ASQC for his contributions to reliability theory and the assurance sciences.

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