High Speed Rail Transport Valuation

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ABSTRACT

The present paper investigates the optimal timing of investment for a high speed rail (HSR) project, in an uncertain environment, using a real options analysis (ROA) framework. It develops a continuous time framework with stochastic demand that allows for the determination of the optimal timing of investment and the value of the option to defer in the overall valuation of the project. The modelling approach used is based on the differential utility provided to railway users by the HSR service.

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1. Introduction

Today's global and dynamic business market is characterized by a growing uncertainty that affects significantly the decision-making processes within business organizations. In this sense, dealing with uncertainty and taking optimal decisions regarding investment opportunities becomes a way of achieving supremacy over competition.

Flexibility is crucial to perform efficiently, for instance, in terms of technological changes, competition's shifts, or even in order to limit potential losses related to unexpected adverse scenarios in the market.

Given the ineffectiveness of the traditional capital budgeting techniques in uncertain environments (Trigeorgis, 1996), the conceptual framework available in order to appraise complex investments in real assets consists in using real options analysis (ROA) techniques (Dixit and Pindyck, 1994).

ROA has changed the investment valuation paradigm, due to its ability to cope with decision makers' flexibility. In the offing period, new information may lead to a total or partial change of the initial plan, including the abandonment of the project.

In spite of having emerged in the academy, (Brennan and Schwartz, 1985; McDonald and Siegel, 1986; Dixit, 1989; Pindyck, 1991; and Dixit and Pindyck, 1994, amongst others), this new paradigm has already made an impact in the business world, since an increasing number of companies and managers are adopting a real options perspective. Especially in capital budgeting decisions and in the assessment of the corresponding strategic positioning and competitiveness (Paddock, Siegel and Smit, 1988; Nichols, 1994; Kallberg and Laurin, 1997; Moel and Tufano, 2002; Smit, 2003; etc.).

The modelling framework proposed in this paper is inspired by a set of projects for the development of high speed rail (HSR) lines in Europe. The structuring nature of the projects for the countries involved; the need to renew the railway sector; the huge amounts of money needed; the uncertainty about the timings to invest and the economic challenge inherent in developing a conceptual setting for a decision that needs to be taken in the interest of the entire set of European taxpayers, all play a part in providing relevance to the study of the embedded option to defer and the optimal timing to invest.

2. Real Options in Major Projects in the Transportation Sector

Although usually linked to political discussion and controversy, transport infrastructures tend to be understood as critical for the sustainable growth and development of any economy. According to Wilson (1986), since 1870 economists have been drawing their attention towards the transport industry in general, and to the railway sector in particular. The same author suggests that wrong transportation policies and the corresponding investment mistakes in transport infrastructures may compromise seriously economic growth. To prevent this type of outcome, it is important to develop and apply suitable decision criteria based upon sound cost/benefit analysis.

Infrastructure investments that are usually understood to provide benefit and leverage to the economic growth of whole regions include investments in seaports, airports and railways links, energy networks, road systems, amongst others.

The size, budget and impact in the global economic activity lead big transportation investments to assume the role of strategic options. Almost all these investments include a portfolio of options intended to, at some extent, protect the enormous funds needed to implement the project from failure.

Rose (1998) has valued the concession of a toll road, considering the existence of two options interacting with each other. The author assumed that the traffic volume followed a geometric Brownian motion and used Monte Carlo simulation to compute *i)* the value of the embedded call option that allowed for the early acquisition of the project by the franchiser and *ii)* the option to defer regarding the payment of the corresponding fees' by the franchisee. Similarly, Brandão (2002) applied the Copeland and Antikarov's (2003) framework to value several options embedded in a project that included the building and operation of highways in Brazil.

More recently, two other empirical ROA works focused on the valuation of structural investments in the transportation sector, were published: Smit (2003) and Bowe and Lee (2004). The first, analyses the expansion of an airport, while the second is apparently pioneer in the analysis of a railway transportation project.

Investments in infrastructure or platform assets generate other investments opportunities that change the competitive standing of the companies involved. Smit (2003) combines ROA and game theory to capture the intrinsic value derived from the company's positioning adjustment inside the industry, with an empirical application to the expansion of a European airport. His work has helped to fill in a gap in the real options literature, where researchers have tended to, either, ignore competition, underestimating the impact of a competitive entry, or assume that the competition is exogenous to the valuation process. Smit (2003) has chosen to develop his work in a discrete time framework, arguing that it makes it more simple and available to management teams, at the same time that includes potential asymmetries amongst competitors and makes possible the definition of alternative stochastic processes.

The main contribution of Smit's (2003) work is due to the fact that he evaluates the growing opportunities generated by an infrastructure as a game of several sequential exercises. Following Trigeorgis' (1996) developments, Smit (2003) starts by valuing the project without expansion opportunities, using traditional capital budgeting decision techniques, and subsequently valued the embedded growth opportunities in a competitive context with other European airports.

Similarly to Smit (2003), Bowe and Lee (2004) apply binomial analysis. However, they use a logarithmic transformation similar to Trigeorgis (1991), to evaluate the high speed train project in Taiwan, comparing the obtained results with a valuation based on traditional capital budgeting decision techniques. The work embraces the valuation of three different options (expand, reduce and defer) and the according interactions, included in a project that does not pay dividends. Nevertheless, as stated by the authors, this type of analysis should incorporate the effect of dividends in order for the framework to become close enough to real life situations to deserve proper consideration by companies.

3. Investment Valuation Using a Real Options Framework

In a HSR project, at any moment in time, the owner of the investment's rights holds the possibility of acquiring the future cash flow generated by the venture, in exchange for the payment of the corresponding implementation costs. Thus, we are dealing with an option to invest.

Considering the investment in a HSR line as an optimal stopping problem allows us to determine the value of the embedded option to defer. Following, the work of McDonald and Siegel (1986) and Salahaldin and Granger (2005), it also permits to determine the optimal timing to invest.

In the present paper it will be assumed not only that the option to defer is perpetual in nature ($T = \infty$), but also that, once implemented, the investment will produce perpetual benefits. Without major technological changes, the impact of these assumptions in the global valuation should not be unreasonable for two reasons. In the first place, because the present value of the more remote cash flows tends naturally to zero. In the second place, because maintenance and conservation - whose expenses are taken into consideration - tend to restore the operational aptitude of the assets in place and the corresponding flow of benefits.

3.1. Optimal Timing to Invest – Investment in one Period

In a context of the nature above mentioned, a decision to implement a project in a non-optimal moment, implies destruction of value. Therefore, finding the optimal timing offers the possibility to study the impact of the ability to delay in the global value of the project.

Thus, it is important to answer the question of when to invest, or at least find a critical value that might support in a rational way the decision of implementing the investment. The irreversibility features of the investment, given that there is no other use for the project rather than the railways, emphasise the importance of estimating the optimal timing to invest.

The model proposed here draws on the work of Salahaldin and Granger (2005) on the valuation of sustainable systems of urban transport aimed at relieving air pollution. It is a model that comprises a unique change from an inactive to an active state, and considers a single stochastic variable.

Because investment in infrastructures, like HSR lines, will affect the economic and social conditions of future generations, it should be assessed considering a global point of view, in terms of economic welfare. In an uncertain environment, it will only make sense to invest in such a project, if the economic value of the utility provided by the resulting benefits is able to surpass the joint value of the option to defer (lost by investing) and of the utility provided by the conventional railway system to its users.

Investing, in a moment other than the corresponding optimal timing, implies a reduction in the global level of utility achieved by the users, compromising seriously the projects' success. In such circumstances, any potential user may always maintain his current level of utility, choosing to travel in the conventional railway line, rather than in the new HSR service. If a suboptimal investment timing is chosen, the ability of the HSR service to attract clients will be strongly distressed.

At any moment users can choose to travel in the conventional railway, without any constraints. Consequently, to maintain the users' utility, the fraction of the new investment supported by each one must be identical to the sum of the benefits earned resulting from the reduction of the travel time and the conventional service fare saved, net of variable and fixed operational costs upheld.

Given a fixed amount to invest, the higher the demand, the higher the expected net benefit per capita. Consequently, higher levels of demand tend to lead to the anticipation of the optimal invest timings. The main source of uncertainty derives obviously from the level of future demand for the HSR service.

We will consider that the demand for the new high speed service, x_t, follows a geometric Brownian motion process:

$$\frac{dx_t}{x_t} = \mu dt + \sigma dw \qquad (3.1)$$

Similar assumptions may be found in Rose (1998), with the purpose of modelling highway traffic; in Salahaldin and Granger (2005), with the purpose of modelling the dynamics of a city' population; and in Marathe and Ryan (2005) and Pereira *et al.* (2006) with the purpose of modelling airline demand.

In equation (3.1) μ and σ represent the growth rate and the standard deviation of the demand for the HSR service. We assume that both parameters are constant in time. The Wiener process, w_t, has zero mean and standard deviation $\sigma\sqrt{dt}$.

Under these circumstances, it is reasonable to expect that, in the future, the natural demand for HSR will reach a level capable of providing a rational reason to invest in such a project.

In order to model such a situation we are going to assume that each user will face a cost for railway travel between two cities, ψ, whose global worth will be a function of the value of time for the user, η, and the travel fare, p. According to the literature, both these variables exhibit a relationship to the global demand for railway services (*vide* Owen e Phillips, 1987; Wardman, 1994; and Wardman 1997).

Considering the relationship between the value of travel time and the demand for faster railway services (Owen e Phillips, 1987; and Wardman, 1994), the following functional form will be used:

$$\eta = \beta x^{\delta_\beta} \tag{3.2}$$

In this functional form, δ_β represents the elasticity between the value of travel time η and the HSR demand x. Consequently, β is the scale parameter between demand, x, and the value of travel time, η, given by:

$$\beta = \eta x^{-\delta_\beta} \tag{3.3}$$

Concerning the relationship between the fare value and the demand for railway services, this will be given by the functional form (Owen e Phillips, 1987):

$$p = \alpha x^{\delta_\alpha} \tag{3.4}$$

The elasticity between the fare value, p, and the HSR demand, x, is represented by the parameter δ_α. The scale parameter α, that relates demand and the fare value, p, is given by:

$$\alpha = p x^{-\delta_\alpha} \tag{3.5}$$

The demand may be inferred from the preferences of a risk neutral representative user, with a utility function $U = c$, in which c represents the mean consumption of all users that constitute the overall demand. The budget constraint is given by $c + \psi = m$, in which ψ represents the travel cost and m the individual disposable income by unit of time. Analytically, we have:

$$c = m - \psi \tag{3.6}$$

Replacing the level of consumption in the utility function, will allow for the determination of the following indirect utility function, V, representative of the value that each user attributes to a railway trip:

$$V = U = m - \psi \tag{3.7}$$

The relationship between demand x_t and value of travel time, *i)* in the period of time that precedes investment, η_0; *ii)* during the period of effective investment, η_1; and *iii)* after the investment's implementation, η_2, is represented, respectively, by β_0, β_1 and β_2. Since the new rail service will save travel time and, in consequence, will reduce the value of travel time from η_0 to η_2, it will be reasonable to expect that from the pre-investment period to the operational phase β_0 will change to β_2 with $\beta_0 > \beta_2$. The difference between β_0 and β_2 reflects the decrease in travel time.

Meanwhile, for the moment, we will assume that the investment will take place during a single period of time. Thus, the relationship between demand and travel costs, during the construction period β_1 is assumed to be equal to β_2.

Analytically, the cost of travelling in a conventional railway, ψ_0, and the cost of travelling in HSR, ψ_2, will be represented by the following equations,

$$\psi_0 = \beta_0 x_t^{\delta_\beta} + \alpha_0 x_t^{\delta_\alpha} \quad (3.8)$$

$$\psi_2 = \beta_2 x_t^{\delta_\beta} \quad (3.9)$$

For modelling purposes, the conventional railway travel cost, ψ_0, includes both the value of the travel time lost and the fare paid. In contrast, the HSR travel cost function here considered, ψ_2, is not affected by the value of the corresponding fare, p_2, because the current valuation framework assumes implicitly that each user will bear his part of the investment expenditure plus the corresponding operating costs per user. In other words, a socially acceptable HSR service fare is already implicitly considered in the valuation framework. Consequently, it does not make sense to duplicate it.

The existing conventional railway service that charges a fare p_0, enables us to identify the relationship between HSR demand, x_t, and the price of a substitute service (Owen and Phillips, 1987; and Wardman, 1997) given by equation (3.4).

As long as the investment is not implemented, the indirect utility function will be given by:

$$V_0 = m - \beta_0 x^{\delta_\beta} - \alpha_0 x^{\delta_\alpha} \quad (3.10)$$

After the investment is implemented, users will continue to face a (smaller) cost in terms of time spent. However, since the analysis performed here takes into consideration all costs and benefits induced by the project (including not only capital investment expenditure, but also all fixed and variable operating costs), the new indirect utility function will be given by:

$$V_2 = m - \beta_2 x^{\delta_\beta} - \omega - \frac{\varphi}{x} - \frac{\rho\gamma}{x} \tag{3.11}$$

with γ representing the capital investment expenditure, ρ the discount rate, ω the variable operating costs and φ the fixed operating costs. Notice that $\frac{\varphi}{x}$ and $\frac{\rho\gamma}{x}$ represent the fixed operating costs and the investment expenditure per unit of time, for each user that integrates the global demand for the HSR service. We assume implicitly that the outcomes of the investment will last for an unlimited time horizon.

The purpose is to carry out the investment without changing the present utility function equilibrium. In order to achieve this outcome, it will be necessary to find the critical demand level for x^*, above which it will be optimal to invest.

Noting that, in these terms, the whole framework might be understood as an intergeneration welfare problem, as previously stated, we may use the objective function of Ramsey-Koopmans adopted by Salahaldin and Granger (2005). Analytically, we have:

$$\sup_{x^*} E_{x_0}\left(\int_0^\tau x_t V_0(x) e^{-\rho t}\, dt + \int_\tau^\infty x_t V_2(x) e^{-\rho t}\, dt\right) \tag{3.12}$$

Where,

τ = Moment of time in which the optimal value is achieved by the first time;

$V_0(x)$ = Indirect utility function per unit of time before investment implementation, given by the equation (3.10);

$V_2(x)$ = Indirect utility function per unit of time after investment implementation, given by the equation (3.11);

x_t = Demand throughout time, given by the equation (3.1); and

x_0 = Estimated demand at present.

Aggregating the utility of all users that constitute the potential demand before and after the investment, and replacing V_0 and V_2 for the corresponding values in (3.10) and (3.11), we get:

$$\sup_{x^*} E_{x_0}\left(\int_0^{\tau} e^{-\rho t}\left[mx_t - \beta_0 x_t^{\theta_\beta} - \alpha_0 x_t^{\theta_\alpha}\right]dt + \int_{\tau}^{\infty} e^{-\rho t}\left[mx_t - \beta_2 x_t^{\theta_\beta} - \omega x_t - \varphi - \rho\gamma\right]dt\right) \tag{3.13}$$

with $\theta_\beta = 1 + \delta_\beta$ and $\theta_\alpha = 1 + \delta_\alpha$.

Applying the Markov propriety as in Oksendal (2003), we will get:

$$E_x\left(\int_{\tau}^{\infty} e^{-\rho t}\left[mx_t - \beta_0 x_t^{\theta_\beta} - \alpha_0 x_t^{\theta_\alpha}\right]dt\right) = E_x\left(e^{-\rho\tau} E_{x^*}\left(\int_0^{\infty} e^{-\rho t}\left[mx_t - \beta_0 x_t^{\theta_\beta} - \alpha_0 x_t^{\theta_\alpha}\right]dt\right)\right) \tag{3.14}$$

in the second element of (3.13), we obtain

$$\begin{aligned} &E_{x_0}\left(\int_{\tau}^{\infty} e^{-\rho t}\left[mx_t - \beta_2 x_t^{\theta_\beta} - \omega x_t - \varphi - \rho\gamma\right]dt\right) = \\ &= E_{x_0}\left(e^{-\rho\tau} E_{x^*}\left(\int_0^{\infty} e^{-\rho t}\left[mx_t - \beta_2 x_t - \omega x_t - \varphi - \rho\gamma\right]dt\right)\right) \end{aligned} \tag{3.15}$$

Replacing the second element of (3.13) by the RHS of (3.15) and additionally adding and subtracting the RHS of (3.14), results in the following objective function,

$$\sup_{x^*}\left[\begin{aligned} &E_{x_0}\left(\int_0^{\infty} e^{-\rho t}\left[mx_t - \beta_0 x_t^{\theta_\beta} - \alpha_0 x_t^{\theta_\alpha}\right]dt\right) + \\ &+ E_{x_0}\left(e^{-\rho\tau} E_{x^*}\left(\int_0^{\infty} e^{-\rho t}\left[(\beta_0 - \beta_2)x_t^{\theta_\beta} + \alpha_0 x_t^{\theta_\alpha} - \omega x_t - \varphi - \rho\gamma\right]dt\right)\right) \end{aligned}\right] \tag{3.16}$$

Since the first component does not depend on x^*, the problem may be rewritten, in the following terms:

$$\sup_{x^*} E_{x_0}\left[e^{-\rho\tau}\left(E_{x^*}\left(\int_0^{\infty} e^{-\rho t}\left[(\beta_0 - \beta_2)x_t^{\theta_\beta} + \alpha_0 x_t^{\theta_\alpha} - \omega x_t - \varphi - \rho\gamma\right]dt\right)\right)\right] \tag{3.17}$$

This objective function maximizes the net gain provided by an investment in a HSR link, in terms of travel costs for the corresponding users.

Simplifying, we have:

$$E_{x^*}\left(\int_0^\infty e^{-\rho t}\left[(\beta_0-\beta_2)x_t^{\theta_\beta}+\alpha_0 x_t^{\theta_\alpha}-\omega x_t-\varphi-\rho\gamma\right]dt\right)= \\ =\int_0^\infty e^{-\rho t}\left[(\beta_0-\beta_2)E_x^*\left(x_t^{\theta_\beta}\right)+\alpha_0 E_x^*\left(x_t^{\theta_\alpha}\right)-\omega E_x^*\left(x_t\right)-\varphi-\rho\gamma\right]dt \tag{3.18}$$

We know that x_t follows a geometric Brownian motion described by (3.1). Thus,

$$E_x^*\left(x_t^\theta\right)=\left(x^*\right)^\theta e^{\left(\theta\mu_x t+\frac{1}{2}\theta(\theta-1)\sigma_x^2 t\right)} \tag{3.19}$$

The existence of a future optimal timing to invest requires the need to respect the following condition $\rho-\theta\mu_x-\frac{1}{2}\theta(\theta-1)\sigma_x^2>0$. This condition imposes the demand growth rate to be lower than discount rate, thus providing a rational economic interpretation to the underlying mathematical developments. Simplifying again and under this new condition, we have:

$$\int_0^\infty e^{-\rho t}\left[(\beta_0-\beta_2)E_x^*\left(x_t^{\theta_\beta}\right)+\alpha_0 E_x^*\left(x_t^{\theta_\alpha}\right)-\omega E_x^*\left(x_t\right)-\varphi-\rho\gamma\right]dt= \\ =\frac{2(\beta_0-\beta_2)\left(x^*\right)^{\theta_\beta}}{2\rho-2\mu_x\theta_\beta-\theta_\beta^2\sigma_x^2+\theta_\beta\sigma_x^2}+\frac{2\alpha_0\left(x^*\right)^{\theta_\omega}}{2\rho-2\mu_x\theta_\alpha-\theta_\alpha^2\sigma_x^2+\theta_\alpha\sigma_x^2}-\frac{\omega\left(x^*\right)}{\rho-\mu_x}-\frac{\varphi}{\rho}-\gamma \tag{3.20}$$

Rewriting (3.17) taking into consideration the result (3.20), we achieve:

$$\sup_{x^*} E_{x_0}\left[e^{-\rho\tau}\left(\begin{array}{l}\frac{2(\beta_0-\beta_2)\left(x^*\right)^{\theta_\beta}}{2\rho-2\mu_x\theta_\beta-\theta_\beta^2\sigma_x^2+\theta_\beta\sigma_x^2}+\frac{2\alpha_0\left(x^*\right)^{\theta_\alpha}}{2\rho-2\mu_x\theta_\alpha-\theta_\alpha^2\sigma_x^2+\theta_\alpha\sigma_x^2}- \\ -\frac{\omega\left(x^*\right)}{\rho-\mu_x}-\frac{\varphi}{\rho}-\gamma\end{array}\right)\right] \tag{3.21}$$

With,

$$A=\frac{2(\beta_0-\beta_2)}{2\rho-2\mu_x\theta_\beta-\theta_\beta^2\sigma_x^2+\theta_\beta\sigma_x^2} \tag{3.22}$$

$$B=\frac{2\alpha_0}{2\rho-2\mu_x\theta_\alpha-\theta_\alpha^2\sigma_x^2+\theta_\alpha\sigma_x^2} \tag{3.23}$$

$$C=-\frac{\varphi}{\rho} \tag{3.24}$$

$$D = -\gamma \tag{3.25}$$

and

$$F = -\frac{\omega}{\rho - \mu_x} \tag{3.26}$$

function (3.21) becomes,

$$\sup_{x^*} E_{x_0}\left[e^{-\rho\tau}\left(A\left(x^*\right)^{\theta_\beta} + B\left(x^*\right)^{\theta_\alpha} + F\left(x^*\right) + C + D\right)\right] \tag{3.27}$$

Given the current demand level, the value of the project, v, is determined through the maximization of the function:

$$v\left(t_0, x_0\right) = E_{x_0}\left[e^{-\rho\tau}\left(A\left(x^*\right)^{\theta_\beta} + B\left(x^*\right)^{\theta_\alpha} + F\left(x^*\right) + C + D\right)\right] \tag{3.28}$$

That satisfies the differential equation,

$$\frac{1}{2}\sigma_x^2 x_0^2 \frac{\partial^2 v}{\partial x_0^2} + \mu_x x_0 \frac{\partial v}{\partial x_0} - \rho v = 0, \text{ for } x \neq x^* \tag{3.29}$$

Resulting from the simplification of the following partial differential equation:

$$\frac{1}{2}\sigma_x^2 x_0^2 \frac{\partial^2 v}{\partial x_0^2} + \mu_x x_0 \frac{\partial v}{\partial x_0} + \frac{\partial v}{\partial t} = 0, \text{ for } x \neq x^* \tag{3.30}$$

Equation (3.29) satisfies the following conditions:

1. Initial condition:

$$v(0) = 0 \tag{3.31}$$

2. Value matching condition:

$$v\left(t_0, x_0\right) = A x_0^{\theta_\beta} + B x_0^{\theta_\alpha} + F x_0 + C + D, \text{ with } x_0 = x^* \tag{3.32}$$

and,

3. Smooth-pasting condition:

$$v'\left(t_0, x_0\right) = \theta_\beta A x_0^{\theta_\beta - 1} + \theta_\omega B x_0^{\theta_\omega - 1} + F, \text{ with } x_0 = x^* \tag{3.33}$$

To solve (3.29) we substitute $v\left(t_0, x_0\right) = e^{-\rho t_0}\Phi\left(x_0\right)$, where Φ represents the projects' value function at any moment in time. Therefore, (3.29) becomes,

$$\frac{1}{2}\sigma_x^2 x_0^2 \Phi''(x_0) + \mu_x x_0 \Phi'(x_0) - \rho \Phi(x_0) = 0, \text{ for } x \neq x^* \tag{3.34}$$

With the following conditions:

1. Initial condition:

$$\Phi(0) = 0 \tag{3.35}$$

2. Value matching condition:

$$\Phi(x_0) = A x_0^{\theta_\beta} + B x_0^{\theta_\alpha} + F x_0 + C + D, \text{ with } x_0 = x^* \tag{3.36}$$

and,

3. Smooth-pasting condition

$$\Phi'(x_0) = \theta_\beta A x_0^{\theta_\beta - 1} + \theta_\alpha B x_0^{\theta_\alpha - 1} + F, \text{ with } x_0 = x^* \tag{3.37}$$

Since equation (3.34) is a *Cauchy-Euler* second order homogeneous differential equation, the solution may be written as,

$$\Phi(x_0) = a_1 x_0^{r_1} + a_2 x_0^{r_2} \tag{3.38}$$

where r_1 and r_2 are the two roots of the quadratic equation:

$$\frac{1}{2}\sigma_x^2 r(r-1) + \mu_x x_0 r - \rho = 0 \tag{3.39}$$

given by,

$$r_1 = \frac{\left(\frac{1}{2}\sigma_x^2 - \mu_x\right) + \sqrt{\left(\mu_x - \frac{1}{2}\sigma_x^2\right)^2 + 2\rho\sigma_x^2}}{\sigma_x^2} \tag{3.40}$$

and

$$r_2 = \frac{\left(\frac{1}{2}\sigma_x^2 - \mu_x\right) - \sqrt{\left(\mu_x - \frac{1}{2}\sigma_x^2\right)^2 + 2\rho\sigma_x^2}}{\sigma_x^2} \tag{3.41}$$

As $a_2 x_0^{r_2}$ tends to the infinity when x_0 tends to zero, according to the initial condition (3.35) and $\Phi(x_0)$ needs to be limited when $x_0 \to 0$, $a_2 = 0$. Thus, equation (3.38) becomes,

$$\Phi(x_0) = a_1 x_0^{r_1} \quad (3.42)$$

Using the condition $\Phi(x^*) = Ax^{*\theta_\beta} + Bx^{*\theta_\alpha} + Fx^* + C + D$ that results from the substitution of x_0 by x^* in equation (3.36), we find the coefficient $a_1 = Ax^{*\theta_\beta - r_1} + Bx^{*\theta_\alpha - r_1} + Fx^{*1-r_1} + Cx^{*-r_1} + Dx^{*-r_1}$, concluding that the solution of (3.29) is,

$$v(t_0, x_0) = e^{-\rho t_0}\left[Ax^{*\theta_\beta - r_1} + Bx^{*\theta_\alpha - r_1} + Fx^{*1-r_1} + Cx^{*-r_1} + Dx^{*-r_1}\right]x_0^{r_1} \quad (3.43)$$

For a given value of x_0 in $t_0 = 0$, the value of x^* that maximizes $v(0, x_0)$ is implicitly given by the equation:

$$Ax^{*\theta_\beta - r_1}(\theta_\beta - r_1) + Bx^{*\theta_\omega - r_1}(\theta_\alpha - r_1) + Fx^{*1-r_1}(1 - r_1) - Cx^{-r_1}r_1 - Dx^{-r_1}r_1 = 0 \quad (3.44)$$

The critical value x^* can only be found through numerical solution of (3.44), except if two assumptions are made. The first assumption related to equality between the HSR demand/value of travel time elasticity and the HSR demand/conventional service fare cross elasticity, conducting to $\theta_\beta = \theta_\alpha = \theta$. The second assumption comes from the possibility of neglecting the operational variable costs, $F = 0$, considering the operational characteristics of the project. Taking these two conditions into account, x^* has the following closed form solution:

$$x^* = \exp\left[\frac{\ln \dfrac{-r_1(C+D)}{(A+B)(r_1-\theta)}}{\theta}\right] \quad (3.45)$$

The critical value x^* represents the level of demand that, when reached, justifies (turns optimal) an immediate implementation of the project.

This solution preserves utility equilibrium between HSR and conventional service for railway users, making the optimal solution independent of the original income m and the initial level of demand for the HSR service x_0. The fact that the whole framework is aimed at achieving a better level of global economic welfare, based on the equilibrium between the utility of two similar services, turns this model especially adequate to analyse governmental scale investment decisions.

3.2. Optimal Timing to Invest – Investment over Several Periods

Large projects normally take time to implement. Thus, it is crucial to include this feature in the ROA's model, allowing the time-to-build effect to be incorporated.

Relaxing the assumption previously made at this level and allowing $\beta_1 \neq \beta_2$, we create a transition period that corresponds to the time needed to build the HSR link.

A new HSR link can only start to operate after all the inherent engineering and development work is finished. Consequently, during this building period n, the cost of travelling is still given by ψ_0, so β_1 remains equal to β_0 ($\beta_1 = \beta_0$). When the HSR starts to operate, the cost of travelling will change to ψ_2, with β_2 incorporating the decrease in travel time.

The new Ramsey-Koopmans objective function becomes,

$$\sup_{x^*} E_{x_0}\left(\int_0^\tau x_{t+n} e^{-\rho n} V_0(x_{t+n}) e^{-\rho t}\, dt + \int_\tau^\infty x_{t+n} e^{-\rho n} V_2(x_{t+n}) e^{-\rho t}\, dt\right) \tag{3.46}$$

Where, now:

$V_0(x_{t+n})$ = Indirect utility function by unit of time before the beginning of the HSR operation;

$V_2(x_{t+n})$ = Indirect utility function by unit of time after the beginning of the HSR operation;

n = Time-to-build (construction) of the investment;

With,

$$V_0(x_{t+n}) = m_{t+n} - \beta_0 x_{t+n}^{\delta_\beta} - \alpha_0 x_{t+n}^{\delta_\alpha} \tag{3.47}$$

and

$$V_2(x_{t+n}) = m_{t+n} - \beta_2 x_{t+n}^{\delta_\beta} - \omega - \frac{\varphi}{x_{t+n}} - \frac{\rho\gamma e^{\rho n}}{x_{t+n}} \tag{3.48}$$

Considering the global utility of all the users that constitute the demand before and after the HSR link starts to operate, and substituting V_0 and V_2 from (3.47) and (3.48) into (3.46), we obtain:

$$\sup_{x^*} E_{x_0}\left(\begin{array}{l}\int_0^\tau e^{-\rho t}\left[\left(m_{t+n} x_{t+n} - \beta_0 x_{t+n}^{\theta_\beta} - \alpha_0 x_{t+n}^{\theta_\alpha}\right)e^{-\rho n}\right]dt + \\ + \int_\tau^\infty e^{-\rho t}\left[\left(m_{t+n} x_{t+n} - \beta_2 x_{t+n}^{\theta_\beta}\right)e^{-\rho n} - \omega x_{t+n} e^{-\rho n} - \varphi e^{-\rho n} - \rho\gamma\right]dt\end{array}\right) \tag{3.49}$$

with $\theta_\beta = 1+\delta_\beta$ and $\theta_\alpha = 1+\delta_\alpha$.

Using again the Markov propriety from Oksendal (2003), with a simplification identical to that performed in the previous section, it is possible to obtain the following objective function,

$$\sup_{x^*}\left[\begin{array}{l} E_{x_0}\left[\int_0^\infty e^{-\rho t}\left[\left(m_{t+n}x_{t+n}-\beta_0 x_{t+n}^{\theta_\beta}-\alpha_0 x_{t+n}^{\theta_\alpha}\right)e^{-\rho n}\right]dt\right]+ \\ +E_{x_0}\left[e^{-\rho\tau}E_{x^*}\left(\int_0^\infty e^{-\rho t}\left[(\beta_0-\beta_2)x_{t+n}^{\theta_\beta}e^{-\rho n}+\alpha_0 x_{t+n}^{\theta_\alpha}-\omega x_{t+n}e^{-\rho n}-\varphi e^{-\rho n}-\rho\gamma\right]dt\right)\right]\end{array}\right]$$

(3.50)

The first element does not depend on x^*, so (3.50) may be rewritten in the following terms:

$$\sup_{x^*} E_{x_0}\left[e^{-\rho\tau}\left[E_{x^*}\left[\int_0^\infty e^{-\rho t}\left[(\beta_0-\beta_2)x_{t+n}^{\theta_\beta}e^{-\rho n}+\alpha_0 x_{t+n}^{\theta_\alpha}-\omega x_{t+n}e^{-\rho n}-\varphi e^{-\rho n}-\rho\gamma\right]dt\right]\right]\right]$$

(3.51)

This objective function, similar to equation (3.17), maximizes the net utility gain, provided by an investment in a HSR link, in terms of travel costs for the corresponding users. In contrast to (3.17), this new formulation considers that after the decision to implement the project a n building period will need to take place, before the HSR link may start to operate.

Simplifying, we have,

$$\begin{aligned} &E_{x^*}\left(\int_0^\infty e^{-\rho t}\left[(\beta_0-\beta_2)x_{t+n}^{\theta_\beta}e^{-\rho n}+\alpha_0 x_{t+n}^{\theta_\alpha}e^{-\rho n}-\omega x_{t+n}e^{-\rho n}-\varphi e^{-\rho n}-\rho\gamma\right]dt\right)= \\ &=\int_0^\infty e^{-\rho t}\left[(\beta_0-\beta_2)E_x^*\left(x_{t+n}^{\theta_\beta}\right)e^{-\rho n}+\alpha_0 E_x^*\left(x_{t+n}^{\theta_\alpha}\right)e^{-\rho n}-\omega E_x^*\left(x_{t+n}\right)e^{-\rho n}-\varphi e^{-\rho n}-\rho\gamma\right]dt\end{aligned}$$

(3.52)

Knowing that x_t follows a geometric Brownian and that $E_x^*\left(x_t^\theta\right)$ is given by (3.19), then,

$$E_x^*\left(x_{t+n}^\theta\right)=\left(x^*\right)^\theta e^{\left(\theta\mu_x+\frac{1}{2}\theta(\theta-1)\sigma_x^2\right)(t+n)} \tag{3.53}$$

Simplifying again and under the condition that $\rho-\mu_x\theta-\frac{1}{2}\theta(\theta-1)\sigma_x^2>0$, we have,

$$\int_0^\infty e^{-\rho t}\left[(\beta_0-\beta_2)E_x^*\left(x_{t+n}^{\theta_\beta}\right)e^{-\rho n}-\omega E_x^*\left(x_{t+n}^{\theta_\omega}\right)e^{-\rho n}-\varphi e^{-\rho n}-\rho\gamma\right]dt=$$
$$\frac{2(\beta_0-\beta_2)\left(x^*\right)^{\theta_\beta}e^{\left(\mu_x\theta_\beta+\frac{1}{2}\theta_\beta(\theta_\beta-1)\sigma_x^2\right)n}e^{-\rho n}}{2\rho-2\mu_x\theta_\beta-\theta_\beta^2\sigma_x^2+\theta_\beta\sigma_x^2}+\frac{2\alpha_0\left(x^*\right)^{\theta_\alpha}e^{\left(\mu_x\theta_\alpha+\frac{1}{2}\theta_\alpha(\theta_\alpha-1)\sigma_x^2\right)n}e^{-\rho n}}{2\rho-2\mu_x\theta_\alpha-\theta_\alpha^2\sigma_x^2+\theta_\alpha\sigma_x^2}-$$
$$-\frac{\omega\left(x^*\right)e^{(\mu_x-\rho)n}}{\rho-\mu_x}-\frac{\varphi e^{-\rho n}}{\rho}-\gamma$$

(3.54)

Rewriting (3.51) considering these simplifications, we get:

$$\sup_{x^*}E_{x_0}\left[e^{-\rho\tau}\left(\begin{array}{l}\frac{2(\beta_0-\beta_2)\left(x^*\right)^{\theta_\beta}e^{\left(\mu_x\theta_\beta+\frac{1}{2}\theta_\beta(\theta_\beta-1)\sigma_x^2-\rho\right)n}}{2\rho-2\mu_x\theta_\beta-\theta_\beta^2\sigma_x^2+\theta_\beta\sigma_x^2}+\frac{2\alpha_0\left(x^*\right)^{\theta_\alpha}e^{\left(\mu_x\theta_\alpha+\frac{1}{2}\theta_\alpha(\theta_\alpha-1)\sigma_x^2\right)n}e^{-\rho n}}{2\rho-2\mu_x\theta_\alpha-\theta_\alpha^2\sigma_x^2+\theta_\alpha\sigma_x^2}-\\-\frac{\omega\left(x^*\right)e^{(\mu_x-\rho)n}}{\rho-\mu_x}-\frac{\varphi e^{-\rho n}}{\rho}-\gamma\end{array}\right)\right]$$

(3.55)

Now with,

$$A_{tc}=\frac{2(\beta_0-\beta_2)e^{\left(\mu_x\theta_\beta+\frac{1}{2}\theta_\beta(\theta_\beta-1)\sigma_x^2-\rho\right)n}}{2\rho-2\mu_x\theta_\beta-\theta_\beta^2\sigma_x^2+\theta_\beta\sigma_x^2} \tag{3.56}$$

$$B_{tc}=+\frac{2\alpha_0e^{\left(\mu_x\theta_\alpha+\frac{1}{2}\theta_\alpha(\theta_\alpha-1)\sigma_x^2\right)n}e^{-\rho n}}{2\rho-2\mu_x\theta_\alpha-\theta_\alpha^2\sigma_x^2+\theta_\alpha\sigma_x^2} \tag{3.57}$$

$$C_{tc}=-\frac{\varphi e^{-\rho n}}{\rho} \tag{3.58}$$

$$F_{tc}=-\frac{\omega e^{(\mu_x-\rho)n}}{\rho-\mu_x} \tag{3.59}$$

and D equal to (3.25). The subscript tc used above refers to solutions for A, B, C and F that apply to situations in which a time-to-build effect is considered.

The function that has to be maximized is similar to (3.28), with the inclusion of the above-mentioned difference in terms of notation:

$$v(t_0,x_0)=E_{x_0}\left[e^{-\rho\tau}\left(A_{tc}\left(x^*\right)^{\theta_\beta}+B_{tc}\left(x^*\right)^{\theta_\alpha}+F_{tc}\left(x^*\right)+C_{tc}+D\right)\right] \tag{3.60}$$

It is solved in the same way, since it satisfies the same differential equation (3.29) and also boundary conditions (3.31), (3.32) and (3.33).

For a given value of x_0 in $t=0$, the value of x^* that maximizes $v(0,x_0)$ is given by the numerical solution of the equation:

$$A_{tc}x^{*\theta_\beta - r_1}\left(\theta_\beta - r_1\right) + B_{tc}x^{*\theta_\alpha - r_1}\left(\theta_\alpha - r_1\right) + F_{tc}x^{*1-r_1}\left(1-r_1\right) - C_{tc}x^{-r_1}r_1 - Dx^{-r_1}r_1 = 0 \tag{3.61}$$

with r_1 given by (3.40).

When $\theta_\beta = \theta_\alpha = \theta$ and $F_{tc}=0$, x^* is given by the following closed form solution similar to (3.45):

$$x^* = \exp\left[\frac{\ln \dfrac{-r_1\left(C_{tc}+D\right)}{\left(A_{tc}+B_{tc}\right)\left(r_1-\theta\right)}}{\theta}\right] \tag{3.62}$$

In this case, the critical value x^* represents the level of demand that, when reached, justifies (turns optimal), an immediate implementation of a project whose HSR link will start to operate n periods afterwards.

Using the traditional capital budgeting analysis, based on the concept of net present value (NPV), the rationale for taking the decision would be structurally similar, except that the decision would not be taken in an uncertain framework: the capital investment should only take place when the reduction in the cost of travelling provided by the HSR link and measured by the difference between ψ_0 and ψ_2 was enough to cover for the investment capital expenditure plus the operating costs. Analytically, for $\theta = \theta_\beta = \theta_\omega$, $F_{tc}=0$ and any $n \geq 0$, we have,

$$\beta_0 x_{t+n}^\theta + \alpha_0 x_{t+n}^\theta > \beta_2 x_{t+n}^\theta + \varphi + \rho\gamma e^{\rho n} \tag{3.63}$$

Considering $x_{t+n}^\theta \equiv x_t^\theta e^{\theta\mu n}$, it would only become optimal to invest if the demand level reached,

$$x_t > \hat{x} = \left[\frac{\varphi + \rho\gamma e^{\rho n}}{\left(\beta_0 - \beta_2 - \alpha_0\right)e^{\theta\mu n}}\right]^{\frac{1}{\theta}} \tag{3.64}$$

with $\hat{x}$ representing the traditional capital budgeting analysis critical demand level, that once reached would justify the investment.

The comparison between the optimal rule of investment given by ROA (3.45) and by traditional capital budgeting analysis becomes evident if in an investment implemented in one single period of time, we consider $\theta = 1$ as well as inexistence of fixed and variable costs ($\varphi = \omega = 0$).

In this case, equations (3.62) and (3.64) would become,

$$x^* = \frac{-r_1 D}{(r_1 - 1)(A_{tc} + B_{tc})} \tag{3.65}$$

$$\hat{x} = \frac{\rho\gamma}{(\beta_0 - \beta_2 + \alpha_0)} \tag{3.66}$$

Equations (3.65) and (3.66) show that $x^* > \hat{x}$. Thus, when $\hat{x} < x_t < x^*$ a decision to implement based on a traditional capital budgeting analysis framework results in a value reduction for the whole project. In this situation, the value of the projects will be smaller than the sum of the capital expenditure and the value of the (sacrificed) option to defer. The ability to delay has value because it allows for uncertainty resolution.

3.3. Valuation of an HSR Investment Using ROA Framework

Considering the investment value function given by the (3.43), for a given level of x_0, with $t_0 = 0$, the value of an investment opportunity when $x_0 < x^*$ is given by:

$$v(x_0) = \left(\frac{x_0}{x^*}\right)^{r_1} \left[A_{tc}\, x^{*\theta_\beta} + B_{tc}\, x^{*\theta_\alpha} + F_{tc}\, x^* + C_{tc} + D\right] \tag{3.67}$$

while for $x_0 \geq x^*$ the value of the investment opportunity is given by:

$$v(x_0) = \left[A_{tc}\, x^{\theta_\beta - r_1} + B_{tc}\, x^{\theta_\alpha - r_1} + F_{tc}\, x^{1-r_1} + C_{tc}\, x^{-r_1} + D\, x^{-r_1}\right] x_0^{r_1} \tag{3.68}$$

Assuming $\theta = \theta_\beta = \theta_\alpha$ and $F_{tc} = 0$, we may replace the critical value, x^*, given by (3.45) in the second part of the RHS of equation (3.67) and simplifying, the solution of the project's value function may be rewritten in the following terms:

$$v(x_0)=\begin{cases}\left(\dfrac{x_0}{x^*}\right)^{r_1}\left[\dfrac{\theta(C_{tc}+D)}{\theta-r_1}\right] & \text{for } x_0 < x^* \\ (A_{tc}+B_{tc})x_0^{\theta}+C_{tc}+D & \text{for } x_0 \geq x^* \end{cases} \tag{3.69}$$

with C_{tc}, D, r_1, A_{tc} and B_{tc} given by (3.58), (3.25), (3.40), (3.56) and (3.57).

In accordance to the literature (vide McDonald and Siegel, 1986; and Dixit and Pindyck, 1994), from the moment τ, in which the optimal number of passengers is reached, x^*, the value of the option to defer is zero, since it is always better to implement the investment and receive in exchange the NPV - given by $A_{tc}x_0^{\theta_\beta}+B_{tc}x_0^{\theta_\alpha}+F_{tc}x_0+C_{tc}+D$ - of the expected decrease in the cost of travelling.

As long as the optimal timing to implement the investment is not reached, $t<\tau$, there is always an inherent value of waiting for new information about demand. In this case, the value of the option to defer is given by the difference between $v(x_0)$ and the NPV calculated using the expected demand in that moment.

In addition for allowing the inclusion of the impacts produced by *i)* the building period, *ii)* the fixed operating costs and *iii)* the variable operating costs, in the global value of the project, these developments take into consideration the elasticity between the value of travel time and demand. As the model is developed in terms of differential utility, factors other than those related to the cost of travelling (*e.g.*, income), are assumed constant and do not influence the final outcome.

Whenever the elasticity between demand and value of travel time is null ($\delta_\beta=0 \Rightarrow \theta_\beta=1$), we are implicitly assuming that, both, the conventional railway service and HSR service will not suffer real changes in terms value of travel time. Real changes in both services' fares imply positive levels of elasticity. Similarly, the conventional railway service fare remains constant in real terms whenever $\delta_\alpha=0 \Rightarrow \theta_\alpha=1$.

If $\theta_\beta>1$, increases in the value of travel time will be directly related to the passengers' growth rate. This type of demand behaviour for a faster rail transportation related to the value of travel time, besides being economically rational, is supported by the work of Owen and Phillips (1987) and Wardman (1994). In this sense it is acceptable that increases in the demand for the HSR service are, at least partially due to raises by the users in the value of travel time.

When $\theta_\alpha > 1$ the cross elasticity between conventional railway fare and HSR demand is positive. Supported by the works of Owen and Phillips (1987) and Wardman (1997), increases in the fare of substitute service justify increases in the railway service demand.

The global value of a project determined by this ROA framework includes the economic worth of the ability to wait for uncertainty resolution, provided by the option to defer. When the ability to delay does not exist, as in the traditional capital budgeting decision analysis, this component is not taken in consideration and the global result will underestimate the corresponding true value. The value embedded in the option to postpone the investment derives from the incorporation of the value inherent in the "good tail" of the uncertainty regarding the demand by the HSR service. Turning parallel, the "bad tail" of demand uncertainty is limited by the option to carry on deferring (not investing), if the situation does not look attractive enough (McDonald and Siegel, 1986 and Dixit and Pindyck, 1994).

4. Numerical Example

Table 1. Base-case parameters for the project

Parameter	Value
x_0 – HSR demand at actual moment	3 M
γ – Present value of the investment expenditures	5,000 M€
η_0 – Value of travel time in conventional railway service	30 €
η_2 – Value of travel time in HSR service	10 €
p_0 – Conventional railway service fare	25 €
ω – Variable operating costs	1 €
φ – Fixed operating costs	90 M€
ρ – Discount rate	0.09
μ_x – Expected growth rate of x	0.035
σ_x – Standard deviation of x	0.20
n – Number of years for the construction	5
δ_β – Elasticity between x and η	0.60
δ_α – Cross elasticity between x and p_0	0.40

Note: M = Millions

We are going to assume a project for the construction of a HSR link connecting two cities. The basic parameters are in Table 1. The conventional railway service operates in the same link. The new HSR service will reduce the travel time to ⅓ comparatively to the conventional railway service.

Table 2 presents the HSR line investment valuation results for the base-case parameters.

Table 2. Project valuation results

Output	Value
x^* – Critical demand for HSR service (n.º passengers)	10.777 M
$v(x)$ – Investment Opportunity Value	3,743.3 M€
npv – Net Present Value	254.2 M€
vod – Value of the Option do Defer	3,489.1 M€

Based on the results obtained, the construction of the HSR line should only start when the demand reaches 10,777 millions of passengers. Although the project registers a slightly positive NPV, shouldn't be implemented at the current time, concerning the uncertainty regarding the number of passengers of the new service. Maintaining "alive" this investment's opportunity has a value of 3,743 millions of euros, of which 93,21% results from the value of the option to defer the investment.

Figure 1. Investment's opportunity value, NPV and value of the option to defer, for the base case

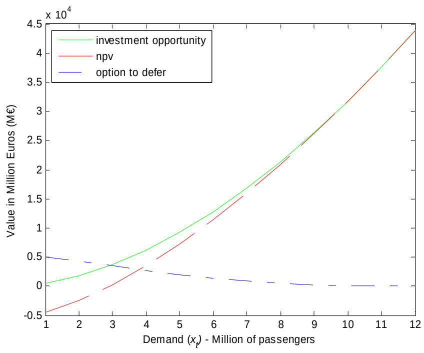


Figure 1 represents the evolution of the investment's opportunity value, the NPV and the option to defer according to the demand x_t increase throughout time. As we may observe, for levels of demand higher than the critical demand level of 10,777 millions passengers, the option to defer the implementation no longer has value. Thus, from this point on, the decision to immediately implement the project is the one which maximizes the value for its owners.

Figure 2 to Figure 7 show the sensibility of the valuation indicators of the project regarding the variation of some parameters. Thus, we may notice that critical demand level x^* varies inversely with the demand growth rate μ_x (Figure 2) and with the reduction of the value of travel time given by $\frac{\eta_0 - \eta_2}{\eta_0}$ which the HSR line enables (Figure 7). For higher demand growth rates μ_x and with major reductions in the value of travel time, the present value of the benefits resulting from the project increases, justifying anticipating its implementation.

Figure 2. The impact of the growth rate

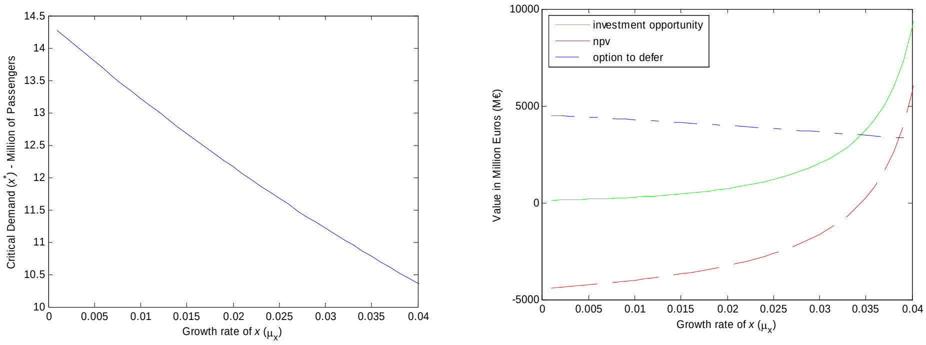


Figure 3. The impact of the discount rate

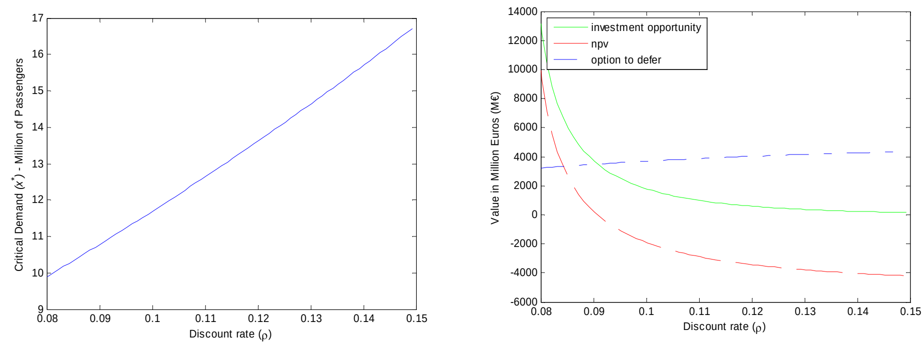


Figure 4. The impact of the investment expenditures

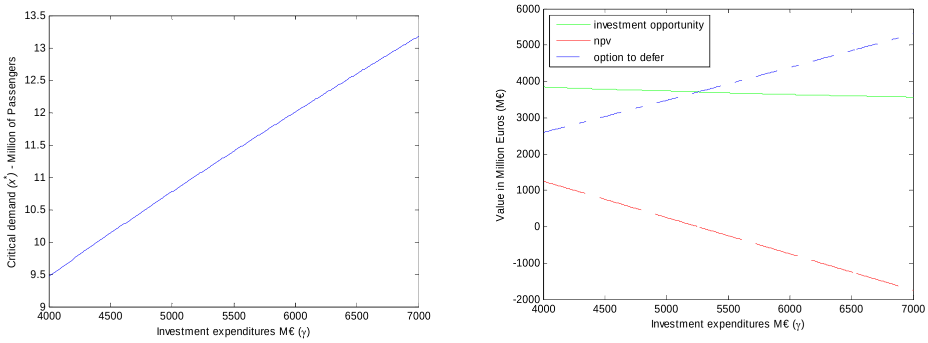


Figure 5. The impact of the volatility of the number of passengers

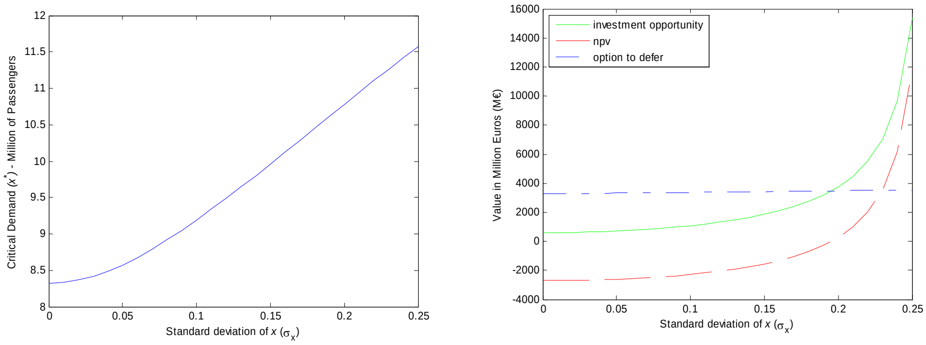
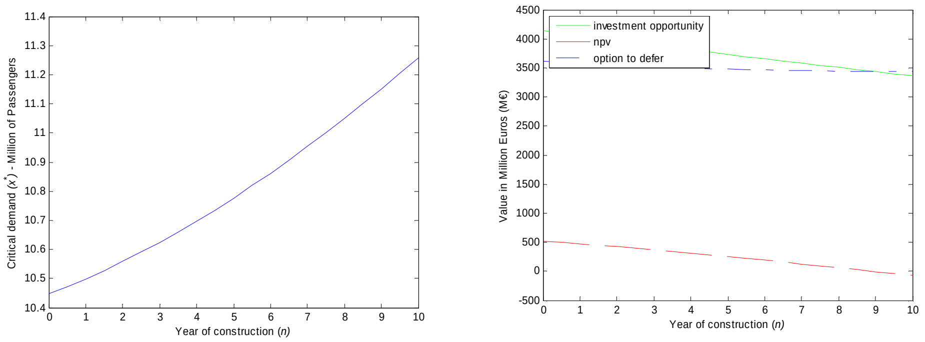
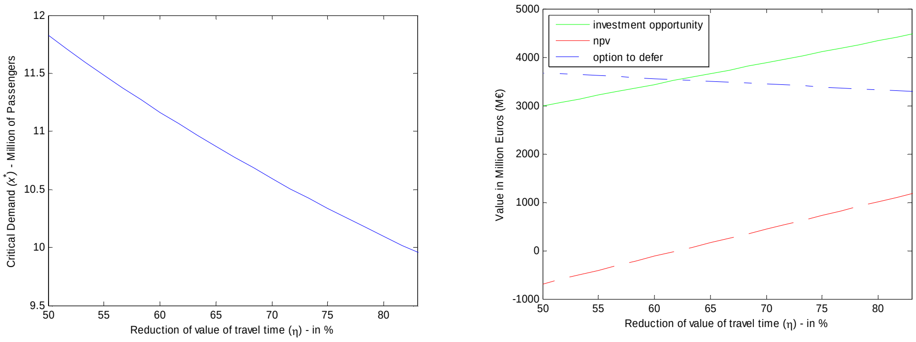


Figure 6. The impact of the time-to-build



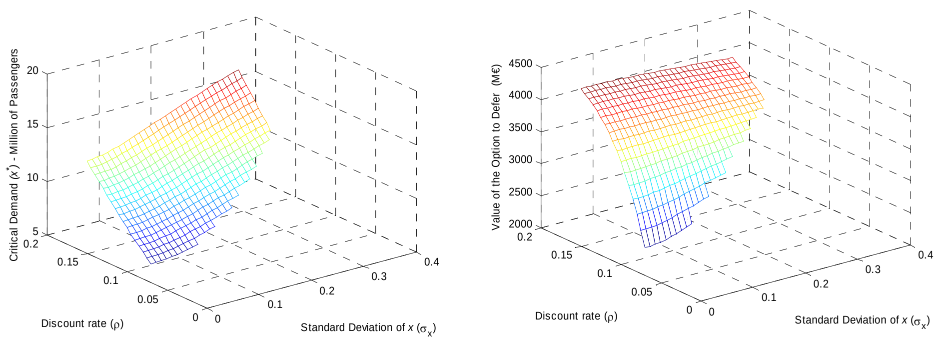
The other parameters analyzed assume a direct relationship with the critical level of demand x^*. Larger discount rates (Figure 3), larger investment's expenditures (Figure 4), larger volatility in the number of passengers (Figure 5) or more construction time needed (Figure 6) instigate significant postponements in the projects' implementation.

Figure 7. The impact of the reduction in the value of travel time



In presence of variations in any of the analyzed parameters, the investment's opportunity value and the NPV always register the same trend, for each one of the parameters, although with different drifts. It is worth noting the behaviour of NPV regarding increments in volatility. Figure 5 shows that NPV increases with uncertainty increase. This result originates from the fact that the valuation model incorporates the elasticity between HSR demand and the value of travel time and the cross elasticity between the HSR demand and the conventional service fare. This specificity of the developed model results in a value of the option to defer that slightly diminishes with the increase of uncertainty. These findings can also be seen in Figure 8. It is always assumed that the discount rate remains unchanged as the volatility of the project changes.

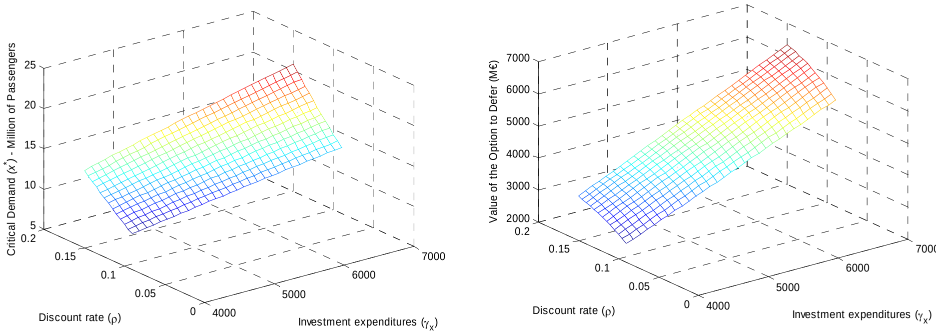
Figure 8. The impact of both the volatility of the number of passengers and the discount rate



In case a larger period of time to implement the project is needed, the increase in uncertainty throughout time and the delay on the benefits from the investment's operation instigate a reduction in the investment's opportunity value and in the NPV (Figure 6).

Figure 9 gives the joint impact of both the discount rate ρ and the investment expenditures γ in the critical demand value x^* and in the value of the option to defer. Both valuation outputs shows a direct relationship with these two parameters, turning the option to defer more valuable as this project parameters value increase. As showed in Figure 3 and Figure 4 this is due to a deeper decrease in NPV than the one registered in the investment's opportunity value.

Figure 9. The impact of both the investment expenditures and the discount rate



5. Conclusion

The present work develops a model aimed at finding the optimal timing to implement a HSR investment, in an uncertain environment. We have introduced several adjustments to the original valuation model of the option to defer (McDonald and Siegel, 1986) and to the optimal stopping model of Salahaldin and Granger (2005), given the need to design a model applicable to an HSR investment in an environment of stochastic demand. As far has we are aware, the development of closed form solution ROA's models to value railway investments was never done before.

The existence of a conventional railway service enables the analysis of the investment in HSR to be performed in an incremental basis, measured in terms of the corresponding utility functions. The indifference in the demand utility between HSR and conventional railway services makes possible for the problem to be equated in terms of finding a critical demand level that justifies the implementation of the investment.

The presented developments, regarding the optimal timing to invest and the investment's opportunity value, have the advantage of offering a clear way to evaluate the utility of the HSR investment in each moment in time, for the set of potential users - the society in general. The numerical example and simulation of some important input parameters demonstrates the consistency of the model concerning the behaviour of the valuation outputs.

In future research, it should be possible to enrich the model in order to include more uncertainty sources – like the fare price and the investment expenditure. Additionally, we expect to perform an empirical application[1] capable of providing the feedback necessary to guide additional improvements in the structure of the modelling framework.

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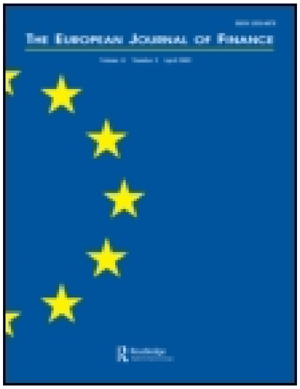

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In this paper, the optimal timing for investing in high-speed rail projects under uncertainty in relation to the utility provided to railway users was investigated. To accomplish this, a continuous time real options analysis framework using a stochastic demand model was developed to determine the optimal time to invest. Uncertainty upon investment expenditures was also added in an extended framework. The value of the option to defer and the investment opportunity value were also assessed.

Keywords: real options; uncertainty; timing; waiting; investment; high-speed rail

JEL Classification: D81; D83; D92

1. Introduction

Under uncertainty, it is important for a firm to be flexible with the products it is creating, to accommodate for technological changes and competition shifts. Flexibility is also crucial to limit potential losses related to unexpected adverse scenarios (Trigeorgis 1996).

Real option analysis (ROA) emerged in the academy to value investments in real assets under uncertainty (Brennan and Schwartz 1985; McDonald and Siegel 1986; Dixit 1989; Pindyck 1991; and Dixit and Pindyck 1994, amongst others). Nowadays ROA has already made an impact in the business world, since an increasing number of companies and managers are adopting a real options perspective. This new paradigm has been used in the area of capital budgeting analysis and in the assessment of strategic positioning and competitiveness (Paddock, Siegel and Smith 1988; Nichols 1994; Kallberg and Laurin 1997; Moel and Tufano 2002; Smit 2003).

Transport infrastructures are critical for sustainable growth and the development of an economy. According to Wilson (1986), since the year 1870, economists have drawn their attention to transportation, where rail transport assumes an important role. The same author suggested that poor transportation policies and investment mistakes in transport infrastructures may compromise economic growth. To prevent this compromise in growth, it is important to develop and apply suitable decision criteria based on a thorough cost/benefit analysis.

Infrastructure investments, such as in seaports, airports, railways, energy networks, and road systems, have provided huge economic benefits and have leveraged economic growth. The size, budget and impact of these investments on the global economic activity led transportation investments to assume the role of strategic options. Almost all transportation investments include a portfolio of options to protect the enormous funds required to implement the investments.

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Rose (1998) valued the concession of a toll road in Australia, considering the existence of two options interacting with each other. Similarly, Brandão (2002) applied the Copeland and Antikarov (2001) framework to value options embedded in a highway investment project in Brazil. Two other empirical ROA studies focused on the valuation of structural investments in the transportation sector: Smit (2003) and Bowe and Lee (2004). The first analyzed the expansion of a European airport. Smit (2003) combined ROA and game theory, in a discrete time framework, to fill a gap in the real options literature regarding competition effects. The second analyzed a railway transportation investment. Bowe and Lee (2004) applied binomial analysis to evaluate high-speed train investment in Taiwan.

The valuation framework proposed in this paper was inspired by a set of investments in high-speed rail (HSR) across Europe. The structural nature of the HSR investments for the countries involved; the need to renew the railway sector; the huge amounts of money required; the uncertainty about the timings to invest; and the economic challenge inherent in developing a conceptual setting for a decision that needs to consider the interest of all European taxpayers, all play a part in providing relevance to the study of the ideal time to invest and the embedded options to be deferred.

The framework proposed draws on the work of Salahaldin and Granger (2005) on the valuation of sustainable systems of urban transport aimed at relieving air pollution. Like their framework, ours will be comprised of a unique change, from an inactive to an active state; it will also consider a single stochastic variable, extended afterwards to two uncertainty factors. However, our framework is distinct, because it incorporates the time to build, combining it with the benefits of travel time saved. In addition, fixed and variable operating costs will be incorporated.

Transportation investment analysis rarely incorporates real option theory. As a result, this paper will introduce the transportation investment analysis of the HSR investment valuation in continuous time, providing some closed form solutions. Although Pereira *et al.* (2006) studied these issues, their work focused on airport construction. Our ROA framework will support the utility balance for the user between different rail speed services.

Compared to classical works in the economic literature, such as McDonald and Siegel (1986), we will extend the methodology regarding closed form solutions for the value of waiting to invest. We also broaden the applications of ROA to HSR investments and incorporate issues related to elasticity in ROA frameworks. We hope insights provided by this new valuation framework will be useful in many areas of the transportation industry. For example, it may help improve the investment flexibility in order to reduce the delay in the optimal time to invest.

The rest of the paper will be structured as follows. In Section 2, we develop the valuation framework. In Section 3, we provide the numerical results. In Section 4 uncertainty upon investment expenditures is added in an extended framework. The paper's primary conclusions and recommendations for future studies are presented in Section 5.

2. Investment valuation using a real options framework

In HSR investment, the owner of the investment holds the possibility of acquiring the future cash flow generated, at any moment in time, in exchange for the payment of the corresponding implementation costs. Thus, we will be investigating an option to invest. Considering the investment in HSR as an optimal stopping problem permits us to determine the value of the embedded option to defer. Following the work of McDonald and Siegel (1986) and Salahaldin and Granger (2005), our model will allow us to determine the optimal time period to invest. The model considers an *a priori* dimensioned HSR project.

In this paper, we will assume that the option to defer is perpetual in nature ($T = \infty$), but also that, once implemented, the investment will produce perpetual benefits. Without major technological changes, the impact of these assumptions in the global valuation should not be unreasonable for two reasons. First, because the present value of the more remote cash flows tends naturally to zero. Second, because maintenance and conservation expenses tend to restore the operational aptitude of the assets and the corresponding flow of benefits.

2.1 *Optimal timing to invest*

The decision to implement an investment in a non-optimal way implies the destruction of value. Finding the optimal timing to invest provides us with the possibility to value the ability to delay the project, as well as also its corresponding impact on the investment opportunity value. The optimal timing to invest may be given by a demand threshold supporting, in a rational way, the decision of implementing the investment. Once implemented, the investment expenditures become sunk costs, since there is no other use for railways.

Because investment in infrastructures, like HSR, will affect the economic and social conditions of future generations, it should be assessed in terms of economic welfare. The value per railway user is calculated based on utility theory, namely on the consumption capacity of each user. Investing in a moment other than the optimal timing implies a reduction in the global level of utility achieved by the users, compromising the HSR investment success. In such circumstances, any potential user may maintain his/her current level of utility, choosing to travel on the conventional railway, rather than in the new HSR. If a suboptimal investment timing is chosen, the ability of the HSR to attract users will be strongly distressed.

The framework does not explicitly account for the competition between railway transportation and other alternative transport modes. Implicitly competition effects are incorporated in railway demand stochastic process parameters. Indeed, any user from other alternative transport modes, such as road or air flights, is a potential railway user. However, facing two similar railway travel services, users will decide for HSR if at least utility remains. Note that competition effects from the conventional railway or other alternative transport modes through its fare, frequency or better service improvements should also be implicitly incorporated in the HSR demand stochastic process parameters.

At any moment, users may choose to travel by the conventional railway, without any constraints. Consequently, to maintain the users' level of utility, the fraction of the new investment supported by each one must be identical to the sum of the benefits earned from the reduction in travel time and the conventional railway travel fare saved, net of the variable and fixed operational costs upheld. Given fixed investment expenditures, the higher the demand, the higher the expected net benefit per capita. Consequently, higher levels of demand tend to lead to the anticipation of the optimal invest timings. The main source of uncertainty is derived from the level of future HSR demand.

We will consider that the demand for the new HSR, x_t, follows a geometric Brownian motion process:

$$dx_t = \mu_x x_t dt + \sigma_x x_t dw_t, \tag{1}$$

where μ_x represents the growth rate and σ_x represents the standard deviation of HSR demand. We assume that both parameters are constant in time. The Wiener process, w_t, has a zero mean and standard deviation of $\sigma_x \sqrt{dt}$. Under these circumstances, it is reasonable to expect that, in the future, the HSR demand will reach a level capable of providing a rational reason to invest.

Similar assumptions were found in Rose (1998), who modeled highway traffic, Salahaldin and Granger (2005), who modeled the dynamics of a city's population, and Marathe and Ryan (2005) and Pereira *et al.* (2006), who modeled airline demand. Emery and McKenzie (1996), on the other hand, implicitly assumed that income from the railway followed a geometric Brownian motion process.[1] Bowe and Lee (2004) implicitly assumed the discrete time analogue of a geometric Brownian motion for the operational cash flows of an HSR investment.

Assuming that each user will face a cost for travel between two cities, ψ, that is a function of the total value of travel time for the user, η, and the travel fare, p. According to the literature, both of these variables exhibit a relationship to railway demand (*vide* Owen and Phillips 1987; Wardman 1994; and Wardman 1997).

The following functional form illustrates the relationship between the total value of travel time and the demand for faster railway travel (Owen and Phillips 1987; Wardman 1994):

$$\eta(x_t) = \beta x_t^{\delta_\beta}, \tag{2}$$

where δ_β represents the elasticity between the total value of travel time, η, and the HSR demand, x. Consequently, β is the scale parameter between HSR demand, x, and the total value of travel time, η.

The relationship between the travel fare and the HSR demand is given by the following functional form (Owen and Phillips 1987):

$$p(x_t) = \alpha x_t^{\delta_\alpha}, \tag{3}$$

where the elasticity between the fare value, p, and the HSR demand, x, is represented by the parameter δ_α. The scale parameter α relates HSR demand, x, and the travel fare, p.

We now assume a risk neutral user with a utility function $U(c) = c$. This utility is solely the function of the mean consumption per user, in which c represents the mean consumption of all users. The budget constraint is given by

$$m_t = c_t - \psi(x_t) \tag{4}$$

where ψ represents the travel cost and m the individual disposable income by unit of time.

Replacing the level of consumption in the utility function, we obtain the following value function, V, representative of the value that each user confers to a railway trip:

$$V(x_t) = U(c_t) = m_t - \psi(x_t). \tag{5}$$

The relationship between HSR demand, x_t, and the total value of travel time, (i) in the period of time that precedes the investment, η_0; (ii) during the investment's implementation, η_1; and (iii) after the investment's implementation, η_2, is represented, respectively, by β_0, β_1 and β_2. Since the HSR will save travel time and, consequently, will reduce the total value of travel time from η_0 to η_2, it will be reasonable to expect that from the pre-investment period to the operational period, β_0 will become β_2, with $\beta_0 > \beta_2$. The difference between β_0 and β_2 reflects the decrease in travel time.

Large investments need time to be implemented. Thus, it is crucial to incorporate the time-to-build in the ROA framework. During the building period, n, the cost of travelling is given by ψ_0 and the relationship between demand and travel costs, β_1, remains equal to β_0 ($\beta_1 = \beta_0$). When the HSR begins to operate, the cost of travelling will change to ψ_2 and β_2 reflects the decrease in travel time. Analytically, the cost of travelling by a conventional railway, ψ_0, and the cost of travelling by the HSR, ψ_2, will be represented by the following equations:

$$\psi_0(x_t) = \beta_0\, x_t^{\delta_\beta} + \alpha_0 x_t^{\delta_\alpha}, \tag{6}$$

$$\psi_2(x_t) = \beta_2 x_t^{\delta_\beta}, \tag{7}$$

where the conventional railway travel cost, ψ_0, includes both the total value of the travel time lost and the travel fare. In contrast, the HSR travel cost function, ψ_2, is not affected by the corresponding travel fare, p_2, because the current valuation framework implicitly assumes that each user will bear his/her part of the investment expenditure plus the corresponding operating costs per user. Hence, a socially acceptable HSR travel fare is already implicitly considered in the valuation framework. Consequently, it does not make sense to duplicate it.

The conventional railway with a travel fare p_0 enables us to identify the relationship between HSR demand, x_t, and the price of a substitute service (Owen and Phillips 1987; Wardman 1997) given by Equation (3).

Until the HSR begins to operate, the value function per user will be given by

$$V_0(x_{t+n}) = m_{t+n} - \beta_0 x_{t+n}^{\delta_\beta} - \alpha_0 x_{t+n}^{\delta_\alpha} \tag{8}$$

After the investment is implemented, the users will continue to face a value for travel time, but it will be a smaller travel time. However, since the analysis performed here takes into consideration all costs and benefits induced by the HSR investment (including the investment expenditure, fixed and variable operating costs), the new value function per user will be given by

$$V_2(x_{t+n}) = m_{t+n} - \beta_2 x_{t+n}^{\delta_\beta} - \omega - \frac{\varphi}{x_{t+n}} - \frac{\rho\gamma\, \mathrm{e}^{\rho n}}{x_{t+n}}, \tag{9}$$

where γ represents the capital investment expenditure, ρ the discount rate, ω the variable operating costs and φ the fixed operating costs. Notice that φ/x_t and $\rho\gamma/x_t$ represent the fixed operating costs and the investment expenditure per unit of time for each HSR user. We implicitly assume that the HSR investment cash flows will last for an unlimited time horizon.

Using the objective function of Ramsey–Koopmans to compute the net benefits generated by the HSR investment, according to the appendix, the investment opportunity value, here denoted by $v(x)$, is given by

$$v(x) = \int_0^\infty \mathrm{e}^{-\rho(t+n)} E[V_2(x_{t+n}) - V_0(x_{t+n})]\mathrm{d}t. \tag{10}$$

The purpose of the model is to calculate the optimal timing to invest preserving utility function balance. For that, it is necessary to locate the HSR demand threshold, x^*, above which it will be optimal to invest. Thus one wants to find the optimal value x^* such that

$$v(x^*) = \sup_x v(x). \tag{11}$$

The investment opportunity value is determined through the maximization of the following equation (appendix):

$$v(x^*) = A_{tc}(x^*)^{\theta_\beta} + B_{tc}(x^*)^{\theta_\alpha} + F_{tc}(x^*) + C_{tc} + D \tag{12}$$

with

$$A_{tc} = \frac{2(\beta_0 - \beta_2)\mathrm{e}^{(\mu_x\theta_\beta + (1/2)\theta_\beta(\theta_\beta - 1)\sigma_x^2 - \rho)n}}{2\rho - 2\mu_x\theta_\beta - \theta_\beta^2\sigma_x^2 + \theta_\beta\sigma_x^2} \tag{13}$$

$$B_{tc} = +\frac{2\alpha_0\,\mathrm{e}^{\left(\mu_x\theta_\alpha + (1/2)\theta_\alpha(\theta_\alpha - 1)\sigma_x^2 - \rho\right)n}}{2\rho - 2\mu_x\theta_\alpha - \theta_\alpha^2\sigma_x^2 + \theta_\alpha\sigma_x^2} \tag{14}$$

$$C_{tc} = -\frac{\varphi\,\mathrm{e}^{-\rho n}}{\rho} \tag{15}$$

$$F_{tc} = -\frac{\omega\,\mathrm{e}^{(\mu_x - \rho)n}}{\rho - \mu_x} \tag{16}$$

$$D = -\gamma \tag{17}$$

To include economic intuition, note that: A_{tc} reflects the present value of travel time savings; B_{tc} reflects the present value of conventional railway travel fare; C_{tc} reflects the present value of fixed operating costs; D represents the present value of investment expenditures; and F_{tc} represents the present value of variable operating costs. The subscript tc, used above, indicates the time-to-build effect.

Now, as the investment opportunity function, $v(\cdot)$, is a function of the demand process $\{x_t\}$ that follows a geometric Brownian motion, if we apply Ito's lemma to $v(x_t)$, we end up with the following ordinary differential equation:

$$\frac{1}{2}\sigma_x^2 x^2 v''(x) + \mu_x\, x\, v'(x) - \rho v(x) = 0 \quad \text{for } x \neq x^* \tag{18}$$

subject to the boundary equations:

$$v(0) = 0 \tag{19}$$

$$v(x) = A_{tc}x^{\theta_\beta} + B_{tc}x^{\theta_\alpha} + F_{tc}x + C_{tc} + D \quad \text{with } x = x^* \tag{20}$$

$$v'(x) = \theta_\beta A_{tc}\,x^{\theta_\beta - 1} + \theta_\alpha B_{tc}x^{\theta_\alpha - 1} + F_{tc} \quad \text{with } x = x^* \tag{21}$$

Note that the first condition means that the process is absorbing when the HSR demand is 0; the second is the value-matching condition; and the third is the smooth-pasting condition.

Therefore the investment opportunity function, $v(\cdot)$, considering the current HSR demand, is given by the supremum of Equation (11), that satisfies the differential Equation (18).

Since Equation (18) is a *Cauchy–Euler* second order homogeneous differential equation, the solution may be written as

$$v(x) = a_1 x^{r_1}, \tag{22}$$

where r_1 is the positive root of the quadratic equation:

$$\frac{1}{2}\sigma_x^2 r(r-1) + \mu_x r - \rho\ = 0 \tag{23}$$

given by

$$r_1 = \frac{((1/2)\sigma_x^2 - \mu_x) + \sqrt{(\mu_x - (1/2)\sigma_x^2)^2 + 2\rho\sigma_x^2}}{\sigma_x^2} \tag{24}$$

Using the condition $v(x^*) = A_{tc}x^{*\theta_\beta} + B_{tc}\,x^{*\theta_\alpha} + F_{tc}x^* + C_{tc} + D$, we calculate the coefficient $a_1 = A_{tc}\,x^{*\theta_\beta - r_1} + B_{tc}\,x^{*\theta_\alpha - r_1} + F_{tc}\,x^{*1-r_1} + C_{tc}\,x^{*-r_1} + D\,x^{*-r_1}$, concluding that the solution of Equation (18) is

$$v(x) = [A_{tc}\,x^{*\theta_\beta - r_1} + B_{tc}\,x^{*\theta_\alpha - r_1} + F_{tc}\,x^{*1-r_1} + C_{tc}\,x^{*-r_1} + D\,x^{*-r_1}]\,x^{r_1} \tag{25}$$

For a given value of x in $t = 0$, the value of x^* that maximizes $v(x)$ is given by the numerical solution of the equation:

$$A_{tc}x^{*\theta_\beta - r_1}(\theta_\beta - r_1) + B_{tc}x^{*\theta_\alpha - r_1}(\theta_\alpha - r_1) + F_{tc}x^{*1-r_1}(1 - r_1) - C_{tc}x^{*-r_1}r_1 - Dx^{*-r_1}r_1 = 0 \tag{26}$$

with r_1 given by Equation (24).

The HSR demand threshold, x^*, may only be found through a numerical solution of Equation (26), except if two assumptions are made. The first assumption, related to the equality between the total value of travel time/HSR demand elasticity and the conventional railway travel fare/HSR demand cross elasticity, equaling $\theta_\beta = \theta_\alpha = \theta$. This assumption means that the conventional railway travel fare and the value of travel time have a similar trend. The second assumption comes from the possibility of neglecting the operational variable costs, $F_{tc} = 0$, considering the operational characteristics of the HSR investment. Operational variable costs include essentially those related to ticket printing, since for an *a priori* dimensioned HSR project with a pre-established operation schedule, all the major operational costs tend to be fixed.

Considering these two assumptions, the HSR demand threshold, x^*, obtains the following closed form solution:

$$x^* = \exp\left[\frac{\ln((-r_1(C_{tc} + D))/((A_{tc} + B_{tc})\,(r_1 - \theta)))}{\theta}\right] \tag{27}$$

When the HSR demand threshold, x^*, is reached, it justifies (becomes optimal) an immediate implementation of the HSR investment, which will begin to operate n periods afterwards. This solution preserves the utility balance for users between the HSR and the conventional railway, making the optimal solution independent of the original income, m, and the initial HSR demand, x_0. Because the framework deals with an economic welfare issue, based on the utility balance for users between two similar transportations, this framework is especially adequate to analyze governmental scale investment decisions.

Using traditional capital budgeting analysis, based on the net present value (NPV), the rationale for making the decision would be similar. The investment should only be implemented when the reduction in the cost of travelling provided by the HSR and measured by the difference between ψ_0 and ψ_2 is enough to cover the investment expenditure plus the operating costs. Analytically, for $\theta = \theta_\beta = \theta_\omega$, $F_{tc} = 0$ and any $n \geq 0$, we have

$$\beta_0 x_{t+n}^{\theta} + \alpha_0 x_{t+n}^{\theta} > \beta_2 x_{t+n}^{\theta} + \varphi + \rho\gamma\,\mathrm{e}^{\rho n} \tag{28}$$

Considering $x_{t+n}^{\theta} \equiv x_t^{\theta} e^{\theta\mu n}$, it would only become optimal to invest when the HSR demand reaches the threshold $\hat{x}$,

$$x_t > \hat{x} = \left[\frac{\varphi + \rho\gamma\,\mathrm{e}^{\rho n}}{(\beta_0 - \beta_2 - \alpha_0)\mathrm{e}^{\theta\mu n}}\right]^{1/\theta} \tag{29}$$

with $\hat{x}$ representing the HSR demand threshold given by the traditional capital budgeting analysis.

The comparison between the optimal decision to invest, given by ROA Equation (27) and by the traditional capital budgeting analysis, becomes evident if in an investment implemented in one single period of time, we consider $\theta = 1$, as well as no fixed and variable costs ($\varphi = \omega = 0$).

In this case, Equations (27) and (29) would become:

$$x^* = \frac{-r_1 D}{(r_1 - 1)(A_{tc} + B_{tc})} \tag{30}$$

$$\hat{x} = \frac{\rho\gamma}{(\beta_0 - \beta_2 + \alpha_0)} \tag{31}$$

Equations (30) and (31) illustrate that $x^* > \hat{x}$. Thus, when $\hat{x} < x_t < x^*$, the decision to invest based on a traditional capital budgeting analysis results in a value reduction for the investment. In this situation, the investment opportunity value will be smaller than the sum of the investment expenditure and the value of the (sacrificed) option to defer. The ability to delay has value, because it allows the uncertainty resolution.

2.2 *Valuation of an HSR investment using an ROA framework*

Consider the investment value function given by Equation (25), for a given HSR demand, x, with $t = 0$. The investment opportunity value when $x < x^*$ is given by

$$v\,(x) = \left(\frac{x}{x^*}\right)^{r_1} \left[A_{tc}\, x^{*\theta_\beta} + B_{tc}\, x^{*\theta_\alpha} + F_{tc}\, x^* + C_{tc} + D\right] \tag{32}$$

while for $x \geq x^*$, the investment opportunity value is given by

$$v(x) = \left[A_{tc}\, x^{\theta_\beta - r_1} + B_{tc}\, x^{\theta_\alpha - r_1} + F_{tc}\, x^{1-r_1} + C_{tc}\, x^{-r_1} + D\, x^{-r_1}\right] x^{r_1} \tag{33}$$

Assuming $\theta = \theta_\beta = \theta_\alpha$ and $F_{tc} = 0$, we may replace the HSR demand threshold, x^*, given by Equation (27) in the second part of the RHS of Equation (32). After simplifying, the investment opportunity value may be rewritten in the following terms:

$$v(x) = \begin{cases} \left(\dfrac{x}{x^*}\right)^{r_1} \left[\dfrac{\theta(C_{tc} + D)}{\theta - r_1}\right] & \text{for } x < x^* \\ (A_{tc} + B_{tc})x^{\theta} + C_{tc} + D & \text{for } x \geq x^* \end{cases} \tag{34}$$

with C_{tc}, D, r_1, A_{tc} and B_{tc} given by Equations (15), (17), (24), (13) and (14).

In accordance to previous studies (vide McDonald and Siegel 1986; and Dixit and Pindyck 1994), from the moment τ, in which the HSR demand threshold is reached, x^*, the value of the option to defer is zero. As a result, it is always better to invest and receive in exchange the NPV – given by $A_{tc}x^{\theta_\beta} + B_{tc}x^{\theta_\alpha} + F_{tc}x + C_{tc} + D$ – of the expected decrease in the cost of travelling.

As long as the optimal timing to invest has not been reached, $t < \tau$, there is always an inherent value of waiting for new information about the HSR demand. In this case, the value of the option to defer is given by the difference between the investment opportunity value, $v(x)$, and the NPV calculated, using the expected HSR demand at that moment. In addition, for allowing the inclusion of the (i) time-to-build, (ii) fixed operating costs and (iii) variable operating costs, in the investment opportunity value, these developments take into consideration the elasticity between the value of travel time and demand. As the framework is developed, considering the users utility balance between the HSR and conventional railway travel, factors other than those related to travelling (e.g. income) are assumed to be constant and do not influence the outcome.

The HSR transport uses clean energy. However the impact of the fuel price expectations on fuel dependent transport modes is captured indirectly thought the HSR demand stochastic process parameters.

Whenever the elasticity between the total value of travel time and the HSR demand is null ($\delta_\beta = 0 \Rightarrow \theta_\beta = 1$), we are implicitly assuming no real changes in the value of travel time. Real changes imply positive levels of elasticity. Similarly, the conventional railway travel fare remains constant in real terms whenever $\delta_\alpha = 0 \Rightarrow \theta_\alpha = 1$. If $\theta_\beta > 1$, increases in the value of travel time will be directly related to the demand growth rate. The demand behavior for faster rail transportation when the value of travel time rises is economically rational, as supported by Owen and Phillips (1987) and Wardman (1994). Therefore, it is acceptable that increases in the HSR demand are, at least partially, due to increases in the value of travel time. When $\theta_\alpha > 1$, the cross elasticity between the conventional railway travel fare and the HSR demand is positive. According to Owen and Phillips (1987) and Wardman (1997), an increase in the travel fare of substitute services justifies increases in the railway demand.

The investment opportunity value determined by this ROA framework includes the ability to wait for an uncertainty resolution, provided by the option to defer. When the ability to delay does not exist, as in the traditional capital budgeting decision analysis, this component is not taken into consideration, underestimating the investment opportunity fair value. The value of the option to postpone the investment comes from the incorporation of the "good tail" of the HSR demand uncertainty. The "bad tail" of demand uncertainty is limited by the option to defer the HSR investment until the situation becomes attractive enough (McDonald and Siegel 1986; Dixit and Pindyck 1994).

3. Numerical illustration

Assume a project for the construction of an HSR connecting two cities. The basic parameters values will include those in Table 1, supported by the released Portuguese Government data on the HSR investment. The construction period is 5 years and the investment expenditure's present value is 15 billion Euros. According to HSR demand studies provided, the actual HSR demand is 3 million passengers and should rise 3.5% per year with 20% standard deviation. However, the expected HSR demand growth rate and standard deviation could be estimated by the mean and variance of demand instantaneous growth rate upon historical data.

Table 1. Base-case parameters for the HSR investment.

Parameter	Value
x – HSR demand at the actual moment	3 M
γ – Present value of the investment expenditures	€5000 M
η_0 – Total value of travel time by the conventional railway	€30
η_2 – Total value of travel time by the HSR	€10
p_0 – Conventional railway travel fare	€25
ω – Variable operating costs	€1
φ – Fixed operating costs	€90 M
ρ – Discount rate	0.09
μ_x – HSR demand expected growth rate	0.035
σ_x – HSR demand standard deviation	0.20
n – Time-to-build (years)	5
δ_β – Elasticity between the total value of travel time and the HSR demand	0.60
δ_α – Cross elasticity between the conventional railway travel fare and the HSR demand	0.40

Note: M, millions.

Table 2. HSR investment valuation results.

Output	Value
x^* – HSR demand threshold	10.777 M
$v(x)$ – Investment opportunity value	€3743.3 M
npv – Net present value	€254.2 M
vod – Value of the option to defer	€3489.1 M

The conventional railway operates between the same two cities. The new HSR will reduce the travel time to one third comparative to the conventional railway travel. Therefore, a 3-hour journey by the conventional railway will be around 1 hour by HSR. Using official data provided by the EU guide to appraise infrastructural investments, the estimated value of travel time per hour in Portugal is 10 Euros. It is also considered that 65% of the passengers travel on non-working time and 35% on working time. The total value of travel time by the conventional railway and HSR is given by the multiplication of the value of travel time per hour and the travel time spent for each railway service.

Table 2 presents the HSR investment valuation results for the base-case parameters.

According to the results (Table 2), the HSR construction should only begin when the demand reaches 10.777 million passengers. Because investment expenditures and operational costs were considered as *a priori* known, the demand threshold obtained is therefore suitable only for an *a priori* dimensioned HSR project. Although the HSR investment has a slightly positive NPV, it should not be implemented at the current time. The HSR demand uncertainty forces a delay in the HSR investment. Maintaining "alive", this investment opportunity has a value of 3743 million Euros. The value of the option to defer the investment represents 93.21% of the investment opportunity value.

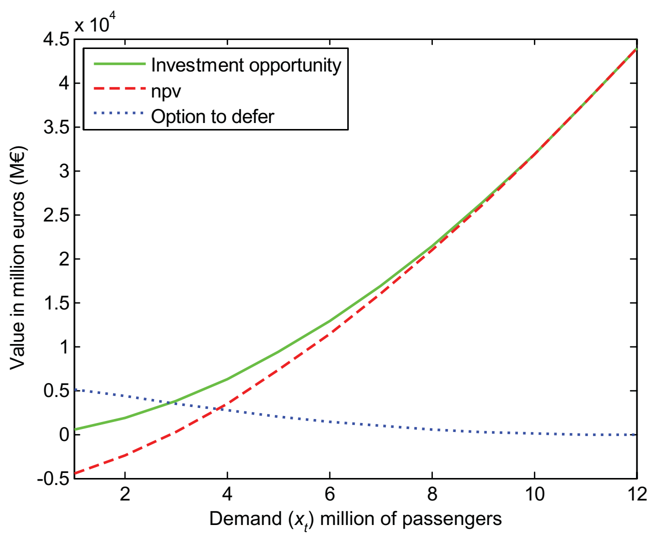

Figure 1. Investment opportunity value, NPV and value of the option to defer.

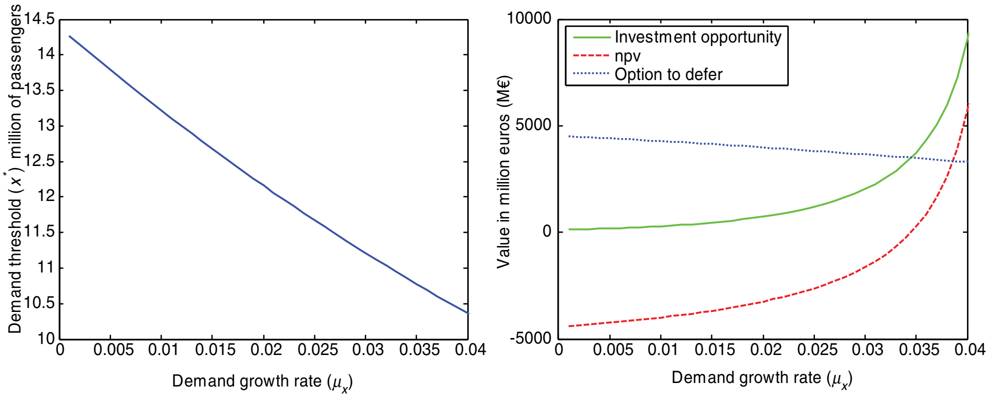

Figure 2. The impact of the growth rate.

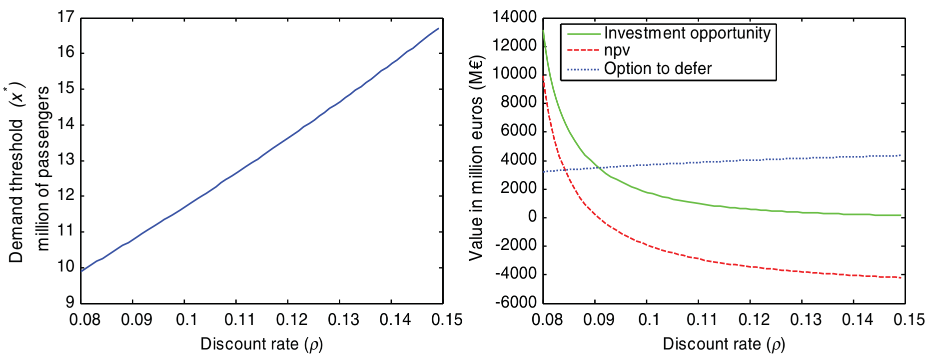

Figure 3. The impact of the discount rate.

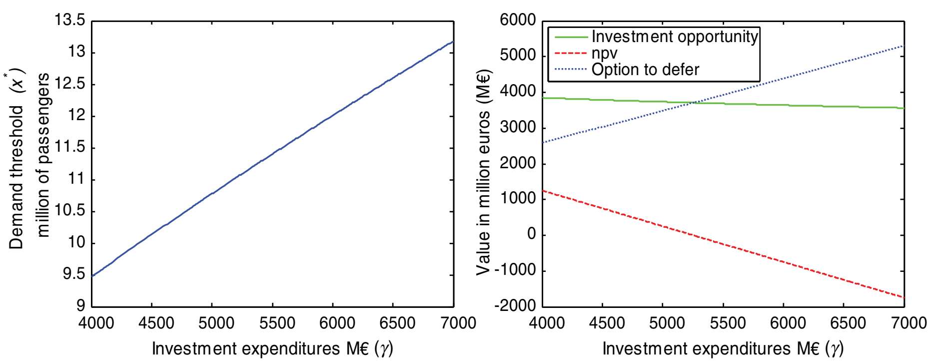

Figure 4. The impact of the investment expenditures.

Figure 1 illustrates the investment opportunity value, the NPV and the option to defer when the HSR demand increases over time. As we may observe, if the demand exceeds 10.777 million passengers, the option to defer the investment no longer has a value. Thus, from this point on, the decision to immediately invest is the one which maximizes the investment value for its owners.

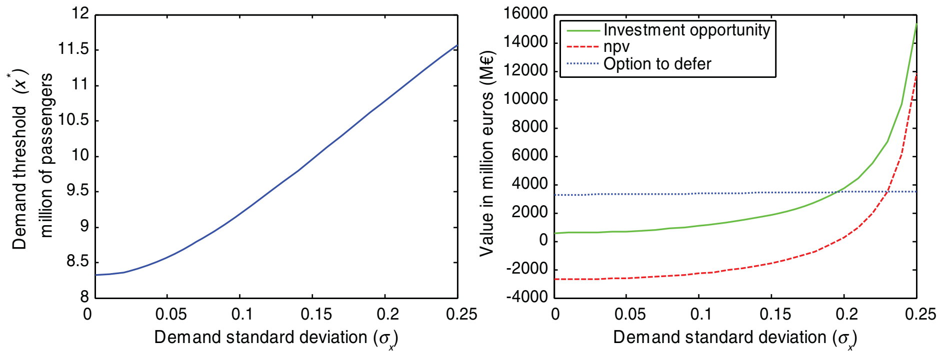

Figure 5. The impact of the HSR demand volatility.

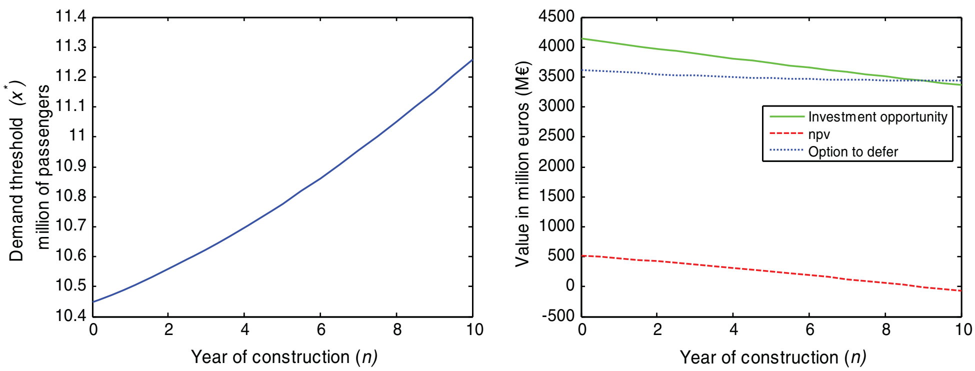

Figure 6. The impact of the time-to-build.

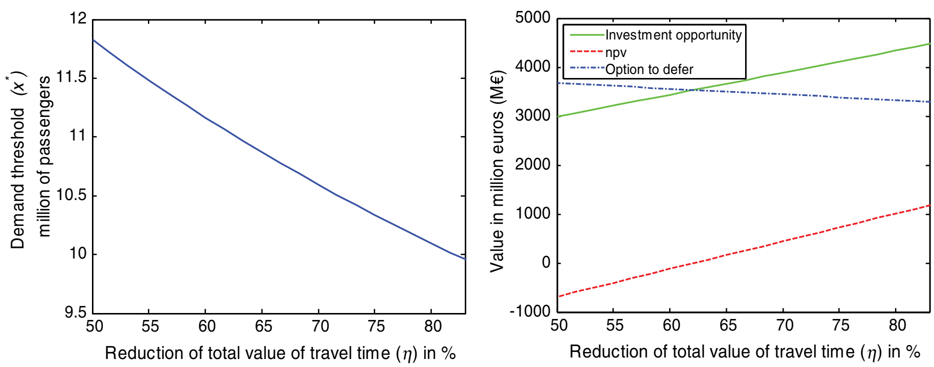

Figure 7. The impact of the total value of travel time savings.

Figure 2–7 illustrate the sensibility of the valuation results regarding the variation of various input parameters.

Thus, the HSR demand threshold, x^*, varies inversely with the HSR demand growth rate, μ_x (Figure 2), and with the total value of travel time savings given by $(\eta_0 - \eta_2)/\eta_0$ (Figure 7). For

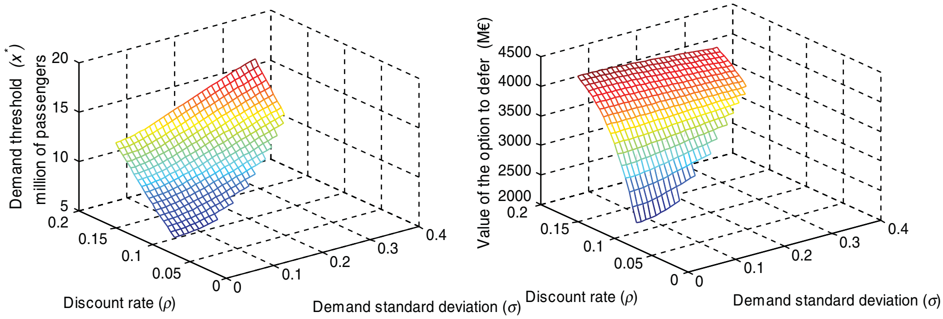

Figure 8. The impact of both the HSR demand volatility and the discount rate.

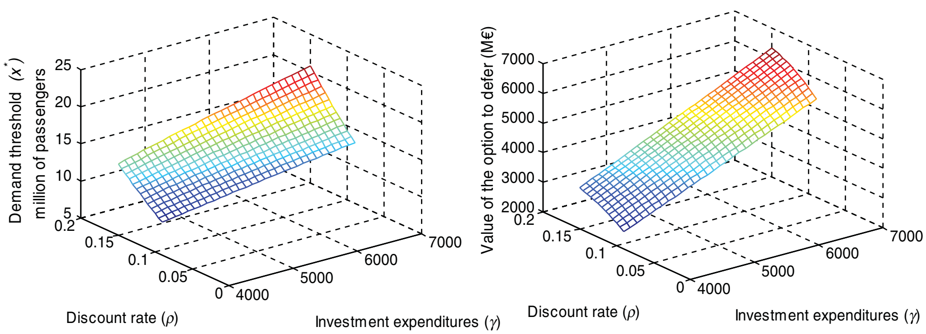

Figure 9. The impact of both the investment expenditures and the discount rate.

higher HSR demand growth rates, μ_x, and with a major total value of travel time savings, the present value of the HSR benefits increases, justifying investment anticipation.

The other parameters analyzed assume a direct relationship with the HSR demand threshold, x^*. Larger discount rates (Figure 3), larger investment expenditures (Figure 4), larger HSR demand volatility (Figure 5) or more construction time needed (Figure 6) instigates significant delays in the optimal time to invest.

Behind variations in any of the input parameters, the investment opportunity value and the NPV have the same trend, for each one of the parameters, although with different drifts. Figure 5 illustrates that the NPV increases with the increase in uncertainty. This finding is related to the elasticity between the total value of travel time and HSR demand and the cross elasticity between the conventional railway travel fare and HSR demand. This specificity of the framework results in a value of the option to defer that slightly diminishes with an increase in uncertainty. These findings are also revealed in Figure 8, where it is assumed that the discount rate remains unchanged when volatility changes.

If a larger time-to-build is required, the increase in uncertainty throughout time and the delay in the HSR operation benefits reduce the investment opportunity value and the NPV (Figure 6).

Figure 9 illustrates the joint impact of both the discount rate ρ and the investment expenditures γ on the HSR demand threshold, x^*, and on the value of the option to defer. In both cases, there is a direct relationship between these two input parameters, turning the option to defer more valuable when the value of these input parameters increases. As illustrated in Figures 3 and 4, this is due to a deeper decrease in NPV than the one registered in the investment opportunity value.

4. Extensions

Let us also consider that investment expenditures follow a geometric Brownian motion process (McDonald and Siegel 1986):

$$d\gamma_t = \mu_\gamma \gamma_t dt + \sigma_\gamma \gamma_t dw_t \tag{35}$$

where μ_γ represents the growth rate and σ_γ represents the standard deviation of investment expenditures. We assume that both parameters are constant in time. The Wiener process, w_t, has a zero mean and standard deviation of $\sigma_\gamma \sqrt{dt}$.

Adding uncertainty upon investment expenditures to the previous framework, the optimal timing to invest is given by

$$q^* = \left[\frac{(l+1)}{(A_{tc}+B_{tc})}\right]\left[\frac{s_1}{s_1-1}\right] \tag{36}$$

Note that in order to get a closed form solution, we must set:

1. The fixed operating costs as a proportion (l) from the investment expenditures; and
2. The optimal decision dependent on the threshold g^*, which represents the optimal ratio between HSR demand and the investment expenditures, given by x^θ/γ.

In Equation (36), A_{tc} and B_{tc} are given by Equations (13) and (14). The positive root, s_1, of the quadratic equation similar to (23), is given by

$$s_1 = \frac{((1/2)\sigma_q^2 - \mu_q) + \sqrt{(\mu_q - (1/2)\sigma_q^2)^2 + 2\sigma_q^2(\rho - \mu_\gamma)}}{\sigma_q^2} \tag{37}$$

with, $\sigma_q^2 = \sigma_x^2\theta^2 - 2\sigma_x\sigma_\gamma \mathrm{corr}_{x,\gamma}\theta + \sigma_\gamma^2$ and $\mu_q = \mu_x\theta + (1/2)\sigma_x^2\theta(\theta-1) - \mu_\gamma$.

Under the same assumptions, this optimal timing threshold is consistent with the one obtained when only HSR demand is stochastic. The difference between Equations (27) and (36) reflects the additional impact from investment expenditure uncertainty.

If a positive growth in the investment expenditures is expected, this extended framework supports an anticipation of the optimal timing to invest, regarding the optimal timing to invest when the uncertainty comes only from HSR demand. The increase in investment expenditures under uncertainty justifies an anticipation of the HSR investment implementation taking advantage from lower investment expenditures (McDonald and Siegel 1986).

With $v(x,\gamma) = \gamma f(q)$, the investment opportunity value can be computed if at any moment of time, the investment expenditure, γ, is known. Hence, we have

$$v(x,\gamma) = \begin{cases} \left(\dfrac{q}{q^*}\right)^{s_1}\left[\dfrac{(l+1)}{(s_1-1)}\right]\gamma & \text{for } q < q^* \\ [(A_{tc}+B_{tc})q - l - 1]\,\gamma & \text{for } q \geq q^*. \end{cases} \tag{38}$$

In the continuation region, while threshold q^* is not reached, Equation (38) incorporates the value of the option to defer, which represents the value of waiting for new information about demand and investment expenditures. When the threshold q^* is reached, the HSR investment should be implemented immediately.

In the stopping region, $q \geq q^*$, the investment opportunity value is given by NPV.

This extended framework assesses simultaneously the impact of investment expenditures and HSR demand uncertainties on the optimal timing to invest and on the investment opportunity value.

5. Conclusions

This paper developed a framework to determine the optimal timing to invest in HSR, in an uncertain environment. We introduced several extension adjustments to the original option to the defer valuation framework by McDonald and Siegel (1986) and to the optimal stopping framework of Salahaldin and Granger (2005). Those extensions were made, given the need to design an adequate framework for HSR investments in an environment of stochastic demand, combined afterwards with stochastic investment expenditures. An ROA closed form solution to value railway investments has never been conducted previously.

The HSR investment analysis was incremental regarding conventional railways. The users' utility balance between the HSR and the conventional railway quantified the benefits to the HSR users. The optimal timing to invest was calculated with the HSR demand threshold model. The developments regarding the optimal timing to invest and the investment opportunity value present the advantage of offering a clear way to evaluate the HSR investment opportunity at each moment in time, for the set of potential users. The numerical illustration and simulation of some important input parameters demonstrated the consistency of the framework.

We recommend that future research enriches the framework to include more uncertainty factors, such as travel fare and demand random shocks. Additionally, we expect to extend the empirical application and use it to carry out additional improvements in the structure of the valuation framework.

Acknowledgement

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Note

1. It is easy to illustrate that if the travel fare is non-stochastic, the number of passengers follows a geometric Brownian motion process.

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Appendix

Note that the entire framework might be understood as an intergenerational welfare problem. Because of this situation, we may use the objective function of Ramsey and Koopmans, adopted by Salahaldin and Granger (2005). Analytically, we have

$$\sup_{x^*} E_x \left(\int_{-n}^{\tau} x_{t+n} \mathrm{e}^{-\rho n} V_0(x_{t+n}) \mathrm{e}^{-\rho t} \mathrm{d}t + \int_{\tau}^{+\infty} x_{t+n} \mathrm{e}^{-\rho n} V_2(x_{t+n}) \mathrm{e}^{-\rho t} \mathrm{d}t \right), \tag{A1}$$

where

τ is the moment of time at which the optimal value is achieved;
$V_0(x_{t+n})$ is the value function per user per unit of time until the HSR begins to operate;
$V_2(x_{t+n})$ is the value function per user per unit of time after the HSR begins to operate;
n is the time-to-build (construction); and
x_t is the HSR demand, given by Equation (1).

Considering the global value of all users before and after the HSR begins to operate, and replacing V_0 and V_2 in Equation (A1) for (8) and (9), we obtain

$$\sup_{x^*} E_x \begin{pmatrix} \displaystyle\int_{-n}^{\tau} \mathrm{e}^{-\rho t} \left[\left(m_{t+n} x_{t+n} - \beta_0 x_{t+n}^{\theta_\beta} - \alpha_0 x_{t+n}^{\theta_\alpha} \right) \mathrm{e}^{-\rho n} \right] \mathrm{d}t \\ \displaystyle + \int_{\tau}^{+\infty} \mathrm{e}^{-\rho t} \left[\left(m_{t+n} x_{t+n} - \beta_2 x_{t+n}^{\theta_\beta} \right) \mathrm{e}^{-\rho n} - \omega x_{t+n} \mathrm{e}^{-\rho n} - \varphi \mathrm{e}^{-\rho n} - \rho \gamma \right] \mathrm{d}t \end{pmatrix} \tag{A2}$$

assuming $\theta_\beta = 1 + \delta_\beta$ and $\theta_\alpha = 1 + \delta_\alpha$.

Simplifying and excluding the components that do not depend on τ and x^*, it is possible to obtain the following objective function:

$$\sup_{x^*} E_x \left(\int_\tau^{+\infty} \mathrm{e}^{-\rho t} [(\beta_0 - \beta_2) x_{t+n}^{\theta_\beta} \mathrm{e}^{-\rho n} + \alpha_0 x_{t+n}^{\theta_\alpha} - \omega x_{t+n} \mathrm{e}^{-\rho n} - \varphi \mathrm{e}^{-\rho n} - \rho \gamma] \mathrm{d}t \right) \tag{A3}$$

Equation (A3) maximizes the net benefits generated by the HSR investment. The decision to implement the investment requires n building periods before the HSR begins to operate.

With v denoting the investment opportunity value, let

$$v(x^*) = E_x \left(\int_\tau^{+\infty} e^{-\rho t} [(\beta_0 - \beta_2) x_{t+n}^{\theta_\beta} e^{-\rho n} + \alpha_0 x_{t+n}^{\theta_\alpha} - \omega x_{t+n} e^{-\rho n} - \varphi e^{-\rho n} - \rho \gamma] dt \right) \tag{A4}$$

Using the strong Markov property from Oksendal (2003) on the RHS, we obtain

$$\begin{aligned} & E_x \left[\int_\tau^{+\infty} \mathrm{e}^{-\rho t} [(\beta_0 - \beta_2) x_{t+n}^{\theta_\beta} \mathrm{e}^{-\rho n} + \alpha_0 x_{t+n}^{\theta_\alpha} - \omega x_{t+n} \mathrm{e}^{-\rho n} - \varphi \mathrm{e}^{-\rho n} - \rho \gamma] \mathrm{d}t \right] \\ & = E_{x^*} \left[\int_0^{+\infty} \mathrm{e}^{-\rho t} [(\beta_0 - \beta_2) x_{t+n}^{\theta_\beta} \mathrm{e}^{-\rho n} + \alpha_0 x_{t+n}^{\theta_\alpha} - \omega x_{t+n} \mathrm{e}^{-\rho n} - \varphi \mathrm{e}^{-\rho n} - \rho \gamma] \mathrm{d}t \right] \end{aligned} \tag{A5}$$

Under the dominated convergence theorem, we have

$$\begin{aligned} & E_{x^*} \left(\int_0^{+\infty} \mathrm{e}^{-\rho t} [(\beta_0 - \beta_2) x_{t+n}^{\theta_\beta} \mathrm{e}^{-\rho n} + \alpha_0 x_{t+n}^{\theta_\alpha} \mathrm{e}^{-\rho n} - \omega x_{t+n} \mathrm{e}^{-\rho n} - \varphi \mathrm{e}^{-\rho n} - \rho \gamma] \mathrm{d}t \right) \\ & = \int_0^{+\infty} \mathrm{e}^{-\rho t} [(\beta_0 - \beta_2) E_{x^*}(x_{t+n}^{\theta_\beta}) \mathrm{e}^{-\rho n} + \alpha_0 E_{x^*}(x_{t+n}^{\theta_\alpha}) \mathrm{e}^{-\rho n} - \omega E_{x^*}(x_{t+n}) \mathrm{e}^{-\rho n} - \varphi \mathrm{e}^{-\rho n} - \rho \gamma] \mathrm{d}t \end{aligned} \tag{A6}$$

Since HSR demand, x_t, follows a geometric Brownian motion described by Equation (1), then

$$E_{x^*}(x_{t+n}^{\theta}) = (x^*)^\theta \mathrm{e}^{(\theta \mu_x + (1/2)\theta(\theta - 1)\sigma_x^2)(t+n)} \tag{A7}$$

To assure the optimal timing to invest, the condition $\rho - \theta\mu_x - (1/2)\theta(\theta - 1)\sigma_x^2 > 0$ is required. This condition also imposes the HSR demand growth rate to be lower than the discount rate, thus providing a rational economic interpretation to the mathematical developments. Simplifying again, we have

$$\begin{aligned} & \int_0^{+\infty} \mathrm{e}^{-\rho t} [(\beta_0 - \beta_2) E_{x^*}(x_{t+n}^{\theta_\beta}) \mathrm{e}^{-\rho n} - \omega E_{x^*}(x_{t+n}^{\theta_\omega}) \mathrm{e}^{-\rho n} - \varphi \mathrm{e}^{-\rho n} - \rho \gamma] \mathrm{d}t \\ & = \frac{2(\beta_0 - \beta_2)(x^*)^{\theta_\beta} \mathrm{e}^{(\mu_x \theta_\beta + (1/2)\theta_\beta(\theta_\beta - 1)\sigma_x^2)n} \mathrm{e}^{-\rho n}}{2\rho - 2\mu_x\theta_\beta - \theta_\beta^2\sigma_x^2 + \theta_\beta\sigma_x^2} + \frac{2\alpha_0 (x^*)^{\theta_\alpha} \mathrm{e}^{(\mu_x \theta_\alpha + (1/2)\theta_\alpha(\theta_\alpha - 1)\sigma_x^2)n} \mathrm{e}^{-\rho n}}{2\rho - 2\mu_x\theta_\alpha - \theta_\alpha^2\sigma_x^2 + \theta_\alpha\sigma_x^2} \\ & \quad - \frac{\omega(x^*)\mathrm{e}^{(\mu_x - \rho)n}}{\rho - \mu_x} - \frac{\varphi \mathrm{e}^{-\rho n}}{\rho} - \gamma \end{aligned} \tag{A8}$$

Rewriting Equation (A3) after simplifying, we obtain

$$\begin{aligned} v(x^*) = & \frac{2(\beta_0 - \beta_2)(x^*)^{\theta_\beta} \mathrm{e}^{(\mu_x \theta_\beta + (1/2)\theta_\beta(\theta_\beta - 1)\sigma_x^2 - \rho)n}}{2\rho - 2\mu_x\theta_\beta - \theta_\beta^2\sigma_x^2 + \theta_\beta\sigma_x^2} + \frac{2\alpha_0 (x^*)^{\theta_\alpha} \mathrm{e}^{(\mu_x \theta_\alpha + (1/2)\theta_\alpha(\theta_\alpha - 1)\sigma_x^2 - \rho)n}}{2\rho - 2\mu_x\theta_\alpha - \theta_\alpha^2\sigma_x^2 + \theta_\alpha\sigma_x^2} \\ & - \frac{\omega(x^*)\mathrm{e}^{(\mu_x - \rho)n}}{\rho - \mu_x} - \frac{\varphi \mathrm{e}^{-\rho n}}{\rho} - \gamma \end{aligned} \tag{A9}$$