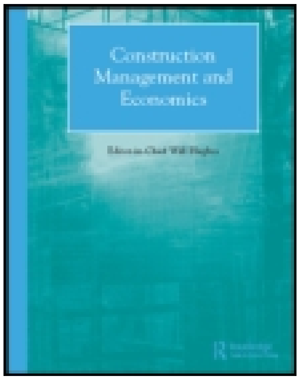

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Variance models for project financial risk analysis with applications to greenfield BOT highway projects

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Variance models for project financial risk analysis with applications to greenfield BOT highway projects

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Assessment of BOT project financial risk is generally performed by combining Monte Carlo simulation with discounted cash flow analysis. The outcomes of this risk assessment depend, to a significant extent, upon the total project uncertainty, which aggregates aleatory and epistemic uncertainties of key risk variables. Unlike aleatory uncertainty, modelling epistemic uncertainty is a rather difficult endeavour. In fact, BOT epistemic uncertainty may vary according to the significant information disclosed during the concession period. Two properties generally characterize the stochastic behaviour of the uncertainty of BOT epistemic variables: (1) the learning property and (2) the increasing uncertainty property. A new family of Markovian processes, the Martingale variance model and the general variance model, are proposed as an alternative modelling tool for BOT risk variables. Unlike current stochastic models, the proposed models can be adapted to incorporate a risk analyst's view of properties (1) and (2). A case study, a hypothetical BOT transportation project, illustrates that failing to properly model a project's epistemic uncertainty may lead to a biased estimate of the project's financial risk. The variance models may support, guide and extend the thinking process of risk analysts who face the challenging task of representing subjective assessments of key risk factors.

Keywords: Build-operate-transfer, Monte Carlo simulation, risk analysis, stochastic models.

Introduction

Over roughly the last three decades, the private sector's involvement in public infrastructure projects has multiplied dramatically. Indeed, the public–private partnership (P3) approach is increasingly viewed as a potential strategy to expand or improve infrastructure systems and services. P3 arrangements can take a variety of forms from availability arrangements to greenfield concessions. A popular approach within the surface transportation domain for expanding a highway network is the build-operate-transfer (BOT) project delivery method (Levy, 1996; Asian Development Bank, 2000). This method is frequently employed to develop toll roads. In this scheme a public entity, the government, and a private entity, the sponsor, enter into an agreement where the sponsor is bound to design, build, finance and operate the toll road on behalf of the government for a predetermined period of time, the concession period. At the end of the concession period, the sponsor transfers its operating and cash flow rights back to the government. The critical success factor for a BOT project is the efficient and effective allocation of project risks and returns among the government, the sponsor and lenders. In such arrangements, proper implementation of risk management instruments such as risk analysis and risk mitigation is crucial for their success (Dailami *et al.*, 1999). Greenfield (or new development) BOT toll road projects are particularly challenging for risk modelling and quantification because project analysts can only collect a limited amount of direct information on critical sources of project uncertainty.

Project uncertainty aggregates uncertainties from key project risk variables. These uncertainties can fall into two groups: aleatory or epistemic. The first type of uncertainty, the aleatory uncertainty, is related to the intrinsic and natural variation of a risk variable. Aleatory uncertainty cannot be reduced because it is inherently variational in nature (Hoffman and

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Hammonds, 1994; Hora, 1996). The second uncertainty, the epistemic uncertainty, is due to the lack of knowledge about the behaviour of a risk variable. Epistemic uncertainty can be reduced or eliminated if further pieces of relevant information are collected (Hoffman and Hammonds, 1994; Hora, 1996).

A review of the literature shows that while much attention has been given to representing the general behaviour of BOT risk variables (Seneviratne and Ranasinghe, 1997; Lam and Tam, 1998; Malini, 1999; Wibowo and Kochendörfer, 2005; Aziz and Russell, 2006) no specific studies have focused upon how the variance (uncertainty) of risk variables changes over time. Certainly, this issue needs some attention as the combined total project variance, which represents the project uncertainty, can dramatically affect both the results of BOT risk analysis assessments (Ye and Tiong, 2000; Shen and Wu, 2005; Aziz and Russell, 2006) and the economic value of BOT risk mitigation tools, such as government-sponsored revenue guarantees (Dailami *et al.*, 1999; Irwin, 2003; Cheah and Liu, 2006; Chiara and Garvin, 2007). Since these assessments and mitigation tools are prone to asymmetry, that is, they tend to focus upon the downside 'spread' of risk, their outcomes are sensitive to the magnitude of the resulting total variance. For instance, Figure 1 depicts a constant expected net present value (NPV) of a project with differing values of the total variance; clearly, the alternative measures of variance can influence a decision maker's choice about rejecting or accepting the project.

In a cash flow risk analysis of a BOT toll road, the traffic volume is one of the key risk variables an analyst must consider. Often, a risk analyst will consult an expert or a group of experts to elicit the parameters for building the stochastic process of the traffic volume. Traffic experts use different methods to model the toll road's market share of the network traffic demand (Hensher and Button, 2000; Ortuzar and Willumsen, 2001). Forecasting methods for highway traffic volumes can be divided into two broad groups: micro-simulation methods, which are based on individuals and households, and macro-simulation methods, which are based on zonal averages (Meyer and Miller, 2001; NCHRP, 2007). Typical traffic demand models, however, are based upon *ex ante* assumptions of, for instance, traveller preferences, trip generation/distribution, and growth of demographic/macroeconomic factors. Eventually, these assumptions are the primary source of the epistemic uncertainty of the predicted traffic volume. Figure 2 illustrates the common traffic demand forecasting methods and their underlying critical assumptions.

In BOT cash flow analysis under uncertainty, the evolution of a risk variable is represented through a stochastic model (Dailami *et al.*, 1999). This stochastic model is usually identified once the following parameters, which are illustrated in Figure 3, are determined: the evolution of the expected value (or the predicted value), the evolution of the variance (or the uncertainty around the expected value) and the probability distribution function that shapes either the

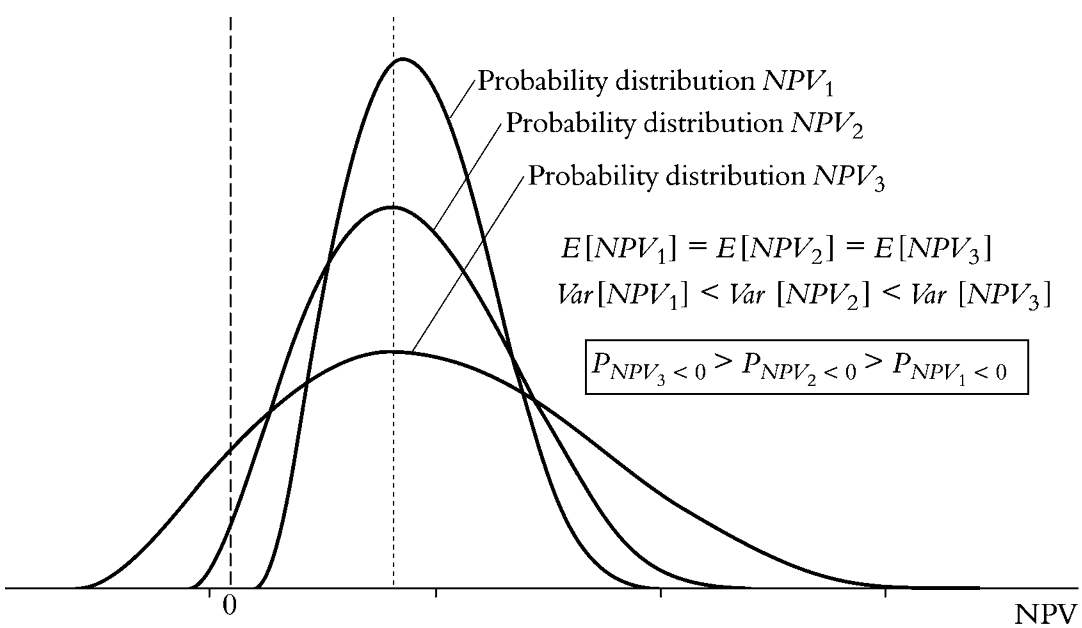

Figure 1 Variance of NPV distribution affects the probability of project financial unfeasibility

Forecasting Methods*	**Critical Assumptions****
1. Urban trasportation modelling system	a. Local growth policies
4 step model	b. The magnitude and distribution of future land use
2. Individual choice models	c. The intensity of future development
Multinomial logit	d. Projected economic growth
Nested logit	e. Changes in traffic patterns
3. Activity based methods	f. Driver's willingness to pays tools
Econometric models	g. New competing roads in the transportation network
Computational process models	

* (Meyer and Miller, 2001)

** (NCHRP, 2007)

Figure 2 Traffic demand forecasting methods and their critical assumptions

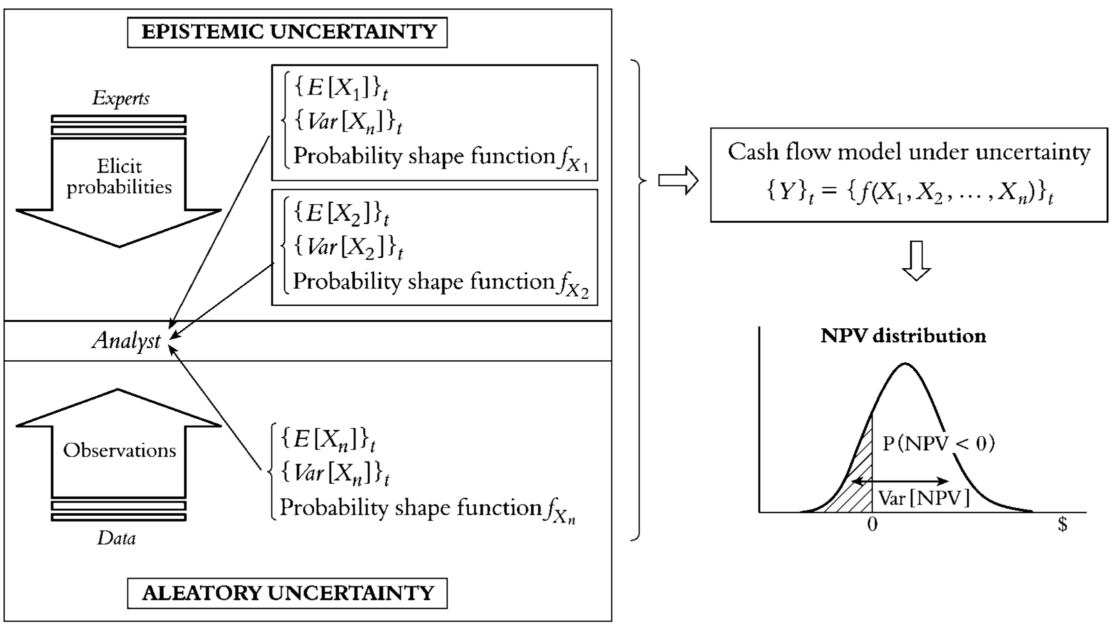

Figure 3 BOT cash flow model under uncertainty

frequency (in case of aleatory variables) or the degree of beliefs (in case of epistemic variables) of a range of potential outcomes (Hoffman and Hammonds, 1994; Paté-Cornell, 1996; Condamin *et al.*, 2007).

A new family of Markovian processes, the Martingale variance model (MVM) and the general variance model (GVM), is presented for modelling BOT risk variables. These variance models give a project risk analyst an extremely flexible and transparent tool to represent the evolution of the expected value, the evolution of the uncertainty (variance), and the shape of the probability distribution function of the project risk variables. The variance models are particularly useful when modelling the traffic volume since they permit the modeller to incorporate two properties that limit the spread of the traffic volume uncertainty over the operational period:

(1) *Learning property.* At the time the risk analysis is performed, the BOT analyst knows that s/he can learn significant information about the traffic flow once the road opens for operation. Through the learning process, s/he will be able to identify the steady traffic demand that follows the end of the traffic ramp-up period. At this stage, stable annual growth figures typically emerge and are usually in line with traffic patterns observed on

other comparable parts of the highway network (Bain, 2002a).

(2) *Increasing uncertainty property*. At the time the risk analysis is performed, the advising experts have usually provided the BOT analyst with a vector of predicted annual traffic volume values: $\underline{\bar{\mathbf{W}} = [\bar{W}_0, \bar{W}_1, \bar{W}_2, \ldots\ldots, \bar{W}_T]}$

and indicated how uncertain they are about the annual predictions. As with any forecasting activity, the shorter the time horizon is the more reliable the prediction (Bain, 2002b). Thus, the analyst's uncertainty about the traffic volume prediction will increase with time starting from the most 'reliable' prediction, i.e. year 1, to the next year's prediction and so forth.

The details of these properties are discussed subsequently.

This paper has four sections. The background section presents the stochastic approach used to frame the *learning* and *increasing uncertainty properties* of the traffic volume uncertainty. Furthermore, significant discrepancies related to the variance evolution embedded in two frequently used stochastic models are highlighted. The variance models section presents the Martingale variance model (MVM) and the general variance model (GVM), a set of Markovian stochastic processes that are well suited for modelling the BOT traffic volume. The case study section presents a hypothetical case that illustrates how to implement the variance models in a greenfield BOT toll road project. This case demonstrates that failure to properly model the variance evolution of the project traffic volume may lead to overestimating the downside project risk which, in turn, may prompt the decision to abandon a BOT project that otherwise might be viable. The conclusions section closes the paper.

Background

One way to express the stochastic evolution of the annual traffic volume $\{Y_0, Y_1, Y_2, \ldots, Y_t, \ldots Y_T\}$ is $Y_t = Y_0 + \sum_{k=1}^{t} \Delta Y_k$ where $Y_0 = 0$ is the traffic volume at time t=0; $\Delta Y_k = Y_k - Y_{k-1}$ is the annual increment in traffic volume.

Assuming that the annual increments of traffic volume are independent, the variance function at time t is given by:

$$Var[Y_t] = Var\left[Y_0 + \sum_{k=1}^{t} \Delta Y_k\right] = \sum_{k=1}^{t} Var[\Delta Y_k] \qquad (1)$$

The *learning* property refers to the capacity to learn significant information once the toll road opens to traffic. In fact, initial prediction about traffic volume is based on the available information, I_0. Once the project starts operating, new information is collected $\{I_1, I_2 \ldots, I_t \backslash I_1 \subset I_2 \ldots \subset I_t\}$. If the information 'learned' over time can improve future predictions, then the uncertainty of predictions relative to the variation of the traffic volume in consecutive periods of time, $Var[\Delta Y_t]$ with $\Delta Y_t = (Y_t - Y_{t-1})$, may decrease. For instance, if the operational phase of a BOT toll road has a lifespan of 40 years, then the prediction of traffic volume made in year 39 for year 40 is likely less uncertain than the prediction made in year 3 for year 4. In probabilistic terms this yields

$$Var[\Delta Y_1] \geq Var[\Delta Y_2] \ldots\ldots \geq Var[\Delta Y_t] \qquad (2)$$

which states that the variance of the annual increments of the traffic volume may decrease with time because of the information learned over time. The amount of uncertainty reduction due to the 'learning' process depends on the specific BOT project, and it can range 'theoretically' from 0 to 100%. Normally, considering no uncertainty reduction is the result of either a conservative choice or the belief that information collected over time cannot be more significant than current information, I_0. On the other hand, a 100% uncertainty reduction occurs at times $t \geqslant t^*$ when the uncertainty about the traffic volume is completely revealed at time $t = t^*$. Generally, the quality of information collected during the project's lifespan will be more significant than that which is received at time $t = 0$ but not revealing enough to warrant a 100% uncertainty reduction. The learning property of the traffic volume, aside from its intuitive appeal, is based on observations of how actual traffic volume progresses over operational periods. In fact, studies show that there is a transitory period after the toll road is open to traffic in which the traffic demand builds, the ramp-up phase (Blanquier, 1997; Bain and Polakovic, 2005). This ramp-up stage, which usually lasts from two to eight years with an average of five years (Wibowo, 2005), precedes the development of a steady traffic demand. The establishment of an observable and quantifiable stable traffic trend suggests that a reduction in prediction uncertainty is quite reasonable. Once the traffic volume reaches the steady phase it should become easier for an analyst to predict the next annual traffic volume increment.

The *uncertainty increasing* property refers to the fact that the uncertainty associated with the traffic volume predictions increases monotonically with time. This is consistent with the well-known notion that what is known two units of time from now is more uncertain

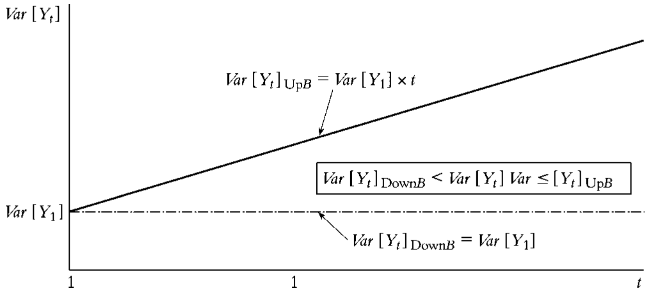

Figure 4 Variance evolution $Var[Y_t]$ consistent with *learning* and *increasing uncertainty* properties

than what is known one unit of time from now. In probabilistic terms this yields

$$Var[Y_1] < Var[Y_2] \ldots < Var[Y_t] \tag{3}$$

or the variance of the process increases with time.

Thus, incorporating the learning property (Equation 2) and the increasing variance property (Equation 3) in Equation 1 results in Equation 1 being a non-convex function (i.e. either a linear function or a concave function) that must lie within the boundaries UpB, $Var[Y_t] \leqslant Var[Y_t]_{UpB} = Var[Y_1] \times t$, and $DownB$, $Var[Y_t] > Var[Y_t]_{DownB} = Var[Y_1]$, as illustrated in Figure 4.

BOT analysts routinely employ two approaches to stochastically represent the traffic volume evolution: the first approach considers a single probability distribution and the second one uses a combination of two probability distributions.

When a single probability distribution is employed to represent the evolution of the traffic volume (Shen and Wu, 2005), the resulting stochastic process is an independent and identically distributed (IID) random process $\{Y_1 = Y_2 = Y_3 = \ldots = Y_t\}$ that is stationary (i.e. both expected value and variance are constant over the years). Accordingly:

$$Var[Y_1] = Var[Y_2] \ldots = Var[Y_t] \tag{4}$$

However, assuming a constant variance (Equation 4) over the operational period contradicts the *increasing uncertainty* property (Equation 3).

When a combination of two probability distributions is employed for modelling the traffic volume (Aziz and Russell, 2006; Cheah and Liu, 2006) then the resulting stochastic process is

$$Y_t = X\left(1 + Z \cdot t^*\right) = X + XZ \cdot t^*$$

where:

X is the random initial traffic volume with probability distribution $f_1(x)$;

Z is the random annual traffic growth rate with probability distribution $f_2(z)$;

$t^* = (t-1)$ is the adjusted time.

Thus, the evolution of the process uncertainty is

$$Var[Y_t] = Var\left[X + XZ \cdot t^*\right] = A + 2Bt^* + C \cdot t^{*2} \tag{5}$$

where $A = Var[X]$, $B = Cov[X,XZ]$, and $C = Var[XZ]$.

The variance function (Equation 5) is a convex parabolic curve. The convexity of Equation 5 implies that the variance of the annual traffic increments increases through time

$$Var[\Delta Y_1] < Var[\Delta Y_2] \ldots\ldots < Var[\Delta Y_t] \tag{6}$$

However, Equation 6 contradicts the *learning* property concept in Equation 2. Moreover, Equation 6 paradoxically suggests that the BOT analyst becomes more and more uncertain (or ignorant) about the next annual traffic volume increment as he/she collects new information over time.

The variance models

The novel family of Markovian stochastic models presented, the Martingale variance model (MVM) and the general variance model (GVM), allow BOT analysts to adjust the variance of a BOT risk variable over time. The adjustment is based upon the analyst's perceptions of how uncertainty evolves in the future and whether or not project learning is possible.

Martingale variance model (MVM)

At the outset of a BOT highway project, or when $t = 0$, the information known is:

$$\begin{cases} \text{– the duration of the BOT operating life-} \\ \text{time, } [0,T] \\ \text{– the expected value vector of the forecasted} \\ \text{traffic volume} \\ \qquad \bar{\mathbf{W}} = [\bar{W}_0, \bar{W}_1, \bar{W}_2, \ldots\ldots, \bar{W}_T] \\ \text{– and, the initial value of the process, } Y_0 = \bar{W}_0 = 0 \end{cases} \tag{7}$$

The traffic volume is represented by the discrete-time stochastic process

$$Y_t = Y_0 + \sum_{k=1}^{t} \Delta Y_k \tag{8}$$

where ΔY_k is the yearly traffic volume increment, i.e. $\Delta Y_k = (Y_k - Y_{k-1})$.

Further, the yearly traffic volume increments $\{\Delta Y_t\}_{t=1,2,\ldots T}$ can be represented as the stochastic process:

$$\Delta Y_t = \Delta \bar{W}_t + X_t \tag{9}$$

where $\Delta \bar{W}_t$ is the non-random component and X_t is the random component.

The random component of Equation 9, $\{X_t\}_{t=1,2,\ldots T}$, can be modelled as a Martingale process:

$$X_t = g(t)\varepsilon_t \tag{10}$$

where $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_t\}$ is an independently distributed random sequence with a mean of zero and a unit variance, and $g(t)$ is the time function:

$$g(t) = \sigma\sqrt{\frac{1}{\sum_{i=1}^{t}\gamma^{i-1}}} \tag{11}$$

with $\gamma \in [0,1]$, the coefficient of variance reduction, and $\sigma^2 = Var[Y_1] = Var[\Delta Y_1]$.

The Martingale process (Equation 10) has an expected value and variance (see Appendix A for details) of:

$$E[X_t] = 0$$
$$Var[X_t] = \sigma^2\left(\frac{1-\gamma}{1-\gamma^t}\right) \tag{12}$$

Accordingly, the expected value and variance of the stochastic process (Equation 9) are, respectively (see Appendix B):

$$\begin{cases} E[\Delta Y_t] = \Delta \bar{W}_t \\ Var[\Delta Y_t] = \sigma^2\left(\frac{1-\gamma}{1-\gamma^t}\right) \end{cases} \tag{13}$$

One can observe from Equation 13 that the variance of the process $\{\Delta Y_t\}_{t=1,2,\ldots T}$ is decreasing with time, and it will approach the 'long-term' yearly variance $Var[\Delta Y_{t^*}]$

$$Var[\Delta Y_{t^*}] = \lim_{t\to t^*} Var[\Delta Y_t] = \lim_{t\to t^*:\gamma^t\to 0} \sigma^2\left(\frac{1-\gamma}{1-\gamma^t}\right) = \sigma^2(1-\gamma) \tag{14}$$

with γ, the reduction coefficient, indicating the percentage variance reduction from the initial yearly variance, $Var[\Delta Y_1] = \sigma^2$.

Thus, the discrete-time traffic volume process represented by the MVM (Equation 8) retains the following properties:

- Equations 8 and 9 illustrate that this is a Markov process.
- The variance of the process, $Var[Y_t]$, is monotonically increasing with time:

$$Var[Y_t] = \sum_{k=1}^{t} Var[\Delta Y_k].$$

- If the coefficient of reduction is equal to zero, $\gamma = 0$, then:

  $$Var[\Delta Y_1] = Var[\Delta Y_2]\ldots Var[\Delta Y_T] = \sigma^2$$

  and

  $$Var[Y_t] = \sum_{k=1}^{t} Var[\Delta Y_k] = \sigma^2 t \tag{15}$$

  that is, the diffusion process characterized by $Var[Y_t] = \sigma^2 t$ is a process that shows no 'learning' over time, as depicted in Figure 5. If the coefficient of reduction is $0 < \gamma \leqslant 1$, then $Var[\Delta Y_t]$ shows 'learning' over time by decreasing toward the 'long-term' yearly variance $\sigma^2(1-\gamma)$, as depicted in Figure 5.

General variance model (GVM)

The approach used to develop the MVM can be generalized to create a general variance model (GVM); this model also retains the three properties that characterize MVM: traffic volume treated as a Markov process, increasing uncertainty with time, and the learning property. In this case, the same information is known at the outset of the project, or at time

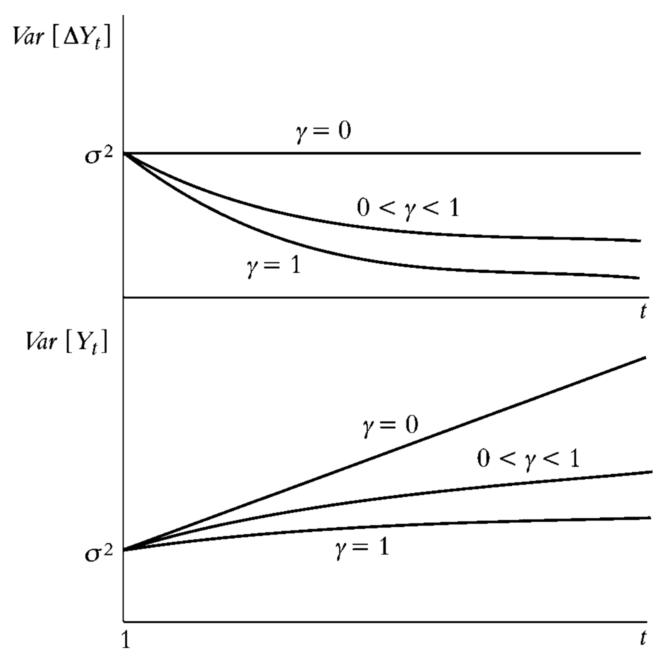

Figure 5 Adjusting the Martingale variance model's 'learning'

$t=0$, as summarized in Equation 7. Similarly, the stochastic process for the traffic volume is represented by:

$$Y_t = Y_0 + \sum_{k=1}^{t} \Delta Y_k \tag{16}$$

with

$$\Delta Y_k = \Delta \bar{W}_k + X_k \tag{17}$$

and

$$X_t = g(t)\varepsilon_t \tag{18}$$

where $\{\varepsilon_t\}$ is an independently distributed random sequence with a mean of zero and a unit variance, and $g(t)$ is a time function. Since $Var[\Delta Y_t] = Var[X_t]$ it yields that

$$Var[Y_t] = \sum_{k=1}^{t} Var[\Delta Y_k] = \sum_{k=1}^{t} Var[X_k] = H(t) \tag{19}$$

where $H(t)$, the variance function of the process, is

$$H(t) = \sum_{k=1}^{t} g(k)^2 \tag{20}$$

The chosen variance function, $H(t)$, should fit the expected evolution of the process variance over time. Once $H(t)$ is known, $g(t)$ can be computed from Equation 20:

$$\begin{cases} g(1) = \sqrt{H(1)} \\ g(2) = \sqrt{H(2) - g(1)^2} = \sqrt{H(2) - H(1)} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ g(t) = \sqrt{H(t) - g(1)^2 \ldots - g(t-1)^2} = \\ \sqrt{H(t) - H(t-1)} \end{cases} \tag{21}$$

to completely define Equation 18 and consequently the stochastic process (Equation 16).

As indicated in Equation 16 and Equation 17, the GVM is a Markov process. Furthermore, a monotonically increasing $H(t)$ function, confined within the boundaries *DownB* and *UpB* of Figure 4, will be consistent with the *learning* and *increasing uncertainty properties*. Accordingly, the 'learning' property of the process is governed by the shape of $H(t)$ where the variance of the yearly traffic volume increment is given by

$$Var[\Delta Y_k] = H(k) - H(k-1) \tag{22}$$

Therefore, in a non-learning process, i.e. $\{Var[\Delta Y_t] = \text{constant}\}_{t=1,2,\ldots T}$, $H(t)$ must be a straight line, and in a learning process, i.e. $Var[\Delta Y_1] \geqslant Var[\Delta Y_2] \geqslant \ldots Var[\Delta Y_t]$, $H(t)$ is an increasing concave function with its first derivative decreasing through time.

Hence, Equation 16 and Equation 17 become a general form of the variance model where alternative variance functions, $H(t)$, may be utilized. Two variance functions are considered in this paper: the variance function derived from the MVM process:

$$H_1(t) = \sigma^2 \sum_{k=1}^{t} \left(\frac{1-\gamma}{1-\gamma^k} \right) \tag{23}$$

and the linear piece-wise variance function:

$$H_2(t) = \sum_{i=1}^{n} I_{i,[a_i,b_i]} \cdot p(t)_{i,[a_i,b_i]} \tag{24}$$

where $I_{i,[a_i,b_i]}$, the indicator function, equals 1 if $t \in [a_i,b_i]$ otherwise it is zero, and $p(t)_{i,[a_i,b_i]}$ is a linear function defined in the domain $[a_i,b_i]$. For instance, a simple linear piece-wise variance function $H_2(t) = I_{1[1,t']} \cdot p_1(t) + I_{2[t',t'']} \cdot p_2(t)$ is shown in Figure 6, where:

$$\begin{cases} I_{1[1,t']} = 1 \text{ for } t \in [1,t'] \text{ otherwise } I_{1[1,t']} = 0 \\ I_{2[t',t'']} = 1 \text{ for } t \in [t',t''] \text{ otherwise } I_{2[t',t'']} = 0. \end{cases}$$

The variance function (Equation 24) is extremely useful and effective due to its flexibility. In fact, it is possible to approximate any other variance function with it, including Equation 23.

Though the annual traffic volume increments (Equation 17) are independent, the annual traffic volume process (Equation 16) modelled through the variance models turns out to be autocorrelated. It should not be unexpected that a stochastic process obtained by summing independent non-overlapping increments may be autocorrelated. For instance, the well-known Brownian motion process, $\{X_1^B, X_2^B, \ldots\ldots, X_t^B\}$, a Gaussian Markovian process with independent non-overlapping increments, is an autocorrelated process with autocorrelation coefficient:

$$C_{s<t}(s,t) = \frac{Cov(X_s^B, X_t^B)}{\sqrt{Var(X_s^B) \times Var(X_t^B)}} = \sqrt{\frac{s}{t}}$$

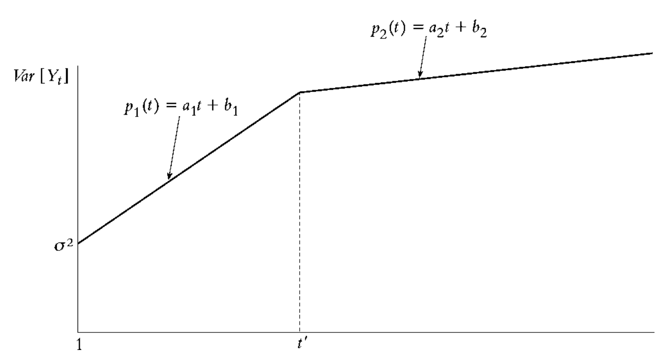

Figure 6 A simple linear piece-wise variance function $H_2(t)$

Period	Financial Projections: Annual traffic $ [1]	Toll per vehicle $ [2]	Gross revenue $ [3]	Total costs (operation, maitenance. etc.) $ [4]	TAX $ [5]	Debt service $ [6]	Capital expenditure $ [7]	Valuation: ECF $ [8]	Discounted ECF $ [9]
1							(70,000,000)		
2							(30,000,000)		
3	6,416,172	3.0	19,248,516	(6,250,385)	0	(10,965,500)		2,032,631	1,336,488
4	6,485,724	3.0	19,457,172	(6,536,376)	(453,289)	(10,684,333)		1,783,173	1,019,535
5	6,555,276	3.5	22,943,466	(6,835,450)	(598,880)	(10,403,167)		5,105,969	2,538,569
6	6,624,828	3.5	23,186,898	(7,148,204)	(1,727,762)	(10,122,000)		4,188,932	1,810,991
7	6,694,380	4.0	26,777,520	(7,475,265)	(1,883,785)	(9,840,833)		7,577,637	2,848,714
8	6,763,932	4.5	30,437,694	(7,817,285)	(3,043,966)	(9,559,667)		10,016,776	3,274,502
9	6,833,484	4.5	30,750,678	(8,174,951)	(4,225,012)	(9,278,500)		9,072,216	2,578,890
10	6,903,036	5.0	34,515,180	(8,548,976)	(4,401,900)	(8,997,333)		12,566,970	3,106,363
11	6,972,588	5.5	38,349,234	(8,940,109)	(5,614,245)	(8,716,167)		15,078,714	3,241,067
12	7,042,140	6.0	42,252,840	(22,723,988)	(6,847,455)	(8,435,833)		4,246,397	793,682
13	7,111,692	6.5	46,225,998	(9,776,864)	(5,401,530)	(8,153,833)		22,893,770	3,720,878
14	7,181,244	7.0	50,268,708	(10,224,160)	(9,376,471)	(7,872,667)		22,795,410	3,221,645
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......	....	....	....	....	....	....		....	....
29	8,224,524	21.0	172,715,004	(19,999,616)	(44,460,164)	(3,655,167)		104,600,057	1,816,747
30	8,294,076	22.0	186,616,710	(20,914,436)	(48,651,073)	(3,374,000)		133,677,201	1,716,873
31	8,363,628	24.0	204,908,886	(21,871,091)	(52,904,579)	(3,092,833)		127,040,383	1,668,433
32	8,433,180	26.0	219,262,680	(22,871,493)	(97,765,448)	(2,811,667)		95,814,072	1,094,204
									69,938,832

NPV (ECF) = PV(ECF) - PV(EQUITY) = 69,938,832 - 30,000,000 = $39,938,832

Cash Model Assumption

a) Contruction period = 2 years
b) Operational period = 30 years
c) Capital expenditure = $100 millions
d) Capital structure
d1) Equity = $30 millions
d2) Senior debt = $700 millions
e) Tax rate = 30%
f) Cost of equity Ke = 15%
g) Borrowing interest rb = 10%

Figure 7 Project's financial projections

Unlike Brownian motion, the variance models do not have a closed-form solution for the autocorrelation coefficient. Consequently, the variance model autocorrelation coefficients must be obtained through statistical computational analysis. More details about the autocorrelation properties of the variance models are addressed in the illustrative case study.

Illustrative case study

The illustrative case study examines a hypothetical greenfield BOT toll road project with a two-year construction phase and a 30-year operational phase. The illustrative case study is divided into two sections. In the first section, a deterministic project analysis, or base case analysis, is presented. The base case analysis is usually performed to assess the toll road's financial feasibility when uncertainty is not considered. The second section, the stochastic financial project analysis, explains how to perform a project financial risk analysis under uncertainty using the variance models, MVM and GVM.

Base case analysis

A main objective in the base case analysis is for the BOT analyst to assess whether or not the toll road can generate the expected sponsor's rate of return. No uncertainty is considered at this stage, i.e. all the values used in the analysis are considered projected or expected (Yescombe, 2003). A simplified cash flow model (Equation 25) to determine the project's annual net revenue available to the sponsors during the period i is given by Esty (1999):

$$ECF_i = \text{Gross Revenue}_i - \text{Total Cost}_i - \text{Tax}_i - \text{Debt Service}_i \qquad (25)$$

where

Step 1: *Estimating the variance at year 1,* $Var[\Delta Y_1] = H(1) = \sigma^2$. The BOT analyst, in concert with the traffic analyst, will select the most appropriate probability distribution shape (triangular, beta, etc.) for $\{\varepsilon_t\}$ in (18). For instance, if a beta distribution is considered a reasonable probabilistic distribution, the variance at year one is estimated by considering a four-parameter beta distribution (Bury, 1999; Park and Sharp-Bette, 1990). In this case, BOT analyst and traffic analysts must specify

1) the expected value $E[\Delta Y_1] = \overline{W}_1$, which is given from the base case analysis;

2) the lower (pessimistic) and the upper (optimistic) bound of the traffic demand relative to $t = 1$, which are obtained by considering a pessimistic and optimistic scenario for the critical assumptions in Figure 2.

3) the two shape parameters α and β.

Once 1, 2 and 3 are specified, the variance at year one, $Var[\Delta Y_1] = H(1) = \sigma^2$, is univocally determined. In the case study, the variance at year one, $Var[\Delta Y_1] = H(1) = \sigma^2 = (5.57E5)^2$ was estimated by considering a four-parameter beta distribution where the expected value is $E[\Delta Y_1] = \overline{W}_1 = 6.41E6$, the two shape parameters are $\alpha = 2.9$ and $\beta = 4.4$, and the lower (pessimistic) and the upper (optimistic) bound of the traffic demand relative to $t = 1$ are Low = 5.10E6 vehicles and High = 8.40E6 vehicles, respectively.

Step 2: *Estimating the duration of the expected traffic rump-up stage*. In this phase traffic analysts will assist BOT analyst in identifying the expected duration of the traffic rump-up phase through a traffic data analysis on greenfield toll road projects with characteristics similar to the one under examination. In the case study, assuming that a data driven analysis on comparable transportation facilities has been performed, the BOT analyst expects that:

a) a steady traffic demand will emerge after an 8-year-ramp-up period.

b) in the "long term", around the year 16 of the operational period, when the steady traffic flow is expected to be well established, the degree of uncertainty related to annual traffic increments is considered as much as 80% less than the uncertainty relative to the prediction of traffic volume increment at year 1,i.e. $Var[\Delta Y_{t>15}] = 0.2 \cdot \sigma^2$.

Step 3: *Modeling the Variance function* (20), $H(t)$. The Variance models give the analyst full flexibility in shaping the expected uncertainty reduction due to the learning capacity of the system in which the process Y evolves. Two approaches for modeling the process variance function $H(t)$ are proposed in this paper: the Martingale Variance function (23) and the linear piece-wise variance function (24). In the case study, the Variance function $H_1(t)$ is built considering (23) with $H_1(1) = \sigma^2 = (5.57E5)^2$ and a reduction factor γ equal to 80% that reduces the "long-term" time-step variance (14) up to 20% of the initial variance.. Figure 9 shows the normalized annual traffic increment variance and the normalized variance function for $H_1(t)$. While Figure 10 shows the evolution of the variance reduction of the annual traffic increments, $\frac{\sigma^2 - Var[\Delta Y_t]}{\sigma^2}$, for $H_1(t)$. The variance function $H_2(t)$ is a linear piece-wise variance function (24) that is built considering annual variance reductions consistent with the step 2 analyst's expectations (a) and (b) as well as $H_2(1) = \sigma^2 = (5.57E5)^2$ (Figure 9 and 10). A third variance function, $H_3(t)$, representing the conservative case where no project learning capacity is anticipated, is obtained using a Martingale function (23) with a reduction factor γ equal to 0% and $H_3(1) = \sigma^2 = (5.57E5)^2$ (Figure 9 and 10).

Figure 8 Procedure to model the variance function H(t)

ECF_i is the equity cash flow accumulated in the time period i;

Gross $Revenue_i$ is the toll revenue, i.e. average traffic volume in time interval i multiplied by toll price;

Total $Cost_i$ represents all project costs in the time interval i;

Tax_i represents the project tax in time interval i; and

Debt $Service_i$ represents debt principal and interest paid during the time interval i.

The simplified model (Equation 25) is presented for illustrative purposes. A more sophisticated cash flow model could also be used such as the one found in Dailami *et al.* (1999).

The hypothetical project's capital structure, capital expenses, tax structure and other cash flow assumptions are shown at the bottom of Figure 7. For the sake of simplicity, it is also assumed that this toll road project has only one vehicle class. A pool of traffic analysts usually computes the projected annual traffic volume using one of the forecasting methods previously shown in Figure 2. Regardless of the forecasting method employed, traffic analysts are asked to determine an expected future scenario for a project by taking into account the critical assumptions listed in Figure 2 (NCHRP, 2007). Once traffic analysts have completed the traffic forecast demand model, they provide the BOT analyst with both the vector of predicted/expected traffic volumes (column [1] on Figure 7), which may

Time t (years)	Leading $H_1(t)$		Leading $H_2(t)$		Leading $H_3(t)$		
	$\frac{Var[\Delta Y_t]}{\sigma^2}$	$\frac{H_1(t)}{\sigma^2}$	$\frac{Var[\Delta Y_t]}{\sigma^2}$	$\frac{H_2(t)}{\sigma^2}$	$\frac{Var[\Delta Y_t]}{\sigma^2}$	$\frac{H_3(t)}{\sigma^2}$	
1	1.00	1.00	1.00	1.00	1.00	1	Expected ramp-up-period
2	0.56	0.56	0.60	1.60	1.00	2	
3	0.41	0.97	0.40	2.00	1.00	3	
4	0.34	2.30	0.35	2.35	1.00	4	
5	0.30	2.60	0.30	2.65	1.00	5	
6	0.27	2.87	0.30	2.95	1.00	6	
7	0.25	3.13	0.30	3.25	1.00	7	
8	0.24	3.37	0.30	3.55		8	
9	0.23	3.60	0.25	3.80	1.00	9	
10	0.22	3.82	0.25	4.05	1.00	10	
11	0.22	4.04	0.25	4.30	1.00	11	
12	0.21	4.25	0.25	4.55	1.00	12	
13	0.21	4.47	0.25	4.80	1.00	13	
14	0.21	4.68	0.25	5.05	1.00	14	
15	0.21	4.88	0.25	5.30	1.00	15	
16	0.21	5.09	0.25	5.55	1.00	16	
17	0.20	5.29	0.20	5.75	1.00	17	Expected period of steady traffic trend
18	0.20	5.50	0.20	5.95	1.00	18	
19	0.20	5.70	0.20	6.15	1.00	19	
20	0.20	5.90	0.20	6.35	1.00	20	
21	0.20	6.10	0.20	6.55	1.00	21	
22	0.20	6.31	0.20	6.75	1.00	22	
23	0.20	6.51	0.20	6.95	1.00	23	
24	0.20	6.71	0.20	7.15	1.00	24	
25	0.20	6.91	0.20	7.35	1.00	25	
26	0.20	7.11	0.20	7.55	1.00	26	
27	0.20	7.31	0.20	7.75	1.00	27	
28	0.20	7.51	0.20	7.95	1.00	28	
29	0.20	7.71	0.20	8.15	1.00	29	
30	0.20	7.91	0.20	8.35	1.00	30	

Figure 9 Evolution of the normalized annual traffic increment variance, $\frac{Var[\Delta Y_t]}{\sigma^2}$ and normalized variance function $\frac{H(t)}{\sigma^2}$ for $H_1(t)$, $H_2(t)$ and $H_3(t)$

take the form of a time function such as Equation 26 (Aziz and Russell, 2006):

$$\text{Annual Vehicles' Demand} = (6.34E6 + 6.95E5 \cdot t), \forall t \geq 1 \tag{26}$$

and the expected toll price time structure (column [2] on Figure 7). It is worth noting that recent research has pointed out that traffic forecast demand for greenfield highway projects may be affected by the so-called 'optimism bias' which may lead, on average, to an overestimation of the forecasted traffic demand (Bain and Polakovic, 2005; NCHRP, 2007). This 'optimism bias' can be corrected provided that explicit, empirically based adjustments to the predicted traffic demand are made (NCHRP, 2007).

Once the BOT analyst has collected all the project information about the financial projections, which is shown in the upper portion of Figure 7, s/he performs a net present value analysis on the sponsor's ECF (Yescombe, 2003). The computed project NPV on the sponsor's ECF yields a positive NPV of roughly \$39.9 million.

Sensitivity analysis

At this stage, it is common for the BOT analyst to perform a simplistic risk assessment by running a sensitivity

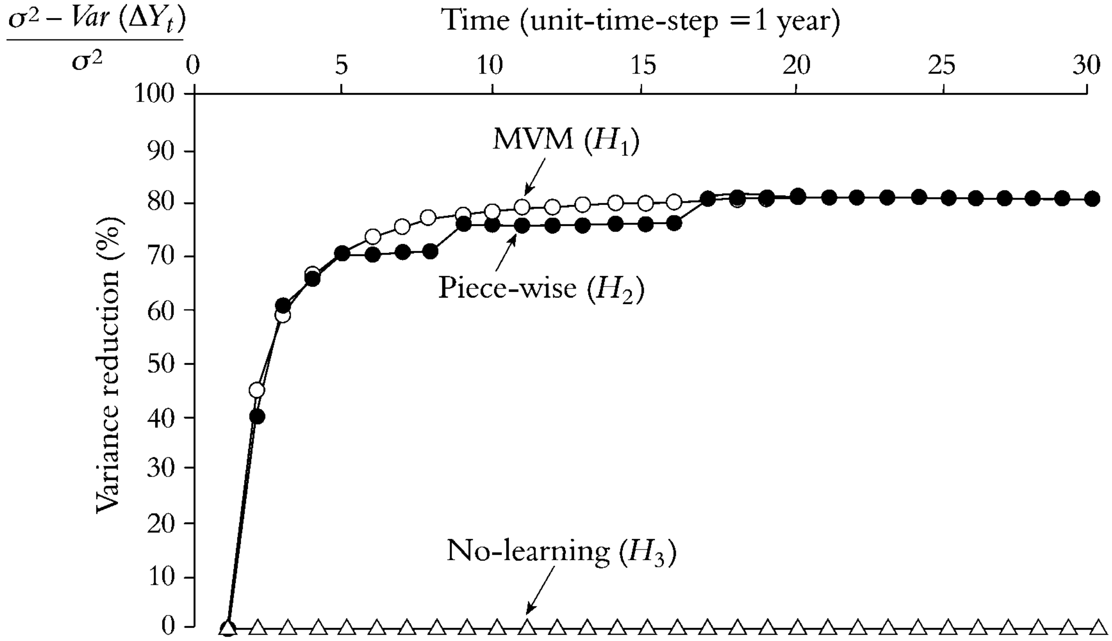

Figure 10 Normalized variance reduction of annual traffic increments over time, $\frac{\sigma^2 - Var[\Delta Y_t]}{\sigma^2}$

analysis on the cash flow model (Equation 25) (NCHRP, 2007). In this respect, traffic analysts will provide the BOT analyst with alternative traffic demand forecast scenarios based on different values of the critical assumptions shown in Figure 2. Though the sensitivity analysis may provide useful insights about critical risk variables, practitioners and academics have recognized that a sensitivity analysis is not generally sufficient to evaluate project risk (NCHRP, 2007). A more effective way to proceed is to carry out a complete risk analysis that can evaluate many different scenarios by considering a wide range of potential outcomes with their related probabilities (Dailami *et al.*, 1999).

Financial risk analysis under uncertainty

In order to single out the effects of the variance models on the project's NPV, only one risk variable is considered, the annual traffic volume, Y, while the

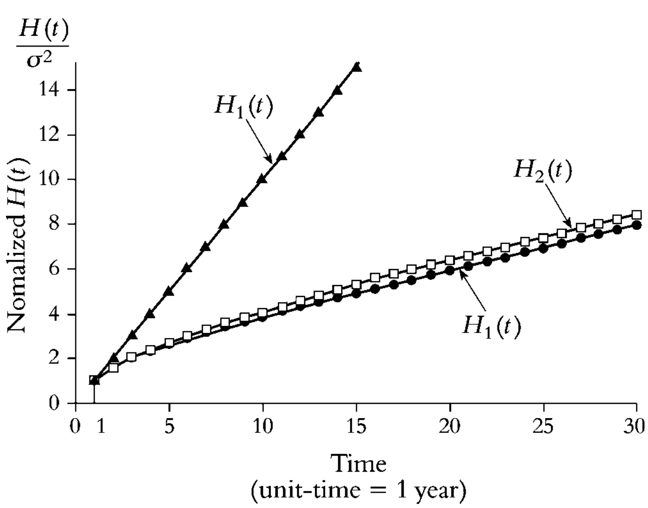

Figure 11 Representation of the MVM variance function $H_1(t)$ and $H_3(t)$ as well as of the linear piece-wise variance function $H_2(t)$

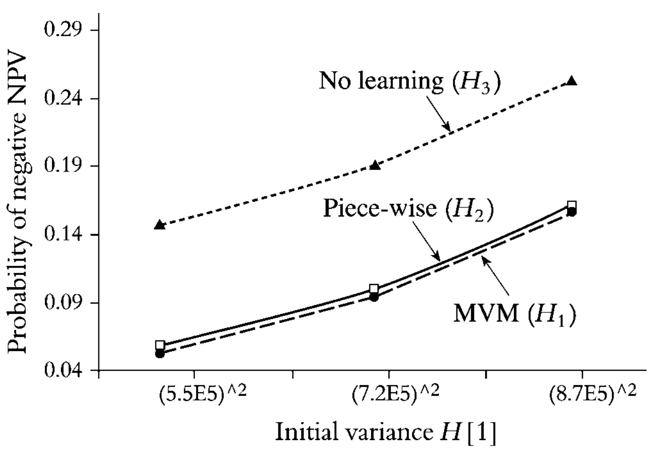

Figure 12 Probability of project financial unfeasibility as the estimated uncertainty at year 1 changes

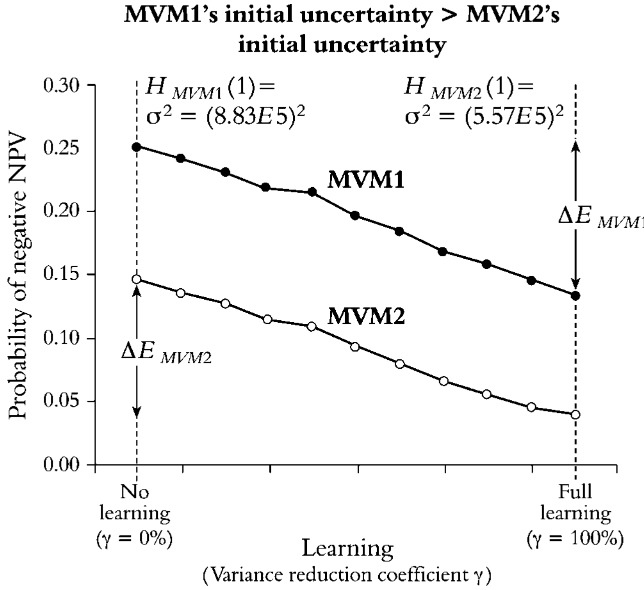

Figure 13 Probability of project financial unfeasibility with and without considering project learning capacity

other parameters in Equation 25 are kept deterministic. The vector of the predicted values of the annual traffic volume, $\bar{\mathbf{W}} = [\bar{W}_0, \bar{W}_1, \bar{W}_2, \ldots\ldots, \bar{W}_T]$, is taken from the financial projections of the base case analysis (column [1] on Figure 7). A three-step procedure, illustrated in detail in Figure 8, is used to build the two variance models with *learning* features, the Martingale variance model $H_1(t)$ and the linear piece-wise variance model $H_2(t)$. These models incorporate the BOT analyst's predictions about the duration of the traffic ramp-up phase, which is assumed to occur in the first eight years, and the establishment of a defined steady traffic trend, which is assumed to occur after 16 years. The assumptions about the length of the ramp-up period and when the steady traffic trend is established come from a data-driven analysis on comparable transportation facilities. Furthermore, another variance model, $H_3(t)$ with no learning feature, is introduced for comparison. Details on how to build $H_3(t)$ are also presented in Figure 8. All the three models are built to have the same initial variance, $H_1(1) = H_2(1) = H_3(1) = \sigma^2$.

Now, Monte Carlo risk analyses are performed using 10 000 simulations for each model. In the first analysis where the Martingale variance function $H_1(t)$ is used, the probability of a negative NPV is 15.2%. In the second analysis where the piece-wise variance function $H_2(t)$ is used, the probability of a negative NPV is 15.6%. When the no-learning function $H_3(t)$ is used, the probability of a negative NPV is 25.0%.

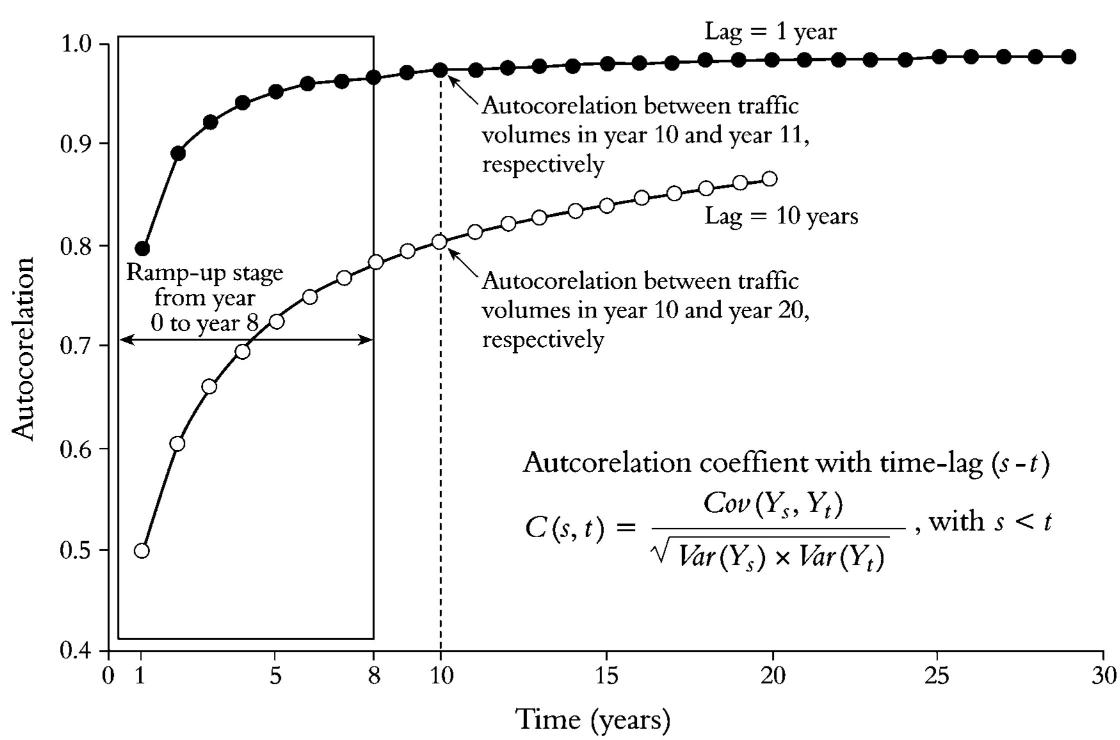

Figure 14 Autocorrelation evolution over time: lag 1 year and lag 10 years

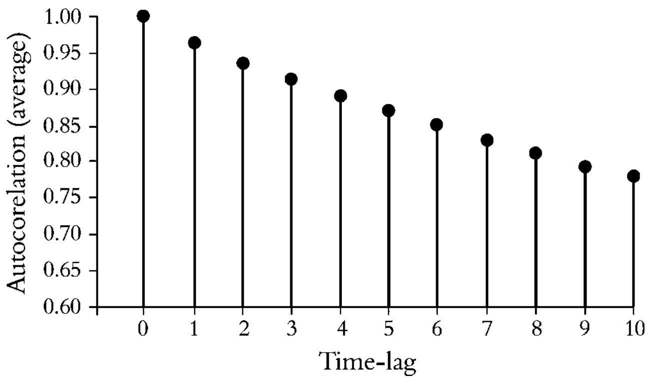

Figure 15 Average autocorrelation for different time lags

The differences in value among the resulting probabilities clearly indicate that the risk analysis is affected by how the analyst models the project's uncertainty. Results from parametric analyses, shown in Figures 12 and 13, also support this conclusion. In fact, Figure 12 shows that different analysts' estimations of the initial variance, all else kept equal, lead to substantially different assessments of the project's financial viability. Figure 13 illustrates that neglecting to incorporate project learning features in the risk variable model, when it may be actually present, generates a risk analysis error that can be equal to or less than ΔE. This in turn may lead to the overestimation of project risks and the eventual rejection of projects that otherwise would be feasible. Curves MVM1 and MVM2 in Figure 13 were both determined using a Martingale variance model (Equation 23) with MVM1's initial uncertainty greater than MVM2's.

From the results of the autocorrelation analysis shown in Figure 14 and 15, two properties about the autocorrelation of the variance models surface:

(1) For a given time lag, the variance models generate a stochastic process with an autocorrelation coefficient that is not constant but increasing with time (Figure 14). This is consistent with the fact that, as time goes, the traffic volume reaches a more stable pattern and consequently traffic volume values become more and more autocorrelated.

(2) The average autocorrelation coefficient, obtained by averaging the autocorrelation coefficients of a given time lag, decreases as the time lag increases (Figure 15). This is consistent with the fact that closer annual traffic volume realizations are more correlated than realizations with a wider time lag. From Figure 14 it is clear that the variance models are also able to model the expected traffic fluctuation during the ramp-up phase. In fact, the results of autocorrelation analysis in Figure 14 show that in the early stage (ramp-up phase) of the toll road the traffic volume values are much less correlated than traffic volume values pertaining to the project period where the traffic volume trend has stabilized.

Conclusions

A new set of Markovian processes, called the general variance models (GVMs), has been developed to model project risk variables. Unlike some black-box stochastic processes, the GVMs give analysts the freedom to directly model the evolution of the expected value, the evolution of the process variance and the type of probability distribution function. In BOT risk analysis under uncertainty, decisions regarding the financial feasibility of a project depend upon the combined project uncertainty which ultimately aggregates aleatory and epistemic uncertainty of key project variables. In BOT projects, modelling epistemic uncertainty effects is complex because the evolution of the epistemic uncertainty is influenced by the disclosure of significant information revealed during the concession period. Notably, the proposed GVM stochastic processes can incorporate the *learning* and the *increasing uncertainty* properties associated with the traffic volume random process. A case study illustrated that failing to properly model the epistemic uncertainty of the traffic volume may lead to a biased assessment of the project financial risk. If the assessment is too conservative, then the decision maker may reject a project that otherwise would be financially feasible.

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Appendix A

The representation of Equation 16 is:

$$\begin{cases} X_0 = 0 \\ X_1 = g(1)\varepsilon_1 = \sigma\sqrt{\frac{1}{\sum_{i=1}^{1}\gamma^{i-1}}}\varepsilon_1 = \sigma\varepsilon_1 \\ X_2 = g(2)\varepsilon_2 = \sigma\sqrt{\frac{1}{\sum_{i=1}^{2}\gamma^{i-1}}}\varepsilon_2 = \sigma\sqrt{\frac{1}{1+\gamma}}\varepsilon_2 \\ X_3 = g(3)\varepsilon_3 = \sigma\sqrt{\frac{1}{\sum_{i=1}^{3}\gamma^{i-1}}}\varepsilon_3 = \sigma\sqrt{\frac{1}{1+\gamma+\gamma^2}}\varepsilon_3 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ X_t = g(t)\varepsilon_t = \sigma\sqrt{\frac{1}{\sum_{i=1}^{t}\gamma^{i-1}}}\varepsilon_t = \sigma\sqrt{\frac{1}{1+\gamma+\gamma^2+\ldots.+\gamma^{t-1}}}\varepsilon_t \end{cases} \quad (A1)$$

From Equation A1 we observe that since:

$$\begin{aligned} E[X_t] &= E[X_{t-1}] = E[X_{t-2}] = \ldots\ldots \\ &= E[X_1] = E[X_0] = 0 \end{aligned} \quad (A2)$$

the process represented in Equation 16 is a Martingale process. Furthermore, the variance of the Martingale process (Equation 16) is given by

$$\begin{cases} Var[X_0] = 0 \\ Var[X_1] = \sigma^2 \\ Var[X_2] = \sigma^2 \frac{1}{1+\gamma} \\ Var[X_3] = \sigma^2 \frac{1}{1+\gamma+\gamma^2} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ Var[X_t] = \sigma^2 \frac{1}{1+\gamma+\gamma^2+\dots+\gamma^{t-1}} \end{cases} \tag{A3}$$

Then, the equivalent representation of $Var[X_t]$ for $0 \leqslant \gamma < 1$ is given by

$$Var[X_t] = \sigma^2 \frac{1}{\sum_{i=1}^{t} \gamma^{i-1}} = \sigma^2 \frac{1}{\left(\frac{1-\gamma^t}{1-\gamma}\right)} = \sigma^2 \left(\frac{1-\gamma}{1-\gamma^t}\right) \tag{A4}$$

Appendix B

$$E[\Delta Y_t] = E[\Delta \bar{W}_t + X_t] = E[\Delta \bar{W}_t] + E[X_t] = \Delta \bar{W} \tag{B1}$$

$$Var[\Delta Y_t] = Var[\Delta \bar{W} + X_t] = Var[X_t] = \sigma^2 \left(\frac{1-\gamma}{1-\gamma^t}\right) \tag{B2}$$