INSPECTION—MAINTENANCE—REPLACEMENT SCHEDULES UNDER MARKOVIAN DETERIORATION*†‡

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A problem of finding optimal policies for the management of systems whose deterioration can be expressed as a Markov chain is considered. Using methods due to Manne and Derman for sequential decision problems involving average cost per unit time criteria, it is shown that the problem can be formulated in linear programming terms.

1. Introduction

Consider a system, in use or in storage, which is deteriorating. Suppose that the deterioration is stochastic and that the condition of the system is known only if it is inspected. After inspection, the manager of the system has two basic alternatives: (a) to replace the system or (b) to keep it. Under (b) further decisions are required; these are: (1) the extent, if any, of immediate repairs and (2) the timing of the next inspection.

In [1] Barlow and Hunter considered the problem of finding an optimal (least cost) inspection schedule for equipment whose life is of random length with either an exponential or a uniform distribution. Their formulation was for the case in which the only information obtained from an inspection was whether or not the equipment was operative. In [2] Derman considered a similar problem without assumptions on the distribution function and demonstrated an optimal minimax inspection schedule. A further condition of both of the studies is that a repair or replacement decision always returns the system to the same "as new" state.

In this paper, our points of departure from the above are as follows. First, we assume that an inspection is capable of supplying more information about the system's condition than just whether it is operative or inoperative.[1] Specifically, our assumption is that the deterioration of the system can be described as a discrete time, finite Markov chain and that the inspection procedure is capable of detecting which state the system is in at the time it takes place. This much of the formulation, with periodic inspection and with replacement as the only decision was considered by Derman [5]. Our second assumption is that repairs, if made, can put the system in one of *many* possible states, the "as new" state being only one of the available alternatives. Our purpose is to show, for an average cost per unit time criterion function, that the problem of finding an optimal

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[1] Derman and Sacks in [3] were apparently the first to consider a problem in which information other than of the operative-inoperative kind was used. Their formulation was for periodic inspection with decision alternatives of replacement or non-replacement only.

inspection-repair-replacement policy can be formulated in linear programming terms. The basic observation that decision problems involving Markov chains with average cost per unit time cost criterion can be treated as linear programming problems is due to Manne [4]. Recently Derman [6] has extended the applicability of linear programming to other decision problems involving Markov chains. Our formulation is not covered by that of Manne and makes use of the type of extension obtained by Derman. The main difference between our problem and Manne's is that the time between successive transitions is under the control of the decision maker rather than being fixed.

2. The Deterioration Process

We assume that the condition of the system at any time t can be completely characterized by classifying it as in one of the states $0, 1, \cdots, L$. State 0 will represent the process before any changes (deterioration) take place, i.e. it is the initial state of the system. State L will represent the terminal state of the deterioration process, i.e. the state after which no further deterioration can take place. No ordering is implied in the labeling of the intermediate states $1, 2, \cdots, L-1$.

The deterioration law of the system is assumed to be Markovian; more specifically, the sequence of successive states of the system forms a discrete Markov chain with stationary transition probabilities q_{ij}, $i, j = 0, 1, \cdots, L$. The process is one of deterioration in that, if left untouched, a system starting in the initial state would eventually reach the terminal state and remain there. This is formalized in the following two conditions on the process:

$$\text{(a)} \qquad \lim_{t\to\infty} q_{iL}^{(t)} = 1, \qquad i = 0, 1, \cdots, L-1,$$

and

$$\text{(b)} \qquad q_{LL} = 1,$$

where $q_{ij}^{(t)}$ represents the t-step transition probability from state i to j; $t = 1, 2, \cdots$. We also assume[2]

$$\text{(c)} \qquad q_{iL} > 0, \qquad i = 0, 1, \cdots, L-1.$$

Aside from mathematical convenience and simplicity, there are two justifications for suggesting the Markov chain description of deterioration: the first lies in the current employment of multi-component electronic systems, each component of which has, approximately, exponential life characteristics. Such a system, in toto, could be expected to yield to a Markovian description. Additionally, as with classical dynamics, the first order approximating description of many physical systems is that in which the state of the system is specified sufficiently well for predictive purposes by information dealing only with the process at the present instant of time, i.e. knowledge of the histories of such systems contains no predictive value. The stochastic equivalent of this kind of process is a Markov process.

[2] This condition is not strictly required; however, its inclusion allows for a simpler formulation than would be the case otherwise. See footnote 3 for further comments on this point.

3. Inspection—Maintenance—Replacement Policies

A basic specification for the class of systems under discussion is that they be in use or available for use over extended periods of time. In the presence of deterioration, this implies a necessity for occasional repair or replacement of parts or of the whole unit. Further, since deterioration can only be detected after inspection, an inspection schedule is also necessary for its operation. After each inspection, a range of alternatives are available to the system manager; these are given formal expression below.

Let $d_{s:k}(i)$ denote a decision to convert the system from its present condition, state i, to that described by state s and to schedule the next inspection k periods from the present.

The ranges of the various indices are:

$$i = 0, 1, \cdots, L, L(1), \cdots, L(K-1); \qquad k = 1, 2, \cdots, K;$$

$$s \in S_i, \quad \text{where} \quad S_i = \{0, 1, \cdots, L-1\} \cup \{i\};$$

where $L(m)$ means that the system has been inoperative for (m) time units. The ordered set of states $L(m)$ arise since the system can become inoperative and remain so, undetected, until the next inspection. We assume that the longest period of time that a system can be allowed to remain inoperative is specified; it is here denoted by $K - 1$. This, together with assumption (c), implies that the longest interval between inspections is $k = K$.[3] This also implies that if $i = s = L(j)$, $j = 0, 1, \cdots, K-1$, where $s = i$ indicates that no maintenance action is taken, then k must satisfy the inequality $j + k < K$. The set of indices S_i is constructed to allow only physically feasible decisions; i.e. repair, replacement, and inaction.

We assume that inspection, repairs, and replacements are made instantaneously with no uncertainty involved in any of these operations. Subsequent transitions take place at later time periods.

The class of all possible randomized decision rules is of the form

$$D_{is:k} = P\{d_{s:k}(i)\},$$

where

$$\sum_{s:k} D_{is:k} = 1, \qquad i = 0, 1, \cdots, L(K-1).$$

(It should be noted that summation over the set of indices $\{s:k\}$ is restricted to those values of $s \in S_i$.) Thus the rules are such that the decision making mechanism at each inspection time depends only on the observed state. Derman [6] has shown that it is sufficient to consider this class of rules.

4. The Controlled Process

The decision rules mentioned in the previous section, in combination with the original deterioration process, gives rise to a new ergodic Markov chain with

[3] If some $q_{sL} = 0$, for $s \in \{0, 1, \cdots, L\text{-}1\}$, the longest allowable interval becomes a function of s and the least number of transitions possible from s to L.

stationary transition probabilities p_{ij}, $i, j = 0, 1, \cdots, L(K-1)$ which we will call the "controlled" process. The transition probabilities take the form:

$$p_{ij} = \sum_{s:k} D_{is:k} v_{s:kj}, \quad i, j = 0, 1, \cdots, L(K-1);$$

where

$$v_{s:kj} = \begin{cases} q_{sj}^{(k)}, & j = 0, 1, \cdots, L, \quad s = 0, 1, \cdots, L-1, \quad k = 1, 2, \cdots, K; \\ q_{sL}^{(k-n)}, & s = 0, 1, \cdots, L-1, \quad j = L(n), \quad n = 1, 2, \cdots, K-1, \\ & (k-n) > 0; \\ 1, & s = L(m), \quad m = 0, 1, \cdots, K-2, \quad j = L(n), \\ & n = 1, 2, \cdots, K-1, \quad k = n - m > 0, \quad j = s + k, \\ & k = 1, 2, \cdots, L(K-1) - s; \\ 0, & \text{otherwise.} \end{cases}$$

It may be noted that the transitions of the controlled process take place at each inspection as contrasted with those of the deterioration process which transpire at each point of time.

5. Cost Structure

Three main classes of costs are relevant to the management of deteriorating systems. The first consists of *maintenance* costs, i.e. repair and replacement costs. Generally, one can assume these to be a function of both the state of the system after inspection and the new state resulting from the maintenance decision. For example, repair to an "as new" condition or replacement can usually be expected to depend importantly on whether the system has failed or not. Repair, even to the extent of replacement, before failure involves savings over similar treatment after failure. Similarly, maintenance, in terms of partial repairs of operative equipment is a standard procedure in both industrial and military fields frequently not only for economic reasons, but also for those of feasibility.

A second class of costs are related to *inspection*. In addition to having a fixed component, these may be partially dependent on the condition of the equipment when inspected; e.g. certain states may be more difficult to detect than others. Additionally, they may involve the use of different inspection procedures which are dependent on the length of time since the previous inspection and the condition of the equipment resulting from the maintenance decision made at that time.

Finally, there may be *penalty* costs related to the length of time between failure of the system and its discovery through inspection. For an operating unit this can be associated with the amount of lost service or production. For a unit in storage this could be related to the unavailability of the unit.

Based on the above, the following notation will be used:

$a_{s:kj}$ = cost of inspection in state j given that k periods in the past the state was s, where $a_{s:kj} = 0$ if $s = L, L(1), \cdots, L(K-1)$; (since the system was known to be inoperative in state s, no inspection is required when it is in state $j = s + k$);

$b_{s:kj}$ = penalty cost associated with state j given that k periods previous the system was in state s,

where

$$b_{s:kj} = \begin{cases} 0 & \text{if } j = 0, 1, \cdots, L-1 \\ b_j \geqq 0 & \text{if } s = 0, 1, \cdots, L-1; \; j = L, L(1), \cdots, L(K-1) \\ b_j - b_s \geqq 0 & \text{if } s = L, L(1), \cdots, L(K-2) \text{ (note: } j = s + k). \end{cases}$$

It is convenient to combine the inspection and penalty costs with the notation $t_{s:kj}$.

Further, let $c_{is:k}$ = cost associated with the decision $d_{s:k}(i)$. It consists of

m_{is} = maintenance cost associated with the decision to change the state from i to s,

where

$$m_{i0} = \text{cost of replacement} \begin{cases} \text{before failure if } i \leqq L-1 \\ \text{after failure if } i > L-1 \end{cases}$$

and

$$\begin{aligned} m_{is} &= \text{cost of repair from } i \text{ to } s \text{ if } 0 < s \leqq L-1 \\ &= 0 \quad \text{if} \quad i = s \neq L(K-1). \end{aligned}$$

As was mentioned earlier our policy choice criterion will be the long run average cost per unit time. Let us denote this quantity by ϕ; it is clear that ϕ is equal to the average cost per inspection divided by the average time between inspections.

The ergodic theorem for Markov chains allows us to compute the average cost per inspection, denoted by EC, according to the formula

$$(1) \qquad EC = \sum_i \sum_{s:k} \sum_j \pi_i D_{is:k} v_{s:kj} t_{s:kj} + \sum_i \sum_{s:k} \pi_i D_{is:k} c_{is:k}$$

where the π_i's ($i = 0, 1, \cdots, L(K-1)$) represent the equilibrium probabilities[4] of the different states for the controlled process and satisfy the conditions

$$(2) \qquad \pi_j = \sum_i \pi_i p_{ij}, \qquad j = 0, 1, \cdots, L(K-1)$$

and

$$(3) \qquad \sum_j \pi_j = 1.$$

If we represent the average time between inspections by EI, it may be computed according to the relationship

$$(4) \qquad EI = \sum_i \sum_{s:k} k \pi_i D_{is:k}$$

with conditions (2) and (3) as above applicable.

[4] For the sake of simplicity, it is assumed that the q_{ij}'s are such that the resulting "controlled process" is always ergodic.

Thus, our criterion becomes

$$\phi = \frac{\sum_i \sum_{s:k} \sum_j \pi_i D_{is:k} v_{s:kj} t_{s:kj} + \sum_i \sum_{s:k} \pi_i D_{is:k} c_{is:k}}{\sum_i \sum_{s:k} k\pi_i D_{is:k}} \tag{5}$$

6. Conversion to a Linear Programming Problem

Following Manne [4], let

$$x_{is:k} = \pi_i D_{is:k} ;$$

then our criterion function (5) may be rewritten in the form

$$\phi = \frac{\sum_i \sum_{s:k} x_{is:k} h_{is:k}}{\sum_i \sum_{s:k} k x_{is:k}} \tag{6}$$

where $h_{is:k} = \sum_j t_{s:kj} v_{s:kj} + c_{is:k}$. Further, since

$$\pi_j = \sum_i \pi_i p_{ij} = \sum_i \pi_i \sum_{s:k} D_{is:k} v_{s:kj}$$
$$= \sum_i \sum_{s:k} x_{is:k} v_{s:kj},$$

and $\pi_j = \sum_{s:k} x_{js:k}$, the $L + K$ conditions (2) become

$$\sum_{s:k} x_{js:k} = \sum_i \sum_{s:k} x_{is:k} v_{s:kj}, \quad j = 0, 1, \cdots, L(K-1) \tag{7}$$

and (3) becomes

$$\sum_j \sum_{s:k} x_{js:k} = 1. \tag{8}$$

Thus, our problem is to find $\{x_{js:k}\}$, $j = 0, 1, \cdots, L(K-1)$, $s \in S_j$, $k = 1, 2, \cdots, K$ to minimize the non-linear function (6) subject to the $L + K + 1$ restraints (7) and (8).

Although ϕ is not linear, its particular form allows us to treat the minimization problem as a sequence of linear programming problems by fixing the denominator and finding the minimum of the numerator subject to (7), (8) and the fixed value of the denominator; the problem would then be to determine the minimum over all possible (or reasonable) values of the denominator. It may be noted that a fixed single value or range of values for the denominator (the average time between inspections) may occur 'almost'[5] naturally within the context of system deterioration problems where there are government or legal inspection requirements to be satisfied.

An alternative direct treatment as a single linear programming problem is also possible via the following development due to C. Derman [6].

[5] 'Almost', since a legal inspection requirement is usually in the form of a *fixed* rather than an average interval.

We first alter our notation to single subscripts so that the problem is of the form

$$\text{Find: } \{x_i\}, \qquad i = 1, \cdots, n,$$

$$\text{(9)} \qquad \text{To minimize: } \phi = \frac{\sum c_i x_i}{\sum d_i x_i}$$

$$\text{Subject to: } \sum_i a_{ji} x_i = 0, \qquad j = 1, \cdots, n,$$

$$\sum x_i = 1,$$

and

$$x_i \geqq 0, \qquad i = 1, \cdots, n;$$

where

$$d_i > 0, \qquad i = 1, \cdots, n.$$

Now,[6] let

$$\text{(10)} \qquad y_i = {}^{+}\sqrt{d_i x_i}$$

and

$$z_i = y_i/t$$

with

$$\sum z_i^2 = 1.$$

Then, (9) becomes

$$\text{(11)} \qquad \phi = \sum \frac{c_i}{d_i} z_i^2,$$

a linear function in z_i^2.

We may now solve the *linear* programming problem of minimizing (11) subject to

$$\sum z_i^2 = 1$$

and

$$\sum \frac{a_{ji}}{d_i} z_i^2 = 0, \qquad j = 1, 2, \cdots, n,$$

with

$$z_i^2 \geqq 0, \qquad i = 1, 2, \cdots, n.$$

Then, by finding a value of t which satisfies

$$t^2 \sum \frac{z_i}{d_i} = 1,$$

we may solve, via the original transformations, for $\{x_i\}$ and obtain an optimal solution to the original problem.

[6] The square root transformation used here is not the only one nor is it the simplest one useful in this connection. (For example, all exponents and the radical in (10) may be omitted.) It has been retained since the possible linearization of this problem was first seen in terms of this transformation. The treatment in [6] employs a different transformation; see, also Charnes and Cooper [7].

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