An Inspection Policy for the Weibull Case

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This paper is concerned with the detection of failure of a system when the time to failure is a Weibull variate. The suggested inspection policy depends on a single meaningful parameter. Graphical aids for computing an appropriate inspection policy on the basis of costs, or on the basis of mean time between failure and its detection are given.

1. INTRODUCTION

THE REACTIONS of a mild and well-mannered addict to a saga which unfolds in a series of television programmes when his television set breaks down in the middle of an eagerly awaited programme will no doubt be quite fascinating to a detached observer. This abrupt end to a service, usually necessary, is the fate of a very large variety of items or systems. Cars break down, bicycles get punctured, glasses break, components fail, alarm systems greet intruders with silence, structures crack, etc. A general description is that there are many items or systems which can be in one of two states, say state 0 and state 1. State 0 represents a properly working system and state 1 represents some form of failure. Some systems gradually deteriorate from 0 to 1 passing through increasingly poorer levels of service, while other systems remain in state 0 for all practical purposes till a sudden change to state 1 occurs. We are concerned with systems of the latter kind; that is, we suppose that initially a system starts in state 0, the service represented by this state being an essential one, and that it may suffer an abrupt change to state 1. After the initial starting time when the system is known to be in state 0, the actual state of the system can only be determined, at a cost, by inspection. We assume that inspection has no harmful effects on the system. While the system is in service, it may change from state 0 to state 1 but the system cannot change on its own from state 1 to state 0. Undetected state 1 means a loss and the problem is to achieve a balance between the cost of inspection and the loss due to an undetected state 1.

Barlow *et al.*[1,2] have considered this problem using linear costs and some mild conditions on the transition time between state 0 and state 1. They have derived a recurrence relation for determining optimal inspection times x_i which minimize the expected value of the total cost. This solution can be computationally difficult and in an earlier paper[3] we have suggested a one-parameter inspection policy in which x_i are such that:

$$\text{Prob}\,[\text{state 1 in } (x_{i-1}, x_i)\,|\,\text{state 0 at } x_{i-1}] = p, i = 1, 2, 3, \ldots.$$

In that paper we give a numerical comparison which supports the intuitive feeling that this inspection policy should compare reasonably well with the optimal policy with linear costs. There are situations where reliable explicit costs are not available and an inspection policy has to be determined by an intuitive balancing of inspection costs and the loss due to undetected state 1. This one-parameter policy provides a very easy guide for this intuitive balancing; the choice of an inspection policy depends on a single meaningful parameter p and the plot of expected number of inspections till the detection of state 1 and the plot of expected time between the transition to state 1 and its detection against p provide a quantitative help. It is convenient to denote this inspection policy by X_p.

A fundamental element for the solution of this inspection problem is a model for the transition from state 0 to state 1. We suppose that this transition time is a Weibull variate, T, with the probability density function $f(t)$, distribution function $F(t)$ and failure rate function $r(t)$, where:

$$f(t) = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} \exp -\left(\frac{t}{\alpha}\right)^{\beta}, \quad \alpha, \beta > 0, \quad t \geqslant 0,$$

$$F(t) = 1 - \exp -\left(\frac{t}{\alpha}\right)^{\beta},$$

$$r(t) = \frac{f(t)}{1-F(t)} = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1}.$$

The parameter α is a scaling parameter which takes the time scale of the system into account. The parameter β is a shape parameter and it gives a variety of shapes to $f(t)$. Thus we have a simple and a flexible family of density functions. Also note that $\beta = 1$ gives constant $r(t)$, $\beta < 1$ gives decreasing $r(t)$ and $\beta > 1$ gives increasing $r(t)$; for most practical cases we would have $\beta \geqslant 1$. Weibull distribution is very widely used in practice, see Weibull,[4] Plait[5] and Johnson and Kotz.[6]

2. X_p POLICY FOR THE WEIBULL CASE

We first consider the case where an optimal X_p policy has to be computed with linear costs. Let c_1 be the cost of each of the inspections. If state 1 occurs at time t and is detected at the ith inspection time x_i, so that $x_i \geqslant t$, let the cost due to undetected state 1 be $c_2(x_i - t)$. An optimal X_p policy is one which minimizes the expected value of the total cost:

$$c = ic_1 + c_2(x_i - t). \tag{1}$$

Using the general results of the earlier paper,[3] we have for any p in $(0, 1)$ and $q = 1-p$:

$$x_i = F^{-1}(1-q^i), \quad i = 1, 2, 3, \ldots,$$

$$x_0 = 0,$$

so that for the particular Weibull case we get:

$$x_i = \alpha\{\ln(1/q)\}^{1/\beta}\, i^{1/\beta}, \quad i = 1, 2, 3, \ldots, \tag{2}$$

$$x_0 = 0.$$

Also:

$$E(c) = c_1/p + c_2\,\tau(p), \tag{3}$$

where:

$$\tau(p) = \sum_{i=1}^{\infty} x_i q^{i-1} p - E(T). \tag{4}$$

The function $\tau(p)$ is the mean time between the transition to state 1 and its detection. Note that the case of no inspection at all can be represented by $x_0 = 0$, $x_1 = \infty$, so that for finite $E(T)$, $\tau(p)$ is infinite. For the particular case of a Weibull distribution we have:

$$E(T) = \int_0^\infty t\frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} \exp-\left(\frac{t}{\alpha}\right)^{\beta} \mathrm{d}t$$

$$= \alpha\int_0^\infty \left[\left(\frac{t}{\alpha}\right)^{\beta}\right]^{1/\beta} \exp-\left(\frac{t}{\alpha}\right)^{\beta}\frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} \mathrm{d}t = \alpha\Gamma\left(1+\frac{1}{\beta}\right), \tag{5}$$

and equations (2), (4) and (5) lead to:

$$\tau(p) = \alpha\{\ln(1/q)\}^{1/\beta}\sum_{i=1}^{\infty} i^{1/\beta} q^{i-1} p - \alpha\Gamma(1+1/\beta). \tag{6}$$

From equations (3) and (6) in order to find an optimal p which minimizes $E(c)$, p has to be found such that with $k = c_1/\alpha c_2$, $g(p)$ is minimized where:

$$g(p) = \frac{k}{p} + \{\ln(1/q)\}^{1/\beta}\sum_{i=1}^{\infty} i^{1/\beta} q^{i-1} p. \tag{7}$$

The optimal value of p has to be determined numerically, and an accurate minimization of $g(p)$ is made easy by the rapid convergence of the infinite sum in $g(p)$. Note that the computation of optimal p needs β and $c_1/\alpha c_2$. The relevance of the cost ratio c_1/c_2 rather than the individual costs c_1, c_2 may be a help in some cases; it may be that the relative cost c_1/c_2 is available while the individual costs c_1, c_2 are not available. For practical purposes a nomogram for finding optimal p would be of obvious help in the determination of the inspection times x_i from equation (2). The methods outlined by Lyle[7] usually lead to quite accurate nomograms and Figure 1 is no exception. Given β, and $k = c_1/\alpha c_2$, the optimal value of p can be read off quite simply in the usual manner by using a straight edge. Table 1 gives an indication of the accuracy of this nomogram.

Given p, the inspection times $x_1, x_2 \ldots$ can be easily computed from equation (2).

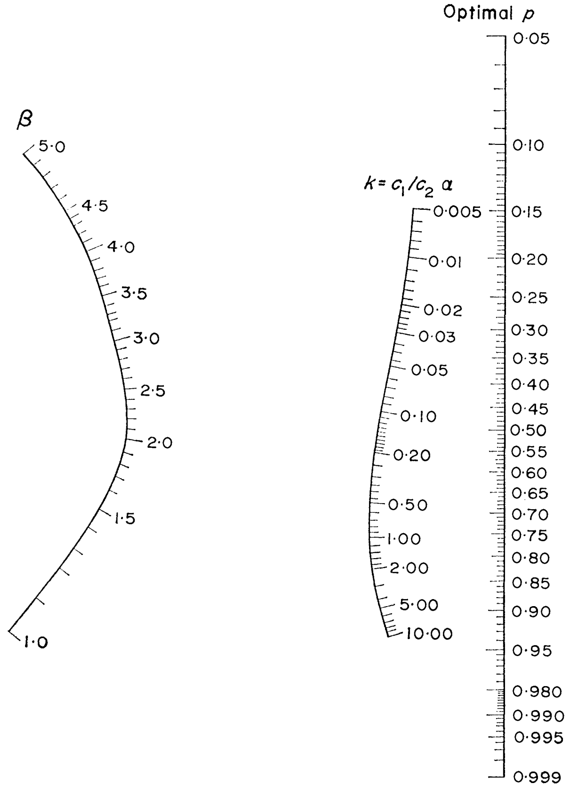

FIG. 1. *Nomogram for optimal* p.

TABLE 1. ACCURACY OF THE NOMOGRAM FOR OPTIMAL p

β	k	Computed p	p from nomogram
1·20	0·30	0·5263	0·528
1·95	0·02	0·2124	0·210
2·50	2·50	0·9338	0·930
3·50	0·01	0·1856	0·185
4·0	0·10	0·5426	0·543
4·95	0·006	0·1641	0·167

We now turn to the case where an inspection policy has to be determined through an intuitive balancing of the inspection effort and consequences of undetected state 1. The mean number of inspections, and the mean time to the transition to state 1 and its detection provide good quantitative guides for this intuitive balancing. Both of these mean values are functions of p. The mean number of inspections until the detection of state 1 is $1/p$, and equation (6) shows that $\tau(p)/\alpha$ is a function of p and β. Figure 2 gives $\tau(p)/\alpha$ for a range of values of β and this Figure or an extended version together with $1/p$ should be of help in the choice of a good value of p. Note that $\tau(p)/\alpha$ increases rapidly and approaches infinity as p tends to 1 but a large p means a small average number of inspections. A reasonable value of p would seem to be near the point where the curve begins to rise sharply. For example, with $\beta = 2$, Figure 2 shows that a suitable value of p is 0·8. With this value, the mean number of inspections is $1/0{\cdot}8 = 1{\cdot}25$, and $\tau(p)/\alpha \simeq 0{\cdot}5$.

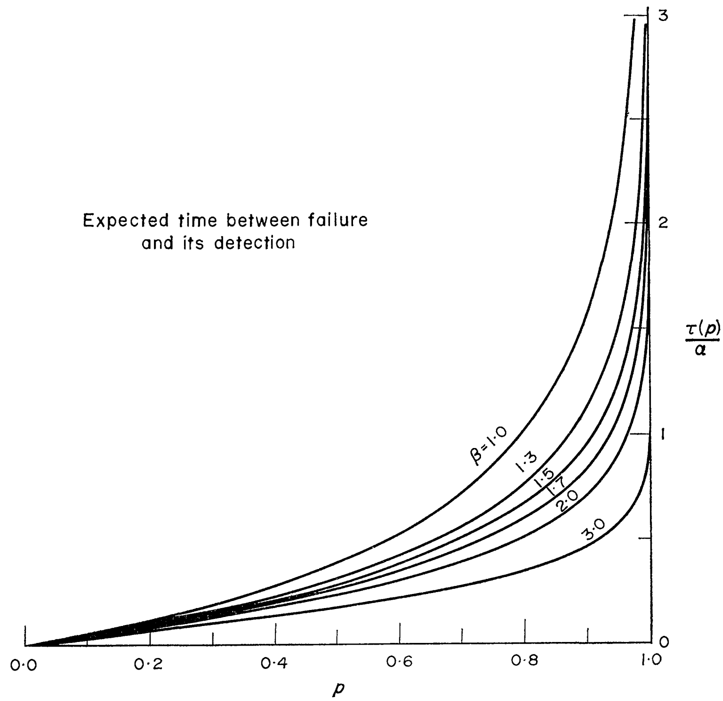

FIG. 2.

Figure 2 can also be used for computing the expected cost. In this particular case the total expected cost is proportional to $(1{\cdot}25k + 0{\cdot}5)$. Given p, the inspection times can be easily computed from equation (2).

ACKNOWLEDGEMENT

It is a pleasure to acknowledge Mr. Newbold's skill in the construction of Figure 1.

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