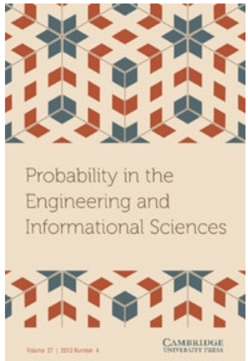

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OPTIMAL AGE AND BLOCK REPLACEMENT FOR A GENERAL MAINTENANCE MODEL

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We continue the study of our general cost structure for a maintained system. Here we focus on the optimization questions for an age or block policy. The notion of a marginal cost function is rigorously formulated and its utility investigated. Various applications are considered, including a new model in which minimal repairs are performed as long as the total accumulated repair costs do not exceed a fixed amount.

1. INTRODUCTION

In this paper we continue our study of a general cost structure for maintained systems. The basic model was introduced in Savits [10]. A discounted version was later studied by Chen and Savits [6]. In the preceding papers, the primary emphasis was on the relationships among the various cost expressions for the age and block replacement maintenance policies. Our focus here is on the optimization question, i.e., on minimizing cost for these policies.

The basic model and associated results are reviewed in Section 2. The notion of a marginal cost function is defined and developed in Section 3. Although

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this concept was first introduced into the reliability literature by Berg [1], it was presented in a heuristic manner. Here we give a rigorous treatment and explore their relationships for the various policies.

The marginal cost function is then used in Section 5 to solve the optimization problem for the various maintained systems already considered. In a certain sense, the role of the marginal cost function in these optimization problems is similar to the role of the infinitesimal operator in Markov theory. This dynamic viewpoint is explored further in Section 4.

Various examples are presented in Section 6, including two new models. Finally, in Section 7, some further generalizations are delineated and an error in a paper by Berg, Bienvenu, and Cleroux [2] is corrected.

In this paper we shall say that a function f is nondecreasing if $x \leq y$ implies $f(x) \leq f(y)$. Furthermore, the notation $E[X;A]$ means $E[XI_A]$, where I_A is the indicator function of the set A.

2. PRELIMINARIES

Before we consider the optimization problem, we first need a quick review of the basic model and results. According to the setup of Savits [10], the basic accumulated operational cost of a unit on line during a time interval $[0,t)$ is specified by a stochastic process $R(t)$, $0 \leq t \leq \zeta$. The nonnegative random variable ζ designates a major unrepairable breakdown; i.e., after aging ζ units of time, the failed item is replaced with a new identical unit. Such replacements are called *unplanned* (or *unscheduled*) and cost c_1.

Another cost is incurred for planned maintenance actions. The two maintenance policies considered here are called age replacement and block replacement. In the case of an age replacement maintenance policy, *planned* (or *scheduled*) replacements occur whenever a functioning unit reaches age T; in the case of a block replacement maintenance policy, a planned replacement occurs at the absolute times $T, 2T, \ldots$. In either case, the cost of such a planned replacement is c_2.

To avoid some technical considerations, we shall make the following assumptions throughout this paper.

ASSUMPTIONS 2.1:

(i) $\{R(t), 0 \leq t \leq \zeta\}$ *is a nonnegative, nondecreasing, left-continuous stochastic process with right-hand limits and* $R(0) = R(0+) = 0$.

(ii) $R(t+)$ *represents the accumulated operational cost of a unit over the time interval* $[0,t]$, $0 \leq t < \zeta$.

(iii) ζ *is a strictly positive random variable with finite expectation.*

(iv) c_1 *and* c_2 *are nonnegative random variables with finite expectation.*

(v) $E[R(\zeta)] < \infty$.

Given these ingredients, one can then compute the objective functions of interest. When there is no discounting, we are interested in the expected long-run cost per unit time for both the age policy and the block policy. These are denoted by $J_A(T)$ and $J_B(T)$, respectively. When there is a discounting factor of $\alpha > 0$, we are interested in the total expected discounted cost, again for both the age policy and the block policy. In this case, they are denoted by $J_A^{(\alpha)}(T)$ and $J_B^{(\alpha)}(T)$, respectively.

Expressions for these quantities were obtained and were related to Savits [10, pp. 791–793] and Chen and Savits [6, pp. 113–114]. The main results are summarized below.

THEOREM 2.2: *Suppose Assumptions* 2.1 *hold. Then*

$$J_A(T) = \frac{A(T)}{E[\zeta \wedge T]}, \qquad J_B(T) = \frac{B(T)}{T}, \tag{2.1a, b}$$

$$J_A^{(\alpha)}(T) = \frac{A^{(\alpha)}(T)}{1 - E[e^{-\alpha(\zeta \wedge T)}]}, \qquad J_B^{(\alpha)}(T) = \frac{B^{(\alpha)}(T)}{1 - e^{-\alpha T}}, \tag{2.2a, b}$$

where

$$A(T) = E[R(\zeta) + c_1; \zeta < T] + E[R(T) + c_2; \zeta \geq T], \tag{2.3}$$

$$B(T) = \int_{[0,T)} A(T-x)\, dU(x), \tag{2.4}$$

$$A^{(\alpha)}(T) = \int_{[0,T)} e^{-\alpha x}\, dA(x) + E[c_2 e^{-\alpha(\zeta \wedge T)}], \textit{ and} \tag{2.5}$$

$$B^{(\alpha)}(T) = \int_{[0,T)} e^{-\alpha x}\, dB(x) + e^{-\alpha T} E[c_2]. \tag{2.6}$$

In the preceding theorem, $U(x)$ is the renewal function (in the sense of Feller [7]) generated by ζ; more precisely, $U(x) = \sum_{k=0}^{\infty} P(\sigma_k \leq x)$ with $\sigma_0 = 0$ and $\sigma_k = \zeta_1 + \cdots + \zeta_k$ for $k \geq 1$, where $\zeta_1, \zeta_2, \ldots$ are independent copies of ζ.

Remarks 2.3:

1. In Chen and Savits [6], the expressions relating $A^{(\alpha)}$ to A and $B^{(\alpha)}$ to B were somewhat different in that the integration range $[0, T)$ was replaced by $(0, T]$. The reason for this discrepancy is that in the former paper we assumed that A and B were right-continuous functions in the derivation, whereas in this paper they can be shown to be left-continuous functions under Assumptions 2.1.
2. It is sometimes useful to note that we can also write

$$B^{(\alpha)}(T) = \int_{[0,T)} e^{-\alpha x} A^{(\alpha)}(T-x)\, dU(x). \tag{2.7}$$

3. MARGINAL COST FUNCTIONS

In the preceding sections, we listed four different cost expressions of interest: $J_A(T)$, $J_B(T)$, $J_A^{(\alpha)}(T)$, and $J_B^{(\alpha)}(T)$. Each expression depends on the maintenance parameter T. An obvious problem of interest then is to minimize these objective functions with respect to T.

In each case, we note that the cost function has the form

$$J(T) = \frac{N(T)}{D(T)}. \tag{3.1}$$

It is an elementary calculus exercise to show that critical points of J (i.e., values of T for which $J'(T)$ vanishes) satisfy the relation

$$\frac{N'(T)}{D'(T)} = J(T). \tag{3.2}$$

It will be shown in Section 5 that the quantity $N'(T)/D'(T)$ plays an important role in the minimization problem. This motivates the following definition.

DEFINITION 3.1: *Consider the objective function $J(T) = N(T)/D(T)$ with N and D continuously differentiable on $[0,\infty)$ and $D'(T) > 0$ there. We then define the marginal cost function η by $\eta(T) = N'(T)/D'(T)$.*

Remark 3.2: The term marginal cost function has been extensively used in the theory of economics for many years (e.g., see Henderson and Quandt [8]). To the best of our knowledge, the term was first introduced into the reliability literature by Berg [1]. His notion of a marginal cost function, however, was never explicitly defined. Instead, a heuristic formulation was presented for each particular application. (For further discussion, see also Section 4).

Our goal now is to determine the marginal cost function for each of the objective functions $J_A(T)$, $J_B(T)$, $J_A^{(\alpha)}(T)$, and $J_B^{(\alpha)}(T)$. These will be denoted by $\eta_A(T)$, $\eta_B(T)$, $\eta_A^{(\alpha)}(T)$, and $\eta_B^{(\alpha)}(T)$, respectively. Since the computations are rather straightforward, we will only illustrate with one example.

For ease of exposition, we will assume that c_1 and c_2 are constants. The more general variable cost case is delineated in Section 7. We let G be the distribution function of ζ and assume that it has a continuous density function g. Also, let $r_G = g/\bar{G}$ be its failure rate, where $\bar{G}$ is the survival probability function.

We now consider the marginal cost function for the age replacement case without discounting. It follows from Eq. (2.1a) that $\eta_A(T) = A'(T)/\bar{G}(T)$, where $A(T)$ is given by Eq. (2.3). To write down the derivative of $A(T)$, we shall assume that there exists a measurable function $\phi(u,y)$ such that

$$E\left[R(x) \mid \zeta = y\right] = \int_0^x \phi(u,y)\, du \tag{3.3}$$

for all $0 \le x \le y$. Then we can write

$$\begin{aligned} A(T) &= \int_0^T \int_0^y \phi(u,y)\,du\,g(y)\,dy + c_1 G(T) \\ &\quad + \int_T^\infty \int_0^T \phi(u,y)\,du\,g(y)\,dy + c_2 \bar{G}(T) \\ &= \int_0^T \int_u^\infty \phi(u,y) g(y)\,dy\,du + (c_1 - c_2)G(T) + c_2. \end{aligned}$$

Thus, if $\int_u^\infty \phi(u,y)g(y)\,dy$ is continuous in u, then $A(T)$ is continuously differentiable and

$$A'(T) = \int_T^\infty \phi(T,y)g(y)\,dy + (c_1 - c_2)g(T).$$

Consequently,

$$\eta_A(T) = \frac{A'(T)}{\bar{G}(T)} = \frac{1}{\bar{G}(T)} \int_T^\infty \phi(T,y)g(y)\,dy + (c_1 - c_2)r_G(T).$$

The other cases are summarized in the next theorem.

THEOREM 3.3: *Assume there exists a measurable function $\phi(u,y)$ such that $E[R(x) \mid \zeta = y] = \int_0^x \phi(u,y)\,du$. Further, suppose that the distribution function G of ζ has a continuous density function g and that $\int_T^\infty \phi(T,y)g(y)\,dy$ is continuous in T. Then, for constants c_1 and c_2, the marginal cost functions are given by*

$$\eta_A(T) = \frac{1}{\bar{G}(T)} \int_T^\infty \phi(T,y)g(y)\,dy + (c_1 - c_2)r_G(T), \tag{3.4}$$

$$\eta_B(T) = \int_{[0,T)} \{\eta_A(T-x)\bar{G}(T-x) + c_2 g(T-x)\}\,dU(x), \tag{3.5}$$

$$\eta_A^{(\alpha)}(T) = \frac{1}{\alpha}\,\eta_A(T) - c_2, \text{ and} \tag{3.6}$$

$$\eta_B^{(\alpha)}(T) = \frac{1}{\alpha}\,\eta_B(T) - c_2. \tag{3.7}$$

Remarks 3.4:

1. A sufficient condition for $\int_T^\infty \phi(T,y)g(y)\,dy$ to be continuous in T is that $\phi(u,y)$ be continuous in u for each y and, for each u, be bounded by a G-integrable function in a neighborhood of u.
2. When $\phi(u,y)$ is independent of y, say $\phi(u,y) = \Phi(u)$, the expression for the marginal cost functions simplify, e.g.,

$$\eta_A(T) = \Phi(T) + (c_1 - c_2)r_G(T).$$

3. It should be noted that the expression for η_B does not depend on c_2. This is so because

$$\eta_A(T)\bar{G}(T) + c_2 g(T) = \int_T^\infty \phi(T,y)g(y)\,dy + c_1 g(T).$$

Perhaps a better way of writing η_B is to define

$$\eta_{AR}(T) = \frac{1}{\bar{G}(T)} \int_T^\infty \phi(T,y)g(y)\,dy.$$

Then, the expression for η_B is more clearly expressed as

$$\eta_B(T) = \int_{[0,T)} \{\eta_{AR}(T-x)\bar{G}(T-x) + c_1 g(T-x)\}\,dU(x). \qquad \textbf{(3.8)}$$

The usefulness of Theorem 3.3 should be emphasized here. As will be shown in Section 5, the optimization results for the various policies require a knowledge of the associated marginal cost functions. Since these marginal cost functions are related through Theorem 3.3, it is thus only necessary to determine one of them. As a general rule, the marginal cost function for the age policy $\eta_A(T)$ is easiest to compute; moreover, it can often be calculated in a simple manner, as the next section shows.

4. A DYNAMIC VIEWPOINT OF η_A

In this section we present an alternative interpretation of the marginal cost function η_A. Our discussion will be mostly heuristic, although the arguments can easily be made rigorous. It is also possible to obtain similar results for the other marginal cost functions, but in view of Theorem 3.3, that seems superfluous.

Recall that the marginal cost function η_A is given by $\eta_A(T) = A'(T)/\bar{G}(T)$. We now write (see Eq. (2.3))

$$\begin{aligned} A(T+\Delta) - A(T) &= E[R(\zeta) - R(T);\ T \le \zeta < T+\Delta] \\ &\quad + E[R(T+\Delta) - R(T);\ \zeta \ge T+\Delta] \\ &\quad + E[c_1 - c_2;\ T \le \zeta < T+\Delta] \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned}$$

But

$$\mathrm{I} = \int_T^{T+\Delta} \int_T^{y} \phi(u,y)\,du\,g(y)\,dy \qquad \text{and}$$

$$\mathrm{III} = (c_1 - c_2)\int_T^{T+\Delta} g(y)\,dy.$$

Hence, under fairly general conditions, $(1/\Delta)\mathrm{I} \to 0$ and $(1/\Delta)\mathrm{III} \to (c_1 - c_2)g(T)$ as $\Delta \to 0$. Consequently,

$$A'(T) = \lim_{\Delta \to 0} \frac{1}{\Delta} E[R(T+\Delta) - R(T);\ \zeta \geq T + \Delta] + (c_1 - c_2)g(T),$$

or

$$\eta_A(T) = \lim_{\Delta \to 0} \frac{1}{\Delta} E\big[R(T+\Delta) - R(T) \,\big|\, \zeta \geq T + \Delta\big] + (c_1 - c_2)r_G(T). \quad \textbf{(4.1)}$$

Equation (4.1) gives an infinitesimal characterization of η_A. This is particularly useful since many times the behavior of the model is known (or even specified) at the infinitesimal level. Consequently, one can obtain η_A without first computing its integrated version $A(T)$ and then differentiating. This formulation is somewhat analogous to specifying a Markov chain through its infinitesimal operator.

The preceding heuristic approach also seems to be the one used by Berg: he first writes down an expression like Eq. (4.1) for the marginal cost function and then computes the objective function from it. For example, Berg [1] defines the marginal cost of an age replacement at age x "as the difference, per unit time, between the cost of an age replacement now and the expected costs associated with waiting an additional short time Δ." Also, Berg and Cleroux [3] define

$$\eta_A(T) = \lim_{\Delta \to 0} \frac{1}{\Delta} \left[\begin{pmatrix} \text{the expected costs in} \\ (T, T+\Delta) \text{ if at age } T \\ \text{we defer the preventive} \\ \text{replacement to } T+\Delta \end{pmatrix} - \begin{pmatrix} \text{the cost of} \\ \text{a preventive} \\ \text{replacement now} \\ \text{i.e., at age } T \end{pmatrix} \right].$$

5. OPTIMIZATION RESULTS

In this section we want to consider the optimization problem for the four objective functions $J_A(T)$, $J_B(T)$, $J_A^{(\alpha)}(T)$, and $J_B^{(\alpha)}$, i.e., minimize these expressions with respect to the maintenance parameter T.

Section 3 noted that all these objective functions have the form $J(T) = N(T)/D(T)$. This minimization problem was considered by Puri and Singh [9]. Their main result is summarized below.

Consider the minimization problem for the objective function of Eq. (3.1) with N and D continuously differentiable on $[0,\infty)$, $N(0) > 0$, $D(0) = 0$, and $D'(T) > 0$ for all $T \geq 0$. Let $\eta(T) = N'(T)/D'(T)$ be the associated marginal cost function.

THEOREM 5.1 (Puri and Singh): *Assume for the preceding setup that* η *is nondecreasing on* $[0,\infty)$. *Then the following statements are true*:

(i) $\eta(T)D(T) - N(T)$ *is nondecreasing on* $[0,\infty)$ *and thus has a limit* L (*which may be* $+\infty$) *as* $T \to \infty$.

(ii) *If $L > 0$, there exists a smallest value T_0, $0 < T_0 < \infty$, where $J(T)$ assumes its global minimum; moreover, at T_0 we have the relation $\eta(T_0) = J(T_0)$.*

(iii) *Any finite solution of the equation $\eta(T) = J(T)$ is a point of global minimum for $J(T)$.*

(iv) *If η is strictly increasing, there is at most one solution of the equation $\eta(T) = J(T)$.*

(v) *If $L = 0$, $J(T)$ is a nonincreasing function and so does not have a minimum value unless η is eventually constant.*

Remarks 5.2:

1. If the functions N and D are also bounded, then $L > 0$ if and only if

$$\eta(\infty) \equiv \lim_{T\to\infty} \eta(T) > \lim_{T\to\infty} J(T) \equiv J(\infty).$$

2. Note that we can recapture J through η and D because

$$N(T) = N(0) + \int_0^T \eta(s)D'(s)\,ds.$$

3. Puri and Singh also showed that if η is nondecreasing and $L < 0$, or if η is nonincreasing, then $J(T)$ is strictly decreasing and so J is minimized only at $T = \infty$.

We shall now apply the results of Theorem 5.1 to each of our objective functions. Our goal is to obtain conditions on R, ζ, c_1, and c_2, which guarantee that the associated marginal cost function η is nondecreasing and that the limiting value L is positive. The results we derive here are intended to be illustrative, not definitive.

In the following discussion we shall assume that the conditions of Theorem 3.3 hold without explicitly mentioning them.

5.1. Age Policy without Discounting

Recall that in this case we have $J_A(T) = A(T)/\int_0^T \bar{G}(u)\,du$ with $A(T) = E[R(\zeta) + c_1;\ \zeta < T] + E[R(T) + c_2;\ \zeta \geq T]$ and

$$\eta_A(T) = \frac{1}{\bar{G}(T)} \int_T^\infty \phi(T,y)g(y)\,dy + (c_1 - c_2)r_G(T).$$

It is not hard to show that the first term of η_A is nondecreasing in T if $\phi(u,y)$ is nondecreasing in both u and y. This follows since $\bar{G}(T_1 \vee t)/\bar{G}(T_1) \leq \bar{G}(T_2 \vee t)/\bar{G}(T_2)$ for all $t \geq 0$ whenever $T_1 \leq T_2$. Hence the corresponding probability measures are stochastically ordered. Thus η_A is nondecreasing if, in addition, $c_1 \geq c_2$ and r_G is nondecreasing (i.e., if ζ is IFR).

Because the numerator and denominator of $J_A(T)$ are bounded, i.e.,

$$A(T) \le E[R(\zeta)] + c_1 + c_2,$$

$$\int_0^T \bar{G}(u)\,du \le E[\zeta] = \mu_1,$$

it follows from Remark 5.2.1 that $L_A > 0$ if and only if $\eta_A(\infty) > J_A(\infty) = (E[R(\zeta)] + c_1)/\mu_1$. One simple sufficient condition for this is that $c_1 > c_2$ and $\lim_{T\to\infty} r_G(T) \equiv r_G(\infty) = \infty$.

Alternately, if we assume that $\phi(T,y) \to c_R$ uniformly in a neighborhood of $y = \infty$ as $T \to \infty$, then we can write an explicit expression for $\eta_A(\infty)$, namely,

$$\eta_A(\infty) = c_R + (c_1 - c_2)r_G(\infty).$$

THEOREM 5.3: *Let $\phi(u,y)$ be nondecreasing in (u,y) with $\phi(T,y) \to c_R$ uniformly in a neighborhood of $y = \infty$ as $T \to \infty$ and assume that ζ is* IFR *and $c_1 \ge c_2$. Then $0 < T_0 < \infty$ exists, which minimizes $J_A(T)$ provided*

$$c_R + (c_1 - c_2)r_G(\infty) > (E[R(\zeta)] + c_1)/\mu_1, \tag{5.1}$$

where $\mu_1 = E[\zeta]$.

5.2. Block Policy without Discounting

For this case we have $J_B(T) = B(T)/T$ with $B(T) = \int_{[0,T)} A(T-x)\,dU(x)$ and (see Eq. (3.8))

$$\eta_B(T) = \int_{[0,T)} \{\eta_{AR}(T-x)\bar{G}(T-x) + c_1 g(T-x)\}\,dU(x).$$

We have not been able to obtain any nice condition for η_B to be nondecreasing. The obvious condition of requiring the integrand $\{\eta_{AR}(T)\bar{G}(T) + c_1 g(T)\}$ to be nondecreasing does not work because

$$\int_0^\infty \{\eta_{AR}(t)\bar{G}(t) + c_1 g(t)\}\,dt = E[R(\zeta)] + c_1$$

is finite. Of course, η_B can be nondecreasing without the integrand being a nondecreasing function; e.g., there are many cases where the renewal density $v_G(T) = \int_{[0,T)} g(T-x)\,dU(x)$ is nondecreasing.

We thus turn our attention to the limit value $L_B = \lim_{T\to\infty}[\eta_B(T)T - B(T)]$ of Theorem 5.1. In this case, however, because the denominator is not bounded, Remark 5.2.1 is not applicable; in fact, we generally have that $\eta_B(\infty) = J_B(\infty)$ by L'Hospital's rule. Thus, we need to compute L_B directly. To do this, we make use of a Tauberian argument.

We first observe that $\eta_B(t)$ satisfies the renewal equation

$$\eta_B(t) = k(t) + \int_0^t \eta_B(t-x)g(x)\,dx,$$

where $k(t) = \eta_{AR}(t)\bar{G}(t) + c_1 g(t)$. Also,

$$\eta_B(t)t - B(t) = \int_0^t [\eta_B(t) - \eta_B(u)]\,du - c_2.$$

Hence, $L_B > 0$ if and only if $\lim_{t\to\infty}\int_0^t[\eta_B(t) - \eta_B(u)]\,du > c_2$.

THEOREM 5.4: *Suppose η satisfies the renewal equation*

$$\eta(t) = k(t) + \int_0^t \eta(t-x)g(x)\,dx,$$

where g is a probability density on $[0,\infty)$, and k is locally bounded. Define $\phi(t) = \int_0^t[\eta(t) - \eta(u)]\,du$ and assume that the following quantities are finite:

$$\mu_i = \int_0^\infty t^i g(t)\,dt, \quad i = 1,2,$$

$$K = \int_0^\infty k(t)\,dt, \qquad \textit{and}$$

$$\nu_1 = \frac{1}{K}\int_0^\infty tk(t)\,dt.$$

Then $\phi(t) \to K[(\nu_1/\mu_1) - (\mu_2/2\mu_1^2)]$ as $t \to \infty$ provided ϕ is ultimately monotonic.

PROOF: Let $\hat{\eta}(s) = \int_0^\infty e^{-st}\eta(t)\,dt$ be the Laplace transform of $\eta (s > 0)$. Similarly, $\hat{k}$, $\hat{g}$, and $\hat{\phi}$ denote their respective Laplace transforms.

Since η satisfies the preceding renewal equation, we get

$$\hat{\eta}(s) = \frac{\hat{k}(s)}{1 - \hat{g}(s)}.$$

Also, from the definition of ϕ,

$$\hat{\phi}(s) = -\frac{d}{ds}\hat{\eta}(s) - \frac{1}{s}\hat{\eta}(s).$$

Recall that the behavior of $\phi(t)$ as $t \to \infty$ is determined by the behavior of $s\hat{\phi}(s)$ as $s \downarrow 0$. Thus, we consider

$$s\hat{\phi}(s) = \frac{-\dfrac{d}{ds}\hat{k}(s)}{\left[\dfrac{1-\hat{g}(s)}{s}\right]} - \frac{\hat{k}(s)\left[\dfrac{1}{s}\left\{\dfrac{1-\hat{g}(s)}{s} + \dfrac{d}{ds}\hat{g}(s)\right\}\right]}{\left[\dfrac{1-\hat{g}(s)}{s}\right]^2}.$$

But, as $s \downarrow 0$,

$$\frac{1-\hat{g}(s)}{s} = \int_0^\infty \int_0^t e^{-su}\,du\,g(t)\,dt \to \int_0^\infty tg(t)\,dt = \mu_1,$$

$$\hat{k}(s) = \int_0^\infty e^{-st}k(t)\,dt \to \int_0^\infty k(t)\,dt = K,$$

$$-\frac{d}{ds}\hat{k}(s) = \int_0^\infty te^{-st}k(t)\,dt \to \int_0^\infty tk(t)\,dt = K\nu_1, \qquad \text{and}$$

$$\frac{1}{s}\left\{\frac{1-\hat{g}(s)}{s} + \frac{d}{ds}\hat{g}(s)\right\} = \int_0^\infty \int_0^t \int_u^t e^{-sv}\,dv\,du\,g(t)\,dt$$

$$\to \int_0^\infty \int_0^t (t-u)\,du\,g(t)\,dt = \frac{1}{2}\mu_2.$$

Consequently,

$$s\hat{\phi}(s) \to \frac{K\nu_1}{\mu_1} - \frac{K\mu_2}{2\mu_1^2}$$

as $s \downarrow 0$. Assuming that ϕ is ultimately monotone, we can invoke a Tauberian theorem to get the desired result (cf. Feller [7, p. 423, Theorem 4]). ∎

If we now apply the preceding result to our situation, we obtain the following theorem.

THEOREM 5.5: *Assume that $\eta_B(T)$ is nondecreasing in T. Then there exists $0 < T_0 < \infty$, which minimizes $J_B(T)$ provided $\mu_1 = E[\zeta]$ and $\mu_2 = E[\zeta^2]$ are finite and*

$$(E[R(\zeta)] + c_1)\left\{\frac{\int_0^\infty \int_0^y t\phi(t,y)\,dt\,g(y)\,dy + c_1\mu_1}{(E[R(\zeta)] + c_1)\mu_1} - \frac{\mu_2}{2\mu_1^2}\right\} > c_2. \qquad \textbf{(5.2)}$$

5.3. Age Policy with Discounting

According to Theorems 2.2 and 3.3, $J_A^{(\alpha)}(T) = A^{(\alpha)}(T)/\{1 - E[e^{-\alpha(\zeta\wedge T)}]\}$ with $A^{(\alpha)}(T) = \int_{[0,T)} e^{-\alpha x}\,dA(x) + c_2E[e^{-\alpha(\zeta\wedge T)}]$ and

$$\eta_A^{(\alpha)}(T) = \frac{1}{\alpha}\eta_A(T) - c_2.$$

Thus, $\eta_A^{(\alpha)}$ is nondecreasing if and only if η_A is.

Since the numerator and denominator of $J_A^{(\alpha)}(T)$ are bounded, $L_A^{(\alpha)} > 0$ if and only if $\eta_A^{(\alpha)}(\infty) > J_A^{(\alpha)}(\infty)$. Clearly, $\eta_A^{(\alpha)}(\infty) = (1/\alpha)\eta_A(\infty) - c_2$. Hence, we need only compute $J_A^{(\alpha)}(\infty)$:

$$J_A^{(\alpha)}(\infty) = \frac{A^{(\alpha)}(\infty)}{1 - E[e^{-\alpha\zeta}]}.$$

To compute $A^{(\alpha)}(\infty)$, we make use of an alternate expression for $A^{(\alpha)}(T)$ given in Chen and Savits [6]:

$$A^{(\alpha)}(T) = E\left[\alpha \int_0^{\zeta} e^{-\alpha v} R(v+)\, dv + e^{-\alpha\zeta}\{R(\zeta) + c_1\};\ \zeta < T\right]$$
$$+ E\left[\alpha \int_0^{T} e^{-\alpha v} R(v+)\, dv + e^{-\alpha T}\{R(T) + c_2\};\ \zeta \geq T\right].$$

Hence,

$$A^{(\alpha)}(\infty) = E\left[\alpha \int_0^{\zeta} e^{-\alpha v} R(v+)\, dv + e^{-\alpha\zeta}\{R(\zeta) + c_1\}\right]$$
$$= \int_0^{\infty} \int_0^{y} e^{-\alpha t} \phi(t,y)\, dt\, g(y)\, dy + c_1 E[e^{-\alpha\zeta}].$$

THEOREM 5.6: *Let $\phi(u,y)$ be nondecreasing in (u,y) with $\phi(T,y) \to c_R$ uniformly in a neighborhood of $y = \infty$ as $T \to \infty$ and assume that ζ is* IFR *and $c_1 \geq c_2$. Then $0 < T_0 < \infty$ exists, which minimizes $J_A^{(\alpha)}(T)$ provided*

$$\left\{\frac{1}{\alpha}[c_R + (c_1 - c_2) r_G(\infty)] - c_2\right\}\{1 - E[e^{-\alpha\zeta}]\}$$
$$> \left\{\int_0^{\infty} \int_0^{y} e^{-\alpha t} \phi(t,y)\, dt\, g(y)\, dy + c_1 E[e^{-\alpha\zeta}]\right\}. \tag{5.3}$$

5.4. Block Policy with Discounting

In this case, $J_B^{(\alpha)}(T) = B^{(\alpha)}(T)/[1 - e^{-\alpha T}]$ with $B^{(\alpha)}(T) = \int_{[0,T)} e^{-\alpha x}\, dB(x) + c_2 e^{-\alpha T}$ and

$$\eta_B^{(\alpha)}(T) = \frac{1}{\alpha}\eta_B(T) - c_2.$$

Again, we have that $\eta_B^{(\alpha)}$ is nondecreasing if and only if η_B is.

We now consider the term $B^{(\alpha)}(T)$. An alternative expression is given in Remark 2.3.2:

$$B^{(\alpha)}(T) = \int_{[0,T)} e^{-\alpha x} A^{(\alpha)}(T - x)\, dU(x).$$

Now $e^{-\alpha x}U(x)$ is the renewal function generated by the subdistribution function $H(x) = \int_{[0,x]} e^{-\alpha y}\, dG(y)$. Hence, it is well known (e.g., see Feller [7, p. 361]) that $B^{(\alpha)}(T) \to B^{(\alpha)}(\infty)$ as $T \to \infty$, where

$$B^{(\alpha)}(\infty) = A^{(\alpha)}(\infty)/\{1 - E[e^{-\alpha\zeta}]\}.$$

Since we have shown that $B^{(\alpha)}$ is bounded, we can conclude that $L_B^{(\alpha)} > 0$ if and only if $\eta_B^{(\alpha)}(\infty) > J_B^{(\alpha)}(\infty) = B^{(\alpha)}(\infty)$. But $\eta_B^{(\alpha)}(\infty) = (1/\alpha)\eta_B(\infty) - c_2$ and so it suffices to compute $\eta_B(\infty)$. Using the facts that $\eta_B(\infty) = \lim_{T\to\infty} B'(T) = \lim_{T\to\infty}[B(T)/T]$ (by L'Hospital's rule) and that $B(T) = \int_{[0,T)} A(T-x)\, dU(x)$, we readily conclude that

$$\eta_B(\infty) = \lim_{T\to\infty} \frac{B(T)}{T} = \frac{1}{\mu_1} A(\infty) = \frac{1}{\mu_1} (E[R(\zeta)] + c_1).$$

THEOREM 5.7: *Assume that η_B is nondecreasing. Then $0 < T_0 < \infty$ exists, which minimizes $J_B^{(\alpha)}(T)$ provided*

$$\left\{\frac{1}{\alpha\mu_1} (E[R(\zeta)] + c_1) - c_2\right\}\{1 - E[e^{-\alpha\zeta}]\}$$
$$> \left\{\int_0^\infty \int_0^y e^{-\alpha t}\phi(t,y)\, dt\, g(y)\, dy + c_1 E[e^{-\alpha\zeta}]\right\}. \tag{5.4}$$

6. APPLICATIONS

Example 6.1 (Classical Case): In this simple situation, $R(t) \equiv 0$ and G is the lifetime of a new unit. Hence,

$$\eta_A(T) = (c_1 - c_2) r_G(T),$$
$$\eta_B(T) = c_1 v_G(T),$$
$$\eta_A^{(\alpha)}(T) = \frac{1}{\alpha} (c_1 - c_2) r_G(T) - c_2, \qquad \text{and}$$
$$\eta_B^{(\alpha)}(T) = \frac{1}{\alpha} v_G(T) - c_2.$$

Thus, for an optimization result, we want G to be IFR, $c_1 > c_2$ and $v_G(T)$ to be nondecreasing with

$$(c_1 - c_2) r_G(\infty) > \frac{c_1}{\mu_1},$$
$$1 - \frac{\mu_2}{2\mu_1^2} > \frac{c_2}{c_1},$$
$$\frac{1}{\alpha} (c_1 - c_2) r_G(\infty) - c_2 > c_1 E[e^{-\alpha\zeta}]/(1 - E[e^{-\alpha\zeta}]), \qquad \text{and}$$
$$\frac{1}{\alpha\mu_1} c_1 - c_2 > c_1 E[e^{-\alpha\zeta}]/(1 - E[e^{-\alpha\zeta}]),$$

respectively, where $\mu_1 = E[\zeta]$ and $\mu_2 = E[\zeta^2]$.

Example 6.2 (Age-Dependent Minimal Repair): This model is described in detail in Block, Borges, and Savits [4]. Intuitively, the model behaves as follows.

Let a new unit have survival distribution $\bar{F}$ and failure rate r_F. If the item fails at age x, it is completely replaced by a new unit (unplanned replacement) with probability $p(x)$, or it is minimally repaired with probability $q(x) = 1 - p(x)$. The cost of the ith minimal repair at age x is $c_0^i(x)$.

It is shown in Block et al. [4] that

(i) ζ is the first arrival time T_1 in a nonhomogeneous Poisson process $\{N(t),\ t \geq 0\}$ with intensity $p(y)r_F(y)$;

(ii) $R(t) = \sum_{i=1}^{M(T-)} c_0^i(S_i)$, where $S_1, S_2, \ldots,$ are the arrival times of a nonhomogeneous Poisson process $\{M(t),\ t \geq 0\}$ with intensity $q(y)r_F(y)$; and

(iii) $\{M(t), t \geq 0\}$ and $\{N(t), t \geq 0\}$ are independent.

From the preceding, it follows that if $h(x) = E[c_0^{M(x)+1}(x)]$,

$$E[R(x) \mid \zeta = y] = E[R(x)] = \int_0^x h(u)q(u)r_F(u)\,du,$$

and so $\phi(u, y) = h(u)q(u)r_F(u)$. Consequently,

$$\eta_A(T) = h(T)q(T)r_F(T) + (c_1 - c_2)p(T)r_F(T).$$

In the special case when $p(y) = p$ and $c_0^i(y) = c_0$ are constants, the results simplify:

$$\eta_A(T) = [c_0 q + (c_1 - c_2)p]\,r_F(T);$$

$$\eta_B(T) = \frac{1}{p}\,[c_0 q + c_1 p]\,v_G(T).$$

Thus, if the preceding marginal cost functions are nondecreasing, the corresponding objective functions are minimized at a finite T_0 provided

$$[c_0 q + (c_1 - c_2)p]\,r_F(\infty) > \frac{1}{\mu_1}\left[\frac{c_0 q}{p} + c_1\right]$$

in the age replacement case and

$$\frac{c_0 q}{p} + c_1 - \frac{\mu_2}{2\mu_1^2} > c_2$$

in the block replacement case.

Example 6.3 (at most k minimal repairs): We consider the same model as that in Example 2 except that now at most k minimal repairs are allowed. Thus $\zeta = T_1 \wedge S_{k+1}$ and $R(t) = \sum_{i=1}^{M(t-)} c_0^i(S_i)$. This model was considered by Chen [5]. In particular, for $k = 1$, and $p(y) = p$, $c_0^i(y) = c_0$ constants, the following results were obtained:

(i) $\eta_A(T) = (c_1 - c_2)r_F(T)\left[1 - \dfrac{q\left(1 - \dfrac{c_0}{c_1 - c_2}\right)}{1 - q\ln\bar{F}(T)}\right]$ is nondecreasing if F is IFR and $c_1 - c_2 > c_0 > 0$;

(ii) ζ has failure rate $r_G(y) = r_F(y)\left[1 - \dfrac{q}{1 - q\ln\bar{F}(y)}\right]$;

(iii) assuming $\eta_A(T)$ is nondecreasing, $J_A(T)$ is minimized at a finite T_0 provided

$$(c_1 - c_2)r_F(\infty) > \frac{1}{\mu_1}[c_1 + qc_0],$$

where $\mu_1 = E[\zeta]$.

A particularly convenient way to derive η_A is to use Eq. (4.1), i.e., $\eta_A(T) = (c_1 - c_2)r_G(T) + \eta_{AR}(T)$, where

$$\eta_{AR}(x) = \frac{1}{\bar{G}(x)}\lim_{\Delta\to 0}\frac{1}{\Delta}E[R(x+\Delta) - R(x); \zeta \geq x + \Delta].$$

Since

$$E[R(x+\Delta) - R(x); \zeta \geq x+\Delta] = c_0 P(T_1 \geq x+\Delta)P(S_2 \geq x+\Delta > S_1 \geq x),$$

it easily follows that

$$\eta_{AR}(x) = \frac{1}{\bar{G}(x)}c_0 q f(x) = \frac{c_0 q}{1 - q\ln\bar{F}(x)}.$$

Remark 6.4: In contrast to the first two examples, we note that R and ζ are obviously dependent; in fact, it is not hard to show that

$$\phi(u,y) = c_0 q r_F(u)\frac{f(y)}{g(y)}, \qquad 0 \leq u \leq y.$$

This is also true for our next example.

Example 6.5 (Cumulative Cost Model): The simplest version of this model is as follows. We consider a Poisson process $\{N(t), t \geq 0\}$ with rate λ, and a collection $\{Y_1, Y_2, \ldots\}$ of independent exponential random variables with mean $1/\beta$, also independent of the Poisson process. Define

$$R(t) = \sum_{j=1}^{N(t-)} Y_j$$

and let $\zeta = \inf\{t \geq 0 : R(t+) \geq \Delta\}$. The process R is sometimes called a Poisson-exponential shock model.

This process was also studied by Chen [5]; he obtained the following results:

(i) ζ is IFR with density

$$g(y) = \lambda e^{-\lambda y} e^{-\beta\Delta} I_0(2\sqrt{\lambda\beta\Delta y});$$

(ii)

$$\phi(u,y) = \frac{1}{y}\,\Delta I_2(2\sqrt{\lambda\beta\Delta y})/I_0(2\sqrt{\lambda\beta\Delta y});$$

(iii) $\eta_A(T) = [1/\bar{G}(T)]\int_T^\infty \phi(T,y)g(y)\,dy + (c_1 - c_2)r_G(T)$ is strictly increasing if $c_1 - c_2 \geq \frac{2}{3}\Delta$ and strictly decreasing if $c_1 \leq c_2$; and

(iv) assuming $\eta_A(T)$ is nondecreasing, $J_A(T)$ admits a global minimum at a finite point T_0 if

$$(1 + \beta\Delta)(c_1 - c_2) > c_1 + (\beta\Delta - 1 + e^{-\beta\Delta})/\beta.$$

In the preceding, $I_\nu(x)$ is the modified Bessel function of the first kind of order ν, i.e.,

$$I_\nu(x) = \sum_{j=0}^{\infty} \frac{1}{j!\Gamma(j+\nu+1)} \left(\frac{x}{2}\right)^{2j+\nu}.$$

We also remark that the random variable $2\lambda\zeta$ has a noncentral chi-square distribution with 2 degrees of freedom and noncentrality parameter $2\beta\Delta$.

7. GENERAL VARIABLE REPLACEMENT COSTS

In this section we briefly mention how the previous results change if we replace constant costs c_1 and c_2 with their stochastic process counterparts $c_1(t)$ and $c_2(t)$, respectively. Now an unplanned replacement at age x costs $c_1(x)$ and a planned replacement at age x costs $c_2(x)$. This mechanism would thus allow us, for example, to incorporate the salvage value of used items into our cost analysis. We shall also use these results to point out an error in Berg et al. [2].

Since the analysis is similar to that in the preceding sections, we shall just write down the resulting expressions. Appropriate smoothness and boundedness conditions are implicitly assumed. For more details, see Chen [5].

$$A(T) = E[R(\zeta) + c_1(\zeta); \zeta < T] + E[R(T) + c_2(T); \zeta \geq T]; \tag{7.1}$$

$$B(T) = \int_{[0,T)} A(T-x)\,dU(x); \tag{7.2}$$

$$A^{(\alpha)}(T) = \int_{[0,T)} e^{-\alpha x}\,dA_R(x) + \int_{[0,T)} e^{-\alpha x}\,dA_1(x) + e^{-\alpha T}A_2(T); \tag{7.3}$$

$$
\begin{aligned}
B^{(\alpha)}(T) &= \int_{[0,T)} e^{-\alpha x}\, dB_R(x) + \int_{[0,T)} e^{-\alpha x}\, dB_1(x) + e^{-\alpha T} B_2(T) \\
&= \int_{[0,T)} e^{-\alpha x} A^{(\alpha)}(T-x)\, dU(x);
\end{aligned} \tag{7.4}
$$

$$
\begin{aligned}
\eta_A(T) &= \frac{1}{\bar{G}(T)} \int_T^{\infty} [\phi(T,y) + \psi(T,y)] g(y)\, dy \\
&\quad + E[\{c_1(T) - c_2(T)\} \mid \zeta = T] r_G(T);
\end{aligned} \tag{7.5}
$$

$$
\eta_B(T) = \int_{[0,T)} \eta_A(T-x)\bar{G}(T-x)\, dU(x) + E[c_2(0)]\, v_G(T); \tag{7.6}
$$

$$
\eta_A^{(\alpha)} = \frac{1}{\alpha}\, \eta_A(T) - \frac{1}{\bar{G}(T)} A_2(T); \qquad \text{and} \tag{7.7}
$$

$$
\eta_B^{(\alpha)}(T) = \frac{1}{\alpha}\, \eta_B(T) - B_2(T). \tag{7.8}
$$

In the preceding, we sometimes separated the costs into their component parts; e.g., $A(T) = A_R(T) + A_1(T) + A_2(T)$, where $A_R(T) = E[R(\zeta); \zeta < T] + E[R(\zeta); \zeta \geq T] = E[R(\zeta > T)]$, $A_1(T) = E[c_1(\zeta); \zeta < T]$, and $A_2(T) = E[c_2(T); \zeta \geq T]$. We also introduced $\psi(u,y)$, which is specified by $\psi(T,y) = (d/dT)E[c_2(T) \mid \zeta = y]$.

Lastly we turn our attention to the paper of Berg et al. [2]. They considered an age replacement model that is somewhat similar to that prescribed in Example 2 of Section 6, except that c_1 and c_2 are replaced with deterministic functions of t. Thus, according to the preceding expression for η_A in Eq. (7.5), the contributions from $c_1(t)$ and $c_2(t)$ should be

$$
c_2'(T) + [c_1(T) - c_2(T)] r_G(T),
$$

whereas they only obtain the latter part, namely, $[c_1(T) - c_2(T)] r_G(T)$. Thus, the expression for their objective function $J(T)$ (our $J_A(T)$) is also incorrect. Their results in Section 3 of that paper are presumably valid, however, because they only consider constant c_1 and c_2 there.

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