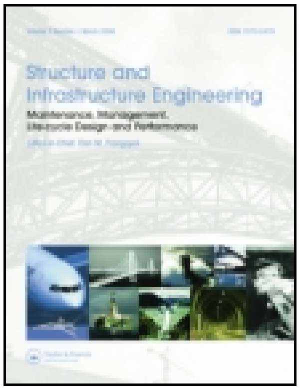

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State transition probabilities in bridge deterioration based on Weibull sojourn times

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State transition probabilities in bridge deterioration based on Weibull sojourn times

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This paper presents an investigation of the Markov property underlying the stochastic deterioration models for highway bridges, including transition probabilities between the condition states. Using historical data of sojourn times for the 'decay' (no improvement intervention) deterioration, hazard functions were developed and 'instantaneous' 1 year transition probabilities estimated for the sojourn times in the condition states, for various bridge categories, by type of material and roadway carried. The rate of transition out of each state was found to be not constant relative to time, as assumed for Markov chain models, but rather, increasing with the time spent in the state. Best-fit distributions of the sojourn times were determined to not be exponential (Markov chain), but Weibull, with the parameters established using the maximum likelihood estimate (MLE) method. A semi-Markov model of the bridge deterioration process was formulated and developed, including kernels of transition probabilities and time-based matrices of multi-step transition probability functions.

Keywords: bridge deterioration; sojourn times; probabilities; semi-Markov; Weibull distribution

1. Introduction

Many bridge management systems predict bridge deterioration using the Markov chain models, based on a discrete-time discrete-state assumption (Jiang *et al*. 1988, Cesare *et al*. 1992, Golabi *et al*. 1993, Scherer and Glagola 1994, Thompson and Johnson 2005, Morcous 2006), while there is an increasing suggestion to use the Markov process, a continuous-time version of the Markov chain model (Mishalani and Madanat 2002, Kallen and Noortwijk 2005). One major issue is that of the Markov property assumption, i.e. assumption of an exponential distribution as the fitted model for the times between transitions among the various deteriorated states; these times are also referred to as holding or sojourn times. The Markov property suggests that the probability of transition between states depends only on the current state occupied by the process, and not the history of the process, i.e. the process is memoryless. Assuming an exponential distribution would imply a constant 'failure' or transition rate with respect to the time spent in the state. There is also the case of the time homogeneity in the Markov chain models, but this can be addressed by using different mean rates for the exponential distributions (in a Markov process) at the various states, or having different transition probability matrices in the Markov chain for various ages of the bridge.

The deterioration process can be considered as a bridge subjected to degradation factors in a variation of aggressive environments. This process is typically documented to include prevention and improvement efforts (rehabilitation, repair, etc.) on the bridge, but the model in this paper is focused on the 'decay' deterioration, i.e. assuming no improvement intervention in the deterioration process. While many mathematical models of bridge deterioration as a stochastic process have used the memoryless Markov property, the nature of bridge deterioration process suggests otherwise. In other words, this paper intends to show empirically that there is ageing or wear-out deterioration in bridges. Despite the popularity of the discrete-time and continuous-time Markov chain models, some concerns have been identified and consequently, many efforts have been made to address these concerns, including the use of non-exponential distributions to model the sojourn times (Ng 1996, Ng and Moses 1998, Mishalani and Madanat 2002, Kallen and Noortwijk 2005). It should be noted that these concerns do not necessarily make the deterioration models in existing bridge management systems irrelevant or useless, but rather suggestions are presented in this paper on making the models better. A possible solution is the use of semi-Markov models.

Though suggested in a preliminary framework by Sobanjo (1993), the first comprehensive efforts to

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recognise these problems and formulate a semi-Markov model, specifically for bridge deterioration, are documented in Ng (1996) and Ng and Moses (1998). Kallen and Noortwijk (2005) also discussed a semi-Markov model for bridges, in its formulation of a general continuous-time Markov process, while Black *et al*. (2005) developed semi-Markov models for asset deterioration, illustrating them with electrical transformers and switchgears. Ng (1996) and Ng and Moses (1998) estimated, using bridge condition data, each state's sojourn time distribution parameters from the mathematical difference between the two age distributions for the respective states. Kallen and Noortwijk (2005) used inspection intervals to determine the sojourn time distributions, while Black *et al*. (2005) determined the sojourn time parameters from a least square fit between the predicted data and the original data. Efforts in these existing semi-Markov models are commendable, and have been actually used to some extent later in this paper to formulate the proposed models. There is, however, a common shortcoming – they lack the use and investigation of the estimated sojourn times, directly from historical condition data, for the transitions between the specific states.

The objective of this paper is to mathematically investigate the bridge deterioration process, with the focus on sojourn times. The historical condition data from the Florida bridge inventory is used to generate time data between changes in states by bridge components, and also to estimate transition probabilities. First, some fundamental theories, regarding the mathematical modelling of bridge deterioration as a stochastic process, including the reliability theories as related to the sojourn times, the semi-Markov process and the probability of transitions between states, are presented. Finally, the proposed semi-Markov model is developed and illustrated with some categories of bridge decks and superstructures.

2. Framework of the stochastic process

If the sojourn time interval in a stochastic process is not necessarily exponential and the distribution has parameters that depend on both the current state and the next state, the process can be classified as a general case of a semi-Markov process (Cinlar 1975). If the sojourn times are exponentially distributed and are independent of the next state, then the semi-Markov process is a Markov process (Heyman and Sobel 1982). If there is only one state, with the sojourn time also exponentially distributed, then it is a Markov renewal process; with a sojourn time of 1 time unit, the semi-Markov process becomes a Markov chain (Heyman and Sobel 1982).

The stochastic process of bridge deterioration can therefore be generalised using the semi-Markov model, in which some flexibility is introduced, allowing the sojourn times to be arbitrarily distributed random variables, and the transition probabilities may depend not only on the current state occupied, but also on the next state. While the semi-Markov model may describe the stochastic process overall, transition between states are primarily governed by the distribution parameters of the sojourn time. Thus, the transition probabilities are discussed next, using some basic theories of reliability.

2.1. *Transition probability rates*

With the non-negative random variable T representing the time spent by the bridge component in a specific state i of deterioration (i.e. the sojourn time), the cumulative distribution function can be given as:

$$F_i(t) = \text{prob}(T \leq t). \tag{1}$$

$F_i(t)$ is related to the probability distribution function, $f_i(t)$, and the reliability or survivor function, $S_i(t)$, as shown below in Equations (2) and (3). $F_i(t)$ describes the probability that the bridge element will transition out of state i by age or time t. The survivor function, $S_i(t)$, is the complement of $F_i(t)$, meaning the probability that the bridge element would still be in state i by time t:

$$F_i(t) = \int_0^t f_i(x)\mathrm{d}x \tag{2}$$

and

$$S_i(t) = prob(T > t) = 1 - F_i(t) \tag{3}$$

The hazard function, $h_i(t)$ can be defined as the failure rate or the rate of change of the conditional probability of failure and is defined as:

$$h_i(t) = \frac{f_i(t)}{1 - F_i(t)} = \frac{f_i(t)}{S_i(t)}. \tag{4}$$

The failure rate in this context of sojourn times means the transition out of the current state i. As presented in some literature, including Mishalani and Madanat (2002), at any time t, the transition probability $p_{ik}(t, \Delta)$ out of a state i into any lower (worse condition) state, say k, within a period Δ after time t, is given as:

$$\begin{aligned} p_{ik}(t,\Delta) &= \text{prob}(t < T < t + \Delta | T > t) \\ &= \frac{F_i(t+\Delta) - F_i(t)}{S_i(t)}, \end{aligned} \tag{5}$$

where $F_i(t)$ is the cumulative distribution function of the duration variable T in state i and $S_i(t)$ is the survivor function in state i. Therefore,

$$p_{ik}(t,\Delta) = \frac{S_i(t) - S_i(t+\Delta)}{S_i(t)} = 1 - \frac{S_i(t+\Delta)}{S_i(t)}. \tag{6}$$

The probability of remaining in the same state i is given as:

$$p_{ii}(t,\Delta) = 1 - p_{ik}(t,\Delta) = \frac{S_i(t+\Delta)}{S_i(t)}. \tag{7}$$

In the case of more than two probable lower state destinations during the transitions, i.e. when the bridge condition can deteriorate to the next lower (worse) state or to even lesser states at one transition, then the transition probability can be estimated based on some approximations, as described in Mishalani and Madanat (2002). First, the one-step transition probability is given as:

$$p_{ij}(t,\Delta) = p_{ik}(t,\Delta)S_j(\Delta/2), \quad j = i+1, \tag{8}$$

and the two-step transition probability is estimated as:

$$p_{ij}(t,\Delta) = p_{ik}(t,\Delta)F_j(\Delta/2), \quad j = i+2. \tag{9}$$

It was assumed that the transition between states i and j actually occurs at the midpoint of the period Δ and the transition probability is the product of the transition probability from states i and the probability of 'surviving' in state j for half the period Δ (Mishalani and Madanat 2002).

2.2. *The semi-Markov process*

As mentioned earlier, the overall deterioration process, involving the various condition states, can be defined using a semi-Markov model. The semi-Markov process with m feasible states may be defined as a process $Y = (Y_t;\ t \geq 0)$ associated with parameters $(X_n,\ TS_n)$ for the state and time in the system at the nth transition. The process Y involves $TS_1,\ TS_2, \ldots$ as successive times of transitions and states $X_0,\ X_1,\ X_2, \ldots$ as successive states visited. The length of a sojourn interval $(TS_n,\ TS_{n+1})$ is a random variable, the same as the random variable T described in Equation (1), whose distribution depends on both the state X_n being visited and the state X_{n+1} to be visited next (Cinlar 1975).

For any $i,j = 1,2,\ldots,\ m$ and $t \geq 0$, a family or matrix of transition probabilities, formally called the semi-Markov kernel, can be defined over the state space $(1,2,\ldots,\ m)$, as:

$$Q_{ij}(t) = P(X_{n+1} = j, TS_{n+1} - TS_n \leq t | X_n = i). \tag{10}$$

The semi-Markov kernel, $Q_{ij}(t)$ is also a one-step transition probability that, after making a transition into state i, the process next makes a transition into state j in an amount of time less than or equal to t. The following conditions must be satisfied:

$$Q_{ij}(t) \geq 0, \quad i,j = 1,2,\ldots,m, \tag{11}$$

and

$$\sum_{j=0}^{m} Q_{ij}(\infty) = 1, \quad i = 1,2,\ldots,m. \tag{12}$$

Now, let

$$p_{ij}^{\mathrm{e}} = \lim_{t\to\infty} Q_{ij}(t) = P\{X_{n+1} = j | X_n = i\}. \tag{13}$$

The term p_{ij}^{e} is the *eventual* transition probability that the process can move from state i to state j, neglecting the sojourn times in state i, and the following conditions must be satisfied:

$$p_{ij}^{\mathrm{e}} \geq 0 \tag{14}$$

and

$$\sum_{i,j=0}^{m} p_{ij}^{\mathrm{e}} = 1. \tag{15}$$

Then, according to Cinlar (1975) and Osaki (1985) we can define:

$$F_{ij}(t) = \frac{Q_{ij}(t)}{p_{ij}^{\mathrm{e}}}, \tag{16}$$

where $F_{ij}(t)$ is the cumulative distribution function of the sojourn time that the process spends in state i, given that the next state is j. In other words, $F_{ij}(t)$ is a *conditional* distribution function, given both the current and next states, i.e.

$$F_{ij}(t) = P\{TS_{n+1} - TS_n \leq t | X_{n+1} = j, X_n = i\}. \tag{17}$$

A matrix having these eventual transition probabilities, p_{ij}^{e}, is called the *embedded* Markov chain. It should be noted however, that for 'decay' deterioration of a bridge with no improvement throughout its service life, the bridge would have a finite service life. All states in such a process would be transient states, except for the last (worst condition) state m, which would be an absorbing state. In other words, the process would *eventually* always transition out of each state, except the last state m, implying the eventual transition probability of staying or remaining in the same state i is given as:

$$p_{ii}^{\mathrm{e}} = \lim_{t\to\infty} Q_{ii}(t) = 0, \qquad i \neq m. \tag{18}$$

Therefore the semi-Markov process will be defined as:

$$Q_{ij}(t) = p_{ij}^{\mathrm{e}} F_{ij}(t), \quad \text{if } p_{ij}^{\mathrm{e}} > 0, \quad i,j = 1,2,\ldots,m; i \neq j. \tag{19}$$

It should be noted that, for example, the Markov process, a special case of a semi-Markov process, with exponential distributions (mean $= \lambda$) for the sojourn times, would have a function of the form:

$$Q_{ij}(t) = p_{ij}^{\mathrm{e}}[1 - \exp(-\lambda t)]. \tag{20}$$

To compute transition probabilities in a semi-Markov process, various formulations are available (Howard 1971, Cinlar 1975) and have been demonstrated in bridge deterioration applications by Ng and Moses (1998) and Kallen and Noortwijk (2005) and also for assessing the durability of building materials (Binda and Molina 1990). Let us consider a system in a semi-Markov process, having spent an initial time t_0 in the state i before the time $t = 0$, with a destination state j and the system could go through an intermediate state k. The term t_0 can be best interpreted as the age of bridge when the observation starts ($t = 0$), given that the bridge was in state i. Let x be a time measured for the stay in state i from time $t = 0$, to the time of transition to state k. The transition probability, $P_{ij}(t,t_0)$, from state i to state j at any time t, is given as (Howard 1971, Binda and Molina 1990):

$$P_{ij}(t,t_0) = \delta_{ij} S_i(t,t_0) + \sum_k p_{ik}^{\mathrm{d}}(t_0) \int_0^t f_{ik}(x,t_0) P_{kj}(t-x)\mathrm{d}x, \tag{21}$$

where δ_{ij} is the Kronecker delta (i.e. $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise); $S_i(t,t_0)$ is the Survivor function for the sojourn time in state i, given the initial elapsed time t_0, and $S_i(t, t_0) = 1 - F_k(t, t_0)$, where $F_k(t, t_0)$ is the cumulative distribution function; $p_{ik}^{\mathrm{d}}(t_0)$is the probability density function describing the sojourn time before the first transition; $f_{ik}(x, t_0)$ is the probability density function describing the sojourn time before the first transition at a time x, given the initial elapsed time t_0; and $P_{kj}(t - x)$ is the transition probability for the system to move from state k to state j in the remaining time $(t - x)$.

In addition,

$$P_{kj}(t) = \delta_{kj} S_k(t) + \sum_r p_{kr}^{\mathrm{d}}(t) \int_0^t f_{kr}(x) P_{rj}(t-x)\mathrm{d}x, \tag{22}$$

where δ_{kj} is the Kronecker delta (i.e. $\delta_{kj} = 1$ if $k = j$ and $\delta_{kj} = 0$ otherwise); $S_k(t)$ is the Survivor function for the sojourn time in state k, and $S_k(t) = 1 - F_k(t)$, where $F_k(t)$ is the cumulative distribution function; $p_{kr}^{\mathrm{d}}(t_0)$is the probability density function describing the sojourn time before the transition from state k to another state r; $f_{kr}(x)$is the probability density function describing the sojourn time before the transition at a time x, from state k to state r; $P_{rj}(t - x)$ is the transition probability for the system to move from state r to state j in the remaining time $(t - x)$.

Equations (21) and (22) above are recursive equations, which, even for a few number states in the process, could be complicated but solvable using Laplace transforms. However, for Weibull (distribution) sojourn times, there are no closed form solutions, but Perman *et al.* (1997), Black *et al.* (2005), and Kallen and Noortwijk (2005) have suggested approximation solutions by use of discrete-time evaluation of the Weibull functions. Such an approximation will be used later in this paper.

To simplify things, the following assumptions are first made regarding the proposed model of the bridge deterioration process:

(1) the elapsed time t_0 in the initial state i is zero, i.e. ignore the age of the bridge in the initial state;
(2) there is no improvement carried out to the bridges throughout their service lives that would increase the condition ratings; and
(3) the deterioration cannot be more than between three states ($i \leq j \leq i + 2$,) i.e. the condition

ratings cannot drop by more than two units during a period time t.

With approximation to discrete state summation for the original integrals in Equations (21) and (22), the transition probability equations for the semi-Markov process can now be rewritten as:

$$P_{ij}(t) = S_i(t), \quad i = j, \tag{23}$$

and

$$P_{ij}(t) = \sum_k \sum_{x=1}^{t} f_{ik}(x) P_{kj}(t-x), \quad i \neq j, \tag{24}$$

where k is an intermediate state between states i and j.

It was shown earlier in Equation (5) that the probability can be estimated for a transition within time period Δ after time t. Applied to the current situation in Equation (24), it is known that, upon entering state k, a transition would occur to state j, within a time period Δ (equal to $t - x$), spent in state k. This transition probability is obtained using the cumulative distributions function, $F_{kj}(t)$, of the sojourn time in state k, before transitioning to state j. In this case, $t = 0$ and period $\Delta = t - x$. Substituting for $P_{kj}(t - x)$ in Equation (24):

$$P_{ij}(t) = \sum_k \sum_{x=1}^{t} f_{ik}(x) \left[\frac{F_{kj}(t-x) - F_{kj}(0)}{1 - F_{kj}(0)} \right], \quad j = k+1. \tag{25}$$

The term $P_{ij}(t)$ can actually be interpreted as the probability that if the bridge element starts in state i, it will be in state j after a period time of t. If a bridge is observed starting from age zero ($t = 0$), given the assumptions listed above, the following equations illustrate the probabilities of finding the bridge in a specified state after a time t. First, we are interested in starting from state 1 and ending up in state 3, implying the intermediate state $k = 2$, i.e.

$$P_{11}(t) = S_1(t), \tag{26}$$

$$P_{13}(t) = \sum_{x=1}^{t} f_{12}(x) P_{23}(t-x) \tag{27}$$

and

$$P_{13}(t) = \sum_{x=1}^{t} f_{12}(x) \left[\frac{F_{23}(t-x) - F_{32}(0)}{1 - F_{23}(0)} \right]. \tag{28}$$

For example, at $t = 4$ and $x = 1$, 2 and 3 (the possible times to be in the intermediate state 2), then:

$$\begin{aligned} P_{13}(t) = {} & f_{12}(1) \left[\frac{F_{23}(3) - F_{23}(0)}{1 - F_{23}(0)} \right] \\ & + f_{12}(2) \left[\frac{F_{23}(2) - F_{23}(0)}{1 - F_{23}(0)} \right] \\ & + f_{12}(3) \left[\frac{F_{23}(1) - F_{23}(0)}{1 - F_{23}(0)} \right] \end{aligned} \tag{29}$$

Or, simply, since $F_{23}(0) = 0$:

$$P_{13}(t) = f_{12}(1) F_{23}(3) + f_{12}(2) F_{23}(2) + f_{12}(3) F_{23}(1). \tag{30}$$

In addition, it is assumed that:

$$P_{12}(t) = 1 - (P_{11}(t) + P_{13}(t)). \tag{31}$$

Only the two-step transition probabilities, i.e. $i \leq j \leq i + 2$, have been formulated so far. To estimate a one-step further drop in condition, i.e. $i \leq j \leq i + 3$, the feasible realisation paths of the process should be identified, and the calculations done as illustrated above. For example, let us consider $P_{14}(t)$, the transition probability starting from state 1 and ending in state 4. There are three feasible paths: (a) state 1 to state 2 and then to state 4 (skipping state 3); (b) state 1 to state 3 and then to state 4 (skipping state 2); and (c) state 1 to state 2, then to state 3 and finally to state 4 (visiting each state). Based on Equations (24) and (25), the mutually exclusive probabilities can be combined and expressed as:

$$\begin{aligned} P_{14}(t) = {} & \sum_{x=1}^{t} f_{12}(x) F_{24}(t-x) + \sum_{x=1}^{t} f_{13}(x) F_{34}(t-x) \\ & + \sum_{x=r+1}^{t} \sum_{r=1}^{x} f_{12}(r) f_{23}(x-r) F_{34}(t-x). \end{aligned} \tag{32}$$

The first two terms on the right-hand side of Equation (32) represent the first two feasible paths mentioned above. These first two terms can be ignored because the probabilities are very small and are negligible for transitions involving multi-state drops in one-step, i.e. skipping the immediate state. Thus, the third term can be used to estimate transition probabilities involving visits to four states. By adjusting Equation (32) appropriately, estimates can also be obtained for the probability $P_{25}(t)$ and similar probabilities, provided the pertinent probability and cumulative distribution functions exist.

3. Data collection and analyses

The bridge condition data, in the National Bridge Inventory (NBI) format, was obtained from the Federal Highways Administration and the Florida Department of Transportation for annual bridge condition reports between the years 1992 and 2005. Table 1 summarises the NBI codes and the respective definition of the condition ratings. Also shown is the transformation of each condition rating to a condition state designation compatible with the deterioration model being discussed in this paper.

Starting with 1992 and moving through the next and consequent years, the duration was estimated for the time spent by the bridge in each condition rating before deteriorating to a worse state. The age of the bridge at transition was assumed to be the average of two estimates: the age (in years) at departure (from the specific rating) and age (in years) at arrival (to the worse condition state). MiniTab (2007) statistical software was then used to perform some reliability analyses on the duration data. Parametric analyses included finding the best-fit probability distributions and generating the survivor (reliability) and hazard functions for the various bridge duration data at the various condition ratings. The maximum likelihood estimate (MLE) method was used to estimate pertinent parameters of the probability distributions.

In modelling the failure times of items during typical reliability tests, some items are observed until complete failure, with their failure times exactly known. However, in certain circumstances, some items may have not failed at the end of the observation. In the latter case, the times observed are termed type I right censored observations. In rare situations, when the item may have failed before the start of observation, this is termed type I left censoring. Type II right censoring occurs when a predetermined number of items reach failure before the end of observation. For the modelling of sojourn times in bridge deterioration, only type I right censoring is practically relevant. There are also cases of uncensored data, i.e. with complete observations. In other words, for uncensored data, the time spent at each bridge deteriorated state was estimated completely, knowing the beginning age of the bridge and the age at transition to a worse state. For right-censored data, these are bridges observed not to have transitioned out of its state at the end of the observation time.

Table 1. NBI condition ratings and condition states.

NBI condition rating*	NBI condition*	Condition state designation
9	Excellent condition	State 1
8	Very good condition	State 2
7	Good condition	State 3
6	Satisfactory condition	State 4
5	Fair condition	State 5
4	Poor condition	State 6
3	Serious condition	State 7
2	Critical condition	State 8

*Based on the NBI standards (FHWA 2005).

In statistical analyses, a probability plot would show how a specific probability distribution (Weibull, etc.) fits the data. If the distribution fits the data, plotted points will appear roughly linear and will fall close to the fitted distribution line. The goodness-of-fit test involves visual observation of the probability plot and conduction of a statistic hypothesis test, such as the Anderson–Darling (A–D) test. The AD statistic measures how well a data fits a specified distribution. The statistic is based on the non-parametric step function using the Kaplan–Meier method (MiniTab 2007). The AD test is also based on the difference between the hypothesised and empirical cumulative distributions, similar to the Kolmogorov–Smirnov test (Blishke and Murthy 2000). The better the fit, the smaller the AD statistic will be. The distribution with the lowest AD statistic has the closest fit to the data. If the distributions have similar AD statistics, then some engineering judgement or experience may be used in choosing the best fit. When testing at a chosen significance level, typically 0.05 or 0.10, the p-values (if computed) are used to determine whether to accept or reject the null hypothesis that the data follows a specified distribution. If the p-value is greater than the significance level, then the null hypothesis is accepted. The p-value is not necessarily computed in the MiniTab software, especially when an adjusted A–D statistic is computed using the MLE method.

In fitting the probability distributions, three feasible types of distributions were primarily considered: Weibull, lognormal and exponential. The basis of selecting the best-fit distribution was using the AD test statistic, as described above. It was observed that the Weibull distribution was the best suited to the times data, except for a very few cases where the lognormal distribution was slightly better. But even then, the AD statistics for both types of distributions were very close. The Weibull probability distribution is mathematically defined as:

$$f(t) = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} \exp - \left[\frac{t}{\alpha}\right]^{\beta}, \tag{33}$$

where α is the scale parameter; β is the shape parameter; and t is the time spent in the state before transition out.

The estimates of time spent in various condition ratings (states) and the fitted Weibull distribution parameters are shown in Table 2, with the probability distribution function for concrete bridge decks and cumulative distribution function for interstate prestressed concrete superstructures shown in Figures 1 and 2, respectively. The results in the table are provided for cases where the data is available for analysis, because for some cases, no bridge condition data were available to track the time spent in the desired states, or to estimate the time to reach some specified condition ratings. For instance, some categories of bridges did not have new bridges, i.e. bridges in condition rating 9 while some bridge categories did not have bridges in bad deteriorated states like condition rating 6 or lower.

Using only the results from uncensored (complete) data, as these gave the best statistical fits, two major types of analysis were carried out in this study: an investigation of the Markov property based on the transition between the various states and the formulation of a semi-Markov model for the bridge deterioration process.

3.1. Review of the Markov property

As shown in Table 2, the Weibull, rather than the exponential, distribution was the best fitting distribution for the sojourn times of each of the various states. The shape parameter was never estimated as 1.0, which would have suggested the exponential distribution. First, using Equations (5) to (9), the 'instantaneous' 1 year transition probabilities were estimated for the various condition states. This means the probability at any time *t*, for the bridge component to transition out of a specific state *i* within a 1 year period, including both a one-step transition to the next worse state and a multi-step transition into even worse states. Also estimated were the probabilities of remaining in the specific state *i*. If the Markov property were valid, the functions for transition out of the states, shown in Figures 3 to 8 for various categories of bridges, should have been straight lines, parallel to the time axis (implying a constant rate of deterioration). However, they are not. All the curves for transition out of state indicated increasing probabilities relative to time spent in the state. This observation supports the suggestion that Markov property assumptions may not hold for bridge deterioration.

Secondly, the hazard functions estimated using Equation (4), are plotted for the various sojourn times, and shown to be nonlinear for concrete decks and prestressed concrete superstructures on the interstate roadways (Figures 9 and 10). These show that the transition rates, or 'failure within each state', are not constant, but increase with respect to time. Interestingly,

Table 2. Weibull-fitted probability distribution parameters for sojourn times on various bridge categories.

Condition rating (state)	Probability distribution parameters				Data size		Goodness-of-fit	
	Mean (yr)	Standard deviation (yr)	Shape (β)	Scale (α)	Complete	Censored	AD coefficient	*p*-value
(a) Cast-in-place (CIP) concrete decks on all roadways								
9 (State 1)	1.26	0.66	1.976	1.417	74	0	14.089	0.009
8 (State 2)	4.81	3.28	1.493	5.323	116	0	1.077	0.009
8 (State 2)	33.67	38.03	0.887	31.748	116	212	1254.087*	N/A
7 (State 3)	6.58	2.34	3.078	7.359	80	0	0.643	0.092
7 (State 3)	83.53	58.21	1.458	92.200	80	1638	1904.586*	N/A
6 (State 4)	5.48	2.03	2.932	6.140	27	0	0.610	0.103
6 (State 4)	21.05	12.46	1.742	23.624	27	211	426.367*	N/A
(b) Prestressed concrete superstructures on interstate roadways								
9 (State 1)	1.47	0.78	1.96	1.661	26	0	2.601	0.009
8 (State 2)	6.74	2.98	2.414	7.603	21	0	0.660	0.079
8 (State 2)	107.56	83.02	1.307	116.58	21	358	497.402*	N/A
7 (State 3)	6.28	2.68	2.512	7.082	16	0	0.429	0.251
7 (State 3)	53.37	33.28	1.646	59.672	16	350	369.537*	N/A
(c) Steel superstructures on interstate roadways								
8 (State 2)	5.85	4.254	1.393	6.413	13	0	1.383	0.009
8 (State 2)	32.94	34.636	0.951	32.209	13	25	151.485*	N/A
7 (State 3)	7.421	1.903	4.418	8.141	6	0	0.371	0.251
7 (State 3)	21.027	9.882	2.251	23.74	6	46	105.024*	N/A
(d) Steel superstructures on non-interstate roadways								
8 (State 2)	9.59	3.75	2.767	10.778	14	0	2.051	0.009
8 (State 2)	17.81	8.46	2.223	20.104	14	27	141.936*	N/A
7 (State 3)	7.02	2.11	3.716	7.778	13	0	0.454	0.246
7 (State 3)	33.89	19.72	1.776	38.077	12	112	262.532*	N/A
6 (State 4)	26.42	12.54	2.228	29.833	3	48	60.183*	N/A

*Adjusted value of AD coefficient based on the maximum likelihood estimates; *p*-value not available.

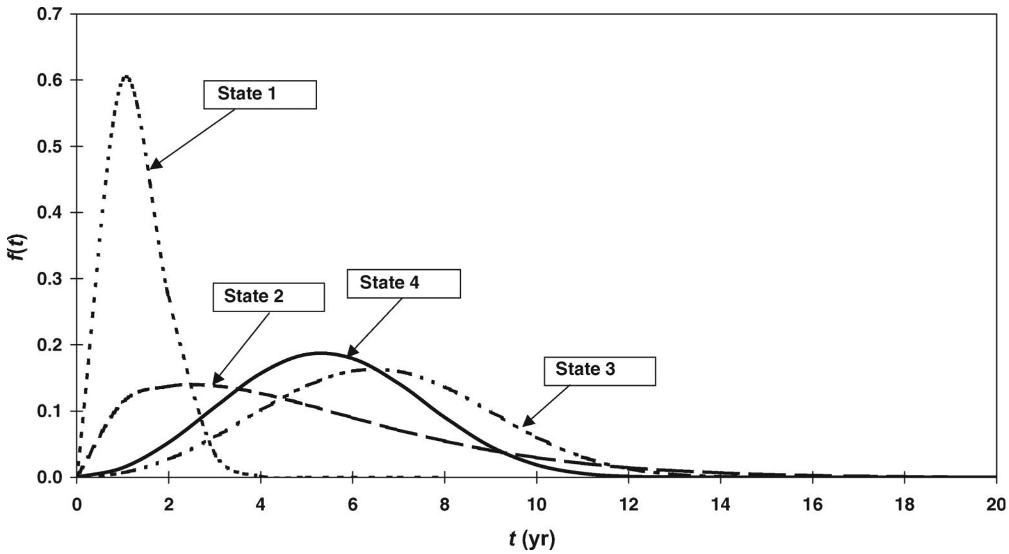

Figure 1. Probability distribution functions for duration in condition ratings for bridge Cast-in-place (CIP) concrete decks.

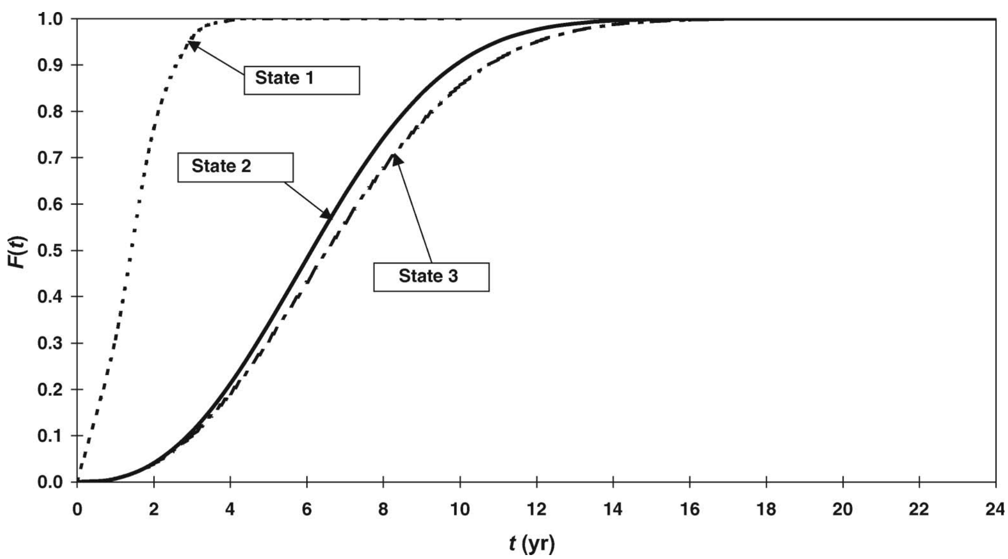

Figure 2. Cumulative distribution functions for duration in condition ratings for bridge prestressed concrete superstructures on interstates.

in Figure 10, the hazard rates show faster age-related deterioration rates for the decks statewide, than the prestressed concrete superstructures on the interstates and non-interstates.

3.2. *Development of the semi-Markov model*

The semi-Markov kernel defined in Equation (10) earlier was developed as shown in Equation (34) for Cast-in-place (CIP) concrete bridge decks. Due to the limited availability of historical condition data for ratings lower than 5, the NBI ratings 9, 8, 7, 6 and 5, which are states 1, 2, 3, 4 and 5, respectively, are used as the states of bridge deterioration. For bridge 'decay' deterioration, the eventual probabilities would be such that $p^{e}_{ii} = 0$, i.e. the bridge element cannot stay in the same state forever, but the eventual probabilities may exist for transition to the next lower state or worse. This implies that the worst state, i.e. rating 5 (state 5) according to Equation (34), is an absorbing state, and $p^{e}_{55} = 1$.

Unlike many assumptions that there are only one-step eventual transitions in the bridge deterioration, i.e. only one-rating drop (Ng and Moses 1998), this study observed a few two-ratings drops in the data, but the relative frequencies were not estimated. For convenience here, it is assumed that only 5% of the transitions to worse states were two-ratings drops and that others are one-rating drops:

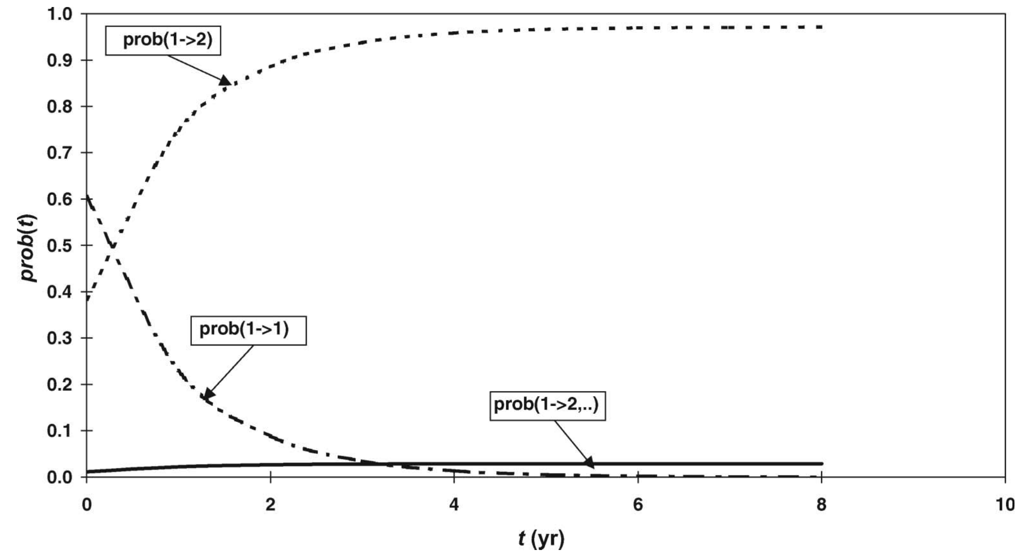

Figure 3. Instantaneous '1 year' transition probability functions from condition rating 9 (state 1) for CIP concrete bridge decks.

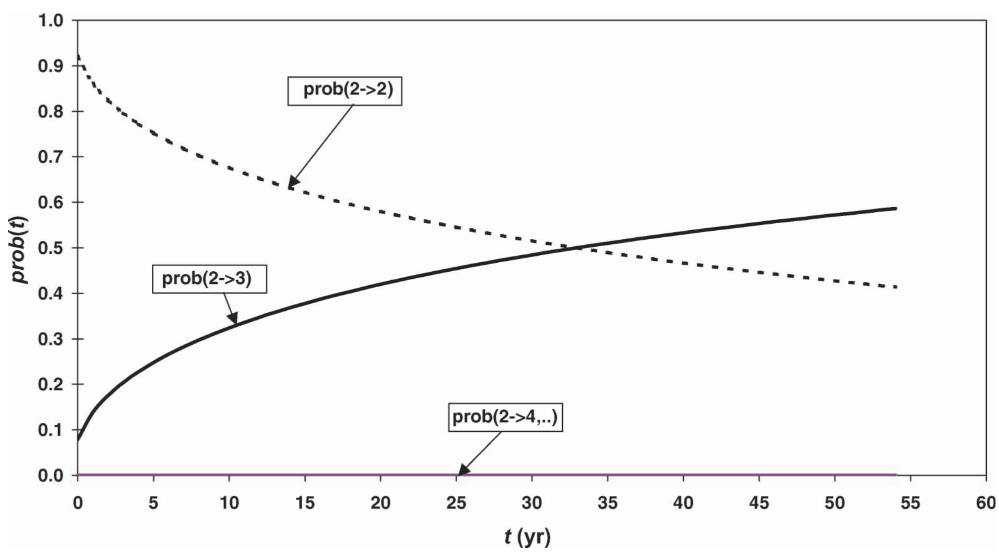

Figure 4. Instantaneous '1 year' transition probability functions from condition rating 8 (state 2) for CIP concrete bridge decks.

$$Q_{ij}(t)=\begin{pmatrix} 0.0 & 0.95\left(1-\exp\left[\frac{-t}{1.416}\right]^{1.976}\right) & 0.05\left(1-\exp\left[\frac{-t}{1.416}\right]^{1.976}\right) & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.95\left(1-\exp\left[\frac{-t}{5.323}\right]^{1.493}\right) & 0.05\left(1-\exp\left[\frac{-t}{5.323}\right]^{1.493}\right) & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.95\left(1-\exp\left[\frac{-t}{7.359}\right]^{3.078}\right) & 0.05\left(1-\exp\left[\frac{-t}{7.359}\right]^{3.078}\right) \\ 0.0 & 0.0 & 0.0 & 0.0 & \left(1-\exp\left[\frac{-t}{6.140}\right]^{2.932}\right) \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \end{pmatrix} \tag{34}$$

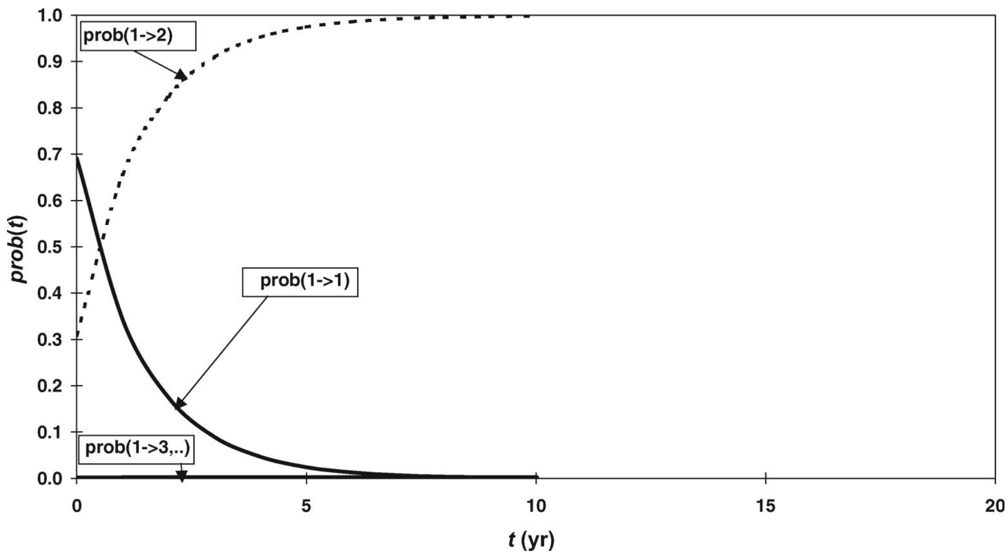

Figure 5. Instantaneous '1 year' transition probability functions from condition rating 9 (state 1) for prestressed concrete bridge superstructures on interstates.

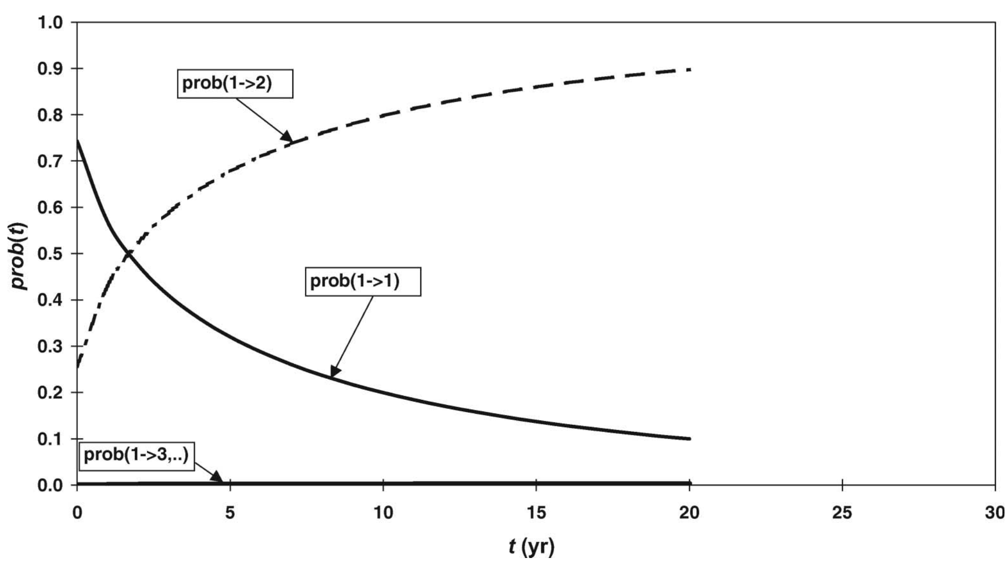

Figure 6. Instantaneous '1 year' transition probability functions from condition rating 9 (state 1) for prestressed concrete bridge superstructures on non-interstates.

For example, consider CIP concrete decks in condition rating 9 (state 1); the bridges upon entering state 1 will have an eventual probability of 0.95 of transitioning to condition rating 8 (state 2) and eventual probability 0.05 of transitioning to rating 7 (state 3). Other transitions are treated similarly, with the last absorbing state 5 (rating 5) given an eventual probability of 1. This kernel of probabilities shown in Equation (34) can be used for more semi-Markov analyses (Cinlar 1975), but an overall semi-Markov model identified earlier in Equations (21) and (22), and more suitable for the bridge deterioration process, would now be formulated and developed.

Using the Florida bridge condition data, the transition probabilities for prestressed concrete superstructures on interstates are shown in Figures 11 and 12. First, in Figure 11, the probability of remaining in state 1, $P_{11}(t)$, is shown as rapidly decreasing from its initial value of 1 in its newly built year, to almost 0 by the fourth year, while the probability of moving to state 2, $P_{12}(t)$, increases sharply, within the same 4 year period. Observing the transition probabilities $P_{12}(t)$ and $P_{13}(t)$ shows that, if a bridge starts at time

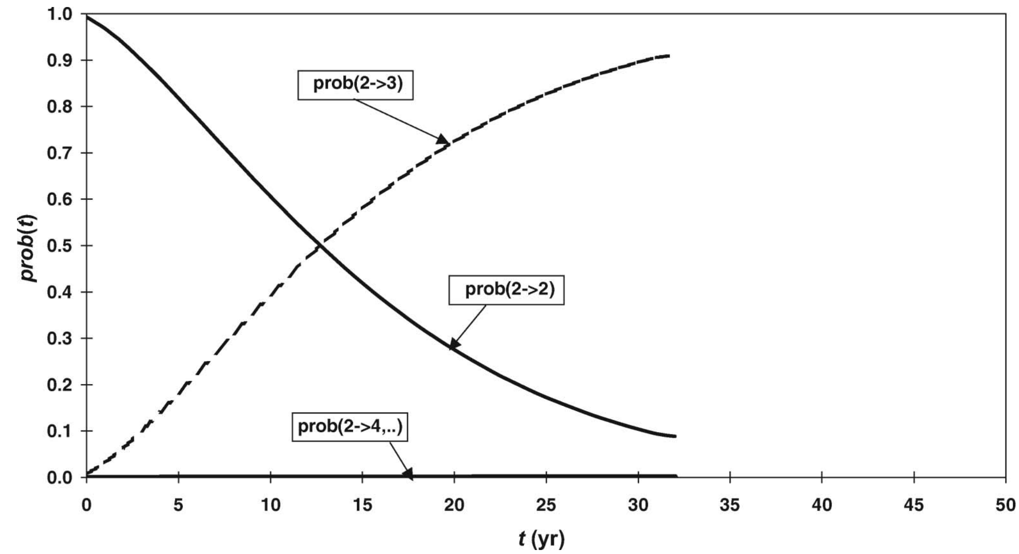

Figure 7. Instantaneous '1 year' transition probability functions from condition rating 8 (state 2) for prestressed concrete bridge superstructures on non-interstates.

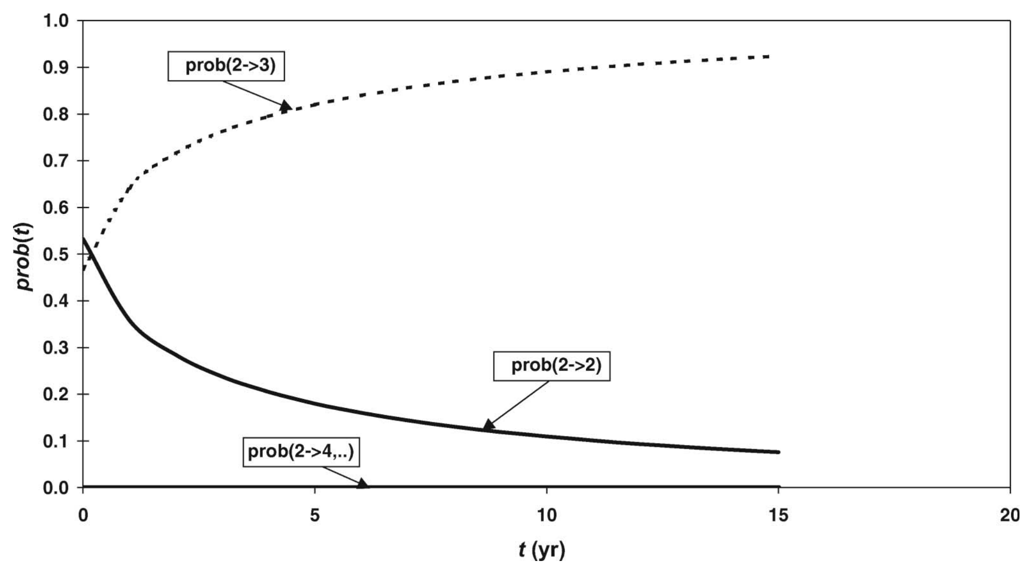

Figure 8. Instantaneous '1 year' transition probability functions from condition rating 8 (state 2) for steel bridge superstructures on interstates.

$t = 0$ in state 1, by the fourth year there is just over a 93% chance of the bridge being in state 2, about a 6% chance of being in state 3 already and almost 0.5% chance of still being in state 1.

After the fourth year, with the bridge element starting in state 1, the chances of being in state 2 gradually decreases with respect to time, while that of state 3 increases, and now there are chances of the bridge being in state 4. Looking at 10 years after being in service, the bridge element has a 67% probability of being in state 3, compared to 25% of being in state 2 and 12% probability of being in state 4, with 0 chance of still being in state 1. At the age of 12 years, three is a 3:1 chance of the bridge being in state 3 compared to state 4, and 0 probability of it still being in state 1 or state 2. After this point in time, the probability decreases for state 3, while it increases for state 4. At the age of 20 years, the bridge element has an 84% probability of being in state 4 and 16% probability of being in state 3.

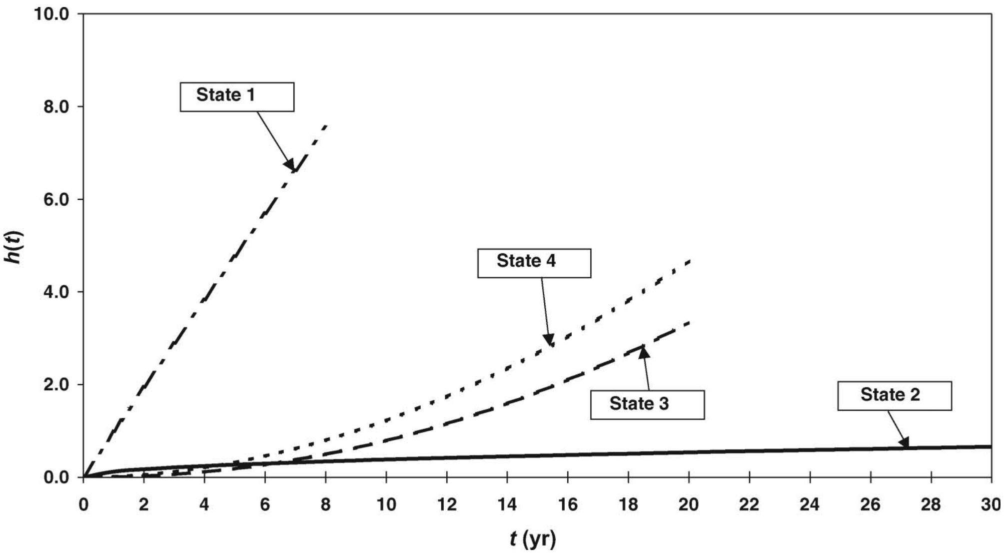

Figure 9. Hazard functions for sojourn times by CIP concrete bridge decks.

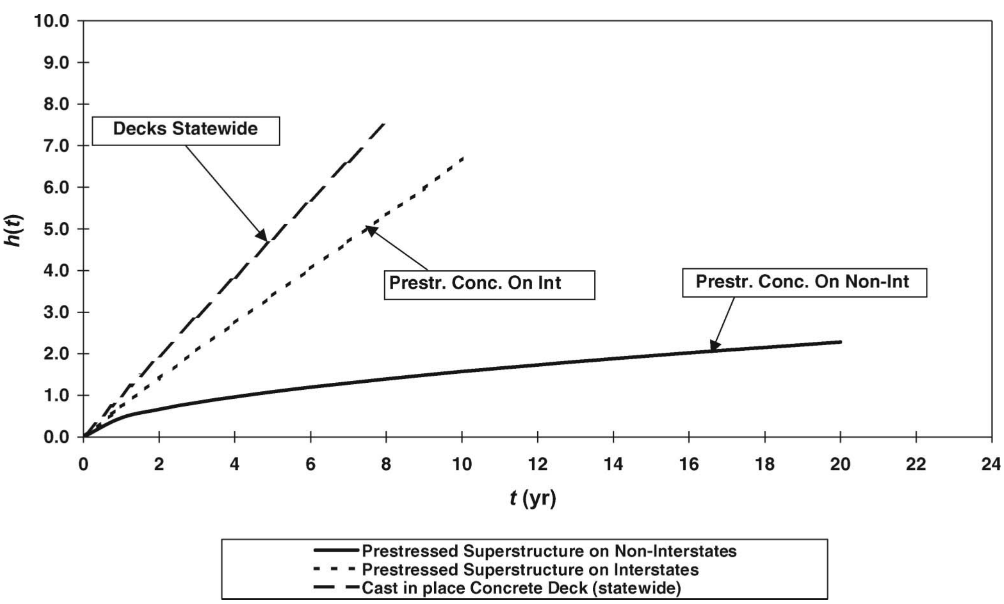

Figure 10. Hazard functions for sojourn times: various bridge types in condition rating 9 (state 1).

As shown in Figure 12 for the transition from condition rating 8 (state 2), first, the bridge has a better chance of remaining in state 2 for a longer time, up to 17 years. If the bridge is observed entering state 2 at a time $t = 0$, then, about 10 years later, it has about 62% and 24% chance of already being in state 3 or state 4, respectively, with about a 14% chance of still being in state 2. At a time 24 years after entering state 2, the bridge is almost sure of being in state 4, with a very low chance of being in state 3 and with 0 chance of still being in state 2.

Similar results are shown for CIP concrete bridge decks for various transitions in Table 3 and Figures 13 to 15. For bridges entering state 4 (rating 6), the distribution parameters are not available for waiting times in state 5 (rating 5), so the transition probabilities are estimated based on the cumulative distribution functions of the waiting times in state 4 (using Equation (5)).

As mentioned earlier, the estimated probabilities shown so far are actually the estimated probabilities of finding the bridge in the various states at any specified

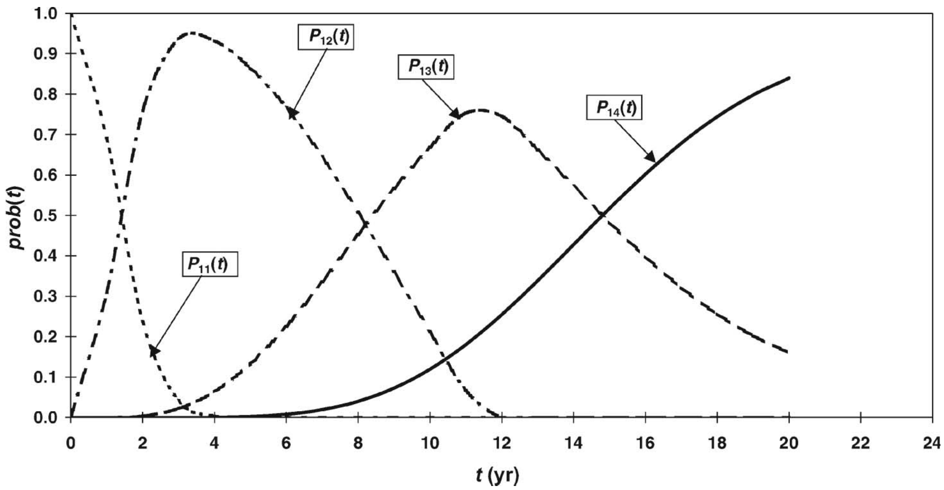

Figure 11. Condition state probability at time t years after starting in state 1 (condition rating 9) for bridge prestressed concrete superstructures on interstates.

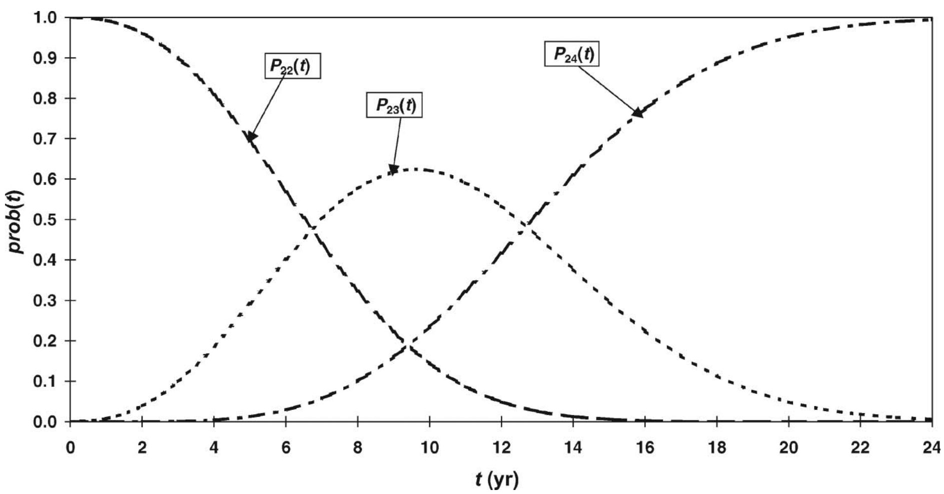

Figure 12. Condition state probability at time t years after starting in state 2 (condition rating 8) for bridge prestressed concrete superstructures on interstates.

time t in the future, given the state it entered a specified state at time $t = 0$. A matrix can therefore be assembled to represent the predicted condition of the bridge at any time t, generally represented as:

$$P_{ij}(t) = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & 0 & a_{33} & a_{34} & a_{35} \\ 0 & 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & 0 & a_{55} \end{pmatrix}, \qquad (35)$$

where a_{ij} is the probability of the bridge (or proportion of the bridge inventory) being in state j after entering state i at time $t = 0$; $i, j = 1, 2, \ldots, 5$.

The predicted condition matrices are shown as follows, as developed for CIP concrete bridge decks:

$$P_{ij}(1) = \begin{pmatrix} 0.605 & 0.395 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.921 & 0.079 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.998 & 0.002 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.995 & 0.005 \\ 0.000 & 0.000 & 0.000 & 0.000 & 1.000 \end{pmatrix}, \quad (36)$$

$$P_{ij}(5) = \begin{pmatrix} 0.000 & 0.605 & 0.389 & 0.006 & 0.000 \\ 0.000 & 0.402 & 0.571 & 0.027 & 0.000 \\ 0.000 & 0.000 & 0.738 & 0.254 & 0.008 \\ 0.000 & 0.000 & 0.000 & 0.578 & 0.422 \\ 0.000 & 0.000 & 0.000 & 0.000 & 1.000 \end{pmatrix}, \quad (37)$$

$$P_{ij}(10) = \begin{pmatrix} 0.000 & 0.000 & 0.790 & 0.210 & 0.000 \\ 0.000 & 0.077 & 0.507 & 0.374 & 0.042 \\ 0.000 & 0.000 & 0.076 & 0.663 & 0.261 \\ 0.000 & 0.000 & 0.000 & 0.015 & 0.985 \\ 0.000 & 0.000 & 0.000 & 0.000 & 1.000 \end{pmatrix}, \quad (38)$$

Table 3. Transition probabilities for CIP concrete bridge decks (states 1–5).

t (yr)	$P_{11}(t)$	$P_{12}(t)$	$P_{13}(t)$	$P_{14}(t)$	$P_{22}(t)$	$P_{23}(t)$	$P_{24}(t)$	$P_{25}(t)$	$P_{33}(t)$	$P_{34}(t)$	$P_{35}(t)$	$P_{44}(t)$	$P_{45}(t)$
0	1.000	0.000	0.000	0.000	1.000	0.000	0.000	0.000	1.000	0.000	0.000	1.000	0.000
1	0.605	0.395	0.000	0.000	0.921	0.079	0.000	0.000	0.998	0.002	0.000	0.995	0.005
2	0.139	0.814	0.048	0.000	0.793	0.207	0.000	0.000	0.982	0.018	0.000	0.963	0.037
3	0.012	0.842	0.146	0.000	0.654	0.344	0.002	0.000	0.939	0.061	0.000	0.885	0.115
4	0.000	0.731	0.267	0.001	0.521	0.470	0.010	0.000	0.858	0.140	0.002	0.752	0.248
5	0.000	0.605	0.389	0.006	0.402	0.570	0.027	0.000	0.738	0.255	0.008	0.578	0.422
6	0.000	0.480	0.501	0.019	0.303	0.636	0.060	0.001	0.587	0.392	0.021	0.393	0.607
7	0.000	0.358	0.598	0.044	0.222	0.662	0.112	0.004	0.424	0.528	0.048	0.230	0.770
8	0.000	0.237	0.678	0.085	0.159	0.646	0.185	0.010	0.274	0.631	0.095	0.114	0.886
9	0.000	0.115	0.741	0.144	0.112	0.592	0.274	0.021	0.156	0.678	0.166	0.047	0.953
10	0.000	0.000	0.790	0.219	0.077	0.507	0.374	0.042	0.076	0.663	0.261	0.015	0.985
11	0.000	0.000	0.694	0.306	0.052	0.396	0.477	0.075	0.032	0.593	0.375	0.004	0.996
12	0.000	0.000	0.602	0.398	0.035	0.269	0.575	0.121	0.011	0.489	0.500	0.001	0.999
13	0.000	0.000	0.512	0.488	0.023	0.133	0.663	0.182	0.003	0.373	0.624	0.000	1.000
14	0.000		0.429	0.571	0.014	0.000	0.730	0.256	0.001	0.264	0.735	0.000	1.000
15	0.000		0.357	0.643	0.009	0.000	0.651	0.340	0.000	0.173	0.826	0.000	1.000
16	0.000		0.297	0.703	0.006	0.000	0.566	0.428	0.000	0.105	0.895	0.000	1.000
17	0.000		0.248	0.752	0.003	0.000	0.479	0.517	0.000	0.059	0.941	0.000	1.000
18	0.000		0.211	0.789	0.002	0.000	0.396	0.602	0.000	0.030	0.970	0.000	1.000
19	0.000		0.183	0.817	0.001	0.000	0.320	0.679	0.000	0.014	0.986	0.000	1.000
20	0.000		0.162	0.838	0.001	0.000	0.254	0.745	0.000	0.006	0.994	0.000	1.000
21									0.000	0.002	0.998		
22									0.000	0.001	0.999		
23									0.000	0.000	1.000		
24									0.000	0.000	1.000		
25									0.000	0.007	0.993		

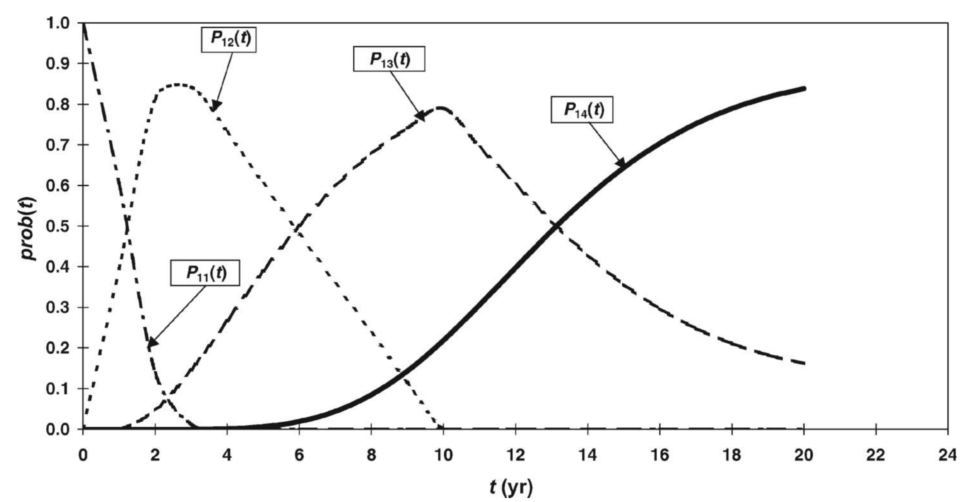

Figure 13. Condition state probability at time t years after starting in state 1 (condition rating 9) for CIP concrete bridge decks.

$$P_{ij}(15) = \begin{pmatrix} 0.000 & 0.000 & 0.357 & 0.643 & 0.000 \\ 0.000 & 0.009 & 0.000 & 0.651 & 0.340 \\ 0.000 & 0.000 & 0.076 & 0.173 & 0.827 \\ 0.000 & 0.000 & 0.000 & 0.000 & 1.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 1.000 \end{pmatrix}, \quad (39)$$

and

$$P_{ij}(20) = \begin{pmatrix} 0.000 & 0.000 & 0.162 & 0.838 & 0.000 \\ 0.000 & 0.001 & 0.000 & 0.254 & 0.745 \\ 0.000 & 0.000 & 0.000 & 0.006 & 0.994 \\ 0.000 & 0.000 & 0.000 & 0.000 & 1.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 1.000 \end{pmatrix}. \quad (40)$$

To predict the bridge condition for any time t, the initial condition at time $t = 0$ is needed in the form of a state vector,

$$\mathrm{COND}^{\mathrm{T}}(0) = [a_1, a_2, a_3, a_4, a_5], \quad (41)$$

where a_i is the probability that the bridge (or proportion of the bridge inventory) will be in state i, at time $t = 0$, $i = 1, 2, \ldots, 5$.

Then, the predicted condition in the form of a transposed vector would simply be a product of the

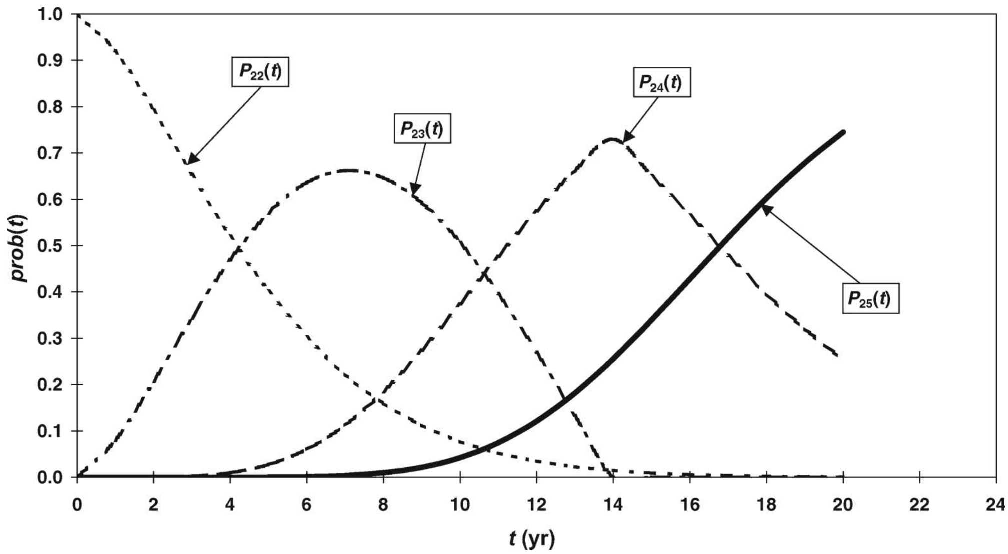

Figure 14. Condition state probability at time t years after starting in state 2 (condition rating 8) for CIP concrete bridge decks.

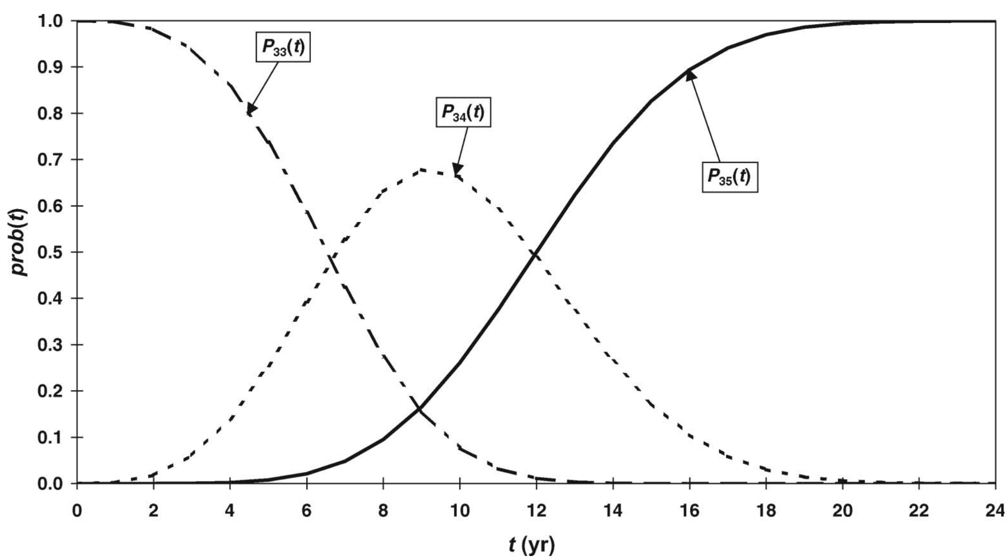

Figure 15. Condition state probability at time t years after starting in state 3 (condition rating 7) for CIP concrete bridge decks.

initial condition vector and the predicted transition condition matrix, i.e.

$$\mathrm{COND}^{\mathrm{T}}(t) = \mathrm{COND}^{\mathrm{T}}(0) \times P_{ij}(t). \tag{42}$$

For illustration, let us assume a bridge has 90% of its elements all in excellent condition (state 1) and 10% in state 2 at the initial time. This could also mean that 90% of the bridges on the network are in state 1, while 10% are in state 2. The bridge condition can be predicted for 5 years time using the following computation:

$$\begin{aligned}\mathrm{COND}^{\mathrm{T}}(5) &= (\,0.9 \quad 0.1 \quad 0.0 \quad 0.0 \quad 0.0\,) \\ &\times \begin{pmatrix} 0.000 & 0.605 & 0.389 & 0.006 & 0.000 \\ 0.000 & 0.402 & 0.571 & 0.027 & 0.000 \\ 0.000 & 0.000 & 0.738 & 0.254 & 0.008 \\ 0.000 & 0.000 & 0.000 & 0.578 & 0.422 \\ 0.000 & 0.000 & 0.000 & 0.000 & 1.000 \end{pmatrix}\end{aligned} \tag{43}$$

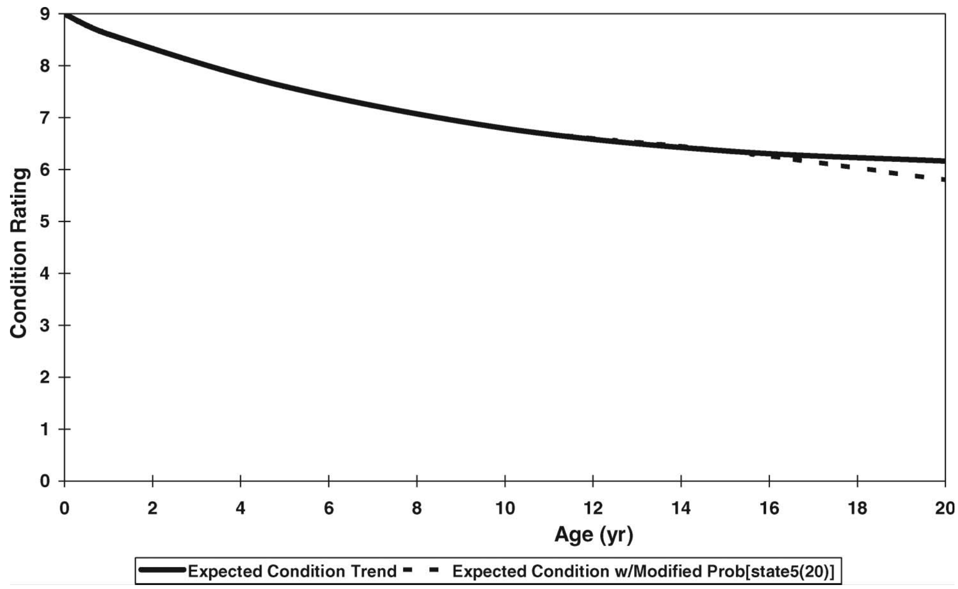

Figure 16. Predicted condition ratings relative to age for bridge CIP concrete decks.

or

$$\mathrm{COND}^{\mathrm{T}}(5) = (0.000\ 0.585\ 0.407\ 0.008\ 0.000). \quad (44)$$

With an NBI rating vector, $\mathrm{RATING}^{\mathrm{T}} = (9\ 8\ 7\ 6\ 5)$, the expected condition rating in 5 years is estimated to be $\mathrm{EXPCOND}(5) = \mathrm{COND}^{\mathrm{T}}(5) \times \mathrm{RATING}$, or

$$\mathrm{EXPCOND}(5) = (0.000 \quad 0.590 \quad 0.407 \quad 0.003 \quad 0.000) \begin{pmatrix} 9 \\ 8 \\ 7 \\ 6 \\ 5 \end{pmatrix}, \quad (45)$$

resulting in an expected condition rating of 8.5.

If the bridge condition was perfect initially, i.e. the initial condition vector was $\mathrm{COND}^{\mathrm{T}}(0) = (1.0\ 0.0\ 0.0\ 0.0\ 0.0)$, then the expected condition rating at each of the various times (in years) can be predicted and plotted as a deterioration curve, as shown in Figure 16. It should be noted here that the curve appears to flatten out from about 15 years on because of the limitations of the three-step computations (e.g. $P_{14}(t)$) used in the illustration above. Given that the bridge elements spend a relatively short time in state 1, an estimate of $P_{15}(t)$ or a four-step drop is needed because there is a reasonable chance that the bridge element will be in state 5 after 20 years. Even with an assumed probability of 33% of being in state 5 after 20 years, there is a slight change in the curve (Figure 16), appearing as expected for a 'decay' deterioration curve. The estimate of $P_{15}(t)$ can be obtained using the same methodology demonstrated earlier, but its computation is beyond the scope of this paper.

4. Conclusions

In this paper, an analysis of Florida's historical data on bridge condition has been conducted to estimate sojourn times at the various condition states, for various categories of bridges. The goal of studying a 'decay' deterioration (without significant improvement intervention) of the bridges resulted in data with both complete data (fully observed) and right-censored data. Using the complete data as they produced best fits for the Weibull distribution of the sojourn times, the hazard functions and transition probabilities between states were determined, with the results indicating that the bridge deterioration rates are not constant, but increase with age, i.e. an ageing wear-out deterioration. The use of semi-Markov models for bridge deterioration was also discussed, including the development of kernels of transition probabilities, as well as an overall semi-Markov model for prediction of future bridge conditions.

A suggestion for improvement on the results in this paper is the need for more accurate fitting of the sojourn time data to probability distributions, particularly the right-censored data. It was demonstrated that the bridge deterioration process could be better modelled using arbitrary distributions for the sojourn times, instead of the restrictive exponential distribution implied in existing models of the Markov chain and Markov process. Overall, it is safe to assume that right-censored data will represent bridge condition data with 'hidden' improvement efforts, i.e. these bridges are not truly in a 'decay' deterioration process, as intended for the scope of this paper.

Finally, a comment on the overall bridge inventory methods. The existing formats of bridge condition data collection at various transportation agencies are not ideally suitable for duration-based modelling (using sojourn times) of bridge deterioration, but the implementation of such models, as presented in this paper, is becoming more realistic, more so because many agencies are starting to have historical condition data for a longer period.

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Appendix 1. Notations

The following symbols were used in this paper.

T	Non-negative random variable representing the sojourn time.
$F_i(t)$	The cumulative distribution function, describes the probability that the bridge element will transition out of state i by age or time t.
$f_i(t)$	The probability distribution function, describes the probability of a sojourn time at state i.
$S_i(t)$	The survivor function, estimates the probability that bridge element would still be in state i by time t.
$h_i(t)$	The hazard or failure rate for state i, i.e. the rate of change of the conditional probability of transition from state i.
$p_{ik}(t, \Delta)$	The 'instantaneous' transition probability out of a state i into any worse condition state, say k, within a period Δ after time t.
$p_{ii}(t, \Delta)$	The probability of remaining in the state i within a period Δ after time t.
(X_n, TS_n)	The parameters of a stochastic process where X_n are the successive states at times TS_n at the nth transitions.

$Q_{ij}(t)$	The semi-Markov kernel, describes one-step transition probability that after making a transition into state i, the process next makes a transition into state j in an amount of time less than or equal to t.
p_{ij}^{e}	The transition probability that the process will *eventually* move from state i to state j,
$F_{ij}(t)$	The cumulative distribution function of the sojourn time that the process spends in state i given that the next state is j.
$P_{ij}(t,\ t_0)$	The transition probability, from state i to state j at any time t, given the elapsed time t_0 at the initial state i.
δ_{ij}	Kronecker delta (i.e. $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise).
$S_i(t,\ t_0)$	Survivor function for the sojourn time in state i, given the initial elapsed time t_0.
$p_{ik}^{d}(t_0)$	The probability density function describing the elapsed time t_0 in initial state i before the first transition.
$f_{ik}(x,\ t_0)$	The probability density function describing the sojourn time before the first transition at a time x, given the initial elapsed time t_0.
$P_{kj}(t-x)$	The transition probability for system to move from state k to state j in the remaining time $(t-x)$.
$F_{kj}(t)$	The cumulative distributions function of the sojourn time in state k, before transitioning to state j.
a_{ij}	Probability of the bridge being (or proportion of the bridge inventory) in state j after entering state i at time $t = 0$, $i, j = 1, 2, \ldots, 5$.
a_i	Probability that the bridge (or proportion of the bridge inventory) that will be in state i, at time $t = 0$, $i = 1, 2, \ldots, 5$.
$\mathrm{COND}^{\mathrm{T}}(t)$	The state probability vector, showing the probability of finding the bridge element in specific condition state at time t.
$\mathrm{RATING}^{\mathrm{T}}$	The vector showing the applicable NBI condition ratings.
$\mathrm{EXPCOND}(t)$	The expected condition rating (NBI) of bridge element at time t.