IMPORTANCE OF SYSTEM COMPONENTS AND FAULT TREE EVENTS

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ABSTRACT

A new measure of the importance of the components in a coherent system and of the basic events in a fault tree is defined and its properties derived. The importance measure is a useful guide during the system development phase as to which components (or alternatively, which basic events) should receive more urgent attention in achieving system reliability growth. The new measure of component importance has certain desirable properties not possessed by the previous measure of component importance proposed by Birnbaum in "On the Importance of Different Components in a Multicomponent System," appearing in MULTIVARIATE ANALYSIS-II, edited by P. R. Krishnaiah, Academic Press, New York (1969).

The measure is extended to minimal cut sets and to systems of components undergoing repair. A number of commonly occurring systems are treated in detail for illustrative purposes.

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1. INTRODUCTION AND SUMMARY.

In attempting to achieve high reliability for a complex system, a basic problem facing the systems analyst is that of evaluating the relative importance of the various components comprising the system. Measuring the relative importance of components may permit the analyst to determine which components merit the most additional research and development to improve overall system reliability at minimum cost or effort.

Birnbaum (1969) defines measures of component importance for coherent systems. (See Birnbaum, Esary and Saunders, 1961, Barlow and Proschan, 1965 and 1974, for definitions and properties of coherent structures.) Let $\underline{p} = (p_1, \ldots, p_n)$ denote the vector of component reliabilities of a coherent system and $h(\underline{p})$ denote the system reliability function. Then Birnbaum defines the *reliability importance* $B_r(i)$ *of component* i by

$$B(i \mid \underline{p}) = \frac{\partial h(\underline{p})}{\partial p_i}, \tag{1.1}$$

Note that $B(i \mid \underline{p})$ depends on $\underline{p}$. For the case in which $\underline{p}$ is unknown, Birnbaum defines the *structural importance* $B(i)$ *of component* i by

$$B(i) = \left.\frac{\partial h}{\partial p_i}\right|_{p_1=\ldots=p_n=1/2} ; \tag{1.2}$$

i.e., $B(i) = B(i \mid \underline{p})$ with each p_i set equal to $1/2$.

We introduce another measure of relative component importance which is essentially the conditional probability that system failure is caused by (i.e., coincides with) the failure of a given component. This new measure reveals more clearly the relative extent to which each component is contributing to system failure. These importance measures also enjoy the property that they sum to one.

We develop the properties of these measures of component importance, and give examples of their computation for commonly occurring systems. We also develop a measure of the importance of minimal cut sets, which play a fundamental role in analyzing systems and obtaining bounds on system reliability. Finally, we extend the measure of component importance to cover the case in which components undergo repair.

2. PRELIMINARIES CONCERNING COHERENT SYSTEMS AND FAULT TREES.

Coherent Systems.

Consider a system of n components. Let the state x_i of component i be defined by

$$x_i = \begin{cases} 1 & \text{if } i \text{ is functioning} \\ 0 & \text{if } i \text{ is failed.} \end{cases}$$

Similarly, let the state ϕ of the system be a deterministic binary function of the vector $\underline{x} = (x_1, \ldots, x_n)$ of component states:

$$\phi(\underline{x}) = \begin{cases} 1 & \text{if the system is functioning} \\ 0 & \text{if the system is failed.} \end{cases}$$

We shall need the following conventions and definitions:

(a) $(1_i, \underline{x}) = (x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)$

(b) $(0_i, \underline{x}) = (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$

(c) Component i is *irrelevant* to ϕ if $\phi(1_i, \underline{x}) = \phi(0_i, \underline{x})$ for all $(\cdot_i, \underline{x})$.

(d) ϕ is *coherent* if ϕ is nondecreasing in each coordinate and each component is relevant.

(e) Let components be stochastically independent. The *reliability function* $h(\underline{p})$ is the probability that the system operates, as a function of component reliabilities $\underline{p} = (p_1, \ldots, p_n)$.

Let component i have life distribution $F_i(t)$, and let $X_i(t) = 1$ if the i^{th} component functions until time t and 0 otherwise. Thus $EX_i(t) = \bar{F}_i(t) \overset{\text{def}}{=} 1 - F_i(t)$ and $h(\underline{\bar{F}}(t)) = P[\phi(\underline{X}(t)) = 1] = E\phi(\underline{X}(t))$.

Fault Trees.

Fault trees have been used by engineers and reliability analysts to represent schematically, basic events and their various logical combinations which may result in a so-called "top event", generally corresponding to system failure. The dependence of the "top event" on the basic events is analogous to the dependence of a coherent system state on the states of the components. Fault trees are more general than coherent structures in the sense that basic events in a fault tree need not necessarily correspond to component failures in a coherent system. For a discussion of fault tree analysis see Barlow and Proschan (1974), Appendix, Barlow and Chatterjee (1973) and Fussell (1973b).

A sample fault tree appears in Figure 1. The events labelled 1 through 7 are basic events. Output event G3 occurs if and only if *both* basic events 6 and 7 occur; the symbol shown signifies logical intersection, and is called an AND gate. Output event G2 occurs if and only if any one of the input events 4, 5, or G3 occurs; the symbol shown signifies logical union, and is called an OR gate. See Fussell (1973a) and Lambert (1973) for discussions of fault tree construction.

A *min cut set* is a set of basic events whose occurrence causes the top event, but which cannot be reduced and still insure the occurrence of the top event. We consider only fault trees for which each basic event is *relevant* to the top event; i.e., each basic event is in the union of min cut sets.

Fault tree analysts list min cut sets in order of size; small min cut sets are generally considered more important than large min cut sets. However, because of replication of basic events among min cut sets, this intuitive ordering can be incorrect. In terms of our definition below of min cut set importance, we show, for example, that a 4 component min cut set can be more important than a 3 component min cut set.

Remark:

In *competing risk theory* (cf. Berman, 1963), competing causes of failure or death can be characterized by a series system or by a fault tree with a

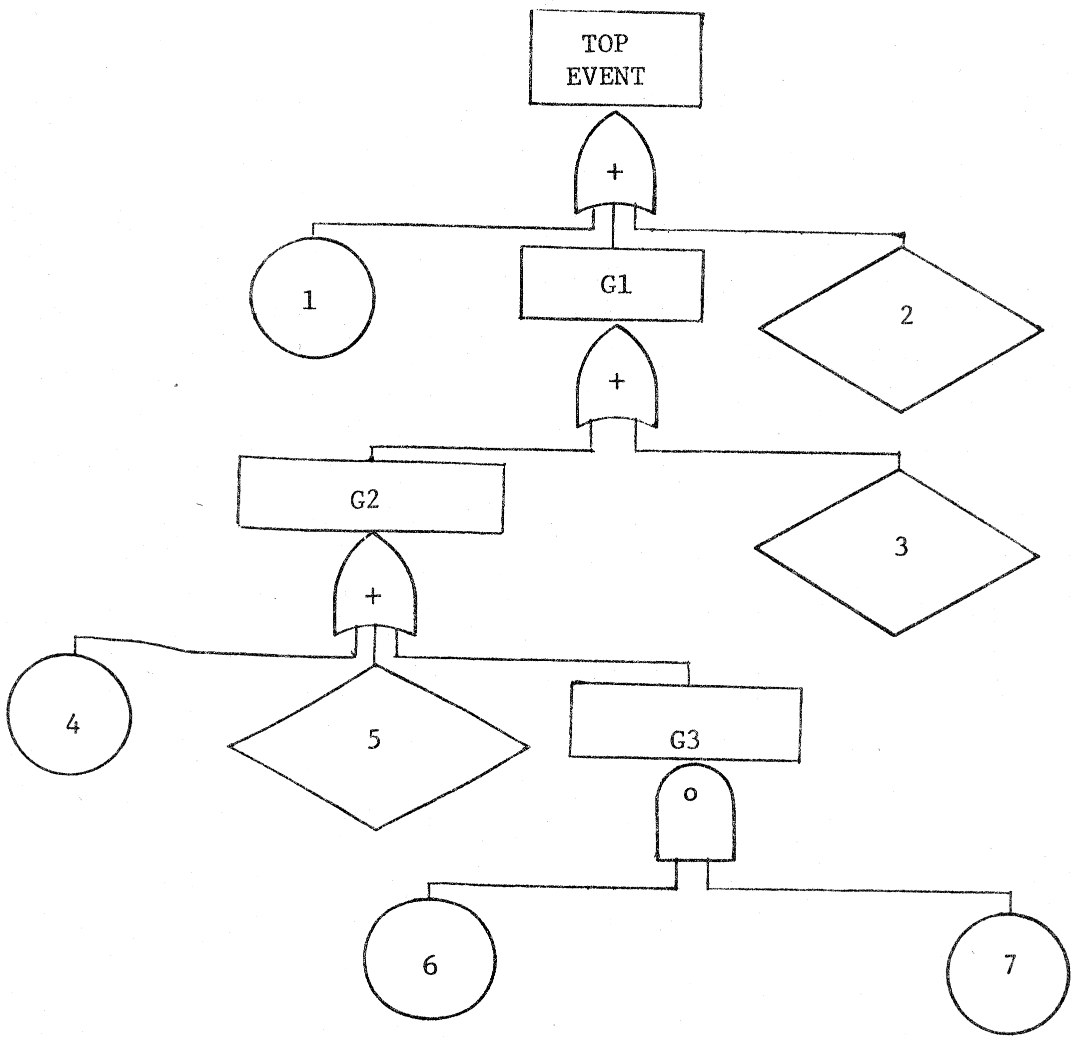

FIGURE 1. SAMPLE FAULT TREE

single OR gate below the top event. The relative importance of these causes is of interest in this theory. Our concepts of component importance in a coherent system (and of event importance in a fault tree) generalize certain aspects of competing risk theory in that we consider systems more general than series systems.

In the discussion that follows we shall use the terminology of coherent structure theory. However all results also apply to fault trees when the proper identifications are made.

We shall also assume throughout that components are stochastically independent to avoid technicalities. Many results can be extended to the more general case with obvious modifications.

3. IMPORTANCE: DEFINITION, PROPERTIES, AND COMPUTATION.

In this and following sections we assume that component i has continuous life distribution F_i, $i = 1,2, \ldots, n$. Hence, we know that a system failure will coincide with the failure of some component, say component i. In this sense, we say that component i has caused system failure. First, consider a system without repair which has failed at some specified time t. Suppose we are interested in determining the most probable cause of failure.

3.1. Proposition.

If component i has distribution F_i with density f_i $(i = 1,2, \ldots, n)$, then the probability that i caused system failure, given the system failed at time t is given by

$$\text{(3.1)} \qquad \frac{[h(1_i,\underline{\bar{F}}(t)) - h(0_i,\underline{\bar{F}}(t))]f_i(t)}{\sum_{j=1}^{n} [h(1_j,\underline{\bar{F}}(t)) - h(0_j,\underline{\bar{F}}(t))]f_j(t)} .$$

Proof:

$h(1_i,\underline{\bar{F}}(t)) - h(0_i,\underline{\bar{F}}(t)) = P[\phi(1_i,\underline{X}(t)) - \phi(0_i,\underline{X}(t)) = 1]$ is the probability that at time t, the system is functioning if i is functioning but is failed otherwise. Thus the numerator (times dt) is the probability that i causes system failure in $[t,t+dt]$, while the denominator (times dt) is the probability of system failure in $[t,t+dt]$. ||

An obvious consequence of (3.1) is

3.2. Proposition.

The probability that i causes system failure in $[0,t]$ given system failure in $[0,t]$ is

$$(3.2) \qquad \frac{\int_0^t [h(1_i,\underline{\bar{F}}(u)) - h(0_i,\underline{\bar{F}}(u))]dF_i(u)}{\sum_{j=1}^{n} \int_0^t [h(1_j,\underline{\bar{F}}(u)) - h(0_j,\underline{\bar{F}}(u))]dF_j(u)} .$$

Letting $t \to \infty$ in (3.2) we have the probability that i causes system failure when the system eventually fails. Note that in this case, the denominator is one. We take this limit as our definition of component importance.

3.3. Definition.

The *importance*, $I_h(i)$, of component i is the probability that i causes system failure, where

$$(3.3) \qquad I_h(i) = \int_0^\infty [h(1_i,\underline{\bar{F}}(t) - h(0_i,\underline{\bar{F}}(t))]dF_i(t) .$$

(3.3) is an immediate consequence of (3.2).

3.4. Properties of the Importance Measure.

(P1) $0 \leq I_h(i) \leq 1$.

(P2) If $n \geq 2$ and the intersection of supports of F_j $(j = 1,2, \ldots, n)$ has positive probability with respect to the product distribution $\prod_{j=1}^{n} F_j(t)$, then $0 < I_h(i) < 1$.

(P3) $\sum_{i=1}^{n} I_h(i) = 1$.

Proof:

(P1) The integrand in (3.3) is between 0 and 1, implying (P1).

(P2) $I_h(i) = 0$ implies that for t in the support of F_i , $h(1_i,\underline{\bar{F}}(t)) - h(0_i,\underline{\bar{F}}(t)) \equiv 0$, since component distributions are continuous. This in turn implies that i is irrelevant to ϕ with positive probability, which contradicts the fact that ϕ is coherent.

Likewise, $I_h(i) = 1$ implies that $h(1_i,\underline{\bar{F}}(t)) - h(0_i,\underline{\bar{F}}(t)) \equiv 1$. Hence all other components are irrelevant with positive probability, again contradicting the fact that ϕ is coherent.

The conclusion now follows. Note that (P2) is false if the supports do not intersect with positive probability with respect to $\prod_{j=1}^{n} F_j(t)$. Consider two compoents in parallel . If component 1 has a life distribution with support on),1] while component 2 has a life distribution with support on [2,3] , thn $I_h(1) = 0$ and $I_h(2) = 1$.

(P3) System failure coincides with the failure of exactly one component. Thus $\sum_{i=1}^{n}{}_{1}(i) = 1$. ||

Importancf a Module.

Intuvely, a module of a coherent system is a subset of components of a coherenystem which behaves like a "supercomponent". More technically, let ϕ b coherent structure of n components, M be a subset of {1, ..., with complement M^c , χ be a coherent system of the components in M , a$\phi(\underline{x}) = \psi\left[\chi(\underline{x}^M), \underline{x}^{M^c}\right]$. Then (M,χ) is a module of ϕ . See Birnbaum Esary (1965) for a discussion of modules.

Let M) , the *importance of the module* (M,χ) , be the probability that the modulses system failure; i.e., that system failure coincides with the failure c module. Letting g denote the reliability function of the module, we prove:

3.5. The

(a) $i \in M$, then

$$(3.4) \qquad = \int_0^\infty [h(\underline{1}^M,\underline{\bar{F}}(t)) - h(\underline{0}^M,\underline{\bar{F}}(t))][g(1_i,\underline{\bar{F}}(t)) - g(0_i,\underline{\bar{F}}(t))]dF_i(t)$$

(b) $I_h(M) = \sum_{i\epsilon M} I_h(i)$.

Proof:

(a) $h(\underline{1}^M,\underline{\bar{F}}(t)) - h(\underline{0}^M,\underline{\bar{F}}(t)) = P[\phi(\underline{1}^M,\underline{X}(t)) - \phi(\underline{0}^M,\underline{X}(t)) = 1]$ represents the probability that at time t , the system is functioning if the module is functioning, but is failed otherwise; similarly, $g(1_i,\underline{\bar{F}}(t)) - g(0_i\underline{\bar{F}}(t)) = P[\chi(1_i,\underline{X}(t)) - \chi(0_i,\underline{X}(t)) = 1]$ represents the probability that at time t , the module is functioning if component i is functioning, but is failed otherwise. Since components are independent, the integral in (3.4) is the probability that component i fails at some time and *causes* the module to fail, and that module failure *causes* the system to fail. Since i cannot cause system failure except through module failure, (a) follows.

(b) $\sum_{i\epsilon M} I_h(i) =$

$$\int_0^\infty [\underline{1}^M,\underline{\bar{F}}(t)) - h(\underline{0}^M,\underline{\bar{F}}(t))] \sum_{i\epsilon M} [g(1_i,\underline{\bar{F}}(t)) - g(0_i,\underline{\bar{F}}(t))]dF_i(t)$$

$$= -\int_0^\infty [h(\underline{1}^M,\underline{\bar{F}}(t)) - h(\underline{0}^M,\bar{F}(t))]\frac{d}{dt}g(\underline{\bar{F}}(t))dt = I_h(M).||$$

Remark:

Using the Birnbaum (1969) definition of importance, the importance of a component of a module is the product of the importance of the component to the module times the importance of the module to the system. Note that this does not hold in general for our definition of importance. This is a consequence of the fact that $w(x) = u(x)v(x)$ for each x does *not* imply that $\int_a^b w(x)dx = \int_a^b u(x)dx \int_a^b v(x)dx$.

A simple criterion is given next for showing one component more important than any of the other components. We first prove two simple lemmas of independent interest.

3.6. Lemma:

Let component i be in series (parallel) with the rest of the system. Then $I_h(i)$ is increasing (decreasing) in $F_i(t)$ and in $\bar{F}_j(t)$, $j \neq i$.

Proof:

First assume i is in series with the rest of the system. Then $I_h(i) = \int_0^\infty h(1_i, \underline{\bar{F}}(t) dF_i(t)$ [since $h(0_i, \underline{\bar{F}}(t)) = 0$ by hypothesis]. But $h(1_i, \underline{p})$ is increasing in each p . Thus $I_h(i)$ is increasing in $\bar{F}_j(t)$, $j \neq i$. Also $h(1_i, \underline{\bar{F}}(t))$ is decreasing in t . Thus $I_h(i)$ is increasing in $F_i(t)$. A similar proof applies when i is in parallel with the rest of the system.||

3.7. Lemma:

Let i be in series (parallel) with the rest of the system. Let all components have common distribution F . Then $I_h(i) \geq I_h(j)$, $j \neq i$.

Proof:

First assume that i is in series with the rest of the system. Since components are stochastically alike, we can compute $I_h(k)$ as the proportion of permutations of $1, \ldots, n$ which correspond to a system failure due to k . By interchanging i and j in each permutation, we see that there are at least as many permutations in which the system fails due to i as there are permutations in which system failure is due to j .

A similar proof applies in the parallel case.||

3.7. Theorem:

Let component i be in series (parallel) with the rest of the system. Let $F_i(t) \geq F_j(t)$ for $j \neq i$, $t \geq 0$. Then $I_h(i) \geq I_h(j)$ for $j \neq i$.

Proof:

By application of Lemmas 3.6 and 3.7, the result follows immediately.||

Computing Importance in the Proportional Hazards Case.

It is difficult to compute $I_h(i)$ for arbitrary failure distributions. However, if we assume *proportional hazards*, i.e., $\bar{F}_i(t) = e^{-\lambda_i R(t)}$ for $i = 1, \ldots, n$, where $R(t)$ is the common hazard, then the calculation becomes more tractable. By a change of variable, it is evident that in computing $I_h(i)$, we may as well assume $\bar{F}_i(t) = e^{-\lambda_i t}$.

Thus, for a series system,

$$I_h(i) = \int_0^\infty e^{-(\sum_{j\neq i} \lambda_j)t} \lambda_i e^{-\lambda_i t}\, dt = \lambda_i / \sum_{j=1}^n \lambda_j .$$

For a parallel system,

$$I_h(i) = \int_0^\infty \left[\prod_{j\neq i} \left(1 - e^{-\lambda_j t}\right)\right] \lambda_i e^{-\lambda_i t}\, dt$$

$$= \lambda_i \left[\lambda_i^{-1} - \sum_{j\neq i} (\lambda_i + \lambda_j)^{-1} + \sum_{\substack{j<k \\ j,k\neq i}} (\lambda_i + \lambda_j + \lambda_k)^{-1} - \ldots + (-1)^{n-1}(\lambda_1 + \ldots + \lambda_n)^{-1} \right].$$

In general, for portional hazards, by a change of variable, we obtain:

$$I_h(i) = \int_0^1 \Big[h\left(p^{\lambda_1}, \ldots, p^{\lambda_{i-1}}, 1, p^{\lambda_{i+1}}, \ldots, p^{\lambda_n}\right) - h\left(p^{\lambda_1}, \ldots, p^{\lambda_{i-1}}, 0, p^{\lambda_{i+1}}, \ldots, p^{\lambda_n}\right)\Big] \lambda_i p^{\lambda_i - 1} dp . \tag{3.3}$$

Example:

Assume the three components in the following structure have proportional hazards.

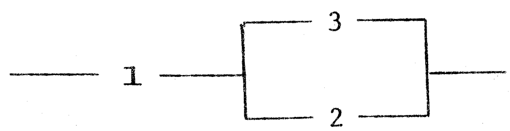

FIGURE 2. THREE-COMPONENT STRUCTURE.

Using (3.3), we compute

$$I_h(1) = \int_0^1 \left[p^{\lambda_2} + p^{\lambda_3} - p^{\lambda_2+\lambda_3}\right] \lambda_1 p^{\lambda_1 - 1} dp = \frac{\lambda_1}{\lambda_1 + \lambda_2} + \frac{\lambda_1}{\lambda_1 + \lambda_3} - \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} .$$

Similarly,

$$I_h(2) = \frac{\lambda_2}{\lambda_1 + \lambda_2} - \frac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3}$$

$$I_h(3) = \frac{\lambda_3}{\lambda_1 + \lambda_3} - \frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3} .$$

Monte Carlo Methods.

For complex systems with large numbers of components, it may be necessary to use Monte Carlo methods even in the case of proportional hazards.

Without loss of generality, assume $\sum_{j=1}^{n} \lambda_j = 1$. To simulate the sequence of successive component failures that ultimately end in system failure, draw successive independent uniform random variables $U_1, U_2, \ldots$ on $[0,1]$. If U_1 falls between $\lambda_1 + \lambda_2 + \ldots + \lambda_{i_1 - 1}$ and $\lambda_1 + \lambda_2 + \ldots + \lambda_{i_1}$ (where λ_0 is defined to be 0), then conclude that component i_1 has failed first. Repeat the process, using U_2 to determine which component failed second. If U_2 should call for the failure of component i_1 again, simply discard U_2, and use U_3 instead. Continue this process until the system has failed. The component causing system failure is then recorded. If component i causes system failure n_i times in n trials, then n_i/n estimates $I_h(i)$.

4. STRUCTURAL IMPORTANCE.

In the absence of information concerning component reliabilities (as might be the case in the early stages of system development), it may be reasonable to assume that component life distributions are the same. The importance of component i, computed from (3.1) with $F_1 = F_2 = \ldots = F_n$, is called the *structural importance of component* i in structure ϕ ; we write $I_\phi(i)$. Making the change of variable $p = \bar{F}_i(t)$ for $i = 1, \ldots, n$, yields:

$$(4.1) \qquad I_\phi(i) = \int_0^1 [h(1_i,p) - h(0_i,p)]dp ,$$

where $(1_i,p)$ is the vector having 1 in the i^{th} position and p in all other positions.

Examples:

Let $\phi_{k/n}$ denote the structure function of a k-out-of-n structure. Then system reliability $h_{k/n}(\underline{p})$ is a symmetric function of component reliabilities $p_1 = p_2 = \ldots = p_n = p$. It follows from (4.1) that all components have equal importance. Since $\sum_{i=1}^{n} I_\phi(i) = 1$, we conclude that $I_\phi(i) = 1/n$ for $i = 1,2, \ldots, n$.

More generally, let ϕ be a symmetric function of component states $x_1, \ldots, x_n$. Then, immediately from (4.1), we conclude that $I_\phi(i) = 1/n$, $i=1, \ldots, n$. Examples of symmetric ϕ are compositions of k-out-of-n structures; i.e., $\phi = \phi_{k_1|n_1} \circ \phi_{k_2|n_2} \circ \cdots \circ \phi_{k_r|n_r}$.

Computation of Structural Importance.

Next we show how to compute the structural importance of a component in terms of the numbers of critical vectors for that component. We need some definitions first.

Definitions:

A *path set* is a set of components whose functioning ensures the functioning of the system. A path set is *minimal* if it cannot be reduced and still be a path set.

Similarly, a *cut set* is a set of components whose failure is sufficient to cause system failure. A cut set is *minimal* if it cannot be reduced and still be a cut set.

A *critical path vector for component* i is a vector $(1_i, \underline{x})$ such that $\phi(1_i, \underline{x}) = 1$, while $\phi(0_i, \underline{x}) = 0$; the corresponding *critical path set for* i is $\{i\} \cup \{j \mid x_j = 1, j \neq i\}$. In this sense, the functioning or failure of component i determines whether the system functions or fails. A *critical path vector (set) for component* i *of size* r is a critical path vector (set) for which $1 + \sum_{j \neq i} x_j = r$, $r = 1, \ldots, n$. The number, $n_r(i)$, of critical path vectors for component i of size r is given by

$$n_r(i) = \sum_{\sum_{j \neq i} x_j = r-1} [\phi(1_i, \underline{x}) - \phi(0_i, \underline{x})] \ .$$

We may now express the structural importance $I_\phi(i)$ of component i in terms of the number $n_r(i)$ of critical path vectors.

4.1. Theorem:

$$I_\phi(i) = \sum_{r=1}^{n} \frac{(r-1)!(n-r)!}{n!} n_r(i) \tag{4.2}$$

Proof:

From (4.1),

$$I_\phi(i) = \int_0^1 [h(1_i, p) - h(0_i, p)] dp$$

$$= \int_0^1 \left\{ \sum_{\underline{x}} [\phi(1_i,\underline{x}) - \phi(0_i,\underline{x})] p^{\sum_{j \neq i} x_j} (1 - p)^{n-1-\sum_{j \neq i} x_j} \right\} dp$$

$$= \int_0^1 \sum_{r=1}^{n} n_r(i) p^{r-1} (1 - p)^{n-r} dp = \sum_{r=1}^{n} n_r(i) \frac{(r - 1)!(n - r)!}{n!} . ||$$

Expression (4.2) may be rewritten to yield an interesting interpretation of structural importance. We may rewrite (4.2) as

(4.3)
$$I_\phi(i) = \frac{1}{n} \sum_{r=1}^{n} n_r(i) / \binom{n - 1}{r - 1} .$$

The numerator $n_r(i)$ represents the number of critical path vectors of size r, while the denominator represents the number of outcomes in which exactly $r - 1$ components are functioning among the $n - 1$ components excluding component i. Thus (4.3) states that the structural importance of component i is the "average probability" of a vector being a critical path vector for component i. The average is taken over the n different possible sizes $r = 1, \ldots, n$ of a critical path vector, where the probability of a vector being a critical path vector for component i of size r is computed as the ratio of the number of critical path vectors for component i of size r to the number of possible critical path vectors of size r.

Another interesting interpretation of structural importance is suggested by the equation

(4.4)
$$I_\phi(i) = \int_0^1 \left\{ \sum_{r=1}^{n} \frac{n_r(i)}{\binom{n - 1}{r - 1}} \binom{n - 1}{r - 1} p^{r-1} (1 - p)^{n-r} \right\} dp ,$$

which is immediate from the last line of the proof of Theorem 4.1. Note that $\binom{n - 1}{r - 1} p^{r-1} (1 - p)^{n-r}$ represents the probability that among the $n - 1$

components excluding component i , $r - 1$ are functioning, while $n_r(i)/\binom{n-1}{r-1}$ represents the probability that the $r - 1$ functioning components together with component i constitute a critical path set for component i . Thus the integrand represents the probability that i causes system failure. Integrating this probability over p is equivalent to assuming that a priori, common component reliability p is uniformly distributed on $[0,1]$.

Remark:

A similar analysis may be performed in terms of critical cut vectors and critical cut sets. To each critical path vector $(1_i,\underline{x})$ for i , there corresponds the *critical cut vector* $(0_i,\underline{x})$ *for* i , possessing of course, the property that $\phi(1_i,\underline{x}) = 1$ while $\phi(0_i,\underline{x}) = 0$. The corresponding cut set $\{i\} \cup \{j \mid x_j = 0, j \neq i\}$ is called a *critical cut set for* i . A *critical cut vector (set) for component* i *of size* r is a critical cut vector (set) for which $1 + \sum_{j \neq i} (1 - x_j) = r$. If $n_r^*(i)$ denotes the number of critical cut vectors for i of size r , then $n_r^*(i) = n_{n-r+1}(i)$.

Comparison with Birnbaum Structural Importance.

It is interesting to compare our definition of structural importance given in Equation (4.1) with the corresponding Birnbaum (1969) definition:

$$B(i) = \frac{\partial h(\underline{p})}{\partial p_i} \Bigg|_{p_1=\dots=p_n=1/2} .$$

Equivalently,

$$B(i) = h(1_i,1/2) - h(0_i,1/2) . \tag{4.5}$$

This may be compared with expression (4.1) representing our measure of the structural importance of component i . Note that Birnbaum computes the difference $h(1_i,p) - h(0_i,p)$ with p set equal to $1/2$, while we average this difference as p ranges over $[0,1]$.

Another comparison between the two measures of structural importance is also suggestive. From (4.5), we may write:

$$B(i) = \sum_{\underline{x}} [\phi(1_i,\underline{x}) - \phi(0_i,\underline{x})]\frac{1}{2^{n-1}} ,$$

so that

$$\sum_{r=1}^{n} n_r(i)\frac{1}{2^{n-1}} , \tag{4.6}$$

Comparing (4.2) and (4.6), we see that $I_\phi(i)$ attaches weight $\frac{(r-1)!(n-r)!}{n!}$ to the term $n_r(i)$, while Birnbaum's measure attaches the common weight $\frac{1}{2^{n-1}}$ to each of the $n_r(i)$. Since $\frac{(r-1)!(n-r)!}{n!}$ is decreasing in r for $r \leq n/2$ and increasing in r for $r \geq n/2$, we see that $I_\phi(i)$ attaches greatest weight to critical paths which are either very small or very large.

Importance numbers are useful for *ordering* the components. Since the weights in (4.2) and (4.6) differ, it is not surprising that different component orderings may be achieved using $I_\phi(i)$ and $B(i)$, as shown in the following example.

4.2. Example:

Let ϕ be a coherent structure with $n = 6$ components and min cut sets: $\{1,2\},\{1,3\},\{2,3,4\},\{2,3,5\},\{2,3,6\},\{2,4,5\},\{2,4,6\},\{2,5,6\},\{3,4,5\},\{3,4,6\},$ $\{3,5,6\},\{4,5,6\}$.

By inspection we find $n_2^*(1) = 2$ and $n_2^*(2) = 1$. By tabulation, we find the size 3 critical cut sets for component 1: $\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,6\},$ $\{1,3,4\},\{1,3,5\},\{1,3,6\}$. Thus $n_3^*(1) = 7$. By tabulation, we find the size

3 critical cut sets for component 2: {1,2,4},{1,2,5},{1,2,6},{2,3,4},{2,3,5},{2,3,6},{2,4,5},{2,4,6},{2,5,6} . Thus $n_3^*(2) = 9$.

It follows that

$$B(1) = \frac{n_2^*(1) + n_3^*(1)}{2^5} = 9/2^5 ,$$

$$B(2) = \frac{n_2^*(2) + n_3^*(2)}{2^5} = 10/2^5 ,$$

while

$$I_\phi(1) = 2\frac{1!4!}{6!} + 7\frac{2!3!}{6!} = 11/60 ,$$

$$I_\phi(2) = 1\frac{1!4!}{6!} + 9\frac{2!3!}{6!} = 11/60 .$$

Hence $B(2) > B(1)$, while $I_\phi(1) = I_\phi(2)$.

A Characterization of $I_\phi(i)$.

One approach to defining structural importance is to assume all components have the same reliability p , with p having a priori distribution $P(p)$, and then to incorporate Birnbaum's measure of structural importance, yielding as a measure of "expected" structural importance:

$$(4.7) \qquad I(i) = \int_0^1 \frac{\partial h(\underline{p})}{\partial p_i} \Bigg|_{p_1 = \ldots = p_n = p} dP(p) .$$

Note that if $P(p) = p$, then $I(i)$ reduces to our measure, $I_\phi(i)$; alternatively, if $P(p) = 0$ for $0 \leq p < 1/2$, and $P(p) = 1$ for $p \geq 1/2$, then $I(i)$ reduces to the Birnbaum measure, $B(i)$. Next we show that if $I(i)$ given in (4.7) satisfies the desirable normalization property that component importances in series systems sum to 1, then $I(i)$ *must* coincide with our measure $I_\phi(i)$.

4.3. Theorem (Sethuraman):

Let $I(i)$ given in (4.7) satisfy $\sum_{i=1}^{n} I(i) = 1$ for a series system of size n , $n = 1,2, \ldots$. Then $I(i) \equiv I_\phi(i)$.

Proof:

For the series system $h(\underline{p}) = p_1 \ldots p_n$, we have

$$\frac{\partial}{\partial p_i} h(\underline{p}) \bigg|_{p_1=\ldots=p_n=p} = p^{n-1} .$$

It follows that

$$1 = \sum_{i=1}^{n} I(i) = n \int_0^1 p^{n-1} dP(p) \quad \text{for} \quad n = 1,2, \ldots . \tag{4.8}$$

By the solution to the Hamburger moment problem (Shohat and Tamarkin, 1943, p. 19), the distribution $P(p) = p$ uniquely satisfies (4.8), and so $I(i)$ coincides with $I_\phi(i)$.||

5. IMPORTANCE OF MIN CUT SETS.

In the analysis of coherent systems and of fault trees, minimal cut sets play a basic role. For example, a lower bound on the reliability of a coherent system can be computed from a knowledge of the reliability of the min cut structures of the coherent system. See Barlow and Proschan (1965) and Esary and Proschan (1963, 1970). Thus it is of considerable value to determine the relative importance of each cut set in a coherent structure.

5.1. Definition:

The *importance of min cut set* K is the probability that the failure of min cut set K coincides with the failure of the system, i.e., that min cut set K "causes" system failure.

It is immediately clear that the sum of the importances of the min cut sets of a coherent system is at least one. This is a consequence of the fact that system failure may coincide with the failure of more than one min cut structure in the system.

The following theorem gives an explicit representation for the importance of a min cut set K in a coherent system.

5.2. Theorem:

Let F_i be the life distribution of component i, $i = 1, \ldots, n$, in the coherent system with reliability function $h(\underline{p})$. Then $I_h(K)$, the importance of min cut set K, is given by:

$$(5.1) \qquad I_h(K) = \sum_{i \in K} \int_0^\infty h\left(1_i, \underline{0}^{K-\{i\}}, \underline{\bar{F}}(t)\right) \prod_{j \in K-\{i\}} F_j(t)\,dF_i(t) .$$

Proof:

First note that $\prod_{j \in K-\{i\}} F_j(t)\,dF_i(t)$ represents the probability of the joint event that component i fails at time t *and* that the remaining components in the cut set K have failed by time t. Next note that $h(1_i, \underline{0}^{K-\{i\}}, \underline{\bar{F}}(t))$ represents the probability that component i is critical at time t (i.e., the system is functioning at time t if component i is functioning, but is failed otherwise). Thus the product yields the probability that component i causes system failure. Summing over $i \in K$ (corresponding to the mutually exclusive ways in which cut structure K can fail) gives the probability that cut set K causes system failure.||

Structural Importance of Min Cut Sets.

We may define the *structural importance of a min cut set* K by setting $F_1 \equiv F_2 \equiv \cdots \equiv F_n$ in (5.1); this is the procedure we followed in defining the structural importance of a component in Section 4. Equation (5.1) then yields, after a change of variables, the following expression for the structural importance, $I_\phi(K)$, of min cut set K in coherent structure ϕ :

$$(5.2) \qquad I_\phi(K) = \sum_{i \in K} \int_0^1 h\left(1_i, \underline{0}^{K-\{i\}}, p\right)(1-p)^{k-1} dp ,$$

where k denotes the number of components in min cut set K.

An alternative method for computing $I_\phi(K)$ is to list the $n!$ permutations, each of which represents a sequence of the n component failures, to find the number $n(K)$ of sequences in which K causes system failure, and then to compute $I_\phi(K) = n(K)/n!$ This method of computing min cut set structural importance is valid since all sequences of component failures are equally likely.

Examples:

For the coherent system shown in Figure 3, $K_1 = \{1,2\}$ and $K_2 = \{1,3\}$ are the min cut sets. Using (5.2), we calculate $I_\phi(K_1) = 4/6 = I_\phi(K_2)$.

Note that $I_\phi(K_1) + I_\phi(K_2) > 1$; this is a consequence of the fact that K_1 and K_2 may *simultaneously* cause system failure (e.g., for the sequence of component failures 2-3-1, K_1 and K_2 simultaneously fail and cause system failure.

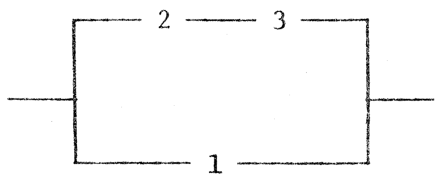

FIGURE 3. COHERENT SYSTEM WITH TWO MIN CUT SETS.

On the other hand, for a k-out-of-n structure we do have $\sum_j I_\phi(K_j) = 1$, even though min cut sets overlap. This is true because two or more min cut sets cannot simultaneously cause system failure. For a k-out-of-n structure, there are $\binom{n}{n-k+1}$ min cut sets. Since all min cut sets have equal structural importance and importances sum to one, we conclude that

$$I_\phi(K_j) = \binom{n}{n-k+1}^{-1} \quad \text{for } j = 1,2, \ldots, \binom{n}{n-k+1} .$$

Fault tree analysts in practice compare different min cut sets to determine the relative importance of basic events and of sets of basic events. It is intuitively reasonable that small min cut sets tend to be structurally more important than large min cut sets. The next theorem shows that such an ordering is true under certain conditions.

5.3. Theorem:

Let min cut set K_i consist of k_i components, $i = 1,2$, with $k_1 < k_2$. Let components in K_1 not appear in any other min cut sets. Then

(5.3) $$I_\phi(K_1) > I_\phi(K_2) .$$

Proof:

Given a sequence of the n component failures for which min cut set K_2 causes system failure, we shall exhibit a corresponding sequence of component failures for which min cut set K_1 causes system failure; moreover, these corresponding sequences will be mutually distinct. This will prove the desired result.

Given a sequence of the n component failures for which min cut set K_2 causes system failure, simply exchange the first k_1 failures of the components of min cut set K_2 by the k_1 failures of the components of min cut set K_1. Since the components in min cut set K_1 do not occur in any other min cut set, for the failure sequence generated, min cut set K_1 will cause system failure. Moreover, it is obvious that failures sequences so generated are mutually distinct since the original failure sequences are mutually distinct.

By the alternative definition of min cut set importance stated following (5.2), we conclude that (5.3) must hold.||

We now show by example that the condition in Theorem 5.3 that no components in K_1 appear in other min cut sets is necessary for the validity of (5.3). Consider a 10 component structure with min cut sets:

$K_1 = \{1, 2, 3, 4\}$	$K_6 = \{5, 7, 8\}$	$K_{11} = \{5, 9, 10\}$	$K_{16} = \{6, 8, 10\}$
$K_2 = \{5, 6, 7\}$	$K_7 = \{5, 7, 9\}$	$K_{12} = \{6, 7, 8\}$	$K_{17} = \{6, 9, 10\}$
$K_3 = \{5, 6, 8\}$	$K_8 = \{5, 7, 10\}$	$K_{13} = \{6, 7, 9\}$	$K_{18} = \{7, 8, 9\}$
$K_4 = \{5, 6, 9\}$	$K_9 = \{5, 8, 9\}$	$K_{14} = \{6, 7, 10\}$	$K_{19} = \{7, 8, 10\}$
$K_5 = \{5, 6, 10\}$	$K_{10} = \{5, 8, 10\}$	$K_{15} = \{6, 8, 9\}$	$K_{20} = \{7, 9, 10\}$
			$K_{21} = \{8, 9, 10\}$

Note that (1) no component of K_1 appears in any other min cut set, and (2) cut sets K_2 to K_{21} are found by taking all combinations of three components from among the remaining six components $5, \ldots, 10$.

From (5.2), we compute $I_{\phi}(K_1) = 4 \int_0^1 \left[\sum_{j=4}^{6} \binom{6}{j} p^j (1 - p)^{6-j} (1 - p)^3 \right] dp$.

Using the formula $\int_0^1 p^{r-1}(1 - p)^{n-r} dp = \frac{(r-1)!(n - r)!}{n!}$, we obtain $I_{\phi}(K_1) = .0714$.

By symmetry, we must have $I_{\phi}(K_2) = \ldots = I_{\phi}(K_{21}) = \frac{1}{20}(1 - .0714) = .0464$ since no two cut sets can fail simultaneously.

It follows that $I_{\phi}(K_1) > I_{\phi}(K_2)$ even though K_1 has more components than does K_2 . The explanation for this result lies in the fact that components in K_2 are highly replicated among the other min cut sets. Examples can also be given in which a three component min cut is structurally more important than a two component min cut.

We can define the importance of a min path set as the probability that it causes system failure, i.e., that its failure coincides with that of the system. We can then obtain results analogous to those above for min cut sets.

6. IMPORTANCE OF COMPONENTS WHEN REPAIR IS PERMITTED.

In this section we consider a coherent system of n stochastically independent components undergoing repair after failure. Let component i repair time be distributed according to continuous distribution G_i, $i = 1, \ldots, n$. Let $A_i(t)$ be the availability of component i at time t, i.e., the probability that component i is functioning at time t. (See Barlow and Proschan, 1973, for a discussion of availability theory.) Let $N_i(t)$ denote the number of failures of component i in $[0,t]$ and $M_i(t) = EN_i(t)$.

Given system failure at time t, the probability that i caused system failure is

$$(6.1) \qquad \frac{[h(1_i,\underline{A}(t)) - h(0_i,\underline{A}(t))]dM_i(t)}{\sum_{j=1}^{n} [h(1_j,\underline{A}(t)) - h(0_j,\underline{A}(t))dM_j(t)} .$$

Note that the numerator represents the probability that i is critical at time t *and* that i then fails at time t, while the denominator represents the probability that *some* component is critical at time t *and* that it fails at time t (i.e., the system fails at time t).

Letting $t \to \infty$, we obtain the stationary probability that the failure of component i is the cause of system failure, given that system failure has occurred:

$$(6.2) \qquad I_h(i) = \frac{[h(1_i,\underline{A}) - h(0_i,\underline{A})]/(\mu_i + \nu_i)}{\sum_{j=1}^{n} [h(1_j,\underline{A}) - h(0_j,\underline{A})]/(\mu_j + \nu_j)} ,$$

where μ_i is the mean life, ν_i the mean repair time of component i, and

$A_i = \lim_{t\to\infty} A_i(t) = \dfrac{\mu_i}{\mu_i + \nu_i}$, the stationary availability of component i,

$i = 1, \ldots, n$. Renewal theory required for the straightforward proofs of (6.1) and (6.2) may be found in Barlow and Proschan (1965), Chapter 3.

We note that, as in the case of no repair, $\sum_{i=1}^{n} I_h(i) = 1$.

Examples.

(1) For a series system with component repair,

(6.3)
$$I_h(i) = \prod_{j \neq i} \mu_j / \sum_{i=1}^{n} \prod_{j \neq i} \mu_j \; .$$

Note that component importance does not depend on component mean repair times.

(2) For a parallel system with component repair,

(6.4)
$$I_h(i) = \prod_{j \neq i} \nu_j / \sum_{i=1}^{n} \prod_{j \neq i} \nu_j \; .$$

Thus component importance does not depend on component mean times to failure.

Remark:

The formula for $I_h(i)$ given by (6.2) can be interpreted in terms of $\tilde{N}_i(t)$, the number of times during $[0,t]$ that component i causes system failure. We claim that

(6.5)
$$E\tilde{N}_i(t) = \int_0^t [h(1_i, \underline{A}(u)) - h(0_i, \underline{A}(u))] dM_i(u) \; .$$

Intuitively, we see that the integrand represents the probability that component i is critical at time u , while $dM_i(u)$ is the probability that component i fails in the small element of time following the instant u . Thus by integrating over $[0,t]$, we are computing the expected number of times during $[0,t]$ that component i causes system failure.

More formally, we may prove (6.5) by forming a partition of $[0,t]$, say $0 = t_0 < t_1 < t_2 < \ldots < t_n = t$. Observe that $[h(1_i, \underline{A}(\xi_j)) - h(0_i, \underline{A}(\xi_j))][M_i(t_{j+1}) -$

$M_i(t_j)]$ is approximately the probability that component i causes system failure in (t_j, t_{j+1}). Summing these probabilities and then letting $n \to \infty$ yields (6.5).

Moreover, we have

$$\lim_{t\to\infty} \frac{E\tilde{N}_i(t)}{\sum_{j=1}^{n} EN_j(t)} = I_h(i) . \tag{6.6}$$

Equation (6.6) follows from an application of the elementary renewal theorem; i.e., $\lim_{t\to\infty} \frac{M_i(t)}{t} = \frac{1}{\mu_i + \nu_i}$, and $A_i \overset{\text{def}}{=} \lim_{t\to\infty} A_i(t) = \frac{\mu_i}{\mu_i + \nu_i}$.

$M_i(t_j)]$ is approximately the probability that component i causes system failure in (t_j, t_{j+1}). Summing these probabilities and then letting $n \to \infty$ yields (6.5).

Moreover, we have

(6.6)
$$\lim_{t\to\infty} \frac{E\tilde{N}_i(t)}{\sum_{j=1}^{n} EN_j(t)} = I_h(i) .$$

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14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Fault trees Coherent structures Component importance Min cuts Competing risk						

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13. ABSTRACT

A new measure of the importance of the components in a coherent system and of the basic events in a fault tree is defined and its properties derived. The importance measure is a useful guide during the system development phase as to which components (or alternatively, which basic events) should receive more urgent attention in achieving system reliability growth. The new measure of component importance has certain desirable properties not possessed by the previous measure of component importance proposed by Birnbaum in "On the Importance of Different Components in a Multi-component System," appearing in MULTIVARIATE ANALYSIS-II, edited by P. R. Krishnaiah, Academic Press, New York (1969).

The measure is extended to minimal cut sets and to systems of components undergoing repair. A number of commonly occurring systems are treated in detail for illustrative purposes.

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