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ScienceDirect

Reliability Engineering and System Safety 94 (2009) 2–21

RELIABILITY ENGINEERING & SYSTEM SAFETY

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A survey of the application of gamma processes in maintenance

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Available online 23 March 2007

Abstract

This article surveys the application of gamma processes in maintenance. Since the introduction of the gamma process in the area of reliability in 1975, it has been increasingly used to model stochastic deterioration for optimising maintenance. Because gamma processes are well suited for modelling the temporal variability of deterioration, they have proven to be useful in determining optimal inspection and maintenance decisions. An overview is given of the rich theoretical aspects as well as the successful maintenance applications of gamma processes. The statistical properties of the gamma process as a probabilistic stress–strength model are given and put in a historic perspective. Furthermore, methods for estimation, approximation, and simulation of gamma processes are reviewed. Finally, an extensive catalogue of inspection and maintenance models under gamma-process deterioration is presented with the emphasis on engineering applications.

Keywords: Gamma process; Deterioration; Maintenance; Renewal theory; Inspection; Markov process; Compound Poisson process; Risk; Brownian motion with drift

1. Introduction

During the last decades, the amount of money spent on maintenance of engineering structures and infrastructures (like roads, railways, bridges, buildings, and industrial plants) has increased continuously. According to Dekker and Scarf [1], this is mainly due to the continuous expansion of the stock of structures and infrastructures, the more severe performance requirements, and the outsourcing of maintenance. In order to lower the costs of maintenance and failure, scientific techniques such as mathematical optimisation models are increasingly applied in the field of maintenance management. For an overview of the applications of maintenance optimisation models, see Dekker [2] and Dekker and Scarf [1].

A characteristic feature of optimising maintenance is that decisions often must be made under uncertainty (such as in deterioration and cost). In maintenance management, the most important uncertainty is generally the uncertainty in the time to failure (lifetime) and/or the rate of deterioration. Up to the early nineties, most mathematical maintenance models were based on describing the uncertainty in ageing using a lifetime distribution. A disadvantage of a lifetime distribution, however, is that it only quantifies whether a component is functioning or not. In order to represent ageing on the basis of lifetime distributions, the celebrated failure rate function can be applied. The failure rate, however, is only useful for making inferences for a large population of components rather than for a single component. As a matter of fact, failure rates cannot be observed or measured for a particular component [3].

For engineering structures and infrastructures it is generally more attractive to base a failure model on the physics of failure and the characteristics of the operating environment [3]. Therefore, it is recommended to model deterioration in terms of a time-dependent stochastic process. In structural engineering, time-dependent functions are advocated for which the coefficients (such as an

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doi:10.1016/j.ress.2007.03.019

average rate of deterioration per unit time) are random quantities (for an overview, see [4]). However, Pandey et al. [5] pointed out that the temporal variability is not taken into account in these random-variable models. For example, in the event of a linear deterioration law with random average deterioration rate, a single inspection already completely reveals the future deterioration evolution. In order to properly model the temporal variability of deterioration, we must rely on other stochastic processes (such as Markov processes). Markov processes include stochastic processes with independent increments like the Brownian motion with drift, the compound Poisson process, and the gamma process. For the stochastic modelling of monotonic and gradual deterioration, the gamma process is most appropriate.

As far as the author knows, the gamma process was first applied by Moran [6–9] in a series of papers and a book published in the fifties of the last century to model water flow into a dam. In 1975, Abdel-Hameed [10] proposed to use the gamma process as a model for deterioration occurring random in time. During the last three decades, gamma processes were satisfactorily fitted to data on creep of concrete [11], fatigue crack growth [12], corroded steel gates [4], thinning due to corrosion [13], and chloride ingress into concrete [14]. Statistical estimation methods that were developed include the maximum-likelihood method and method of moments [11], as well as the Bayesian method with perfect inspection [15] and imperfect inspection [13]. A method for estimating a gamma process by means of expert judgement is proposed by Nicolai et al. [16,17].

On the basis of the gamma deterioration processes, case studies have been performed to determine optimal dike heightenings [18], optimal sand nourishment sizes [19], optimal maintenance decisions for steel coatings [20], and optimal inspection intervals for high-speed railway tracks [21], berm breakwaters [22], steel pressure vessels [13], and automobile brake pads [23], as well as optimal inspection intervals for the block mats and rock dumping of the Eastern-Scheldt barrier in the Netherlands [24,25].

Since the initial paper of Abdel-Hameed [10], the same author used the gamma wear process as a building block for developing mathematical models for optimising time-based maintenance [26] and condition-based maintenance [27,28]. The main characteristics of the latter model are periodic inspection, failure detected only by inspection, random failure level, non-discounted cost, and a single component. Recently, Abdel-Hameed [29] included aperiodic inspection and discounted cost into this model as well. Extensions of Abdel-Hameed's condition-based maintenance model [27,28], but with fixed failure level, include aperiodic inspection [30–32] combined with the possibilities of partial repair [21,33] and a two-component series system [34]. Variations of Abdel-Hameed's condition-based maintenance model [27,28] in which failures are detected immediately were proposed by Park [35] (with fixed failure level) and Kong and Park [36] (with random failure level). Extensions to Park's model with fixed failure level [35,37] include distinction between first and subsequent inspection intervals [38], discounted cost [25,39], imperfect inspection [40–42], aperiodic inspection [39, 43,44], two failure modes [45], and two deterioration modes [46]. An extension to Kong and Park's model with random failure level [36] includes imperfect inspection [13]. In addition, we mention an aperiodic inspection model to guarantee a required availability [47] and models for continuous monitoring with perfect repair [30,48–52] and partial repair [53,54].

The outline of this article is as follows. Section 2 treats the possibilities of modelling stochastic deterioration and explains why the gamma process is the most appropriate candidate. Section 3 gives an overview of the most important statistical and mathematical properties of the gamma process. Section 4 surveys the applications of gamma processes in optimising time-based and condition-based maintenance. Conclusions are formulated in Section 5.

2. Stochastic deterioration processes

For modelling stochastic deterioration, we can use either a failure rate function or a stochastic process such as a random deterioration rate, Markov process, Brownian motion with drift, and non-decreasing jump process (of which the gamma process is a special case).

2.1. Failure rate function

A lifetime distribution represents the uncertainty in the time to failure of a component or structure. Let the lifetime have a cumulative probability distribution $F(t)$ with probability density function $f(t)$, then the failure rate (or hazard rate) function is defined as

$$r(t) = \frac{f(t)}{1 - F(t)} = \frac{f(t)}{\bar{F}(t)}, \quad t > 0 \text{ [55, Chapter 2]}. \tag{1}$$

A probabilistic interpretation of the failure rate function is that $r(t)\,\mathrm{d}t$ represents the probability that a component of age t will fail in the time interval $[t, t + \mathrm{d}t]$. For deteriorating components or structures, the failure rate is increasing. By integrating both sides of Eq. (1) and then exponentiating, we obtain the survival function

$$\bar{F}(t) = \exp\left\{ -\int_0^t r(\tau)\,\mathrm{d}\tau \right\}. \tag{2}$$

Lifetime distributions and failure rate functions are especially useful in mechanical and electrical engineering. In these fields, one often considers equipment which can assume at most two states: the functioning state and the failed state. A structure, on the other hand, can be in a range of states depending on its degrading condition.

A serious disadvantage of failure rates is that they cannot be observed or measured for a particular component [3]. Due to the usual lack of failure data, a reliability approach solely based on lifetime distributions and their unobservable failure rates is unsatisfactory. According to Singpurwalla [3], "a more appealing approach would be to choose a model based on the physics of failure and the characteristics of the operating environment". Dynamic environments—such as the applied stresses and loads—influence failure and vary over time. Therefore, it is recommended to model deterioration in terms of a time-dependent stochastic process $\{X(t), t \geqslant 0\}$ where $X(t)$ is a random quantity for all $t \geqslant 0$.

2.2. Random deterioration rate

The most simple stochastic process is defined as a time-dependent function for which the average rate of deterioration per unit time is a random quantity. An example of this type of stochastic processes is the cumulative amount of deterioration at time t being defined as $X(t) = At$ for all $t \geqslant 0$, where the average deterioration rate A has a probability distribution. Reliability models on the basis of a random deterioration rate have been developed by, amongst others, Ellingwood and Mori [56] and Mori and Ellingwood [57,58]. For an overview of random deterioration-rate models, see Frangopol et al. [4]. However, according to Pandey and van Noortwijk [5], the sample paths of such models are straight lines and a single inspection thus fixes the future deterioration beforehand. Although the random deterioration-rate model can be used as an approximation, one should be careful as soon as inspections are involved. For the purpose of inspection and maintenance modelling, we therefore have to rely on other stochastic processes (such as Markov processes with gamma processes as special case) which properly capture the temporal variability associated with evolution of deterioration. Pandey et al. [5] investigated the differences between the random deterioration-rate model and the gamma-process model by comparing the influences of the different assumptions on probability distributions of the time to failure, optimal time-based maintenance policies, and optimal condition-based maintenance policies.

2.3. Markov processes

According to Barlow and Proschan [55, Chapter 5], deterioration is usually assumed to be a Markov process. Roughly speaking, a Markov process is a stochastic process with the property that, given the value of $X(t)$, the values of $X(\tau)$, where $\tau > t$, are independent of the values of $X(u)$, $u < t$. That is, the conditional distribution of the future $X(\tau)$, given the present $X(t)$ and the past $X(u)$, $u < t$, is independent of the past. Classes of Markov processes which are useful for modelling stochastic deterioration are discrete-time Markov processes having a finite or countable state space (called Markov chains) and continuous-time Markov processes with independent increments such as the Brownian motion with drift (also called the Gaussian or Wiener process), the compound Poisson process, and the gamma process. The assumption of independent increments is more restrictive than the Markov property. Because the increment $X(\tau) - X(t)$ is independent of $X(t)$ and $X(\tau) = X(t) + [X(\tau) - X(t)]$, the stochastic process $\{X(t), t \geqslant 0\}$ is Markovian. Hence, damage accumulation with independent increments naturally leads to the Markov property.

The Brownian motion with drift is a stochastic process $\{X(t), t \geqslant 0\}$ with independent, real-valued increments and decrements having a normal distribution with mean μt and variance $\sigma^2 t$ for all $t \geqslant 0$ [59, Chapter 7]. The compound Poisson process is a stochastic process with independent and identically distributed jumps which occur according to a Poisson process [60, Chapter 2]. A gamma process is a stochastic process with independent, non-negative increments having a gamma distribution with an identical scale parameter. Like the compound Poisson process, the gamma process is a jump process. According to Singpurwalla and Wilson [61], the main difference between these two jump processes is as follows. Compound Poisson processes have a finite number of jumps in finite time intervals, whereas gamma processes have an infinite number of jumps in finite time intervals. The former are suitable for modelling usage such as damage due to sporadic shocks and the latter are suitable for describing gradual damage by continuous use.

Roughly speaking, continuous-time stochastic processes with independent and stationary increments and with right-continuous sample paths having left limits are Lévy processes. A stochastic process has stationary increments if the probability distribution of the increments $X(t+h) - X(t)$ depends only on h for all $t, h \geqslant 0$. Every Lévy process may be written as a sum of a Brownian motion, a deterministic part (linear in time), and an integral of compound Poisson processes, where all the contributing processes are mutually independent. The sample paths of a Lévy process are discontinuous with probability one if the process is monotone, because such a process can be decomposed into a linear part plus an integral of compound Poisson processes [62]. The sample paths of a Brownian motion are continuous with probability one. For the mathematical aspects of stochastic processes in general, we refer the reader to Itô and McKean [63], de Finetti [64, Chapter 8], and Karlin and Taylor [65, Chapter 16]. For the mathematical aspects of gamma processes in particular, see Ferguson and Klass [62], Dufresne et al. [15], Singpurwalla [66], and van der Weide [67].

2.4. Brownian motion with drift

The Brownian motion with drift is a continuous-time stochastic process $\{X(t), t \geqslant 0\}$ with drift parameter μ and variance parameter σ^2, $\sigma > 0$, having the following properties: $X(t)$ is normally distributed with mean μt and variance $\sigma^2 t$ for all $t \geqslant 0$, $X(t)$ has independent increments, $X(0) = 0$ with probability one, and $X(t)$ is continuous at $t = 0$ [59, Chapter 7]. This stochastic process has been used for modelling the exchange value of shares and the movement of small particles in fluids and air. A characteristic feature of this process—in the context of structural reliability—is that a structure's resistance alternately increases and decreases, similar to the exchange value of a share. For this reason, the Brownian motion is inadequate in modelling deterioration which is monotone. For example, a dike of which the height is subject to a Brownian deterioration can, according to the model, spontaneously rise up, which of course cannot occur in practice.

Although the Brownian motion with drift is inadequate in modelling monotonically increasing ageing, it has certain mathematical advantages. For example, explicit expressions are available when stress and strength are assumed to be independent Brownian motions with drift. For this stress–strength model, Basu and Ebrahimi [68] derived the probability distribution of the first-passage time to zero of strength minus stress. Maximum-likelihood and Bayesian estimation of the parameters of the Brownian stress–strength model was studied by Ebrahimi and Ramallingam [69] and Basu and Lingham [70], respectively.

Doksum and Hóyland [71] applied the Brownian motion with drift to a variable-stress accelerated life testing experiment. Their fatigue failure model corresponds to making a transformation of a non-stationary Wiener process to a stationary Wiener process. This is done by performing a monotonic transformation from the (accelerated) clock or calendar time to the transformed or operational time. In engineering, this transformation is often a power-time transformation, where the power is a known constant based on the physical properties of the system. Doksum and Hóyland [71] present the probability distribution of the first time at which the degradation process crosses a critical boundary. This is an inverse Gaussian distribution, which can be time transformed as well. The time-scale transformation was successfully applied by Doksum and Normand [72] in a biomarker process (a biomarker is a human-health indicator representing the immunological progression of a disease such as HIV infection) and by Whitmore and Schenkelberg [73] in a degradation model of self-regulating heating cables.

Whitmore [74] extended the Wiener degradation process with the possibility of imperfect inspections. The measurement errors are assumed to be independent, identically distributed, random quantities having normal distributions and they are independent of the degradation process. Another interesting extension is the bivariate Wiener process of Whitmore et al. [75] in which the processes of the degradation and a marker (such as a covariate in medical applications) are combined.

2.5. *Monotonically increasing jump processes*

According to the overview paper of Singpurwalla [3] on stochastic deterioration processes, Mercer [76,77] may have been the first to describe deterioration by a stochastic process. Mercer and Smith [76] suggested to model the deterioration of a conveyor belt as a one-dimensional random walk process in which the deterioration steps occur randomly in time at a constant mean rate with step sizes being non-negative, independent, and identically distributed. They define failure as the event in which the deterioration reaches such a level that the belt is considered to be worn out completely. The probability distribution of the time taken to reach this level is derived.

Esary et al. [78] study lifetime distributions of devices subjected to a sequence of shocks occurring randomly in time as events in a homogeneous Poisson process. For example, they derived the probability that damage resulting from shocks accumulates until exceedance of a threshold results in failure. In addition to models for wear that accumulates in discrete amounts at isolated points in time, they regard models for continuous wear as a limit of discrete shock models. The latter models represent stochastic processes with increments which are non-negative, stationary, and independent (such as the compound Poisson processes and gamma processes). Random failure thresholds are considered as well. Abdel-Hameed and Proschan [79] extend the stationary shock models of Esary et al. [78] to non-stationary shock models by considering shocks governed by a non-homogeneous Poisson process rather than a homogeneous Poisson process. Sobczyk [80] and Sobczyk and Spencer [81] use compound Poisson processes, amongst others, for modelling random fatigue.

Abdel-Hameed [82] derives life distribution properties (such as failure rate properties) of devices subject to a non-decreasing pure jump damage process. Shaked and Shanthikumar [83] study the probabilistic properties of the time it takes for a non-decreasing pure jump process, which starts at zero, to cross a non-negative threshold (called the first-passage time). A special case of a pure jump process is the one-sided Lévy process (having independent and stationary increments), which was studied by Abdel-Hameed [84]. Both papers study the age-replacement policy. Zuckerman [52] also regarded deterioration as a one-sided Lévy process, but he defined the control-limit level at which preventive replacement is performed as a decision variable rather than the age-replacement interval. This paper extends the replacement model of Zuckerman [85] in which the deterioration is described as a compound Poisson process. A condition-based maintenance model with failure discovered only by periodic inspection and deterioration described as a compound Poisson process can be found in Zuckerman [86].

3. Gamma processes

The purpose of this section is to review the most important mathematical properties of the gamma process as a probabilistic stress–strength model. It includes, amongst others, the history and definition of the gamma process, as well as methods for parameter estimation and simulation.

3.1. Historical background

In order for the stochastic deterioration process to be monotonic, we can best consider it as a gamma process [25,87]. In words, a gamma process is a stochastic process with independent, non-negative increments (e.g., the increments of crest-level decline of a dike) having a gamma distribution with an identical scale parameter. In the case of a gamma deterioration, dikes can only decrease in height due to crest-level decline. In particular, it has been shown that a gamma deterioration process follows directly from the minimal hypothesis of l_1-isotropy; that is, if the order in which non-negative deterioration increments occur is irrelevant (because we are only interested in their sum), the increments are exchangeable and stationary [87]. Note that independence is a special case of exchangeability.

As far as the author knows, Abdel-Hameed [10] was the first to propose the gamma process as a proper model for deterioration occurring random in time. In his two-page paper he called this stochastic process the "gamma wear process". An advantage of modelling deterioration processes through gamma processes is that the required mathematical calculations are relatively straightforward.

The gamma process is suitable to model gradual damage monotonically accumulating over time in a sequence of tiny increments, such as wear, fatigue, corrosion, crack growth, erosion, consumption, creep, swell, degrading health index, etc. Other examples of the application of gamma processes are the theory of water storage by dams [6–9] and the theory of risk of ruin due to aggregate insurance claims [15]. Damage accumulation is the unifying property which connects the models of dam storage, ruin risk, and deteriorating systems.

3.2. Definition of non-stationary gamma process

In mathematical terms, the gamma process is defined as follows. Recall that a random quantity X has a gamma distribution with shape parameter $v>0$ and scale parameter $u>0$ if its probability density function is given by

$$\mathrm{Ga}(x|v,u) = \frac{u^v}{\Gamma(v)} x^{v-1} \exp\{-ux\} I_{(0,\infty)}(x),$$

where $I_A(x)=1$ for $x\in A$ and $I_A(x)=0$ for $x\notin A$, and $\Gamma(a)=\int_{z=0}^{\infty} z^{a-1}\mathrm{e}^{-z}\,\mathrm{d}z$ is the gamma function for $a>0$. Furthermore, let $v(t)$ be a non-decreasing, right-continuous, real-valued function for $t\geqslant 0$, with $v(0)\equiv 0$. The gamma process with shape function $v(t)>0$ and scale parameter $u>0$ is a continuous-time stochastic process $\{X(t), t\geqslant 0\}$ with the following properties:

(1) $X(0)=0$ with probability one;
(2) $X(\tau)-X(t)\sim \mathrm{Ga}(v(\tau)-v(t),u)$ for all $\tau>t\geqslant 0$;
(3) $X(t)$ has independent increments.

Most of the cited publications in this survey deal with stationary gamma processes. The following references consider non-stationary gamma processes: Abdel-Hameed [10], Bagdonavičius and Nikulin [88], Bakker et al. [89], Bakker and van Noortwijk [14], Çinlar [90,91], Çinlar et al. [11], Heutink et al. [20], Kallen and van Noortwijk [45], Lawless and Crowder [12], Nicolai and Frenk [54], Nicolai et al. [16,17], Singpurwalla [66], Singpurwalla and Wilson [61], van der Weide [67], van Noortwijk and Frangopol [92], van Noortwijk and Klatter [24], and Wang et al. [93].

3.3. Mean and variance of gamma process

Let $X(t)$ denote the deterioration at time t, $t\geqslant 0$, and let the probability density function of $X(t)$, in accordance with the definition of the gamma process, be given by

$$f_{X(t)}(x) = \mathrm{Ga}(x|v(t),u) \tag{3}$$

with expectation and variance

$$E(X(t)) = \frac{v(t)}{u}, \quad \mathrm{Var}(X(t)) = \frac{v(t)}{u^2}. \tag{4}$$

The coefficient of variation is defined by the ratio of the standard deviation and the mean; that is,

$$\mathrm{CV}(X(t)) = \frac{\sqrt{\mathrm{Var}(X(t))}}{E(X(t))} = \frac{1}{\sqrt{v(t)}} \tag{5}$$

which decreases as time increases. On the other hand, the ratio of the variance and the mean equals $1/u$ and therefore does not depend on time.

3.4. Cumulative distribution function of time to failure

A component is said to fail when its deteriorating resistance, denoted by $R(t)=r_0-X(t)$, drops below the stress s. We assume both the initial resistance r_0 and the stress s to be fixed (known). Define $y=r_0-s$ and let the time at which failure occurs be denoted by the lifetime T_y (also called the first hitting time of level y). Due to the gamma-distributed deterioration in Eq. (3), the lifetime distribution can then be written as

$$\begin{aligned} F(t) &= \Pr\{T_y \leqslant t\} = \Pr\{X(t)\geqslant y\} \\ &= \int_{x=y}^{\infty} f_{X(t)}(x)\,\mathrm{d}x = \frac{\Gamma(v(t),yu)}{\Gamma(v(t))}, \end{aligned} \tag{6}$$

where $\Gamma(a,x)=\int_{z=x}^{\infty} z^{a-1}\mathrm{e}^{-z}\,\mathrm{d}z$ is the incomplete gamma function for $x\geqslant 0$ and $a>0$. Eq. (6) resembles a striking duality between space (deterioration) and time (lifetime) that makes the gamma-process model amenable to analysis.

3.5. Probability density function of time to failure

Using the chain rule for differentiation, the probability density function of the lifetime is

$$f(t) = \frac{\partial}{\partial t}\left[\frac{\Gamma(v(t),yu)}{\Gamma(v(t))}\right] = \frac{\partial}{\partial \tilde{v}}\left[\frac{\Gamma(\tilde{v},yu)}{\Gamma(\tilde{v})}\right]\Bigg|_{\tilde{v}=v(t)} v'(t) \tag{7}$$

under the assumption that the shape function $v(t)$ is differentiable. Hence,

$$f(t) = \frac{v'(t)}{\Gamma(v(t))} \int_{yu}^{\infty} \{\log(z) - \psi(v(t))\} z^{v(t)-1} \mathrm{e}^{-z} \, \mathrm{d}z, \tag{8}$$

where the function $\psi(a)$ is the derivative of the logarithm of the gamma function:

$$\psi(a) = \frac{\Gamma'(a)}{\Gamma(a)} = \frac{\partial \log \Gamma(a)}{\partial a}$$

for $a > 0$. The function $\psi(a)$ is called the digamma function and can be accurately computed using the algorithm developed by Bernardo [94]. Using a series expansion and a continued fraction expansion, the first partial derivative of $\Gamma(\tilde{v}, yu)/\Gamma(\tilde{v})$ with respect to $\tilde{v}$ in Eq. (7) can be calculated by the algorithm of Moore [95]. Other derivations are given by Newby and Dagg [37] in terms of a Meijer's G-function and by Park and Padgett [96] in terms of a generalised hypergeometric series. These expressions can be derived using the Maple software package. For the definitions of Meijer's G-function and the generalised hypergeometric series, see Erdélyi et al. [97].

In a similar manner as in Liao et al. [53], it can be proven that the lifetime distribution based on a gamma-process deterioration given by Eq. (7) has an increasing failure rate as long as $v'(t) > 0$.

3.6. Gamma wear process combined with random load

In his proposal to use the gamma process as a model for stochastic deterioration, Abdel-Hameed [10] even took the stress–strength model in Eq. (6) a stage further by regarding the deterioration failure level $y = r_0 - s$ as a random quantity $Y > 0$ as well. For each $t \geqslant 0$, the probability of failure in time interval $(0, t]$ can then be written as the convolution integral

$$\begin{aligned}\Pr\{X(t) \geqslant Y\} &= \int_{x=0}^{\infty} f_{X(t)}(x) \Pr\{Y \leqslant x\} \, \mathrm{d}x \\ &= \int_{x=0}^{\infty} \int_{y=0}^{x} f_{X(t)}(x) f_Y(y) \, \mathrm{d}y \, \mathrm{d}x,\end{aligned} \tag{9}$$

where $X(t)$ has a gamma distribution with shape parameter $v(t)$ and scale parameter u, and Y has probability density function $f_Y(y)$ and cumulative distribution function $G(y) = \Pr\{Y \leqslant y\}$. The probability $\Pr\{Y > x\} = 1 - G(x) = \bar{G}(x)$ can be interpreted as the probability that a device survives x units of deterioration. Abdel-Hameed [10] showed that the cumulative distribution function (9) has an increasing failure rate if $v(t)$ is convex and $G(x)$ has an increasing failure rate.

3.7. Parameter estimation for the gamma process

Under the assumption of modelling the temporal variability in the deterioration with a gamma process, the question which remains to be answered is how its expected deterioration increases over time. Empirical studies show that the expected deterioration at time t is often proportional to a power law:

$$E(X(t)) = \frac{v(t)}{u} = \frac{ct^b}{u} = at^b \propto t^b \tag{10}$$

for some physical constants $a > 0$ (or $c > 0$) and $b > 0$. With some effort, there is often engineering knowledge available about the shape of the expected deterioration in terms of the parameter b in Eq. (10). Some examples of expected deterioration according to a power law are the expected degradation of concrete due to corrosion of reinforcement (linear: $b = 1$; [56]), sulphate attack (parabolic: $b = 2$; [56]), diffusion-controlled ageing (square root: $b = 0.5$; [56]), creep ($b = \frac{1}{8}$; [11]), and the expected scour-hole depth ($b = 0.4$; [98,24]). The gamma process is called stationary if the expected deterioration is linear in time, i.e., when $b = 1$ in Eq. (10), and non-stationary when $b \neq 1$.

Çinlar et al. [11] use a non-stationary gamma process as a model for the deterioration of concrete due to creep. They give a comprehensive justification of the gamma process from a physical and practical point of view. In doing so, the expected deterioration agrees with a deterministic creep law in the form of a power law in time. Çinlar et al. [11] show how a non-stationary gamma process can be transformed into a stationary gamma process and how the parameters of a stationary gamma process can be estimated using the method of moments and the method of maximum likelihood. They performed a statistical analysis of data on concrete creep.

In order to apply the gamma-process model to practical examples, statistical methods for the parameter estimation of gamma processes are required. A typical data set consists of inspection times t_i, $i = 1, \ldots, n$, where $0 = t_0 < t_1 < t_2 < \cdots < t_n$, and corresponding observations of the cumulative amounts of deterioration x_i, $i = 1, \ldots, n$, where $0 = x_0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$. Consider a gamma process with shape function $v(t) = ct^b$ and scale parameter u. We assume that the value of the power b is known, but c and u are unknown. The two most common methods of parameter estimation, namely, maximum likelihood and method of moments, are discussed in Sections 3.7.1 and 3.7.2. Both methods for deriving the estimators of c and u were initially presented by Çinlar et al. [11]. To account for statistical uncertainties, Dufresne et al. [15] propose to use a conjugate Bayesian analysis in which the scale parameter of the gamma process is assumed to have an inverted gamma distribution as prior. The Bayesian estimation method is presented in Section 3.7.3. Finally, a method for estimating a gamma process by means of expert judgement is discussed in Section 3.7.4.

3.7.1. Method of maximum likelihood

The maximum-likelihood estimators of c and u can be obtained by maximising the logarithm of the likelihood function of the increments. The likelihood function of the observed deterioration increments $\delta_i = x_i - x_{i-1}$,

$i = 1, \ldots, n$, is a product of independent gamma densities

$$\ell(\delta_1, \ldots, \delta_n | c, u) = \prod_{i=1}^{n} f_{X(t_i)-X(t_{i-1})}(\delta_i) = \prod_{i=1}^{n} \frac{u^{c[t_i^b - t_{i-1}^b]}}{\Gamma(c[t_i^b - t_{i-1}^b])} \delta_i^{c[t_i^b - t_{i-1}^b]-1} \exp\{-u\delta_i\}. \quad (11)$$

By computing the first partial derivatives of the loglikelihood function of the increments with respect to c and u, Çinlar et al. [11] show that the maximum-likelihood estimates $\hat{c}$ and $\hat{u}$ can be solved from

$$\hat{u} = \frac{\hat{c} t_n^b}{x_n}, \quad \sum_{i=1}^{n} [t_i^b - t_{i-1}^b]\{\psi(\hat{c}[t_i^b - t_{i-1}^b]) - \log \delta_i\} = t_n^b \log\left(\frac{\hat{c} t_n^b}{x_n}\right). \quad (12)$$

Given the maximum-likelihood estimator of u in Eq. (12), the expected deterioration at time t can be written as

$$E(X(t)) = x_n \left[\frac{t}{t_n}\right]^b.$$

Because cumulative amounts of deterioration are measured, the last inspection contains the most information. This is confirmed by the fact that the expected deterioration at the last inspection (at time t_n) equals x_n; that is, $E(X(t_n)) = x_n$.

The maximum-likelihood method to estimate the parameters c and u can be extended to estimate the power-law parameter b as well. The parameter b then must be determined by numerically maximising the likelihood function (11). Note that this likelihood function can be extended to include the increments of more than one component as well. Nicolai et al. [17] successfully applied this approach to fit a gamma process to inspection data of the Dutch Haringvliet storm-surge barrier representing percentages of steel-gate surfaces that have been corroded due to ageing of the coating.

In an accelerated crack-growth test, Park and Padgett [96] also included the likelihood function of the time to failure (7) in the likelihood function of (stationary) gamma-process increments (11). Park and Padgett [99] extended this accelerated test model to the case with more than one accelerating variable.

3.7.2. *Method of moments*

Recall that the expected value and variance of the accumulated deterioration at calendar time t are given by

$$E(X(t)) = \frac{ct^b}{u}, \quad \mathrm{Var}(X(t)) = \frac{ct^b}{u^2}. \quad (13)$$

When the power b is known, the non-stationary gamma process can be easily transformed to a stationary gamma process by performing a monotonic transformation from the clock or calendar time t to the transformed or operational time $z(t) = t^b$. Recall that a stochastic process has stationary increments if the probability distribution of the increments $X(t+h) - X(t)$ depends only on h for all $t, h \geqslant 0$. Substituting the inverse time transformation $t(z) = z^{1/b}$ in Eq. (13) yields

$$E(X(t(z))) = \frac{cz}{u}, \quad \mathrm{Var}(X(t(z))) = \frac{cz}{u^2}. \quad (14)$$

This results in a stationary gamma process with respect to the transformed time z. Similarly, the transformed inspection times are $z_i = t_i^b$, $i = 1, \ldots, n$. Let us further define the transformed times between inspections as $w_i = t_i^b - t_{i-1}^b$ and, for mathematical convenience, $D_i = X_i - X_{i-1}$ for $i = 1, \ldots, n$. The deterioration increment D_i has a gamma distribution with shape parameter cw_i and scale parameter u for all $i = 1, \ldots, n$, and the increments $D_1, \ldots, D_n$ are independent. Note that X_i and D_i denote random quantities and x_i and δ_i the corresponding observations. According to Çinlar et al. [11], the method-of-moments estimates $\hat{c}$ and $\hat{u}$ can be solved from

$$\frac{\hat{c}}{\hat{u}} = \frac{\sum_{i=1}^{n} \delta_i}{\sum_{i=1}^{n} w_i} = \frac{x_n}{t_n^b} = \bar{\delta},$$

$$\frac{x_n}{\hat{u}} \left(1 - \frac{\sum_{i=1}^{n} w_i^2}{[\sum_{i=1}^{n} w_i]^2}\right) = \sum_{i=1}^{n} (\delta_i - \bar{\delta} w_i)^2. \quad (15)$$

Clearly, the method of moments leads to simple formulae for parameter estimation which can be easily computed. Note that the first equation in the maximum-likelihood estimation (12) is the same as the first equation in the method-of-moments estimation (15).

3.7.3. *Method of Bayesian statistics*

The theorem of Bayes [100] provides a solution to the problem of how to learn from data. In the framework of estimating the unknown parameters c and u of the time-transformed gamma process, the Bayesian approach assumes these parameters to have a probability distribution. Bayes' theorem can then be written as

$$\pi(c, u | \delta_1, \ldots, \delta_n) = \frac{\ell(\delta_1, \ldots, \delta_n | c, u) \pi(c, u)}{\int_0^\infty \int_0^\infty \ell(\delta_1, \ldots, \delta_n | c, u) \pi(c, u) \, \mathrm{d}c \, \mathrm{d}u}, \quad (16)$$

where $\ell(\delta_1, \ldots, \delta_n | c, u)$ is the likelihood function of the inspection data $\delta_1, \ldots, \delta_n$ when the parametric vector (c, u) is given, $\pi(c, u)$ is the prior density of (c, u) before observing the inspection data, $\pi(c, u | \delta_1, \ldots, \delta_n)$ is the posterior density of (c, u) after observing the inspection data, and $\pi(\delta_1, \ldots, \delta_n)$ is the marginal density of the inspection data. Using Bayes' theorem, we can update the prior distribution to the posterior distribution as soon as new inspection data become available. In this respect, the question arises as what type of probability distribution should be chosen as a prior of c and u. Let us first focus on the prior distribution of the scale parameter u when the parameter c is given; that is, consider the prior density $\pi(u|c)$. It is well known that the family of gamma distributions is a conjugate family with respect to the gamma likelihood function with unknown scale parameter, as both prior and posterior distributions belong to the family of gamma distributions [101, Chapter 9]. If the prior distribution of the scale parameter u is given by a gamma distribution with shape parameter α and scale parameter β when the value of c is given, then the posterior distribution is also a gamma distribution with shape parameter $\alpha + \sum_{i=1}^{n} c[t_i^b - t_{i-1}^b] = \alpha + ct_n^b$ and scale parameter $\beta + \sum_{i=1}^{n}\delta_i = \beta + x_n$. Dufresne et al. [15] applied this Bayesian approach to determine the posterior distribution of the scale parameter of a stationary gamma process (i.e., $b = 1$) under perfect inspection. Kallen and van Noortwijk [13] extended this Bayesian estimation from perfect to imperfect inspection.

Bayesian estimation of the scale parameter of the gamma process can be extended to Bayesian estimation of both the scale parameter and shape function. Recall that $w_i = t_i^b - t_{i-1}^b$, $i = 1, \ldots, n$. In combination with the prior density of $\pi(c)$, Bayes' theorem (16) can be rewritten as

$$\begin{aligned}
\pi(c, u|\delta_1, \ldots, \delta_n) &= \pi(u|c, \delta_1, \ldots, \delta_n)\pi(c|\delta_1, \ldots, \delta_n) \\
&\propto \prod_{i=1}^{n} \frac{u^{cw_i}}{\Gamma(cw_i)} \delta_i^{cw_i - 1} \exp\{-u\delta_i\} \\
&\quad \times \frac{\beta^{\alpha}}{\Gamma(\alpha)} u^{\alpha-1} \exp\{-u\beta\}\pi(c) \\
&= \mathrm{Ga}(u|\alpha + ct_n^b, \beta + x_n) \\
&\quad \times \left[\frac{1}{\beta + x_n}\right]^{\alpha + ct_n^b} \frac{\beta^{\alpha}}{\Gamma(\alpha)} \frac{\Gamma(\alpha + ct_n^b)}{\prod_{i=1}^{n}\Gamma(cw_i)} \\
&\quad \times \prod_{i=1}^{n} \delta_i^{cw_i - 1}\pi(c).
\end{aligned}$$

When the parameter c is unknown, the parameters of the prior density of u can depend on c; that is, the prior density of u given c is a gamma distribution with shape parameter $\alpha(c)$ and scale parameter $\beta(c)$. An interesting choice for the shape and scale parameter is $\alpha(c) = c\tau^b$, $\tau > 0$, and $\beta(c) = \beta$. Under this assumption, the posterior mean of the scale parameter of the gamma process u when the value of c is given can be written as

$$E(U|c, \delta_1, \ldots, \delta_n) = \frac{\alpha(c) + ct_n^b}{\beta(c) + x_n} = \frac{c[\tau^b + t_n^b]}{\beta + x_n},$$

where c is known and U is unknown (and therefore treated as a random quantity). Using Eq. (10), the predictive mean of the cumulative amount of deterioration at time t has the form

$$E\left(\frac{Ct^b}{U}\middle|\delta_1, \ldots, \delta_n\right) = E\left(\frac{C[\beta(C) + x_n]t^b}{\alpha(C) + Ct_n^b - 1}\middle|\delta_1, \ldots, \delta_n\right),$$

where C and U are random quantities.

3.7.4. Method of expert judgement

Nicolai et al. [16,17] fitted a gamma process on deterioration data of coating systems protecting steel structures against corrosion. Due to the lack of inspection data, they had to estimate a non-stationary gamma process with expected power-law deterioration according to Eq. (10) on the basis of expert judgement. The expert data were given by percentiles of the times at which three different deterioration levels were exceeded. The fit was determined by minimising the sums of squared differences of the fit and the expert data. The authors also compared the gamma-process fit with a Brownian motion fit. Although the gamma process is preferable from the physics point of view (monotonic deterioration), the Brownian motion with drift appeared to be less computationally expensive.

3.8. Definition of stationary gamma process

When the expected deterioration is linear over time, it is convenient to rewrite the probability density function of $X(t)$ in Eq. (3) using the following reparameterisation:

$$f_{X(t)}(x) = \mathrm{Ga}(x|[\mu^2 t]/\sigma^2, \mu/\sigma^2) \tag{17}$$

for $\mu, \sigma > 0$ with

$$E(X(t)) = \mu t, \quad \mathrm{Var}(X(t)) = \sigma^2 t.$$

A gamma process with $\mu = \sigma = 1$ is called a standardised gamma process. Due to the stationarity of the above stochastic deterioration process, both the mean value and the variance of the deterioration are linear in time. By using Eq. (6), the lifetime distribution can then be rewritten as

$$\begin{aligned}
F(t) &= \Pr\{T_y \leqslant t\} = \Pr\{X(t) \geqslant y\} \\
&= \int_{x=y}^{\infty} f_{X(t)}(x)\,\mathrm{d}x = \frac{\Gamma([\mu^2 t]/\sigma^2, [y\mu]/\sigma^2)}{\Gamma([\mu^2 t]/\sigma^2)}.
\end{aligned} \tag{18}$$

The stationarity of the gamma process basically follows from the property that increments are independent and have the same type of distribution as their sum. In mathematical terms, this property is covered by the so-called infinite divisibility. A random variable X is infinitely divisible if for any integer $n \geqslant 2$, there are n independent and identically distributed random variables $D_1^{(n)}, \ldots, D_n^{(n)}$ such that their sum $\sum_{i=1}^{n} D_i^{(n)}$ has the same distribution as X. In terms of Laplace transforms, the definition of infinite divisibility can be formulated as

$$E(\mathrm{e}^{-sX}) = \prod_{i=1}^{n} E(\mathrm{e}^{-sD_i^{(n)}}), \quad n \geqslant 2.$$

For computing the probability density function of the time to failure for a stationary gamma process, Park and Padgett [96] propose to approximate the cumulative distribution function (18) with the Birnbaum–Saunders distribution [102]:

$$\begin{aligned}
F(t) &= \Pr\{X(t) \geqslant y\} \approx \Phi\left(\frac{\mu t - y}{\sigma\sqrt{t}}\right) \\
&= \Phi\left(\sqrt{\frac{y\mu}{\sigma^2}}\left[\sqrt{\frac{\mu t}{y}} - \sqrt{\frac{y}{\mu t}}\right]\right),
\end{aligned}$$

where Φ is the cumulative distribution function of the standard normal distribution. This approximation can be applied when $\mu \geqslant \sigma$.

3.9. Discrete-time approximation of stationary gamma process

A useful property of the gamma process with stationary increments is that the gamma density in Eq. (17) transforms into an exponential density if $t = (\sigma/\mu)^2$. When the unit-time length is chosen to be $(\sigma/\mu)^2$, the increments of deterioration are exponentially distributed with mean σ^2/μ. For this unit time, many probabilistic properties can be expressed in analytic form. For example, the probability of failure in unit time i reduces to a shifted Poisson distribution with mean $1 + [y\mu]/\sigma^2$ [87]:

$$q_i = \frac{1}{(i-1)!}\left[\frac{y\mu}{\sigma^2}\right]^{i-1} \exp\left\{-\frac{y\mu}{\sigma^2}\right\}, \quad i = 1,2,3,\ldots . \tag{19}$$

This unit time facilitates the algebraic manipulations considerably and, moreover, often results in a very good approximation of the optimal decision. A physical explanation for the appearance of the Poisson distribution (19) is that it represents the probability that exactly i exponentially distributed jumps with mean σ^2/μ cause the component to fail; that is, cause the cumulative amount of deterioration to exceed $r_0 - s$. Note that the smaller the unit-time length for which the increments are exponentially distributed, i.e., the smaller $t = (\sigma/\mu)^2$, the less uncertain the deterioration process.

3.10. Gamma process as limit of compound Poisson process

An important property of the gamma process is that it is a jump process. The gamma process can be regarded as a compound Poisson process of gamma-distributed increments in which the Poisson rate tends to infinity and increment sizes tend to zero in proportion [12]. Using the technique of Laplace transforms, it can be shown that the gamma process can be reformulated in terms of a limit of a compound Poisson process. In doing so, we first give the Laplace transform of a compound Poisson process representing jumps with intensity μ having random size. Then, we shall show that the Laplace transform of the gamma process can be rewritten in the same form as the Laplace transform of the compound Poisson process. For convenience, the proof is given for a compound Poisson process and a gamma process with stationary increments.

3.10.1. Laplace transform of compound Poisson process

Let the continuous-time stochastic deterioration process $\{X(t), t\geqslant 0\}$ be a compound Poisson process given by $X(t) = \sum_{i=1}^{N(t)} D_i$, where

(1) the number of jumps $\{N(t), t\geqslant 0\}$ is a Poisson process with jump intensity μ,
(2) the jumps $\{D_i, i = 1,2,\ldots\}$ are independent and identically distributed random quantities having distribution $\Pr\{D\leqslant\delta\} = F_D(\delta)$,
(3) the process $\{N(t), t\geqslant 0\}$ and the sequence $\{D_i, i = 1,2,\ldots\}$ are independent.

The Laplace transform of the compound Poisson process $X(t)$ is

$$E(\mathrm{e}^{-sX(t)}) = \exp\{\mu t(E(\mathrm{e}^{-sD}) - 1)\}. \tag{20}$$

According to de Finetti [64, Chapter 8], this Laplace transform can be rewritten as

$$\begin{aligned} E(\mathrm{e}^{-sX(t)}) &= \exp\left\{\mu t \int_{\delta=0}^{\infty} (\mathrm{e}^{-s\delta} - 1)\,\mathrm{d}F_D(\delta)\right\} \\ &= \exp\left\{t \int_{\delta=0}^{\infty} (\mathrm{e}^{-s\delta} - 1)[-\mathrm{d}Q(\delta)]\right\}, \end{aligned} \tag{21}$$

where $s>0$ and

$$Q(\delta) = \mu[1 - F_D(\delta)] = \mu \int_{z=\delta}^{\infty} f_D(z)\,\mathrm{d}z$$

represents the intensity of jumps whose magnitude is larger than δ for $\delta>0$. The negative derivative of $Q(\delta)$, $q(\delta) = -Q'(\delta)$, is also called the Lévy measure of $\{X(t), t\geqslant 0\}$. The measure $q(\delta) = \mu f_D(\delta)$ is a positive measure, but not a probability measure because $\int_{\delta=0}^{\infty} q(\delta)\,\mathrm{d}\delta = \mu \neq 1$. Note that the expected number of jumps of all sizes per unit time (jump intensity) is finite; that is, $Q(0) = \mu$ is finite. Indeed, for a compound Poisson process, the number of jumps in any time interval is finite with probability one. Furthermore, the expected value of $X(t)$ is also finite and has the value

$$E(X(t)) = t\int_{\delta=0}^{\infty} \delta[-\mathrm{d}Q(\delta)] = \mu t \int_{\delta=0}^{\infty} \delta f_D(\delta)\,\mathrm{d}\delta = \mu E(D)t.$$

3.10.2. Laplace transform of gamma process

On the other hand, the Laplace transform of $X(t)$ being a stationary gamma process is

$$\begin{aligned} E(\mathrm{e}^{-sX(t)}) &= \left[\frac{u}{u+s}\right]^{ct} \\ &= \exp\left\{ct \int_{\delta=0}^{\infty} (\mathrm{e}^{-s\delta} - 1)\frac{\mathrm{e}^{-u\delta}}{\delta}\,\mathrm{d}\delta\right\} \\ &= \exp\left\{t \int_{\delta=0}^{\infty} (\mathrm{e}^{-s\delta} - 1)[-\mathrm{d}Q(\delta)]\right\}, \end{aligned} \tag{22}$$

where $s>0$ and

$$Q(\delta) = c\int_{z=\delta}^{\infty} \frac{\mathrm{e}^{-uz}}{z}\,\mathrm{d}z = cE_1(u\delta)$$

represents the intensity of jumps whose magnitude is larger than δ for $\delta>0$ (Dufresne et al. [15]) and the exponential integral is given by

$$E_1(x) = \int_{t=x}^{\infty} \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t.$$

The Lévy measure of the gamma process is $q(\delta) = -Q'(\delta) = c\delta^{-1}\mathrm{e}^{-u\delta}$. Because $\int_{\delta=0}^{\infty} q(\delta)\,\mathrm{d}\delta = \infty$, this measure is infinite. Hence, the expected number of jumps of all sizes per unit time (jump intensity) is infinite as well; that is,

$$Q(0) = \lim_{\delta\downarrow 0} Q(\delta) = \infty.$$

Indeed, for a gamma process, the number of jumps in any time interval is infinite with probability one. Nevertheless, $E(X(t))$ is finite, as the majority of jumps are extremely small:

$$E(X(t)) = t\int_{\delta=0}^{\infty}\delta\,\mathrm{d}Q(\delta) = ct\int_{\delta=0}^{\infty}\delta\frac{\mathrm{e}^{-u\delta}}{\delta}\,\mathrm{d}\delta = \frac{ct}{u}.$$

The agreement between Eqs. (21) and (22) shows us that the gamma process indeed is a jump process. As a matter of fact, the gamma process can be regarded as a limit of a compound Poisson process in the following manner. Let the probability density function of the jump sizes be a gamma distribution with shape parameter $v>0$ and scale parameter $u>0$ and let the Poisson jump intensity be $\mu = c\Gamma(v)/u^v$. According to Dufresne et al. [15], the Laplace transform of this compound Poisson process is

$$E(\mathrm{e}^{-sX(t)}) = \exp\left\{ct\frac{\Gamma(v)}{u^v}\int_{\delta=0}^{\infty}(\mathrm{e}^{-s\delta}-1)\times\frac{u^v}{\Gamma(v)}\delta^{v-1}\exp\{-u\delta\}\,\mathrm{d}\delta\right\}. \tag{23}$$

Clearly, the Laplace transform of the compound Poisson process with jump intensity $c\Gamma(v)/u^v$ and jump sizes being gamma distributed with shape parameter v and scale parameter u approaches the Laplace transform of a gamma process with shape function ct and scale parameter u as v tends to zero from above.

3.11. Simulation of gamma processes

The purpose of this section is to present methods for simulating gamma processes. Although we can approximate a gamma process with a limit of a compound Poisson process, Dufresne et al. [15] and Singpurwalla and Wilson [61] claim that it is not efficient to simulate a gamma process in such a way. This is because there are infinitely many jumps in each finite time interval. A better approach for simulating a gamma process is simulating independent increments with respect to very small units of time. In doing so, we first define the following time grid: $0, t_1, t_2, \ldots, t_{n-1}, t_n$, where $t_i = (i/n)t$ for some $t>0$ and $i = 0,\ldots,n$. Next, we simulate sample paths of the gamma process by randomly drawing independent increments $X(t_1), X(t_2)-X(t_1), \ldots, X(t_n)-X(t_{n-1})$ using Monte Carlo simulation. For sampling independent gamma-process increments, there are basically two simulation methods: gamma-increment sampling and gamma-bridge sampling (for an overview, see [103]).

3.11.1. Gamma-increment sampling

Gamma-increment sampling is defined as drawing independent samples $\delta_i = x_i - x_{i-1}$, where $x_0 = 0$, from the gamma density

$$\mathrm{Ga}(\delta|v(t_i)-v(t_{i-1}),u) = \frac{u^{v(t_i)-v(t_{i-1})}}{\Gamma(v(t_i)-v(t_{i-1}))}\delta^{[v(t_i)-v(t_{i-1})]-1}\exp\{-u\delta\} \tag{24}$$

for every $i = 1,2,\ldots,n$. Avramidis et al. [103] term this discrete-time simulation approach gamma sequential sampling (GSS).

3.11.2. Gamma-bridge sampling

Gamma-bridge sampling is described in Avramidis et al. [103], Dufresne et al. [15], and Singpurwalla and Wilson [61]. It consists of the following steps. First, draw a sample of the cumulative amount of deterioration $X(t)$ at the end of the time interval $(0,t]$ that we are considering. Second, draw a sample of the cumulative deterioration $X(t/2)$ from the conditional distribution of $X(t/2)$ given $X(t) = x$:

$$\begin{aligned} f_{X(t/2)|X(t)}(y|x) &= \frac{1}{x}\,\mathrm{Be}\left(\frac{y}{x}\middle|v(t/2), v(t)-v(t/2)\right) \\ &= \frac{\Gamma(v(t))}{\Gamma(v(t/2))\Gamma(v(t)-v(t/2))} \\ &\quad\times\left[\frac{y}{x}\right]^{v(t/2)-1}\left[1-\frac{y}{x}\right]^{[v(t)-v(t/2)]-1}\left[\frac{1}{x}\right] \end{aligned} \tag{25}$$

for $0\leqslant y\leqslant x$. This is a transformed beta density on the interval $[0,x]$ with parameters $v(t/2)$ and $v(t)-v(t/2)$. Third, subdivide this time interval $(0,t]$ into two intervals $(0,t/2]$ and $(t/2,t]$, and sample $X(t/4)$ given the value of $X(t/2)$ and $X(3t/4)$ given the values of $X(t/2)$ and $X(t)$. Similarly, we can simulate the values of $X(t/8)$, $X(3t/8)$, $X(5t/8)$, $X(7t/8)$, and so on. In summary, this leads to a discrete-time equal-length partition of $(0,t]$ for which a gamma-process path is sampled at the following 2^m time points: $t, t/2, t/4, 3t/4, t/8, 3t/8, 5t/8, 7t/8, \ldots, 2^{-m}t, \ldots, (1-2^{-m})t$ for some positive integer m. An example of gamma-bridge sampling is given in Fig. 1, where $m = 10$.

3.11.3. Compound Poisson simulation

Other—maybe less applicable—simulation methods are based on approximating a gamma process as a limit of a compound Poisson process in the sense of Ferguson and

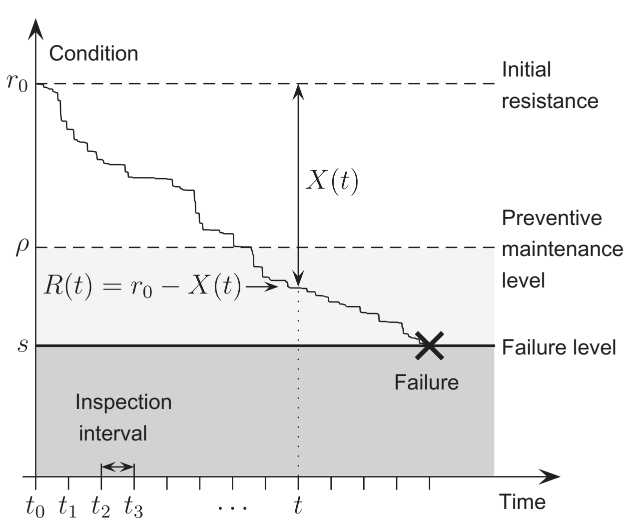

Fig. 1. Basic condition-based maintenance model with gamma-process deterioration.

Klass [62] and Wolpert and Ickstadt [104]. These two approximations follow from representations of Lévy processes without Gaussian components in which a finite number of jump times and sizes is sampled. On the basis of simulation experiments, Walker and Damien [105] conclude that the representation of Ferguson and Klass [62] is generally more accurate and less computational expensive than the representation of Wolpert and Ickstadt [104]. In the former approach, the n largest jumps are sampled so that the total size of the missing jumps is relatively small. In the latter approach, however, n jumps of random size are drawn so that there is no guarantee that the missing jumps are all going to be small. Despite the limited applicability of the Wolpert and Ickstadt [104] sampling strategy, it was successfully applied by Pievatolo et al. [106] in a reliability study of underground trains.

3.12. Extensions of gamma processes

Extensions of the gamma process are local gamma processes, extended gamma processes, Hougaard processes, and multivariate gamma processes. An alternative definition of the gamma process was also found in the literature.

Çinlar [90] claims that creep and fatigue can very well be modelled as a stochastic process which is locally gamma with the shape and scale function depending on a time-dependent random environment. The non-stationary gamma process with expected deterioration being a power law in time is a special case of this local gamma process. The definition and rigorous mathematical theory for local gamma processes may be found in Çinlar [91].

Dykstra and Laud [107] proposed a non-parametric Bayesian approach to reliability. Their key idea is to model the failure rate function $r(t)$ in Eqs. (1)–(2) as an extended gamma process, which is defined as follows. Let $v(t)$ and $u(t)$ be two non-decreasing, right-continuous, real-valued functions. Let $\{Y(t), t \geqslant 0\}$ be a gamma process with shape function $v(t)$ and unit scale parameter. The extended gamma process is then defined as the stochastic process $r(t) = \int_0^t u(s)\,\mathrm{d}Y(s)$, where the integration is done with respect to the sample paths of $Y(t)$. When $u(s) = u$ for all $s \geqslant 0$, the gamma process arises as a special case of the extended gamma process. Ishwaran and James [108] extended the work of Dykstra and Laud [107] from a non-parametric setting on the real line to non-parametric and semi-parametric settings over general spaces. Extended gamma processes are also called "weighted gamma processes". Because the failure rate function is non-negative and non-decreasing, the (extended) gamma process is a good candidate for modelling it. Gaver [109] was the first to regard the failure rate as a stochastic process; in particular, a gamma process. Wang et al. [93] modelled the failure rate of water pumps at a large soft-drinks manufacturing plant by a non-stationary gamma process.

Lee and Whitmore [110] consider the Hougaard process, which is another extension of the gamma process. The Hougaard process has the gamma, positive stable, and inverse Gaussian processes as special cases.

Buijs et al. [111] proposed a bivariate gamma process for which the dependence is modelled using the double-gamma distribution. They applied this bivariate gamma-process model to determine the time-dependent reliability of a flood defence along the Thames, which is subject to motor-cross damage causing deterioration of both the crest level and the vegetation quality. A multivariate gamma process (of which the above-mentioned bivariate gamma process is a special case) is proposed by Ebrahimi [112].

An alternative definition of a gamma process is given by Berman [113], who considers a Poisson process with occurrence rate $\lambda(t)$. He supposes that an event occurs at time zero and that thereafter only every kth event of the Poisson process is observed. He calls this process a non-homogeneous gamma process with rate function $\lambda(t)$ and shape parameter k. If $k = 1$, the non-homogeneous Poisson process arises as a special case. The random quantities studied are the occurrence times of every kth event. The non-homogeneous gamma process was used by Hurley [114] to model the occurrence of bedload transport events and by Hall et al. [115] to model the occurrence of coastal landslides.

3.13. Failure models indexed by time and usage

In the examples above, failure models indexed by a single scale (such as the cumulative deterioration) were considered. Meaningful extensions are failure models indexed by two scales being time and a time-dependent quantity such as usage. According to Singpurwalla [3], the first published paper on failure models with two scales is Mercer [77] who describes the failure of a conveyor belt as a function of both time and usage. Although not explicitly stated, he models wear as a compound Poisson process. Mercer [77] assumed the failure rate to be the sum of a rate which is solely dependent on time and a rate which is linearly dependent on the wear. The composite failure rate can be interpreted as the killing rate defined by Wenocur [116].

Wenocur [116] considers a very interesting extension of a stationary gamma process for the deterioration state under two different failure modes. A system is said to fail either when its condition reaches a failure level or when a traumatic event (such as an extreme load) destroys it. Suppose that the traumatic events occur as a Poisson process with a rate (called the "killing rate") which depends on the system's condition. The latter is a meaningful assumption, because the worse the condition, the more vulnerable it is to failure due to trauma. In a time-dependent reliability analysis of a dike, van Noortwijk et al. [117] show that the killing rate can be interpreted as the frequency of the load exceeding the resistance when the variability of the random loads is modelled by a peaks-over-threshold distribution (such as the generalised Pareto distribution) and the stochastic process of load threshold exceedances is generated by a Poisson process. Wenocur [116] presents the stochastic differential equation from which the gamma process follows. In Lemoine and Wenocur [118,119] and Wenocur [120] the Brownian motion with drift plays the role of the stationary gamma process. The advantage of the Brownian motion with drift is that it becomes a diffusion, leading to much easier and better understood analytic expressions. Its disadvantage is that deterioration is no longer monotonically non-decreasing.

Singpurwalla and Wilson [61] present a failure model indexed by the two scales time and usage. Their basic assumption is that the failure rate can be written as a sum of a baseline rate which accounts for failure due to natural causes and a failure rate which is regarded as a stochastic process (such as the compound Poisson process and the gamma process). They applied their two-scale model for the setting of a warranty—both in time and usage—on locomotive traction motors.

Bagdonavičius and Nikulin [88] model the deterioration by a gamma process and include possibly time-dependent covariates as well as degradation-dependent traumatic events. They define the shape function to be dependent on a covariate. Alternatively, Lawless and Crowder [12] and Crowder and Lawless [23] assume the scale parameter to depend on a covariate and to be proportional to a random effect. For the random effect being gamma distributed, they obtained analytic results which were successfully applied to crack-growth data.

Pievatolo et al. [106] studied the reliability of door-opening systems of underground trains in their warranty period. In their two-scale failure model, the gamma process was used to model the distance run (in kilometres) as a function of time.

3.14. Gamma-process applications other than deterioration

In the literature, the gamma process was applied to model the temporal variability in two phenomena other than deterioration: dam storage and insurance risk. First, Moran [6–9] used gamma processes to model water inflow in his theory of dams. Second, Dufresne et al. [15] introduced the gamma process into the actuarial literature. In the classical model of risk theory developed as early as in 1903 by Lundberg [121,122], the surplus of an insurance company at time t is given by the expression

$$U(t) = u + ct - X(t), \quad t \geqslant 0, \tag{26}$$

where $u \geqslant 0$ is the initial surplus, c is the premium income per unit time, and $X(t)$ is the aggregate claims up to time t being a compound Poisson process. The main difference between the risk-ruin model and the stress–strength model is that the former has a moving barrier in terms of ct. Dufresne et al. [15] defined a gamma process as a limit of compound Poisson processes and proposed it as a model for the aggregate claims process. They discussed many properties of this process and they showed how to calculate the probability of ultimate ruin over an unbounded time horizon. Dickson and Waters [123] derived the formulae for the probability of ruin or survival over a bounded time horizon.

4. Maintenance optimisation

The gamma deterioration process was successfully applied to model time-based preventive maintenance as well as condition-based preventive maintenance. Time-based preventive maintenance is carried out at regular intervals of time, whereas condition-based maintenance is carried out at times determined by inspecting or monitoring a structure's condition. Models for time- and condition-based maintenance under gamma-process deterioration are summarised in Table 1.

4.1. Renewal theory

All maintenance models surveyed in this article use cost-based criteria which are defined over an unbounded time horizon such as the expected average cost per unit time, the expected discounted cost over an unbounded horizon, or the asymptotic (un)availability. Because these cost-based criteria are all based on renewals bringing a component or structure back to its original condition, renewal (reward) theory was used to compute them. See Rackwitz [125] and van Noortwijk [126] for the continuous-time and discrete-time renewal process, respectively. Let $F(t)$ be the cumulative probability distribution of the continuous renewal time $T \geqslant 0$ and let $c(t)$ be the cost associated with a renewal at time t. Using renewal reward theory [60, Chapter 3], the expected average cost per unit time is

$$\lim_{t\to\infty} \frac{E(K(t))}{t} = \frac{\int_0^\infty c(t)\,\mathrm{d}F(t)}{\int_0^\infty t\,\mathrm{d}F(t)} = \frac{E(c(T))}{E(T)}, \tag{27}$$

where $E(K(t))$ represents the expected non-discounted cost in the bounded time interval $(0, t]$, $t > 0$. Let a renewal cycle be the time period between two renewals, and recognise the numerator of Eq. (27) as the expected cycle cost and the denominator as the expected cycle length (mean lifetime). Using a terminating renewal argument [127, Chapter 11], the expected discounted cost over an unbounded horizon can then be written as

$$c_0 + \lim_{t\to\infty} E(K(t,\gamma)) = c_0 + \frac{\int_0^\infty \mathrm{e}^{-\gamma t} c(t)\,\mathrm{d}F(t)}{1 - \int_0^\infty \mathrm{e}^{-\gamma t}\,\mathrm{d}F(t)}, \tag{28}$$

where c_0 is the investment cost, $\gamma > 0$ is the discount rate, $E(K(t,\gamma))$ is the expected discounted cost in the bounded time interval $(0, t]$ for $t > 0$, and the numerator is the expected discounted cycle cost. Eq. (28) can be proven using Laplace transforms [125]. Note that the expression $\int_0^\infty \mathrm{e}^{-\gamma t}\,\mathrm{d}F(t)$ in the denominator of Eq. (28) can be recognised as the Laplace transform $E(\mathrm{e}^{-\gamma T})$ representing the expected discounted cycle length.

Table 1
Summary of models for time- and condition-based maintenance under gamma-process deterioration (CBM = condition-based maintenance, SRC = semi-regeneration cycle)

Reference	Inspection schedule	Failure detection	Failure level	Decision criterion	Time axis	Characteristic feature
[26]	No	Immediately	Random	Average cost	Continuous/ discrete	Age replacement
[5]	No	Immediately	Fixed	Average cost	Continuous	Age replacement
[92,124]	No	Immediately	Fixed	Discounted cost	Continuous/ discrete	Age replacement
[20,89,92]	No	Immediately	Fixed	Discounted cost	Continuous	Life extension
[23]	Single	Immediately/by inspection	Fixed	Average cost	Continuous	Single inspection
[27,28]	Periodic	By inspection	Random	Average cost	Continuous	Wear-dependent maintenance cost
[29]	Aperiodic	By inspection	Random	Average/ discounted cost	Continuous	Wear-dependent inspection cost
[36]	Periodic	Immediately	Random	Average cost	Continuous	Basic CBM model
[5,35]	Periodic	Immediately	Fixed	Average cost	Continuous	Basic CBM model
[38]	Periodic	Immediately	Fixed	Average cost	Continuous	Optimal first inspection
[25]	Periodic	Immediately	Fixed	Average/ discounted cost	Discrete	Basic CBM model
[13]	Periodic	Immediately	Random	Average cost	Discrete	Imperfect inspection
[45]	Periodic	Immediately	Fixed	Average cost	Discrete	2 failure modes
[37,44]	Periodic	Immediately	Fixed	Average cost	Continuous	Perfect inspection, dynamic programming
[39,44]	Aperiodic	Immediately	Fixed	Discounted cost	Continuous	Perfect inspection, dynamic programming
[40–42]	Aperiodic	Immediately	Fixed	Average/ discounted cost	Continuous	Imperfect inspection, dynamic programming
[30–32]	Aperiodic	By inspection	Fixed	Average cost	Continuous	SRC
[43]	Aperiodic	Immediately	Fixed	Average cost	Discrete	SRC
[34]	Aperiodic	By inspection	Fixed	Average cost	Discrete	SRC, 2 components
[33]	Aperiodic	By inspection	Fixed	Average cost/ availability	Discrete	SRC, partial repair
[21]	Periodic	By inspection	Fixed	Average cost	Continuous	SRC, partial repair, maintenance delay
[46]	Periodic	By inspection	Fixed	Average cost	Continuous	SRC, 2 deterioration modes
[47]	Aperiodic	By inspection	Random	Availability	Continuous	Required availability
[50,52]	Continuous	Immediately	Random	Average cost	Continuous	Control-limit policy
[30]	Continuous	Immediately	Fixed	Average cost	Continuous	Maintenance delay
[48]	Continuous	Immediately	Fixed	Unavailability	Continuous	Maintenance delay and duration
[51]	Continuous	Immediately	Fixed	Unavailability	Continuous	Maintenance delay and duration, 2 deterioration modes
[49]	Continuous	Immediately	Fixed	Asymptotic failure rate	Continuous	Maintenance delay
[53]	Continuous	Immediately	Fixed	Availability	Continuous	Maintenance duration, partial repair
[54]	Continuous	Immediately	Fixed	Average cost	Continuous	Partial repair, dynamic programming
[24]	Periodic	By inspection	Fixed	Average cost	Continuous	Damage-initiation time
[22]	Periodic	Immediately	Fixed	Average/ discounted cost	Discrete	Damage-initiation time
[19]	Periodic	Immediately	Fixed	Average/ discounted cost	Discrete	Optimal design
[18]	Periodic	By inspection	Fixed	Discounted cost	Discrete	Optimal design

4.2. *Time-based maintenance*

A well-known time-based maintenance strategy is age replacement, which can be combined with lifetime-extending maintenance.

4.2.1. *Age replacement*

Abdel-Hameed [26] studied an age-replacement policy for which a renewal is defined as either a corrective replacement upon failure *or* upon a preventive replacement reaching a predetermined age k, whichever occurs first. The cost of a corrective replacement is c_{F}, whereas the cost of a preventive replacement is a function of the amount of deterioration $X(k)$ being of the form $c_{\mathrm{P}}(X(k))$. The long-term expected average cost of replacement per unit time can be calculated using renewal reward theory [60, Chapter 3]. The failure level is assumed to be random and failure is detected immediately. Abdel-Hameed [26] studied the stationary gamma process. He also presented the results for the discrete-time case. As a special case he studied the discrete-time version of the ordinary age-replacement model (with fixed preventive and corrective replacement cost c_P and c_F) for which the increments of deterioration are exponentially distributed and the failure level is fixed. van Noortwijk [124] applied this special case to age replacement of a cylinder on a swing bridge. As a cost criterion, he considers the expected discounted cost over an unbounded time horizon. Pandey et al. [5] compared optimal age-replacement intervals with minimal average cost per unit time for deterioration modelled as a gamma process and a gamma-distributed deterioration rate.

4.2.2. *Lifetime extension*

The age-replacement model with fixed failure level has been extended for the possibility of non-stationary gamma-process deterioration and lifetime-extending maintenance by van Noortwijk [124] and Bakker et al. [89]. For an overview of this lifetime-extending maintenance model, see van Noortwijk and Frangopol [92]. With the extended model both the interval of lifetime extension and the interval of preventive replacement can be optimised. Lifetime-extending maintenance is also referred to as imperfect or partial repair: a structure's condition is improved, but not to its initial condition. Through lifetime extension, the deterioration can be delayed so that failure is postponed and the lifetime of a component is extended (e.g., a coating protecting steel). Possible effects of lifetime-extending measures are the increase in the damage-initiation period (time interval in which no deterioration occurs) and the condition improvement. Through replacement, the condition of a component is restored to its original condition and is subject to the same deterioration process. The deterioration under lifetime-extending maintenance is approximated by a non-stationary gamma process with the expected deterioration proportional to a power law of time. The lifetime-extending maintenance model is used in a case study on maintaining the coating on the steel gates of the Haringvliet storm-surge barrier in the Netherlands [20].

4.3. *Condition-based maintenance*

Modelling the deterioration as a gamma process especially suits well when inspections are involved. Since the last three decades, several inspection models with gamma-process deterioration have been developed for the purpose of optimising condition-based maintenance. In this section, we give an overview of these condition-based maintenance models and their differences and similarities.

4.3.1. *Periodic inspection*

Abdel-Hameed [27,28] studied condition-based maintenance of a system subject to stochastic deterioration. The two decision variables are the inspection interval and the preventive maintenance level. In the operations research literature, such a policy is called a "control-limit policy" with the preventive maintenance level called the "control limit" [128]. The system deterioration is inspected periodically and is regarded as a non-decreasing jump process—with compound Poisson processes having positive jumps and gamma processes as special cases (see Fig. 1). A failure is defined as the event in which the stochastic deterioration process exceeds a failure level. This deterioration failure level is assumed to be random having a probability distribution. A failure is detected only by inspection. The system is renewed when an inspection reveals either that the preventive maintenance level ρ is crossed while no failure has occurred (preventive replacement) or that the failure level s is crossed (corrective replacement). A renewal brings the system back to its "as good as new" condition. The cost of preventive replacement is a function of the deterioration and the cost of corrective replacement is fixed, where the former is less than or equal to the latter. Also, the cost of inspection per unit time is a function of the inspection interval and the cost of system operation per unit time is a function of the deterioration. Inspection does not degrade the system and is perfect in the sense that deterioration will be observed with certainty. Both inspection and replacement take negligible time. The optimal maintenance decision is determined by minimising the long-term average cost per unit time. This cost is computed by applying the renewal reward theory. The paper mainly focusses on the mathematical conditions on cost functions and deterioration parameters for which an optimal control-limit policy exists. Recently, Abdel-Hameed [29] included aperiodic inspection and discounted cost into his condition-based maintenance model [27,28]. In [27–29], no algorithms and numerical examples are given.

Abdel-Hameed [27,28] ends his paper with an example in which the deterioration is modelled as a stationary gamma process and all cost functions are constants. That is, the cost of inspection is c_I, the cost of preventive replacement is c_P, and the cost of corrective replacement is c_F. This gamma-process example was also studied by Kong and Park [36]. However, they assumed failure to be immediately detected without inspection. A special case of [36] is the model of Park [35], who considered the failure level to be fixed rather than random. Newby and Dagg [37,39,44] extended this model to determine an optimal multi-level control-limit rule against minimal expected average or discounted cost. Jia and Christer [38] proposed to use [35] for modelling functional checking procedures in reliability-centred maintenance (RCM). In the RCM context, the preventive maintenance level is called the "threshold of potential failure" (or warning level) and the failure level "the threshold of functional failure". Jia and Christer [38] considered periodic functional checks performed at $t_i = t_1 + (i-1)k$, $i = 1, 2, \ldots$, where t_1 is the length of the first inspection interval and k is the length of the repeating inspection interval. In the other papers that deal with periodic inspection, the length of the first inspection interval t_1 equals k. A single-inspection policy is studied by Crowder and Lawless [23], who estimated the preventive replacement time on the basis of a single inspection.

The models of Kong and Park [36] and Park [35] are illustrated with numerical examples in which a cost-optimal preventive maintenance level is computed when the inspection interval is given. On the other hand, dynamic programming is used by Newby and Dagg [37,39,44] to determine a cost-optimal inspection interval when the preventive maintenance level is given. Jia and Christer [38] give a numerical example of a cost-optimal combination of the preventive maintenance level and the lengths of the first and subsequent inspection intervals. van Noortwijk et al. [25,87] approximated the stationary gamma process with a discrete-time stochastic process having exponentially distributed increments. They give a case study on condition-based maintenance of the rock dumping of the Eastern-Scheldt barrier for which the expected average cost per unit time or the expected discounted cost over an unbounded time horizon is minimal. The deterioration process that they consider is current-induced rock displacement near the rock dumping. Pandey et al. [5] compared optimal inspection intervals for condition-based maintenance with minimal average cost per unit time for deterioration modelled as a gamma process and a gamma-distributed deterioration rate. Crowder and Lawless [23] illustrated their single-inspection policy to maintenance of automobile brake pads.

4.3.2. Imperfect inspection

Optimal inspection intervals for a steel pressure vessel subject to corrosion for which the expected average costs of inspection and maintenance per unit time are minimal have been determined by Kallen and van Noortwijk [13]. Using Bayesian statistics, they also deal with imperfect inspections. Following Whitmore [74], they assume the measurement errors to be independent, normally distributed, random quantities being independent of the deterioration process. Failures are detected immediately and the failure level is considered to be random. A similar model for condition-based maintenance under imperfect inspection is presented by Newby and Dagg [40,41,44] and Newby and Barker [42]. They formulated their deterioration model in terms of a Lévy process and identified the actual deterioration process, which cannot be observed, as opposed to a covariate process (such as imperfect inspections), which can be observed. For a stationary gamma deterioration process with a fixed failure level, the resulting integral and functional equations were solved using dynamic programming. Recently, Kallen and van Noortwijk et al. [45] extended the above-mentioned imperfect-inspection model to two independent failure modes with fixed failure levels. The two-failure-mode model is applied to a pipeline elbow, which is susceptible to thinning and cracking due to corrosion.

4.3.3. Aperiodic inspection

Bérenguer et al. [30], Dieulle et al. [31] and Grall et al. [32] studied the following variation of the inspection model of Park [35]. Inspection is aperiodic and it is scheduled by means of a function of the deterioration state. Failure is detected only by inspection and a cost of "inactivity of the system" per unit time is incurred as soon as failure occurs. In order to deal with aperiodic inspections, the long-term expected average cost per unit time is computed on the basis of the semi-regenerative properties of the maintained system condition. A semi-regeneration interval (or semi-renewal cycle) is defined as the interval between two inspections, whereas a regeneration interval (or renewal cycle) is defined as the interval between two renewals. Using a semi-regeneration argument, the long-term expected average cost per unit time equals the ratio of the expected semi-regeneration cycle cost over the expected semi-regeneration cycle length. Both expectations are derived with respect to the steady-state stationary probability distribution of the maintained system state. Grall et al. [43] and Castanier et al. [33,34] approximated the stationary gamma process with a discrete-time stochastic process having independent, identically and exponentially distributed increments. Grall et al. [43] consider a maintenance policy using a multi-level control-limit rule, where failures are detected immediately. Castanier et al. [34] built a model for a two-component series system, where failure is detected only by inspection. Such an optimisation model pays off when actions on the two components can be combined so that set-up cost has to be charged only once. Castanier et al. [33] developed a discrete-time model with the possibility of partial repair, where failure is detected only by (aperiodic) inspection. The condition improvement (called "operation recovery") is a random function of the observed condition and the corresponding maintenance durations are random functions of both the observed and improved conditions. In almost all the above-mentioned papers based on the semi-regeneration property (i.e., in [31–34,43]), numerical examples are given. Meier-Hirmer et al. [21] successfully applied the inspection model of the above-cited French authors to optimise maintenance of high-speed railway tracks. They studied the deterioration of railway tracks by means of longitudinal levelling and they assumed, amongst others, periodic inspection, failure detected only by inspection, maintenance delay time, and uncertain partial repair.

Yang and Klutke [47] propose an aperiodic inspection policy for which a minimally required asymptotic availability is guaranteed. They consider stochastic deterioration in terms of a Lévy process (with the compound Poisson process and gamma process as special cases), a random failure level, and detection of failure only by inspection.

4.3.4. Continuous monitoring

Several condition-based maintenance models were developed for situations in which the deterioration is continuously monitored and is regarded as a stationary gamma process. Park [50] determined the preventive maintenance level for which the long-term expected average cost per unit time is minimal. Failure is detected immediately and the failure level is random. The system is replaced instantaneously when its deterioration is beyond the preventive maintenance level or beyond the failure level, whichever occurs first. The same model emerges as a special case from the replacement model of Zuckerman [52]. Both models differ from the ordinary age-replacement model in the sense that they treat the preventive maintenance level (or control limit) as a decision variable rather than the age-replacement interval.

Bérenguer et al. [30,48] built models of continuous monitoring and perfect repair to find the preventive maintenance level for which the asymptotic unavailability is minimal [48] or to find the preventive maintenance level for which the long-term expected average cost per unit time is minimal [30]. The delay time until maintenance after exceedance of the preventive maintenance level and the maintenance duration are assumed to be random quantities. As soon as failure occurs and maintenance has not yet been performed, the system is unavailable. For these continuous-monitoring models, Grall et al. [49] computed the asymptotic failure rate of the maintained system.

Liao et al. [53] considered continuous monitoring with partial repair rather than perfect repair. They assumed the condition state after partial repair to be random and called their maintenance policy a condition-based availability-limit policy. A similar approach is proposed by Nicolai and Frenk [54] to optimise imperfect and perfect repair of coatings protecting steel by minimising the expected non-discounted cost over a bounded time horizon using dynamic programming. They model imperfect repair (spot repair or repainting) by means of an uncertain condition improvement and a possible change in the parameters of the associated non-stationary gamma deterioration process.

4.3.5. Detection of change of deterioration mode

Saassouh et al. [51] and Fouladirad et al. [46] propose online maintenance policies for a system subject to deterioration that can be in two modes: the nominal mode and the accelerated mode. The time at which the system's deterioration starts to accelerate, and changes from the nominal mode to the accelerated mode, is random. Per mode, the deterioration is modelled as a stationary gamma process. In [51] the change of mode is supposed to be self-announced, whereas in [46] a change detection algorithm has been used for the time estimation of the change of mode (to combine maintenance and detection). The adaptive maintenance policies are illustrated to minimise the asymptotic unavailability of the system [51] or the long-term expected average cost per unit time [46].

4.3.6. Inspection of damage initiation

van Noortwijk and Klatter [24] developed a mathematical model to optimise maintenance of a part of the seabed protection of the Eastern-Scheldt barrier, namely the block mats. This model enables optimal inspection decisions to be determined on the basis of the uncertainties in the process of occurrence of scour holes and, given that a scour hole has occurred, of the process of current-induced scour erosion. The continuous-time stochastic processes of scour-hole initiation and scour-hole development have been regarded as a homogeneous Poisson process and a non-stationary gamma process, respectively.

The model of van Noortwijk and van Gelder [22] studies berm breakwaters under uncertain rock transport caused by extreme waves. It computes inspection intervals having either minimal expected average cost per unit time or minimal expected discounted cost over an unbounded horizon. The discrete-time processes of the occurrence of initial breaches in the armour layer and, given a breach has occurred, of the process of longshore rock transport have been regarded as a geometric distribution and a gamma process with exponentially distributed increments, respectively.

Both models include the cost of inspection, repair and failure; the cost of repair depends on the amount of deterioration. In the former model failure is detected only by inspection and a safety constraint is introduced, whereas in the latter model failure is detected immediately.

4.3.7. Optimal design

Apart from using inspection models under gamma-process deterioration in the use phase of the life-cycle of a structure, they can also be used in the design phase. Only two inspection models have been devoted to optimally balancing the initial investment cost against the future maintenance cost in the design phase. The first model deals with determining optimal sand nourishment sizes for which the expected discounted cost over an unbounded horizon is minimal, while the stochastic process of permanent coastal erosion of dunes is regarded as a stationary gamma process [19]. The second model deals with determining optimal dike heightenings for which the expected discounted cost over an unbounded horizon is minimal, while the stochastic process of crest-level decline is regarded as a stationary gamma process [18]. Both models are defined with respect to discrete-time units for which the deterioration is exponentially distributed and assume condition-dependent maintenance cost. In the former model failure is detected immediately (at discrete units of time), whereas in the latter model failure is detected only by inspection. Speijker et al. [18] also deal with a non-stationary stochastic process for which the increments are independent, non-identically, exponentially distributed. In this situation, the probability distribution of the cumulative amount of deterioration is the general Erlang or general gamma distribution.

5. Conclusions

This article surveys the application of gamma processes in maintenance. Since the introduction of the gamma process in the area of reliability in 1975, it has been increasingly used as a deterioration process in maintenance optimisation models. As gamma processes are well suited for modelling the temporal variability of deterioration, they serve as building blocks in various kinds of inspection models. Gamma processes played a crucial part in case studies on inspection and maintenance of dikes, beaches, steel coatings, berm breakwaters, steel pressure vessels, automobile brake pads, sea-bed protections, underground trains, and high-speed railway tracks. Using statistical estimation techniques (such as maximum likelihood, method of moments, Bayesian updating, and expert judgement), gamma processes were satisfactorily fitted to real-life data on creep of concrete, fatigue crack growth, corrosion of steel protected by coatings, corrosion-induced thinning, chloride ingress into concrete, and longitudinal levelling of railway tracks. The many variations of time- and condition-based maintenance models include lifetime extension, periodic and aperiodic inspection, failures detected immediately or only by inspection, fixed and random failure levels, discounted and non-discounted cost, single- and two-component systems, single- and two-failure-mode analyses, perfect and partial repair, perfect and imperfect inspection, availability requirement, continuous monitoring, and covariates. Up to now, gamma processes are mainly applied to maintenance decision problems for single components rather than for systems. Less developed aspects in the modelling of maintenance under gamma-process deterioration are, for example, spatial variability and dependence, as well as multi-component and multi-failure-mode models including their statistical dependencies.

Acknowledgements

The author gratefully acknowledges the helpful comments from an anonymous reviewer, Christophe Bérenguer, Antoine Grall, Maarten-Jan Kallen, Leo Klatter, and Hans van der Weide. The work reported in this article was partly supported by the Civil Engineering Division of the Netherlands Ministry of Transport, Public Works, and Water Management.

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