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Reliability analysis and maintenance decision for railway sleepers using track condition information

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This paper describes the development of a model which can be used to evaluate the reliability of a sleeper system when the sleeper condition is known. Two cases that are of particular interest to the railway industry are investigated. These are the failure of a sleeper system caused by at least two consecutive failed sleepers and by at least three consecutive failed ones, respectively. A model to optimize sleeper maintenance is proposed, which minimizes the number of sleepers restored during immediate maintenance subject to meeting the requirements of reliable and safe sleeper operation. Finally, an example is given to illustrate the model.

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1. Introduction

Sleepers are a major component of the railway track and perform the functions of maintaining the track gauge within permissible limits and spreading train-induced loads to the underlying substructure. Sleeper failure can have a number of undesirable consequences including the derailment of the train. For example, according to the Federal Railroad Administration (1997), nearly 21% of total derailments from 1988 to 1995 in the USA were attributed to gauge widening as a result of failed sleepers or fasteners.

Consequently, in order for sleepers to be able to perform their intended functions adequately they are inspected periodically to identify possible defects (Railtrack, 2000; Zarembski *et al*, 2003). Using the results of these inspections, an important task for asset managers is to forecast the reliable operation of the sleepers so that future decisions regarding maintenance may be made. Ideally, these forecasts should be made on the basis of continued reliable operation. To date many railway companies have established their own standards for reliable operation of sleepers in terms of various parameters such as the permissible speed and tonnage of trains operating on the network (Railtrack, 2000; Department of Transportation, 2002). In general, the standards can be classified into two groups. One specifies the number of failed sleepers within a group of sleepers while the other is related to the number of consecutive sleepers that are failed. For the former case, the reliability of the system is relatively simple to model and has been well studied (Barlow and Proschan, 1996), and therefore it will not be discussed further herein. The latter case, however, represents a higher risk of a railway accident than that of dispersive failures and is the focus of this paper. An extreme case was given by Lake *et al* (2000a) based on Australian railway experience. He showed that in a system where each failed sleeper lies between two sound ones the railway track can still operate, even though the number of dispersive failures is 50%. The reliability and maintenance of a sleeper system in which consecutive failed sleepers are considered in the system has been the focus of some research. For example, Lake *et al* (2000a,b) employed a Monte Carlo simulation model to predict the distribution of failed sleepers and number of clusters that compose of consecutive failed sleepers. Since its development their model has been used to determine various sleeper replacement strategies (Yun and Ferreira, 2003). However, a Monte Carlo analysis is itself an approximate method and a large amount of computing time is required to solve a problem of optimization using this technique. Simon *et al* (2000) presented an empirical model to predict the percentage of failed sleepers in a segment of track. However, his model was verified only by limited data of a specific case and therefore it is not possible to determine if it is appropriate in other cases, for example, when the probability of sleeper failure is changed.

Consequently, an alternative analytical approach which can give an exact solution to the problem of determining the reliability of a sleeper system is proposed herein. To this end, the sleeper system is modelled as a linear consecutive k out of n system (Kontoleon, 1978). Such a system consists of n linearly ordered components (ie sleepers) and is considered to have failed if at least k consecutive components have failed.

In the field of reliability engineering, initiating from the work by Kontoleon (1978), comprehensive studies have been

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carried out on the reliability of consecutive k out of n systems (Derman, 1982; Bollinger and Salvia, 1982; Lambiris and Papastavridis, 1985; Fu, 1985). However, as far as the knowledge of the authors, little has been done in analysing a consecutive k out of n system given that a number of the components that form part of the system have been identified as failed.

2. Problem statement

Sleeper failures can affect their capability in resisting vertical or lateral train-induced forces. The failure modes related to the former are mainly cracks, splits and broken off pieces, and those related to the later usually include the faults in shoulders or rail seats and cracks in the vicinity of fastening systems. Both types of sleeper failures may have implications for safety. Therefore on a railway network, inspections are carried out to identify the number and location of defective sleepers. In the UK, railway sleepers are inspected at a frequency specified by railway standards. The specified inspection frequencies vary according to the speed and tonnage of the trains permitted to use a particular line. When defective sleepers are discovered they are dealt with in two ways. If the failed sleepers lie consecutively in the track or are considered to be a safety hazard, then they must be replaced immediately (herein this process is known as immediate maintenance). Otherwise, the replacement of failed sleepers may be deferred until it is convenient to replace them using mechanized means. Such major maintenance may also occur in conjunction with other types of railway track maintenance. Both immediate and major maintenance will require the railway track to be closed to traffic; however, the latter is properly planned, will involve the use of machinery and may be of higher quality, therefore it is more efficient in terms of the number of sleepers replaced in a given time period (Jarvis *et al*, 2004).

For this reason, in practice it is preferable to replace failed sleepers during major maintenance. Consequently, the railway maintenance engineer should try to minimize the number of sleepers to be replaced in immediate maintenance. A way in which this may be achieved is the subject of this paper.

There are two possible situations immediately after a group, or system, of sleepers has been inspected. One is that the system is considered to be satisfactory, albeit with any number of failed sleepers that are not located consecutively. The other case is that the system has failed and maintenance will follow to remove the consecutive failed ones. In the latter case, the sleeper system will be restored. Consequently, it is only necessary to analyse the state that, at time t_0 when an inspection is carried out, the system is functional and there are a number of failed sleepers that are not located consecutively in the system. The problem therefore is to evaluate the reliability at given time $t_0 + \tau$ where $t_0 + \tau$ is the time of next inspection or maintenance activity. On this basis, sleeper replacement is optimized by minimizing the number of sleepers to be replaced in immediate maintenance subject to satisfying the requirement of the reliability of the sleeper system.

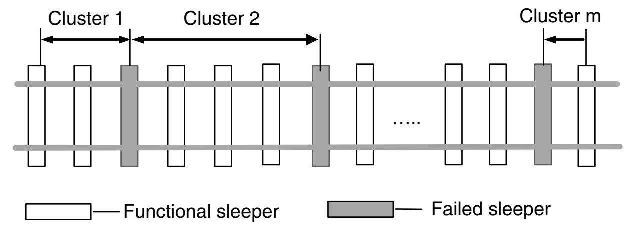

Figure 1 A segment of sleepers with several failed ones at time t_0.

In the work described herein, each sleeper, and the system, may either be in a sound (ie working or functional) or failed condition. The reliability of the system is defined as the probability that no more than $k - 1$ consecutive sleepers fail in the system within a given time during which no maintenance work is carried out to restore failed sleepers. The cases where $k = 2$ and $k = 3$ are only considered as these correspond to the requirements specified in railway standards (Railtrack, 2000; Lake *et al*, 2000a). This system may be divided into m clusters that are separated by dispersed failed sleepers (see Figure 1). Each cluster contains at least one functional sleeper, and has one or two failed sleepers at its two ends that are referred to as bounding sleepers. It should be noted that each bounding sleeper belongs to both of the two adjacent clusters.

When a sleeper is unable to support a train-induced load adequately, the adjacent sleepers will be required to carry a correspondingly higher load than normal and consequently the useful life of adjacent sleepers may be reduced (Yun and Ferreira, 2003). Thus in general, sleeper failures may be considered as being dependent on the condition of adjacent sleepers. However, in accordance with railway practice, the interval between major maintenance activities is short in comparison to the life of a sleeper and consequently the time for sleepers that may fail during this period, to have an impact on adjacent sound ones is small. Thus during this periods the effect of failed sleepers on adjacent ones can be neglected. This is discussed further in Section 4.5. In addition, both immediate and major maintenance activities are assumed to repair the failed sleepers with serviceable ones with a useful remaining life equal to the other sound sleepers in the system. Thus, both maintenance activities are assumed to minimally repair the failed sleepers. Therefore, for the purposes of this research sleepers can be assumed to be identical and their failures statistically independent.

It should be noted that although there are many types of sleeper failure modes, existing data on the failure distribution of sleepers is concerned with the failure of whole sleepers rather than precise modes of failure and therefore individual failure modes are not considered in this work.

3. Reliability of a single sleeper

Several studies have shown (Tucker, 1985; Lake *et al*, 2000a) that a Weibull distribution may be used to describe the distribution of the lifetimes of sleepers. Accordingly, the reliability of an individual sleeper at time t can be expressed as

$$r(t) = \exp[-(t/\beta)^{\alpha}], \qquad \alpha, \beta > 0 \tag{1}$$

where α and β, shape and scale parameters, respectively, may be determined from the analysis of existing sleeper failure data.

The reliability of a single sleeper at $t_0 + \tau$, p, and its corresponding probability of failure q, given that it is functional after inspection at time t_0, may be given by (Barlow and Proschan, 1975):

$$p = r(t_0 + \tau)/r(t_0) \tag{2}$$

$$q = [r(t_0) - r(t_0 + \tau)]/r(t_0) \tag{3}$$

where τ is the time interval between the current inspection and the next major maintenance.

4. Reliability of a cluster of sleepers

Consider the ith cluster of sleepers in the system, which contains n_i functional sleepers and is bounded by x_i ($x_i \leqslant 2$) failed sleepers at its two ends at time t_0. For the first and last cluster in the system, x may be equal to 1 or 2, and for all other clusters x_i is equal to 2. Let k be the minimum number of consecutive failed sleepers that cause system failure then in terms of k and x, the reliability, $R^{(x)}(n_i, k)$, of a cluster at time $t_0 + \tau$ with the constraint described earlier that $k < 4$, can be determined for four cases (ie cases 1–4). These are discussed below:

4.1. Case 1: $k = 2$, $x_i = 2$

The reliability, $R(n, k)$, of a consecutive k out of n system is given by (Derman, 1982)

$$R(n, k) = \sum_{j=0}^{n} H(j, n - j + 1, k - 1) p^{n-j} q^{j} \tag{4}$$

where $H(j, n - j + 1, k - 1)$ is the number of combinations that the system is functional with j failed units.

Writing $s = n - j + 1$ and $z = k - 1$, then (Derman, 1982)

$$H(j, s, 1) = \begin{cases} \binom{s}{j}, & 0 \leqslant j \leqslant s \\ 0, & j > s \end{cases} \tag{5a}$$

and

$$H(j, s, z) = \sum_{i=0}^{s} \binom{s}{i} H(j - zi, s - i, z - 1), \quad z \geqslant 2 \tag{5b}$$

Using Equations (4), (5a) and (5b), $H(j, s, z)$ can be solved by calculating sequentially for z from $z = 1$ to $z = k - 1$.

With the condition that the sleeper next to each failed bounding sleeper is functional, the reliability, $R^{(2)}(n_i, k)$, of the cluster is equivalent to a linear consecutive 2 out of $n_i - 2$ system as shown in Figure 2. Letting A represent the event that the cluster is reliable, and letting B be the event that the sleeper next to each bounding sleeper is sound, then it follows that:

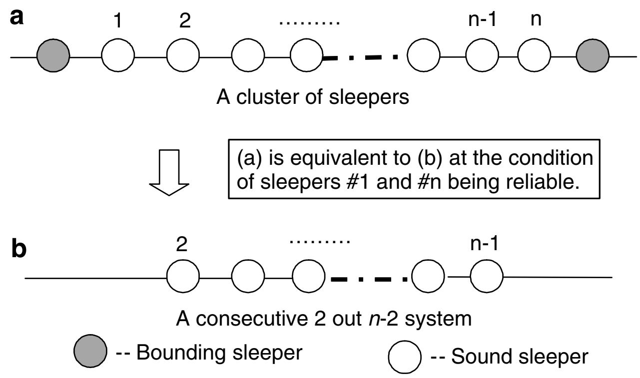

Figure 2 Graphical illustration of reliability analysis for $k = 2$.

$$\begin{aligned} R^{(2)}(n_i, 2) &= P(A|B)P(B) \\ &= R(n_i - 2, 2)p^2 \end{aligned} \tag{6a}$$

Then using Equations (5a), (5b) and (6a) the reliability of a cluster of sleepers with more than one functional sleeper (ie $n_i \geqslant 2$) is given by

$$\begin{aligned} R^{(2)}(n_i, 2) &= p^2 \sum_{j=0}^{n_i-2} H(j, n_i - 2 - j + 1, 1) p^{n_i-2-j} q^{j} \\ &= \sum_{j=0}^{n_i-2} H(j, n_i - j - 1, 1) p^{n_i-j} q^{j} \end{aligned} \tag{6b}$$

For the special case, the cluster contains only one functional sleeper ($n_i = 1$), the reliability of the system is the probability of the single sleeper being reliable and is given by

$$R^{(2)}(1, 2) = p \tag{7}$$

4.2. Case 2: $k = 2$, $x_i = 1$

The reliability, $R^{(1)}(n_i, 2)$ of a sleeper cluster with a failed sleeper at either end (ie $x_i = 1$) can be determined in a similar manner to the case described above for $x = 2$. For $n_i \geqslant 2$, by analogy with Equations (6a) and (6b):

$$R^{(1)}(n_i, 2) = pR(n_i - 1, 2) \tag{8a}$$

Using Equation (4),

$$\begin{aligned} R^{(1)}(n_i, 2) &= p \sum_{j=0}^{n_i-1} H(j, n_i - j, 1) p^{n_i-j-1} q^{j} \\ &= \sum_{j=0}^{n_i-1} H(j, n_i - j, 1) p^{n_i-j} q^{j} \end{aligned} \tag{8b}$$

With only one functional sleeper in the cluster at time t_0, the reliability of the cluster is given by

$$R^{(1)}(1, 2) = p \tag{9}$$

By inspection of Equations (6a) and (8a), the reliability of the sleeper cluster for $k = 2$ in general terms is given by

$$R^{(x_i)}(n_i, 2) = p^{x_i} R(n_i - x_i, 2) = \sum_{j=0}^{n_i - x_i} H(j, n_i - j - x_i + 1, 1) p^{n_i - j} q^j \quad \text{for } n_i \geqslant 2 \tag{10}$$

4.3. *Case 3: $k = 3$, $x_i = 2$*

Where the number of sound sleepers, $n_i \leqslant 3$ the reliability of the cluster may be determined by analysis of possible events from probability theory. For $n_i = 1$, in order for the cluster to be reliable at time $t_0 + \tau$, the sole functional sleeper should be in working condition. Hence,

$$R^{(2)}(1, 3) = p \tag{11}$$

When $n_i = 2$, the reliable cluster requires at least one of the sound sleepers is working during the time interval $[t_0, t_0 + \tau]$. Thus, the reliability is given by

$$R^{(2)}(2, 3) = p^2 + 2pq \tag{12a}$$

and using the fact that $p + q = 1$,

$$R^{(2)}(2, 3) = p + pq \tag{12b}$$

For the case of $n_i = 3$, if the sleeper in the middle of the three sound ones is working in $[t_0, t_0 + \tau]$, then the cluster is reliable. Otherwise, the other two must be functional during the time period. Therefore,

$$R^{(2)}(3, 3) = p + p^2 q \tag{13}$$

For the case where the number of sound sleepers, $n_i \geqslant 4$, the reliability of the system given that $k = 3$ and $x = 2$, can be determined by considering all the possible states of the two sleepers next to the bounding ones during the interval $[t_0, t_0 + \tau]$. These states are as follows:

(a) Both of them fail. If the two sleepers adjacent to the bounding sleepers are considered to have failed, then for the cluster to function both of the sleepers next to the failed sleepers must be in a sound condition (Figure 3(a)). Thus, the reliability of the cluster is that of linear consecutive 3 out of $n_i - 4$ system given that the two sleepers adjacent to the bounding sleepers have failed and that the two sleepers next to these have not. Consequently, the reliability of the cluster corresponding to this scenario can be written as

$$P_1 = p^2 q^2 R(n_i - 4, 3) = \sum_{j=0}^{n_i - 4} H(j, n_i - j - 3, 2) p^{n_i - j - 2} q^{j+2} \tag{14a}$$

(b) Either of the sleepers adjacent to the bounding one fails. Similarly to case (a) above, in order for the cluster to be sound, the sleeper adjacent to the failed sleeper must be sound (Figure 3(b)). For the event that either of the sleepers adjacent to the bounding sleepers are in a failed condition and the sleepers next to this failed sleeper and the other bounding one are sound, the probability is $2p^2q$. Then subject to the event occurring, the reliability of the cluster is that of linear consecutive 3 out of $n_i - 3$ system. Hence, the reliability of the cluster corresponding to this scenario is given by

$$P_2 = 2p^2 q R(n_i - 3, 3) = 2\sum_{j=0}^{n_i - 3} H(j, n_i - j - 2, 2) p^{n_i - j - 1} q^{j+1} \tag{14b}$$

(c) Both the sleepers are sound. This case is similar to that described above for case (a), except that the cluster is a linear 3 out of $n_i - 2$ system given both of the sleepers are in a sound condition (Figure 3(c)). Therefore, by analogy with Equation (14a), the reliability of the cluster corresponding to this scenario can be given by

$$P_3 = p^2 R(n_i - 2, 3) = \sum_{j=0}^{n_i - 2} H(j, n_i - j - 1, 2) p^{n_i - j} q^j \tag{14c}$$

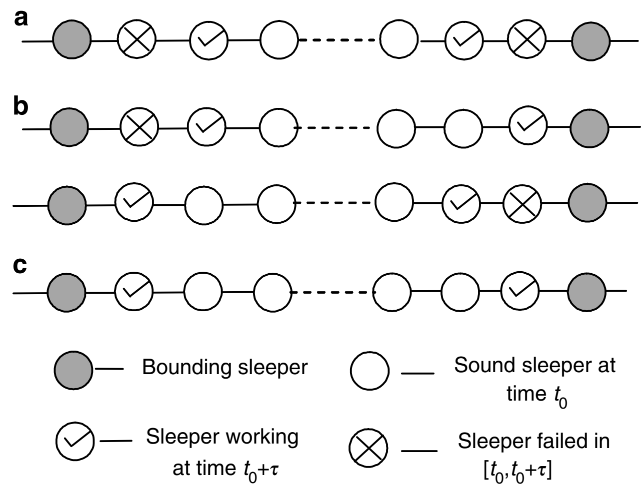

Figure 3 Graphical illustration of reliability analysis for $k = 3$, $x = 2$.

Combining the analysis given in (a), (b) and (c) above, the reliability of the cluster for $n_i > 4$ can be expressed as

$$R^{(2)}(n_i, 3) = P_1 + P_2 + P_3 \tag{15}$$

4.4. *Case 4: $k = 3$, $x_i = 1$*

When the number of initially sound sleepers, n_i, is less than 4, a cluster can be analysed by inspection of all possible states. For $n_i = 1$, according to the previous definition of the reliability of cluster, it is always functional, that is,

$$R^{(1)}(1, 3) = 1 \tag{16}$$

For the cases of $n_i = 2$ and $n_i = 3$, similar to the derivation for Equations (11)–(13), the reliability of the cluster is, respectively, given by

$$R^{(1)}(2, 3) = p + pq \tag{17}$$

$$R^{(1)}(3, 3) = p + pq \tag{18}$$

When $n_i \geqslant 4$, with the condition that there is one bounding sleeper at either end of the cluster ($x = 1$), the sleeper next to the bounding one can either remain in a sound condition or it can fail at time $t_0 + \tau$. These scenarios are similar to case 3 described above and by inspection of Equations (14a)–(14c), the reliabilities of the cluster corresponding to each scenario are

$$\begin{aligned} P_4 &= pR(n_i - 1, 3) \\ &= \sum_{j=0}^{n_i - 1} H(j, n_i - j, 2) p^{n_i - j} q^j \end{aligned} \tag{19}$$

$$\begin{aligned} P_5 &= pqR(n_i - 2, 3) \\ &= \sum_{j=0}^{n_i - 2} H(j, n_i - j - 1, 2) p^{n_i - j - 1} q^{j+1} \end{aligned} \tag{20}$$

Combining Equations (19) and (20), the reliability of the cluster for $n_i \geqslant 4$ is

$$R^{(1)}(n_i, 3) = P_4 + P_5 \tag{21}$$

A general expression for the reliability of the cluster, when $k = 3$, can be obtained by combining Equations (15) and (21) as follows:

$$\begin{aligned} R^{(x_i)}(n_i, 3) = \sum_{i=0}^{x_i} \binom{x_i}{i} \sum_{j=0}^{n_i - 2x_i + i} H(j, n_i - j - 2x_i + i + 1, 2) \\ \times p^{n_i - j - x_i + i} q^{j + x_i - i}, \qquad \text{for } n_i \geqslant 4 \end{aligned} \tag{22}$$

4.5. *Effect of failed sleepers on its neighbours*

In order to test the applicability of the assumption of independence, a Monte Carlo simulation was carried out to address the case of the dependence of sleeper failure. In the simulation, it was assumed that the residual life of a sleeper is reduced by 50% if it is adjacent to a failed one, and by 75% if it is bounded by two failed sleepers (Yun and Ferreira, 2003). The effect on the results of the simulation of varying a number of parameters, in accordance with their usual range of values, was investigated. The parameters varied were the length of time of the planning horizon, τ, which was varied between 0.5 and 2.0 years; the Weibull scale parameter of sleeper failure distribution, β, which was varied between 15 and 25; the shape parameter, α, which was varied between 2.5 and 3.5; the initial time of the planning horizon, t_0, which was varied from 0 to 70% of sleeper service life; and the number of sound sleepers in a cluster, m, which was varied between 2 and 20.

By comparing the result of the Monte Carlo simulation with that of the model proposed, the maximum difference between the two methods was found to be only 2.8%, that is, the assumption of independence is valid when the planning horizon is short (less than two years).

5. Reliability of a sleeper system

The reliability of a sleeper system, defined as m clusters of sleepers (see Section 2), can be determined by analysing the relationship between the reliability of each cluster and that of the system.

Firstly, consider the case when at least two consecutive failed sleepers cause the system to fail (ie $k = 2$). Evidently in this case, all of the clusters in the system must be reliable for the system to be reliable. Furthermore, as sleeper failures are previously assumed to occur independently of each other, it follows that the failure of clusters must also be independent of each other. Consequently, the system can be treated as a series reliability system and its reliability is given by (Barlow and Proschan, 1996)

$$R_s(\varphi_s, 2) = \prod_{i=1}^{m} R^{(x_i)}(n_i, 2) \tag{23}$$

where φ_s is the set of all the sleepers in the sleeper system.

For the case where three or more consecutively failed sleepers cause the system to fail (ie $k = 3$), the system can be in a failed state (unreliable) even if all clusters are still in working order. Let φ_i be the subsystem consisting of the set of sleepers in the clusters from 1 to i, and φ_{i+1} be the subsystem consisting of the sleepers in the clusters from 1 to $i + 1$ (see Figure 4(a)). Denoting $E_{i,i+1}$ as the event that the subsystem φ_i and cluster $i + 1$ are both reliable when considered independently, but the subsystem, φ_{i+1} has failed. Thus, a recursive relationship between the reliability of φ_{i+1} and φ_i can be written as follows:

$$R_s(\varphi_{i+1}, 3) = R_s(\varphi_i, 3) R^{(x_{i+1})}(n_{i+1}, 3) - P\{E_{i,i+1}\} \tag{24}$$

where $P\{E_{i,i+1}\}$ is the probability of the event $E_{i,i+1}$ and can be determined as follows.

Let φ_i' represent the set of sleepers in φ_i excluding the two consecutive sleepers next to right bounding sleeper of the ith cluster under the condition of $n_i > 2$ (see Figure 4(a)). By definition $E_{i,i+1}$ can only occur when the two sleepers that are adjacent to the common bounding one of ith and $(i+1)$th clusters (see Figure 4(b)) fail during the time interval $[t_0, t_0 + \tau]$. At the same time, the two sleepers adjacent to these three failed ones on either sides must be in a sound condition to ensure that the subsystem, φ_i and the cluster $i + 1$ are reliable (see Figure 4(b)). With this condition, φ_i is equivalent to φ_i', and the reliability of cluster $i + 1$ is

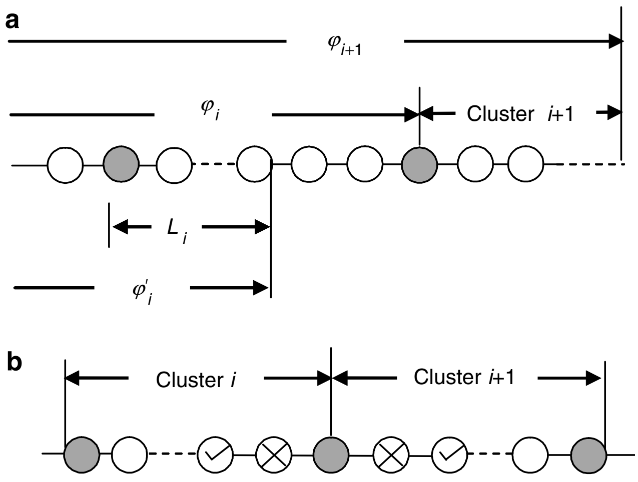

Figure 4 Graphical illustration for analysis of reliability of sleeper systems.

$R^{(x_{i+1}-1)}(n_{i+1}-2, 3)$. Therefore,

$$P\{E_{i,i+1}\} = p^2q^2R_s(\varphi'_i, 3)R^{(x_{i+1}-1)}(n_{i+1}-2, 3) \quad \text{for } n_{i+1} \geqslant 3 \tag{25a}$$

For the case of $n_{i+1} = 1$, $x_{i+1} = 2$, $E_{i,i+1}$ cannot occur. Consequently,

$$P\{E_{i,i+1}\} = 0 \qquad \text{for } n_{i+1} = 1 \text{ and } x_{i+1} = 2 \tag{25b}$$

When $n_{i+1} = 2$, $E_{i,i+1}$ can only occur if the sleeper next to the left bounding sleeper of cluster $i + 1$ fails and the other one is sound during the time interval $[t_0, t_0 + \tau]$. Thus,

$$P\{E_{i,i+1}\} = p^2q^2R_s(\varphi'_i, 3) \qquad \text{for } n_{i+1} = 2 \tag{25c}$$

For the case of $n_{i+1} = 1$ and $x_{i+1} = 1$, the cluster $i + 1$ can never fail as the cluster size is less than 3 for the case of $k=3$. Similarly to the analysis used to determine for Equation (25c):

$$P\{E_{i,i+1}\} = pq^2R_s(\varphi'_i, 3) \quad \text{for } n_{i+1} = 1 \text{ and } x_{i+1} = 1 \tag{25d}$$

Let L_i be the set of sleepers in cluster i excluding the two consecutive sleepers and the right bounding sleeper, as shown in Figure 4(a). Denoting $D_{i-1,i}$ as the event that φ_{i-1} and L_i are reliable when considered separately, but the subsystem consisting of φ_{i-1} and L_i fails. Using a similar analysis as that used to derive Equations (24) and (25a) above, the reliability of φ'_i can be written as

$$R_s(\varphi'_i, 3) = R_s(\varphi_{i-1}, 3)R^{(x_i-1)}(n_i-2, 3) - P\{D_{i-1,i}\} \tag{26}$$

$$P\{D_{i-1,i}\} = p^2q^2R_s(\varphi'_{i-1}, 3)R^{(x_i-1)}(n_i-4, 3) \qquad \text{for } n_i \geqslant 5 \tag{27a}$$

When $n_i=1$, $P\{E_{i,i+1}\}=0$, for $n_i=2$, $\varphi'_i=\varphi_{i-1}$ and therefore, for $n_i \leqslant 2$, it is not necessary to calculate $R_s(\varphi'_i, 3)$ in order to obtain $R_s(\varphi_{i+1}, 3)$. For the case $n_i = 3$, $D_{i-1,i}$ can occur only if the only sleeper next to the bounding one in L_i fails during the time period $[t_0, t_0 + \tau]$. Thus,

$$P\{D_{i-1,i}\} = pq^2R_s(\varphi'_{i-1}, 3) \qquad \text{for } n_i = 3 \tag{27b}$$

Similarly for $n_i = 4$,

$$P\{D_{i-1,i}\} = p^2q^2R_s(\varphi'_{i-1}, 3) \qquad \text{for } n_i = 4 \tag{27c}$$

The above recursive equations can be solved sequentially for $R_s(\varphi_1, 3)$, $R_s(\varphi'_1, 3)$ and then $R_s(\varphi_2, 3)$, $R_s(\varphi'_2, 3)$ and finally for $R_s(\varphi_m, 3)$. Evidently, since $\varphi_m = \varphi_s$, the reliability of the sleeper system can be obtained by $2m$ steps of recursive calculation. Usually, the number of failed sleepers in a track segment is not large and thus the computational effort to determine the reliability $R_s(\varphi_s, 3)$ is small. For example, the UK railway standards specify (Railtrack, 2000) that no more than six failed sleepers are allowed to remain within an 18 m length of railway track (approximately 26 sleepers). In this case, it would take no more than 12 recursive calculations to obtain $R_s(\varphi_s, 3)$.

6. Optimization of immediate maintenance

During an inspection if any failed sleepers are detected, a decision must be made whether to conduct maintenance immediately or to postpone the maintenance so that it is included as part of the next major maintenance treatment. Since the cost of immediate maintenance per sleeper is more expensive than for major maintenance, the number of failed sleepers replaced immediately should be kept to a minimum. The number of failed sleepers that are to be replaced immediately can be minimized by determining whether the system will be reliable until the next scheduled maintenance if the sleepers are not replaced immediately.

Several railway organizations use two different sets of criteria to judge the performance and safety of a sleeper system, respectively (eg Railtrack, 2000). In the UK for example, performance standards are based on at least two consecutive sleepers failing (ie $k = 2$) whereas safety criteria are specified according to three or more consecutive sleepers failing (ie $k = 3$). Both requirements are discussed below.

If h is the number of sleepers replaced immediately, then the optimal decision for sleeper maintenance can be expressed as

$$\text{minimize} \quad h \tag{28a}$$
$$\text{subject to} \quad R_s(\varphi_s, 2) \geqslant f_r \tag{28b}$$
$$R_s(\varphi_s, 3) \geqslant f_s \tag{28c}$$

where f_r is the performance requirement for reliable operation of a sleeper system and f_s is the standard for safe sleeper service. Note that $f_s > f_r$.

The problem represented by Equations (28a)–(28c) is complex as they are integer decision variables with nonlinear constraints. To obtain a near-optimal solution, in this paper, a mathematical technique known as the steepest gradient method is employed, which has been used in the optimal allocation of redundancy (Barlow and Proschan, 1996). In addition, during the analysis the case of at least three consecutive failed sleepers is considered to have a higher priority than two consecutive failed ones. This is because system failure caused by three consecutive sleepers could potentially result in more severe consequences than for the case where two consecutive sleepers have failed.

In the procedure used to obtain a solution, one failed sleeper is replaced during one step. For each step during the procedure the failed sleeper to be replaced is that which will increase $R_s(\varphi_s, 3)$ by greatest amount. Failed sleepers are replaced until Equation (28c) has been satisfied. At this stage, if the condition represented by Equation (28b) has been satisfied, the procedure is stopped, and the solution is deemed to be found. If the condition is not satisfied then the sleeper whose removal will increase $R_s(\varphi_s, 2)$ by the largest amount is replaced successively during each step until the condition represented by Equation (28b) is satisfied.

At time t_0 a failed sleeper, i, will bound two clusters. Suppose one cluster contains n_i sleepers and the other n_{i+1} sleepers. When the maximum number of consecutive failed sleepers that causes system failure is equal to 2 (ie $k = 2$), the reliability of the two clusters of sleepers may be given by Equation (23) as: $R^{(x_i)}(n_i, k)R^{(x_{i+1})}(n_{i+1}, k)$.

However, if the failed sleeper is replaced the two clusters become a single cluster. Consequently, the total number of sleepers in this new cluster is $n_i + n_{i+1} + 1$ and the number of bounding sleepers is $x_i + x_{i+1} - 2$. Then the reliability of the new cluster becomes: $R^{(x_i+x_{i+1}-2)}(n_i + n_{i+1} + 1, k)$.

Let R_c be the reliability of the sleeper system excluding the two clusters. Hence, the change in system reliability by the replacement is given by

$$\Delta R_s(\varphi_s, k) = [R^{(x_i+x_{i+1}-2)}(n_i + n_{i+1} + 1, k) - R^{(x_i)}(n_i, k)R^{(x_{i+1})}(n_{i+1}, k)]R_c \quad (29)$$

An interesting case to examine is that two consecutive clusters in a system are partitioned by a failed sleeper and just one functional sleeper is contained in each of the clusters (ie $k = 2$, $x_i = x_{i+1} = 2$, $n_i = n_{i+1} = 1$). From Equations (6b) and (7) we have $R^{(2)}(3, 2) = p^2$, and $R^{(2)}(1, 2) = p$ and therefore, $\Delta R_s(\varphi_s, 2) = 0$. Consequently, and counter-intuitively, in this case system reliability cannot be improved by only replacing the failed sleeper which bounds the two clusters.

However, this result does not hold generally for the case $k = 3$, $x_1 = x_2 = 2$, $n_1 = n_2 = 1$. For simplicity, we consider a system only containing two clusters with $n_1 = 1$, $n_2 = 1$ and $x_1 = x_2 = 2$. Since $P\{E_{1,2}\} = 0$, Equation (29) is valid. From Equations (11), (13) and (29), we have

$$\begin{aligned}\Delta R_s(\varphi_s, 3) &= R^{(2)}(3, 3) - [R^{(2)}(1, 3)]^2 \\ &= p^2 q + pq\end{aligned} \quad (30)$$

Since $p > 0$ and $q > 0$ for $\tau > 0$, then $\Delta R_s(\varphi_s, 3) > 0$. Consequently, if the failed sleeper in the middle is replaced, the safety-related reliability will be improved. For example, when $p = 0.9$, the reliabilities before and after the replacement of the failed sleeper in the middle are 0.81 and 0.981, respectively (ie a 21% increase in system reliability is obtained by the replacement). The result, in turn, supports the notion that both reliability and safety requirements should be considered in the decision-making process for the case of $k = 2$.

7. Illustrative example

As an example of the proposed approach, an analysis of reliability and maintenance for a system of timber sleepers has been undertaken. Relevant data for the study are presented in Table 1, in which the values α and β have been taken from Lake *et al* (2000a).

Figure 5 shows the reliability of the sleeper system for the cases when $k = 2$ and 3, as a function of the time period, τ.

Table 1 Parameters and sleeper conditions

Parameter and sleeper condition	*Parameter value*
Number of total sleepers in the system	56
Number of failed sleepers	10
Age of sleepers, t_0, (year)	5
Sequential numbers of failed sleepers at t_0	3, 11, 13, 15, 26, 32, 35, 39, 41, 49
Parameter of lifetime distribution α	3.0
Parameter of lifetime distribution β	20.0
Time to the next major maintenance (year)	1
Requirement of reliable operation	0.90
Requirement of safe operation	0.99

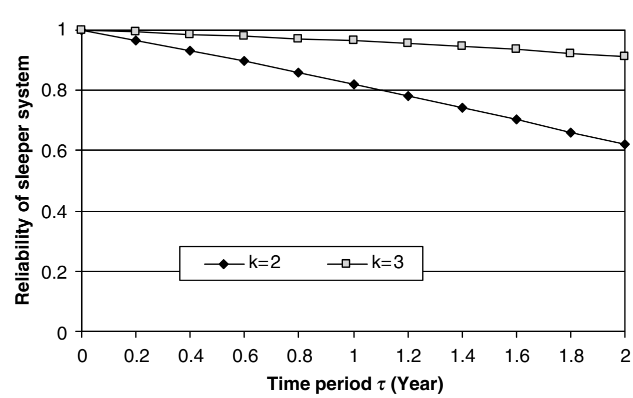

Figure 5 Reliability of the sleeper system within a period of time.

Table 2 Optimization example

Step	*Sequential number of sleeper replaced at each step*	$R_S(\varphi_s, 2)$	$R_S(\varphi_s, 3)$
0	Not applicable	0.822	0.965
1	13	0.822	0.986
2	39	0.831	0.998*
3	35	0.85	0.998
4	11	0.869	0.999
5	3	0.889	0.999
6	32	0.909*	0.999

*Means the requirement of reliability or safety is first satisfied at the corresponding step.

It may be seen that the reliability of a sleeper cluster presented in Section 4, the reliability of a system for $k = 2$ is lower and decreases faster than $k = 3$. The reliabilities within 1 year are 0.822 for $k = 2$ and 0.965 for $k = 3$. Obviously, both the requirements of reliable and safe operation are not met at time t_0.

The optimization of sleeper maintenance is summarized in Table 2, which gives the system reliability for $k = 2$ and 3 after each step of the process described above. In the first step, the 13th sleeper in the system is replaced, in which a maximum improvement of safety-related reliability (ie $k = 3$) is obtained in the current situation. This situation is that discussed in Section 6, that is $k = 3$, $x_i = x_{i+1} = 2$, $n_i = n_{i+1} = 1$. As can be seen from Table 2 for such a case, the reliability for $k = 2$ is unchanged by making the replacement, but the reliability for $k = 3$ is increased significantly. When the second failed sleeper (number 39) is removed, the requirement of safe operation, f_s, is satisfied. In the subsequent search, the next sleeper to be replaced is the one that will maximize the improvement in reliability for $k = 2$ (ie maximize $\Delta R_s(\varphi_s, 2)$ in Equation (29)). Finally, the requirements are satisfied after the sixth failed sleeper is removed from the track.

8. Concluding discussion

In this paper, firstly, using the theory of a consecutive k out of n system, a model has been developed to evaluate the reliability of a segment of sleepers when the sleeper conditions are known. Two cases of particular interest to the railway industry were considered. These are failures of a sleeper system caused by at least two consecutive failed sleepers ($k = 2$) and by at least three consecutive ones ($k = 3$). For the case of $k = 2$, it was shown that the reliability of the sleeper system can be treated as a series system of clusters, whereas this does not generally hold for the case $k = 3$. Therefore, in order to evaluate the system reliability for $k = 3$, a recursive procedure was presented.

Based on the proposed model an optimal decision making process which may be used to determine whether maintenance should be carried out immediately after an inspection or deferred was formulated. To investigate the use of the procedure, a special case was analysed where a system contains two consecutive clusters partitioned by a failed sleeper with only one functional sleeper in each of the clusters. The result of the analysis highlighted the need to consider the reliability of a system in terms of three consecutive failed sleepers even when railway standards specify the need to consider only two consecutive failed sleepers (ie $k = 2$). Finally, an example showing the decision-making process was given to illustrate the performance of the model.

A number of assumptions were made in the research presented, the main ones are that the failures of sleepers are statistically independent during a given time period and that maintenance is assumed to be a minimal repair. The former assumption was validated by a Monte Carlo simulation, while the latter may be regarded as a conservative assumption for the case where failed components are replaced by new ones rather than serviceable ones.

Although a focus is given to the reliability and maintenance of railway sleepers in this paper, the approach presented may be also applicable to other practical consecutive k out of n systems, where the assumptions presented in Section 2 are valid.

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