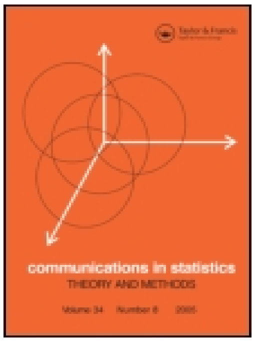

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Bayesian Estimation for the Exponentiated Weibull Model

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ABSTRACT

In this article, Bayes estimates of the two shape parameters, reliability, and failure rate functions of the exponentiated Weibull lifetime model are derived from complete and type II censored samples. When the Bayesian aproach is considered, conjugate priors for either the one or the two shape parameters cases are considered. An approximation form due to Lindely [Lindely, D. V. (1980). Approximate Bayesian method. *Trabajos de Estadistica* 31:223–237] is used for obtaining the Bayes estimates under the squared error loss and LINEX loss functions. The root mean square errors of the estimates

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are computed. Comparisons are made between these estimators and the maximum likelihood ones using Monte Carlo simulation study.

Key Words: Bayesian analysis; Informative prior; Squared error loss function; LINEX loss function; Maximum likelihood estimates; EW distribution.

1. INTRODUCTION

The exponentiated Weibull (EW) distribution was introduced by Mudholkar and Srivastava (1993). This distribution is an extension of the well-known Weibull distribution. The EW family contains distributions with nonmonotone failure rates besides a broader class of monotone failure rates. Practically, many lifetime data are of bathtub shape or upside-down bathtub shape failure rates and so the exponentiated Weibull distribution as a failure model is more realistic than that of monotone failure rates and plays an important role to represent such data.

The two shape parameter EW distribution has a probability density function (pdf) of the form

$$f(t) = \alpha\theta t^{\alpha-1} e^{-t^{\alpha}} \left(1 - e^{-t^{\alpha}}\right)^{\theta-1}, \quad t > 0, \ \ \alpha > 0, \ \ \theta > 0 \tag{1.1}$$

and a cumulative distribution function

$$F(t) = \left(1 - e^{-t^{\alpha}}\right)^{\theta}, \quad t > 0 \tag{1.2}$$

where α and θ are the two shape parameters. It is important to mention that when $\theta = 1$ the EW pdf is that of the Weibull distribution. For $\theta > 1$, the EW distribution has a unique mode, $[2(\alpha\theta - 1)/\alpha(\theta + 1)]^{1/\alpha}$. The median of the distribution is $[-\ell n(1 - 2^{-1/\theta})]^{1/\alpha}$. Other statistical properties of this distribution are discussed by Mudholkar and Hutson (1996) and Nassar and Eissa (2003).

The reliability and failure rate functions of the EW distribution are given, respectively, by

$$R(t) = 1 - \left(1 - e^{-t^{\alpha}}\right)^{\theta}, \quad t > 0, \tag{1.3}$$

$$H(t) = \alpha\theta t^{\alpha-1} e^{-t^{\alpha}} \left(1 - e^{-t^{\alpha}}\right)^{\theta-1} \left[1 - \left(1 - e^{-t^{\alpha}}\right)^{\theta}\right]^{-1}, \quad t > 0 \tag{1.4}$$

Currently, there are little studies for the use of the EW distribution in reliability estimation and so this article is to promote its use in applications. The EW distribution can be widely and effectively used in reliability applications because it has a wide variety of shapes in its density and failure rate functions, making it useful for fitting many types of data. Maximum likelihood estimations (besides testing of hypotheses) for the distribution are discussed by Mudholkar et al. (1995), using several sets of data.

This article is concerned with Bayesian estimation using complete and type II censored data from the EW distribution. Maximum likelihood and Bayes estimates of the shape parameters α and θ, reliability, and failure rate functions of this distribution are derived. The cases in which one of the two shape parameters is known and both of them are unknown and dependent are considered. Finally, numeric illustration and comparisons are presented.

It is important to state that, in Bayesian estimation, we consider two types of loss functions. The first is the squared error loss function (quadratic loss), which is classified as a symmetric function and associates equal importance to the losses for overestimation and underestimation of equal magnitude. The second, the LINEX (linear-exponential) loss function, which is asymmetric, was introduced by Varian (1975). These loss functions were widely used by several authors, among them Canfield (1970), Zellner (1986), Rojo (1987), Basu and Ebrahimi (1991), Pandey and Rai (1992), Pandey et al. (1996), Calabria and Pulcini (1996), Pandey (1997), and Soliman (2000, 2001).

The quadratic loss for Bayes estimate of a parameter β, say, is the posterior mean, assuming it exists and is denoted by $\hat{\beta}_S$. The LINEX loss function may be expressed as

$$L(\delta) \propto e^{c\delta} - c\delta - 1, \quad c \neq 0,$$

where $\delta = \hat{\beta} - \beta$. The sign and magnitude of c reflects the direction and degree of asymmetry, respectively. The Bayes estimator for β relative to LINEX loss function, denoted by $\hat{\beta}_L$, is given by

$$\hat{\beta}_L = -\frac{1}{c} \ln E_\beta(e^{-c\beta}), \tag{1.5}$$

provided that $E_\beta(e^{-c\beta})$ exists and is finite, where E_β denotes the expected value.

2. MAXIMUM LIKELIHOOD ESTIMATION

Suppose a type II censored sample $t = (t_1, t_2, \ldots, t_r)$ is considered where t_i is the time of the ith component to fail. This sample of failure times are obtained and recorded from a life test of n items whose lifetimes have an $EW(\alpha, \theta)$ distribution with density and distribution functions given, respectively, by (1.1) and (1.2). The likelihood function (LF) in this case can be written as:

$$\ell(t; \alpha, \theta) \propto \alpha^r \theta^r e^{-T} (1 - v^{\theta})^{n-r} \tag{2.1}$$

where

$$T = \sum_{i=1}^{r} [t_i^{\alpha} - (\alpha - 1)\ln t_i - (\theta - 1)\ln u_i],$$

$$u_i = 1 - e^{-t_i^{\alpha}} \quad \text{and} \quad v = 1 - e^{-t_r^{\alpha}}.$$

The logarithm of the LF is given by

$$L = \log \ell(t; \alpha, \theta) \propto r \ln \alpha + r \ln \theta - T + (n - r) \ln(1 - v^{\theta}) \tag{2.2}$$

Assuming a case in which α is known, the maximum likelihood estimator (MLE), of θ, denoted by $\hat{\theta}_M$, is given by

$$\hat{\theta}_M = r \Big/ \left[(n - r) V \ln v - \sum_{i=1}^{r} \ln u_i \right] \tag{2.3}$$

where

$$V = (v^{-\theta} - 1)^{-1}. \tag{2.4}$$

If both of the parameters α and θ are unknown, their MLEs $\hat{\alpha}_M$ and $\hat{\theta}_M$ can be obtained by solving the following likelihood equations:

$$\begin{aligned} &\frac{r}{\alpha} + \theta T_2 - (n - r)\theta V T_3 - (T_1 + T_2) = 0, \\ &\frac{r}{\theta} - (n - r) V \ln v + \sum_{i=1}^{r} \ln u_i = 0 \end{aligned} \tag{2.5}$$

where

$$T_1 = \sum_{i=1}^{r} (t_i^{\alpha} - 1)\ln t_i, \quad T_2 = \sum_{i=1}^{r} \gamma_i \ln t_i, \quad T_3 = \gamma \ln t_r,$$

$$\gamma_i = t_i^{\alpha} e^{-t_i^{\alpha}} u_i^{-1} \quad \text{and} \quad \gamma = t_r^{\alpha} e^{-t_r^{\alpha}} v^{-1}. \tag{2.6}$$

The required estimates $\hat{\alpha}_M$ and $\hat{\theta}_M$ are to be found by solving simultaneously the two equations (2.5). Clearly these equations are transcendental equations in α and θ and no closed from solutions are known and so it may be solved using an iterative numeric technique, such as Newton–Raphson iteration, to get the estimates. For a given t, the MLEs of $R(t)$ and $H(t)$ may be obtained by replacing α and θ by $\hat{\alpha}_M$ and $\hat{\theta}_M$ in equations (1.3) and (1.4), respectively.

3. BAYESIAN ESTIMATION WITH KNOWN α

Under the assumption that the parameter α is known, the natural family of conjugate prior for θ is a gamma distribution with density function

$$g(\theta) = \frac{\delta^{\nu}}{\Gamma(\nu)} \theta^{\nu-1} e^{-\delta\theta}, \quad \theta > 0, \ \ \nu \ \& \ \delta > 0 \tag{3.1}$$

from which the prior mean and variance of θ are given, respectively, by ν/δ and ν/δ^2.

Applying Bayes theorem, we obtain from Eqs. (2.1) and (3.1), the posterior density of θ as

$$g(\theta/t, \alpha) = \frac{(\delta + q)^{r+\nu}}{k\Gamma(r+\nu)} \theta^{r+\nu-1} e^{-(\delta+q)\theta} (1 - v^{\theta})^{n-r}, \quad \theta > 0 \tag{3.2}$$

where

$$k = \sum_{j=0}^{n-r} \omega(j) \left(1 - \frac{jp}{\delta + q}\right)^{-(r+\nu)}, \tag{3.3}$$

$$\omega(j) = (-1)^j \binom{n-r}{j}, \quad q = -\ln u, \quad p = \ln v,$$

$$u = \prod_{i=1}^{r} (1 - e^{-t_i^{\alpha}}) \quad \text{and} \quad v = 1 - e^{-t_r^{\alpha}}. \tag{3.4}$$

Note that at $r = n$, the constant $k = 1$ and $g(\theta/t, \alpha)$ approaches to the posterior density in a complete sample situation.

3.1. Estimation of θ

The Bayes estimate $\hat{\theta}_S$ of θ relative to squared error loss function is given by

$$\hat{\theta}_S = k^{-1}\frac{r+\nu}{\delta+q}\xi_1, \tag{3.5}$$

where

$$\xi_1 = \sum_{j=0}^{n-r}\omega(j)\left(1-\frac{jp}{\delta+q}\right)^{-(r+\nu+1)}.$$

Under LINEX loss function, the Bayes estimate $\hat{\theta}_L$ of θ using Eq. (1.5) can be obtained as

$$\hat{\theta}_L = -\frac{1}{c}\ln\xi_2, \quad c \neq 0 \tag{3.6}$$

where

$$\xi_2 = k^{-1}\sum_{j=0}^{n-r}\omega(j)\left(1+\frac{c-jp}{\delta+q}\right)^{-(r+\nu)}.$$

3.2. Estimation of $R(t)$

Let the reliability $R = R(t)$ be a parameter itself. Replacing θ in terms of R by that of Eq. (3.2), we obtain the posterior density function R as

$$g(R/t,\alpha) = k^{-1}\frac{Q^{r+\nu}}{\Gamma(r+\nu)}[\varphi_1(R)]^{r+\nu-1}e^{-(Q-1)\varphi_1(R)}\left(1-v^{Z\varphi_1(R)}\right)^{n-r}, \quad 0 < R < 1 \tag{3.7}$$

where

$$Q = Z(\delta+q), \quad \varphi_1(R) = \ln(1-R)^{-1}, \quad Z = 1/\ln z^{-1}, \quad z = z(t) = 1 - e^{-t^{\alpha}}. \tag{3.8}$$

Assuming the quadratic loss is the appropriate, the Bayes estimate of the reliability function R is

$$\widehat{R}_S = k^{-1}\sum_{j=0}^{n-r}\omega(j)(\zeta_1 - \zeta_2) \tag{3.9}$$

where

$$\zeta_1 = \left(1 - \frac{jpZ}{Q}\right)^{-(r+\nu)}, \quad \zeta_2 = \left(1 - \frac{jpZ-1}{Q}\right)^{-(r+\nu)},$$

and k and $\omega(j)$ are given, respectively, by Eqs. (3.3) and (3.4).

Under LINEX loss function, the Bayes estimate of R using Eq. (1.5) is

$$\widehat{R}_L = 1 - \frac{1}{c}\ln\zeta_3, \quad c \neq 0 \tag{3.10}$$

where

$$\zeta_3 = k^{-1}\sum_{s=0}^{\infty}\frac{c^s}{s!}\sum_{j=0}^{n-r}\omega(j)\left(1 + \frac{s - jpZ}{Q}\right)^{-(r+\nu)}.$$

3.3. Estimation of $H(t)$

To derive the Bayes estimate of the failure rate function $H(t)$, we first obtain the posterior density function of $H = H(t)$, which can be given by

$$g(H/t,\alpha) = k^{-1}\frac{Q^{r+\nu}}{\Gamma(r+\nu)}\frac{e^{-H}}{1-e^{-H}}[\varphi_2(H)]^{r+\nu-1}e^{-Q\varphi_2(H)}\left(1 - v^{Z\varphi_2(H)}\right)^{n-r}, \quad H > 0 \tag{3.11}$$

where

$$\varphi_2(H) = \ln\left(1 - e^{-H}\right)^{-1}.$$

The Bayes estimate of the failure rate function H relative to quadratic loss is

$$\widehat{H}_S = k^{-1}\frac{Q^{r+\nu}}{\Gamma(r+\nu)}G_1 \tag{3.12}$$

where

$$G_1 = \int_0^{\infty} \ln(1 - e^{-x})^{-1} x^{r+\nu-1} e^{-Qx} (1 - v^{Zx})^{n-r} dx.$$

When the LINEX loss function is the appropriate, the Bayes estimate of H is

$$\widehat{H}_S = -\frac{1}{c} \ln\left(\frac{k^{-1} Q^{r+\nu}}{\Gamma(r+\nu)} G_2\right), \quad c \neq 0 \tag{3.13}$$

where

$$G_2 = \int_0^{\infty} \ln(1 - e^{-x})^{c} x^{r+\nu-1} e^{-Qx} (1 - v^{Zx})^{n-r} dx.$$

Resorting to numeric integration, we may determine the nonclosed form integral G_1 and G_2 and then the Bayes estimates $\widehat{H}_S$ and $\widehat{H}_L$ are evaluated.

4. BAYESIAN ESTIMATION WITH UNKNOWN α AND θ

In this section, we consider the typical case in which the two shape parameters α and θ of an EW process are unknown. We suppose some information on the shape parameters α and θ is available priori. Formulation of a joint prior density would be constructed. We suggest a conditional prior distribution of θ given α, which may appropriately be the conjugate gamma with density function that possesses a hypershape parameter $\nu > 0$ and is given by

$$g_1(\theta/\alpha) = \frac{\alpha^{-\nu}}{\Gamma(\nu)} \theta^{\nu-1} e^{-\theta/\alpha}, \quad \theta > 0. \tag{4.1}$$

The scale parameter of this density is α, which is assumed to become known previously with knowledge that may be translated into an exponential distribution with density function

$$g_2(\alpha) = \frac{1}{d} e^{-\alpha/d}, \quad \alpha > 0, \tag{4.2}$$

where d is a positive hyperscale parameter and a mean value of α.

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Multiplying $g_1(\theta/\alpha)$ by $g_2(\alpha)$, we obtain the bivariate prior density of α and θ by

$$g(\alpha,\theta) = (d\Gamma(\nu))^{-1}\alpha^{-\nu}\theta^{\nu-1}e^{-(\alpha^2+d\theta)/d\alpha}, \quad \alpha>0, \;\; \theta>0. \tag{4.3}$$

Combining the likelihood equation (2.1) and the prior density function (4.3), the joint posterior density of α and θ is

$$g(\alpha,\theta/t) \propto \alpha^{r-\nu}\theta^{r+\nu-1}(1-v^{\theta})^{n-r}e^{-(\frac{\alpha^2+d\theta}{d\alpha}+T)}, \quad \alpha>0, \;\; \theta>0. \tag{4.4}$$

4.1. Estimation Under Squared Error Loss Function

Under squared error loss function, the Bayes estimate of a function $w = w(\alpha,\theta)$, $\hat{w}_S$, is the posterior s-expectation given as

$$\begin{aligned}\hat{w}_S &= E(w/t)\\ &= \int_\alpha\int_\theta w(\alpha,\theta)\ell(t;\,\alpha,\theta)g(\alpha,\theta)d\alpha d\theta \Big/ \int_\alpha\int_\theta \ell(t;\,\alpha,\theta)g(\alpha,\theta)d\alpha\, d\theta\end{aligned} \tag{4.5}$$

where $\ell(t;\alpha,\theta)$ and $g(\alpha,\theta)$ are the likelihood and the prior functions given, respectively, by Eq. (2.1) and (4.3). The ratio of the two integrals given by (4.5) cannot, in general, be obtained in a closed form. Therefore, in such situation, we resort to use a numeric integration technique. In what follows, we consider Lindely's approximation method to evaluate the Bayes estimate $\hat{w}_S$.

Lindely's procedure was developed by Lindely (1980) to evaluate the ratio of two integrals such as (4.5). This procedure has been used by many authors to obtain Bayes estimators of parameters of various distributions, such as Sinha and Sloan (1988), Howlader and Weiss (1989), AL-Hussiani and Jaheen (1992, 1994), Ahmad et al. (1997), and Soliman (2001).

In a two parameter case, $\lambda = (\lambda_1,\, \lambda_2)$, say, Lindely's approximation form reduces $\hat{w}_S$ to the following:

$$\hat{w}_S = w(\lambda) + \frac{1}{2}(A + \ell_{30}B_{12} + \ell_{03}B_{21} + \ell_{21}C_{12} + \ell_{12}C_{21}) + p_1A_{12} + p_2A_{21} \tag{4.6}$$

where

$$A = \sum_{i=1}^{2}\sum_{j=1}^{2} w_{ij}\tau_{ij}; \quad \ell_{\eta\zeta} = \frac{\partial^{\eta+\zeta}\ell}{\partial\lambda_1^{\eta}\partial\lambda_2^{\zeta}}; \quad \eta \;\&\; \zeta = 0, 1, 2, 3; \quad \eta + \zeta = 3,$$

$$\text{for } i, j = 1, 2, \;\; p_i = \frac{\partial p}{\partial\lambda_i}, \;\; w_i = \frac{\partial w}{\partial\lambda_i}; \;\; w_{ij} = \frac{\partial^2 w}{\partial\lambda_i\partial\lambda_j},$$

where $p = \ln g(\alpha, \theta)$ and for $i \neq j$,

$$A_{ij} = w_i\tau_{ii} + w_j\tau_{ji}, \quad B_{ij} = (w_i\tau_{ii} + w_j\tau_{ij})\tau_{ii},$$

$$C_{ij} = 3w_i\tau_{ii}\tau_{ij} + w_{\mathrm{j}}(\tau_{ii}\tau_{jj} + 2\tau_{ij}{}^2),$$

where τ_{ij} is the $(i,\; j)$th element in the inverse of the matrix $\{-\ell_{ij}\}$, $i, j = 1,2$ such that $\ell_{ij} = \frac{\partial^2\ell}{\partial\lambda_i\partial\lambda_j}$. Expression (4.6) is to be evaluated at the maximum likelihood estimates of λ_1 and λ_2.

Now, we apply Lindely's form (4.6) in our case in which $(\lambda_1, \lambda_2) \equiv (\alpha, \theta)$, $w(\lambda) \equiv w(\alpha, \theta)$, and we first obtain the elements τ_{ij}, which can be obtained as

$$\tau_{11} = H/N, \quad \tau_{22} = G/N, \quad \tau_{12} = \tau_{21} = -I/N \tag{4.7}$$

where

$$N = GH/I^2,$$

$$G = \frac{r}{\alpha^2} + \sum_{i=1}^{r}[t_i^{\alpha} - (\theta - 1)\gamma_i(1 - t_i^{\alpha}u_i^{-1})](\ln t_i)^2$$
$$- (n - r)\theta V\gamma(t_r^{\alpha}v^{-1} - \theta V\gamma v^{-\theta} - 1)(\ln t_r^{\alpha})^2,$$

$$H = \frac{r}{\theta^2} + (n - r)V^2 v^{-\theta}(\ln v)^2,$$

$$\text{and} \quad I = (n - r)V\gamma(\theta V v^{-\theta}\ln v + 1)\ln t_r - \sum_{i=1}^{r}\gamma_i \ln t_i.$$

where V and γ_i, γ are given by Eqs. (2.4) and (2.6), respectively.

The values of $\ell_{\eta\zeta};\ \eta,\ \zeta = 0, 1, 2, 3$ can be obtained as follows

$$\ell_{12} = -(n-r)V^2 v^{-\theta}\gamma(2+2\theta v^{-\theta}V\ln v - \theta\ln v)\ln t_r \ln v$$

$$\begin{aligned}\ell_{21} = {} & \sum_{i=1}^{r}\gamma_i(1-t_i^{\alpha}u_i^{-1})(\ln t_i)^2 \\ & -(n-r)V\gamma\{1-t_r^{\alpha}v^{-1}+\theta v^{-\theta}V[\gamma+\ln v(1-\theta\gamma-t_r^{\alpha}v^{-1} \\ & \qquad +2\theta\gamma v^{-\theta}V)]\}(\ln t_r)^2,\end{aligned}$$

$$\begin{aligned}\ell_{30} = {} & \frac{2r}{\alpha^3} - \sum_{i=1}^{r} t_i^{\alpha}(\ln t_i)^3 + (\theta-1)\sum_{i=1}^{r}\gamma_i[1-t_i^{\alpha}u_i^{-1}(3-t_i^{\alpha}-2\gamma_i)](\ln t_i)^3 \\ & +(n-r)\theta\{\gamma V(3t_r^{\alpha}v^{-1}-t_r^{\alpha}v^{-1}-1)+\gamma^2[\theta v^{-\theta}V^2(t_r^{\alpha}v^{-1}-3) \\ & \qquad +2t_r^{\alpha}(\theta v^{-\theta}-v^{-1}V)+\gamma^3\theta v^{-\theta}(2+\theta-2\theta v^{-\theta}V^3\}(\ln t_r)^3,\end{aligned}$$

and

$$\ell_{03} = \frac{2r}{\theta^3} - (n-r)v^{-\theta}V^2(2v^{-\theta}V-1)(\ln v)^3. \tag{4.8}$$

Using prior density (4.3), we obtain

$$p = \ln g(\alpha, \theta) = \text{consnt.} - \nu\ln\alpha + (\nu-1)\ln\theta - \frac{\theta}{\alpha} - \frac{\alpha}{d}$$

from which we get

$$p_1 = \frac{\partial p}{\partial p} = \frac{\theta}{\alpha^2} - \frac{\nu}{\alpha} - \frac{1}{d}, \quad p_2 = \frac{\partial p}{\partial\theta} = \frac{\nu-1}{\theta} - \frac{1}{\alpha}. \tag{4.9}$$

Using equations (4.7) to (4.9), the Bayes estimate of the function $w(\alpha,\theta)$ relative to squared error loss function given by Eq. (4.6), can be shown to be

$$\hat{w}_S = w(\alpha,\theta) + \Phi + \Psi_1 w_1 + \Psi_2 w_2 \tag{4.10}$$

where

$$\Phi = \frac{1}{2N}[Hw_{11} - I(w_{12}+w_{21}) + Gw_{22}],$$

$$\Psi_1 = \frac{1}{N}(Hp_1 - Ip_2) + \frac{1}{2N^2}[H^2\ell_{30} - IG\ell_{03} + (GH+2I^2)\ell_{12} - 3IH\ell_{21}],$$

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and

$$\Psi_2 = \frac{1}{N}(Gp_2 - Ip_1) + \frac{1}{2N^2}[G^2\ell_{03} - IH\ell_{30} + (GH + 2I^2)\ell_{21} - 3IG\ell_{12}].$$

All functions of the right-hand side of Eq. (4.10) are to be evaluated at $\hat{\alpha}_M, \hat{\theta}_M$.

From equation (4.10), we can deduce the values of the Bayes estimates of various parameters in what follows.

(I) If $w(\alpha, \theta) = \alpha$, then

$$\hat{\alpha}_S = \alpha + \Psi_1. \tag{4.11}$$

(II) If $w(\alpha, \theta) = \theta$, then

$$\hat{\theta}_S = \theta + \Psi_2. \tag{4.12}$$

(III) If $w(\alpha, \theta) = R(t)$ given by Eq. (1.3), then

$$\widehat{R}_S = R(t)(1 - W_1 + W_2) + W_1 - W_2 \tag{4.13}$$

where

$$W_1 = \frac{1}{2N}\{2\beta I[1 + \ln(1 - R(t))]\ln t - \theta\beta H[1 - e^{-t^\alpha}\beta + \theta\beta](\ln t)^2 - \frac{G}{\theta^2}[\ln(1 - R(t))]^2\},$$

$$W_2 = \theta\beta \ln t.\Psi_1 + (1/\theta)\ln(1 - R(t)).\Psi_2,$$

$$\text{and} \quad \beta = t^\alpha e^{-t^\alpha}(1 - e^{-t^\alpha})^{-1}. \tag{4.14}$$

(IV) If $w(\alpha, \theta) = H(t)$ given by Eq. (1.4), then

$$\widehat{H}_S = H(t)\Big[1 + \frac{1}{2N}(HW_3 - 2IW_4 + GW_5) + W_6\Big] \tag{4.15}$$

where

$$W_3 = 2\alpha^{-1}a_1\ln t + \{a_1^2 + a_1[\alpha^{-1}tH(t) - (\theta-1)\beta + 2] + b_1 + t^\alpha - 2\alpha^{-1}tH(t) - 2\}(\ln t)^2,$$

$$W_4 = \alpha^{-1}(\theta^{-1} + a_2\ln(1-e^{-t^\alpha}) + b_2\ln t + (b_3+b_4)\ln(1-e^{-t^\alpha})\ln t,$$

$$W_5 = a_2\{2\theta^{-1} + [a_2 + (\alpha\theta\beta)^{-1}tH(t)]\ln(1-e^{-t^\alpha})\}\ln(1-e^{-t^\alpha}),$$

$$W_6 = (\alpha^{-1} + a_1\ln t)\Psi_1 + [\theta^{-1} + a_2\ln(1-e^{-t^\alpha})]\Psi_2,$$

$$a_1 = 1 - t^\alpha + (\theta-1)\beta + \alpha^{-1}tH(t), \tag{4.16}$$

$$a_2 = 1 + (\alpha\theta\beta)^{-1}tH(t),$$

$$b_1 = (\theta-1)\beta[\alpha^{-1}tH(t) + (\theta-2)\beta - 2t^\alpha],$$

$$b_2 = \theta^{-1}(a_1 + \alpha^{-1}tH(t) + \beta),$$

$$b_3 = a_2[a_1 + \alpha^{-1}tH(t) - (\theta-1)\beta],$$

and $b_4 = (\theta-1)[(\alpha\theta)^{-1}tH(t) + \beta]$.

4.2. Estimation Under LINEX Loss Function

On the basis of the LINEX loss function (1.5), the Bayes estimate of a function $w = w(\alpha, \theta)$ of the unknown parameters α and θ is given by

$$\hat{w}_L = -\frac{1}{c}\ln E(e^{-cw}/\underline{t}), \quad c \neq 0 \tag{4.17}$$

where

$$E(e^{-cw}/\underline{t}) = \int_\alpha\int_\theta e^{-cw}g(\alpha,\theta/\underline{t})d\alpha\, d\theta \Big/ \int_\alpha\int_\theta g(\alpha,\theta/\underline{t})d\alpha\, d\theta \tag{4.18}$$

We can apply Lindely's approximation cited previously as it was used to evaluate Eq. (4.5). So we obtain the following:

(i) If $w(\alpha,\theta) = e^{-c\alpha}$, then

$$\hat{\alpha}_L = \alpha - \frac{1}{c}\ln\left(\frac{c^2H}{2N} - c\Psi_1 + 1\right), \quad c \neq 0. \tag{4.19}$$

(ii) If $w(\alpha,\theta) = e^{-c\theta}$, then

$$\hat{\theta}_L = \theta - \frac{1}{c}\ln\left(\frac{c^2 G}{2N} - c\Psi_2 + 1\right), \quad c \neq 0. \tag{4.20}$$

(iii) If $w(\alpha,\theta) = e^{-cR(t)}$, then

$$\widehat{R}_L = R(t) - \frac{1}{c}\ln[c(1-R(t))(W_7 + W_2)], \quad c \neq 0 \tag{4.21}$$

where

$$\begin{aligned} W_7 = \frac{1}{2N}\{&\theta\beta H[1 - e^{-t^\alpha}\beta + \theta\beta + c\theta\beta(1-R(t))](\ln t)^2 \\ &- 2\beta I[1 + c(1-R(t))\ln(1-R(t)) + \ln(1-R(t))]\ln t \\ &+ (G/\theta^2)[1 - c(1-R(t))](\ln(1-R(t)))^2, \end{aligned} \tag{4.22}$$

β and w_2 are given by Eq. (4.14).

(iv) If $w(\alpha,\theta) = e^{-cH(t)}$, then

$$\widehat{H}_L = H(t) - \frac{1}{c}\ln\left\{1 + cH(t)\left[\frac{1}{2N}(HW_8 - 2IW_9 + GW_{10}) - W_6\right]\right\}, \quad c \neq 0 \tag{4.23}$$

where

$$\begin{aligned} W_8 &= cH(t)(\alpha^{-1} + a_1 \ln t)^2 - 2\alpha^{-1}a_1 \ln t \\ &\quad - \{1 + (a_1 - 1)(t^\alpha + 2\alpha^{-1}tH(t) - 3) + b_1\}(\ln t)^2, \\ W_9 &= (\alpha^{-1} + a_1 \ln t)[\theta^{-1} + a_2\ln(1 - e^{-t^\alpha})] - (\alpha\theta)^{-1} \\ &\quad - \alpha^{-1}a_2\ln(1 - e^{-t^\alpha}) - b_2 \ln t - (b_3 + b_4)\ln(1 - e^{-t^\alpha})\ln t, \\ W_{10} &= cH(t)[\theta^{-1} + a_2\ln(1 - e^{-t^\alpha}]^2 - W_5, \end{aligned} \tag{4.24}$$

and $a_1, a_2, b_1, b_2, b_3, b_4, w_5, w_6$, are given by (4.16). Keep in mind that the estimators given by Eq. (4.19) to (4.21) and (4.23) are evaluated at $\hat{\alpha}_M$ and $\hat{\theta}_M$.

5. SIMULATION STUDY

We obtained, in the previous sections, the approximate Bayes estimates for the two shape parameters α and θ, reliability, $R(t)$, and failurerate, $H(t)$, functions of the EW(α, θ) distribution. We adopted the squared error loss and LINEX loss functions.The MLEs are also obtained.

To assess the statistical performances of these estimates, a simulation study is conducted. The rootmean square errors (RMSEs) using generated random samples of different sizes are computed for each estimator. The random samples are generated as follows:

(1) For a given value of d, generate values of α and θ from the joint prior density given by (4.3).
(2) Using (α, θ), obtained in (1), generate random samples of sizes: $n = 25$, 50, 100, 200 from the generation random variable $T = [-\ln(1 - \ln U^{1/\theta})]^{1/\alpha}$, where T is EW(α, θ) and U is a uniform (0,1) distribution.
(3) The MLEs $\hat{\alpha}_M$ and $\hat{\theta}_M$ are obtained by iteratively solving the nonlinear equations (2.5). The estimators $\widehat{R}_M(t_0)$ and $\widehat{H}_M(t_0)$ are then computed at some t_0.
(4) The Bayes estimates $\hat{\alpha}_S$, $\hat{\theta}_S$, $\hat{\alpha}_L$, and $\hat{\theta}_L$ are computed, respectively, using Eq. (4.11), (4.12), (4.17), and (4.18). The estimates $\widehat{R}_S, \widehat{H}_S, \widehat{R}_L$, and $\widehat{H}_L$ are computed, respectively, using Eq. (4.13), (4.15), (4.21), and (4.23).
(5) The previous steps are repeated 1,000 times, and the means and the RMSEs are computed for different sample sizes n and censoring sizes r, where the had symbol $\wedge$ stands for an estimate $(\hat{\cdot})_M$, $(\hat{\cdot})_S$, or $(\hat{\cdot})_L$.

Table 1. Estimated means and RMSEs (in parentheses) of various estimators of θ for different sample sizes (known α).

n	R	$\hat{\theta}_M$	$\hat{\theta}_S$	$\hat{\theta}_L : c = 2.5$
15	15	1.2902(0.364)	1.2418(0.249)	1.1426(0.222)
15	10	1.2921(0.362)	1.2181(0.258)	1.1216(0.238)
30	30	1.2599(0.239)	1.2313(0.194)	1.1760(0.181)
30	25	1.2683(0.247)	1.2325(0.203)	1.1770(0.188)
50	50	1.2428(0.180)	1.2239(0.163)	1.1894(0.156)
50	40	1.2454(0.191)	1.2134(0.161)	1.1793(0.157)

Table 2. Estimated means and RMSEs (in parentheses) of various estimators of $R(t)$ for different sample sizes (known α).

n	r	$\widehat{R}_M$	$\widehat{R}_S$	$\widehat{R}_L : c = -25$
15	15	0.9216(0.048)	0.9041(0.046)	0.9278(0.030)
15	10	0.9210(0.049)	0.9021(0.049)	0.9264(0.031)
30	30	0.9247(0.033)	0.9125(0.036)	0.9261(0.027)
30	25	0.9255(0.035)	0.9130(0.036)	0.9265(0.028)
50	50	0.9251(0.027)	0.9185(0.026)	0.9270(0.022)
50	40	0.9250(0.028)	0.9181(0.027)	0.9267(0.022)

The computational results for the means and RMSEs (in parentheses) are displayed in Tables (1–3) for the case of known parameter α ($\alpha = 3$) and the prior parameters $\nu = 3$ and $\delta = 3$ which yield the value of $\theta = 1.2156$ (as the true value). The true values of $R(t)$ and $H(t)$ at $t = t_0 = 0.5$ are $R(0.5) = 0.9259$ and $H(0.5) = 0.0769$. Tables 3a, 3b, 4a, and 4b propose the results for the case of unknown α and θ, where the values of the parameters used are $d = 4$ and $\nu = 3$ yield $\alpha = 1.1678$ and $\theta = 0.4811$ (as true values) and then the true values of $R(t)$ and $H(t)$ are computed to be $R(0.5) = 0.3889$ and $H(0.5) = 1.4018$. The computations are achieved under complete and censored samples.

6. CONCLUDING REMARKS

In this article, we present the Bayes estimates of the two shape unknown parameters, reliability, and failure rate functions of the lifetimes following the EW distribution. The estimation are conducted on

Table 3. Estimated means and RMSEs (in parentheses) of various estimators of $H(t)$ for different sample sizes (known α).

n	r	$\widehat{H}_M$	$\widehat{H}_S$	$\widehat{H}_L : c = 25$
15	15	0.0841(0.053)	0.1034(0.054)	0.0742(0.032)
15	10	0.0837(0.055)	0.1037(0.056)	0.0743(0.032)
30	30	0.0790(0.037)	0.0918(0.037)	0.0756(0.027)
30	25	0.0781(0.038)	0.0929(0.039)	0.0764(0.028)
50	50	0.0782(0.030)	0.0872(0.030)	0.0769(0.024)
50	40	0.0784(0.031)	0.0870(0.030)	0.0767(0.024)

Table 4a. Estimated means and RMSEs (in parentheses) of various estimators of α and θ for different complete sample sizes (unknown α and θ).

n	r	$\hat{\alpha}_M$	$\hat{\alpha}_S$	$\hat{\alpha}_L : c = -5$	$\hat{\theta}_M$	$\hat{\theta}_S$	$\hat{\theta}_L : c = 20$
25	25	1.3892	1.1045	1.2567	0.4611	0.6337	0.4645
		(0.524)	(0.198)	(0.170)	(0.183)	(0.177)	(0.034)
50	50	1.2589	1.0835	1.1687	0.4736	0.5836	0.4758
		(0.277)	(0.128)	(0.100)	(0.104)	(0.118)	(0.019)
100	100	1.2173	1.1211	1.1722	0.4762	0.5314	0.4772
		(0.177)	(0.073)	(0.048)	(0.076)	(0.059)	(0.015)
200	200	1.1869	1.1436	1.1694	0.4815	0.5070	0.4805
		(0.119)	(0.031)	(0.013)	(0.099)	(0.028)	(0.003)

Table 4b. Estimated means and RMSEs (in parentheses) of various estimators of $R(t)$ and $H(t)$ for different complete sample sizes (unknown α and θ).

n	r	$\widehat{R}_M$	$\widehat{R}_S$	$\widehat{R}_L : c = -10$	$\widehat{H}_M$	$\widehat{H}_S$	$\widehat{H}_L : c = -15$
25	25	0.3904	0.4029	0.3949	1.4928	1.3750	1.4732
		(0.067)	(0.051)	(0.027)	(0.280)	(0.206)	(0.084)
50	50	0.3910	0.4109	0.3896	1.4420	1.3251	1.4049
		(0.046)	(0.029)	(0.010)	(0.181)	(0.124)	(0.045)
100	100	0.3901	0.4045	0.3887	1.4254	1.3490	1.4047
		(0.032)	(0.017)	(0.002)	(0.126)	(0.063)	(0.029)
200	200	0.3895	0.3967	0.3889	1.4100	1.3708	1.4014
		(0.025)	(0.008)	(0.000)	(0.102)	(0.035)	(0.010)

the basis of complete and type II censored samples. Bayes estimators, under squared error loss and LINEX loss functions, are derived in approximate forms by using Lindely's method.

Our observations about the results are stated in the following points:

(1) For the case of known α, Tables 1–3 show that the Bayes estimates under the LINEX loss function have the smallest estimated RMSEs as compared with the estimates under quadratic loss or MLEs. This is true for both complete and censored samples. It is immediate to note that the RMSEs decrease as sample size increases.

Table 5a. Estimated means and RMSEs (in parentheses) of various estimators of α and θ for different censored sample sizes (unknown α and θ).

n	r	$\hat{\alpha}_M$	$\hat{\alpha}_S$	$\hat{\alpha}_L : c = 25$	$\hat{\theta}_M$	$\hat{\theta}_S$	$\hat{\theta}_L : c = 15$
25	20	1.5308	0.8174	1.3448	0.4464	0.6267	0.5271
		(0.800)	(0.589)	(0.178)	(0.199)	(0.186)	(0.098)
50	40	1.3867	0.8720	1.2134	0.4643	0.6758	0.5148
		(0.616)	(0.498)	(0.054)	(0.209)	(0.242)	(0.085)
100	80	1.3151	1.0425	1.1641	0.4635	0.5944	0.4781
		(0.404)	(0.277)	(0.029)	(0.195)	(0.145)	(0.046)
200	160	1.2602	1.1344	1.1368	0.4655	0.5328	0.4673
		(0.330)	(0.147)	(0.040)	(0.089)	(0.072)	(0.033)

Table 5b. Estimated means and RMSEs (in parentheses) of various estimators of $R(t)$ and $H(t)$ for different censored sample sizes (unknown α and θ).

n	r	$\widehat{R}_M$	$\widehat{R}_S$	$\widehat{R}_L : c = -5$	$\widehat{H}_M$	$\widehat{H}_S$	$\widehat{H}_L : c = -15$
25	20	0.3805	0.3389	0.3939	1.5335	2.0189	1.5760
		(0.226)	(0.079)	(0.021)	(0.396)	(0.907)	(0.181)
50	40	0.3874	0.3586	0.3790	1.4810	1.8647	1.5087
		(0.052)	(0.072)	(0.036)	(0.312)	(0.801)	(0.114)
100	80	0.3863	0.3815	0.3815	1.4701	1.7602	1.5038
		(0.041)	(0.053)	(0.035)	(0.240)	(0.656)	(0.113)
200	160	0.3855	0.3854	0.3741	1.4473	1.5352	1.4726
		(0.032)	(0.042)	(0.040)	(0.188)	(0.434)	(0.081)

(2) For the case of unknown α and θ, the results in Tables 4a and 4b show that the performances of the LINEX loss function are better than the quadratic loss and MLEs based on the RMSEs of the estimates. This is true for both complete and censored samples.
(3) It is shown from Tables 5a and 5b, that the performances of MLEs are the worst as a rule and especially in the case of small censored samples ($n \leq 30$). The estimates under quadratic loss (as a symmetric) function are quite severe in the same case.

From the previous observations, the estimation from EW data is possible and flexible using Bayes approach, especially using asymmetric loss functions such as LINEX function, which is the most appropriate as shown from this article.

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