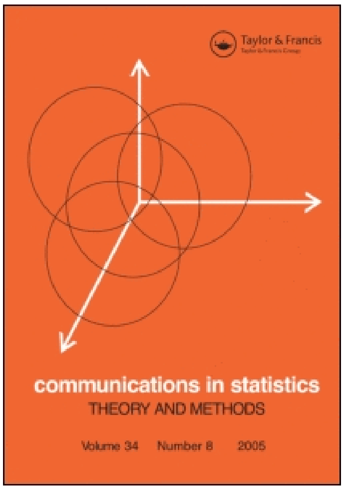

Communications in Statistics - Theory and Methods
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713597238

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Online publication date: 04 August 2004

COMMUNICATIONS IN STATISTICS
Theory and Methods
Vol. 33, No. 3, pp. 681–696, 2004

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ABSTRACT

In this paper, a general reward model for a non homogeneous semi-Markov system will be provided. Both discrete and continuous time models will be studied considering reward to be a random variable of the economic type associated with the state occupancies and transitions. Basic equations will be exhibited and the main formulas for the expected reward that the system will generate in some interval of time (with and without discounting) will be determined for the discrete and continuous cases and a method is provided in order to find it in closed analytic form in relation with the basic parameters of the system. Also, results will be obtained for the expected reward per time period for the steady state.

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DOI: 10.1081/AGB-120028691 0361-0926 (Print); 1532-415X (Online)
 www.dekker.com

Key Words: Stochastic population model; Semi-Markov process; Membership; Non homogeneity; Rewards.

I. INTRODUCTION

The definition of the non homogeneous semi-Markov process is provided in Iosifescu-Manu (1972) for the continuous time case, in Janssen and De Dominics (1984) for the discrete case and in De Dominics and Manca (1985). A general definition of rewards can be found in Limnios and Oprisan (2001) and the study of the asymptotic behaviour of semi-Markov reward process in Reza Soltani and Khorshidian (1998). Later on the definition of a non homogeneous semi-Markov system in discrete time is provided in Vassiliou and Papadopoulou (1992) and the asymptotic behavior of the same model is studied in Papadopoulou and Vassiliou (1994). Important theoretical results and applications for semi-Markov models can be found in the work of Cinlar (1969, 1975a,b), Teugels (1976), Pyke and Schaufele (1964), Keilson (1969, 1971), McLean and Neuts (1967), Howard (1971), McClean (1976, 1978, 1980, 1986, 1998), Taylor et al. (2000), Janssen (1986) and in Janssen and Limnios (1999). Continuing this effort in the present, the usefulness of the probabilistic structure of the semi-Markov model is extended by inserting a feature of great practical importance i.e., the possibility of attaching rewards to the semi-Markov process. The addition of the reward structure allows us to compute the operational characteristics of a wide variety of systems. In the present, reward is considered to be a random variable of the economic type associated with the state occupancies and transitions. So the model provided in the present is a useful tool for economic applications in manpower systems. The introduction of a reward structure in a semi-Markov system on the one hand increases the complexity of analysis for one more variable is added in a complex model but on the other hand increases also the modelling convenience to real life problems because for every action a reward is always expected. To provide some answers while limiting the complexity of analysis, we shall focus our attention on determining the expected reward that the system will generate in some interval of time. Since we often encounter situations where the payment of rewards stretches a long period of time we have to provide a capability for discounting of future income or expenditures. The rationale is that a sum of money that is to be received or paid in the future is worth less today because a smaller amount placed at interest beginning today could generate that larger sum in the future. In Sec. II of the present is studied the discrete time case with and without discounting and basic equations for the expected reward generated by the system during an interval of time are provided in relation with the basic sequences. In Sec. III some results for the limiting behavior of the system defined in Sec. II are given. In Sec. IV similar results with that of Sec. II are exhibited for the continuous time case. Finally, some of the theoretical results are illustrated in Sec. V.

II. THE DISCRETE TIME CASE

Let us consider a population which is stratified into a set of states according to various characteristics and $S = \{1, 2, \ldots, N\}$ be the set of states that assumed to be exclusive and exhaustive, so that each member of the system may be in one and only one state at any given time. In the present section, time t is considered to be a discrete parameter and the state of the system at any given time is described by the vector $\mathbf{N}(t) = [N_1(t), N_2(t), \ldots, N_N(t)]'$ where $N_i(t)$ is the expected number of members of the system in the i state at time t. The expected number of members of the system at time t is denoted by $T(t)$ and $N_{N+1}(t)$ is the expected number of leavers during the time interval $(t-1, t]$. Also we assume that $T(t)$ is a known sequence for every t and that the individual transitions between the states occur according to a non homogeneous semi-Markov chain (*embedded non homogeneous semi-Markov chain*). In this respect we denote by $\mathbf{F}(t)_{t=0}^{\infty}$ the sequence of matrices, the (i,j) element of which is the probability of a member of the system making its next transition to state j, given that it entered state i at time t. Let also $\mathbf{p}_{N+1}(t)$ be the $1 \times N$ vector whose jth element is the probability of leaving the system from j, given that the entrance in state j occurred at time t and $\mathbf{p}_o(t)$ the $1 \times N$ vector the jth element of which is the probability of entering the system in state j as a replacement of a member who entered his last state at time t. Let there initially be $T(0)$ "memberships" (Vassiliou and Papadopoulou, 1992) in the system and a member entering the system holds a particular membership which moves within the states with the members. When a member decides to leave the system, the empty membership is taken by a new recruit. If the system is expanding new memberships are created in the system with behavior similar to the initial ones. Denote by $\mathbf{P}(t)_{t=0}^{\infty}$ the sequence of matrices described by the following relation

$$\mathbf{P}(t) = \mathbf{F}(t) + \mathbf{p}'_{N+1}(t)\mathbf{p}_o(t) \tag{1}$$

Obviously $\mathbf{P}(t)_{t=0}^{\infty}$ is a sequence of stochastic matrices with the (i,j)th element equal to the probability a membership of the system which entered state i at time t makes its next transition to state j. Thus, whenever a membership enters state i at time t, it selects state j for its next transition according to probabilities $p_{ij}(t)$. However before the entrance into j, the membership 'holds' for a time in state i. Holding times for the memberships are described by the holding time mass function $h_{ij}(m)$ which equals the probability, a membership which entered state i at its last transition holds for m time units in i before its next transition, given that state j has been selected. Denote by $w_i(m,t)$ the probability $\sum_{j=1}^{N} p_{ij}(t)h_{ij}(m)$ for every i,t,m. Define also as $y_{ij}(t)$ the reward that a membership earns at time t after entering state i for occupying state i during the interval $[t,t+1)$ when its successor state is j and $b_{ij}(m)$ as the bonus reward that the membership earns for making a transition from state i to j, after holding time m time units in state i. Thus if a membership enters state i at time s and decides to make a transition to j after m time units in i the total reward that earns equals:

$$b_{ij}(m) + \sum_{t=s}^{s+m-1} y_{ij}(t) \tag{2}$$

If now we define as β to be a discount factor then the present value of a unit sum paid n time units in the future is equal to β^n, where $0 < \beta \leq 1$. In what follows, the basic equations for the discrete time reward process are exhibited. Define as

$$v_i(s,n,\beta) = \{\text{the expected present value of rewards that a membership earns during the interval } [s,s+n)/\text{entered state } i \text{ at time } s \text{ the discount factor equals } \beta\}.$$

Then

$$\begin{aligned} v_i(s,n,\beta) &= \sum_{k=1}^{N} p_{ik}(s) \sum_{m=n+1}^{\infty} h_{ik}(m)\left[\sum_{l=0}^{n-1} \beta^{l+s} y_{ik}(l+s) + \beta^{n+s} v_i(s)\right] \\ &\quad + \sum_{k=1}^{N} p_{ik}(s) \sum_{m=1}^{n} h_{ik}(m)\left[\sum_{l=0}^{m-1} \beta^{l+s} y_{ik}(l+s) + \beta^{m+s} b_{ik}(m) \right. \\ &\qquad \left. + \beta^{m+s} v_k(s+m,n-m,\beta)\right] \end{aligned} \tag{3}$$

where $v_i(s)$ is the reward that a membership earns if it occupies state i (in which entered at time s) at the end of the interval. Let now

$$y_{ik}(s,m,\beta) = \sum_{l=0}^{m-1} \beta^{l+s} y_{ik}(l+s), \qquad m = 1,2,\ldots \tag{4}$$

$$^{>}y_i(s,n,\beta) = \sum_{k=1}^{N}\sum_{m=n+1}^{\infty} p_{ik}(s)h_{ik}(m)y_{ik}(s,n,\beta) \tag{5}$$

which is the contribution to the expected present value of the reward when the first transition is made after n time units,

$$r_i(s,n,\beta) = \sum_{k=1}^{N}\sum_{m=1}^{n} p_{ik}(s)h_{ik}(m)[y_{ik}(s,m,\beta) + \beta^{m+s}b_{ik}(m)] \tag{6}$$

which is the contribution to the expected present value of the reward until the time of the first transition when it takes place during the interval $(s, s+n]$. Equation (3) is equivalently written as

$$\begin{aligned} v_i(s,n,\beta) ={}^{>} y_i(s,n,\beta) +{}^{>} w_i(n,s)\beta^{n+s}v_i(s) + r_i(s,n,\beta) \\ + \sum_{k=1}^{N}\sum_{m=1}^{n} p_{ik}(s)h_{ik}(m)[\beta^{m+s}v_k(s+m,n-m,\beta)] \end{aligned} \tag{7}$$

In matrix notation Eq. (7) is written as

$$\begin{aligned} \mathbf{V}(s,n,\beta) ={}^{>} \mathbf{Y}(s,n,\beta) +{}^{>} \mathbf{W}(n,s)\beta^{n+s}\mathbf{V}(s) + \mathbf{R}(s,n,\beta) \\ + \sum_{m=1}^{n} \mathbf{P}(s) \diamond \mathbf{H}(m)\beta^{s+m}\mathbf{V}(s+m,n-m,\beta) \end{aligned} \tag{8}$$

where $\mathbf{V}(s,n,\beta)$ is a $N \times 1$ vector, $\mathbf{H}(m) = \{h_{ij}(m)\}_{i,j\in\Sigma}$, $^{>}\mathbf{W}(n,s)$ is a $N \times N$ diagonal matrix with its ith element equal to $\sum_{m=n+1}^{\infty} w_i(m,s)$, $\mathbf{V}(s)$ and $\mathbf{R}(s,n,\beta)$ are $N \times 1$ vectors and $\mathbf{P}(s)\diamond\mathbf{H}(m)$ is the Hadamard (elementwise) product of the matrices $\mathbf{P}(s)$, $\mathbf{H}(m)$. Also, from Eq. (8) and if we define as $^{>}\mathbf{W}(n,s,\beta) ={}^{>} \mathbf{W}(n,s)\beta^{n+s}$ and $\mathbf{C}(s,m,\beta) = \mathbf{P}(s)\diamond\mathbf{H}(m)\beta^{s+m}$ for every s, m, n we have

$$\begin{aligned} \mathbf{V}(s,n,\beta) ={}^{>} \mathbf{Y}(s,n,\beta) +{}^{>} \mathbf{W}(n,s,\beta)\mathbf{V}(s) + \mathbf{R}(s,n,\beta) \\ + \sum_{m=1}^{n} \mathbf{C}(s,m,\beta)\mathbf{V}(s+m,n-m,\beta) \end{aligned} \tag{9}$$

with initial condition $\mathbf{V}(s,0,\beta) = \beta^{s}\mathbf{V}(s)$, for every s.

Let now $TV_i(s,n,\beta)$ be the expected present value of rewards earned by the memberships of the system during the interval $[s, s+n)$ given that the memberships entered the system at time s in state i. Then $TV_i(0,t,\beta) =$ {the expected present value of rewards earned by the initial memberships of the system until time t/memberships entered state i at time 0} $+ \sum_{m=1}^{t}$ {the expected present value of rewards earned by the

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memberships, created by the expansion of the system's population until time t/memberships entered state i at time $m\}$. Thus

$$TV_i(0,t,\beta) = N_i(0)v_i(0,t,\beta) + \sum_{m=1}^{t} \Delta T(m) r_{oi}(m) v_i(m,t-m,\beta) \tag{10}$$

In matrix notation the above relation is written equivalently as

$$\mathbf{TV}(0,t,\beta) = \mathbf{N}(0) \diamond \mathbf{V}(0,t,\beta) + \sum_{m=1}^{t} \Delta T(m) \mathbf{r}_o(m) \diamond \mathbf{V}(m,t-m,\beta) \tag{11}$$

where $\mathbf{N}(0)=[N_1(0),N_2(0),\ldots,N_N(0)]'$, $\mathbf{r}_o(m)=[r_{o1}(m),r_{o2}(m),\ldots,r_{oN}(m)]'$ the recruitment vector for new memberships, $\mathbf{V}(s,t,\beta)=[v_1(s,t,\beta), v_2(s,t,\beta),\ldots,v_N(s,t,\beta)]'$, $\mathbf{TV}(0,t,\beta)=[TV_1(0,t,\beta), TV_2(0,t,\beta),\ldots, TV_N(0,t,\beta)]'$.

Finally, if we define by $TV(t,\beta)$ the expected present value of the total reward earned by the system until time t we have that

$$TV(t,\beta) = \mathbf{N}'(0)\mathbf{V}(0,t,\beta) + \sum_{m=1}^{t} \Delta T(m) \mathbf{r}_o'(m) \mathbf{V}(m,t-m,\beta). \tag{12}$$

Obviously we need a solution for the recursive equation in relation (9). Now if we define as

$$\mathbf{A} = \begin{pmatrix} -\mathbf{I} & \mathbf{C}(s,1,\beta) & \mathbf{C}(s,2,\beta) & \ldots & \mathbf{C}(s,n-1,\beta) \\ \mathbf{0} & -\mathbf{I} & \mathbf{C}(s+1,1,\beta) & \ldots & \mathbf{C}(s+1,n-2,\beta) \\ \mathbf{0} & \mathbf{0} & -\mathbf{I} & \ldots & \mathbf{C}(s+2,n-3,\beta) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{C}(s+n-2,1,\beta) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & -\mathbf{I} \end{pmatrix},$$

$\mathbf{b} = -[b_0,b_1,b_2,\ldots,b_{n-2},b_{n-1}]'$ where b_i is equal to ${}^{>}\mathbf{Y}(s+i, n-i,\beta) + {}^{>}\mathbf{W}(n-i,s+i,\beta)\mathbf{V}(s+i) + \mathbf{R}(s+i,n-i,\beta) + \mathbf{C}(s+i,n-i,\beta) \mathbf{V}(s+n)\beta^{s+n}$ for every $i=0,1,\ldots,n-1$, $\overline{\mathbf{V}} = [\mathbf{V}(s,n,\beta), \mathbf{V}(s+1,n-1, \beta),\ldots,\mathbf{V}(s+n-1,1,\beta)]'$ and develop Eq. (9) we get the system $\mathbf{A}\overline{\mathbf{V}} = \mathbf{b}$. The matrix $\mathbf{A}^{-1}$ exists since $\mathbf{A}$ is upper diagonal and its determinant is equal to $\det\mathbf{A} = [-\mathbf{I}_{N\times N}]^n \neq 0$. Thus $\overline{\mathbf{V}} = \mathbf{A}^{-1}\mathbf{b}$. From the above equation it is evident that in order to find the analytic relation for $\mathbf{V}(s,n,\beta)$ it is sufficient to find the $N \times N$ block elements of the first line of $\mathbf{A}^{-1}$. Let us define by $\mathbf{a_{1j}}^{-1}$, $j=1,2,\ldots,n$ these $N\times N$ block elements and obviously $\mathbf{a_{11}}^{-1} = -\mathbf{I}$. Using now Lemmas 3.1, 3.2 from Vassiliou and Papadopoulou (1992) it can be proved that

$$\mathbf{a_{1j}}^{-1} = \mathbf{C}(s, j-1, \beta) + \sum_{k=1}^{j-2} \mathbf{S}_j(k, s, m_k, \beta), \quad j = 1, 2, \ldots, n \tag{13}$$

where

$$\mathbf{S}_j(k, s, m_k, \beta) = \sum_{m_k=2}^{j-k} \sum_{m_{k-1}=1+m_k}^{j-k+1} \cdots \sum_{m_1=1+m_2}^{j-1} \prod_{r=-1}^{k-1} \mathbf{C}(s + m_{k-r} - 1, m_{k-r-1} - m_{k-r}, \beta)$$

for every $j \geq k+2$ and $\mathbf{S}_j(k, s, m_k, \beta) = 0$ for every $j > k+2$.

Thus from equations $\overline{\mathbf{V}} = \mathbf{A}^{-1}\mathbf{b}$ and (13) we get that

$$\begin{aligned} \mathbf{V}(s, n, \beta) &= {}^{>}\mathbf{Y}(s, n, \beta) + {}^{>}\mathbf{W}(n, s, \beta)\mathbf{V}(s) + \mathbf{C}(s, n, \beta)\mathbf{V}(s+n)\beta^{s+n} \\ &\quad + \mathbf{R}(s, n, \beta) + \sum_{j=2}^{n} \left[\mathbf{C}(s, j-1, \beta) + \sum_{k=1}^{j-2} \mathbf{S}_j(k, s, m_k, \beta) \right] \\ &\quad \times [{}^{>}\mathbf{Y}(s+j-1, n-j+1, \beta) + {}^{>}\mathbf{W}(n-j+1, s+j-1, \beta) \\ &\quad \times \mathbf{V}(s+j-1) + \mathbf{R}(s+j-1, n-j+1, \beta) \\ &\quad + \mathbf{C}(s+j-1, n-j+1, \beta)\mathbf{V}(s+n)\beta^{s+n}] \end{aligned} \tag{14}$$

Relations (12), (14) provide the expected present value of the total reward earned by the system as a function of the basic sequences, which apart from prediction purposes, is also useful for answering problems related to the asymptotic behavior of the model.

In modelling many operational systems the importance of discounting is negligible. So, instead of studying the expected present values we can examine the total expected reward. Thus, let $v_i(s, n)$ be the expected reward that a membership will earn during the interval $[s, s+n)$ given that it entered state i at time s. Then

$$\begin{aligned} v_i(s, n) &= {}^{>} y_i(s, n) + {}^{>} w_i(n, s) v_i(s) + r_i(s, n) \\ &\quad + \sum_{k=1}^{N} \sum_{m=1}^{n} p_{ik}(s) h_{ik}(m) v_k(s+m, n-m) \end{aligned} \tag{15}$$

where

$$y_{ik}(s, m) = \sum_{l=0}^{m-1} y_{ik}(l+s), \quad m = 1, 2, \ldots \tag{16}$$

$$ {}^{>}y_i(s,n) = \sum_{k=1}^{N} \sum_{m=n+1}^{\infty} p_{ik}(s)h_{ik}(m)y_{ik}(s,n) \tag{17} $$

$$ r_i(s,n) = \sum_{k=1}^{N} \sum_{m=1}^{n} p_{ik}(s)h_{ik}(m)[y_{ik}(s,m) + b_{ik}(m)] \tag{18} $$

In matrix notation Eq. (15) is written as

$$ \begin{aligned} \mathbf{V}(s,n) &= {}^{>}\mathbf{Y}(s,n) + {}^{>}\mathbf{W}(n,s)\mathbf{V}(s) + \mathbf{R}(s,n) \\ &\quad + \sum_{m=1}^{n} \mathbf{C}(s,m)\mathbf{V}(s+m, n-m) \end{aligned} \tag{19} $$

and if we follow similar reasoning with that resulted in Eq. (14) we can finally get

$$ \begin{aligned} \mathbf{V}(s,n) &= {}^{>}\mathbf{Y}(s,n) + {}^{>}\mathbf{W}(n,s)\mathbf{V}(s) + \mathbf{C}(s,n)\mathbf{V}(s+n) + \mathbf{R}(s,n) \\ &\quad + \sum_{j=2}^{n} \left[\mathbf{C}(s,j-1) + \sum_{k=1}^{j-2} \mathbf{S}_j(k,s,m_k) \right] \\ &\qquad \times [{}^{>}\mathbf{Y}(s+j-1, n-j+1) + {}^{>}\mathbf{W}(n-j+1, s+j-1) \\ &\qquad\quad \times \mathbf{V}(s+j-1) + \mathbf{R}(s+j-1, n-j+1) \\ &\qquad\quad + \mathbf{C}(s+j-1, n-j+1)\mathbf{V}(s+n)] \end{aligned} \tag{20} $$

where the i, r element of $\mathbf{S}_j(k,s,m_k)$ is the probability that a membership which entered state i at time s makes a transition to state r after $j-1$ time units and k intermediate transitions during the interval $(s, s+j-1)$.

III. LIMITING BEHAVIOR

We begin now with the analysis of the limiting behavior of Eq. (20) which provides the expected reward that a membership earns during an interval of n time units. Since there is no discounting we expect $\mathbf{V}(s,n)$ to grow without bound as n increases. So the question is about the type of growth. It can be easily proved that the terms ${}^{>}\mathbf{Y}(s,n), {}^{>}\mathbf{W}(n,s)\mathbf{V}(s)$ and $\mathbf{C}(s,n)\mathbf{V}(s+n)$ vanish as $n \to \infty$. Also if we look thoroughly at the term $\sum_{k=1}^{j-2} \mathbf{S}_j(k,s,m_k) + \mathbf{C}(s,j-1)$ we can identify that the i, r element of the matrix equals the probability that a membership which entered state i at time s will enter state r at time $s+j-1$. Thus the above matrix equals the matrix $\mathbf{E}(s,j-1)$ of the entrance probabilities (Papadopoulou, 1997). So, for very large n we can say that

Eq. (20) reduces to

$$\mathbf{V}(s,n) = \mathbf{R}(s,n) + \sum_{j=2}^{n} [\mathbf{E}(s,j-1)]\mathbf{R}(s+j-1, n-j+1) \tag{21}$$

If now we assume that $\lim_{n\to\infty,s\to\infty} \mathbf{R}(s,n) = \mathbf{R}$ where the i-element of $\mathbf{R}$ equals the expected reward from the first transition out of state i regardless of when it occurs and $\lim_{n\to\infty,s\to\infty} \mathbf{E}(n,s) = \mathbf{E}$ the matrix of the entrance probabilities in the steady state, then from (21) we get the type of growth for the reward vector and for very large n, given below

$$\mathbf{V}(n) = n\mathbf{E}\mathbf{R} \tag{22}$$

Also from (Papadopoulou, 1997) we can get the relation between the entrance and the interval transition probabilities of a non homogeneous semi-Markov system in the steady state

$$\mathbf{Q} = \mathbf{E}\mathbf{W} \tag{23}$$

where $\mathbf{Q} = \lim_{n\to\infty,s\to\infty} \mathbf{Q}(n,s)$ and $\mathbf{Q}(n,s)$ is the matrix with the (i,j)th element equals the probability that a membership which entered state i at time s will be in state j after n time units and $\mathbf{W}$ is a diagonal matrix whose elements are the limiting expected waiting times in the states for the memberships. From relations (22), (23) we get $\mathbf{V}(n) = n\mathbf{Q}\mathbf{W}^{-1}\mathbf{R}$. Thus, the gain for a membership per time period for the steady state is described by the vector $\mathbf{Q}\mathbf{W}^{-1}\mathbf{R}$.

IV. THE CONTINUOUS TIME CASE

In the present section, time t for the system is considered to be a continuous parameter. The continuous time case for non homogeneous semi-Markov systems is studied in Papadopoulou and Vassiliou (1999). Following these definitions we have as $\mathbf{P}(s,t)$ the transition probability matrix for the memberships of the system, $\mathbf{H}(t)$ the matrix with the probability density functions of the holding times $h_{ij}(.)$ and ${}^{\leq}\mathbf{W}(s,t)$ the diagonal matrix with the cumulative distributions of the waiting times ${}^{\leq}w_i(s,t)$ in the main diagonal where

$${}^{\leq}w_i(s,t) = \sum_{k\in S} \int_0^t p_{ik}(s,s+x)h_{ik}(x)dx.$$

Let us also denote with ${}^{>}\mathbf{W}(s,t)$ the diagonal matrix with elements the complementary cumulative distributions of the waiting times. We

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assume continuous discounting at a rate α i.e., a unit sum of money at time t in the future has a present value of e^{-at} today, $a \geq 0$. Let us now define by $v_i(s,t,a)$ the expected present value of the reward that a membership will generate during a time interval of length t if it is placed in state i at the beginning of the interval $[s, s+t)$. Also let $y_{ij}(x)$ the "yield rate" of state i at time x when the successor state is j, $b_{ij}(t)$ the bonus reward that the membership earns for making a transition from state i to j after time t and $v_i(s)$ is the additional reward payment for the membership if it is still occupies state i at the end of the interval. From all of the above we have that

$$v_i(s,t,a) = {}^{>}y_i(s,t,a) + {}^{>}w_i(s,t,a)v_i(s) + r_i(s,t,a) + \sum_{j=1}^{N}\int_0^t c_{ij}(s,x,a)v_j(s+x,t-x,a)dx \tag{24}$$

where

$${}^{>}y_i(s,t,a) = \sum_{j=1}^{N}\int_t^\infty p_{ij}(s,s+x)h_{ij}(x)dx\left[\int_0^t y_{ij}(x)e^{-ax}\right]dx,$$

$${}^{>}w_i(s,t,a) = {}^{>}w_i(s,t)e^{-at},$$

$${}^{>}w_i(s,t,a)v_i(s) = \sum_{j=1}^{N}\int_t^\infty p_{ij}(s,s+x)h_{ij}(x)dx[v_i(s)e^{-at}],$$

$$r_i(s,t,a) = \sum_{j=1}^{N}\int_0^t p_{ij}(s,s+x)h_{ij}(x)\left[\int_0^x e^{-ar}y_{ij}(r)dr + e^{-ax}b_{ij}(x)\right]dx$$

and $c_{ij}(s,x,a) = p_{ij}(s,s+x)h_{ij}(x)e^{-ax}$.

Equation (24) in matrix notation is as follows

$$\mathbf{V}(s,t,a) = {}^{>}\mathbf{Y}(s,t,a) + {}^{>}\mathbf{W}(s,t,a)\mathbf{V}(s) + \mathbf{R}(s,t,a) + \int_0^t \mathbf{C}(s,x,a)\mathbf{V}(s+x,t-x,a)dx \tag{25}$$

where $\mathbf{V}(s,t,a)$, ${}^{>}\mathbf{Y}(s,t,a), \mathbf{V}(s)$ and $\mathbf{R}(s,t,a)$ are $N \times 1$ vectors, ${}^{>}\mathbf{W}(s,t,a)$ is a $N \times N$ diagonal matrix and $\mathbf{C}(s,x,a)$ is a $N \times N$ matrix. Then if we follow similar reasoning with that of Sec. II, it can be shown that the expected present value of the total reward earned by the system until time t is given by

$$TV(t,a) = \mathbf{N}'(0)\mathbf{V}(0,t,a) + \int_0^t \Delta T(x)\mathbf{r}'_o(x)\mathbf{V}(x,t-x,a)dx \tag{26}$$

where

$$\begin{aligned}
&\mathbf{V}(s,t,a)\\
&\quad = {}^{>}\mathbf{Y}(s,t,a) + {}^{>}\mathbf{W}(s,t,a)\mathbf{V}(s) + \mathbf{C}(s,t,a)\mathbf{V}(s+t)e^{-a(s+t)} + \mathbf{R}(s,t,a)\\
&\qquad + \int_0^t \mathbf{C}(s,x_1,a)\big[{}^{>}\mathbf{Y}(s+x_1,t-x_1,a) + {}^{>}\mathbf{W}(s+x_1,t-x_1,a)\mathbf{V}(s+x_1)\\
&\qquad + \mathbf{R}(s+x_1,t-x_1,a) + \mathbf{C}(s+x_1,t-x_1,a)\mathbf{V}(s+t)e^{-a(s+t)}\big]dx_1\\
&\qquad + \sum_{k\geq 2}\int_0^t \int_0^{t-x_1} \int_0^{t-x_1-x_2} \cdots \int_0^{t-x_1\cdots-x_{k-1}} \mathbf{C}(s,x_1,a)\\
&\qquad \times \mathbf{C}(s+x_1,x_2,a)\cdots\mathbf{C}(s+\cdots+x_{k-1},x_k,a)\\
&\qquad \times \{{}^{>}\mathbf{Y}(s+\cdots+x_k,t-x_1-\cdots-x_k,a)\\
&\qquad\quad + {}^{>}\mathbf{W}(s+\cdots+x_k,t-x_1-\cdots-x_k,a)\mathbf{V}(s+\cdots+x_k)\\
&\qquad\quad + \mathbf{R}(s+\cdots+x_k,t-x_1-\cdots-x_k,a)\\
&\qquad\quad + \mathbf{C}(s+\cdots+x_k,t-x_1-\cdots-x_k,a)\\
&\qquad\quad \times \mathbf{V}(s+t)e^{-a(s+t)}\}dx_k dx_{k-1}\cdots dx_2 dx_1
\end{aligned} \tag{27}$$

Finally, if we follow the same reasoning as the one in Sec. III and consider Eq. (27) we can conclude to similar results for the type of growth for the reward vector in the steady state.

V. ILLUSTRATION

In this section, some of the previous theoretical results are illustrated numerically with hypothetical data which are representative of the kind of data we usually get from manpower systems. In this respect, let us assume that we have an organization with four grades and the transition probability matrix for the memberships is of the form:

$$\mathbf{P}(0) = \begin{pmatrix} 9/16 & 5/16 & 1/16 & 1/16 \\ 1/16 & 9/16 & 5/16 & 1/16 \\ 1/10 & 1/10 & 7/10 & 1/10 \\ 1/20 & 1/20 & 7/20 & 11/20 \end{pmatrix},$$

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$$\mathbf{P}(t) = \begin{pmatrix} 9/16 + 1/100t^3 & 5/16 - 1/100t^3 & 1/16 & 1/16 \\ 1/16 & 9/16 - 1/150t^3 & 5/16 + 1/150t^3 & 1/16 \\ 1/10 & 1/10 & 7/10 & 1/10 \\ 1/20 & 1/20 & 7/20 - 1/120t^3 & 11/20 + 1/120t^3 \end{pmatrix}$$

for every $t = 1, 2, 3, \ldots$.

Also we assume that holding times for the memberships follow geometric distributions and that the matrix $\mathbf{H}(m)$ is as follows:

$$\mathbf{H}(m) = \begin{pmatrix} (0.8)(0.2)^{m-1} & (0.9)(0.1)^{m-1} & (0.7)(0.3)^{m-1} & (0.6)(0.4)^{m-1} \\ (0.7)(0.3)^{m-1} & (0.8)(0.2)^{m-1} & (0.9)(0.3)^{m-1} & (0.8)(0.2)^{m-1} \\ (0.6)(0.4)^{m-1} & (0.8)(0.2)^{m-1} & (0.6)(0.4)^{m-1} & (0.9)(0.1)^{m-1} \\ (0.8)(0.2)^{m-1} & (0.6)(0.4)^{m-1} & (0.7)(0.3)^{m-1} & (0.8)(0.2)^{m-1} \end{pmatrix}$$

for every $m = 1, 2, 3, \ldots$ and $\mathbf{H}(m) = \mathbf{0}$.

Let the initial population structure be $\mathbf{N}(0) = [50, 100, 150, 300]$, $\Delta T(1) = 75, \Delta T(2) = 35, \Delta T(3) = 15, \Delta T(x) = 0$ for every $x \geq 4$ and the recruitment vector is of the form

$$\mathbf{r}_o(0) = [0.1, 0.2, 0.3, 0.4]',$$

$$\mathbf{r}_o(1) = [0.15, 0.25, 0.25, 0.35]',$$

$$\mathbf{r}_o(t) = [0.20, 0.20, 0.30, 0.30]' \text{ for every } t \geq 3.$$

Also we assume that the reward for occupying state i during $[t, t+1)$ when the successor state is j is of the form $y_{ij}(t) = 100 + 100i + ti/100$, the matrix $\mathbf{B}(m)$ with elements the bonus rewards is equal to

$$\mathbf{B}(m) = \begin{pmatrix} 0 & 1000/m & 2000/m & 3000/m \\ 0 & 0 & 1000/m & 2000/m \\ 0 & 0 & 0 & 1000/m \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$\mathbf{V}(s) = \mathbf{0}$ and the discount factor is equal to 1. Using the above data and simulating the system we get results for the expected reward that a membership earns during time intervals. Some of these results

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are given below:

$$\mathbf{V}(0,1) = [591.25, 501.25, 220.00, 140.00]'$$
$$\mathbf{V}(1,1) = [582.26, 507.27, 220.03, 140.04]'$$
$$\mathbf{V}(2,1) = [590.15, 502.04, 220.06, 140.08]'$$
$$\mathbf{V}(3,1) = [590.95, 501.53, 220.09, 140.12]'$$
$$\mathbf{V}(4,1) = [591.15, 501.42, 220.12, 140.16]'$$
$$\ldots$$
$$\mathbf{V}(19,1) = [591.44, 501.63, 220.57, 140.76]'$$
$$\mathbf{V}(20,1) = [591.45, 501.65, 220.60, 140.80]'$$

$$\mathbf{V}(0,2) = [1081.13, 868.95, 450.54, 328.34]'$$
$$\mathbf{V}(1,2) = [1074.08, 871.84, 450.66, 328.11]'$$
$$\mathbf{V}(2,2) = [1082.31, 867.42, 450.72, 328.61]'$$
$$\mathbf{V}(3,2) = [1083.19, 866.99, 450.78, 328.73]'$$
$$\mathbf{V}(4,2) = [1083.43, 866.91, 450.84, 328.82]'$$
$$\ldots$$
$$\mathbf{V}(19,2) = [1083.96, 867.47, 451.72, 329.96]'$$
$$\mathbf{V}(20,2) = [1083.99, 867.51, 451.78, 330.04]'$$

$$\mathbf{V}(0,3) = [1508.45, 1195.63, 701.03, 546.35]'$$
$$\mathbf{V}(1,3) = [1503.01, 1197.92, 701.31, 546.03]'$$
$$\mathbf{V}(2,3) = [1510.31, 1194.40, 701.41, 546.89]'$$
$$\mathbf{V}(3,3) = [1511.13, 1194.08, 701.51, 547.08]'$$
$$\mathbf{V}(4,3) = [1511.37, 1194.05, 701.59, 547.21]'$$
$$\ldots$$

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$$\mathbf{V}(19,3) = [1512.17, 1194.99, 702.90, 548.85]'$$
$$\mathbf{V}(20,3) = [1512.21, 1195.06, 702.99, 548.96]'$$
$$\ldots$$
$$\mathbf{V}(0,19) = [6358.31, 5822.52, 5222.30, 4982.63]'$$
$$\mathbf{V}(1,19) = [6355.65, 5824.73, 5223.05, 4982.24]'$$
$$\mathbf{V}(2,19) = [6360.06, 5822.97, 5223.59, 4984.44]'$$
$$\mathbf{V}(3,19) = [6360.94, 5823.23, 5224.12, 4985.17]'$$
$$\mathbf{V}(4,19) = [6361.50, 5823.66, 5224.64, 4985.76]'$$
$$\ldots$$
$$\mathbf{V}(19,19) = [6368.44, 5831.01, 5232.39, 4994.00]'$$
$$\mathbf{V}(20,19) = [6368.89, 5831.51, 5232.90, 4994.54]'$$

$$\mathbf{V}(0,20) = [6644.68, 6108.85, 5508.63, 5268.91]'$$
$$\mathbf{V}(1,20) = [6642.05, 6111.09, 5509.41, 5268.55]'$$
$$\mathbf{V}(2,20) = [6646.49, 6109.36, 5509.98, 5270.78]'$$
$$\mathbf{V}(3,20) = [6647.39, 6109.64, 5510.53, 5271.53]'$$
$$\mathbf{V}(4,20) = [6647.98, 6110.10, 5511.08, 5272.16]'$$
$$\ldots$$
$$\mathbf{V}(19,20) = [6655.32, 6117.85, 5519.22, 5280.79]'$$
$$\mathbf{V}(20,20) = [6655.80, 6118.37, 5519.77, 5281.37]'$$

Also we get results for the gain that a membership earns per time period for the steady state. If we compare the above values of $\mathbf{V}(s,n)$ we find that the limiting gain converges to $[286.91, 286.86, 286.87, 286.83]$. This result is in agreement with relation (22) of Sec. III since $\mathbf{ER} = [285.9, 285.9, 285.9, 285.9]$ where $\mathbf{R} = [680.9, 554.2, 299.2, 187.1]$ and

$$\mathbf{E} = \begin{pmatrix} 0.1109 & 0.1665 & 0.3255 & 0.1109 \\ 0.1109 & 0.1665 & 0.3255 & 0.1109 \\ 0.1109 & 0.1665 & 0.3255 & 0.1109 \\ 0.1109 & 0.1665 & 0.3255 & 0.1109 \end{pmatrix}.$$

Finally, we give for some values of t the total expected reward earned by the system until time t.

$$TV(1) = 154687.5, TV(2) = 330899.02, TV(3) = 520964.54,$$
$$TV(4) = 718844.3, TV(5) = 919181.51, TV(6) = 1121609.07,$$
$$\ldots$$
$$TV(18) = 3600737.09, TV(19) = 3808288.28, TV(20) = 4015866.45.$$

ACKNOWLEDGMENT

The author wishes to thank the referees for their valuable comments on the submitted version of the present paper.

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