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Characterizing metro networks: state, form, and structure

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Abstract The broad goal of this paper is to characterize the network feature of metro systems. By looking at 33 metro systems in the world, we adapt various concepts of graph theory to describe characteristics of *State*, *Form* and *Structure;* these three characteristics are defined using new or existing network indicators. *State* measures the complexity of a network; we identify three phases in the development of transit networks, with mature systems being 66% completely connected. *Form* investigates the link between metro systems and the built environment, distinguishing networks oriented towards regional accessibility, local coverage or regional coverage. *Structure* examines the intrinsic properties of current networks; indicators of connectivity and directness are formulated. The method presented is this paper should be taken as a supplement to traditional planning factors such as demand, demography, geography, costs, etc. It is particularly useful at the strategic planning phase as it offers information on current and planned systems, which can then be used towards setting a vision, defining new targets and making decision between various scenarios; it can also be used to compare existing systems. We also link the three characteristics to transit line type and land-use; overall the presence of tangential and/or (semi)-circumferential lines may be key. In addition, we have been able to identify paths of development, which should be strongly considered in future projects.

Keywords Network design · Transit planning · Graph theory · Accessibility · Connectivity · Directness

Introduction

Transit systems are composed of stations and stops all linked by rails or roads; they are in essence physical networks. The study of networks is emerging as an area of fundamental

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focus in the twenty-first century (Barabási 2003; Watts 2003). Whether it is the human body, the World Wide Web or even the stock market, all can be studied using analytical tools dedicated to networks. Transit networks, although being structurally simpler than many other networks, present some specific challenges, due to the existence of *lines*, as well as *overlapping* between the *lines*.

Furthermore, transit systems are bound to grow, given the world is constantly urbanizing (United Nations 2008), creating mega-cities that rely more and more on public transportation. While traditional transit planning methods consider such characteristics as demography, geography, demand, cost and others, none seem to address the network design in a direct manner, which becomes increasingly important as systems grow.

The method presented in this paper looks at networks from a holistic perspective and can prove to be particularly useful at the strategic planning phase; however, it does not consider the operations (e.g., local vs. express lines, capacity, etc.) or aesthetics of the systems, but focuses solely on the design on the network (e.g., number of lines, transfers, etc.). The purpose is not to comprehensively plan a new metro system (e.g., find an optimal corridor), but to reveal new and substantial information on the network aspect of transit systems that can supplement more traditional planning tools. The method also offers a means to compare systems in the world, as well as help planners in their decision-making and target-setting processes. For instance, several systems were built in an organic manner, in response to commercial competition or in concordance with political will; the method therefore first informs on the typology of the existing system and can be used to set new strategies addressing new challenges; it can also be used to forecast the impact of different scenarios. Overall, the broad goal is to create a systematic method to establish the characteristics of metro networks.

For this research, we use principles of graph theory to define three particular characteristics of transit networks: *State*, *Form*, and *Structure*. *State* describes the development of networks, whether they are relatively simple or have grown to be more complex. *Form* describes the spatial relationship between transit networks and the built environment, looking at network design and identifying the type of uses (regional or local). *Structure* describes networks through two indicators: connectivity and directness; connectivity refers to the affluence and importance of links and node; directness determines the ease to travel within a network so as to avoid unnecessary transfers. All three characteristics are particularly relevant to transit network design, planning, and also to acquire a more comprehensive understanding of networks.

The objectives of this research are to:

- Present and explain graph theory tools that are useful in the analysis of transit networks.
- Adapt the tools according to transit specificities to acquire an enhanced understanding of the three network characteristics: *State*, *Form* and *Structure*.
- Characterize transit systems according to their network properties.

In this study we narrow the focus to the metro mode only, developing data for 33 networks. In this case, metro means urban rapid rail transit with its own right-of-way, whether it is underground, at grade or elevated. We observe a wide arena of metro networks; over six major world areas (North and Latin America, Eastern and Western Europe, Africa and Asia), hence accounting for different cultures and specificities. By including networks of all sizes (from 2 to 14 lines) we are also able to get a sense of the patterns of development of metro systems.

The rest of this paper proceeds as follows. First, previous work on the subject will be reviewed. We will then look at the methodology applied to translate metro systems into graphs. Emphasis will then be put to explain the three characteristics and apply them to the 33 networks. At this point, we can start to depict and characterise metro systems by plotting two-dimensional graphs with respect to the characteristics' indicators. Finally, we will integrate the three characteristics in one diagram to better examine the relationship between them and identify paths of development.

Previous work

Graph theory first came about in the eighteen century to solve a transportation problem; Euler (1741) famously analysed the Seven Bridges of Königsberg and showed it was not possible to cross them all once consecutively. The theory later evolved as a distinct mathematical field; Claude Berge (1962) was notably one of the founders of the field, his work can be found in the book *The theory of graphs and its applications*. Today, it is applied mostly to large and complex networks (Newman 2003).

The application of graph theory to road transportation systems emerged in the late 1950's and lasted through the 1970's; it was mainly tied to economics. Afterwards, with the development of computers, research was then redirected towards exploiting more intensive models, e.g., the four-stage model. Avondo Bodino (1962) formulated an application of the theory of graphs to transport systems, but did not consider network design. In contrast, Garrison and Marble (1962, 1964) really pioneered the field by introducing three graph theory measures/indicators directly linked to network design (circuits α, degree of connectivity γ, and complexity β). At the same time, Kansky (1963) also made a contribution by defining new indicators related to complexity and network specificities (η, π, θ, ι). Morlok (1970) dedicated a book to the different aspects of transport technology and network structures, where he reviews the mentioned indicators. More recently, Black (2003) reviews the early stages of the research, explaining clearly all the different concepts.

The application of graph theory to urban transit systems emerged in the early 1980's. Lam and Schuler (1981, 1982) applied Garrison's indicators to bus networks and introduced a new measure related to travel time. Nevertheless, the most significant progress was realised by Musso and Vuchic (1988) and Vuchic and Musso (1991) in the late 1980's and 1990's. They adapted the indicators to public transport specifically, as well as introducing new ones (i.e., overlapping lines). However, their approach is different from traditional methods. They tackle networks as computational systems as opposed to mathematical, following the general trend towards computerisation. While this is preferable to carry out in-depth and automated studies, it is wise to first understand the fundamentals behind the work. In this paper, we use both approaches. More recently, Gattuso and Miriello (2005) applied Garrison's and Kansky's indicators to 13 metro networks; however, the usefulness of these indicators in transit studies is limited. Derrible and Kennedy (2009) introduced new indicators suited for transit-specific applications. They applied these to 19 metro networks in the world, showing a strong correlation between network design and ridership.

Transit networks as graphs

In this section, we explain how transit networks can be translated into graphs, where nodes become vertices and links become edges. Additionally, we introduce a new measure that is referred to as maximum number of transfers.

One of the main fields developed to comprehend and analyse networks is graph theory. Largely developed by Claude Berge (1962) in the second half of the twentieth century, graph theory is essentially a formal mathematical method, by which networks are transformed into graphs composed of edges (links) and vertices (nodes).

The first step towards drawing a graph from a metro network is to define what the vertices (nodes) and edges (links) are. Two approaches can be followed when studying transit networks. Either all stations are considered as vertices (this is particularly useful to study flows of people in the network), or only transfer stations and terminals are included. As we are more interested in network design, we feel the latter is more appropriate. For this research, a metro station is not a node if it does not offer a transfer. Nevertheless, the total number of stations n_{S} is still a relevant measure when considering aspects of coverage. In addition, three more transit network characteristics will be useful: number of lines n_{L}, number of branches n_{B} and total route length L (km), that is the kilometres of track existing in the network.

The second step is to identify and count the number of edges, vertices and other attributes present in the graphs. As an example, Fig. 1 represents the Brussels metro (not to scale); Fig. 1a shows an updated version of the original map as found on the Brussels Transit Authority website (STIB 2008); Fig. 1b shows a representation of the metro as a graph. The network has three lines (represented as solid, dash and dot). We will return to the figure as we explain some of the concepts.

Vertices v

We define two types of vertices (nodes): transfer and end-vertices. Transfer-vertices are transfer stations, where it is possible to switch lines without exiting the system, whether the transfer is a simple cross platform interchange (e.g., Mong Kok station in Hong-Kong) or requires a longer walk (e.g., lines 4 and 13 at the Montparnasse Bienvenüe station in Paris, or between the Central and Bakerloo line at Oxford Circus station in London). End-vertices are the line terminals, where it is not possible to switch to another metro line. Note that if a terminal actually hosts two lines, it is considered a transfer-vertex (e.g., vertex 2 on Fig. 1b); this is also true if two terminals connect (the resulting station is considered to be a transfer-vertex as opposed to two end-vertices); in other words, the ability to transfer is the determining factor here. For the example of Brussels, the metro has a total of nine vertices (5 end and 4 transfer-vertices). Mathematically, the total number of vertices v is equal to the sum of the transfer-vertices v^{t} plus the end-vertices v^{e}:

$$v = v^{\mathrm{t}} + v^{\mathrm{e}} \tag{1}$$

where

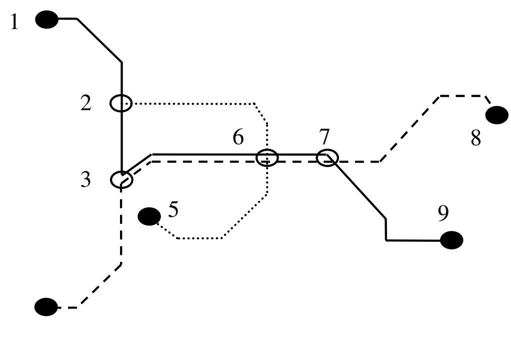
$$v = \sum_{i} v_i \tag{2}$$

$$v^{\mathrm{t}} = \sum_{i,l \neq 1} v_{i,l} \tag{3}$$

$$v^{\mathrm{e}} = \sum_{i,l = 1} v_{i,l} \tag{4}$$

In Eqs. 2, 3 and 4, 'i' is the vertex label (the order is arbitrary) and 'l' is the line attribute (related to the number of lines crossing a vertex, i.e., $l = 1$ if the station hosts only one line, $l = 2$ is the station hosts two lines, etc.). The line attribute is quite relevant, it allows us to measure the connectivity of a network. Table 1a shows how this attribute can be used in practise by considering the Brussels example.

A. Map of the Brussels Metro Network (Metro lines only), updated from STIB (2008)



B. Representation of Brussels Metro Network as a Graph

Fig. 1 Example of the Brussels metro. **a** Map of the Brussels metro network (*metro lines* only), updated from STIB (2008). **b** Representation of Brussels metro network as a graph

Edges *e*

Edges are non-directional links. We define two types: single and multiple edges. If two consecutive vertices are linked by two edges, then we consider there is one single edge and one multiple edge. For instance in Brussels, the edge between vertices 1 and 2 is single; however, there are two edges connecting vertices 3–6, one is considered single and the other multiple (again, the order is arbitrary). The introduction of multiple edges accounts for the overlapping property of certain transit lines; in other words, it helps us to track down redundancies. Mathematically, the total number of edges e is equal to the sum of the single edges e^{s} and the multiple edges e^{m}. Either each edge is labelled separately, similar to the case of vertices, or each edge can be viewed as linking vertex 'i' to 'j'. The latter approach offers the possibility to create a matrix (Table 1b); it is therefore preferable.

$$e = e^{\mathrm{s}} + e^{\mathrm{m}} \tag{5}$$

where

$$e = \frac{1}{2}\sum_{ij} e_{ij} \tag{6}$$

$$e^{\mathrm{s}} = \frac{1}{2}\left(\sum_{ij}\frac{e_{ij}}{e_{ij}}\right), \quad \text{for all } e_{ij} \neq 0 \tag{7}$$

$$e^{\mathrm{m}} = \frac{1}{2}\left(\sum_{ij}\left(e_{ij} - \frac{e_{ij}}{e_{ij}}\right)\right), \text{ for all } e_{ij} \neq 0 \tag{8}$$

In this case, e_{ij} is the number of edges linking vertex 'i' to 'j' (i.e., on Fig. 1, $e_{12} = 1$ and $e_{36} = 2$). In Eqs. 6, 7 and 8, edges are counted twice, hence the need to divide the sum by two (i.e., Fig. 1b, $e_{36} = 2$ and $e_{63} = 2$ though there are the same). Equations 7 and 8 are atypical in the way that we divided one item by itself; this is to make sure that we add only 'one' edge when considering a possible link between two vertices. The downside is that most of the matrix is filled with zeroes (Table 1b), and this can complicate the

Table 1a Brussels metro network characteristics: Brussels metro vertices

Vertex	1	2	3	4	5	6	7	8	9	Transfer vertices v^{t}	End vertices v^{e}
Line attribute	1	2	2	1	1	3	2	1	1	4	5

Table 1b Brussels metro network characteristics: Matrix of edges for the Brussels metro

Vertex	1	2	3	4	5	6	7	8	9	Total edges e	Single edges e^{s}	Multiple edges e^{m}
1	0	1	0	0	0	0	0	0	0	1	1	0
2	1	0	1	0	0	1	0	0	0	3	3	0
3	0	1	0	1	0	2	0	0	0	4	3	1
4	0	0	1	0	0	0	0	0	0	1	1	0
5	0	0	0	0	0	1	0	0	0	1	1	0
6	0	1	2	0	1	0	2	0	0	6	4	2
7	0	0	0	0	0	2	0	1	1	4	3	1
8	0	0	0	0	0	0	1	0	0	1	1	0
9	0	0	0	0	0	0	1	0	0	1	1	0
Total	1	3	4	1	1	6	4	1	1	22	18	4
Effective total										11	9	2

Table 2 Metro network measures

Measure	Notation	Symbol	Value in Brussels metro
Route length (km)	L		39.5
Stations	n_S		59
Lines	n_L		3
Branches	n_B		0
Vertices	v		9
End	v^e		5
Transfer	v^t		4
Edges	e		11
Single	e^s		9
Multiple	e^m		2
Max number of transfers	δ		1

computing task. Also, note that Eq. 8 basically calculates the remaining number of edges that were not included in e^s.

Maximum number of transfers δ

Another important measure is the maximum number of transfers δ to go from one vertex to another by taking the shortest path; it is commonly referred to as the *network diameter* in graph theory. The *network diameter* is the total number of edges linking the two "furthest" vertices, using the shortest path. However, metro networks have lines. In Fig. 1b, the *network diameter* is 4, e.g., to go from vertex 4–8 (e_{43} to e_{36} to e_{67} to e_{78}), but these edges are part of the same metro line and no transfers are needed. In fact, the maximum number of transfers is only 1, since vertex 6 hosts all three lines.

There is no easy method to find δ. Nevertheless, Dong and Chen (2005) have found a way to simplify the problem. They build what they call a *transit line network* model; essentially, it is a graph where transit lines become vertices and transfer stations become edges. Applying Dong's approach to Brussels, we can clearly see that δ equals 1 (Fig. 2).

Finally, we can establish a Table 2 recapitulating the different measures that were introduced here, along with their symbols, images (graphical representation on Fig. 1b)

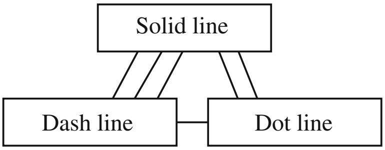


Fig. 2 Line network model for the Brussels metro

and values in the Brussels example. We could also derive equations to calculate the remaining measures, but they would not be of use. These measures are normally readily available (stations, lines and branches), are not related to the graph (route length), or there are simply no formal methods available to calculate them (maximum number of transfers).

At this point, we have introduced a method to translate networks into graphs. We can use these results and compile a set of indicators to define the three network characteristics: *State, Form,* and *Structure.*

Network characteristics

State

State refers to the current development phase of a network, whether it is a relatively simple network or a more complex one. The first step is therefore to identify the number of different possible phases. To do this, we use two network indicators that were developed by Garrison and Marble (1962); the first one relates to complexity β and the second to degree of connectivity γ.

Despite its name, the β indicator is fairly simple; it expresses the ratio of edges to vertices:

$$\beta = \frac{e}{v} \tag{9}$$

From β, we are able to extract some significant properties by looking at how the ratio evolves as networks grow, i.e., how many edges are created when introducing new vertices. Predicting the behaviour of the number edges as we increase the number of vertices (also described as the average number of edges per vertex) is not a trivial exercise. In fact, it depends greatly on the type of network (e.g., road, rail, air, electric, social, etc.); there are no systematic methods to determine it.

The degree of connectivity γ indicator calculates the degree of connectivity as opposed to what we call structural connectivity (section on *Structure*). It describes how much a network is connected relative to how much it could be connected. Therefore, γ calculates the ratio between the actual number of edges to the potential number of edges; that is if the network is 100% connected. In a multi-dimensional graph, the potential number of edges is calculated as $\frac{1}{2}v(v-1)$; it simply calculates all the possible links between the vertices, and then halves the result not to count the potential edges twice. Nevertheless, metro networks are almost always planar; this means that two edges crossing each other will automatically create a vertex (station). Therefore, for planar networks with $v \geq 3$, the potential number of edges becomes $3v - 6$. As result, γ becomes:

$$\gamma = \frac{e}{e_{\max}} = \frac{e}{3v-6} \tag{10}$$

Form

Form illustrates how networks are integrated in the built environment. Metro networks can play a different role depending on how they were planned. Some networks act at the regional level, by bringing people from residential areas to employment areas (e.g., suburbs to CBD), while others tend to focus on servicing the people living in one specific location, so most trips are made via the metro. To take another perspective, *Form* relates to the strategy that was envisioned when the network was conceptually created. In this section, we look at three typical metro network indicators: number of lines n_{L}, total number of stations n_{S} and total route length L; these measures are usually the first to be mentioned when analysing transit networks.

The goal is to identify whether a network is regionally or locally focused. This is done by looking at the average line length (Eq. 11). Typically, longer lengths imply the lines are trying to reach further out in the suburbs, therefore representing a regionally oriented network.

$$A = L/n_{\mathrm{L}} \tag{11}$$

The number of stations n_{S} is also important as a network with small average line length and many stations will tend to increase local coverage.

Finally, another indicator can be incorporated in the study; it is the inter-station spacing, defined as:

$$S = L/n_{\mathrm{s}} \tag{12}$$

The interpretation of the inter-station spacing may be limited due to the fact it says very little on the expanse of the network; nevertheless S adds a level of detail. Although, there is one well-known trade-off related to inter-station spacing (shorter spacings usually imply lower speeds, hence longer travel time), it can still strengthen the results. For instance, a network with large inter-station spacing is likely to be rather regionally oriented.

Structure

Two significant measures of network structure are presented here: connectivity ρ and directness τ. In this case, we talk about structural connectivity as opposed to degree of connectivity (section State). In other words, the ρ indicator measures the affluence and importance of connections (i.e., transfers) in the system. But first, we have developed a new indicator that is central to this research; it is called the number of transfer possibilities $v_{\mathrm{c}}^{\mathrm{t}}$ and is defined as:

$$v_{\mathrm{c}}^{\mathrm{t}} = \sum_{i} (l-1) \cdot v_{i,l} \tag{13}$$

where 'l' is the line attribute as introduced in the section Transit Networks as Graphs.

It counts all the transfer possibilities in a network; it is the sum of the number of lines at a transfer-station minus one. For instance, from Fig. 1b, if traveling along the solid line, starting from vertex 1 and arriving at vertex 2, there is one transfer possibility, then, arriving at vertex 6, there are 2 transfer possibilities. The Brussels metro has a total $v_{\mathrm{c}}^{\mathrm{t}}$ of 5. However, $v_{\mathrm{c}}^{\mathrm{t}}$ does not consider the travel paths redundancies and is not standardized. Considering these issues, the connectivity indictor ρ is:

$$\rho = \frac{v_c^t - e^m}{v^t} = \frac{\sum_{i}(l-1)\cdot v_{i,l} - \frac{1}{2}\left(\sum_{ij}\left(e_{ij} - \frac{e_{ij}}{e_{ij}}\right)\right)}{\sum_{i,l\neq 1} v_{i,l}}, \text{ for all } e_{ij} \neq 0 \tag{14}$$

The numerator calculates the total number of net transfer possibilities; the net value was preferred to avoid false information due to the overlapping line property of metro networks (e.g., northern part of the Circle line for the London Underground). The denominator is used as a means to standardize the indicator; in practise, it actually provides information about the structure of the network itself, which in turn makes the indicator independent of network size; this is a rather valuable property. Another advantage of this indicator is that it emphasizes "hubs" (transfer-stations hosting more than two lines). Indeed, adding one edge between two new vertices will not improve connectivity since the two new transfer-possibilities at the numerator will be offset by the two new transfer-vertices at the denominator. It therefore becomes clearer that ρ effectively measures a rather structurally focused connectivity.

Directness is certainly the hardest indicator to conceptualize. In a world where transit has to be as attractive as possible, avoiding transfers is crucial. Directness is also relatively specific to transit networks.

The directness indicator τ obviously has to be related to the maximum number of transfers δ. However, δ cannot be considered as a raw measure since larger networks tend to have larger δ but still perform adequately given their structure. We therefore need to derive an appropriate expression.

We first saw a parallel between directness and the indicator π introduced by Kansky (1963). He explains π as being "a number expressing the relationship between the circumference of a circle and its diameter … let us assume that the total mileage of a transportation system is analogous to the circumference of a circle and the total mileage of all edges of the diameter of a network is analogous to the diameter of the circle". Here, π does not deal with line type (circle, radial, spinal, etc.); it measures the ratio between the total route length and the length of the diameter (i.e., the "longest" route).

For transit, the total route length L is used as the numerator. The length of the diameter is concomitant with the maximum number of transfers δ since it is the "longest" route. Therefore, we can assume that the "longest" route is equal to δ times a given length. The given length can be a portion k of the average line length $A = L/n_L$. The denominator includes $k \times L/n_L$.

$$\tau = \frac{L}{\frac{k\cdot L}{n_L}\cdot\delta} = \frac{n_L}{k\cdot\delta} \tag{15}$$

Equation 15 shows how directness simplifies into a measure of number of lines and maximum number of transfers. The portion k can be omitted; first because there are no sensible values for it (half, a third of a line, etc.); second because it is a constant (i.e., it does not add information) and it simply gets absorbed by the slope of a line during a regression analysis. This measure of directness remains simple whilst accounting for network size (number of lines). Consequently, the directness indicator can be defined as:

$$\tau = \frac{n_L}{\delta} \tag{16}$$

Results

The data collected and results calculated are shown in Table 3. By studying 33 metro systems in the world, ranging all sizes (from 2 to 14 lines), we can effectively scrutinize metro networks and draw some conclusions on their characteristics. Table 3 includes all the data used and the various indicators calculated. Data was mostly collected from each individual transit authority website; the book *Transit Maps of the World* by Ovenden (2007) was also useful for that purpose. It should be noted that the New York City subway presents a particular problem due to the importance of branches within the system, although the value of nine was chosen for the number of lines. This issue is dealt with separately in the following sections.

State

As it was introduced in Eq. 9, the measure β calculates the ratio between the number of edges and the number of vertices. When performing a regression analysis between e and v, we calculate a slope of 1.94 and an intercept of -10, a goodness-of-fit (adjusted R^2) of 0.99, with t-tests of 46.55 and -6.92 respectively, both being higher than the 95% confidence value. The value 1.94 of the slope is particularly interesting. It means that by adding one vertex to a network, almost two edges are created, thus close to a 1:2 relationship; this is not trivial.

First of all, the relationship found is linear and of the form:

$$e = a \cdot v - b \tag{17}$$

where a and b are constants.

To take a closer look at the ratio β, Eq. 17 becomes:

$$\beta = a - b/v \tag{18}$$

It is obvious that as the number of vertices increases, β tends to the value a. We find empirically, through the regression analysis, that the value of a is just under two. Figure 3 strengthens the important character of the value "two".

Theoretically, the value of a could differ from two. We recall that for planar graphs with $v \geq 3$, the maximum number of edges in a planar graph is: $3v - 6$. If we therefore substitute e by $3v - 6$ in Eq. 9, we get $\beta_{\max} = (3v - 6)/v$. Consequently, as the number of vertices increases, β tends to the value three (i.e., $a = 3$). Relating this phenomenon to the degree of connectivity γ (Eq. 10), the actual to potential number of edges tend to 2/3, or 66% completely connected.

$$\gamma = \frac{e}{3v - 6} \rightarrow \frac{2}{3} \tag{19}$$

We therefore see that as networks grow, they seem to adopt a 1:2 relationship between their number of vertices and edges; this results in networks that are 66% completely connected. The reason for this phenomenon is unclear; nevertheless we can try to tentatively explain how edges are formed.

Taking the example of a small two-line network (e.g., Rome), the two lines are likely to cross only once, hence the number of edges equals the number of vertices minus one, a rough 1:1 relationship. This phenomenon slowly decreases as the number of edges and vertices increase (for small networks, the regression is influenced by the intercept). Taking the example of a larger network, there are three possible scenarios. Firstly, the end-vertices

Table 3 Values of basic indicators and characteristics for the 33 networks studied

Basic indicators																	
Metro networks	Route length	Stations	Lines	Branches	Vertices			Transfer with x lines					Edges			Max. number of transfers
					Total	End	Transfer	2	3	4	5	6	Total	Single	Multiple	
	L (km)	n_s	n_L	n_b	v	v^e	v^t	v_2^t	v_3^t	v_4^t	v_5^t	v_6^t	e	e^s	e^m	δ
Brussels	39.50	59	3	0	9	5	4	3	1	0	0	0	11	9	2	1
Washington, DC	171.14	86	5	0	17	9	8	4	3	1	0	0	25	19	6	1
Toronto	68.75	69	4	0	10	5	5	5	0	0	0	0	11	10	1	3
Montreal	60.86	68	4	0	10	6	4	3	1	0	0	0	11	11	0	2
Boston	102.56	117	5	6	29	16	13	10	3	0	0	0	32	31	1	3
Marseille	19.00	24	2	0	6	4	2	2	0	0	0	0	6	6	0	1
Delhi	68.00	59	3	0	8	6	2	2	0	0	0	0	7	7	0	2
Singapore	89.40	64	4	1	12	6	6	6	0	0	0	0	13	12	1	3
Cairo	65.50	53	2	0	6	4	2	2	0	0	0	0	6	6	0	1
Rome	34.94	47	2	0	5	4	1	1	0	0	0	0	4	4	0	1
Milan	74.06	87	3	2	14	8	6	6	0	0	0	0	15	15	0	1
Athens	52.00	44	3	0	9	5	4	4	0	0	0	0	10	10	0	1
Stockholm	109.48	100	3	5	20	11	9	8	1	0	0	0	22	20	2	1
Prague	59.10	54	3	0	9	6	3	3	0	0	0	0	9	9	0	1
Bucharest	62.31	45	4	1	11	5	6	6	0	0	0	0	13	12	1	2
St Petersburg	104.17	54	4	0	13	7	6	6	0	0	0	0	15	15	0	1
Hong-Kong	90.97	53	7	0	17	7	10	10	0	0	0	0	20	18	2	3
Buenos Aires	48.94	63	5	0	12	8	4	2	2	0	0	0	13	13	0	2
Lyon	29.30	39	4	0	10	6	4	4	0	0	0	0	10	10	0	3
Lisbon	37.03	44	4	0	11	7	4	4	0	0	0	0	11	11	0	2
Mexico City	177.10	151	11	0	35	12	23	15	7	1	0	0	47	47	0	2
Barcelona	102.60	123	9	0	29	11	18	14	2	2	0	0	44	42	2	3

Table 3 continued

Basic indicators																
Metro networks	Route length	Stations	Lines	Branches	Vertices			Transfer with x lines					Edges			Max. number of transfers
					Total	End	Transfer	2	3	4	5	6	Total	Single	Multiple	
	L (km)	n_s	n_L	n_b	v	v^e	v^t	v_2^t	v_3^t	v_4^t	v_5^t	v_6^t	e	e^s	e^m	δ
Berlin	151.70	170	9	0	32	14	18	15	2	1	0	0	45	43	2	3
Osaka	125.42	121	8	0	36	12	24	22	2	0	0	0	53	53	0	3
Paris	211.30	297	14	2	77	23	54	37	11	3	2	0	137	126	11	3
Madrid	226.70	190	13	0	46	10	36	27	8	1	0	0	82	81	1	3
Chicago	173.08	151	8	1	24	11	13	5	4	2	1	1	47	29	18	3
London	438.73	306	13	18	83	27	56	36	13	5	1	1	155	125	30	2
Shanghai	225.00	162	8	0	22	10	12	7	5	0	0	0	32	29	3	3
Moscow	282.50	173	12	0	42	14	28	20	7	1	0	0	67	65	2	2
New York City	368.05	422	9	14	73	26	47	33	7	5	2	0	130	109	21	3
Tokyo	292.38	202	13	2	61	16	45	31	9	3	2	0	119	111	8	2
Seoul	287.00	286	11	4	71	17	54	50	4	0	0	0	135	134	1	3

	State		Form		Structure	
Metro networks	Complexity β	Degree of connectivity γ	Av. line length A (km)	Av. station spacing S (km)	Structural connectivity ρ	Directness τ
Brussels	1.22	0.52	13.17	0.67	0.75	3.00
Washington, DC	1.47	0.56	34.23	1.99	0.88	5.00
Toronto	1.10	0.46	17.19	1.00	0.80	1.33
Montreal	1.10	0.46	15.21	0.89	1.25	2.00
Boston	1.10	0.40	20.51	0.88	1.15	1.67
Marseille	1.00	0.50	9.50	0.79	1.00	2.00
Delhi	0.88	0.39	22.67	1.15	1.00	1.50

Table 3 continued

Metro networks	State		Form		Structure	
	Complexity β	Degree of connectivity γ	Av. line length A (km)	Av. station spacing S (km)	Structural connectivity ρ	Directness τ
Singapore	1.08	0.43	22.35	1.40	0.83	1.33
Cairo	1.00	0.50	32.75	1.24	1.00	2.00
Rome	0.80	0.44	17.47	0.74	1.00	2.00
Milan	1.07	0.42	24.69	0.85	1.00	3.00
Athens	1.11	0.48	17.33	1.18	1.00	3.00
Stockholm	1.10	0.41	36.49	1.09	0.89	3.00
Prague	1.00	0.43	19.70	1.09	1.00	3.00
Bucharest	1.18	0.48	15.58	1.38	0.83	2.00
St Petersburg	1.15	0.45	26.04	1.93	1.00	4.00
Hong-Kong	1.18	0.44	13.00	1.72	0.80	2.33
Buenos Aires	1.08	0.43	9.79	0.78	1.50	2.50
Lyon	1.00	0.42	7.33	0.75	1.00	1.33
Lisbon	1.00	0.41	9.26	0.84	1.00	2.00
Mexico City	1.34	0.47	16.10	1.17	1.39	5.50
Barcelona	1.52	0.54	11.40	0.83	1.22	3.00
Berlin	1.41	0.50	16.86	0.89	1.11	3.00
Osaka	1.47	0.52	15.68	1.04	1.08	2.67
Paris	1.78	0.61	15.09	0.71	1.20	4.67
Madrid	1.78	0.62	17.44	1.19	1.25	4.33
Chicago	1.96	0.71	21.63	1.15	0.77	2.67
London	1.87	0.64	33.75	1.43	1.00	6.50
Shanghai	1.45	0.53	28.13	1.39	1.17	2.67

Table 3 continued

Metro networks	State		Form		Structure	
	Complexity β	Degree of connectivity γ	Av. line length A (km)	Av. station spacing S (km)	Structural connectivity ρ	Directness τ
Moscow	1.60	0.56	23.54	1.63	1.25	6.00
New York City	1.78	0.61	40.89	0.87	1.04	3.00
Tokyo	1.95	0.67	22.49	1.45	1.29	6.50
Seoul	1.90	0.65	26.09	1.00	1.06	3.67

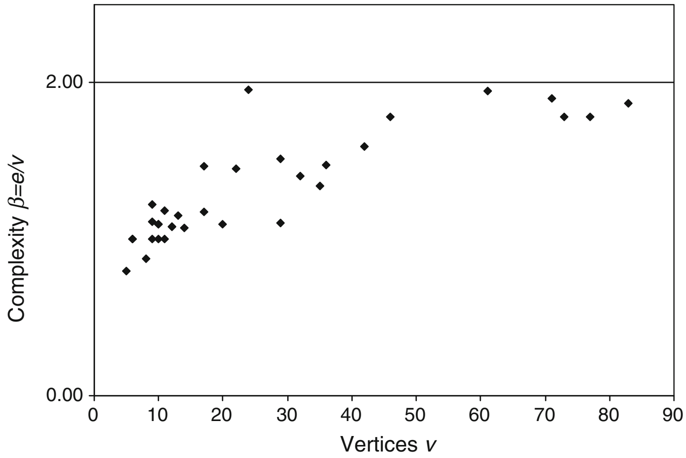


Fig. 3 Relationship between complexity β and vertices v

solely create one edge, a 1:1 relationship. Secondly, if a new line crosses an already existing vertex, then two edges are created, but not a vertex, a 0:2 relationship. Thirdly, if a new line crosses an already existing line and thus create a new transfer-vertex, then three edges are created (four edges minus one existing), a 1:3 relationship. A certain number of variations between these three scenarios are also possible. Furthermore, about half of these edges are actually created; since edges connect two vertices, they should not be counted twice. As a result, it is possible to see that edges can be created without creating vertices. Moreover, by adding a line, new vertices do not only add edges to this new line, but also to pre-existing lines. This is why the 1:1 relationship rapidly evolves.

Overall, *State* appears to approximately follow a three-phase process (Fig. 4). The first phase consists of creating a network and developing it (up to $\beta = 1.3$ and $\gamma = 0.5$). Then new vertices are added to the network, which gradually increases the number of edges to the second phase (up to $\beta = 1.6$ and $\gamma = 0.6$). Once the network is significantly expanded, we see the third and final phase (up to $\beta = 1.96$ and $\gamma = 0.7$) where the ratio of edges to vertices seems to stay constant (a bit lower than 2) and the degree of connectivity approximates 66% fully connected.

Form

Form is assessed using three typical attributes of metro networks: route length L, number of lines n_L and number of stations n_S. First, it is obvious that the three measures are inter-dependent (as can be shown by single and multiple regressions). Indeed, as networks grow, so do the number of lines, stations and route length. However, not all networks evolve in the same way. Figure 5 shows that the relationship between the average line length ($A = L/n_L$) and the number of stations (n_S) is diverse.

The figure shows three zones; note that the figure is planar (i.e., the dashed lines may appear to add another dimension to the figure but this is not the case). The first one is labelled "Regional Accessibility". By increasing the average line length and the distance between the metro stations, some networks focus on connecting people living in the outer layers of the city to the city core, acting more like a regional rail system; this therefore

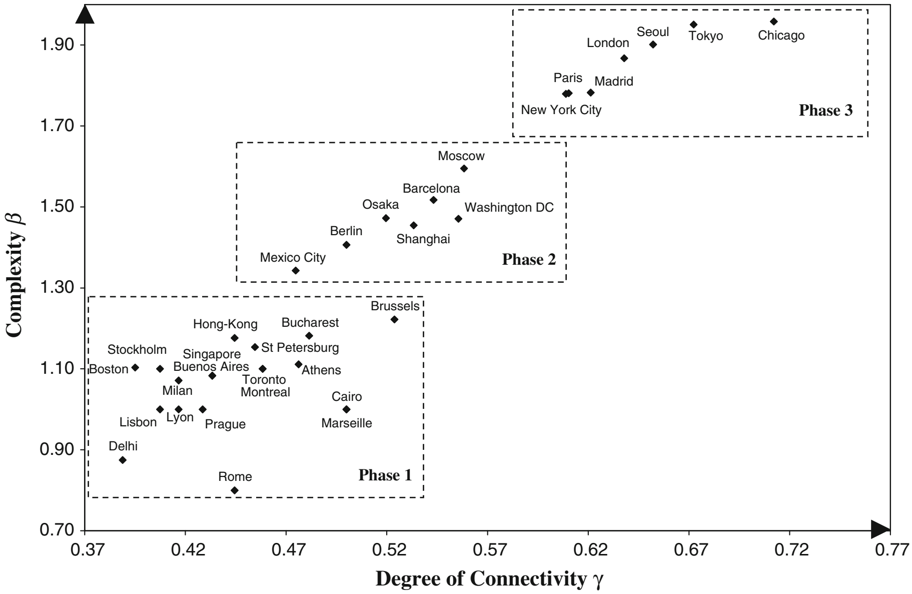


Fig. 4 State of metro networks

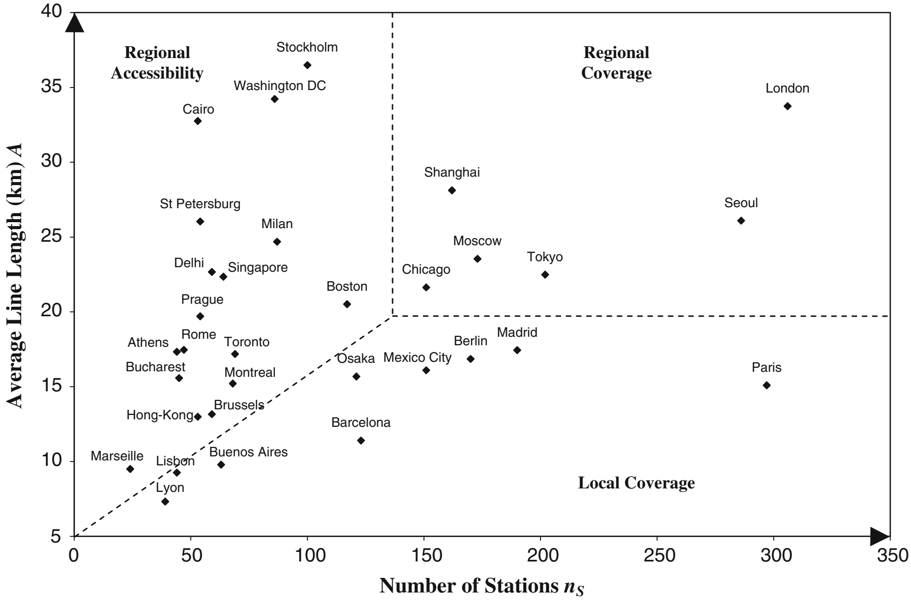


Fig. 5 Form of metro networks

enhances regional accessibility by allowing the suburban population to use the metro. The second zone is labelled "Local Coverage". By having many stations and keeping lines short, networks focus on servicing the city core; hence making the metro a prime travel mode downtown. The third zone is a mix of the other two and is labelled "Regional Coverage". The emphasis is two fold; the city core is generally well serviced, while still connecting the surrounding regions. It should be noted that the term accessibility is widely used in the literature; for a short review, see Geurs and van Wee (2004). Here, we are more interested in the presence of a metro at the regional level, rather than its physical accessibility by transit users.

From Fig. 5, it is possible to see that networks such as the Washington DC and St Petersburg metros emphasize on connecting people, who live further away, to the city core; Stockholm also is notably known for connecting its satellite towns to the downtown. Meanwhile, other cities such as Paris and Barcelona have an extensive network in the city core. Finally, metros seeking to do both tend to service the whole region as are London, Tokyo and Moscow. Also, note that New York City was omitted from Fig. 5 due to its large number of branches (the average line length A cannot be calculated accurately); however, it is in the regional coverage zone.

There are no optimal systems here. Regional accessibility may be neglecting important areas of the city core. Local coverage may not link all inhabitants of the region; even if it has a regional rail service, transfers between modes can be burdensome. Moreover, regional coverage may hinder speed of service by making frequent stops. Last, it should be noted that design of metro systems is also concomitant to other existing transit technology (e.g., bus, streetcar, light rail, etc.), which is why it is not possible at this stage to favour one of the three zones.

Finally, the average station spacing should also be considered (Table 3). The average station spacing between all the networks was found to be 1.25 km. We can automatically link it with the previous analysis. The Washington, DC and St Petersburg metros have spacings of 1.99 and 1.93 km, respectively, again sustaining the fact they act as regional rail systems, while Paris' and Barcelona's remain significantly small with 0.71 and 0.83 km, respectively. On the other hand, the London (1.43 km), Tokyo (1.45 km) and Moscow (1.63 km) networks have higher than average spacings accounting for the fact they extend into the suburbs whilst still servicing locally.

Structure

Structure is represented by two different indicators: structural connectivity ρ and directness τ. Connectivity ρ was introduced in Eq. 14, being related to transfer possibilities. Table 3 shows the values of ρ for each network. It seems the most connected network is the five-line Buenos Aires metro with $\rho = 1.50$. The least connected is the Brussels metro, with a value of 0.75. The Brussels metro has three lines and four transfer-vertices. In comparison, the Delhi metro also has three lines but only two transfer-vertices, and yet it appears to be more connected than Brussels, with $\rho = 1$. This is due to the configuration of the transfer-vertices and not the network design as such.

Directness τ was introduced in Eq. 16. Tokyo and London have the highest values ($\tau = 6.50$ each) since they have a large number of lines but were able to sustain a small maximum number of transfers (Table 3). On the other hand, Singapore, Lyon and Toronto do quite poorly in terms of directness ($\tau = 1.33$ each) given they do not have many lines but a large number of maximum transfers. As a matter of comparison, London has 13 lines and $\delta = 2$ as opposed to Toronto that has four lines and $\delta = 3$.

When we plot one against the other (Fig. 6), it is possible to distinguish three zones as well; some networks are directness-oriented; others are connectivity-oriented; and finally others are, what we call, "integrated", they couple both indicators. It seems most

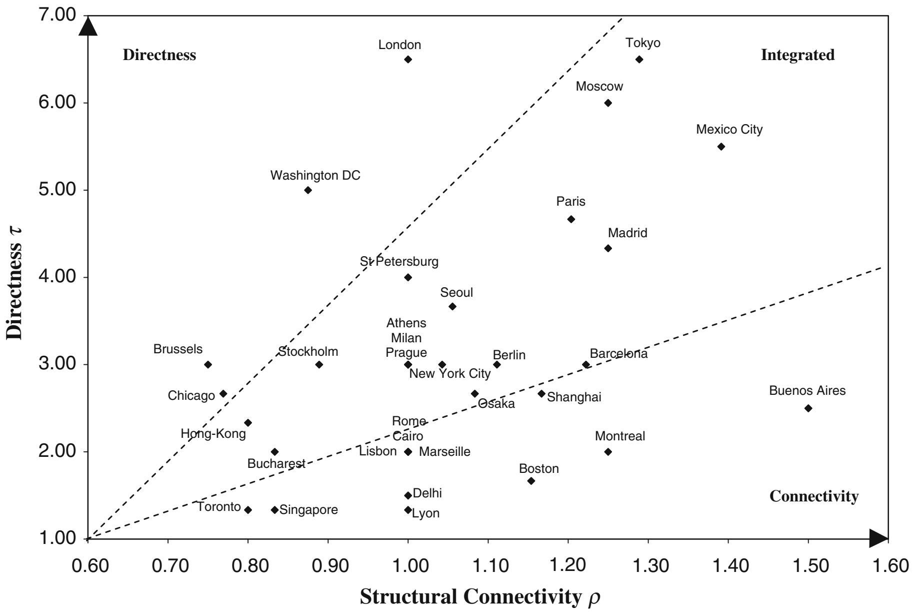


Fig. 6 Structure of metro networks

directness-oriented networks are also regionally oriented. The relationship is not as clear for connectivity-oriented networks. Nonetheless, it seems that the Buenos Aires metro is well connected but has poor directness, which can be spotted when looking at its metro map; it has five lines, four well located transfer-stations, but a maximum number of transfers of two. In comparison, Mexico City achieves high connectivity and directness; it is not surprising when considering the fact it has 11 metro lines that are dense and well connected while keeping a maximum number of transfers to a value of two as well.

Yet, a good practise can be clearly defined from this graph. Networks should aim to be strong in both connectivity and directness, hence be integrated. There is no trade-off since networks can be well connected and direct, no matter their size.

Characterizing metro networks

Network characteristics, line type and land-use

At this point, we have been able to look at *State*, *Form* and *Structure* individually for all metro networks. From *State*, it seems that networks tend towards a 1:2 relationship between the number of vertices and edges. At the same time, the degree of connectivity seems to increase to a maximum of 66% completely connected. This is all the more interesting when we consider that Bon (1979) dedicated an entire paper to the γ ratio, but looking at road networks (highways) of 13 islands. He found that road networks tended to be 50% connected. This means that transit networks have more actual edges than road networks. This may be due to the fact that the road networks Bon looked at were covering entire islands instead of cities. Or it could also be because transfers on roads are not considered negative as with transfers in transit systems.

Our analysis of *Form* has identified three types of networks: regional accessibility, local coverage and regional coverage. A regional accessibility strategy will prefer long lines connecting people with major employment centres; these networks can also extend outside the city administrative boundaries, thus acting also as a regional rail system. The lines are most often radial and diametrical. On the contrary, local coverage strategies will keep lines relatively short and focus on providing a comprehensive level of service in a defined part of the city (e.g., Barcelona), hence the emergence of (semi)-circumferential and tangential lines (e.g., Paris). Finally, some systems couple both strategies to design a region-wide and extensive metro system (e.g., Seoul). This type of network also depends on the availability of other transit technologies: bus, bus rapid transit, light rail transit, and especially regional rail. The type of strategy can have a great impact on some city components such as land use. Enhancing regional accessibility favours the development of a few corridors (e.g., Toronto and North York Centre) that can strengthen the city core (Cervero 1998). However, having long lines reaching the suburbs can also enhance other negative effects such as urban sprawl, resulting in heavily used metro systems during peak hour only (Vuchic 2005). In contrast, local coverage tends to develop larger areas of a city; it concentrates population and employment density whilst not being contained in one corridor.

We have also analysed metro networks in relation with their *Structure*. Networks emphasising directness can have long radial and diametrical lines that are well connected in the downtown so as not to create many unnecessary transfers, thus keeping the travel times reasonably short (e.g., Washington DC). Directness-oriented networks can also have tangential and/or (semi)-circumferential lines to reduce the number of transfers. On the contrary, other networks can be well connected but do poorly in terms of directness. This is especially true when, despite the presence of “hubs”, networks do not avoid transfers. For instance, the Buenos Aires metro lines are of the radial and diametrical type but do not meet at a central point, which makes it difficult to go from one terminal to the terminal of another line. One systematic approach to increase directness is to plan circumferential (e.g., Moscow) or tangential (e.g., Mexico City) lines. Furthermore, connectivity and directness can be coupled effectively without impeding each other, hence integrated networks. In Fig. 6, we see that networks achieving high connectivity and directness all have radial and/or diametrical lines that are inter-connected by (semi)-circumferential and/or tangential lines. In addition, achieving integrated networks does not depend on *Form*. Longer lines can still be connected to other long lines via tangential lines.

Network characteristics and development

Figure 7 displays all strategies taken from the 33 networks. First networks are divided according to their *Form* and then according to their *Structure*. Finally, we enlist networks according to their *State*.

We see that most relatively simple networks (first phase) belong to the regional accessibility zone. Simple networks normally have a small number of lines emphasising on moving people to one particular area of the downtown (e.g., CBD, central station, etc.). As they grow, they are likely to be integrated and/or move to the regional coverage zone. Nevertheless, three simple networks (Buenos Aires, Lyon and Lisbon) have managed to emphasize local coverage.

This can be due to the geographical features of the city; for instance, Lyon’s CBD is located by the Gare Part-Dieu but its major commercial and recreational sectors are located

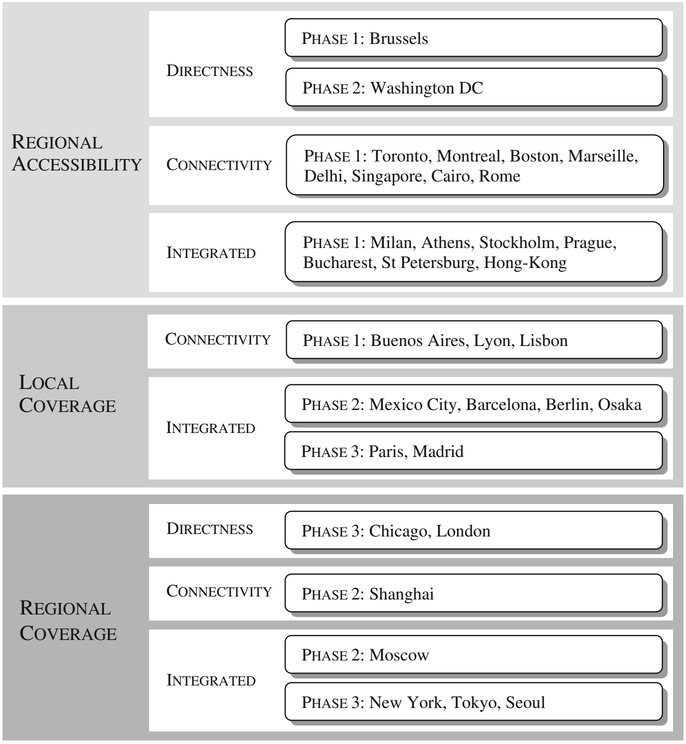


Fig. 7 Characteristics of metro networks studied

on the Presqu'ile between the Place des Terreaux and Bellecour. If the initial strategy is kept, as they grow, these networks are likely to become integrated whilst remaining in the local coverage zone. Only Washington DC belongs to the second phase but still is part of the regional accessibility zone; as it grows, it should include tangential and/or (semi)-circumferential lines (i.e., build new lines, create many new stations and keep the same maximum number of transfers), and thus move to the regional coverage zone either as integrated or directness-oriented.

Some complex networks (2nd/3rd phase) are in the local coverage zone; they all have an integrated type of *Structure*. This is almost automatic. Indeed, as networks grow in a restricted area, lines are likely to inter-connect more often; this normally enhances connectivity (by creating and reinforcing stations hosting more than two lines) and keeps the number of transfers to a minimum relative to number of lines (directness). Future development is likely to remain limited unless they take another approach and extend their lines into the suburbs to become more regionally coverage oriented.

Finally, some complex networks are in the regional coverage zone. Those focusing on either directness or connectivity already have extensive lines. As they grow, they are likely to become integrated by building tangential and (semi)-circumferential lines. Networks that are integrated and part of the regional coverage zone can only remain in this zone no matter how much more they grow; these networks are also likely to develop other transit technologies, potentially to act as "feeder" to the metro, or to serve corridors with lower demand.

Conclusion

For this research, we studied 33 metro networks located throughout the world. The main objective was to adapt graph theory to transit-specific applications and use this method to characterize networks. We first introduced a way to translate transit networks into graphs, using the concepts of edges and vertices. Afterwards, we developed a set of indicators to measure the characteristics: *State*, *Form* and *Structure*.

The *State* of a network is determined by the ratio of edges to vertices β and its degree of connectivity γ. There are three existing phases in the formation of metro networks; networks in the third and final phase have a 1:2 relationship between vertices and edges and tend to be 66% completely connected.

Form was then analysed. We found that a trade-off sometimes exists between regional accessibility and local coverage. Networks emphasizing regional accessibility have longer lines reaching to the outer layers of cities, while local coverage is enhanced by shorter lines being present through out the city core. A combination of the two resulted in a regional coverage type network. There are no good practises on this matter; it is rather dependent on other transit technologies and strategies set by transit planning authorities.

We then defined and analysed the *Structure* of metros. Networks can simultaneously achieve good performance levels in connectivity and directness. This is important for creating a well designed network. The key to this success may be held by the presence of (semi)-circumferential and/or tangential lines.

The study of networks can be a valuable component of transit network design and planning. It should be incorporated in future projects as it can add information along with demand-side variables such as the Origin–Destination pairs.

Overall, graph theory seems to grant the tools to effectively study metro networks, and it could be used to further exploit the field. We are currently working on applying this method to the proposed Toronto plans for the next 15 and 25 years. The results clearly show how the system could be improved in the first place; it is also interesting to compare the results with the vision that was originally imagined; a publication should follow in 2010.

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References

Avondo Bodino, G.: Economic applications of the theory of graphs. Gordon and Breach Science Publishers, New York (1962)

Barabási, A.: Linked: how everything is connected to everything else and what it means for business, science, and everyday life. Plume, New York (2003)

Berge, C.: The theory of graphs and its applications. Methuen, London (1962)

Black, W.R.: Transportation : a geographical analysis. Guilford Press, New York (2003)

Bon, R.: Allometry in topologic structure of transportation networks. Qual. Quant. **13**(4), 307–326 (1979)

Cervero, R.: The transit metropolis: a global inquiry. Island Press, Washington, DC (1998)

Derrible, S., Kennedy, C.A.: A network analysis of subway systems in the world using updated graph theory. Transp. Res. Rec. **2112**, 17–25 (2009)

Dong, J., Chen, L.: Algorithm for the optimal riding scheme problem in public traffic. In: Proceedings of 2005 international conference on neural networks and brain proceedings, Inst. Elec. Elec. Eng. Computer Soc. ICNNB'05 1: 62–66 (2005)

Euler, L.: Solutio problematis ad geometriam situs pertinentis. Comm. Acad. Sci. Petropi **8**, 128–140 (1741)

Garrison, W.L., Marble, D.F.: The structure of transportation networks. Transportation Center Northwestern University, Evanston, IL (1962)
Garrison, W.L., Marble, D.F.: Factor-analytic study of the connectivity of a transportation network. Reg. Sci. **12**(1), 231–238 (1964)
Gattuso, D., Miriello, E.: Compared analysis of metro networks supported by graph theory. Netw. Spat. Econ. **5**(4), 395–414 (2005)
Geurs, K.T., van Wee, B.: Accessibility evaluation of land-use and transport strategies: review and research directions. J. Transp. Geogr. **12**(2), 127–140 (2004)
Kansky, K.J.: Structure of transportation networks: relationships between network geometry and regional characteristics. The University of Chicago Press, Chicago (1963)
Lam, T.M., Schuler, H.J.: Public transit connectivity volume 1. Institute of Transportation Studies, University of California, Irvine (1981)
Lam, T.M., Schuler, H.J.: Connectivity index for systemwide transit route and schedule performance. Transp. Res. Rec. **854**, 17–23 (1982)
Morlok, E.K.: An analysis of transport technology and network structure. Transportation Center Northwestern University, Evanston, IL (1970)
Musso, A., Vuchic, V.R.: Characteristics of metro networks and methodology for their evaluation. Transp. Res. Rec. **1162**, 22–33 (1988)
Newman, M.E.J.: The structure and function of complex networks. SIAM Rev. **45**(2), 167–256 (2003)
Ovenden, M.: Transit maps of the world. Penguin Books, London (2007)
STIB.: Network maps of Brussels. Société des Transports Intercommunaux de Bruxelles (STIB). Available via http://www.stib.be (2008). Accessed 5 Nov 2008
United Nations.: World urbanization prospects: the 2007 revision. United Nations, New York (2008)
Vuchic, V.R.: Urban transit: operations, planning, and economics. Wiley, New Jersey (2005)
Vuchic, V.R., Musso, A.: Theory and practice of metro network design. Public Trans. Int. **40**(3), 298 (1991)
Watts, D.: Six degrees: the science of a connected age. W. W. Norton & Company, New York (2003)

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