A Nearly Optimal Inspection Policy

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This paper is concerned with an easily computable inspection policy for the detection of the failure of a system. A one-parameter policy is suggested and it is shown that this policy has the property of decreasing (increasing) intervals between successive inspection times if the system has an increasing (decreasing) failure rate. Cost comparisons, using linear cost functions, with a computationally difficult optimal policy which has a minimum expected cost show that the proposed policy compares quite well with the optimal policy.

1. INTRODUCTION

MANY items or systems can be described as being in one of two states, one of which is preferable to the other. This preferred state can be described as working, whilst the other might represent some form of failure. Some examples of this type are shelf life of goods, health of a human being, life testing of components and standby systems.

A stock holder of tinned meat will regularly open tins to see if their quality has been maintained and then have to dispose of the opened tins. The more frequently he opens tins the closer control he has on the quality but the more he wastes. Alternatively, if he does not examine the tins frequently there is a real danger of losses caused by the quality deteriorating. He has therefore a balance problem. In this example the only sure method of determining the quality is to carry out an inspection. Similarly, when the health of a human being is characterized by a satisfactory or an unsatisfactory recording, an inspection (examination) must be carried out on for example, pulse rate, temperature, reaction time. Certain components may be regularly monitored for the acceptability of a number of parameters. A typical test situation for the quality of a television tube involves examining the various features at predefined times. The group of equipment in the fields of fire fighting, alarm and weapon systems form stand-by systems which have similar characteristics to the tinned meat example.

It is convenient to label the working state as state 0, and to call the unsatisfactory or failed state as state 1. While the system is in service, it may lead to a cost by changing from state 0 to state 1. A transition from state 1 to state 0, i.e. failed to working, cannot occur while the system is in service; the failed state must be detected by inspection and the system will then be either repaired or replaced. These two states may be defined by a single characteristic (which may be a qualitative judgement) or by a vector θ. When $\theta = \theta_0$, we have state 0 and when $\theta = \theta_1$ we have state 1. The health of a human being characterized by a vector θ = (general sense of well being, pulse rate, temperature, reaction time ...) is an example. The transition from state 0 to state 1 can only be detected by inspecting the system, and we wish to determine an appropriate inspection policy $X = (x_1, x_2, x_3, \ldots)$ which defines the times x_i, $i = 1, 2, 3, \ldots$, at which the system should be inspected. There is a cost associated with inspection, so that the problem of determining an appropriate policy is the problem of achieving a balance between the cost due to inspection and the cost due to leaving the system in the undesirable state.

We need to specify or determine the appropriate cost functions for determining an appropriate inspection policy. We suppose that the cost of each of the inspections is C_1. If the system fails at time t and the failure is detected by an inspection at a later time x_i then the cost due to leaving the system in service in state 1, i.e. failed, is proportional to the length of time from failure to discovery, say, $C_2(x_i - t)$. Note that if state 1 is detected at x_i, we must have $x_{i-1} < t \leqslant x_i$. Let C be the total cost until a failure is detected then:

$$C = iC_1 + C_2(x_i - t). \tag{1}$$

We suppose that initially at time $t = 0$, the system is "new" and it is known to be in state 0. The time, T, for the transition from state 0 to state 1 is a continuous random variable with the probability density function $f(t)$, $t > 0$. Using a fairly standard notation, we take:

$$F(t) = \int_0^t f(u)\,\mathrm{d}u,$$

$$r(t) = \frac{f(t)}{1 - F(t)}.$$

The function $r(t)$ is known by a number of names, and we will call it the failure rate function. A physical interpretation of $r(t)\,\mathrm{d}t$ is that it is the probability that the transition will occur in $(t, t + \mathrm{d}t)$ given that the system was in state 0 at time t. We suppose that $r(t)$ is monotonic. In practice $r(t)$ is commonly taken to be non-decreasing.

Barlow *et al.*[1,2] define an optimal policy X_0 to be one which minimizes the expected value of C as defined in equation (1); they have derived a necessary condition in the form of a recurrence relation for X_0 to be optimal. We shall refer to such an optimal policy as a B.P. policy. Unfortunately the practical evaluation of a B.P. policy can lead to computational difficulties. One feels that the computational difficulties of a B.P. policy, which arise from using the inspection times x_i, $i = 1, 2, 3, \ldots$, as the control variables for minimizing $E(C)$, could be avoided if the determination of the inspection policy X depended on a single parameter or control variable. The obvious solution of inspection at constant intervals, say $x_i = ih$, is not very appealing in general, for it seems reasonable to expect that the intervals between successive inspection times x_i

would decrease (increase) if $r(t)$ is increasing (decreasing). We suggest an inspection policy X_p which has a meaningful single parameter p. We show that X_p is easy to compute and that the intervals between successive inspection times do behave in the desirable fashion. Some cost comparisons with a particular case of a B.P. policy indicate that it is reasonable to suppose that X_p is nearly optimal.

2. AN X_p POLICY

2.1. *Definition*

The probability of a transition to state 1 from state 0 during the interval (x_{i-1}, x_i) given that the system was in state 0 at time x_{i-1} is:

$$\frac{F(x_i)-F(x_{i-1})}{1-F(x_{i-1})}, \quad i = 1, 2, 3, \ldots,$$

where we take $x_0 = 0$ and $F(0) = 0$. We define an inspection policy X to be an X_p policy if for all i, and for a constant p in the interval $(0, 1)$:

$$\frac{F(x_i)-F(x_{i-1})}{1-F(x_{i-1})} = p. \tag{2}$$

Note that $F(x_1) = p$. Equation (2) may be rewritten as:

$$F(x_i) = p+(1-p)\,F(x_{i-1}). \tag{3}$$

Equation (3) can be easily solved for x_i, and we get:

$$F(x_i) = 1-(1-p)^i \tag{4}$$

$$= 1-q^i, \quad \text{where } q = 1-p. \tag{5}$$

Thus for a given p, we can find x_i from:

$$x_i = F^{-1}(1-q^i). \tag{6}$$

In most cases X_p will be easy to compute from equation (6).

2.2. *Choice of p*

Let a random variable I denote the number of inspections necessary for the detection of state 1. It is easy to see that for an X_p policy we have

$$\Pr(I = i) = q^{i-1}p, \quad i = 1, 2, 3, \ldots \tag{7}$$

so that:

$$E(I) = \sum_i iq^{i-1}p = 1/p. \tag{8}$$

If the transition occurs at time t and it is detected by an inspection at time x_i, then $(x_i - t)$ is the time for which the system was left in service in state 1.

The mean time for which the system will be left in service in state 1 is:

$$\tau = \sum_i \int_{x_{i-1}}^{x} (x_i - t)\, f(t)\, \mathrm{d}t$$

$$= \sum_i x_i[F(x_i) - F(x_{i-1})] - \int_0^\infty t f(t)\, \mathrm{d}t.$$

Equation (5) gives:

$$\tau = \sum_i x_i q^{i-1} p - E(T). \tag{9}$$

Note that τ is a function of p and we emphasize this by writing:

$$\tau = \tau(p). \tag{10}$$

The cost C, as shown in equation (1) is:

$$C = iC_1 + C_2(x_i - t).$$

So that from equations (8) and (10) we get:

$$E(C) = C_1/p + C_2\, \tau(p). \tag{11}$$

We choose p such that $E(C)$ as defined in (11) is minimized.

2.3. *Intervals between successive inspection times*

Let $d_i = x_i - x_{i-1}$, $i = 1, 2, 3, \ldots$, be the intervals between successive inspection times (x_0 being equal to 0) of an X_p policy; the value of p used can be any value in $0 < p < 1$ and not necessarily the one which minimizes $E(C)$ as defined in (11). Note that $d_i > 0$.

Suppose that the failure rate function, $r(t)$, of the system is monotonic increasing. In this case P[transition to state 1 in $(x_i - d_i, x_i)$| state 0 at $x_i - d_i$] is less than P[transition to state 1 in $(x_i, x_i + d_i)$| state 0 at x_i]. That is:

$$\frac{F(x_i + d_i) - F(x_i)}{1 - F(x_i)} > \frac{F(x_i) - F(x_i - d_i)}{1 - F(x_i - d_i)} = p. \tag{12}$$

Now $[F(x_{i+1}) - F(x_i)]/[1 - F(x_i)]$ is also equal to p. So that from (12):

$$\frac{F(x_i + d_i) - F(x_i)}{1 - F(x_i)} > \frac{F(x_{i+1}) - F(x_i)}{1 - F(x_i)}$$

so that:

$$F(x_i + d_i) > F(x_{i+1}),$$

which means:

$$x_i + d_i > x_{i+1}$$

or:

$$d_i > x_{i+1} - x_i = d_{i+1}.$$

Thus in the case of an increasing failure rate function, we have shown that the intervals between the successive inspection times x_i of an X_p policy form a decreasing sequence. Similarly, it can be seen that if the failure rate function is decreasing then the intervals d_i form an increasing sequence. Also if the failure rate function is constant then the inspection times are equally spaced; in other words $d_i = d$ for all i.

2.4. *An example; exponential case*

As an example of an X_p policy we consider $f(x) = \lambda e^{-\lambda x}$, $x>0$, $\lambda>0$. In this case the failure rate function is constant, $r(x) = \lambda$, and the inspection times are equally spaced. We can define an X_p policy by:

$$x_i = ix_1$$

and the problem is to choose x_1 in an appropriate manner.

Now:

$$p = F(x_1) = 1 - e^{-\lambda x_1}, \tag{13}$$

so that:

$$x_1 = -\ln(q)/\lambda. \tag{14}$$

Also:

$$\begin{aligned} E(C) &= C_1/p + C_2 x_1 \sum i q^{i-1} p - C_2/\lambda \\ &= C_1/p + C_2 \ln(q)/\lambda p - C_2/\lambda. \end{aligned} \tag{15}$$

The appropriate values of p is determined from $\partial E(C)/\partial p = 0$, and we get:

$$p/q + \ln(q) = \lambda C_1/C_2. \tag{16}$$

From (13), (14) and (16) we see that the appropriate value of x_1 will be determined from:

$$e^{\lambda x_1} - \lambda x_1 = 1 + (\lambda C_1/C_2).$$

It is interesting to note that this is the solution derived by Barlow *et al.*[1]

3. A COMPARISON OF B.P. AND X_p POLICIES

3.1. *B.P. policy*

A B.P. inspection policy $X_0 = (x_1, x_2, x_3, \ldots)$ is such that the expected value of the cost C, as defined in equation (1) is minimized. Now:

$$E(C) = \sum_1^\infty \int_{x_{i-1}}^{x_i} [iC_1 + C_2(x_i - t)]\, f(t)\, dt. \tag{17}$$

We would determine X_0 from the solution of the equations $\partial E(C)/\partial x_i = 0$, $i = 1, 2, 3, \ldots$. Barlow *et al.*[1,2] have shown that this leads to the recurrence relation:

$$x_{i+1} - x_i = \frac{F(x_i) - F(x_{i-1})}{f(x_i)} - \frac{C_1}{C_2}. \tag{18}$$

Since $x_0 = 0$, we only need to find optimal x_1 in order to determine X_0. Barlow *et al.*[1,2] have suggested an iterative method for the case of a particular class of density functions, called Polya frequency function of order 2 (P.F.2). A density f is a member of the class P.F.2 if:

$$\frac{f(t)}{F(t+h)-F(t)}$$

is non-decreasing in t. A P.F.2 density is thus a slight generalization of the density corresponding to the increasing failure rate function. The iterative method depends on two results:

(i) $d_i > 0$ for all i

(ii) For the optimal inspection policy X_0, $\{d_i\}$ is a decreasing sequence if the density f is P.F.2.

The suggested iterative method is

(a) Start with a value of x_1.

(b) Compute $x_2, x_3, x_4, \ldots$.

(c) If for some i, $d_i > d_{i-1}$, decrease the current value of x_1. Re-compute $x_2, x_3, \ldots$.

(d) If for some i, $d_i < 0$, increase the current value of x_1. Re-compute $x_2, x_3, \ldots$.

It is easy to appreciate the computational difficulties involved in this method. As an example Barlow *et al.*[2] take the case of a normal distribution with mean 500 and standard deviation 100; $C_1 = 10, C_2 = 1$. The time is measured in hours. A value $x_1 = 422{\cdot}5$ leads to the conclusion that 422·5 is too large when x_{11} is computed. It is eventually seen that the optimal x_1 lies between 422·4 and 422·5 for when 422·4 is tried, $x_{16} < x_{15}$. We shall use this case of a normal distribution for a comparison of B.P. and X_p policies.

3.2. *A comparison of X_p and B.P. policies*

Let X be a normal variate with mean μ and variance σ^2. Let Z be a standard normal variate, so that:

$$Z = (X-\mu)/\sigma.$$

The expected cost of an X_p policy is:

$$\left.\begin{aligned} E(C) &= C_1/p + C_2 \sum_1^{\infty} x_i q^{i-1} p - C_2 \mu \\ &= C_1/p + \sigma C_2 \sum \frac{(x_i-\mu)}{\sigma} q^{i-1} p \end{aligned}\right\} \tag{19}$$

$$= \sigma C_2 \left[\gamma/p + \sum_1^{\infty} z_i q^{i-1} p \right], \tag{20}$$

where $\gamma = C_1/\sigma C_2$.

The expression in the brackets of (20) is the expected cost of a Z_p policy with the costs γ and 1. The value of p which will minimize the expected cost of a Z_p policy will also minimize the expected cost of the X_p policy. Thus the problem of finding an appropriate value of p for an X_p policy with costs C_1, C_2 and any μ, σ^2 is the problem of finding an appropriate value of p for a Z_p policy with costs $C_1/\sigma C_2$, 1. Also the substitution of $x_i = \sigma z_i + \mu$ in (18) shows that the problem of finding a B.P. X_0 policy with the cost ratio C_1/C_2 is the problem of finding a B.P. Z_0 policy with the cost ratio $C_1/\sigma C_2$. Thus in the case of a normal distribution we can compare B.P. X_0 and X_p policies through the standard normal variate Z. Table 1 shows the optimal p for a Z_p policy, the minimum

TABLE 1. A COMPARISON OF Z_p AND Z_0 POLICIES

$\gamma = C_1/\sigma C_2$	Suggested P	$E(C)$ for Z_p	$E(C)$ for Z_0	$100E_{Z_0}(c)/E_{Z_p}(c)$
0·01	0·0985	0·2155	0·1988	92·28
0·03	0·1734	0·3625	0·3434	94·73
0·05	0·2234	0·4632	0·4436	95·77
0·07	0·2628	0·5455	0·5259	96·41
0·09	0·2956	0·6171	0·5977	96·86
0·10	0·3103	0·6501	0·6308	97·04
0·30	0·4927	1·1413	1·1256	98·63
0·50	0·5897	1·5092	1·4962	99·14
0·70	0·6538	1·8302	1·8191	99·40
0·90	0·7001	2·1252	2·1156	99·56
1·00	0·7189	2·2661	2·2572	99·60
2·00	0·8278	3·5471	3·5417	99·85
3·00	0·8769	4·7170	4·7133	99·92
4·00	0·9049	5·8381	5·8353	99·95
5·00	0·9229	6·9317	6·9295	99·97

costs for Z_p and Z_0 policies and the ratio of the minimum costs. This ratio is an obvious measure of the efficiency of a Z_p policy relative to a Z_0 policy and we see that a Z_p policy may be said to be nearly optimal. It is interesting to take the normal distribution and the costs of Section 3.1; the value of γ in this case is 0·1 and the first check would be scheduled at $x_1 = 450{\cdot}5$, which is to be compared with a value in the range (422·4, 422·5) for the B.P. policy. Note that for $\gamma = 0{\cdot}1$, the efficiency of a Z_p policy is 97·04 per cent.

REFERENCES

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