Slackforce: Documentation on the Theoretical Background

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Contents

1 Introduction

This document describes the physical background and derivation of the formulas and calculations of the App Slackforce. The App has the following three main functionalities:

Measure Force: Measure the pretension of the slackline from the sound you get after slapping the line with your finger.

Static Calculations: Calculate the Forces on the Line at static conditions from the geometric properties of the line.

Dynamic Simulation: Simulate the movement of the line and the forces involved in dynamic moves like bouncing or jumping.

This document is structured in three chapters corresponding to the named app functionalities. Note that there might be errors in the presented formulas. So use the formulas careful and do not trust the calculated values. The creator of this document is not responsible of any harm resulting from the use of the presented calculations and formulas.

2 Measure Force

When you slap a tensioned Slackline with your finger you hear a rhythmical sound on the line multiple times. The idea is to calculate the pretension of the line from the frequency of that sound. When slapping the line a mechanical wave is propagating along the line and reflected at the end. The propagation speed is dependend on the tension of the line.

2.1 Physical Background

The propagation speed of a wave on a tensioned rope or slackline can be described with the following formula.

$$v = \sqrt{\frac{F}{\mu_0}} \tag{2.1}$$

v is the propagation speed, F the force and μ_0 the weight per unit length of the line. During two sounds you can hear the mechanical wave goes from one anchor to the other and back, so it covers a distance two times of the length l of the line. If you measure the time of oscillation t_{osc} the propagation speed can be calculated with the following formula:

$$v = \frac{2 \cdot l}{t_{osc}} \tag{2.2}$$

If you combine equation 2.1 and 2.2 and solve for the force you get the following formula:

$$F_0 = \frac{4 \cdot l^2 \cdot \mu_0}{t_{osc}^2} \tag{2.3}$$

F_0 is the pretension of the line. As you can see the also the weight and the length of the line have to be known. In the App the time of oscillation can be measured automatically with the microphone and some simple audio processing or manually with hearing to the sound of the slackline and typing the rhythm on a button.

2.2 The Influence of the Stretch Behavior

If you tension the line, the line is also stretching. As the total weight of the line does not change this leads to a decrease of weight per unit length. Equation 2.3 will therefore make a systematically mistake on stretchy webbings. To take this into account a linear stretch behavior corresponding to the following formula is assumed.

$$\frac{\Delta l}{l} = \alpha \cdot F \tag{2.4}$$

α is the stretch coefficient and F the force on the line. The corrected weight per unit length is than

$$\mu = \mu_0 \cdot \frac{1}{1 + \alpha \cdot F} \tag{2.5}$$

If you replace μ_0 in formula 2.3 with the corrected value μ and solve that equation for F you get the following formula:

$$F = \sqrt{\frac{1}{4\alpha^2} + \frac{\mu_0 \cdot 4l^2}{\alpha \cdot t_{osc}^2}} \tag{2.6}$$

In the App this more exact formula is used whenever a stretch coefficient greater than zero is entered. It can be shown, that formula 2.6 equals formula 2.3 for $\alpha \to 0$.

3 Calculate Force

The idea of calculating forces is to use the geometric information of the line and a known force to calculate unknown forces with simple vector algebra. If the tension of the line at a given sag is known the tension at a different sag can easily be calculated from the stretch behavior.

3.1 Basics

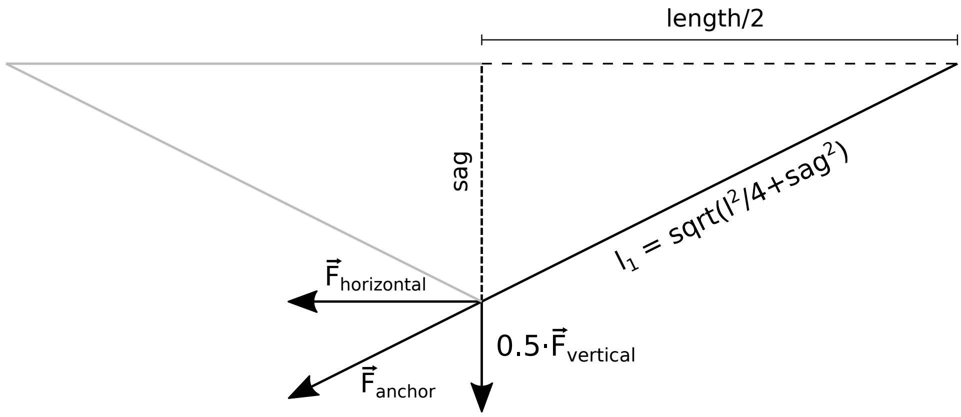

Figure 3.1: Schematical slackline with horizontal and vertical components of force

Figure 3.1 shows a schematical slackline with a given length and sag. As you can see it is possible to divide the force that affects the anchor of the line into a horizontal and a vertical component. The forces of the left part of the line will look the same. The total force will be the sum of forces on the left and the right side. So the horizontal components will eliminate each over and the vertical ones get added. If a slackliner is standing on the line the total vertical force have the amount corresponding to the slackliner's weight. From the geometric properties shown in figure 3.1 the following equation can easily get extracted:

$$\frac{F_a}{0.5 \cdot F_v} = \frac{l_1}{s} = \frac{\sqrt{\frac{l^2}{4} + s^2}}{s^2} \tag{3.1}$$

Together with the gravity acceleration $g = 9.81\frac{m}{s^2}$ and the weight of the slackliner m this leads to the following four equations:

$$F = \frac{\sqrt{s^2 + \frac{l^2}{4}}}{2 \cdot s} \cdot m \cdot g \tag{3.2}$$

$$m = \frac{2 \cdot s \cdot F}{g \cdot \sqrt{s^2 + \frac{l^2}{4}}} \tag{3.3}$$

$$s = \sqrt{\frac{m^2 \cdot l^2 \cdot g^2}{4 \cdot (4F^2 - m^2 g^2)}} \tag{3.4}$$

$$l = \sqrt{\frac{4 \cdot s \cdot (4F^2 - m^2 g^2)}{m^2 g^2}} \tag{3.5}$$

With Equation 3.2 - 3.5 it is possible to calculate any of the parameters force, weight of slackliner, sag and length if the other parameters are known.

3.2 Calculating the Pretension

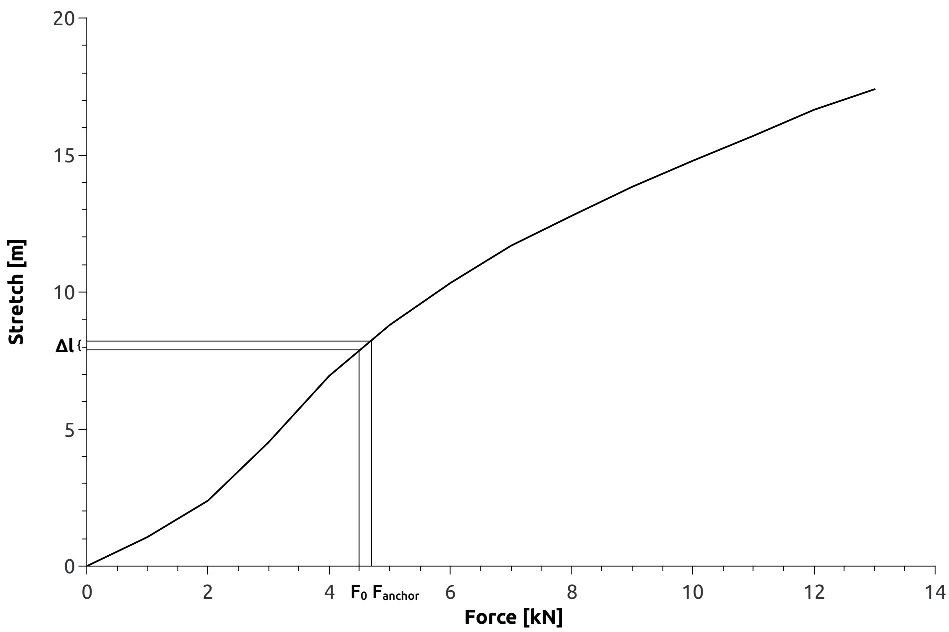

Figure 3.2: Force-Stretch-Diagramm of a 100 m Type 18 Line with 4,5 kN Pretension

If you know the tension with a person and the stretch behavior of a line it is possible to calculate the pretension. Figure 3.2 shows the stretch behavior of a Type 18 MKII line. Is can be seen that the stretch is not linear. To get the pretension of the line you need to know the increase of length Δl due to the sag of the line. This can be calculated with the following formula:

$$\Delta l = 2 \cdot l_1 - l = 2 \cdot \sqrt{s^2 + \frac{l^2}{4}} - l \tag{3.6}$$

To get the pretension you have to do the following steps:

1. Calculate the anchor force with a slackliner on the line
2. read the corresponding stretch from the force-stretch-diagram
3. calculate the new stretch with formula 3.6
4. read the pretension from the force-stretch-diagram

If a linear stretch behavior $\frac{\Delta l}{l} = \alpha \cdot F$ is assumed it is possible to put that relation into an analytic formula:

$$F = F_0 + \frac{2 \cdot \sqrt{s^2 + \frac{l^2}{4}} - l}{\alpha \cdot l} \tag{3.7}$$

However, the App always uses the force-stretch-diagram as the results are better with that method. The diagram is interpolated linear between the provided data points. So the accuracy always depends on the number of available data points.

3.3 Rodeo Lines

Rodeo Lines are slacklines a lot of sag and no pretension. So the pretension F_0 is always zero and the line has an initial sag s_0 without any slackliner on the line. When a slackline steps on the line, it might stretch further and lead to a sag greater than the initial sag. Figure 3.3 shows the forces on an rodeo line.

As it can be seen there is no change to the static forces with a slackliner on the line from figure 3.1. So all the calculations of section Basics are still valid. If you want to do calculations based on the force-stretch-diagram you have to be aware the equation 3.6 now changes to

$$\Delta l = 2 \cdot (l_2 - l_2) = 2 \cdot \left(\sqrt{s^2 + \frac{l^2}{4}} - \sqrt{s_0^2 + \frac{l^2}{4}} \right) \tag{3.8}$$

In addition there is no pretension to calculate, but you can calculate the initial sag of the line using the force-stretch-diagram. Therefore you have to read the stretch from the diagram and use the following formula:

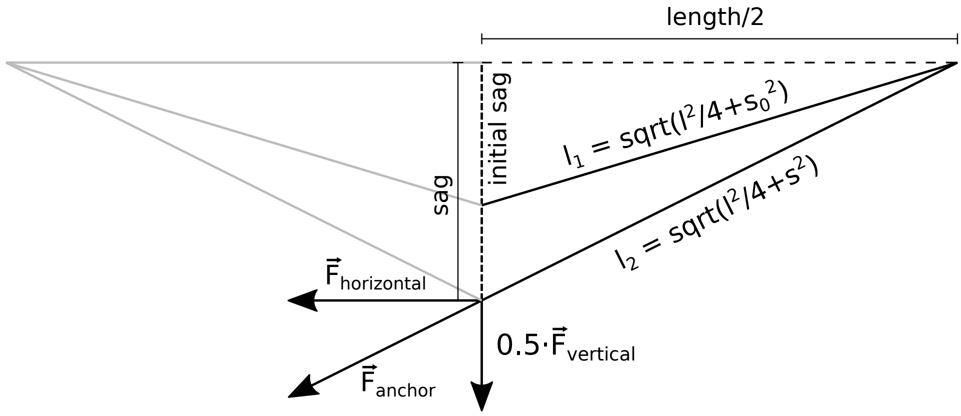

Figure 3.3: Schematical rodeo line with horizontal and vertical components of force

$$s_0 = \sqrt{l_1^2 - \frac{l^2}{4}} = \sqrt{(l_2 - \Delta l)^2 - \frac{l^2}{4}} \tag{3.9}$$

$$= \sqrt{\left(\sqrt{s^2 + \frac{l^2}{4}} - \Delta l\right)^2 - \frac{l^2}{4}} \tag{3.10}$$

3.4 Calculating Slackline Forces from the Pretension

Unfortunately it is very difficult for linear stretch behavior and impossible for a custom stretch behavior to get an analytic expression of the slackline forces dependent on the pretension. Therefore some kind of approximation algorithm has to be used. The App is currently using the Illinois algorithm, that seems to work quite well.

3.5 Slackliner not in the middle of the Line (not implemented yet)

In the following it is assumed that the slackliner is standing at position $p \in [0, 1]$ on the line. The forces always get calculated for the left anchor point. The forces at the right anchor point equal the forces at the left anchor point at position $1 - p$. Figure 3.4 shows the forces on this new setup. If the slackliner is not moving to the left or right the horizontal forces have to be the same on both sides. That leads to different vertical forces and anchor forces on both sides.

From the geometrical properties follows

$$\frac{p \cdot l}{s} = \frac{F_{h,l}}{F_{v,l}} = \frac{\sqrt{F_{a,l}^2 - F_{v,l}^2}}{F_{v,l}} \tag{3.11}$$

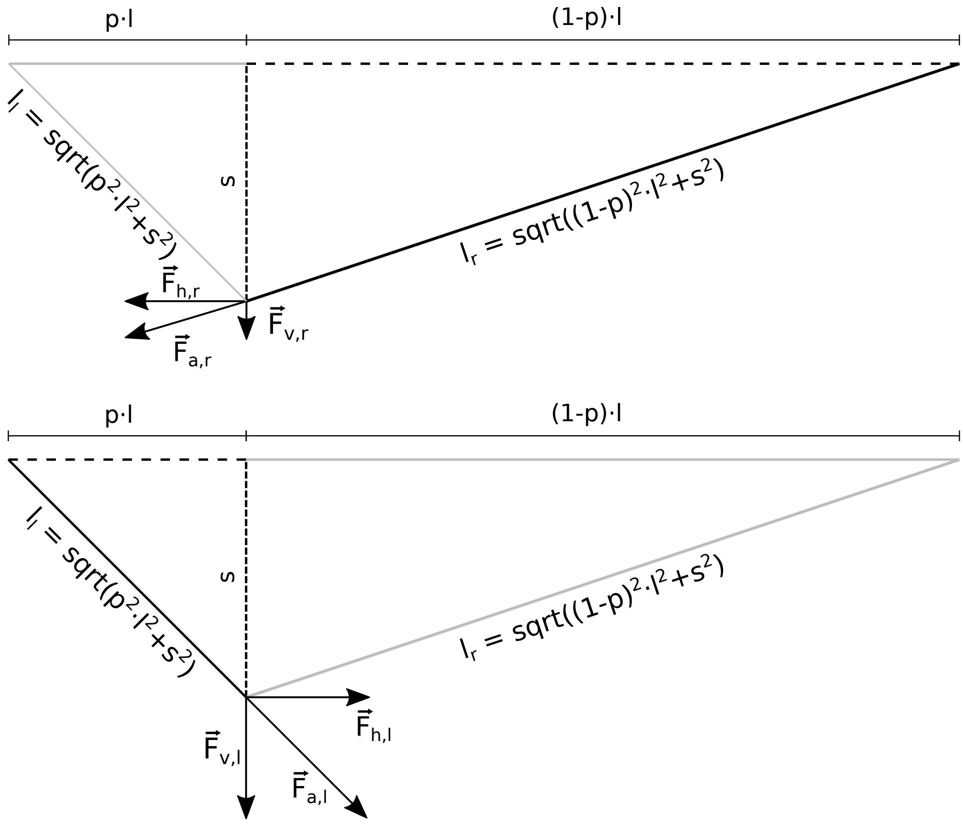

Figure 3.4: Force Components on Unsymmetric Conditions

The the total vertical Force is still the sum of both sides:

$$F_v = F_{v,l} + F_{v,r} \tag{3.12}$$

The equality of the horizontal forces leads to

$$\frac{p \cdot l}{s} \cdot F_{v,l} = \frac{(1-p) \cdot l}{s} \cdot F_{v,r} \tag{3.13}$$

The combination of equation 3.13 and 3.12 gives

$$F_{v,l} = (1-p) \cdot F_v \tag{3.14}$$

Inserting equation 3.14 in 3.11 gives the final equation

$$F_{a,l} = \frac{(1-p) \cdot F_v}{s} \cdot \sqrt{(p \cdot l)^2 + s^2} \tag{3.15}$$

For $p = 0.5$ this equation is similar to equation 3.2 from the previous section. Note that equation 3.6 now changes to

$$\Delta l = \sqrt{(p \cdot l)^2 + s^2} + \sqrt{(1-p)^2 l^2 + s^2} - l \tag{3.16}$$

4 Dynamic Force Simulation

In the previous section the basic calculations at static conditions were described. However, often it is interesting what the maximum forces are during bouncing or jumping. On highlines the forces involved in a leashfall are also very important. As the calculation of dynamic behavior is much more complicated than on static conditions finding analytic expressions is not constructive. Therefore an simulation approach is used.

4.1 Basic Idea of the simulation

To get the forces on dynamic movements the movement of the slackliner gets separated in very small time steps and for a hole period of bouncing or jumping the current speed and position get calculated for every time step. If the time steps are small enough the acceleration of the slackliner can be assumed to be constant. That leads to the following formulas:

$$v(t) = at + v_0 \tag{4.1}$$

$$s(t) = \frac{a}{2}t^2 + v_0 t + s_0 \tag{4.2}$$

In addition the following relations exists:

$$F = a \cdot m \tag{4.3}$$

a is the accelaration, m the mass, v the speed and s the path. Then for every time step the following pattern is done:

1. Set the sag of the slackline to the current position of the slackliner
2. Calculate the force affecting the slackliner $F = m \cdot g - F_{v,slackline}$
3. Calculate the acceleration of the slackliner $a = F/m$
4. Calculate change in speed $\Delta v = a \cdot \Delta t$
5. Calculate change in position $\Delta s = \frac{1}{2} \cdot a \cdot \Delta t^2 + v \cdot \Delta t$
6. Add those values to the current position and speed

When the sign of the speed changes the slackliner is at the lowest point of the line. At this point the maximum forces on the line and the slackliner appear. Therefore the simulation can be stopped here.

4.2 Improved Dynamic Model (not implemented)

In the section above it is assumed, that the stretch behavior of the line is always the same as in the static case. In reality this is not the case and the line behaves more like a viscoelastic material. Therefore it might lead to better results assuming a model based on viscoelastic behavior for the dynamic simulations.

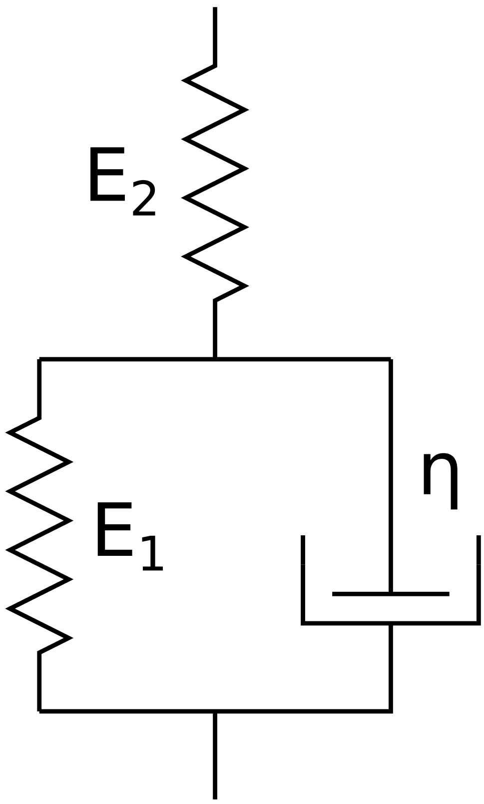

Figure 4.1: Standard linear solid model

Figure 4.1 shows the standard linear solid model. This leads to the following differential equation:

$$\sigma + \frac{\eta}{E_1 + E_2}\dot{\sigma} = \frac{E_1 E_2}{E_1 + E_2}\epsilon + \frac{E_2 \eta}{E_1 + E_2}\dot{\epsilon} \tag{4.4}$$

E_1, E_2 are the elastic modulus of the two springs, σ is the applied stress, ϵ is the stretch $\frac{\Delta l}{l}$ and η is the dynamic viscosity.

For the calculation of slacklines I like to use the stretch coefficient $\alpha = \frac{\epsilon}{F}$ instead of the elastic modulus. The relation between those parameters is:

$$E = \frac{1}{\alpha \cdot A} \tag{4.5}$$

The sum of the stretch coefficients $\alpha_1 + \alpha_2$ is the static stretch coefficient α of the line from the previous chapters. As for slacklines the cross section A is not very important it makes sense to multiply equation 4.4 with A:

$$F(t) + \frac{\alpha_1\alpha_2}{\alpha} \cdot (\eta A) \cdot \dot{F}(t) = \frac{\epsilon(t)}{\alpha} + \frac{\alpha_1}{\alpha} \cdot (\eta A) \cdot \dot{\epsilon(t)} \tag{4.6}$$

Rearranging this equation leads to

$$\dot{F}(t) = \frac{-\alpha}{\alpha_1\alpha_2 \cdot (\eta A)} \cdot F(t) + \frac{\epsilon(t)}{\alpha_1\alpha_2 \cdot (\eta A)} + \frac{\dot{\epsilon}(t)}{\alpha_2} \tag{4.7}$$

Now we have a linear inhomogeneous differential equation. Solving this equation for the force F gives:

$$F(t) = e^{\frac{-\alpha t}{\alpha_1\alpha_2 \cdot (\eta A)} dt} \cdot \left[\int_0^t \left(\frac{\epsilon(t')}{\alpha_1\alpha_2 \cdot (\eta A)} + \frac{\dot{\epsilon}(t')}{\alpha_2} \right) \cdot e^{\frac{\alpha t'}{\alpha_1\alpha_2 \cdot (\eta A)}} dt' + C \right] \tag{4.8}$$

For the simulation the change of force ΔF between to time steps Δt is of interest. Therefore $\Delta F = F(t + \Delta t) - F(t)$ has to be calculated:

$$\Delta F = F(t) \cdot \left(e^{\frac{-\alpha \cdot \Delta t}{\alpha_1\alpha_2 \cdot (\eta A)}} - 1 \right) + e^{\frac{-\alpha(t+\Delta t)}{\alpha_1\alpha_2 \cdot (\eta A)}} \cdot \int_t^{t+\Delta t} \left(\frac{\epsilon(t')}{\alpha_1\alpha_2 \cdot (\eta A)} + \frac{\dot{\epsilon}(t')}{\alpha_1} \right) \cdot e^{\frac{\alpha \cdot t'}{\alpha_1\alpha_2 \cdot (\eta A)}} dt' \tag{4.9}$$

If the time steps are small enough a constant stretch $\epsilon(t) = \epsilon = const$ can be assumed. The derivation of the stretch changes to $\dot{\epsilon}(t) = \frac{\Delta\epsilon}{\Delta t}$. With this simplification the remaining integral can be solved. The result is:

$$\Delta F = \left(1 - e^{\frac{-\alpha \cdot \Delta t}{\alpha_1\alpha_2 \cdot (\eta A)}} \right) \cdot \left(\frac{\epsilon}{\alpha} - F(t) + \frac{\alpha_1 \cdot (\eta A)}{\alpha} \cdot \frac{\Delta\epsilon}{\Delta t} \right) \tag{4.10}$$

Equation 4.10 can now easily be used for the simulation. The calculation of the current forces works with the following procedure.

1. Initialize the force with the pretension and the stretch with the force-stretch-diagram
2. Calculate the new position of the slackliner and sag of the line according to the previous section
3. Calculate the new stretch and $\Delta\epsilon$ with the geometric properties of the line
4. Calculate the new force $F(t + \Delta t) = F(t) + F(\Delta t)$ with equation 4.10
5. Repeat steps 2 - 4

To get good simulation results dynamic characterizations of some real lines are necessary. However, as a starting point I could find some dynamic parameters of climbing ropes at the following webpage (unfortunately in german): `http://www.sigmadewe.com/fileadmin/`

user_upload/pdf-Dateien/SEILPHYSIK.pdf. For the ropes tested there the ratio $\frac{\alpha_1}{\alpha_2}$ was always about 3. The viscosity was varying between $\eta A = 4.8 \ldots 9\, kN \cdot s$. I can imagine that slacklines might have a lower damping than climbing ropes, but that is only speculation. When I have time I will implement this model and play around with the parameters to see if I can get a good match to existing dynamic force measurements.