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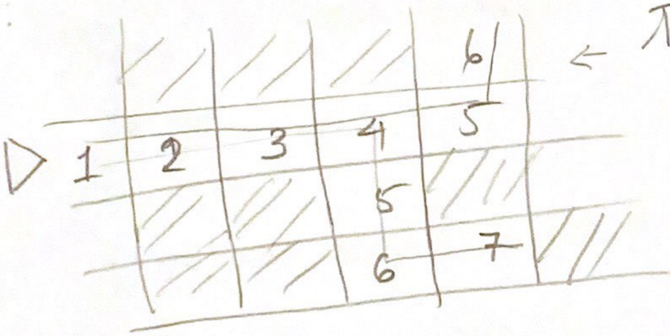
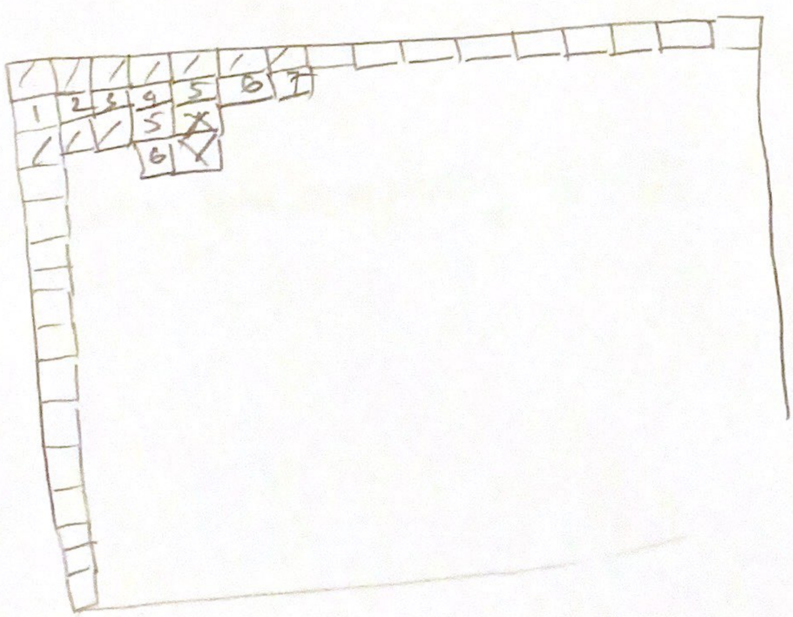
Problem 1

We approach the problem by breadth-first-search algorithm. Given the maze; at the starting point (blue triangle) we put 1 at this location. The agent will move in 4 possible dirrection (up, down, left, right). If the step is valid (no wall) then we put +1 to the current step.

Example:



← There are 2 possible paths

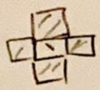


Thus the shortest path is 28.

Algorithm:

Initialize: $(i_{start}; J_{start})$

$S_0 = 1$ (or 1)

For x in Neighborhood - 

If x is wall

$S \leftarrow S$.

Elseif (x is valid

$S \leftarrow S+1$

Elseif $x = (i_{end}, i_{send})$

Return S.

Complexity: If we form a graph, which intersections are nodes, and $E \leftarrow$ possible path; (Graph $G(V, E)$)

thus time complexity is $O(V+E)$.

Question 2, We need to develop optimal choice for filter $\underline{w} = [w[n]]$; with respect to the criterion MSE

So that we hope to minimize the error

$$\varepsilon = E\left\{\left(\overbrace{\sum_{k=1}^{n} w[k]\, x[n-k]}^{\hat{x}[\,]} - y[n]\right)^2\right\} \Leftrightarrow$$

$$\frac{\partial E}{\partial w[k]} = 0 \text{ for all } k \text{ ; or mathematically}$$

$$\frac{\partial E}{\partial w[k]} = E\left\{2\underbrace{\left(\sum w[k]\, x[n-k] - y[n]\right)}_{\varepsilon}\underbrace{\left(x[n-k]\right)}\right\} = 0$$

auto correlation

$$\Leftrightarrow E\{\varepsilon[n]\, x[n-k]\} = 0$$

$$\Leftrightarrow R_{\varepsilon x}[m] = 0 \Leftrightarrow R_{\hat{x}x}[m] - R_{yx}[m] = 0$$

or $R_{\hat{x}x}[m] = R_{yx}[m]$; on the other hand

$$R_{\hat{x}x}[m] = w[m]\, R_{xx}[m]$$

Thus $w[\cancel{x}m] R_{xx}[m] = R_{yx}[m]$

$$\Leftrightarrow \sum_{i=1} w[i] R_{xx}[m-i] = R_{yx}[m]$$

<u>In matrix form</u>:

$$\underbrace{\begin{bmatrix} R_{xx}[0] & \cdots & R_{xx}[N-1] \\ \vdots & & \\ R_{xx}[N-1] & \cdots & R[0] \end{bmatrix}}_{R_x} \underbrace{\begin{bmatrix} w[1] \\ \vdots \\ w[N] \end{bmatrix}}_{w} = \underbrace{\begin{bmatrix} R_{yx}[0] \\ \vdots \\ R_{yx}[N-1] \end{bmatrix}}_{r_x}$$

Thus $R_x w = r_x$ is the solution for minimizing ε

thus $\hat{\omega} = R_x^{-1} r_x$.

Question 5,

a, RADAR is a method that sending radio waveform to a direction. If there is obstacle in the traveling path of the wave; it will reflect.

b, The distance can be found by $d = c(2\Delta t)$ where Δt is the diff. between sending/receiving wave.

Question 3

c) $x(t) = \sqrt{2} A \cos(2\pi f_c t + \theta) + w(t)$

Find θ

Fist of all, the constant $\sqrt{2}$ does not affect the process of finding MLE. We simplify it to

$$x(t) = A \cos(\omega t + \theta) + \varepsilon(t); \quad \varepsilon \sim N(0, \sigma^2)$$

Assume that we have N observations (data points), we need to assume that (1) all obs are i.i.d

and (2) have normadistribution with mean $A\cos(\omega t + \theta)$

We have the pdf of x_i is:

$$p(x_i) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-1}{\sigma^2}\left(x_i - A\cos(\omega t + \theta)\right)^2\right)$$

The likelihood function is $L(\theta) = p(\theta \mid \underset{\sim}{x}) \underset{\text{iid}}{=} \prod p(\theta \mid x_i)$

(function of θ)

Thus: $$L(\theta) = \frac{1}{(2\pi\sigma^2)^{N/2}} \prod_{i=1}^{N} \exp\left(\frac{-1}{\sigma^2}\left(x_i - A\cos(\omega t + \theta)\right)^2\right)$$

$$\log L(\theta) = \frac{N}{2}\log \frac{1}{2\pi\sigma^2} - \frac{1}{\sigma^2}\sum_{i=1}^{N}\left(x_i - A\cos(\omega t + \theta)\right)^2$$

We need to find $\theta_{MLE} = \theta^* = \underset{\theta}{\text{argmax}} \log L(\theta)$

Thus we need to find $\theta^* = \underset{\theta}{\text{argmin}} \sum_{i=1}^{N}\left(x_i - A\cos(\omega t+\theta)\right)^2$

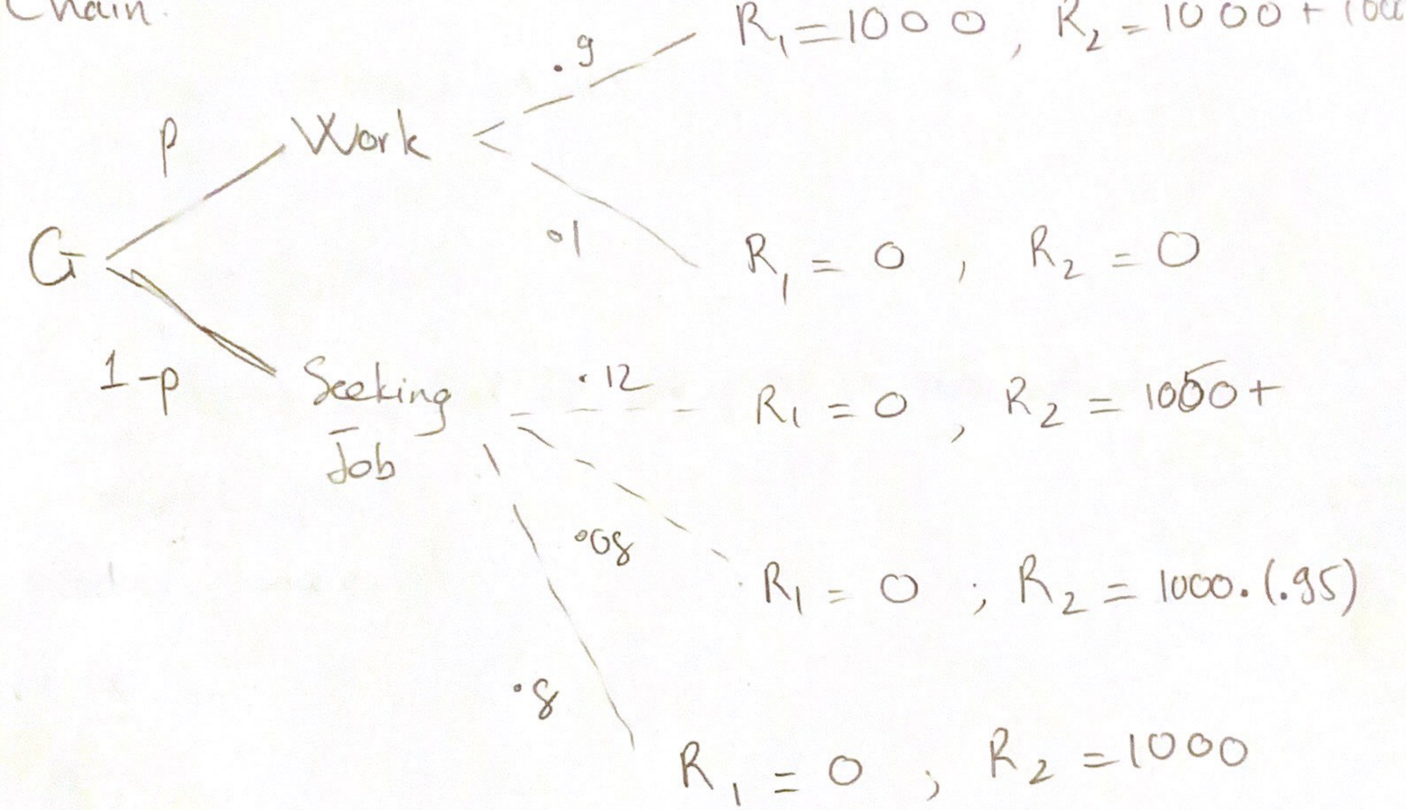
Denote $f(\theta) = \sum_{i=1}^{N}\left(x_i - A\cos(\omega t + \theta)\right)^2$

We have 2 ways to find $\hat{\theta}_{MLE}$ (1) Mathematical and (2) Numerical

(1) Find $\frac{\partial f}{\partial \theta}$ and set $\frac{\partial f}{\partial \theta} = 0$ and $\frac{\partial f^2}{\partial \theta^2} > 0$.

(2) For mumerical, we can use gradiont descent.

Question 4. We first represent the problem as a Hinden Marko Chain.



Thus, we need to find the expected income of Geoffrey.

$$E(R) = p\left(.9 \times 2w + .1 \times 0\right) + (1-p)\Big(.12(w(1+\varepsilon)) + .08(w(1-\varepsilon)) + .8(w)\Big)$$

$$= p(.18w) + (1-p)(w - .04\varepsilon)$$

$$= .18pw + w - .04\varepsilon - pw + .04p\varepsilon$$

$$= .08pw + w - .04\varepsilon(1-p)$$

$= 80p + 1000 - 0.04 \times \varepsilon(1-p)$

If $p = 1$, we see that the term $80p$ is max

and $0.04\varepsilon(1-p) = 0$

There are 2 senarios

(1) The chance that G are fired is independent each week, is 0.1

Question 5 , $x_n = \theta + \varepsilon_{(n)}$ where $\varepsilon_n \sim G(0, \sigma^2)$

. We first need to assume that x_n are all i.i.d., with pdf

$$p(x|\theta) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x_n - \theta)^2\right)$$

To find MLE of θ, we need $\hat{\theta}_{MLE} = \theta^* = \underset{\theta}{\arg\max}\, L(\theta)$

where $L(\theta)$ is the likelihood function (function of θ, realization by x)

$$L(\theta) = \frac{1}{(2\pi\sigma^2)^{5/2}} \prod_{i=1}^{5} \exp\left(\frac{-1}{2\sigma^2}(x_i - \theta)^2\right)$$

$$\log L(\theta) = \frac{5}{2}\log\left(\frac{1}{2\pi\sigma^2}\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{5}(x_i - \theta)^2$$

We need to find $\left(\underset{\theta}{\arg\min} \sum^{5} (x_i - \theta)^2\right)$

It is exactly the Minimizing the L_2-norm or MSE, thus

thus $\hat{\theta}_{MLE}$ is the sample mean $\hat{\theta}_{MLE} = \bar{x}_n = \frac{1}{5}\sum_{i=1}^{5} x_i$

• We now have $X \sim N(\mu; \sigma^2)$, thus, $E(X_i) = \mu$ and

$Var(X_i) = \sigma^2$

$$\Rightarrow E(\bar{X}) = E\left(\frac{1}{5}\sum_{i=1}^{5} x_i\right) = \frac{1}{5}\sum^{5} E(x_i) = \frac{5}{5}\mu = \mu$$

→ Thus $\bar{X}$ is <u>unbiased</u>.

$$E(\bar{x}^2) - E(\bar{x})^2$$

$$\uparrow$$

• For $n \to \infty$, $Var(\bar{X}) = \hat{\sigma}^2 = \left(\frac{1}{n}\sum X_i^2\right) - \bar{X}^2$

We have $E(\hat{\sigma}^2) = E\left[\frac{1}{n}\sum x_i^2 - \bar{x}^2\right]$

$$= \frac{1}{n} E\left(\sum x_i^2\right) - E(\bar{X}^2) = \frac{1}{n}\sum^{n}(\sigma^2 + \mu^2) - \left(\frac{\sigma^2}{n} + \mu^2\right)$$

$$= \frac{1}{n}(n\sigma^2 + n\mu^2) - \left(\frac{\sigma^2}{n} + \mu^2\right)$$

$$= \sigma^2 - \frac{\sigma^2}{n} = \frac{(n-1)\sigma^2}{n}$$ ← Biased