Quantum Quiz 7

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1 Introduction

In the tutorial of quantum stabilizer formalism, Dr. Min-Hsiu Hsieh went through some basic concepts of quantum computing. On top of familiar postulates in the course, there are several new definitions that are very interesting.

1. Quantum Ensemble: Given $\epsilon = \{p_i, |\psi_i\rangle\}$, the quantum ensemble is $\sigma_\epsilon = \sum_i p_i |\psi_i\rangle \langle\psi_i|$. It is noted that the matrix is not longer rank 1 matrix, since it is the sum of density matrix. This can be consider as the most general form of a quantum system.

2. Open system: The most general quantum evolution is a CPTP map. Any state or density matrix can be transformed into another density matrix. $\rho_A \underset{\epsilon_A}{\longrightarrow} \sigma_A$ while ϵ_A has to preserve trace and positivity.

2 Classical Linear Code

2.1 Binary Symmetric Channel

In a binary symmetric channel (BSC), we can protect the message on BSC by adding the redundancy. $0 \to 000$ and $1 \to 111$ and the output is 001 and 110. We can then decode by majority code, for example $\{000, 001, 010, 100\}$ is 0, otherwise is 1. The generatir matrix is $G = [1, 1, 0]$, so $bG = 000$ when $b = 0$, $= 111$ when $b = 1$. On the other hand, we can also define a parity check matrix $H = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, then the **error syndrom** is $Hy = s$. The receive is $y = x + n$, where x is the code and n is the noise.

2.2 Hamming Weights

The weights is $wt(x) :=$ number of nonzeror bit in x. The distance between two vectors is $d(x, y) = wt(x - y)$ and the minimum distance is $e = \min wt(x), x \in e$. The dual code C^T is defined as $C^T := \{u : uv = 0, v \in C\}$; If C is $[n, k]$, then $C^T = [n, n-k]$. We also have E is dual-containing if $C \subset C^T$.

2.3 Symplectic Code

Consider the bit string of size $2n$: $u = (z|x)$, both z, x is n-bit string $x, z \in z_2^n$. The symplectic product is $z^{2n} \times z^{2n} \to z_2$. For example, $u = z(x)$ and $v = z'(x')$ then $u \odot v = zx' + z'x$. The structure of symplectic space in the standard basis in Z_2^{2n}, then the standard basis is $g_i = (e_i|0)$ and $h_j = (0|e_j)$. Then the symplectic product $g_i \odot g_j = 0$, $h_i \odot h_j = 0$ and $g_i \odot h_j = 0, i, = 1, i = j$. We call (g_i, h_i) (same index) a hyperbolic pair and the basis satisfied above assumptions is called a symplectic basis.

Now we define the operation $[S] = \{\beta S | \beta \in C, |\beta| = 1\}$, the Pauli group is

$$[G_1] = \{[I], [X], [Y], [Z]\} \tag{1}$$

with the map $N : z_2^2 \to g_1$. Thus, we have corresponding maps $00 \to I$, $01 \to X$, $10 \to Z$ and $11 \to Y$. For instance, the map $[N_{(z|x)}] = [Z^z X^x]$. This leads us to two properties

1. $[N_u][N_v] = [N_{u+v}]$

2. $N_u N_v = (-1)^{\odot} N_u N_v$. The symplectic product of u, v characterizes the commutation relation of $N_u N_v$

The definition can be extended to n-fold Pauli group. It is a very interesting point of view on the Pauli group.

3 Quantum Stabilizer Code

This is the main part of the tutorial. We first introduce the elementary coding, which involves two people Alice and Bob. It follows the common scheme that Alice prepares $|0\rangle$ based on $b \in \{0, 1\}$, the encoded qubit is $x^b|0\rangle = |b\rangle$. Alice sends the code through an noiseless channel and then Bob measures it. Extending to m-qubit, we have the encoded state is $x^b|0\rangle^{\otimes m}$. Bob decodes the message by measuring $z^{e_1}, ..., z^{e_m} \rightarrow b$.
Given $|\phi\rangle$ is a k-qubit state. We encode the state by unitary (adding redundancy)

$$U_0|\phi\rangle = |0\rangle^{\otimes(n-k)|\phi\rangle} = |\Psi_0\rangle \tag{2}$$

The state $||phi_0\rangle$ is the codeword. We have $E_0 = \{x^a z^b \otimes x^{\alpha(a)} z^{\beta(b)}\}$, the first component is the ground state and the latter is the exciting state. Thus, we derive

$$x^a z^b |0\rangle^{\otimes(n-k)} = x^a |0\rangle^{\otimes(n-k)} = |a\rangle \tag{3}$$

and decoding by $z^{e_1}, ..., z^{e_{m-k}}$.
The stabilizer formalism involves the codeword $\phi_0 = |0\rangle^{\otimes n-k}|\phi\rangle$ is the simultaneous +1 eigenstate of $S_0 =< z^{e_1}, ..., z^{e_{n-k}} >$, S_0 is Ablelian. The stabilizer is defined as

$$T \in S_0, T|\Psi_0\rangle = +1 \times |\Psi_0\rangle \tag{4}$$

The error set can be described by $S_0 \bigcup (g_n - Z(s_0))$

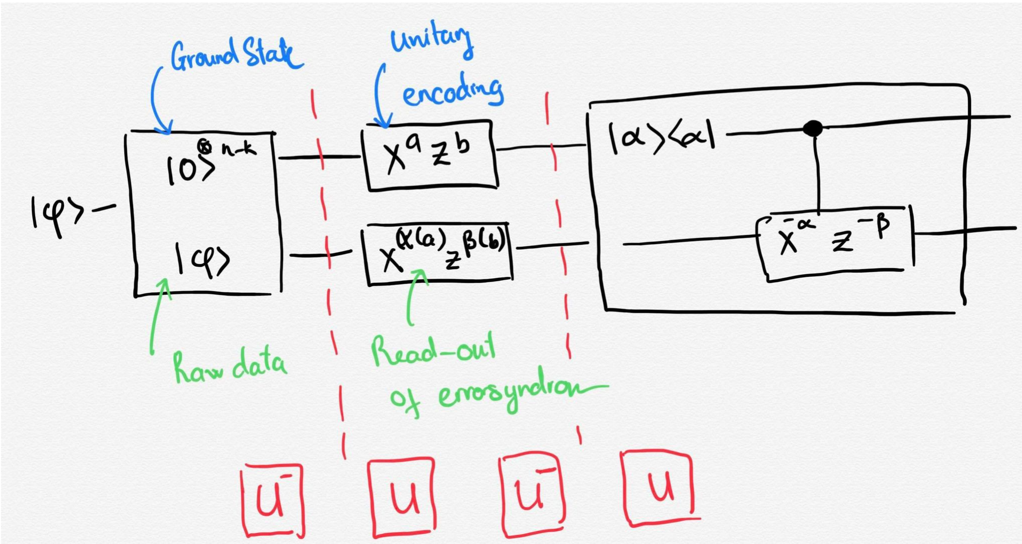

Figure 1: Overview of Quantum Stabilizer Code

Relation to symplectic code: We have $S =< g_1, ..., g_{n-k} >\subset g_n$ through the map N we have corresponding $< N_{u_1}, ..., N_{u_{n-k}} >$, where $u_1, ..., u_{n-k}{}_2^{2n}$ and $u_i \otimes u_j = 0$.