Question 1

(a) $$|\varphi_{entangled}\rangle = \frac{1}{L}\sum_{a=0}^{L-1}\sum_{l=0}^{L-1} e^{2\pi i \frac{al}{L}} |l; f(a)\rangle$$

$$= \frac{1}{8}|0\rangle\Big(|f(0)\rangle + |f(1)\rangle + |f(2)\rangle + |f(3)\rangle + |f(4)\rangle + |f(5)\rangle + |f(6)\rangle + |f(7)\rangle\Big)$$

$$+ \frac{1}{8}|1\rangle\Big(|f(0)\rangle + e^{\frac{2\pi i.1.1}{8}}|f(1)\rangle + e^{\frac{2\pi i}{4}}|f(2)\rangle + e^{\frac{3\pi i}{4}}|f(3)\rangle + e^{\frac{4\pi i}{4}}|f(4)\rangle + e^{\frac{5\pi i}{4}}|f(5)\rangle + e^{\frac{6\pi i}{4}}|f(6)\rangle + e^{\frac{7\pi i}{4}}|f(7$$

$$+ \frac{1}{8}|2\rangle\Big(|f(0)\rangle + e^{\frac{\pi i}{2}}|f(1)\rangle + e^{\frac{2\pi i}{2}}|f(2)\rangle + e^{3\pi i/2}|f(3)\rangle + e^{\frac{4\pi i}{4}}|f(4 + e^{\frac{5\pi i}{4}}|f(5)\rangle + e^{\frac{6\pi i}{4}}|f(6)\rangle + e^{\frac{7\pi i}{4}}|f(7)\rangle$$

$$+ \frac{1}{8}|3\rangle\Big(|f(0)\rangle + e^{\frac{3\pi i}{4}}|f(1)\rangle + e^{\frac{6\pi i}{4}}|f(2)\rangle + e^{\frac{9\pi i}{4}}|f(3)\rangle + e^{\frac{12\pi i}{4}}|f(4)\rangle + e^{\frac{15\pi i}{4}}|f(5)\rangle + e^{\frac{18\pi i}{4}}|f(6)\rangle + e^{\frac{21\pi i}{4}}|f(7)\rangle\Big)$$

$$+\frac{1}{8}|4\rangle\Big(|f(0)\rangle + e^{i\pi}|f(1)\rangle + e^{2\pi i}|f(2)\rangle + e^{3i\pi}|f(3)\rangle$$
$$+e^{4i\pi}|f(4)\rangle + e^{5i\pi}|f(5)\rangle + e^{6i\pi}|f(6)\rangle + e^{7i\pi}|f(7)\rangle\Big)$$

$$+\frac{1}{8}|5\rangle\Big(|f(0)\rangle + e^{\frac{5\pi i}{4}}|f(1)\rangle + e^{\frac{10\pi i}{4}}|f(2)\rangle + e^{\frac{15\pi i}{4}}|f(3)\rangle$$
$$e^{\frac{20\pi i}{4}}|f(4)\rangle + e^{\frac{25\pi i}{4}}|f(5)\rangle + e^{\frac{30\pi i}{4}}|f(6)\rangle + e^{\frac{35\pi i}{4}}|f(7$$

$$+\frac{1}{8}|6\rangle\Big(|f(0)\rangle + e^{\frac{3\pi i}{2}}|f(1)\rangle + e^{\frac{6\pi i}{2}}|f(2)\rangle + e^{\frac{9\pi i}{2}}|f(3)\rangle$$
$$e^{\frac{12\pi i}{2}}|f(4)\rangle + e^{\frac{15\pi i}{2}}|f(5)\rangle + e^{\frac{18\pi i}{2}}|f(6) + e^{\frac{21\pi i}{2}}|f(7)$$

$$+\frac{1}{8}|7\rangle\Big(|f(0)\rangle + e^{\frac{7\pi i}{4}}|f(1)\rangle + e^{\frac{14\pi i}{4}}|f(2)\rangle + e^{\frac{21\pi i}{4}}|f(3)\rangle$$
$$e^{\frac{28\pi i}{4}}|f(4)\rangle + e^{\frac{35\pi i}{4}}|f(5)\rangle + e^{\frac{42\pi i}{4}}|f(6)\rangle + e^{\frac{49\pi i}{4}}|f(7$$

(c) If $r = 2$ the $f(a) = f(a+2)$, the

$f(0) = f(2) = f(4) = f(6)$ and $e^{i(2\pi k + \theta)} = e^{i\theta}$; $k \in \mathbb{Z}$

$f(1) = f(3) = f(5) = f(7)$

thus; $|\varphi_{entangled}\rangle = \frac{1}{8}|0\rangle\left(4|f(0)\rangle + 4|f(1)\rangle\right)$

$+ \frac{1}{8}|1\rangle\left(|f(0)\rangle\left(1 + e^{\frac{2\pi}{4}i} + e^{\pi i} f(4) + e^{\frac{6\pi i}{4}}\right)\right.$

$\left. + |f(1)\rangle\left(1 + e^{\frac{\pi i}{4}} + e^{\frac{3\pi i}{4}} + e^{\frac{5\pi i}{4}}\right)\right)$

$+ \ldots$ we have $e^{i\pi} = -1 \iff e^{i\pi + \theta i} = -e^{i\theta}$

thus terms with $|1\rangle, |2\rangle, |3\rangle; |5\rangle; |6\rangle, |7\rangle$ canceld out

$\Rightarrow |\varphi_{entangled}\rangle = \frac{1}{8}|0\rangle\left(4|f(0)\rangle + 4|f(1)\rangle\right)$

$+ \frac{1}{8}|4\rangle\left(4|f(0)\rangle - 4|f(1)\rangle\right)$

$= \frac{1}{2}\left(|0, f(0)\rangle + |0, f(1)\rangle + |4; f(0)\rangle - |4; f(1)\rangle\right)$

$\rightarrow$ Then the outcome state is $|0\rangle$ or $|4\rangle$ with equal prob $\rightarrow r = 2$

Question 2

(a) Quantum teleportation involves two channels: (1) quantum channel and (2) classical channel. Quantum teleportation canbe describe in 3 steps:

(1) With two particles B and C with maximum entanglement

$$|B_{10}(B;C)\rangle = \frac{1}{\sqrt{2}}\left(|0_B 0_C\rangle - |1_B 1_C\rangle\right)$$

We send B to Alice, so she now has A and B

C to Bob; so he has only C.

then if $|\varphi_A\rangle = c_0|0_A\rangle + c_1|1_A\rangle$

the joint state of the system is

$$|\varphi_A\rangle|B_{10}(B,C)\rangle = \left(c_0|0_A\rangle + c_1|1_A\rangle\right)\left(\frac{1}{\sqrt{2}}|0_B 0_C\rangle - |1_B 1_C\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left[c_0\left(|0_A 0_B 0_C\rangle - |0_A 1_B 1_C\rangle\right) + c_1\left(|1_A 0_B 0_C\rangle - |1_A 1_B 1_C\rangle\right)\right.$$

answer for part (c)

5

We now write $|\varphi_{ABC}\rangle$ in Bell 's state basis.

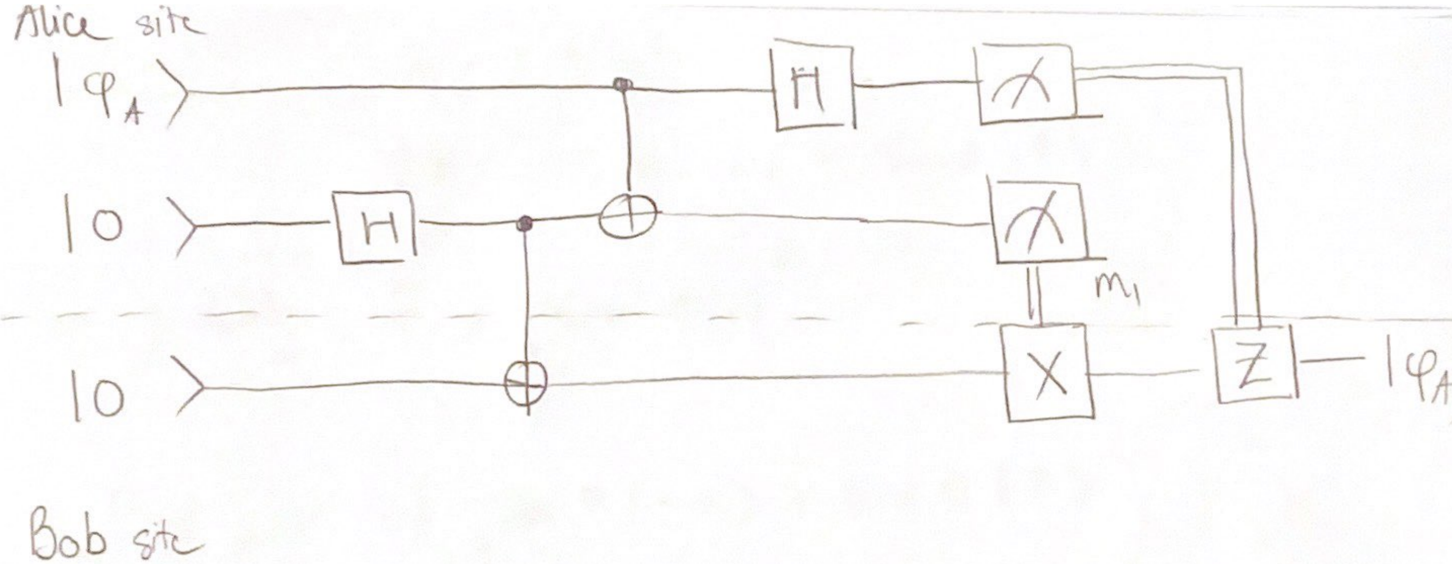
(2) Alice now measures her particlds A and B to obtain one of Bell state.

(3) Alice tells Bob her measurment through classical channel; Bob adjusts and get Alice's orinal state.

(b) The state $|\varphi_A\rangle$ disappear at step (2) when Alice performs the measurement over joint system A and B

(c) The joint state $|\varphi_{ABC}\rangle = \frac{1}{\sqrt{2}}$

$$|\varphi_{ABC}\rangle = \frac{1}{\sqrt{2}}\left[c_0\left(|0_A 0_B 0_C\rangle - |0_A 1_B 1_C\rangle\right) + c_1\left(|1_A 0_B 0_C\rangle - |1_A 1_B 1_C\rangle\right) \right]$$


Alice site
|φA⟩
|0⟩
|0⟩
H
H
X
Z
m1
|φA
Bob site

Question 3

$$|\varphi\rangle = |\rightarrow\rangle$$

$$|\phi\rangle = \cos\theta\,|\rightarrow\rangle + \sin\theta\,|\uparrow\rangle$$

The fidelity $F = |\langle\varphi|\phi\rangle|^2$

$$\Leftrightarrow F = \left| \langle\rightarrow| \left(\cos\theta\,|\rightarrow\rangle + \sin\theta\,|\uparrow\rangle \right) \right|^2$$

$$= \left| \cos\theta \langle\rightarrow|\rightarrow\rangle + \sin\theta \langle\rightarrow|\uparrow\rangle \right|^2$$

0 since orthogonal

$$= (\cos\theta)^2$$

Thus, $0 \leq F \leq 1$, $\begin{cases} F = 0 \text{ when } \theta = 0 \\ F = 1 \text{ when } \theta = \frac{\pi}{2} \end{cases}$

Geometric meaning, $|\phi\rangle$ is a perfect good copy of $|\varphi\rangle$ if $\theta = \frac{\pi}{2}$

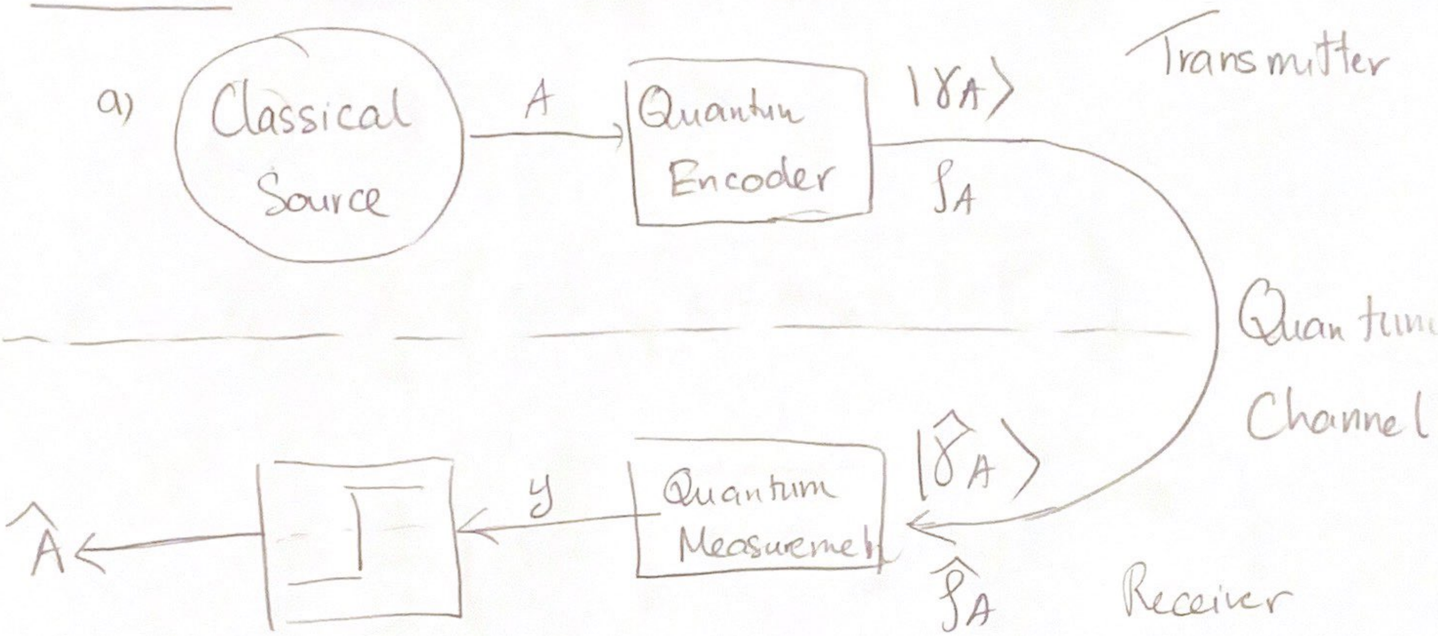
or $|\phi\rangle$ and $|\varphi\rangle$ have the same polarization, otherwise

$\theta = 0$, $|\phi\rangle$ and $|\varphi\rangle$ have contrast polarization

($|\phi\rangle$ is $|\uparrow\rangle$ while $|\varphi\rangle$ is $|\rightarrow\rangle$)

Question 4

a)



b) We have $|\gamma_0\rangle = |0\rangle$ and $|\gamma_1\rangle = |\Delta\rangle$

$\rightarrow \langle 0|\Delta\rangle = e^{-\Delta^2}$. The number of photon with "0"

is $A_0 = 0 = N_R(0)$; with "1" is $N_R(1) = \Delta^2$

. With equally probable signaling, the avg number of photo[ns]

per bit is $N_R = q_0 \underbrace{N_R(0)}_{0} + q_1 N_R(1) = \frac{1}{2} N_R(1)$

(c) The quadratic superposition of $|\gamma_0\rangle$ and $|\gamma_1\rangle$

is $|\Delta|^2 = e^{-2N_R}$

(d) $$P_e = \frac{1}{2}\left(1 - \sqrt{1 - 4q_0q_1|\Delta|^2}\right)$$

$$= \frac{1}{2}\left(1 - \sqrt{1 - 4 \times \frac{1}{2} \times \frac{1}{2} \times e^{-2N_R}}\right)$$

$$= \frac{1}{2}\left(1 - \sqrt{1 - e^{-2N_R}}\right)$$

Question 5

Alice Bit	0	1	0	1	1	0	0	1
Alice Bases	R	R	D	D	R	D	D	R
Alice sent state	$\|0\rangle$	$\|1\rangle$	$\|+\rangle$	$\|-\rangle$	$\|1\rangle$	$\|+\rangle$	$\|+\rangle$	$\|1\rangle$
Bob Basis	D	R	R	D	D	D	R	R
Bob's measured state	$\begin{bmatrix} \|+\rangle \\ \|-\rangle \end{bmatrix}$	$\|1\rangle$	$\begin{bmatrix} \|0\rangle \\ \|1\rangle \end{bmatrix}$	$\|-\rangle$	$\begin{bmatrix} \|+\rangle \\ \|-\rangle \end{bmatrix}$	$\|+\rangle$	$\begin{bmatrix} \|0\rangle \\ \|1\rangle \end{bmatrix}$	$\|1\rangle$
Bob's bit	0/1	1	0/1	1	0/1	0	0/1	1

(a) The out come of Bob's bits is

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad 1 \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad 1 \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad 0 \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad 1$$

where $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is 0 or 1 with equal chance $= \frac{1}{2}$.

- When Alice's and Bob's bases are difference; the out come bit of Bob = Alice's bit with probability $= \frac{1}{2}$.

In BB84 protocol, we discard these bits.

(b) The delived key is $1\ 1\ 0\ 1$; key length $= 4$

(c) Let denote $|\varphi_{ab}\rangle_{b'}$ be the quantum state of Alice

where $a = 0, 1$ ← Alice bit

$b = 0, 1$ ← Alice's basis — 0 = Basis-R, 1 = Basis-D

$b' = 0, 1$ ← Bob's basis

. All posible cases is $|\varphi_{00}\rangle_0$; $|\varphi_{01}\rangle_0$, $|\varphi_{00}\rangle_1$ $|\varphi_{01}\rangle_1$

$|\varphi_{10}\rangle_0$; $|\varphi_{11}\rangle_0$, $|\varphi_{10}\rangle_1$, $|\varphi_{11}\rangle_1$

• The qubit can be used as an element of the final key if $b = b'$ (Alice and Bob used the same basis)

hence there are $|\varphi_{00}\rangle_0$, $|\varphi_{01}\rangle_1$

$|\varphi_{10}\rangle_0$; $|\varphi_{11}\rangle_1$

So $P = \frac{4}{8} = \frac{1}{2}$

• Anothe way to think is that each qubit has only two disjoint events, (1) included in final key and (2) excluded in final key

$\longrightarrow p = 1/2$

(d) Based on (c), the probability of each qubit included in the final key is $p = \frac{1}{2}$

First; we clearly see the events corresponding to each qubit are independent (the events q_i included does not affect the event q_j included)

So the key length will follow Binomial distribution $B(n, \frac{1}{2})$

$$P(\text{lentgth} = k) = \binom{n}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{n-k}$$

$$= \binom{n}{k}\left(\frac{1}{2}\right)^n$$

Mean of key length is $np = \frac{n}{2}$