Homework 5 _ Nam Nguyen U 7084 0587

Question 1:

1, Show that $R_y(\theta) = \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} Y$

$$R_y(\theta) = e^{-i\frac{\theta}{2}Y} = \sum_{k=0}^{\infty} \frac{\left(-i\frac{\theta}{2}\right)^n Y^n}{n!}$$

$$= \sum_{k=0}^{\infty} \frac{\left(-i\,\theta/2\right)^{2k} Y^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{\left(-i\,\theta/2\right)^{2k+1} Y^{2k+1}}{(2k+1)!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k \left(\theta/2\right)^{2k}}{(2k)!} I - i \sum_{k=0}^{\infty} \frac{(-1)^k \left(\theta/2\right)^{2k+1}}{(2k+1)!} Y$$

$$= \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} Y$$

2) $$X^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$Y^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} \overset{=1}{-i^2} & 0 \\ 0 & -i^2 \end{pmatrix} = I$$

$$Z^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$-iXYZ = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -i^2 & 0 \\ 0 & +i^2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -i^2 & 0 \\ 0 & -i^2 \end{pmatrix} = I$$

b, $$XY = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = iZ$$

$$YX = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$= -1\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = -iZ$$

Thus, $XY = -YX = iZ$

c) $YZ = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

$= iX$

$ZY = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$

$= -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -iX$

Thus, $YZ = -ZY = iX$

d) $ZX = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & -i^2 \\ i^2 & 0 \end{pmatrix} = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = iY$

$XZ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$= -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -iY$

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3) $R_y(\theta) X R_y(\theta)^\dagger = \left(\cos\frac{\theta}{2} I - i\sin\frac{\theta}{2} Y\right) X \left(\cos\frac{\theta}{2} I + i\sin\frac{\theta}{2} Y\right)$

$= \left(\cos\frac{\theta}{2} X - i\sin\frac{\theta}{2} YX\right)\left(\cos\frac{\theta}{2} I + i\sin\frac{\theta}{2} Y\right)$

$= \cos^2\frac{\theta}{2} X + i\sin\frac{\theta}{2}\cos\frac{\theta}{2} XY - i\sin\frac{\theta}{2}\cos\frac{\theta}{2} YX + \sin^2\frac{\theta}{2}(YX)Y$

$= \cos^2\frac{\theta}{2} X + \sin\frac{\theta}{2}\cos\frac{\theta}{2}(i^2 Z) + \sin\frac{\theta}{2}\cos\frac{\theta}{2}(i^2 Z) + \sin^2\frac{\theta}{2}(-iZ)Y$

$= \cos^2\frac{\theta}{2} X + \sin^2\frac{\theta}{2}(i^2 X) - \sin\frac{\theta}{2}\cos\frac{\theta}{2} Z - \sin\frac{\theta}{2}\cos\frac{\theta}{2} Z$

$= \left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right) X - 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} Z$

$= \cos\theta X - \sin\theta Z$

4) $R_y(\theta) Z R_y(\theta)^\dagger = \left(\cos\frac{\theta}{2} I - i\sin\frac{\theta}{2} Y\right) Z \left(\cos\frac{\theta}{2} I + i\sin\frac{\theta}{2} Y\right)$

$= \left(\cos\frac{\theta}{2} Z - i\sin\frac{\theta}{2} YZ\right)\left(\cos\frac{\theta}{2} I + i\sin\frac{\theta}{2} Y\right)$

$= \left[\cos\frac{\theta}{2} Z - \sin\frac{\theta}{2}(i^2 X)\right]\left(\cos\frac{\theta}{2} I + i\sin\frac{\theta}{2} Y\right)$

$= \cos^2\frac{\theta}{2} Z + i\cos\frac{\theta}{2}\sin\frac{\theta}{2} ZY + \sin\frac{\theta}{2}\cos\frac{\theta}{2} X + \sin^2\frac{\theta}{2}\, i XY$

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$$= \cos^2\frac{\theta}{2} Z + i\cos\frac{\theta}{2}\sin\frac{\theta}{2}(-iX) + \sin\frac{\theta}{2}\cos\frac{\theta}{2} X - \sin^2\frac{\theta}{2} Z$$

$$= \left(\cos^2\frac{\theta}{2} Z - \sin^2\frac{\theta}{2}\right) Z + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} X$$

$$= \cos\theta Z + \sin\theta X$$

5) $$R_y(\theta) Y R_y(\theta)^{\dagger} = \left(\cos\frac{\theta}{2} I - i\sin\frac{\theta}{2} Y\right) Y \left(\cos\frac{\theta}{2} I + i\sin\frac{\theta}{2} Y\right)$$

$$= \left(\cos\frac{\theta}{2} Y - i\sin\frac{\theta}{2} \overset{I}{\cancel{Y^2}}\right)\left(\cos\frac{\theta}{2} I + i\sin\frac{\theta}{2} Y\right)$$

$$= \cos^2\frac{\theta}{2} Y + i\sin\frac{\theta}{2}\cos\frac{\theta}{2} \overset{I}{\cancel{Y^2}} - i\sin\frac{\theta}{2}\cos\frac{\theta}{2} I + \sin^2\frac{\theta}{2} Y$$

$$= \left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right) Y = 1\, Y = Y$$

6) $$R_y(\theta)\rho R_y(\theta)^{\dagger} = \left(\cos\frac{\theta}{2} I - i\sin\frac{\theta}{2} Y\right)\frac{1}{2}\left(I + r_x X + r_y Y + r_z Z\right)\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2} Y\right)$$

$$= \frac{1}{2}\left[R_y(\theta) + r_x R_y(\theta) X + r_y R_y(\theta) Y + r_z R_y(\theta) Z\right] R_y(\theta)^{\dagger}$$

$$= \frac{1}{2}\left[R_y(\theta)R_y(\theta)^{\dagger} + r_x R_y(\theta) X R_y(\theta)^{\dagger} + r_y R_y(\theta) Y R_y(\theta)^{\dagger} + r_z R_y(\theta) Z R_y(\theta)^{\dagger}\right]$$

$$= \frac{1}{2}\left[I + r_x\left(\cos\theta X - \sin\theta Z\right) + r_y Y + r_z\left(\cos\theta Z + \sin\theta X\right)\right]$$

$$= \frac{1}{2}\left[I + \left(r_x\cos\theta + r_z\sin\theta\right) X + r_y Y + \left(-r_x\sin\theta + r_z\cos\theta\right) Z\right]$$

Thus, $r_x' = r_x \cos\theta + r_z \sin\theta$

$r_y' = r_y$

$r_z' = -r_x \sin\theta + r_z \cos\theta$

$$A = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

$\rightarrow$ Rotation around y-axis by an angle θ on (r_x, r_y, r_z)

Question 2,

1, $H^{\otimes 2}|00\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$

$= \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$

$\varphi_1 = \left(H \otimes H \otimes I\right)|0\rangle|0\rangle|\varphi\rangle$

$= H^{\otimes 2}|00\rangle \; I|\varphi\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)|\varphi\rangle$

2) $\varphi_2 = CCNOT(\varphi_1)$

$$= \frac{1}{2}\Big(|00\rangle|\psi\rangle + |10\rangle|\psi\rangle + |01\rangle|\psi\rangle + |11\rangle X|\psi\rangle \Big)$$

Since Toffoli ($|11\rangle$) is equivalence to applying X-gate

Toffoli($|00\rangle$) = Toffoli($|01\rangle$) = Toffoli($|10\rangle$) = I

$$\rightarrow \varphi_2 = \frac{1}{2}\Big[\big(|00\rangle + |01\rangle + |10\rangle\big)|\psi\rangle + |11\rangle X|\psi\rangle\Big]$$

3) $\varphi_3 = \frac{1}{2}\Big[|00\rangle S|\psi\rangle + |10\rangle S|\psi\rangle + |01\rangle S|\psi\rangle + |11\rangle SX|\psi\rangle \Big]$

(Since apply S on $|\psi\rangle$)

$$= \frac{1}{2}\Big[\big(|00\rangle + |10\rangle + |01\rangle\big)S|\psi\rangle + |11\rangle SX|\psi\rangle\Big]$$

4) We again apply Toffoli on the current gate,

$$\varphi_4 = \frac{1}{2}\Big[\big(|00\rangle + |10\rangle + |01\rangle\big)S|\psi\rangle + |11\rangle XSX|\psi\rangle\Big]$$

5) $H^{\otimes 2}|01\rangle = H|0\rangle \otimes H|1\rangle$

$$= \frac{1}{\sqrt{2}}\big(|0\rangle + |1\rangle\big) \otimes \frac{1}{\sqrt{2}}\big(|0\rangle - |1\rangle\big)$$

$$= \frac{1}{2}\big(|00\rangle - |01\rangle + |10\rangle - |11\rangle\big)$$

$$H^{\otimes 2}|10\rangle = H|1\rangle \otimes H|0\rangle$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$= \frac{1}{2}\left(|00\rangle + |01\rangle - |10\rangle - |11\rangle\right)$$

$$H^{\otimes 2}|11\rangle = H|1\rangle \otimes H|1\rangle$$

$$= \frac{1}{2}\left(|0\rangle - |1\rangle\right) \otimes \left(|0\rangle - |1\rangle\right)$$

$$= \frac{1}{2}\left(|00\rangle - |01\rangle - |10\rangle + |11\rangle\right)$$

6) We apply $H \otimes H \otimes I = H^{\otimes 2}$ on φ_4, yielding

$$\varphi_5 = H^{\otimes 2}\left(\frac{1}{2}\left[\left(|00\rangle + |01\rangle + |10\rangle\right) S|\psi\rangle + |11\rangle XSX|\psi\rangle\right]\right)$$

$$= \frac{1}{2} \cdot \left\{ H^{\otimes 2}\left(|00\rangle + |01\rangle + |10\rangle\right) S|\psi\rangle + H^{\otimes 2}|11\rangle XSX|\psi\rangle \right\}$$

$$= \frac{1}{2}\left\{ \frac{1}{2}\left(\begin{array}{l} |00\rangle + |10\rangle + |01\rangle + |11\rangle + |00\rangle + |10\rangle - |01\rangle - |11\rangle \\ + |00\rangle - |10\rangle + |01\rangle - |11\rangle \end{array} \right) S|\psi\rangle + \right.$$

$$\left. + \frac{1}{2}\left(|00\rangle - |10\rangle - |01\rangle + |11\rangle\right) XSX|\psi\rangle \right\}$$

$$= \frac{1}{4}\Big[3|00\rangle S|\psi\rangle + |00\rangle XSX|\psi\rangle + \big(|01\rangle + |10\rangle - |11\rangle\big) S|\psi\rangle - \big(|01\rangle + |10\rangle - |11\rangle\big) XSX|\psi\rangle\Big]$$

$$= \frac{1}{4}\Big[|00\rangle(3S + XSX)|\psi\rangle + \big(|01\rangle + |10\rangle - |11\rangle\big)\big(S - XSX\big)|\psi\rangle\Big]$$

7) $\cos\theta = \frac{3}{5}$, $\theta > 0$ $\rightarrow \begin{cases} \sin\theta = \pm\sqrt{1 - \left(\frac{3}{5}\right)^2} = \pm\frac{4}{5} \\ \sin\theta > 0 \end{cases}$

$\Rightarrow \sin\theta = \frac{4}{5}$

$$\begin{cases} \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \frac{3}{5} \\ 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{4}{5} \end{cases} \Leftrightarrow \begin{cases} 2\cos^2\frac{\theta}{2} = \frac{8}{5} \\ \sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{2}{5} \end{cases} \Leftrightarrow \begin{cases} \cos\frac{\theta}{2} = \frac{2}{\sqrt{5}} \\ \sin\frac{\theta}{2} = \frac{1}{\sqrt{5}} \end{cases}$$

8) Find; $\alpha \in \mathbb{R}$ such that:

$$3S + XSX = \sqrt{10}\, e^{i\alpha} \begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{pmatrix}$$

$$3S + XSX = \begin{pmatrix} 3 & 0 \\ 0 & 3i \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 \\ 0 & 3i \end{pmatrix} + \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3i \end{pmatrix} + \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3+i & 0 \\ 0 & 1+3i \end{pmatrix}$$

Thus; $\begin{cases} \sqrt{10}\, e^{i\alpha - i\frac{\theta}{2}} = 3+i \\ \sqrt{10}\, e^{i\alpha + i\frac{\theta}{2}} = 1+3i \end{cases}$

$$\Leftrightarrow \begin{cases} e^{i\left(\alpha - \frac{\theta}{2}\right)} = \frac{3}{\sqrt{10}} + \frac{1}{\sqrt{10}} i = \cos\phi + i\sin\phi \\ e^{i\left(\alpha + \frac{\theta}{2}\right)} = \frac{1}{\sqrt{10}} + \frac{3}{\sqrt{10}} i = \sin\phi + i\cos\phi \end{cases}$$

$$\Leftrightarrow \begin{cases} \cos\left(\alpha - \frac{\theta}{2}\right) + i\sin\left(\alpha - \frac{\theta}{2}\right) = \frac{3}{\sqrt{10}} + \frac{1}{\sqrt{10}} i \\ \cos\left(\alpha + \frac{\theta}{2}\right) + i\sin\left(\alpha + \frac{\theta}{2}\right) = \frac{1}{\sqrt{10}} + \frac{3}{\sqrt{10}} i \end{cases}$$

$\rightarrow \cos\left(\alpha - \frac{\theta}{2}\right) = \frac{3}{\sqrt{10}} \quad \rightarrow \quad \sin\left(\alpha - \frac{\theta}{2}\right) = \frac{1}{\sqrt{10}}$

$\cos\left(\alpha + \frac{\theta}{2}\right) = \frac{1}{\sqrt{10}} \qquad \sin\left(\alpha + \frac{\theta}{2}\right) = \frac{3}{\sqrt{10}}$

$\Rightarrow \cos\alpha \cos\frac{\theta}{2} + \sin\alpha \sin\frac{\theta}{2} = \frac{3}{\sqrt{10}}$

$\cos\alpha \cos\frac{\theta}{2} - \sin\alpha \sin\frac{\theta}{2} = \frac{1}{\sqrt{10}}$

$\sin\alpha \cos\frac{\theta}{2} - \cos\alpha \sin\frac{\theta}{2} = \frac{1}{\sqrt{10}}$

$\sin\alpha \cos\frac{\theta}{2} + \cos\alpha \sin\frac{\theta}{2} = \frac{3}{\sqrt{10}}$

$\sin\frac{\theta}{2} = \frac{1}{\sqrt{5}}$ and $\cos\frac{\theta}{2} = \frac{2}{\sqrt{5}}$

then;

$$\begin{cases} \frac{2}{\sqrt{5}}\cos\alpha + \frac{1}{\sqrt{5}}\sin\alpha = \frac{3}{\sqrt{10}} \\ \frac{2}{\sqrt{5}}\cos\alpha - \frac{1}{\sqrt{5}}\sin\alpha = \frac{1}{\sqrt{10}} \\ \frac{2}{\sqrt{5}}\sin\alpha - \frac{1}{\sqrt{5}}\cos\alpha = \frac{1}{\sqrt{10}} \\ \frac{2}{\sqrt{5}}\sin\alpha + \frac{1}{\sqrt{5}}\cos\alpha = \frac{3}{\sqrt{10}} \end{cases}$$

$$\Rightarrow \begin{cases} \frac{4}{\sqrt{5}}\cos\alpha = \frac{4}{\sqrt{10}} \\ \frac{4}{\sqrt{5}}\sin\alpha = \frac{4}{\sqrt{10}} \end{cases} \Rightarrow \begin{cases} \cos\alpha = \frac{1}{\sqrt{2}} \\ \sin\alpha = \frac{1}{\sqrt{2}} \end{cases}$$

Thus, $\alpha = \frac{\pi}{4}$

9) $R_z(\theta)|\psi\rangle = \left(\cos\frac{\theta}{2} I - i\sin\frac{\theta}{2} Z\right)|\psi\rangle$

We have $|\psi\rangle = \cos\frac{\theta}{2} + e^{i\phi}\sin\frac{\theta}{2}$

then $\rho = |\psi\rangle\langle\psi| = \frac{1}{2}\left(I + X\cos\phi\sin\theta + Y\sin\phi\sin\theta + Z\cos\theta\right)$

Thus $R_z(\theta)\rho R_z(\theta)^\dagger = \frac{1}{2}\left[I + r_x\left(\cos\theta + \sin\theta Y\right) + r_y\left(\cos\theta Y - \sin\theta X\right) + r_z Z\right]$

$$= \frac{1}{2}\left[I + \left(r_x\cos\theta - r_y\sin\theta\right)X + \left(r_x\sin\theta + r_y\cos\theta\right)Y + r_z Z\right]$$

$$\Rightarrow \begin{aligned} r'_x &= \cos\theta\, r_x - \sin\theta\, r_x \\ r'_y &= \sin\theta + \cos\theta\, r_y \\ r'_z &= r_z \end{aligned} \rightarrow T = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus, up to a phase, the rotation of angle θ around z has been applied to the target qbit $\rightarrow$ the probability of measuring 00 on the first 2 qbits is $5/8$.

The probability of measuring $|00\rangle$ at the first two qubits is given by

$$\left|\frac{1}{4}\left(3S + XSX\right)\right|^2 = \left|\frac{1}{4}\sqrt{10}\, e^{i\frac{\pi}{4}}\begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}\right|^2 = \frac{10}{16} = \frac{5}{8}$$

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