# Homework 3

## Question 1,

(C) $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

, (H) $|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ $|-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

a). $|+-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}$

. $|101\rangle = \left(\begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}\right) \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}$

$= \begin{pmatrix}0\\0\\0\\0\\0\\1\\0\\0\end{pmatrix} = |5\rangle_{\mathbb{C}^8}$

. $|1\rangle \otimes |-\rangle \otimes |1\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}$

$= \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\-1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\0\\0\\0\\1\\0\\-1\end{pmatrix}$

b. $|110\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}$

$$= \begin{pmatrix}0\\0\\0\\0\\0\\0\\1\\0\end{pmatrix} = |6\rangle \quad \mathbb{C}^8$$

$|101\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}$

$$= \begin{pmatrix}0\\0\\0\\0\\0\\1\\0\\0\end{pmatrix} = |5\rangle \quad \mathbb{C}^8$$

Thus $\langle 110 | 101 \rangle = \langle 6 | 5 \rangle = 0$

$\in \mathbb{C}^8$

- $$|++-\rangle = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$|+--\rangle = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

Thus, $\langle ++- | +-- \rangle = \frac{1}{4 \times 2} \left( 1 + 1 - 1 - 1 + 1 + 1 - 1 - 1 \right) = 0$

- $$\langle ++- | 101 \rangle = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \frac{-1}{2\sqrt{2}}$$

Question 2,

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

a) • $H \otimes I = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

• $H \otimes H = H^{\otimes 2}$

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

TÜL

b) • $H \otimes H |+-\rangle = \left(H|+\rangle\right) \otimes \left(H|-\rangle\right)$

( $|+\rangle$ and $|-\rangle$ are pure tensor )

$$H|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} = |0\rangle$$

$$H|-\rangle = |1\rangle$$

Thus, $H \otimes H |+-\rangle = |0\rangle \otimes |1\rangle = \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$

• $\left(I \otimes H \otimes I\right)|111\rangle = \left(I|1\rangle\right) \otimes \left(H|1\rangle\right) \otimes \left(I|1\rangle\right)$

$$= |1\rangle \otimes |-\rangle \otimes |1\rangle = |1-1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\0\\0\\0\\1\\0\\-1\end{pmatrix}$$

Question 3

a, Show that, $f: V \times W \longrightarrow V \otimes W$ is bilinear
$(v, w) \longmapsto v \otimes w$

$f(v, w) = v \otimes w$

+, $f(v_1 + v_2, w) = (v_1 + v_2) \otimes w$
$= (v_1 \otimes w) + (v_2 \otimes w)$
$= f(v_1, w) + f(v_2, w)$

+, $f(v, w_1 + w_2) = v \otimes (w_1 + w_2)$
$= (v \otimes w_1) + (v \otimes w_2)$
$= f(v, w_1) + f(v, w_1)$

+, $f(cv, w) = ((cv) \otimes w) = c(v \otimes w) = cf(v, w)$
$= (v \otimes (cw)) = f(v, cw)$

TÜL

Question 3c;

1, Show that $f_2(S) = 0 \Rightarrow f_2(X) = f_2(Y)$

Since $X \ominus Y \in S \rightarrow f_2(S) = f_2(X \ominus Y) \underbrace{=}_{Q3b} f_2(X) - f_2(Y)$

Thus, $f_2(X) = f_2(Y)$

2, $f_2\left[(v_1 + v_2, w) \ominus (v_1, w) \ominus (v_2, w)\right]$

$= f_2(v_1 + v_2, w) \ominus f_2\left[(v_1, w) \oplus (v_2, w)\right]$

By def 5.1-4; $(v_1, w) \oplus (v_2, w) \sim (v_1 + v_2, w)$

$= f_2(v_1 + v_2, w) \ominus f_2(v_1 + v_2, w)$

By part (1)

$= 0$

3, Similaly; $f_2\left[(v, w_1 + w_2) \ominus (v, w_1) \ominus (v, w_2)\right)$

$= f_2\left[(v, w_1 + w_2) \ominus f_2\left((v, w_1) \oplus (v, w_2)\right]\right.$

$= f_2(v, w_1 + w_2) \ominus f_2(v, w_1 + w_2) = 0$

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4, $c(v,w) \sim (cv, w) \sim (v, cw)$ (By 5.1-6)

Thus, $f_2\left[c(v,w) \ominus (cv, w)\right]$

$= f_2\left[(cv, w) \ominus (cv, w)\right]$

$= f_2(A) - f_2(A) = 0$

and $f_2\left[c(v,w) \ominus (v, cw)\right]$

$= f_2\left[\underbrace{(v, cw)}_{=A} \ominus (v, cw)\right]$

$= f_2(A) - f_2(A) = 0$

5; Show that $f_2(S) = 0 \Rightarrow \overline{f_2}$ is well-defined

- From (d), we have $f_2(S) = 0 \Rightarrow f_2(X) = f_2(Y)$

and recall that

$$\overline{f_2} : V \otimes W \longrightarrow U$$
$$\overline{X} \longmapsto f_2(X)$$

There 2 things we have

1) For $\forall\ \overline{X} \in V \otimes W$, we have $f_2(X) \in U$

2, $X \ominus Y$, $X, Y \in F(V \times W)$, $f_2(X \ominus Y) = 0$

$\rightarrow f_2(X) = f_2(Y)$

$\rightarrow \overline{f_2}$ is well-defined

Question 3, part 4.

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We have $f : V \times W \longrightarrow U$ ; $f_1 \; V \times W \longrightarrow V \otimes W$

$\overline{f_2} \; V \otimes W \longrightarrow U$

$f = \overline{f_2} \circ f_1$

- $f_1(v, w) = v \otimes w$ hence $\overline{f_2} \circ f_1 = \overline{f_2}(v, w) = f(v, w)$

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$f \; V \times W \longrightarrow U$
$(v, w) \longmapsto f(v, v)$