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a) • $\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

We find λ satisfying $\left| \sigma_x - \lambda I \right| = 0$

$\Leftrightarrow \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \Leftrightarrow \lambda^2 - 1 = 0 \Leftrightarrow \left[\begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = -1 \end{array} \right.$

With $\lambda_1 = 1 \rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} v_1 = 0 \Rightarrow v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = -1 \rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} v_2 = 0 \Rightarrow v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

• $\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$\left| \sigma_y - \lambda I \right| = 0 \Leftrightarrow \begin{vmatrix} -\lambda & -i \\ i & -\lambda \end{vmatrix} = 0$

$\Leftrightarrow \lambda^2 + i^2 = 0 \Leftrightarrow \lambda^2 - 1 = 0 \Leftrightarrow \left[\begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = -1 \end{array} \right.$

+, $\lambda_1 = 1 \Rightarrow \begin{bmatrix} -1 & -i \\ i & -1 \end{bmatrix} v_1 = 0 \rightarrow v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$

+, $\lambda_2 = -1 \rightarrow \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} \longrightarrow v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$

- $\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

We have $\left| \sigma_z - \lambda I \right| = 0 \Leftrightarrow \begin{vmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = 0$

$\Leftrightarrow (1-\lambda)(-1-\lambda) = 0$

$\Leftrightarrow \left[\begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = -1 \end{array} \right.$

With $\lambda_1 = 1 \rightarrow \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} v_1 = 0 \rightarrow v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda_2 = -1 \rightarrow \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} v_2 = 0 \rightarrow v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

b) $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}$

We need to find $\left| M - \lambda I \right| = 0$

$\Leftrightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & i \\ 0 & i & -\lambda \end{vmatrix} = 0$

TÜL

$$\Leftrightarrow (1-\lambda)\begin{vmatrix} -\lambda & i \\ i & -\lambda \end{vmatrix} = 0 \Leftrightarrow (1-\lambda)(\lambda^2 - i^2) = 0$$

$$\Leftrightarrow \lambda_1 = 1 \, , \lambda_2 = i \, , \lambda_3 = -i$$

+, $\lambda_1 = 1 \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & i \\ 0 & i & -1 \end{bmatrix} v_1 = 0 \Rightarrow v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

+, $\lambda_2 = i \Rightarrow \begin{bmatrix} 1-i & 0 & 0 \\ 0 & -i & i \\ 0 & i & -i \end{bmatrix} v_2 = 0 \Rightarrow v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

+, $\lambda_3 = -i \Rightarrow \begin{bmatrix} 1+i & 0 & 0 \\ 0 & i & i \\ 0 & -i & -i \end{bmatrix} v_3 = 0 \Rightarrow v_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

<u>Question 2,</u>

a, $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \quad (1)$

$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \quad (2)$

Take (1) + (2) $\rightarrow$ $|+\rangle + |-\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|0\rangle = \sqrt{2}|0\rangle$

$\Rightarrow |0\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle \rightarrow \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

Take (1) - (2)

$\Rightarrow |+\rangle - |-\rangle = \sqrt{2}|1\rangle \Rightarrow |1\rangle = \frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle \rightarrow \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

b) $|\psi\rangle = \frac{1}{\sqrt{2}}|+\rangle - \frac{\sqrt{3}}{2}|-\rangle$

$$= \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)\right] - \frac{\sqrt{3}}{2}\left[\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)\right]$$

$$= \frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle - \frac{\sqrt{3}}{2\sqrt{2}}|0\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|1\rangle$$

$$= \frac{\sqrt{2}-\sqrt{3}}{2\sqrt{2}}|0\rangle + \frac{\sqrt{2}+\sqrt{3}}{2\sqrt{2}}|1\rangle$$

$$= \frac{2-\sqrt{6}}{4}|0\rangle + \frac{2+\sqrt{6}}{4}|1\rangle$$

TÜL

Question 3,

a, $P = \sum_{i=1}^{n} |i\rangle\langle i|$ $\left(\underline{\text{Case 1:}}\ \{|i\rangle\}_{n=0}^{n-1} \text{ are bases of } \underline{\mathbb{C}^n}\right)$

in Vector space of $\underline{n}$ dimension

$$P^2 = P\,P = \left(\sum_{i=1}^{n} |i\rangle\langle i|\right)\left(\sum_{i=1}^{n} |i\rangle\langle i|\right)$$

$$= \sum_{i=1}^{n} \left(|i\rangle\langle i|\right)\left(|i\rangle\langle i|\right) + \underbrace{\sum_{i\neq j} \left(|i\rangle\langle i|\right)\left(|j\rangle\langle j|\right)}_{0}$$

$$= \sum_{i=1}^{n} |i\rangle \underbrace{\left(\langle i|i\rangle\right)}_{1} \langle i|$$

$$= \sum_{i=1}^{n} |i\rangle\langle i| = P \quad \square$$

(Another way is to show $P = I_n$)

b, First; $|i\rangle\langle i| = \begin{array}{cc} & i \\ i & \begin{bmatrix} 0 & & \\ & 1 & \\ & & 0 \end{bmatrix} \end{array}$

Thus, $P = \sum_{i=1}^{n} |i\rangle\langle i| = \mathbb{I}_n$

The eigenvalues of I_n is defined by

$$\begin{vmatrix} 1-\lambda & & & (0) \\ & 1-\lambda & & \\ & & 1-\lambda & \\ (0) & & & \ddots & 1-\lambda \end{vmatrix} = 0 \to (1-\lambda)^n = 0$$

or, $\lambda_1 = \dots = \lambda_n = 1$

Case 2 $\{|i\rangle\}_{i=0}^{n}$ are bases of $\mathbb{C}^n$

Then and the vector space has m dimensions $(n<m)$

$$P = \begin{array}{c} \left[\begin{matrix} 1 & & (0) \\ & \ddots & \\ (0) & & 1 \\ 0 & 0 \ldots & 0 \\ 0 & 0 \ldots & 0 \end{matrix}\right] \end{array} \begin{matrix} \Big\} n \\ \\ \} m-n \end{matrix}$$

In this case, (a) hold, and for (b), we have addition zero eigenvalue

$\rightarrow$ Thus the eigenvalue now are

$$\begin{cases} \lambda_1, \ldots, \lambda_n = 1 \\ \lambda_{n+1}, \ldots \lambda_m = 0 \end{cases}$$