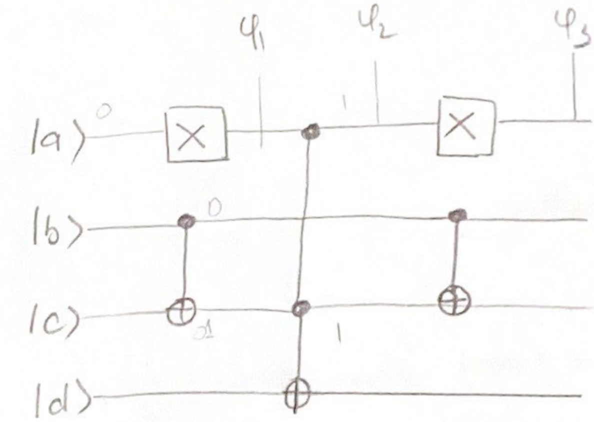
Question 1

a) $U_f =$



$\lvert abc\rangle\lvert d\rangle$	φ_1	φ_2	φ_3
$\lvert 000\rangle\lvert d\rangle$	$\lvert 100\rangle\lvert d\rangle$	$\lvert 100\rangle\lvert d\rangle$	$\lvert 000\rangle\lvert d\rangle$
$\lvert 001\rangle\lvert d\rangle$	$\lvert 101\rangle\lvert d\rangle$	$\lvert 101\rangle\lvert d\oplus 1\rangle$	$\lvert 001\rangle\lvert d\oplus 1\rangle$
$\lvert 010\rangle\lvert d\rangle$	$\lvert 111\rangle\lvert d\rangle$	$\lvert 111\rangle\lvert d\oplus 1\rangle$	$\lvert 010\rangle\lvert d\oplus 1\rangle$
$\lvert 011\rangle\lvert d\rangle$	$\lvert 110\rangle\lvert d\rangle$	$\lvert 110\rangle\lvert d\rangle$	$\lvert 011\rangle\lvert d\rangle$
$\lvert 100\rangle\lvert d\rangle$	$\lvert 000\rangle\lvert d\rangle$	$\lvert 000\rangle\lvert d\rangle$	$\lvert 100\rangle\lvert d\rangle$
$\lvert 101\rangle\lvert d\rangle$	$\lvert 001\rangle\lvert d\rangle$	$\lvert 001\rangle\lvert d\rangle$	$\lvert 101\rangle\lvert d\rangle$
$\lvert 110\rangle\lvert d\rangle$	$\lvert 011\rangle\lvert d\rangle$	$\lvert 011\rangle\lvert d\rangle$	$\lvert 110\rangle\lvert d\rangle$
$\lvert 111\rangle\lvert d\rangle$	$\lvert 010\rangle\lvert d\rangle$	$\lvert 010\rangle\lvert d\rangle$	$\lvert 111\rangle\lvert d\rangle$

b) $U_f |abc\rangle|d\rangle = |abc\rangle|d \oplus f(a,b,c)\rangle$

Base on the table, $\begin{cases} f(001) = f(010) = 1 \\ f(i) = 0 \quad ; \text{ otherwise.} \end{cases}$

Question 2; a) There are two marked elements, $|001\rangle$ and $|010\rangle$

We know that $|001\rangle = |1\rangle$ and $|010\rangle = |2\rangle$; then

$$\begin{cases} f(i) = 1 , & i = 1, 2 \\ f(i) = 0 , & i = 0, 3, 4, 5, 6, 7 \end{cases}$$

So there are 2 marked elements $i = 1, 2; \rightarrow M = 2.$

b) $|\beta\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + |2\rangle\right) \Rightarrow |1\rangle + |2\rangle = \sqrt{2}\,|\beta\rangle$

$$|\alpha\rangle = \frac{1}{\sqrt{8-2}}\begin{pmatrix} |0\rangle + |3\rangle + |4\rangle \\ + |5\rangle + |6\rangle + |7\rangle \end{pmatrix} \Rightarrow \sum_{i \notin S} |i\rangle = \sqrt{6}\,|\alpha\rangle$$

Thus $\sum_{i \in S} |i\rangle = \sqrt{2}|\beta\rangle$; $\sum_{i \notin S} |i\rangle = \sqrt{6}\,|\alpha\rangle$,

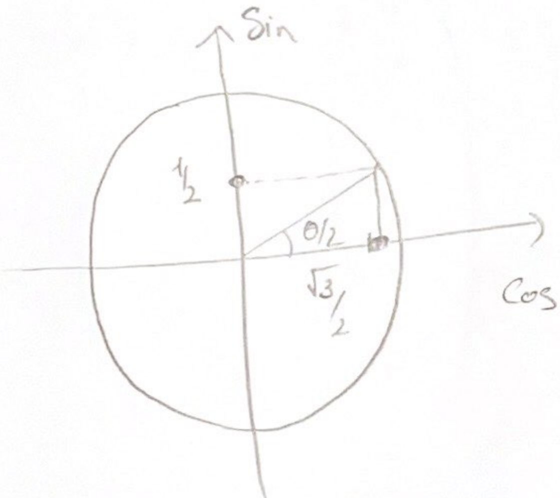
$$\Rightarrow \sum_{i<8} |i\rangle = \sqrt{6}\,|\alpha\rangle + \sqrt{2}\,|\beta\rangle \Rightarrow |\varphi\rangle = \frac{\sqrt{6}}{2\sqrt{2}}|\alpha\rangle + \frac{\sqrt{2}}{2\sqrt{2}}|\beta\rangle$$

So $|\varphi\rangle = \dfrac{\sqrt{3}}{2}|\alpha\rangle + \dfrac{1}{2}|\beta\rangle$

Check out: $\left(\dfrac{\sqrt{3}}{2}\right)^2 + \left(\dfrac{1}{2}\right)^2 = 1$ ← satisfying def of amplitude

c) We have $\cos\dfrac{\theta}{2} = \dfrac{\sqrt{3}}{2}$ and $\sin\dfrac{\theta}{2} = \dfrac{1}{2}$

then $\dfrac{\theta}{2} = \dfrac{\pi}{6} \rightarrow \boxed{\theta = \dfrac{\pi}{3}}$



d) $$G = \frac{1}{4}\left(\begin{array}{cccccccc} -3 & -1 & -1 & 1 & & & (1) & \\ 1 & 3 & -1 & 1 & & & & \\ 1 & -1 & 3 & 1 & & & & \\ 1 & -1 & -1 & -3 & & & & \\ 1 & -1 & -1 & & \ddots & & & \\ 1 & -1 & -1 & & & \ddots & & \\ 1 & -1 & -1 & & (1) & & \ddots & -3 \\ 1 & -1 & -1 & & & & & \end{array}\right)$$

$$|\alpha\rangle = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\0\\0\\1\\1\\1\\1\\1\end{pmatrix} \quad \text{and} \quad |\beta\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\\0\\0\\0\\0\end{pmatrix}$$

$$G|\alpha\rangle = \frac{1}{4\sqrt{6}}\begin{pmatrix}2\\6\\6\\2\\2\\2\\2\\2\end{pmatrix} = \frac{1}{2\sqrt{6}}\begin{pmatrix}1\\3\\3\\1\\1\\1\\1\\1\end{pmatrix}$$

$$= \frac{1}{2\sqrt{6}}\sum_{i\notin S}|i\rangle + \frac{3}{2\sqrt{6}}\sum_{i\in S}|i\rangle$$

$$= \frac{1}{2}|\alpha\rangle + \frac{\sqrt{3}}{2}|\beta\rangle = \cos\frac{\pi}{3}|\alpha\rangle + \sin\frac{\pi}{3}|\beta\rangle$$

$$G|\beta\rangle = \frac{1}{4\sqrt{2}}\begin{pmatrix} -2 \\ 2 \\ 2 \\ -2 \\ -2 \\ -2 \\ -2 \\ -2 \end{pmatrix} = \frac{2}{4\sqrt{2}}\left(\sum_{i\in S}|i\rangle\right) - \frac{2}{4\sqrt{2}}\left(\sum_{i\notin S}|i\rangle\right)$$

$$= \frac{1}{2}|\beta\rangle - \frac{\sqrt{3}}{2}|\alpha\rangle$$

$$= \cos\frac{\pi}{3}|\beta\rangle - \sin\frac{\pi}{3}|\alpha\rangle$$

So; $G|\alpha\rangle = \cos\theta|\alpha\rangle + \sin\theta|\beta\rangle$

$G|\beta\rangle = -\sin\theta|\alpha\rangle + \cos\theta|\beta\rangle$

thus; G as in $|\alpha\rangle, |\beta\rangle$ basis is $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

where $\theta = \frac{\pi}{3}$

e) $2|\varphi\rangle\langle\varphi| - I$

$$= 2 \times \frac{1}{2\sqrt{2}} \times \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & & & & & & & 1 \\ 1 & & & & & & & 1 \\ 1 & & & (1) & & & & 1 \\ 1 & & & & & & & 1 \\ 1 & & & & & & & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} - I_8$$

$$= \frac{1}{4} \begin{pmatrix} -3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -3 & & & & & & \\ & & -3 & & & (1) & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & (1) & & & & & & \\ & & & & & & & -3 \end{pmatrix}$$

1)

$$O_f = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$G = \frac{1}{4} \begin{pmatrix} -3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -3 & & & & (1) & & \\ & & \ddots & & & & & \\ & (1) & & & & & \ddots & \\ & & & & & & & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & & & & & & \\ & & -1 & & & (0) & & \\ & & & 1 & & & & \\ & (0) & & & 1 & & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} -3 & -1 & -1 & 1 & & & & \\ 1 & 3 & -1 & 1 & & (1) & & \\ 1 & -1 & 3 & 1 & & & & \\ 1 & -1 & -1 & -3 & & & & \\ 1 & -1 & -1 & 1 & -3 & & & \\ 1 & -1 & -1 & 1 & & -3 & & \\ 1 & -1 & -1 & 1 & (1) & & -3 & \\ 1 & -1 & -1 & 1 & & & & \end{pmatrix}$$

g) $H^{\otimes 3}|0\rangle^{\otimes 3} = \frac{1}{2\sqrt{2}}\left(|0\rangle + |1\rangle\right) \otimes \left(|0\rangle + |1\rangle\right)\left(|0\rangle + |1\rangle\right)$

$$= \frac{1}{2\sqrt{2}}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)\left(|0\rangle + |1\rangle\right)$$

$$= \frac{1}{2\sqrt{2}}\left(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle\right)$$

$$= \frac{1}{2\sqrt{2}}\begin{pmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{pmatrix}$$

$$GH^{\otimes 3}|0\rangle^{\otimes 3} = \frac{1}{8\sqrt{2}}\begin{pmatrix}0\\8\\8\\0\\0\\0\\0\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\\0\\0\\0\\0\end{pmatrix}$$

h) We have $G H^{\otimes 3} |0\rangle^{\otimes 3} = \frac{1}{\sqrt{2}} |1\rangle + \frac{1}{\sqrt{2}} |2\rangle$

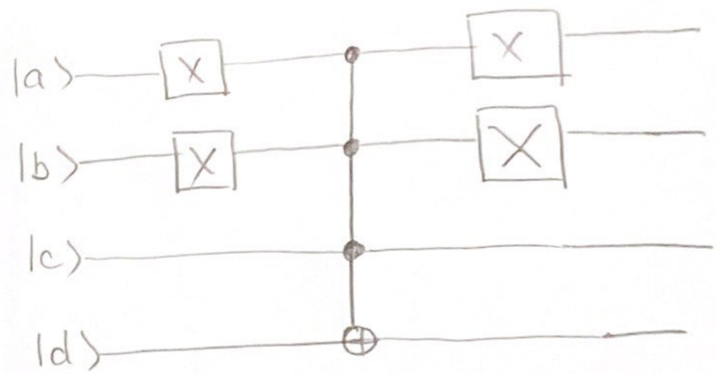
thus the probability of measureng $|i\rangle$ where $i \in S$ is 1

(since output states is superposition of $|1\rangle$ and $|2\rangle$ in S)

- The probability of measuring $|001\rangle$ is $\frac{1}{2}$, so as $|010\rangle$.

Question 3;

$U_g =$



a)

abc	φ_1	φ_2	φ_3
$\|000\rangle\|d\rangle$	$\|110\rangle\|d\rangle$	$\|110\rangle\|d\rangle$	$\|000\rangle\|d\rangle$
$\|001\rangle\|d\rangle$	$\|111\rangle\|d\rangle$	$\|111\rangle\|d\oplus 1\rangle$	$\|001\rangle\|d\oplus 1\rangle$
$\|010\rangle\|d\rangle$	$\|100\rangle\|d\rangle$	$\|100\rangle\|d\rangle$	$\|010\rangle\|d\rangle$
$\|011\rangle\|d\rangle$	$\|101\rangle\|d\rangle$	$\|101\rangle\|d\rangle$	$\|011\rangle\|d\rangle$
$\|100\rangle\|d\rangle$	$\|010\rangle\|d\rangle$	$\|010\rangle\|d\rangle$	$\|100\rangle\|d\rangle$
$\|101\rangle\|d\rangle$	$\|011\rangle\|d\rangle$	$\|011\rangle\|d\rangle$	$\|101\rangle\|d\rangle$
$\|110\rangle\|d\rangle$	$\|000\rangle\|d\rangle$	$\|000\rangle\|d\rangle$	$\|110\rangle\|d\rangle$
$\|111\rangle\|d\rangle$	$\|001\rangle\|d\rangle$	$\|001\rangle\|d\rangle$	$\|111\rangle\|d\rangle$

So $\begin{cases} g(1) = 1 \\ g(i) = 0 \ ; \quad i = 0, 2, 3, 4, 5, 6, 7 \end{cases}$

b) $G \circ H^{\otimes 3} = A$; then $|\varphi\rangle = A|0\rangle^{\otimes 3} = G H^{\otimes 3} |0\rangle^{\otimes 3}$

the $|\varphi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

when measuring $|\varphi\rangle$; the probability of measuring $i = 1 \in S = g^{-1}\{1\}$

is $\frac{1}{2}$.

c) $2|\varphi\rangle\langle\varphi| - I = 2 \cdot \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & \\ 0 & 1 & 1 & 0 & \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & (0) \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \end{pmatrix} - I_8$

$$= \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & & & (0) & \\ & & & & -1 & & & \\ & (0) & & & & -1 & & \\ & & & & & & & -1 \end{pmatrix}$$

d) $$O_g = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & & & & (0) & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & (0) & & & & & 1 & \\ & & & & & & & 1 \end{pmatrix}$$

$$Q = (2|\varphi\rangle\langle\varphi| - I)\, O_g = \begin{pmatrix} -1 & 0 & 0 & & & & & \\ 0 & 0 & 1 & & & (0) & & \\ 0 & -1 & 0 & & & & & \\ 0 & 0 & 0 & -1 & & & & \\ 0 & 0 & 0 & & -1 & & & \\ 0 & 0 & 0 & & & -1 & & \\ 0 & 0 & 0 & & (0) & & -1 & \\ 0 & 0 & 0 & & & & & -1 \end{pmatrix}$$

e) $Q \cdot A|0\rangle^{\otimes 3} = Q|\varphi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$= \frac{1}{\sqrt{2}}|1\rangle - \frac{1}{\sqrt{2}}|2\rangle$

Bonus: No, the chance of measing $|1\rangle$ or $|2\rangle$ is $\frac{1}{}$ in (Q1)

and the chance of measing $|1\rangle$ in Q3 is $\frac{1}{2}$, (same as in Q1)

- the input state of U_g is $|\varphi\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$ and the marked element of U_g is only $|1\rangle$; now: let

$|\alpha\rangle = |2\rangle$ and $|\beta\rangle = |1\rangle$, so: $|\varphi\rangle = \frac{1}{\sqrt{2}}|\alpha\rangle + \frac{1}{\sqrt{2}}|\beta\rangle$

thus $\left.\begin{aligned} \cos\frac{\theta}{2} &= \frac{1}{\sqrt{2}} \\ \sin\frac{\theta}{2} &= \frac{1}{\sqrt{2}} \end{aligned}\right\} \rightarrow \frac{\theta}{2} = \frac{\pi}{4} \Rightarrow \theta = \frac{\pi}{2}$

As in proposition 9.6, we need to perform the minimum of iterations k such as $\sin\left(\frac{2k+1}{2}\theta\right) \simeq \sin\frac{\pi}{2} \rightarrow$ If $\theta = \frac{\pi}{2}$; adding one more iteration does not impact to the final measurement.

- As in Proposition 9.7; $\cos\left(\frac{\theta}{2}\right) = \sqrt{1-p} = \frac{1}{\sqrt{2}} \Rightarrow 1-p = \frac{1}{2}$

and $p = \frac{1}{2}$ (exactly)

then $k \leqslant \frac{\pi}{4}\sqrt{p} = \frac{\pi}{4}\sqrt{\frac{1}{2}} = 0.55 \longrightarrow$ no need for one more iteration