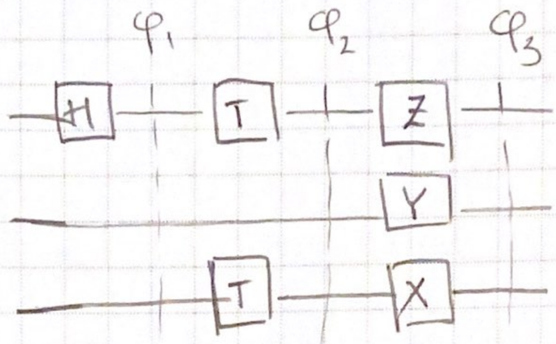
Homework 4

Question



1)

$$\varphi_1 = H \otimes I \otimes I = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

as block matrix

$$\varphi_1 = \begin{pmatrix} I & O & I & O \\ O & I & O & I \\ I & O & -I & O \\ O & I & O & -I \end{pmatrix}$$

$$\varphi_2 = T \otimes I \otimes T$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^2 \end{pmatrix}$$

as block matrix,

$$\varphi_2 = \begin{pmatrix} T & 0 & 0 & 0 \\ 0 & T & 0 & 0 \\ 0 & 0 & \mu T & 0 \\ 0 & 0 & 0 & \mu T \end{pmatrix}$$

$$\varphi_3 = Z \otimes Y \otimes X$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -iX & 0 & 0 \\ iX & 0 & 0 & 0 \\ 0 & 0 & 0 & iX \\ 0 & 0 & -iX & 0 \end{pmatrix}$$

2) To make the multiplication process less tedious;

we multiply $\varphi_1 \varphi_2 \varphi_3$ as block matrixes

$$\varphi_1 \varphi_2 = \begin{pmatrix} I & 0 & I & 0 \\ 0 & I & 0 & I \\ I & 0 & -I & 0 \\ 0 & I & 0 & -I \end{pmatrix} \begin{pmatrix} T & 0 & 0 & 0 \\ 0 & T & 0 & 0 \\ 0 & 0 & \mu T & 0 \\ 0 & 0 & 0 & \mu T \end{pmatrix}$$

$$= \begin{pmatrix} T & 0 & \mu T & 0 \\ 0 & T & 0 & \mu T \\ T & 0 & -\mu T & 0 \\ 0 & T & 0 & -\mu T \end{pmatrix} \begin{pmatrix} 0 & -iX & 0 & 0 \\ iX & 0 & 0 & 0 \\ 0 & 0 & 0 & iX \\ 0 & 0 & -iX & 0 \end{pmatrix}$$

$$\rightarrow \varphi_1 \varphi_2 \varphi_3 =$$

$$= \begin{pmatrix} 0 & T(-iX) & 0 & (\mu T)(iX) \\ T(iX) & 0 & (\mu T)(-iX) & 0 \\ 0 & T(-iX) & 0 & -\mu T(iX) \\ T(iX) & 0 & (\mu T)(iX) & 0 \end{pmatrix}$$

- $T(iX) = i\,TX = i\begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$= i\begin{pmatrix} 0 & 1 \\ \mu & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i\mu & 0 \end{pmatrix}$$

- $(\mu T)(iX) = \mu i\begin{pmatrix} 0 & 1 \\ \mu & 0 \end{pmatrix} = \begin{pmatrix} 0 & i\mu \\ i\mu^2 & 0 \end{pmatrix}$

Thus, $\varphi_1 \varphi_2 \varphi_3 = \begin{pmatrix} 0 & 0 & 0 & -i & 0 & 0 & 0 & i\mu \\ 0 & 0 & -i\mu & 0 & 0 & 0 & i\mu^2 & 0 \\ 0 & i & 0 & 0 & 0 & -i\mu & 0 & 0 \\ i\mu & 0 & 0 & 0 & -i\mu^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & -i\mu \\ 0 & 0 & -i\mu & 0 & 0 & 0 & -i\mu^2 & 0 \\ 0 & i & 0 & 0 & 0 & i\mu & 0 & 0 \\ i\mu & 0 & 0 & 0 & i\mu^2 & 0 & 0 & 0 \end{pmatrix}$

3, $\bullet$ $ZTH|1\rangle = ZT\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)$

$$= Z\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{\mu}{\sqrt{2}}|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}|0\rangle + \frac{\mu}{\sqrt{2}}|1\rangle$$

- $Y|1\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -i \\ 0 \end{pmatrix} = -i|0\rangle$

- $XT|1\rangle = X\,\mu|1\rangle = \mu|0\rangle$

Output state: $\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{\mu}{\sqrt{2}}|1\rangle\right) \otimes \left(-i|0\rangle\right) \otimes \left(\mu|0\rangle\right)$

$$= \frac{iM}{\sqrt{2}}|000\rangle + \frac{iM^2}{\sqrt{2}}|100\rangle$$

- The probability of measuring $|000\rangle$ is $|\alpha|^2$

where $\alpha = \frac{-iM}{\sqrt{2}}$.

$$= \frac{-i}{\sqrt{2}} e^{i\frac{\pi}{4}} = \frac{-i}{\sqrt{2}}\left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right)$$

$$= \frac{-i}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \frac{-i}{2} + \frac{-i^2}{2}$$

$$= \frac{1}{2} - \frac{1}{2}i$$

Thus, $|\alpha|^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

- Check the probability of measuring $|100\rangle$

Let $\beta = \frac{iM^2}{\sqrt{2}} = \frac{i}{\sqrt{2}}\left(e^{i\pi/2}\right)$

$$= \frac{i}{\sqrt{2}}\left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right)$$

$$= \frac{i}{\sqrt{2}}\left(0 + i\right) = \frac{i^2}{\sqrt{2}} = \frac{-1}{\sqrt{2}}$$

then $|\beta|^2 = \frac{1}{2}$

The identity $|\alpha|^2 + |\beta|^2 = 1$ held

4) - G- Cost = 6
- D - cost = 3
- DW - cost = 9

5)