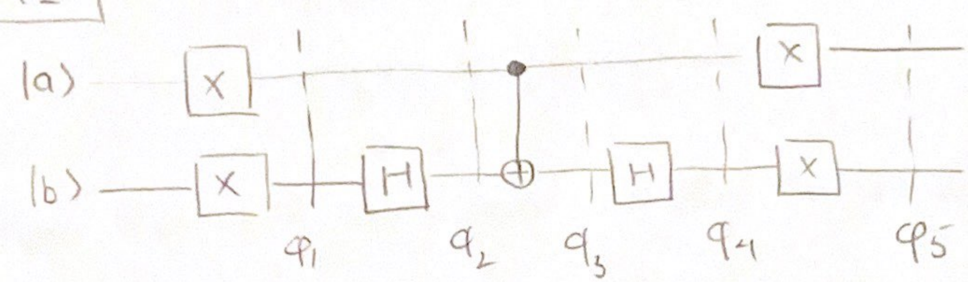
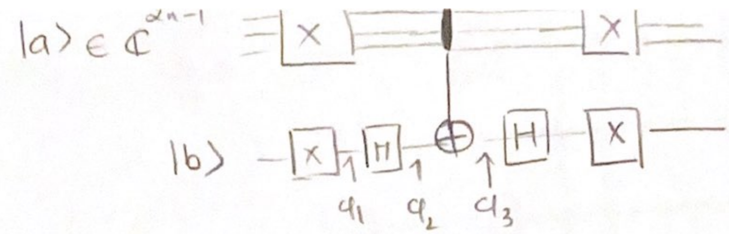
Question 1



ab	φ_1	φ_2	φ_3	φ_4	φ_5
$\|00\rangle$	$\|11\rangle$	$\|1\rangle\|-\rangle$	$-\|1\rangle\|-\rangle$	$-\|1\rangle\|1\rangle$	$-\|00\rangle$
$\|01\rangle$	$\|10\rangle$	$\|1\rangle\|+\rangle$	$\|1\rangle\|+\rangle$	$\|1\rangle\|0\rangle$	$\|01\rangle$
$\|10\rangle$	$\|01\rangle$	$\|0\rangle\|-\rangle$	$\|0\rangle\|-\rangle$	$\|0\rangle\|1\rangle$	$\|10\rangle$
$\|11\rangle$	$\|00\rangle$	$\|0\rangle\|+\rangle$	$\|0\rangle\|+\rangle$	$\|0\rangle\|0\rangle$	$\|11\rangle$

Thus, the action on $|00\rangle$ is flipping the sign to $-|00\rangle$ otherwise it acts as identity on $|01\rangle$, $|10\rangle$, $|11\rangle$

b)



It's clear that with $|a\rangle = |000\ldots0\rangle$ then the $|b\rangle$ will be flipped at q_3. Then, at q_3: , starting with $|b\rangle = |0\rangle$,

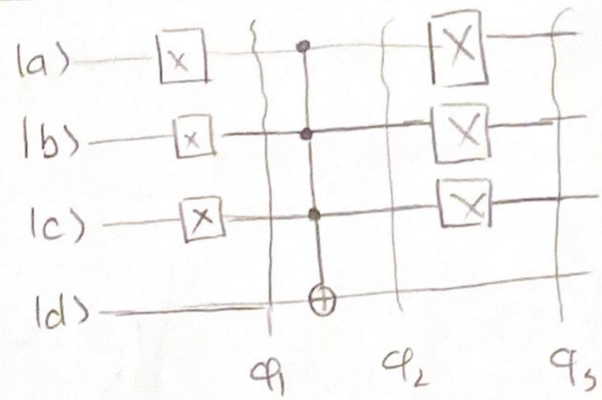
$$q_2 = |11\ldots1\rangle|-\rangle \longmapsto q_3 = -|11\ldots1\rangle|-\rangle$$
$\llcorner$ all are 1's $\lrcorner$

$$q_2 = |0\ldots1\;0\rangle|-\rangle \longmapsto q_3 = |0\ldots1\ldots0\rangle|-\rangle$$
$\llcorner$ at least 1 0's $\lrcorner$

Thus, at the end of the circuits, if $|a\rangle = |0\ldots0\rangle$ and $|b\rangle = |0\rangle$ the sign will be flipped; ow the sign remains the same

3; This circuit is the inversion about the mean or diffusion operator

Question 2, $U_f =$



abc	φ_1	φ_2	φ_3
$\|000\rangle$	$\|111\rangle\|d\rangle$	$\|111\rangle\|0\rangle, \|d\rangle=\|1\rangle$ $\|111\rangle\|1\rangle; \|d\rangle=\|0\rangle$	$\|000\rangle\|0\rangle, d=1$ $\|000\rangle\|1\rangle; d=0$
100			
010			
001	Toffoli at φ_3 acts as identity		
110			
101			
011			
111			

Thus, $f(000) = 1$, $f(x) = 0$; ow or

$$U_f|abc\rangle|d\rangle = |abc\rangle|d \oplus f(abc)\rangle$$

$$= \begin{cases} |abc\rangle|d \oplus 1\rangle & \text{if } |abc\rangle = 0 \\ |abc\rangle|d\rangle & \text{if o.w} \end{cases}$$

Question 3

$$f(abc) = \begin{cases} 1 & \text{when } a=b=c=0 \\ 0 & \text{otherwise} \end{cases}$$

a) $$2|\varphi\rangle\langle\varphi| - I = 2 \times \frac{1}{8} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & & & & & & & \\ 1 & & & & & & & \\ 1 & & & & & & & \\ 1 & & & & & & & \\ 1 & & & & & & & \\ 1 & & & & & & & \\ 1 & & & & & & & 1 \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & & & & & & & \\ 1 & & & & & & & \\ 1 & & & & & & & \\ \vdots & & & & & & & \\ 1 & & & & & & & \end{pmatrix} - I$$

$$= \frac{1}{4} \begin{pmatrix} -3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & -3 & & & & & & \\ & & -3 & & & & & \\ & & & \ddots & & & & \\ (1) & & & & & & & -3 \end{pmatrix}$$

$$O_1 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 1 & & & (0) & & \\ & & & \ddots & & & & \\ & (0) & & & & & & \\ & & & & & & & 1 \end{pmatrix}$$

thus

$$G = \frac{1}{4} \begin{pmatrix} -3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -3 & & & & & & 1 \\ 1 & 1 & -3 & & & & & \\ & & & \ddots & & & & \\ & & & & & & & -3 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & & & & & & \\ 0 & & 1 & & & (0) & & \\ 0 & & & \ddots & & & & \\ 0 & & & & & & & \\ \vdots & & (0) & & & & & \\ 0 & & & & & & & 1 \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} 3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -3 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & -3 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -3 & & (1) & & \\ -1 & & & & \ddots & & & \\ -1 & & & & & & & \\ -1 & & & (1) & & & & \\ -1 & & & & & & & -3 \end{pmatrix}$$

c) $H^{\otimes 3}|0\rangle^{\otimes 3} = \frac{1}{2\sqrt{2}}\left(|0\rangle + |1\rangle\right)\left(|0\rangle + |1\rangle\right)\left(|0\rangle + |1\rangle\right)$

$= \frac{1}{2\sqrt{2}}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)\left(|0\rangle + |1\rangle\right)$

$= \frac{1}{2\sqrt{2}}\left(\begin{array}{l}|000\rangle + |001\rangle + |100\rangle + |101\rangle \\ + |010\rangle + |011\rangle + |110\rangle + |111\rangle\end{array}\right)$

$= \frac{1}{2\sqrt{2}}\begin{pmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{pmatrix}$

$GH^{\otimes 3}|0\rangle^{\otimes 3} = \frac{1}{8\sqrt{2}}\begin{pmatrix}10\\2\\2\\2\\2\\2\\2\\2\end{pmatrix} = \frac{1}{4\sqrt{2}}\begin{pmatrix}5\\1\\1\\1\\1\\1\\1\\1\end{pmatrix}$

$$G^2 = \frac{1}{16}\begin{pmatrix} 2 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\ -6 & 14 & -2 & -2 & \cdots & & & -2 \\ \vdots & -2 & 14 & & & & & \\ \vdots & -2 & -2 & 14 & & & & \\ & \vdots & \vdots & & \ddots & & & \\ -6 & -2 & -2 & & (-2) & & & 14 \end{pmatrix}$$

Then; $G^2 \cdot H^{\otimes 3} |0\rangle^{\otimes 3} = \frac{1}{16} \times \frac{1}{2\sqrt{2}} \cdot \begin{pmatrix} \end{pmatrix} \begin{pmatrix} \vdots \end{pmatrix}$

$$= \frac{1}{32\sqrt{2}} \begin{pmatrix} 44 \\ -4 \\ -4 \\ -4 \\ \vdots \\ -4 \end{pmatrix} = \frac{1}{8\sqrt{2}} \begin{pmatrix} 11 \\ -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}$$