(Q1)

+) $\frac{31}{13} = 2 + \cfrac{1}{2 + \cfrac{3}{5}} = 2 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{2}{3}}}$

$= 2 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}}} \rightarrow [2, 2, 1, 1, 2]$

+) $\frac{19}{17} = 1 + \cfrac{1}{8 + \cfrac{1}{2}} \rightarrow [1, 8, 2]$

+) $\frac{77}{65} = 1 + \cfrac{1}{5 + \cfrac{5}{12}} = 1 + \cfrac{1}{5 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}}}$

$[1, 5, 2, 2, 2]$

Question 2

$$[a_0, a_1, \dots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\dots + a_n}}} = \frac{p_n}{q_n}$$

- $\dfrac{p_0}{q_0} = a_0 \iff \dfrac{p_0}{q_0} = \dfrac{a_0}{1} \Rightarrow \begin{cases} p_0 = a_0 \\ q_0 = 1 \end{cases}$

- $\dfrac{p_1}{q_1} = a_0 + \dfrac{1}{a_1} = \dfrac{a_0 a_1 + 1}{a_1} \Rightarrow \begin{cases} p_1 = 1 + a_0 a_1 \\ q_1 = a_1 \end{cases}$

- $\dfrac{p_2}{q_2} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2}} = a_0 + \cfrac{1}{\cfrac{a_1 a_2 + 1}{a_2}}$

$$= a_0 + \frac{a_2}{a_1 a_2 + 1} = \frac{a_0(a_1 a_2 + 1) + a_2}{a_1 a_2 + 1}$$

$$= \frac{a_0 a_1 a_2 + a_0 + a_2}{a_1 a_2 + 1} = \frac{a_2(a_0 a_1 + 1) + a_0}{a_1 a_2 + 1}$$

$$= \frac{a_2 p_1 + p_0}{a_2 q_1 + q_0}$$

(2)

So for $n = 0, 1, 2,$ $\begin{cases} p_2 = a_2 p_1 + p_0 \\ q_2 = a_2 q_1 + q_0 \end{cases}$

- $$[a_0; \dots; a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \underbrace{a_{n-1} + \cfrac{1}{a_n}}_{= a'_{n-1}}}}}$$

$$= a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \; a_{n-2} + \cfrac{1}{a'_{n-1}}}}}$$

Let $a'_{n-1} = a_{n-1} + \frac{1}{a_n}$

$$= [a_0, a_1; \dots; a_{n-2}; a'_{n-1}]$$

(a_i not nesessary integers)

$$= \left[a_0; \dots; a_{n-2}, a_{n-1} + \frac{1}{a_n}\right]$$

- $$\frac{\tilde{p}_n}{\tilde{q}_n} = \left[a_0; \dots; a_{n-1}; a_n + \frac{1}{a_{n+1}}\right]$$

$$\underset{(\text{part} 4)}{=} \left[a_0; \dots; a_{n-1}; a_n, a_{n+1}\right]$$

- $$\frac{\tilde{p}_n}{\tilde{q}_n} = \left[a_0; \dots; a_{n+1}\right] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots\, a_n + \cfrac{1}{a_{n+1}}}}$$

$$= \frac{p_{n+1}}{q_{n+1}}$$

- $$\frac{p_{n+1}}{q_{n+1}} = \frac{a_{n+1} p_n + p_{n-1}}{a_{n+1} q_n + q_{n-1}} \quad (\text{part } 2)$$

thence, $[a_0, \dots; a_{n+1}] =$

Question 3

- $q_1 p_0 - p_1 q_0 = a_1 a_0 - (1 + a_0 a_1) 1$

$$= a_0 a_1 - 1 - a_0 a_1 = (-1)^1$$

Thus, the equality holds with $n = 1$.

- Assume that $n \geq 1$; $q_n p_{n-1} - p_n q_{n-1} = (-1)^n$

We have $q_{n+1} p_n - p_{n+1} q_n$

$$= (a_{n+1} q_n + q_{n-1}) p_n - (a_{n+1} p_n + p_{n-1}) q_n$$

$$= a_{n+1} \left(q_n p_n - q_n p_n \right) + p_n q_{n-1} - q_n p_{n-1}$$

$$= (-1) \left(q_n p_{n-1} - p_n q_{n-1} \right) = (-1)(-1)^n$$

$$= (-1)^{n+1}$$

- Let $e | p_n$ and $e | q_n$ then

$$e \mid q_n(p_{n-1}) - p_n(q_{n-1})$$

or $e \mid (-1)^n \to$ e divides 1

- $1 \mid e$ thus $1 = \gcd(q_n, p_n)$