Homework 10,

Question 1,

- In the between of φ_1 and φ_2; there are $(n-1)$ Rotation gates $R_2, \ldots, R_n$
- There are $(n-2)$ rotation gates between φ_2 and φ_3
- There are $(n-k)$ rotation gates between φ_k and φ_{k+1}
- The total number of rotation gates is:

$$N = (n-1) + (n-2) + \ldots + (n-k) + \ldots + 1$$

$$= \frac{(n-1+1)(n-1)}{2} = \frac{n(n-1)}{2}$$

• We need to prove that U is isometry; we have

$$U^\dagger U = U U^\dagger.$$

For any $|\varphi\rangle$ and $|\phi\rangle$; we have

- $\langle \varphi | \phi \rangle = \langle \psi | U^\dagger U \phi \rangle = \langle U\varphi | U\phi \rangle \rightarrow U$ is isometry

then $\| U|\varphi\rangle \| = \| |\varphi\rangle \|$

- So $1 = \| \langle \psi | \varphi \rangle \| = \| \langle U\varphi | U\varphi \rangle \| = \| U \|$

- ~~For all x; $\|A\| := \max_{\|x\|=1} \|A|x\rangle\| \geq \|A|x\rangle\|$~~

~~$(-)\quad -\|A|x\rangle\| \geq -\|A\|.$~~

~~On the other hand, $d(Ax - A) \geq$~~

• For any $x \neq 0$

$$\|Ax\| = \|x\| \cdot \left\| A \cdot \underbrace{\frac{1}{\|x\|} x}_{1} \right\| \leqslant \|x\| \|A\|$$

For $x = 0 \Rightarrow \|0\| = \|0\|$ (Trivial)

So for any x; $\quad \|A|x\rangle\| \leqslant \|A\| \, \||x\rangle\|$

$$\|AB\| = \max_{\||x\rangle\|=1} \|AB|x\rangle\| = \max_{\||x\rangle\|=1} \|AB|x\rangle\|$$

$$\leqslant \|A\| \max_{\||x\rangle\|=1} \|B|x\rangle\| = \|A\| \|B\|.$$

- $d(AB, A'B') = \|AB - A'B'\|$

$= \|AB - A'B + A'B - A'B'\|$

$\leq \|AB - A'B\| + \|A'B - A'B'\|$

$\leq d(A, A') + d(B; B') = \ell_1 + \ell_2$

- $d(QFT_n; \widetilde{QFT_n}) = d\big(R_2; \cdots; R_k; \cdots R_n; \tilde{R}_2; \cdots; \tilde{R}_k; \cdots \tilde{R}_n$
$; R_2; \cdots R_{n-1}; \tilde{R}_2 \cdots \tilde{R}_{n-1}$
$R_2 ; \tilde{R}_2 \big)$

$\leq \underbrace{d(R_2, \tilde{R}_2) + \cdots d(R_k; \tilde{R}_k) + \cdots}_{\frac{n(n-1)}{2} \text{ rotations gates}}$

$\leq \ell \dfrac{n(n-1)}{2}.$

Question 2 $N = 15$; $a = 7$; $n = 11$ qubits

$a^0 \bmod 15 = 1$ $a^2 \bmod 15 = 4$ $a^4 \bmod 15 =$

$a^1 \bmod 15 = 7$ $a^3 \bmod 15 = 13$ $\rightarrow r = 4$

- $$\frac{1}{\sqrt{2^{11}}} \sum_{x=0}^{2^{11}-1} |x\rangle |1 \bmod 15\rangle$$

$$\xrightarrow[C-U_a^x]{} \frac{1}{\sqrt{2^{11}}} \sum_{x=0}^{2^{11}-1} |x\rangle |a^x \bmod 15\rangle$$

and $7^x \bmod 15 \in \{1, 7, 4, 13\}$

$$\Rightarrow \sum_{b=0}^{3} \left(\frac{1}{\sqrt{2^{11}}} \sum_{k=0}^{mb-1} |kr + b\rangle \right) |a^b \bmod 15\rangle$$

$$mb = \left\lfloor \frac{2^{11} - b - 1}{r} \right\rfloor$$

- We measure $4 \iff b = 2 \iff mb = 511$

$$\Rightarrow \frac{1}{\sqrt{511}} \sum_{k=0}^{510} |kr+2\rangle$$

- We have $QFT_n^{-1}: |x\rangle \longmapsto \frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^n-1} e^{-2\pi i x y} |y\rangle$

and $|\phi\rangle = \frac{1}{\sqrt{mb}} \sum_{k=0}^{mb-1} |kr+b\rangle$

thus; $QFT_n^{-1} |\phi\rangle = \frac{1}{\sqrt{2^n mb}} \sum_{k=0}^{mb-1} \sum_{y=0}^{2^n-1} e^{\frac{-2\pi i (kr+b) y}{2^n}} |y\rangle$

$\underline{\underline{\text{Reorder}}}$ terms under summation

$$\frac{1}{\sqrt{2^{11} 511}} \cdot \sum_{y=0}^{2^{11}-1} e^{-\frac{2\pi i b y}{2^{11}}} \left(\sum_{k=0}^{mb-1} e^{\frac{-2\pi i k r y}{2^{11}}} \right) |y\rangle$$

let $\xi = e^{\frac{-2\pi i r y}{2^{11}}}$ $= \frac{1}{\sqrt{2^{11} 511}} \sum_{y=0}^{2^{11}-1} e^{\frac{-2\pi i b y}{2^{11}}} \left(\sum_{k=0}^{511} \xi^k \right) |y\rangle$

- $\xi = e^{-\frac{2\pi i r y}{2^n}}$ $\quad n = 11 \; ; \; r = 4.$

$$\Rightarrow \xi_{1024} = e^{-\frac{2\pi i (4).1024}{2^{11}}} = e^{-4\pi i} = 1$$

$$\xi_0 = e^{-\frac{2\pi i .4.0}{2^{11}}} = e^{-0} = 1$$

$$\xi_{512} = e^{-\frac{2\pi i 4.512}{2^{11}}} = e^{-2\pi i} = 1$$

$$\xi_{1536} = e^{-\frac{2\pi i 4.1536}{2^{11}}} = e^{-6\pi i} = 1$$

- State is $\frac{1}{\sqrt{2^{11}.511}} \sum_{y=0}^{2^{11}-1} e^{-\frac{2\pi i b y}{2^{11}}} \underbrace{\left(\sum_{k=0}^{510} \xi^k \right)}_{=511} |y\rangle$

$\xi_{0,512,1024,1536} = 1$

$$\text{Prob}(y = 512) = \text{Prob}(y = 0) = \text{Prob}(y = 1024)$$
$$= \text{Prob}(y = 1536)$$

$$= \frac{511^2}{2^{11} \cdot 511} = \frac{511}{2^{11}} \simeq \frac{1}{4} - \frac{1}{2048}$$

- $$d\left(QFT_{11}^{-1} ; \widetilde{QFT}_{11}^{-1}\right) \leq \frac{1}{2^{11}} \cdot \frac{11(11-1)}{2} = \frac{5}{2048}$$

$$\tilde{P}\left(y = 0, 512 ; 1024 ; 1536\right) \geqslant \frac{1}{4} - \frac{1}{2048} - \frac{5}{2048}$$

$$= \frac{1}{4} - \frac{3}{1024}$$