Question 1

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

• $UU^{\dagger} = I, \Leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\Leftrightarrow \begin{pmatrix} aa^* + bb^* & ac^* + bd^* \\ a^*c + b^*d & cc^* + dd^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} |a|^2 + |b|^2 = 1 \\ |c|^2 + |d|^2 = 1 \\ ca^* = -db^* \quad (ac^* = -bd^*) \end{cases} \qquad (1)$$

• $U^{\dagger}U = I \Leftrightarrow \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\Leftrightarrow \begin{cases} |a|^2 + |c|^2 = 1 \\ |b|^2 + |d|^2 = 1 \\ ab^* = -cd^* \end{cases} \qquad (2)$$

• We have $U^{+}U = UU^{+}$, so

$$|a|^2 + |b|^2 = |a|^2 + |c|^2 \Rightarrow |b|^2 = |c|^2 \Rightarrow |b| = |c|$$

and from (1) & (2) $\rightarrow$ $\begin{cases} |a|^2 + |b|^2 = 1 \\ |c|^2 + |d|^2 = 1 \end{cases}$

$$\Rightarrow \begin{cases} |a|^2 = 1 - |b|^2 \\ |d|^2 = 1 - |c|^2 \end{cases} \rightarrow |a|^2 = |d|^2$$

$$\rightarrow |a| = |d|$$

• $ca^* = -db^* \Leftrightarrow c = \dfrac{-db^*}{a^*} = \dfrac{-adb^*}{aa^*}$

$$\Leftrightarrow c = \frac{-\rho_1 e^{ix_a} \rho_1 e^{ix_d} \rho_2 e^{-ix_b}}{(\rho_1)^2} = -\rho_2 e^{i(x_a + x_d)} e^{-ix_b}$$

• $ca^* = -db^* \Leftrightarrow d = -\dfrac{ca^*}{b^*} = -\dfrac{(\rho_2 e^{i(x_a+x_d)} e^{-ix_b}) \rho_1 e^{-ix_a} \rho_2 e^{ix_b}}{\rho_2^2}$

$$= \rho_1 e^{i(x_a + x_d)} e^{-ix_a}$$

(2)

• Let $\phi = x_a + x_d \Rightarrow c = -\rho_2 e^{i\phi} e^{-ix_b} = -e^{i\phi} b^*$

$d = \rho_1 e^{i\phi} e^{-ix_a} = e^{i\phi} a^*$

Thus, $U = \begin{pmatrix} a & b \\ -e^{i\phi} b^* & e^{i\phi} a^* \end{pmatrix}$

Question 2

- We have $U = \begin{pmatrix} a & b \\ -e^{i\phi} b^* & e^{i\phi} a^* \end{pmatrix}$

Let $\phi_1 = x_a - \frac{\phi}{2}$ and $\phi_2 = x_b - \frac{\phi}{2}$

$$U = \begin{pmatrix} \rho_1 e^{i(\phi_1 + \frac{\phi}{2})} & \rho_2 e^{i(\phi_2 + \frac{\phi}{2})} \\ -e^{i\phi} \rho_2 e^{-i(\phi_2 + \frac{\phi}{2})} & e^{i\phi} \rho_1 e^{-i(\cancel{x}\phi_1 + \frac{\phi}{2})} \end{pmatrix}$$

$$= e^{i\frac{\phi}{2}} \begin{pmatrix} \rho_1 e^{i\phi_1} & \rho_2 e^{i\phi_2} \\ -\rho_2 e^{-i\phi_2} & \rho_1 e^{-i\phi_1} \end{pmatrix}$$

- Let break down U into:

$$U = e^{i\frac{\phi}{2}} \begin{pmatrix} e^{i\phi_1} & e^{i\phi_2} \\ e^{-i\phi_2} & e^{-i\phi_1} \end{pmatrix} \begin{pmatrix} \rho_1 & -\rho_2 \\ \rho_2 & \rho_1 \end{pmatrix}$$

and $\rho_1^2 = |a|^2$, $\rho_2^2 = |c|^2$ and $|a|^2 + |c|^2 = 1 \rightarrow \rho_1^2 \overset{\text{plus}}{\cancel{-}} \rho_2^2 = 1$

(4)

Thus, there exist θ s.t : $\begin{cases} \cos\theta^2 + \sin\theta^2 = 1 \\ \rho_1 = \cos\theta \\ \rho_2 = -\sin\theta \end{cases}$

- Let $\psi = \frac{1}{2}(\phi_1 + \phi_2)$ and $\Delta = \frac{1}{2}(\phi_1 - \phi_2)$

$$e^{i\frac{\phi}{2}} \begin{pmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} e^{i\Delta} & 0 \\ 0 & e^{-i\Delta} \end{pmatrix}$$

$$= e^{i\frac{\phi}{2}} \begin{pmatrix} e^{i\psi}\cos\theta & -e^{i\psi}\sin\theta \\ e^{-i\psi}\sin\theta & e^{-i\psi}\cos\theta \end{pmatrix} \begin{pmatrix} e^{i\Delta} & 0 \\ 0 & e^{-i\Delta} \end{pmatrix}$$

$$= e^{i\phi/2} \begin{pmatrix} e^{i\psi}\cos\theta\, e^{i\Delta} & -e^{i\psi}\sin\theta\, e^{-i\Delta} \\ e^{-i\psi}\sin\theta\, e^{i\Delta} & e^{-i\psi}\cos\theta\, e^{-i\Delta} \end{pmatrix}$$

$$= e^{i\phi/2} \begin{pmatrix} \rho_1 e^{i(\psi+\Delta)} & \rho_2 e^{i(\psi-\Delta)} \\ -\rho_2 e^{-i(\psi-\Delta)} & \rho_1 e^{-i(\psi+\Delta)} \end{pmatrix}$$

We have $\psi + \Delta = \phi_1$

$\psi - \Delta = \phi_2$

$$RHS = e^{i\frac{\Phi}{2}} \begin{pmatrix} \rho_1 e^{i\phi_1} & \rho_2 e^{i\phi_2} \\ -\rho_2 e^{-i\phi_2} & \rho_1 e^{-i\phi_1} \end{pmatrix} = U$$

- We have shown that:

$$U = e^{i\frac{\Phi}{2}} \begin{pmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} e^{i\Delta} & 0 \\ 0 & e^{-i\Delta} \end{pmatrix}$$

- $\alpha = \frac{\Phi}{2}$
- $\frac{\gamma}{2} = \theta \rightarrow \gamma = 2\theta$
- $\beta = -2\psi = \cancel{-(\phi_1 + \phi_2)}$

  $= -2(\phi_1 + \phi_2)$
- $\frac{\delta}{2} = -\Delta \rightarrow \delta = -2\Delta$

  $= -2(\phi_1 - \phi_2)$

Question 3

$$A = R_z(\beta)\, R_y\left(\frac{\gamma}{2}\right)$$

$$B = R_y\left(-\frac{\gamma}{2}\right) R_z\left(-\frac{\delta+\beta}{2}\right)$$

$$C = R_z\left(\frac{\delta-\beta}{2}\right)$$

- $$ABC = R_z(\beta)\, \overbrace{\cancel{R_y\left(\frac{\gamma}{2}\right) R_y\left(-\frac{\gamma}{2}\right)}}^{I} R_z\left(-\frac{\delta+\beta}{2}\right) \cdot C$$

$$= R_z\left(\beta - \frac{\delta}{2} - \frac{\beta}{2}\right) \cdot R_z\left(\frac{\delta-\beta}{2}\right)$$

$$= R_z\left(\beta - \frac{\delta}{2} - \frac{\beta}{2}\right) R_z\left(\frac{\delta}{2} - \frac{\beta}{2}\right)$$

$$= R_z\left(\frac{\beta-\delta}{2}\right) R_z\left(\frac{\delta-\beta}{2}\right) = I$$

• $$X R_y(\theta) X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ +\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}$$

$$= R_y(-\theta)$$

- $$XBX = X\, R_y\!\left(-\frac{\gamma}{2}\right) R_z\!\left(-\frac{\delta+\beta}{2}\right) X$$

$I = X^2$

$$= X\, R_y\!\left(-\frac{\gamma}{2}\right)(X \cdot X)\, R_z\!\left(-\frac{\delta+\beta}{2}\right) X$$

$$= \left(X\, R_y\!\left(-\frac{\gamma}{2}\right) X\right)\left(X\, R_z\!\left(-\frac{\delta+\beta}{2}\right) X\right)$$

$$= R_y\!\left(\frac{\gamma}{2}\right) R_z\!\left(\frac{\delta+\beta}{2}\right)^{*}$$

(*) Since $X R_z(\theta) X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 0 & e^{i\theta} \\ e^{-i\theta} & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

$$= R_z(-\theta) \quad \forall \theta.$$

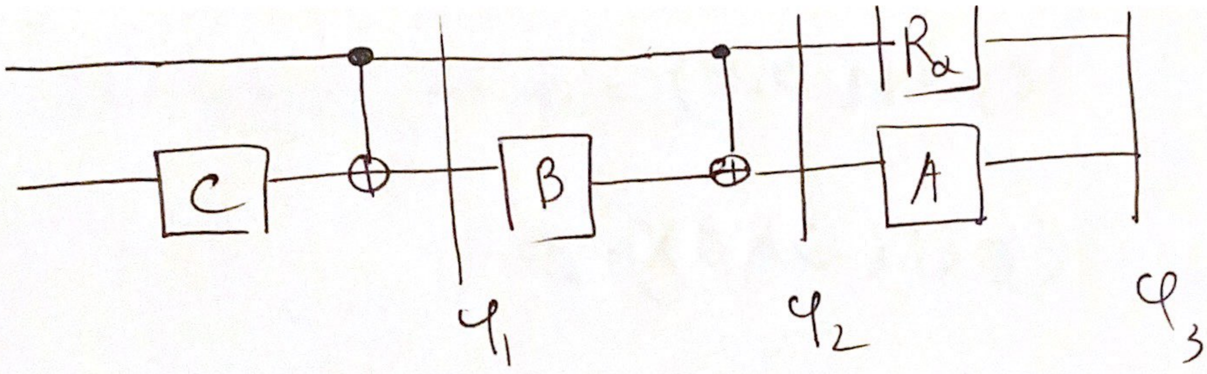
- $ABC = I$

$$AXBXC = A(XBX)C$$

$$= R_z(\beta)\, R_y\left(\frac{\gamma}{2}\right) R_y\left(\frac{\gamma}{2}\right) R_z\left(\frac{\delta+\beta}{2}\right) \cdot C$$

$$= R_z(\beta)\, R_y(\gamma)\, R_z\left(\frac{\delta+\beta}{2}\right) R_z\left(\frac{\delta-\beta}{2}\right)$$

$$= R_z(\beta)\, R_y(\gamma)\, R_z(\delta)$$



- $|0\rangle|\varphi\rangle \rightarrow \varphi_1 = (I \times C)|0\varphi\rangle = C|0\varphi\rangle$

$$\varphi_2 = (I \times B \times C)|0\varphi\rangle = BC|0\varphi\rangle$$

$$\varphi_3 = (R_\alpha A B C)|0\varphi\rangle = R_\alpha |0\rangle I |\varphi\rangle$$

$$R_\alpha|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

Thus; $|0\rangle|\varphi\rangle$ is the out put.

• $|1\rangle|\varphi\rangle \rightarrow \varphi_1 = (X.C)|1\varphi\rangle$

$$\varphi_2 = (XBXC)|1\varphi\rangle$$

$$\varphi_3 = (R_\alpha A X B X C)|1\varphi\rangle$$

$$\downarrow$$

$$= \cancel{R_\alpha} R_\alpha . U \; |1\varphi\rangle$$

$$= R_\alpha|1\rangle \otimes U|\varphi\rangle$$

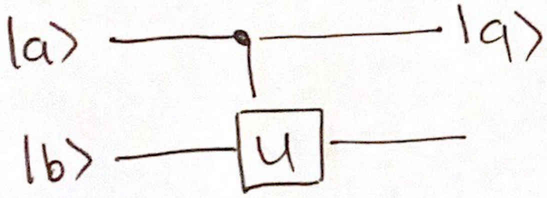
$$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ e^{i\alpha} \end{pmatrix} = e^{i\alpha}|1\rangle$$

$$= \cancel{e^{i\alpha}}|1\rangle . U|\varphi\rangle$$

$$= \cancel{e^{i\alpha}} U|1\rangle|\varphi\rangle$$

The output state is: $\cancel{e^{i\alpha}} U|1\varphi\rangle$

• We create controlled rotation gate U ; s.t.



if $|a\rangle = 0 \rightarrow$ output state at second qubit is the same $= |b\rangle$

• If $|a\rangle = 1 \rightarrow$ Rotate the second qubit by angel

$$U = R_z(\beta)\, R_y(\gamma)\, R_z(\delta)$$