

Nam Nguyen - 1170840587

$$a) \cdot \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

We find λ satisfying $|\sigma_x - \lambda I| = 0$

$$\Leftrightarrow \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \Leftrightarrow \lambda^2 - 1 = 0 \Leftrightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = -1 \end{cases}$$

$$\text{With } \lambda_1 = 1 \rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} v_1 = 0 \Rightarrow v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -1 \rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} v_2 = 0 \Rightarrow v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\cdot \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},$$

$$|\sigma_y - \lambda I| = 0 \Leftrightarrow \begin{vmatrix} -\lambda & -i \\ i & -\lambda \end{vmatrix} = 0$$

$$\Leftrightarrow \lambda^2 + i^2 = 0 \Leftrightarrow \lambda^2 - 1 = 0 \Leftrightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = -1 \end{cases}$$

$$+, \lambda_1 = 1 \Rightarrow \begin{bmatrix} -1 & -i \\ i & -1 \end{bmatrix} \bar{v}_1 = 0 \Rightarrow v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$+, \lambda_2 = -1 \Rightarrow \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} \bar{v}_2 = 0 \Rightarrow v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

We have $|A_2 - \lambda I| = 0 \Leftrightarrow \begin{vmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = 0$

$$\Leftrightarrow (1-\lambda)(-1-\lambda) = 0$$

$$\Rightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = -1 \end{cases}$$

With $\lambda_1 = 1 \rightarrow \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} v_1 = 0 \rightarrow v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\lambda_2 = -1 \rightarrow \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} v_2 = 0 \rightarrow v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

b) $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}$

We need to find $|M - \lambda I| = 0$

$$\Leftrightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & i \\ 0 & i & -\lambda \end{vmatrix} = 0$$

$$\Leftrightarrow (1-\lambda) \begin{vmatrix} -\lambda & i \\ i & -\lambda \end{vmatrix} = 0 \Leftrightarrow (1-\lambda)(\lambda^2 - 1^2) = 0$$

$$\Leftrightarrow \lambda_1 = 1, \lambda_2 = i, \lambda_3 = -i$$

$$+), \lambda_1 = 1 \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & i \\ 0 & i & -1 \end{bmatrix} v_1 = 0 \Rightarrow v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$+), \lambda_2 = i \Rightarrow \begin{bmatrix} 1-i & 0 & 0 \\ 0 & -i & i \\ 0 & i & -1 \end{bmatrix} v_2 = 0 \Rightarrow v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$+), \lambda_3 = -i \Rightarrow \begin{bmatrix} 1+i & 0 & 0 \\ 0 & i & i \\ 0 & -i & -1 \end{bmatrix} v_3 = 0 \Rightarrow v_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Question 2,

$$a), |+\rangle = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \quad (1)$$

$$|-\rangle = \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle \quad (2)$$

$$\text{Take (1) + (2)} \rightarrow |+\rangle + |-\rangle = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |0\rangle = \sqrt{2} |0\rangle$$

$$\Rightarrow |0\rangle = \frac{1}{\sqrt{2}} |+\rangle + \frac{1}{\sqrt{2}} |-\rangle \rightarrow \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

Take (1) - (2)

$$\Rightarrow |+\rangle - |-\rangle = \sqrt{2} |1\rangle \Rightarrow |1\rangle = \frac{1}{\sqrt{2}} |+\rangle - \frac{1}{\sqrt{2}} |-\rangle \rightarrow \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)$$

$$b, \quad |\psi\rangle = \frac{1}{\sqrt{2}} |+\rangle - \frac{\sqrt{3}}{2} |-\rangle$$

$$= \frac{1}{\sqrt{2}} \left[\frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \right] - \frac{\sqrt{3}}{2} \left[\frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \right]$$

$$= \frac{1}{2} |0\rangle + \frac{1}{2} |1\rangle - \frac{\sqrt{3}}{2\sqrt{2}} |0\rangle + \frac{\sqrt{3}}{2\sqrt{2}} |1\rangle$$

$$= \frac{\sqrt{2}-\sqrt{3}}{2\sqrt{2}} |0\rangle + \frac{\sqrt{2}+\sqrt{3}}{2\sqrt{2}} |1\rangle$$

$$= \frac{2-\sqrt{6}}{4} |0\rangle + \frac{2+\sqrt{6}}{4} |1\rangle$$

Question 3,

a, $P = \sum_{i=1}^n |i\rangle\langle i|$

$$P^2 = P \cdot P = \left(\sum_{i=1}^n |i\rangle\langle i| \right) \left(\sum_{i=1}^n |i\rangle\langle i| \right)$$

$$= \sum_{i=1}^n (|i\rangle\langle i|)(|i\rangle\langle i|) + \sum_{i \neq j} \underbrace{(|i\rangle\langle i|)(|j\rangle\langle j|)}_{=0}$$

$$= \sum_{i=1}^n |i\rangle (\underbrace{\langle i|i\rangle}_1) \langle i|$$

$$= \sum_{i=1}^n |i\rangle\langle i| = P$$

(Another way is to show $P = I_n$)

b, First; $|i\rangle\langle i| = \begin{bmatrix} 0 & \vdots & 0 \\ \vdots & 1 & \vdots \\ 0 & \vdots & 0 \end{bmatrix}$

Thus, $P = \sum_{i=1}^n |i\rangle\langle i| = \mathbb{I}_n$

The eigenvalues of \mathbb{I}_n is defined by

$$\begin{vmatrix} 1-\lambda & & (0) \\ & 1-\lambda & \\ (0) & & 1-\lambda \end{vmatrix} = 0 \rightarrow (1-\lambda)^n = 0$$

or: $\lambda_1 = \dots = \lambda_n = 0$