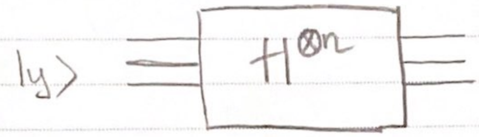
Homework 6

Question 1)



$$|y\rangle \longmapsto \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{x \cdot y} |x\rangle$$

1) For $n = 1$; we need to prove $H|y\rangle = \frac{1}{\sqrt{2}} \sum_{x \in \{0,1\}} (-1)^{x \cdot y} |x\rangle$

Clearly, $H|y\rangle = \frac{1}{\sqrt{2}} \left((-1)^{0 \cdot y} |0\rangle + (-1)^{1y} |1\rangle \right)$

if $y = 0 \rightarrow H|0\rangle = \frac{1}{\sqrt{2}} \left((-1)^{0.0} |0\rangle + (-1)^{1.0} |1\rangle \right)$

$= \frac{1}{\sqrt{2}} \left(|0\rangle + |1\rangle \right)$ ← true

if $y = 1 \rightarrow H|1\rangle = \frac{1}{\sqrt{2}} \left((-1)^{0.1} |0\rangle + (-1)^{1.1} |1\rangle \right)$

$= \frac{1}{\sqrt{2}} \left(|0\rangle - |1\rangle \right)$ ← also true.

Thus, the identity holds when $n = 1$.

2) Assume that $H^{\otimes n} |y\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{x \cdot y} |x\rangle$, $y < 2^n$

We have $H^{\otimes n+1} |y\rangle = H \left(\frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{x \cdot y} |x\rangle \right)$

$= \cancel{\frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} H \left((-1)^{x \cdot y} |x\rangle \right)} = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{x \cdot y} \Big(\frac{1}{\sqrt{2}}$

$$H^{\otimes n+1}|y\rangle = \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} H\left((-1)^{x\cdot y}|x\rangle\right)$$

$$= \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} (-1)^{x\cdot y}\left(\frac{1}{\sqrt{2}} \underbrace{\sum\left((-1)^{x'0}|0\rangle + (-1)^{x'1}|1\rangle\right)}_{\sum_{x'\in\{0,1\}} (-1)^{x'} |x'\rangle}\right)$$

$$= \frac{1}{\sqrt{2^{n+1}}} \sum_{x\in\{0,1\}^{n+1}} (-1)^{xy} |x\rangle$$

Question 2,

1) We have from Q1.1; $H^{\otimes n}|0\rangle^{\otimes n} = \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} (-1)^{x.0} |x\rangle$

$\rightarrow \varphi_1 = \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} |x\rangle$

2) $\varphi_2 = U_f \varphi_1 = U_f \left(\frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} |x\rangle \right)$

$= \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} (-1)^{f(x)} |x\rangle$; $f : \{0,1\}^n \longmapsto x.s$

3) $\varphi_3 = H^{\otimes n} \varphi_2 = H^{\otimes n} \left(\frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} (-1)^{f(x)} |x\rangle \right)$

$= \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} (-1)^{f(x)} H^{\otimes n} |x\rangle$

$= \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} (-1)^{f(x)} \sum_{y\in\{0,1\}^n} \left(\frac{1}{\sqrt{2^n}} (-1)^{y.x} |y\rangle \right)$

$= \frac{1}{2^n} \sum_{y\in\{0,1\}^n} \left(\sum_{x\in\{0,1\}^n} (-1)^{f(x) \oplus y.x} \right) |y\rangle$

typo in Q (ass?)

4, For all y, $\sum_{x\in\{0,1\}^n} (-1)^{f(x) \oplus y.x} = \sum_{x\in\{0,1\}^n} (-1)^{x.s \oplus y.x}$

$= \sum_{x\in\{0,1\}^n} (-1)^{x(s\oplus y)}$

5) For $y = s$

$$\sum_{x\in\{0,1\}^n} (-1)^{x.(s\oplus s)} = \sum_{x\in\{0,1\}^n} (-1)^{x.0} \quad \left(\begin{array}{l}\text{Since } s\oplus s = s+s \bmod 2 \\ = 2s \bmod 2 = 0\end{array}\right)$$

$$= \sum_{x\in\{0,1\}^n} (-1)^0 = \sum_{x\in\{0,1\}^n} 1 = 2^n$$

6) For $s \neq y \rightarrow s\oplus y = s+y \bmod 2 = 1$

$$\sum_{x\in\{0,1\}^n} (-1)^{(s\oplus y)} = \sum_{x\in\{0,1\}^n} (-1)^{x.1} = \sum_{x\in\{0,1\}^n} (-1)^x$$

If $x = 0 \rightarrow (-1)^0 = 1$

$x = 1 \rightarrow (-1)^1 = -1$

Thus, the numbers will be canceled out $\rightarrow$ the result is 0.

7) $$|\psi\rangle = \frac{1}{2^n} \sum_{y\in\{0,1\}^n} \left(\sum_{x\in\{0,1\}^n} (-1)^{x(s\oplus y)} \right) |y\rangle \quad \text{(from part 4)}$$

$$= \frac{1}{2^n} \sum_{y\in\{0,1\}^n} \left(\underbrace{\sum}_{y=s} + \underbrace{\sum}_{y\neq s} \right) |y\rangle$$

$$= \frac{1}{2^n} \sum_{y\in\{0,1\}^n} \left(2^n + 0 \right) |y\rangle \quad \text{(from part 5 \& 6)}$$

$\Rightarrow |\psi\rangle = |s\rangle$ So after the measurment; we have at each qbit q_i measured as s_i ; where $f(x) = x.s = x_1.s_1 \oplus x_2.s_2 \oplus \dots \oplus x_n.s_n$