Question 1,

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1, $S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$

. $S^{+}S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$

$= \frac{1}{2} \begin{pmatrix} 1 - i^2 & i - i \\ -i + i & -i^2 + 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = I$

. $SS^{+} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$

$= \frac{1}{2} \begin{pmatrix} 1 - i^2 & -i + i \\ i - i & -i^2 + 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = I$

$\rightarrow S^{+}S = SS^{+} = I \rightarrow S$ is unitary

2, $P(\uparrow) = \left| \langle \uparrow | S | \rightarrow \rangle \right|^2$

$S|\rightarrow\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$

$\langle \uparrow | S | \rightarrow \rangle = \begin{pmatrix} 0 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}} (i) = \frac{i}{\sqrt{2}}$

$P(\uparrow) = \left| \frac{i}{\sqrt{2}} \right|^2 = \frac{1}{2}$

3) • State after the first BS is $|\varphi_1\rangle = \frac{1}{\sqrt{2}}\left(|\rightarrow\rangle + i|\uparrow\rangle\right)$

or $|\varphi_1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right)$

• State after P is

$$P|\varphi_1\rangle = \frac{1}{\sqrt{2}} P\left(|0\rangle + i|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}} \cdot \left[\begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix}|0\rangle + i\begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix}|1\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ e^{i\varphi} \end{pmatrix}\right]$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle + ie^{i\varphi}|1\rangle\right) = |\varphi_2\rangle$$

• State after the mirror is

$$M|\varphi_2\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + ie^{i\varphi}|0\rangle\right) = \varphi_3$$

• State after the second BS is $S|\varphi_3\rangle$

$$S|\varphi_3\rangle = S\,\frac{1}{\sqrt{2}}\left(|\uparrow\rangle + ie^{i\varphi}|\rightarrow\rangle\right)$$

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$$= \frac{1}{\sqrt{2}} \left(S|\uparrow\rangle + ie^{i\varphi} S|\rightarrow\rangle \right)$$

$$S|\uparrow\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \left(i|\rightarrow\rangle + |\uparrow\rangle \right)$$

$$S|\rightarrow\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}} \left(|\rightarrow\rangle + i|\uparrow\rangle \right)$$

Thus, the state is $\frac{1}{\sqrt{2}} \left[\frac{1}{\sqrt{2}} \left(i|\rightarrow\rangle + |\uparrow\rangle \right) + \frac{1}{\sqrt{2}} ie^{i\varphi} \left(|\rightarrow\rangle + i|\uparrow\rangle \right) \right.$

$$= \frac{1}{2} \left(i|\rightarrow\rangle + |\uparrow\rangle + ie^{i\varphi}|\rightarrow\rangle - e^{i\varphi}|\uparrow\rangle \right)$$

$$= \frac{1}{2} \left[\left(i + ie^{i\varphi} \right)|\rightarrow\rangle + \left(1 - e^{i\varphi} \right)|\uparrow\rangle \right] = u$$

3, The probability of measuring a photon on the horizontal path is $P(\rightarrow) = \left| \langle\rightarrow| SXPS |\rightarrow\rangle \right|^2$

$$= \left(\frac{1}{2} \left(i + ie^{i\varphi} \right) \langle\rightarrow|\rightarrow\rangle + \left(1 - e^{i\varphi} \right) \cancelto{0}{\langle\rightarrow|\uparrow\rangle} \right)^2$$

$$= \left[\frac{1}{\sqrt{2}} \left(i + ie^{i\varphi} \right) \right]^2 = \frac{1}{2} \left(i + ie^{i\varphi} \right)^2$$

Thus, $P(\rightarrow) = \frac{1}{2}\left(i + i e^{i\varphi} \right)^2$

$$= \frac{1}{2}\left(i + i\cos\varphi + i^2 \sin\varphi \right)^2$$

$$= \frac{1}{2}\left[i\left(\cos^2\frac{\varphi}{2} - \sin^2\frac{\varphi}{2}\right) + i - 2\sin\frac{\varphi}{2}\cos\frac{\varphi}{2} \right]^2$$

$$= \frac{1}{2}\left[-2\sin\frac{\varphi}{2}\cos\frac{\varphi}{2} + \left(\cos^2\frac{\varphi}{2} - \frac{\sin^2\varphi}{2} + \cos^2\frac{\varphi}{2} + \sin^2\frac{\varphi}{2}\right) i \right]^2$$

$$= \frac{1}{2}\left(-2\sin\frac{\varphi}{2}\cos\frac{\varphi}{2} + 2\cos^2\frac{\varphi}{2}\, i \right)^2$$

$$= \frac{1}{2}\left[2\cos\frac{\varphi}{2}\left(-\sin\frac{\varphi}{2} + \cos\frac{\varphi}{2}\, i \right) \right]^2$$

$$= 2\cos^2\frac{\varphi}{2}\left(-\sin\frac{\varphi}{2} + \cos\frac{\varphi}{2}\, i \right)^2$$

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Question 2) $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 1 \\ 0 & 1 & i \end{pmatrix}$

1) Solve $\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & i-\lambda & 1 \\ 0 & 1 & i-\lambda \end{vmatrix} = 0$

$\Leftrightarrow \quad (1-\lambda)\left[(i-\lambda)^2 - 1\right] = 0$

$\Leftrightarrow \quad \lambda_1 = 1, \quad \lambda_2 = -1+i \; ; \; \lambda_3 = 1+i$

2) • $\lambda_1 = 1 \Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & i-1 & 1 \\ 0 & 1 & i-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$\Rightarrow \begin{cases} (-1+i)x_2 + x_3 = 0 \\ x_2 + (-1+i)x_3 = 0 \end{cases} \Rightarrow x_2 = x_3 = 0$

$\Rightarrow \quad v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

• $\lambda_2 = -1+i \Rightarrow \begin{pmatrix} 2-i & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$\Rightarrow \quad x_1 = 0, \quad x_2 = 1, \quad x_3 = -1$

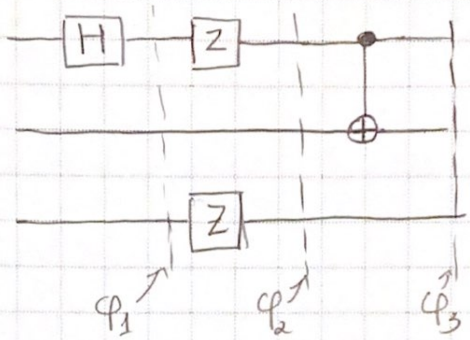
$\Rightarrow \quad v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

- $\lambda_3 = 1 + i \Rightarrow \begin{pmatrix} -i & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$\Leftrightarrow \begin{cases} x_1 = 0 \\ x_2 = x_3 \end{cases} \Leftrightarrow x_1 = 0 \ ; \ x_2 = 1 \ , \ x_3 = 1$

$\Rightarrow \quad v_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Question 4,



1, $\varphi_1 = H \otimes I \otimes I$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

• $\varphi_2 = Z \otimes I \otimes Z$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

• $\varphi_3 = CNOT \otimes I$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

2, $\quad U = \varphi_1 \varphi_2 \varphi_3$

$$\varphi_1 \varphi_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$U = \cdot\, \varphi_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \end{pmatrix}$$

3) The probability of measuring 00 in the first 2 qubits is $P(000) + P(001)$

- $P(000) = \left| \langle 000 | U | 111 \rangle \right|^2$

$$|000\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}$$

$$= \begin{pmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}$$

$$|111\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}$$

$$= \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\1\end{pmatrix}$$

$$|001\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\\0\\0\\0\\0\end{pmatrix}$$

$P(|000\rangle) = 0$.

$U|111\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$P(001) = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$

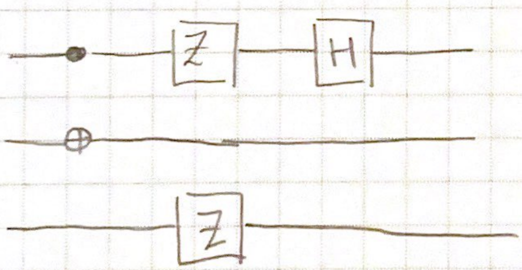
Thus, the probability of measuring 00 first 2 qubits is $\frac{1}{2}$

4) G-cost = 4, (1 H, 2 Z, 1 CNOT)

D-cost = 3

DW-cost = 9 (3 x 3)

5)



Since $Z^{\dagger} = Z$

$H^{\dagger} = H$

6, DW-cost is always larger than G-cost

or G-cost $\leq$ DW-cost

- Simply, a circuit of depth D, width W can contain a maximum D x W gates (there is D x W spaces to place a gate). However, some time we do not put a gate in the place

Question 3,

1, We first summarize the process as follows.

A wants to send (a)	b	A sends B receives	b'	State measure	a'
0	0	$\|0\rangle$	0	$\|0\rangle$	0
1	0	$\|1\rangle$	0	$\|1\rangle$	1
0	1	$\|+\rangle$	0	$\frac{1}{\sqrt{2}}\|0\rangle + \frac{1}{\sqrt{2}}\|1\rangle$	0 or 1
1	1	$\|-\rangle$	0	$\frac{1}{\sqrt{2}}\|0\rangle - \frac{1}{\sqrt{2}}\|1\rangle$	—
0	0	$\|0\rangle$	1	$1/\sqrt{2}\,\|+\rangle + 1/\sqrt{2}\,\|-\rangle$	—
1	0	$\|1\rangle$	1	$1/\sqrt{2}\,\|+\rangle - 1/\sqrt{2}\,\|-\rangle$	—
0	1	$\|+\rangle$	1	$\|+\rangle$	0
1	1	$\|-\rangle$	1	$\|-\rangle$	1

We denote a state Alice sending is $|\varphi_{ab}\rangle$, where $a, b \in \{0,1\}^2$

thus. $|\varphi_{00}\rangle = |0\rangle \quad |\varphi_{10}\rangle = |1\rangle$

$|\varphi_{01}\rangle = |+\rangle \quad |\varphi_{11}\rangle = |-\rangle$

We denote $|\varphi_{ab}\rangle_{b'}$ is the outcome when Bob measures the state at bases

thus, $|\varphi_{00}\rangle_0 = |0\rangle_0 = |0\rangle \qquad |\varphi_{00}\rangle_1 = |0\rangle_1 = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$

$|\varphi_{10}\rangle_0 = |1\rangle_0 = |1\rangle \qquad |\varphi_{10}\rangle_1 = |1\rangle_1 = \frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle$

$|\varphi_{01}\rangle_0 = |+\rangle_0 = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \qquad |\varphi_{01}\rangle_1 = |+\rangle_1 = |+\rangle$

$|\varphi_{11}\rangle_0 = |-\rangle_0 = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \qquad |\varphi_{11}\rangle_1 = |-\rangle_1 = |-\rangle$

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So if $b = b' = 0$, we have $|\varphi_{00}\rangle_0 = |0\rangle_0 = |0\rangle$

$|\varphi_{10}\rangle_0 = |1\rangle_0 = |1\rangle$

$b = b' = 1$, we have $|\varphi_{01}\rangle_1 = |+\rangle_1 = |+\rangle$

$|\varphi_{11}\rangle_1 = |-\rangle_1 = |-\rangle$

So if $b = b' \Rightarrow P(a = a') = 1$

b) If $b = 0$, $b' = 1 \rightarrow |\varphi_{00}\rangle_1 = |0\rangle_1 = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$

$\Rightarrow |\varphi_{10}\rangle_1 = |+\rangle_1 = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$

$|\varphi_{00}\rangle_1 = \frac{1}{2}(|+\rangle + |-\rangle) \Rightarrow P(a' = 0) = P(a' = 1) = \frac{1}{2}$

same as $|\varphi_{10}\rangle_1 \rightarrow P(a' = 0) = P(a' = 1) = \frac{1}{2}$

Thus $P(a = a') = \frac{1}{2}$

$P(a \neq a') = \frac{1}{2}$

Similar to $b = 1$, $b' = 0 \rightarrow |\varphi_{01}\rangle_0 = |+\rangle_0 = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$P(a' = 0) = P(a' = 1) = \frac{1}{2}$ ← same

Similar to $|\varphi_{11}\rangle_0 = |-\rangle_0 = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

Thus $P(a = a') = 1/2$

c) Overall, by the total rule of probability, if $B_1, .., B_N$ is disjointed event, and $B_1 \cup .. \cup B_N = \Omega$, we have

$$P(A) = P(A|B_1)P(B_1) + .. + P(A|B_N)P(B_N)$$

We have $P(a = a') = P(a = a' | b = b')P(b = b') + P(a = a' | b \neq b') P(b \neq b')$

$$= \frac{1}{2} \times 1 + \frac{1}{2} \times \frac{1}{2}$$

$$= 0.75$$

2) When $b_i = b_i'$, we have: Bob measures the state using the same basis that Alice created the state $\Rightarrow a = a'$

Thus, for these indices $P(a_i = a_i') = 1$

The expected number is 4, since there are 4 cases in 8 posibilities

3) a) We summarize the problem as follows

State Eve heare	If Eve get $b_1 = 0$	If Eve get $b_1 = 1$
$\vert 0\rangle$	$\vert 0\rangle$	$\frac{1}{\sqrt{2}}(\vert +\rangle + \vert -\rangle)$
$\vert 1\rangle$	$\vert 1\rangle$	$1/\sqrt{2}(\vert +\rangle - \vert -\rangle)$
$\vert +\rangle$	$1/\sqrt{2}(\vert 0\rangle + \vert 1\rangle)$	$\vert +\rangle$
$\vert -\rangle$	$1/\sqrt{2}(\vert 0\rangle - \vert 1\rangle)$	$\vert -\rangle$

The chance of guessing a_i correctly is $\frac{1}{2}$

Thus; If Eve guesses the state correctly, the ~~ste~~ resulting state stay the same (since Eve measures it on the same basis Alice created the state). However; if Eve guesses it wrong, the state will be disturbed, and Bob will guess a_i wrong. So after compare with Alice; if there is a mistake, that means there is a eavesdropper then Bod & Alice will canal the protocol

b) Similaly:

Eve hears	If $b_1 = 0$ (measure in H)	If $b_1 = 1$ (measure in C)
$\|0\rangle$	$\frac{1}{\sqrt{2}}(\|+\rangle + \|-\rangle)$	$\|0\rangle$
$\|1\rangle$	$\frac{1}{\sqrt{2}}(\|+\rangle - \|-\rangle)$	$\|1\rangle$
$\|+\rangle$	$\|+\rangle$	$\frac{1}{\sqrt{2}}(\|0\rangle + \|1\rangle)$
$\|-\rangle$	$\|-\rangle$	$\frac{1}{\sqrt{2}}(\|0\rangle - \|1\rangle)$

Chance learn a_i is $\frac{1}{2}$. If guessing correctly, the state stay the same, otherwise; the state will be disturbed to a vector in oppose basis

c) If Eve guesses b_i wrongs, Bob receives a state in wrong basis, for exampl

Alice sent	b	Eve disturbed	b'
$\|0\rangle$	0	$1/\sqrt{2}(\|+\rangle + \|-\rangle)$	0
$\|1\rangle$	0	$1/\sqrt{2}(\|+\rangle - \|-\rangle)$	0
$\|+\rangle$	1	$1/\sqrt{2}(\|0\rangle + \|1\rangle)$	1
$\|-\rangle$	1	$1/\sqrt{2}(\|0\rangle + \|1\rangle)$	1

- If $b = b' = 0$; Bob receives $|\varphi_{00}\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right)$

Bob measures this state in $C = \{|0\rangle, |1\rangle\}$, thus

$$|\varphi_{00}\rangle_0 = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) + \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)\right] = |0\rangle$$

Similaly, $b = b' = 1$, $|\varphi_{01}\rangle_1 = |+\rangle \to a = a' = 0$

$|\varphi_{11}\rangle_1 = |-\rangle \quad a = a' = 1$

It is suprising that even though the state is disturb, if Bob guesses b_i right, he will guess a_i correctly ???

4, In the perfect case, when there is no evesdropper, $b_i = b_i' \Rightarrow a_i = a_i'$. However, the evesdroppere induces noices to the communication chanels. We summanze it as follows

a	b	$\|\varphi_{ab}\rangle$	Eves measure and Bob receive	a'
0	0	$\|0\rangle$	$\|0\rangle$	0
1	0	$\|1\rangle$	$\|1\rangle$	1
0	1	$\|+\rangle$	$1/\sqrt{2}(\|0\rangle + \|1\rangle)$	$P(a=0) = P(a=1) = 1/2$
1	1	$\|-\rangle$	$1/\sqrt{2}(\|0\rangle - \|1\rangle)$	$P(a=0) = P(a=1) = 1/2$

The $P(a_i = a_i') = 1/2$

b, We have $\frac{k}{2}$ indices, foreach index i, $P(a_i = a_i') = \frac{1}{2}$

thus the probability of $a_i = a_i'$ is

$$P(a_1 = a_1', \dots, a_{k/2} = a_{k/2}') \underset{\text{independent event}}{=} \prod_{i=1}^{k/2} \left(1/2\right)$$

$$= \left(\frac{1}{2}\right)^{k/2}$$

$$\lim_{k \to \infty} P(a_1 = a_1', \dots, a_{k/2} = a_{k/2}') = 0$$