Nam P Nguyen U70840587 Homework 1

Question 1;

a) We have the stade of linear polarization of the incoming photon is $\frac{1}{\sqrt{2}}|\leftrightarrow\rangle + \frac{1}{\sqrt{2}}|\updownarrow\rangle$; Thus,

$$S|\frac{\pi}{4};\rightarrow\rangle = \frac{1}{\sqrt{2}}|\leftrightarrow;\rightarrow\rangle + \frac{1}{\sqrt{2}}|\updownarrow,\uparrow\rangle = \frac{1}{\sqrt{2}}|\updownarrow,\uparrow\rangle + \frac{1}{\sqrt{2}}|\leftrightarrow;\rightarrow\rangle$$

Therefore $a = b = \frac{1}{\sqrt{2}}$

b) Now, since the beam splitter acts on the momentum as a unitary operator in $\mathbb{C}^2$. We have.

$$\begin{pmatrix} 1/\sqrt{2} & c \\ 1/\sqrt{2} & d \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ c^* & d^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{cases} \frac{1}{2} + |c|^2 = 1 \\ \frac{1}{2} + |d|^2 = 1 \\ \frac{1}{2} + c^* d = 0 \end{cases} \Leftrightarrow \begin{cases} c = \frac{1}{\sqrt{2}} e^{i x_c} \\ d = \frac{1}{\sqrt{2}} e^{i x_d} \end{cases}$$

$$\rightarrow \frac{1}{2} + \frac{1}{2} e^{-i x_c + i x_d} = 0 \Leftrightarrow e^{i(x_d - x_c)} = -1$$

$$\Leftrightarrow e^{i(x_d - x_c)} = e^{i\pi} \leftrightarrow x_d = x_c + \pi$$

Thus $\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} e^{i x_c} \\ \frac{1}{\sqrt{2}} e^{i(x_c+\pi)} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i x_c} \\ -e^{i x_c} \end{pmatrix}$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\varphi} \\ e^{-i\varphi} \end{pmatrix} = e^{i\varphi} \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

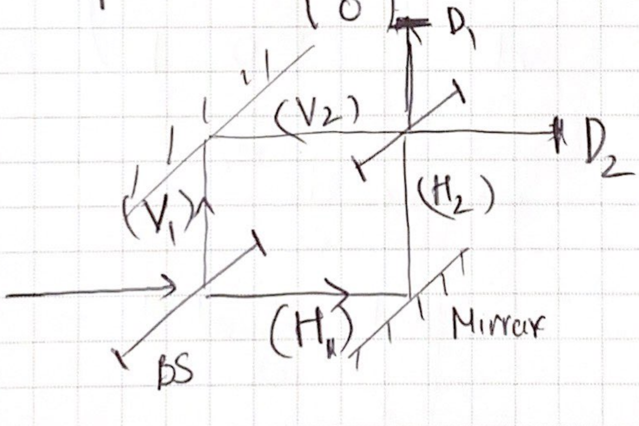
Thus: $U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ (a Hadarmard Matrix)

By and large, there $x_c = \varphi \in \mathbb{R}$; such that

$$S\left|\frac{\pi}{4};\uparrow\right\rangle = e^{i\varphi}\left(\frac{1}{\sqrt{2}}|\updownarrow,\uparrow\rangle - \frac{1}{\sqrt{2}}|\leftrightarrow;\rightarrow\rangle\right)$$

__Question 2:__

a) We now denote the state vector a photon in the horizontal path is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and in vertical path is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$



- We first compute the state vector of photon in (H_1) path. Since the beam splitter is placed at 45° and with equal distance, the BS acts on the state vector as a operator $S = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$ and $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Then the state after the mirrors and before the second beam splitter is:

$$M\left(S\left|\tfrac{\pi}{4};\rightarrow\right\rangle\right) = M\left(\frac{1}{\sqrt{2}}\left|\updownarrow;\uparrow\right\rangle + \frac{1}{\sqrt{2}}\left|\leftrightarrow;\rightarrow\right\rangle\right)$$

$$= \frac{1}{\sqrt{2}} M\left|\updownarrow;\uparrow\right\rangle + \frac{1}{\sqrt{2}} M\left|\leftrightarrow;\rightarrow\right\rangle$$

$$= \frac{1}{\sqrt{2}}\left|\updownarrow;\rightarrow\right\rangle + \frac{1}{\sqrt{2}}\left|\cancel{\updownarrow}\leftrightarrow;\uparrow\right\rangle \quad \blacksquare$$

Another approach is by matrix multiplication; the upcoming photo has state $\begin{pmatrix}1\\0\end{pmatrix}$ (horizontal); thus the state before second BS and after Mirrors are

$$M\left(S\begin{pmatrix}1\\0\end{pmatrix}\right) = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\left[\frac{1}{\sqrt{2}}\begin{bmatrix}1 & i\\ i & 1\end{bmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right]$$

$$= \frac{i}{\sqrt{2}}\underset{\leftrightarrow}{\begin{pmatrix}1\\0\end{pmatrix}} + \frac{1}{\sqrt{2}}\underset{\updownarrow}{\begin{pmatrix}0\\1\end{pmatrix}}$$

$$= \frac{1}{\sqrt{2}}\left|\leftrightarrow;\uparrow\right\rangle + \frac{1}{\sqrt{2}}\left|\updownarrow;\rightarrow\right\rangle$$

b) In case $\theta = 0$; $U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; then

The state after the first BS:

$$S|\theta = 0, \rightarrow\rangle = |\updownarrow, \uparrow\rangle$$

The state after mirrors is

$$M|\updownarrow; \uparrow\rangle = |\updownarrow; \rightarrow\rangle$$

The state after second beam splitter is.

$$S|\updownarrow; \rightarrow\rangle = |\updownarrow; \uparrow\rangle \ (?)$$

So there is 100% chance measuring the photo in the top detector (D_1) ☒

• In case $\theta = \frac{\pi}{2}$; $U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Thus, the state after the 1^{st} BS is:

$$S|\tfrac{\pi}{2}; \rightarrow\rangle = 0|\updownarrow, \uparrow\rangle + 1|\leftrightarrow; \rightarrow\rangle$$

$$= |\leftrightarrow; \rightarrow\rangle$$

After the mirror, the state is

$$M|\leftrightarrow, \rightarrow\rangle = |\leftrightarrow; \uparrow\rangle$$

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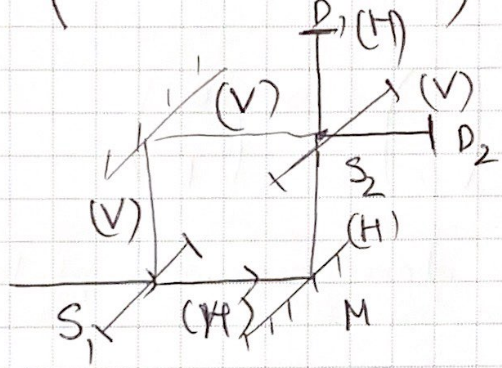
And the state after the second BS is:

$$S\,|\leftrightarrow;\uparrow\rangle = |\leftrightarrow;\rightarrow\rangle$$

Thus, there is 100% chance we have a photon in D_2 detector ($H_1 \rightarrow H_2 \rightarrow D_2$) ☒

Question 3:



We denote the paths as above figure.

Recall; the beam splitter has matrix representation of

$$S = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$

and Mirror is

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The first beam splitter make the photon go to a superposition of going to both (V) and (H) path.

If the photo go to (H) path, the state is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(V) path, the state is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

then after the first beam spliter, the superposition of the photon is

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = \alpha_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

with $|\alpha_0|^2$ is the chance measuring photon in (H)

an $|\alpha_1|^2$ ——————————— (V)

Thus the state after the ~~first~~beam splitter is

$$|S_1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

The state after the mirors and before 2^{nd} BS is

$$|S_2\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix}$$

The state after the second bs is

$$|S_3\rangle = \frac{1}{2} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \times \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix} = \frac{1}{2} \begin{bmatrix} 2i \\ i^2 + 1 \end{bmatrix} = \begin{pmatrix} i \\ 0 \end{pmatrix}$$

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Thus, the chance we measure the photon in D_2 is

$$\left| \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} i \\ 0 \end{pmatrix} \right|^2 = |i|^2 = 1$$

the chance we measure the photon in D_1 is

$$\left| \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} i \\ 0 \end{pmatrix} \right|^2 = |0|^2 = 0$$

Thus, we have 100% chance that the photon is detected by the detector D_2 (in vertical path) and hence 0% it comes out D_1 (in horizontal path) ✱

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