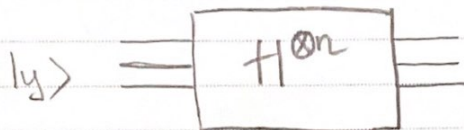


# homework 6

Question 1,



$$|y\rangle \mapsto \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{x \cdot y} |x\rangle$$

1) For  $n=1$ ; we need to prove  $H|y\rangle = \frac{1}{\sqrt{2}} \sum_{x \in \{0,1\}} (-1)^{x \cdot y} |x\rangle$

Clearly,  $H|y\rangle = \frac{1}{\sqrt{2}} \left( (-1)^{0 \cdot y} |0\rangle + (-1)^{1 \cdot y} |1\rangle \right)$

if  $y=0 \rightarrow H|0\rangle = \frac{1}{\sqrt{2}} \left( (-1)^{0 \cdot 0} |0\rangle + (-1)^{1 \cdot 0} |1\rangle \right)$   
 $= \frac{1}{\sqrt{2}} \left( |0\rangle + |1\rangle \right) \leftarrow \text{true}$

if  $y=1 \rightarrow H|1\rangle = \frac{1}{\sqrt{2}} \left( (-1)^{0 \cdot 1} |0\rangle + (-1)^{1 \cdot 1} |1\rangle \right)$   
 $= \frac{1}{\sqrt{2}} \left( |0\rangle - |1\rangle \right) \leftarrow \text{also true.}$

Thus, the identity holds when  $n=1$ .

2) Assume that  $H^{\otimes n} |y\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{x \cdot y} |x\rangle$   $y \in \{0,1\}^n$

We have  $H^{\otimes n+1} |y\rangle = H \left( \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{x \cdot y} |x\rangle \right)$

$$= \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} H \left( (-1)^{x \cdot y} |x\rangle \right) = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{x \cdot y} \left( \frac{1}{\sqrt{2}} \right)$$

$$H^{\otimes n+1} |y\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} H((-1)^{x \cdot y} |x\rangle)$$

$$= \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{x \cdot y} \left( \frac{1}{\sqrt{2}} \left( (-1)^{x'_0} |0\rangle + (-1)^{x'_1} |1\rangle \right) \right)$$

$\sum_{x' \in \{0,1\}} (-1)^{x'_x} |x'\rangle$

$$= \frac{1}{\sqrt{2^{n+1}}} \sum_{x \in \{0,1\}^{n+1}} (-1)^{xy} |x\rangle$$



## Question 2,

1) We have from Q1.1;  $H^{\otimes n} |0\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{x \cdot 0} |x\rangle$

$$\rightarrow \varphi_1 = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle$$

2)  $\varphi_2 = U_f \varphi_1 = U_f \left( \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle \right)$   
 $= \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} |x\rangle$  ;  $f: \{0,1\}^n \rightarrow \mathbb{R}$

3)  $\varphi_3 = H^{\otimes n} \varphi_2 = H^{\otimes n} \left( \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} |x\rangle \right)$   
 $= \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} H^{\otimes n} |x\rangle$

$$= \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} \sum_{y \in \{0,1\}^n} \left( \frac{1}{\sqrt{2^n}} (-1)^{y \cdot x} |y\rangle \right)$$

$$= \frac{1}{2^n} \sum_{y \in \{0,1\}^n} \left( \sum_{x \in \{0,1\}^n} (-1)^{f(x) \oplus y \cdot x} \right) |y\rangle$$

type in Q1.5?

4, For all  $y$ ,  $\sum_{x \in \{0,1\}^n} (-1)^{f(x) \oplus y \cdot x} = \sum_{x \in \{0,1\}^n} (-1)^{x \cdot (S \oplus y)}$   
 $= \sum_{x \in \{0,1\}^n} (-1)^{x \cdot (S \oplus y)}$



5) For  $y = s$ ,

$$\sum_{x \in \{0,1\}^n} (-1)^{x \cdot (s \oplus s)} = \sum_{x \in \{0,1\}^n} (-1)^{x \cdot 0} \quad \left( \begin{array}{l} \text{since } s \oplus s = s + s \pmod{2} \\ = 2s \pmod{2} = 0 \end{array} \right)$$

$$= \sum_{x \in \{0,1\}^n} (-1)^0 = \sum_{x \in \{0,1\}^n} 1 = 2^n$$

6) For  $s \neq y \rightarrow s \oplus y = s + y \pmod{2} = 1$

$$\sum_{x \in \{0,1\}^n} (-1)^{(s \oplus y) \cdot x} = \sum_{x \in \{0,1\}^n} (-1)^{x \cdot 1} = \sum_{x \in \{0,1\}^n} (-1)^x$$

$$\text{If } x = 0 \rightarrow (-1)^0 = 1$$

$$x = 1 \rightarrow (-1)^1 = -1$$

Thus, the numbers will be canceled out  $\rightarrow$  the result is 0.

$$7) |\psi\rangle = \frac{1}{2^n} \sum_{y \in \{0,1\}^n} \left( \sum_{x \in \{0,1\}^n} (-1)^{x \cdot (s \oplus y)} \right) |y\rangle \quad (\text{from part 4})$$

$$= \frac{1}{2^n} \sum_{y \in \{0,1\}^n} \left( \sum_{y=s} + \sum_{y \neq s} \right) |y\rangle$$

$$\stackrel{\text{TOL}}{=} \frac{1}{2^n} \sum_{y \in \{0,1\}^n} (2^n + 0) |y\rangle \quad (\text{from part 5 \& 6})$$

$\Rightarrow |\psi\rangle = |s\rangle$  So after the measurement, we have at each qubit  $q_i$  measured as  $s_i$ , where  $f(x) = x \cdot s = x_1 \cdot s_1 \oplus x_2 \cdot s_2 \oplus \dots \oplus x_n \cdot s_n$



5) For  $y=s$ ,

$$\sum_{x \in \{0,1\}^n} (-1)^{x \cdot (s \oplus s)} = \sum_{x \in \{0,1\}^n} (-1)^{x \cdot 0} \quad \left( \text{Since } s \oplus s = s + s \bmod 2 = 0 \right)$$

$$= \sum_{x \in \{0,1\}^n} (-1)^0 = \sum_{x \in \{0,1\}^n} 1 = 2^n$$

6) For  $s \neq y \rightarrow s \oplus y = s + y \bmod 2 = 1$

$$\sum_{x \in \{0,1\}^n} (-1)^{(s \oplus y) \cdot x} = \sum_{x \in \{0,1\}^n} (-1)^{x \cdot 1} = \sum_{x \in \{0,1\}^n} (-1)^x$$

$$\text{If } x=0 \rightarrow (-1)^0 = 1$$

$$x=1 \rightarrow (-1)^1 = -1$$

Thus, the numbers will be canceled out  $\rightarrow$  the result is 0.

$$7) |\psi\rangle = \frac{1}{2^n} \sum_{y \in \{0,1\}^n} \left( \sum_{x \in \{0,1\}^n} (-1)^{x \cdot (s \oplus y)} \right) |y\rangle \quad (\text{from part 4})$$

$$= \frac{1}{2^n} \sum_{y \in \{0,1\}^n} \left( \sum_{y=s} + \sum_{y \neq s} \right) |y\rangle$$

$$\stackrel{\text{TOL}}{=} \frac{1}{2^n} \sum_{y \in \{0,1\}^n} \left( 2^n + 0 \right) |y\rangle \quad \underset{s}{=} \quad (\text{from part 5 \& 6})$$