Question 1

We need to compute the eigen values and eigen vectors of the following matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

The characteristic polynomial is.

$$\begin{bmatrix} (1-\lambda) & 2 & 3 \\ 4 & (5-\lambda) & 6 \\ 7 & 8 & (9-\lambda) \end{bmatrix} = (1-\lambda)\left[(5-\lambda)(9-\lambda) - 48\right] - 2\left[4(9-\lambda) - 42\right] + 3\left[32 - 7(5-\lambda)\right]$$

$$(1-\lambda)\left[45+\lambda^2-14\lambda-48\right] - 2(36-4\lambda-42) + 3\left[32-35+7\lambda\right]$$

$$= (1-\lambda)\left[\lambda^2-14\lambda-3\right] - 2(-4\lambda-6) + 3(7\lambda-3)$$

$$= \lambda^2-14\lambda-3-\lambda^3+14\lambda^2+3\lambda+8\lambda+12+21\lambda-9$$

$$= -\lambda^3+15\lambda^2+18\lambda$$

$$= -\lambda(\lambda^2-15\lambda-18)$$

$$\lambda^2-15\lambda-18=0.$$

$$D = b^2-4ac = 225+72 = 297. = 3\times3\times33$$

$$\sqrt{D}. = 3\sqrt{33}$$

$$\lambda = \frac{15\pm3\sqrt{33}}{2.}$$

Hence, the three λ values are.

$$\lambda = 0.$$

$$\lambda = \frac{15+3\sqrt{33}}{2}$$

$$\lambda = \frac{15-3\sqrt{33}}{2.}$$

$\Rightarrow$ for $\lambda = 0$.

$A - \lambda I = 0$.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{array}\right] \quad R_2 - 4R_1 = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \\ 7 & 8 & 9 & 0 \end{array}\right]$$

$R_3 \Rightarrow$

$$R_3 - 7R_1 \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & -6 & -12 & 0 \end{array}\right] \xrightarrow{R_2 = R_2/(-3)} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -6 & -12 & 0 \end{array}\right]$$

$$R_3 \rightarrow R_3 + 6R_2 \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 - 2R_2 \rightarrow R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

$x_1 - x_3 = 0. \Rightarrow x_1 = x_3.$

$x_2 + 2x_3 = 0. \Rightarrow x_2 = -2x_3$

hence

$$X = \text{span}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$\Rightarrow$ for $\lambda = \dfrac{15 - 3\sqrt{33}}{2}$

$$A - \lambda I = \begin{bmatrix} \left(\frac{3\sqrt{33} - 13}{2}\right) & 2 & 3 \\ 4 & \left(\frac{3\sqrt{33} - 5}{2}\right) & 6 \\ 7 & 8 & \left(\frac{3\sqrt{33} + 3}{2}\right) \end{bmatrix} \longrightarrow R_1 \rightarrow \left(\frac{3\sqrt{33} + 13}{64}\right) R_1$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & \frac{3\sqrt{33} + 13}{32} & \frac{9\sqrt{33} + 39}{64} & 0 \\ 4 & (3\sqrt{33} - 5)/2 & 6 & 0 \\ 7 & 8 & (3\sqrt{33} + 3)/2 & 0. \end{array}\right]$$

$$R_2 - 4R_1 \longrightarrow \left[\begin{array}{ccc|c} 1 & \frac{(3\sqrt{33}+13)}{32} & \frac{(9\sqrt{33}+39)}{64} & 0 \\ 0 & (9\sqrt{33}-33)/8 & (-9\sqrt{33}+57)/16 & 0 \\ 7 & 8 & (3\sqrt{33}+3)/2 & 0 \end{array}\right]$$

$$R_3 - 7R_1 \longrightarrow \left[\begin{array}{ccc|c} 1 & (3\sqrt{33}+13)/32 & (9\sqrt{33}+39)/64 & 0 \\ 0 & (9\sqrt{33}-33)/8 & (-9\sqrt{33}+57)/16 & 0 \\ 0 & (-21\sqrt{33}+165)/2 & (33\sqrt{33}-177)/64 & 0 \end{array}\right]$$

$$R_2 \to R_2\left(\frac{3\sqrt{33}+11}{66}\right) \longrightarrow \left[\begin{array}{ccc|c} 1 & (3\sqrt{33}+13)/32 & (9\sqrt{33}+39)/64 & 0 \\ 0 & 1 & (3\sqrt{33}-11)/44 & 0 \\ 0 & (-21\sqrt{33}+165)/32 & \frac{(33\sqrt{33}-177)}{64} & 0 \end{array}\right]$$

$$R_3 + \left(\frac{21\sqrt{33}-165}{32}\right)R_2 \longrightarrow \left[\begin{array}{ccc|c} 1 & (3\sqrt{33}+13)/32 & (9\sqrt{33}+39)/64 & 0 \\ 0 & 1 & (3\sqrt{33}-11)/44 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_1 \to R_1 - (3\sqrt{33}+13)/32\, R_2 \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & (3\sqrt{33}+11)/22 & 0 \\ 0 & 1 & (3\sqrt{33}-11)/44 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 + \frac{3\sqrt{33}+11}{22}\cdot x_3 = 0.$$
$$x_2 + \frac{3\sqrt{33}-11}{44}x_3 = 0.$$
$$\Rightarrow \quad X = \text{span}\begin{pmatrix} \frac{-3\sqrt{33}-11}{22} \\ \frac{-3\sqrt{33}+11}{44} \\ 1 \end{pmatrix}$$

$\Rightarrow$ for $\lambda = (3\sqrt{33}+15)/2$

$A - \lambda I = 0.$

$$\left[\begin{array}{ccc|c} (-3\sqrt{33}-13)/2 & 2 & 3. & 0 \\ 4 & (-3\sqrt{33}-5)/2 & 6 & 0 \\ 7 & 8 & \frac{(-3\sqrt{33}+3)}{2} & 0. \end{array}\right] \longrightarrow R_1 \to R_1 \times \left(\frac{-3\sqrt{33}+13}{64}\right)$$

$$\left[\begin{array}{ccc|c} 1 & \frac{(-3\sqrt{33}+13)}{32} & \frac{-9\sqrt{33}+39}{64} & 0 \\ 4 & (-3\sqrt{33}-5)/2 & 6 & 0 \\ 7 & 8 & \frac{(-3\sqrt{33}+3)}{2} & 0 \end{array}\right] \longrightarrow R_2 \to R_2 - 4R_1$$

$$\left[\begin{array}{ccc|c} 1 & (-3\sqrt{33}+13)/32 & \frac{(-9\sqrt{33}+39)}{64} & 0 \\ 0 & \frac{(-9\sqrt{33}-33)}{8} & \frac{(9\sqrt{33}+57)}{16} & 0 \\ 7 & 8 & (-3\sqrt{33}+3)/2 & 0 \end{array}\right] \longrightarrow R_3 - 7R_1 \to R_3$$

$$\left[\begin{array}{ccc|c} 1 & (-3\sqrt{33}+13)/32 & (-9\sqrt{33}+39)/64 & 0 \\ 0 & (-9\sqrt{33}-33)/8 & (9\sqrt{33}+57)/16 & 0 \\ 0 & (21\sqrt{33}+165)/32 & -33\sqrt{33}-177/64 & 0 \end{array}\right] \longrightarrow R_2 \to R_2 \times \left(\frac{-3\sqrt{33}+11}{66}\right)$$

$$\left[\begin{array}{ccc|c} 1 & (-3\sqrt{33}+13)/32 & (-9\sqrt{33}+39)/64 & 0 \\ 0 & 1 & (-3\sqrt{33}-11)/44 & 0 \\ 0 & (21\sqrt{33}+165)/32 & (-33\sqrt{33}-177)/64 & 0 \end{array}\right] \longrightarrow R_3 - \frac{21\sqrt{33}+165}{32} R_2 \to R_3$$

$$\left[\begin{array}{ccc|c} 1 & (-3\sqrt{33}+13)/32 & (-9\sqrt{33}+39)/64 & 0 \\ 0 & 1 & (-3\sqrt{33}-11)/44 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow R_1 + (3\sqrt{33}-13)/32\, R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & (-3\sqrt{33}+11)/22 & 0 \\ 0 & 1 & (-3\sqrt{33}-11)/44 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

$$x_1 - (3\sqrt{33}-11)/22\, x_3 = 0.$$
$$x_2 - (3\sqrt{33}+11)/22\, x_3 = 0.$$

$$X = \operatorname{Span}\begin{pmatrix} (3\sqrt{33}-11)/22 \\ (3\sqrt{33}+11)/22 \\ 1 \end{pmatrix}.$$

⇒ DETERMINANT OF A.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = 1(45-48) - 2(36-42) + 3(32-35).$$

Hence determinant = 0.

$$-3 + 12 - 9 = \underline{\underline{0}}$$

⇒ TRACE OF MATRIX

Sum of diagonals = sum of eigen values

$$15 = 0 - \frac{3\sqrt{33}+15}{2} + \frac{3\sqrt{33}+15}{2} = \underline{\underline{15}}$$

⇒ RANK OF MATRIX.

Rank is the number of non-zero eigen values we get. Hence the rank here is $\underline{\underline{2}}$.

→ The determinant is the product of the eigen values, and since one is 0, the determinant here is 0.

⇒ The trace of the matrix is the sum of eigen values.

⇒ The rank of the matrix is the number of non-zero eigen values.

Question 2

i) Dimensions of matrix A.

$y_i = Ax_i$

$y_i = p\times 1 \qquad A = d\times b \qquad x_i = q\times 1$

$d\times 1 = p\times 1$

$\Rightarrow d = p$

For Ax_i to be possible.

$b = q$

Hence A is a $p\times q$ matrix.

ii) Can there exist a matrix A such that the Euclidean distance between y_1 & y_2 is same as x_1 & x_2

Yes, the euclidean distance between x_1 & x_2 can be preserved even after linear transformation. For this, the matrix A should be ~~the~~ orthogonal matrix, ie $AA^T = I$

$Y_1 = Ax_1$

$Y_2 = Ax_2$

$\|x_1\|^2 = x_1 . x_1 = x_1^T x_1$

$\|Ax_1\|^2 = Ax_1 . Ax_1 = (Ax_1)^T Ax_1$

$x_1^T A^T A x_1$

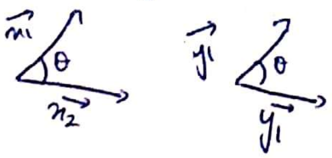
$= x_1^T I x_1 = x_1^T x$

$\therefore \|Ax_1\|^2 = \|x_1\|^2 \quad (1)$

Similarly $\|Ax_2\|^2 = \|x_2\|^2 \quad (2)$

From (1) & (2)

we can say that the lengths are preserved



$$\cos\theta = \frac{x_1 . x_2}{\|x_1\| \|x_2\|}$$

$$\cos\alpha = \frac{y_1 . y_2}{\|y_1\| \|y_2\|}$$

$$= \frac{Ax_1 . Ax_2}{\|Ax_1\| \|Ax_2\|} = \frac{(Ax_1)^T Ax_2}{\|x_1\| \|x_2\|}$$

$$\frac{x_1^T A^T A x_2}{\|x_1\|\|x_2\|}$$

$$\frac{x_1^T x_2}{\|x_1\|\|x_2\|} = \frac{x_1 x_2}{\|x_1\|\|x_2\|} = \cos\theta$$

Angle is preserved.

As angle between the vectors and lengths of vectors are preserved. the distances between x_1 & x_2 will also be preserved.

In case A is not a matrix with $n\times n$ dimension, and with the dimension $m\times n$ dimensions.

We need $A^T A = I$ for the distance between x_1 & x_2 to be preserved.

ie A should be semi-orthogonal (left-invertible)

ie $m > n$. for ($A_{m\times n}$)

iii) a) $A_{p\times q}$, $p=2$, $q=2$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

$A^T A = I$

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$a^2+c^2 = 1$, $b^2+d^2 = 1$

$ab+cd = 0$ $\quad ab = -cd.$

$$A = \begin{bmatrix} 3/5 & 4/5 \\ 4/5 & -3/5 \end{bmatrix}$$

b) $A_{p\times q}$, $p=1$, $q=2$.

Here A is not a square matrix.

For the euclidean distance to be preserved, A should be a left-invertible matrix.

ie $A^T A = I$

for that $p > q$ is required.

but here $p < q$

Hence there are no examples feasible $= A_{1\times 2}$

c $p = 2$, $q = 4$.

Same as in the case of (b). Here $p < q$.

Hence there are no example feasible for $A_{2\times 4}$.

Question 3

line: $w_1 x_1 + w_2 x_2 + w_3 = 0.$

a. Given N points on the line, $D = \{x_i = [x_1^i, x_2^i]^T\}$.

$$w_1 x_1^i + w_2 x_2^i + w_3 = 0.$$

$$\frac{1}{N}\sum_i (w_1 x_1^i + w_2 x_2^i + w_3) = 0.$$

$$w_1 \mu_1 + w_2 \mu_2 + w_3 = 0$$

(Hence, $[\mu_1 - \mu_2]^T$ lies on the line)

$$A' = \begin{bmatrix} x_1^1 - \mu_1 \\ x_2^1 - \mu_2 \end{bmatrix} \begin{bmatrix} x_1^1 - \mu_1 & x_2^1 - \mu_2 \end{bmatrix} = \begin{bmatrix} (x_1^1-\mu_1)^2 & (x_1^1-\mu_1)(x_2^1-\mu_2) \\ (x_1^1-\mu_1)(x_2^1-\mu_2) & (x_2^1-\mu_2)^2 \end{bmatrix}$$

$$R_1 \longrightarrow R_1 / (x_1^1-\mu_1)^2 = \begin{bmatrix} 1 & (x_2^1-\mu_2)/(x_1^1-\mu_1) \\ (x_1^1-\mu_1)(x_2^1-\mu_2) & (x_2^1-\mu_2)^2 \end{bmatrix}$$

$$R_2 \longrightarrow R_2/(x_1^1-\mu_1) - (x_2^1-\mu_2) R_1 = \begin{bmatrix} 1 & (x_2^1-\mu_2)/(x_1^1-\mu_1) \\ 0 & 0 \end{bmatrix}$$

The number of eigen values (non-zero) gives the rank, ie rank$(A') = 1$.

$$A = \frac{1}{N}\sum_{i=1}^{N} \begin{bmatrix} x_1^i - \mu_1 \\ x_2^i - \mu_2 \end{bmatrix} \begin{bmatrix} x_1^i - \mu_1 & x_2^i - \mu_2 \end{bmatrix}.$$

$$= \frac{1}{N}\sum_{i=1}^{N} \begin{bmatrix} (x_1^i-\mu_1)^2 & (x_1^i-\mu_1)(x_2^i-\mu_2) \\ (x_1^i-\mu_1)(x_2^i-\mu_2) & (x_2^i-\mu_2)^2 \end{bmatrix}$$

$$R_1 \rightarrow R_1/(x_1^i-\mu_1)^2 = \frac{1}{N}\sum_{i=1}^{N} \begin{bmatrix} 1 & (x_2^i-\mu_2)/(x_1^i-\mu_1) \\ (x_1^i-\mu_1)\times(x_2^i-\mu_2) & (x_2^i-\mu_2)^2 \end{bmatrix}.$$

$$R_2 \longrightarrow R_2/(x_1^i-\mu_1) - (x_2^i-\mu_2) R_2 \longrightarrow \frac{1}{N}\sum_{i=1}^{N} \begin{bmatrix} 1 & (x_2^i-\mu_2)/(x_1^i-\mu_1) \\ 0 & 0 \end{bmatrix}$$

The number of non-zero eigen values = Rank$(A) = 1$

The eigen vectors of covariance matrix sorted in descending order of their corresponding eigen values represent the direction/component along which the data is in the order of principal components.
Therefore variance is maximized along these vectors. Here, since all data is on the same line, rank = 1, and the max variance is along the line.

b). $B' = \begin{bmatrix} x_1' - \mu_1 \\ x_2' - \mu_2 \end{bmatrix} \begin{bmatrix} x_1' - \mu_1 & x_2' - \mu_2 \end{bmatrix}$

Follow the same steps as you've done for A'. We get B' as.

$$\begin{bmatrix} 1 & (x_2' - \mu_2)/(x_1' - \mu_1) \\ 0 & 0 \end{bmatrix} \Rightarrow \text{rank}(B') = 1 \text{ (no. of non-zero eigen values)}$$

$$B = \frac{1}{N} \sum_{i=1}^{N} \begin{bmatrix} x_1^i - \mu_1 \\ x_2^i - \mu_2 \end{bmatrix} \begin{bmatrix} x_1^i - \mu_1, & x_2^i - \mu_2 \end{bmatrix}$$

Follow the same steps you've followed for A. We get A as.

$$B = \frac{1}{N} \sum_{i=1}^{N} \begin{bmatrix} 1 & (x_2^i - \mu_2)/(x_1^i - \mu_1) \\ 0 & 0 \end{bmatrix} \rightarrow \text{rank}(B) = 1 \text{ (no. of non-zero eigen values)}$$

c) Consider a vector. u to be principal component and D be the data points. and μ is the mean of the points. To determine the component along which maximum information from the data is preserved.

$\max(\text{variance}(u^T D))$ $\quad u^T u = 1$

$$E[(u^T D - u^T \mu)(u^T D - u^T \mu)^T] = E[u^T (D-\mu)(D-\mu)^T u]$$

$$= E[u^T S u]$$

$\max(u^T S u)$, where $u^T u = 1$ } putting both condition together, we get the following.

$$\max[(u^T S u) - \lambda(u^T u - 1)]$$

Differentiating w.r.t u

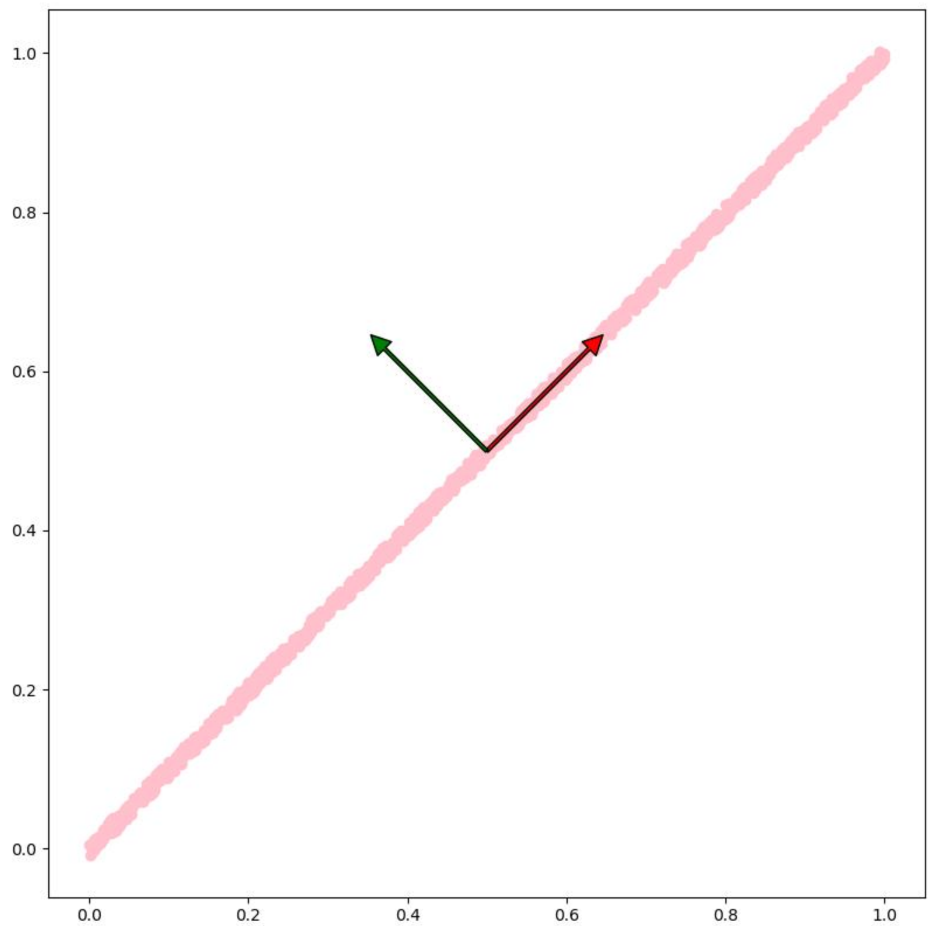
$$\Rightarrow Su = \lambda u$$

where λ = eigen value, u = eigen vector.

Eigen vector corresponding to larger eigen value captures more information than one with smaller eigen value.

Given that we have points around a line with mean μ $\Rightarrow$ we have 2 eigen vectors, perpendicular to each other.

Hence, $\Sigma \Rightarrow$ covariance matrix will have rank = 2 as there are 2 components and hence 2 non-zero eigen values.


1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0