HOMEWORK-6

PROBLEM 2

→ microscopic image (M×N array)

↳ x_{ij} are the entries in the array which indicate intensity measurements.

$P(x|BG)$ — intensity of Background cell — $N(\mu_1, \sigma_1^2)$
$P(x|FG)$ — intensity of foreground cell — $N(\mu_2, \sigma_2^2)$

if $x_{ij} < \theta$, then BG
else FG.

a) if $P(BG) = P(FG)$

for x to be classified as BG

$$P(BG)\,P(x|BG) \geq P(FG)\,P(x|FG)$$

given $P(BG) = P(FG)$
and $\sigma_1 = \sigma_2$.

$$N(\mu_1, \sigma_1^2) > N(\mu_2, \sigma_2^2)$$

$$\frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(\frac{-1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2\right) > \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left(\frac{-1}{2}\left(\frac{x-\mu_2}{\sigma_2}\right)^2\right)$$

$$= \exp\left(\frac{-1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2\right) > \exp\left(\frac{-1}{2}\left(\frac{x-\mu_2}{\sigma_2}\right)^2\right)$$

$$-\left(\frac{x-\mu_1}{\sigma_1}\right)^2 > \frac{-(x-\mu_2)^2}{\sigma_2^2}$$

$$(x-\mu_1)^2 < (x-\mu_2)^2$$

$$\mu_1^2 - 2\mu_1 x + x^2 < \mu_2^2 + x^2 - 2\mu_2 x.$$

$$\mu_1^2 - 2x\mu_1 - \mu_2^2 + 2x\mu_2 < 0$$

$$\mu_1^2 - \mu_2^2 - 2x(\mu_1 - \mu_2) < 0$$

$$(\mu_1 - \mu_2)(\mu_1 + \mu_2 - 2x) < 0.$$

$$2x(\mu_2 - \mu_1) - (\mu_1 + \mu_2)(-\mu_1 + \mu_2) < 0.$$

$$2x(\mu_2 - \mu_1) < (\mu_2 - \mu_1)(\mu_1 + \mu_2)$$

$$2x < (\mu_2 + \mu_1)$$

$$x < \frac{\mu_1 + \mu_2}{2}$$

The optimal value of $\theta^* = \frac{(\mu_1 + \mu_2)}{2}$

b) If we relax the above conditions, under what conditions of $P(BG)$, $P(FG)$, σ_1 and σ_2 can we obtain $\theta^* = \frac{(\mu_1 + \mu_2)}{2}$

We know that when $x = \theta^*$

$$P(x|BG)P(BG) = P(x|FG)P(FG)$$

$$\frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(\frac{-1}{2}\frac{(x-\mu_1)^2}{2\sigma_1^2}\right) P(BG) = \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left(\frac{-1}{2}\frac{(x-\mu_2)^2}{2\sigma_2^2}\right) P(FG)$$

$$\log\left[\frac{1}{\sigma_1}\exp\left(\frac{-(x-\mu_1)^2}{2\sigma_1^2}\right)P(BG)\right] = \log\left[\frac{1}{\sigma_2}\exp\left(-\frac{(x-\mu_2)^2}{2\sigma_2^2}\right)P(FG)\right].$$

$$-\log\sigma_1 - \frac{(x-\mu_1)^2}{2\sigma_1^2} + \log P(BG) = -\log\sigma_2 - \frac{(x-\mu_2)^2}{2\sigma_2^2} + P(FG)$$

$$\log\left(\frac{\sigma_2}{\sigma_1}\right) + \log\left(\frac{P(BG)}{P(FG)}\right) = \frac{(x-\mu_1)^2}{2\sigma_1^2} - \frac{(x-\mu_2)^2}{2\sigma_2^2}$$

let $x = \theta^* = \frac{\mu_1 + \mu_2}{2}$

$$\log\left(\frac{\sigma_2}{\sigma_1}\right) + \log\left(\frac{P(BG)}{P(FG)}\right) = \frac{\left(\frac{\mu_1+\mu_2-2\mu_1}{2}\right)^2}{2\sigma_1^2} - \frac{\left(\frac{\mu_1+\mu_2-2\mu_2}{2}\right)^2}{2\sigma_2^2}$$

$$\log\left(\frac{\sigma_2}{\sigma_1}\right) + \log\left(\frac{P(BG)}{P(FG)}\right) = \frac{\frac{(\mu_2-\mu_1)^2}{4}}{2\sigma_1^2} - \frac{\frac{(\mu_2-\mu_1)^2}{4}}{2\sigma_2^2}$$

$$\log\left(\frac{\sigma_2}{\sigma_1}\right) + \log\left(\frac{P(BG)}{P(FG)}\right) = \frac{(\mu_2-\mu_1)^2}{4}\left[\frac{1}{2\sigma_1^2} - \frac{1}{2\sigma_2^2}\right]$$

$$\log\left(\frac{\sigma_2}{\sigma_1}\right) + \log\left(\frac{P(BG)}{P(FG)}\right) = \frac{(\mu_2-\mu_1)^2}{8}\left[\frac{1}{\sigma_1^2} - \frac{1}{\sigma_2^2}\right]$$

$$8\log\left(\frac{\sigma_2}{\sigma_1}\right) + 8\log\left(\frac{P(BG)}{P(FG)}\right) = (\mu_2-\mu_1)^2\left[\frac{1}{\sigma_1^2} - \frac{1}{\sigma_2^2}\right]$$

This is the required expression to obtain $\theta^* = \frac{\mu_1+\mu_2}{2}$

c). Assuming background has an area 4 times of foreground.

$P(BG) = 4P(FG)$

$\mu_1 = 100$, $\mu_2 = 200$ and $\sigma_1 = \sigma_2$

Find θ^*

For x to be classified as BG.

$$P(BG)\,P(x|BG) > P(FG)\,P(x|FG)$$

$$4P(FG)\,N(100,\sigma_1^2) > P(FG)\,N(200,\sigma_1^2)$$

$$4 \times \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{(x-100)^2}{2\sigma_1^2}\right) > \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{(x-200)^2}{2\sigma_1^2}\right).$$

$$4\exp\left(-\frac{(x-100)^2}{2\sigma_1^2}\right) > \exp\left(-\frac{(x-200)^2}{2\sigma_1^2}\right)$$

$$\log 4 - \frac{(x-100)^2}{2\sigma_1^2} > -\frac{(x-200)^2}{2\sigma_1^2}$$

$$\log 4 > \frac{(x-100)^2}{2\sigma_1^2} - \frac{(x-200)^2}{2\sigma_1^2}$$

$$2\sigma_1^2 \log 4 > (x-100+x-200)(x-100-x+200)$$

$$2\sigma_1^2 \log 4 > (2x-300)(100)$$

$$\sigma_1^2 \log 4 > 100(x-150)$$

$$x < \frac{\sigma_1^2 \log 4}{100} + 150.$$

Hence here

$$\boxed{\theta^* = \frac{\sigma_1^2 \log 4}{100} + 150}$$