HOMEWORK 6

PROBLEM 3

w_1 has 7 points.

(0,0), (0,1), (2,0), (3,2), (3,3), (2,2), (2,0)

w_2 has 7 points.

(7,7), (8,6) (9,7) (8,10) (7,10) (8,9), (7,11)

a) Prior probabilities

$P(w_1) = 7/14 = 1/2$

$P(w_2) = 7/14 = 1/2$

b) $w_1 \longrightarrow$

$$\frac{\sum_{i=1}^{7} x_i}{7} = 1.7143 = \bar{x}$$

$$\frac{\sum_{i=1}^{7} y_i}{7} = 1.1428 = \bar{y}$$

$$\frac{\Sigma(x_i - \bar{x})(y_i - \bar{y})}{7} = 0.8809 = \text{covariance}(X, Y)$$

Mean = $\begin{bmatrix} 1.7143 & 1.1428 \end{bmatrix}$

Covariance matrix = $\begin{bmatrix} 1.5714 & 0.8809 \\ 0.8809 & 1.4762 \end{bmatrix}$

As $\text{var}(X) = \frac{\Sigma(x_i - \bar{x})^2}{7} = 1.5714$

$\text{var}(Y) = \frac{\Sigma(y_i - \bar{y})}{7} = 1.4762$

$$w_2 \rightarrow \quad \frac{\sum_{i=1}^{7} x_i}{7} = 7.7143$$

$$\frac{\sum_{i=1}^{7} y_i}{7} = 8.5714$$

$$\text{covariance} = \frac{\sum (x_i-\bar{x})(y_i-\bar{y})}{7} = -0.6429$$

Since.

$$Var(x) = \frac{\sum (x_i-\bar{x})^2}{7} = 0.5714$$

$$Var(y) = \frac{\sum (y_i-\bar{y})^2}{7} = 3.6190$$

$$\text{mean} = \begin{bmatrix} 7.7143 & 8.5714 \end{bmatrix}.$$

$$\text{Covariance matrix} = \begin{bmatrix} 0.5714 & -0.6429 \\ -0.6429 & 3.6190 \end{bmatrix}$$

c) The decision boundary is given as the function of two classes.

$v = (x, y)$.

$$\cancel{e^{-1/2 (x-\mu_1)^T C_1^{-1}(x-\mu_1)}}$$

$$P(\omega_1) \frac{e^{-1/2 (v-\mu_1)^T C_1^{-1} (v-\mu_1)}}{(\sqrt{2\pi})^2 |C_1|^{1/2}} = P(\omega_2) \frac{e^{-1/2 (v-\mu_2)^T C_2^{-1} (v-\mu_2)}}{(\sqrt{2\pi})^2 |C_2|^{1/2}}$$

$$\ln(P(\omega_1)) - \frac{1}{2}(v-\mu_1)^T C_1^{-1} (v-\mu_1) - \frac{1}{2}\ln|C_1|$$

$$= \ln(P(\omega_2)) - \frac{1}{2}(v-\mu_2)^T C_2^{-1} (v-\mu_2) - \frac{1}{2}\ln|C_2|$$

$$\ln(P(\omega_1)) = \ln(P(\omega_2))$$

$\therefore$

$$-\frac{1}{2}(v-\mu_1)^T C_1^{-1}(v-\mu_1) - \frac{1}{2}\ln|C_1| = -\frac{1}{2}(v-\mu_2)^T C_2^{-1}(v-\mu_2) - \frac{1}{2}\ln|C_2|$$

$$\frac{1}{2}(v-\mu_1)^T C_1^{-1}(v-\mu_1) + \frac{1}{2}\ln|C_1| = \frac{1}{2}(v-\mu_2)^T C_2^{-1}(v-\mu_2) + \frac{1}{2}\ln|C_2|$$

$$(v-\mu_1)^T C_1^{-1}(v-\mu_1) + \ln|C_1| = (v-\mu_2)^T C_2^{-1}(v-\mu_2) + \ln|C_2|$$

$$\ln|C_1| - \ln|C_2| = 0.0695$$

$$\begin{bmatrix} x-1.714 & y-1.142 \end{bmatrix} \begin{bmatrix} 0.956 & -0.571 \\ -0.571 & 1.018 \end{bmatrix} \begin{bmatrix} x-1.714 \\ y-1.142 \end{bmatrix} - 0.0695$$

$$\begin{bmatrix} x-1.714 & y-1.142 \end{bmatrix} \begin{bmatrix} 0.956x - 1.639 - 0.571y + 0.652 \\ -0.571x + 0.979 + 1.018y - 1.163 \end{bmatrix}$$

$$\begin{bmatrix} x-1.714 & y-1.142 \end{bmatrix} \begin{bmatrix} 0.956x - 0.571y - 0.987 \\ -0.571x + 1.018y - 0.184 \end{bmatrix} - 0.0695$$

$$\Rightarrow 0.956x^2 + 0.987x - 0.571xy - 1.638x + 1.692 + 0.979y - 0.571xy + 1.018y^2 - 0.184y + 0.652x - 1.163y + 0.2101 - 0.0695$$

$$\begin{bmatrix} x-7.7142 & y-8.5714 \end{bmatrix} \begin{bmatrix} 2.187 & 0.388 \\ 0.388 & 0.345 \end{bmatrix} \begin{bmatrix} x_1 - 7.7142 \\ y_2 - 8.5714 \end{bmatrix}$$

$$\begin{bmatrix} x-7.7142 & y-85714 \end{bmatrix} \begin{bmatrix} 2.187x - 16.871 + 0.388y - 3.308 \\ 0.388x - 2.996 + 0.345y - 2.941 \end{bmatrix}$$

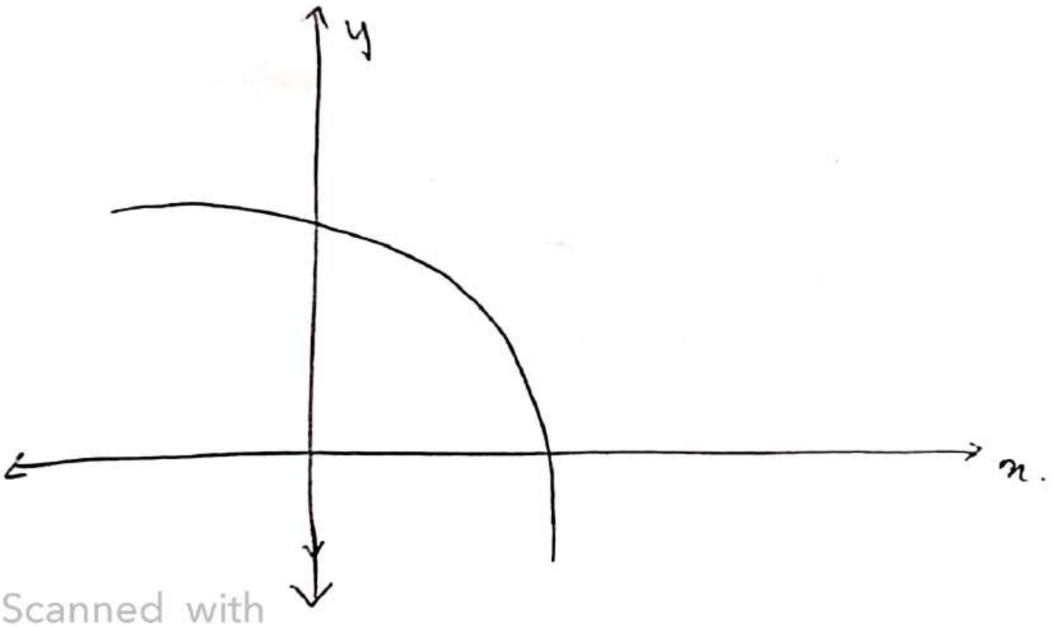
$$\begin{bmatrix} x-7.7142 & y-8.571 \end{bmatrix} \begin{bmatrix} 2.181x + 0.388y - 20.179 \\ 0.388x + 0.345y - 5.937 \end{bmatrix}$$

$$\Rightarrow \quad 2.181x^2 + 0.388xy - 20.179x - 16.811x + 155.655 - 2.996y$$
$$+ \; 0.388xy - 5.937y + 0.345y^2 - 3.329x + 50.888 - 2.9597y$$

The final equation is.

$$1.231x^2 - 0.6727y^2 + 1.9188xy - 40.3801x - 11.5247y + 204.7203 = 0$$

The decision boundary is as follows:



c) Misclassification of w_1 into w_2 is two times costlier than misclassification of w_2 into w_1

To fix this issue, we use prior probabilities

$$P(w_1) = 2P(w_2)$$
$$P(w_1) = 2/3, \quad P(w_2) = 1/3$$

From the previous equation.

$$\ln(P(w_1)) - \ln(P(w_2)) = \ln\left(\frac{P(w_1)}{P(w_2)}\right) = \ln(2)$$

The new decision boundary is

$$\left(\frac{1}{2}\right)(1.231x^2 - 0.6727y^2 + 1.9188xy - 40.3801x - 11.5747y + 204.7203) + \ln(2) = 0.$$

Hence the boundary shifts, and the size is contracted, while it's shape and orientation is preserved.

Mathematically, define a discriminant function say g:

$$g(x) = \begin{cases} w_2 & \text{when } x \in w_1 \\ w_1 & \text{when } x \in w_2 \end{cases}$$

Are our possible errors defined in terms of g.

w_1, w_2 are 2 classes in pattern space Ω with continuous PDFs $p_1(x)$ and $p_2(x)$ respectively. We can see this as dividing space Ω into Ω_1 & Ω_2 $[\Omega_1 \cup \Omega_2 = \Omega$ & $\Omega_1 \cap \Omega_2 \neq \emptyset]$

$$\text{Probability of error 1} = \int_{\Omega_2} p_1(x)\,dx$$

$$\text{Probability of error 2} = \int_{\Omega_1} p_2(x)\,dx$$

The cost of misclassification of w_1 into w_2 is twice as costly as misclassifying w_2 into w_1

C_1 = cost of w_1 into w_2 = $2/3$

C_2 = cost of w_2 into w_1 = $1/3$

now, the total expected error.

$$E = \frac{2}{3} P(\omega_1) \int_{\Omega_2} P_1(x)\,dx + \frac{1}{3} P(\omega_2) \int_{\Omega_1} P_2(x)\,dx$$

$$= \frac{1}{3}\left[2P(\omega_1) \int_{\Omega_2} p_1(x)\,dx + (1 - P(\omega_1)) \int_{\Omega_1} p_2(x)\,dx. \right]$$

which we have to minimize.

The variables here are. $P(\omega_1)$, $P(\omega_2)$. Ω_1, Ω_2 depend on P_1 & P_2

We can see from the graphs that a change in decision boundary is caused by the change in error function

```
(d) Equation: x1 = -0.779366368805849*x2 - 4.06173842404549e-5*sqrt(699416824.0*x2**2 - 9821504896.0*x2 + 62250
971881.0) + 16.4013403736799
(e) Equation: x1 = -0.779366368805849*x2 - 4.59533245287764e-7*sqrt(5464193937500.0*x2**2 - 76730507000000.0*x2
+ 481002816699879.0) + 16.4013403736799
```