Integration

Fundamental Formulae

Basic:

$$\int x^n \, \mathrm{d}x = \frac{x^{n+1}}{n+1} + C, \; n \neq -1$$

$$\int (ax+b)^n \, \mathrm{d}x = \frac{(ax+b)^{n+1}}{a(n+1)} + C \,, n \neq -1$$

$$\int \frac{1}{x} \, \mathrm{d}x = \log|x| + C$$

$$\int \frac{1}{ax+b} \, \mathrm{d}x = \frac{\log|ax+b|}{a} + C$$

$$\int \mathrm{e}^x \, \mathrm{d}x = \mathrm{e}^x + C$$

$$\int a^x \, \mathrm{d}x = \frac{a^x}{\log a} + C$$

Special Forms:

$$\int [f(x)]^n \; f'(x) \, \mathrm{d}x = \frac{[f(x)]^{n+1}}{n+1} + C \,, n \neq -1$$

$$\int \frac{f'(x)}{f(x)} \, \mathrm{d}x = \log|f(x)| + C$$

$$\int \frac{f'(x)}{\sqrt{f(x)}} \, \mathrm{d}x = 2\sqrt{f(x)} + C$$

Trigonometry:

$$\int \sin x \, \mathrm{d}x = -\cos x + C$$

$$\int \cos x \, \mathrm{d}x = \sin x + C$$

$$\int \sec^2 x \, \mathrm{d}x = \tan x + C$$

$$\int \csc^2 x \, \mathrm{d}x = -\cot x + C$$

$$\int \sec x \tan x \, \mathrm{d}x = \sec x + C$$

$$\int \csc x \cot x \, \mathrm{d}x = -\csc x + C$$

$$\int \tan x \, \mathrm{d}x = -\log|\cos x| + C = \log|\sec x| + C$$

$$\int \cot x \, \mathrm{d}x = \log|\sin x| + C$$

$$\int \csc x \, \mathrm{d}x = \log|\csc x - \cot x| + C = \log\left|\tan \frac{x}{2}\right| + C$$

$$\int \sec x \, \mathrm{d}x = \log|\sec x + \tan x| + C = \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C$$

Integration By Parts:

$$\int u \cdot v \, \mathrm{d}x = u \int v \, \mathrm{d}x - \int \frac{\mathrm{d}}{\mathrm{d}x} u \textstyle\int v \, \mathrm{d}x$$

To decide first function use: **L**ogarithmic, **I**nverse Trig, **A**lgebraic, **T**rigo, **E**xponential

For integrals of the form $\int e^{ax} \cos bx$ or $\int e^{ax} \sin bx$:
Integrate by parts taking e^{ax} as the first function, and integrate by parts again, substituting the original integral as I and then simplifying.

$$\int e^x \cdot [f(x) + f'(x)] = e^x f(x) \, \mathrm{d}x$$

Special Integrals:

$$\int \frac{1}{x^2+a^2} \, \mathrm{d}x = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$$

$$\int \frac{1}{x^2-a^2} \, \mathrm{d}x = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + C$$

$$\int \frac{1}{a^2-x^2} \, \mathrm{d}x = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + C$$

For integrals of the form $\int \frac{x^2+1}{x^4+kx^2+1}\,\mathrm{d}x$ or $\int \frac{x^2-1}{x^4+kx^2+1}\,\mathrm{d}x$,

Divide the nummerator and denominator by x^2 and put $\left(x-\frac{1}{x}\right)=t$ and $\left(x+\frac{1}{x}\right)=t$ respectively.

For integrals of the form $\int \frac{1}{ax^2+bx+c}\,\mathrm{d}x$

1. Make coefficient of x^2 unity
2. Complete the square by Adding and Subtracting the square of half the coefficient of x ($^{b^2}/_4$)
3. Use standard form for denominator as $(x+\alpha)^2 \pm \beta^2$

For integrals of the form $\int \frac{px+q}{ax^2+bx+c}\,\mathrm{d}x$,

1. Put $px+q$ as $\frac{\mathrm{d}}{\mathrm{d}x}(\mathit{Denominator})\cdot k_1 \pm k_2$ where k_1 and k_2 are constants.
2. Split numerator taking individual denominators and evaluate

Irrational Forms:

$$\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,\mathrm{d}x = \log\left|x+\sqrt{x^2+a^2}\right| + C$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,\mathrm{d}x = \log\left|x+\sqrt{x^2-a^2}\right| + C$$

For integrals of the form $\int \frac{1}{\sqrt{ax^2+bx+c}}\,\mathrm{d}x$,

1. Make coefficient of x^2 unity
2. Complete the square by Adding and Subtracting the square of half the coefficient of x ($^{b^2}/_4$)
3. Use standard form for denominator as $\sqrt{(x+\alpha)^2 \pm \beta^2}$

For integrals of the form $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,\mathrm{d}x$,

1. Put $px+q$ as $\frac{\mathrm{d}}{\mathrm{d}x}(\mathit{Denominator})\cdot k_1 \pm k_2$ where k_1 and k_2 are constants.
2. Split numerator taking individual denominators and evaluate

For integrals of the form $\int \frac{1}{a+b\cos x}\,\mathrm{d}x$, $\int \frac{1}{a+b\sin x}\,\mathrm{d}x$, $\int \frac{1}{a\cos x+b\sin x}\,\mathrm{d}x$

1. Put $\sin x = \frac{2\tan(^x/_2)}{1+\tan^2(^x/_2)}$, $\cos x = \frac{1-\tan^2(^x/_2)}{1+\tan^2(^x/_2)}$
2. Replace $1+\tan^2(^x/_2)$ in numerator with $\sec^2(^x/_2)$
3. Put $\tan(x/2)=t$ so that $\frac{1}{2}\sec^2(^x/_2)\mathrm{d}x = \mathrm{d}t$
4. Evaluate using standard integrals

More Irrational Integrals:

$$\int \sqrt{a^2-x^2}\,\mathrm{d}x = \frac{x}{2}\cdot\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \sqrt{x^2+a^2}\,\mathrm{d}x = \frac{x}{2}\cdot\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + C$$

$$\int \sqrt{x^2-a^2}\,\mathrm{d}x = \frac{x}{2}\cdot\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C$$