Determinants

For a triangle with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$, Area is given by $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

If $\Delta = 0$, the points are collinear.

Properties of determinants:

1. If all elements of a row or column are zero, value of the determinant is zero.

2. If any two rows or columns of a determinant are identical, result is zero.

3. $|A| = |A'|$, where A' is the transpose of A.

4. If any row of column of a determinant is multiplied by a non-zero constant k, the result is k times.

5. By interchanging two rows or columns of a determinant, the resulting determinant is the negative of the original determinant.

6. $\begin{vmatrix} a_1 + x_1 & b_1 & c_1 \\ a_2 + x_2 & b_2 & c_2 \\ a_3 + x_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x_1 & b_1 & c_1 \\ x_2 & b_2 & c_2 \\ x_3 & b_3 & c_3 \end{vmatrix}$

7. $|AB| = |A||B|$

8. If a row or column is multiplied by a non-zero constant k and added to/subtracted from another row or column, the result does not change.

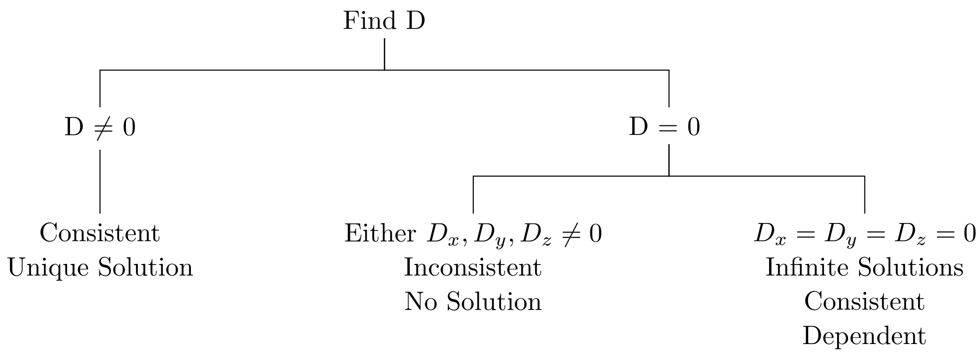
Cramer's Rule:

To solve a system of linear, non-homogenous equations using determinants.

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

$$x = \frac{D_x}{D}, y = \frac{D_y}{D}, z = \frac{D_z}{D}$$



Homogeneous Equations

Equations in which the constant term is always 0. These are always consistent and $D_x = D_y = D_z = 0$.

Find D

D $\neq$ 0
$x = y = z = 0$
Trivial Solution

D = 0
Infinite Solutions
Non-Trivial Solutions
(Includes Trivial Solutions)

Concurrency Of Lines:

For equations of the form $ax + by + c = 0$,

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Find D

D $\neq$ 0
No Solution

D = 0
Consistent
Unique Solution

Matrices

Transpose of a Matrix:

1. $(A')' = A$
2. $(A+B)' = A' + B'$
3. $(AB)' = B'A'$
4. $(kA)' = kA'$

Symmetric Matrix: $A = A'$
Skew-Symmetric Matrix: $-A = A'$

For square matrices, $A + A' =$ Symmetric and $A - A' =$ Skew-Symmetric

$A = \dfrac{A+A'}{2} + \dfrac{A-A'}{2}$

Singular Matrix: $|A| = 0$

Adjoint of a Matrix:

An adjoint matrix is the tranpose of the cofactor matrix.

For 2×2 matrix, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \text{adj}\, A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
Exchange the elements of the principal diagonal and change the sign of the rest.

$\text{adj}\,(AB) = (\text{adj}\, B)(\text{adj}\, A)$

Inverse of a Matrix:

A and B are the inverse of each other if $AB = I$
$\dfrac{\text{adj}\, A}{|A|} = A^{-1}$

$(AB)^{-1} = B^{-1}A^{-1}$

$(A')^{-1} = (A^{-1})'$

Martin's Rule:

To solve linear simultaneous equations using matrices.

$$
\begin{array}{l} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{array}
\qquad
\underset{A}{\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}}
\underset{X}{\begin{bmatrix} x \\ y \end{bmatrix}}
=
\underset{B}{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}
$$

$$AX = B \to X = A^{-1}B = \frac{(\text{adj}\, A)B}{|A|}$$

Find $|A|$

$|A| \neq 0$

Unique Solution

$|A| = 0$

Find $(\text{adj}\, A)B$

$(\text{adj}\, A)B \neq 0$

Inconsistent
No Solution

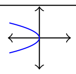
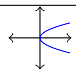
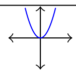
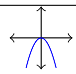
$(\text{adj}\, A)B = 0$

Infinite Solutions
Dependent

Conics

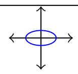
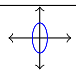
Parabola

A parabola is the locus of a point which moves so that its distance from a fixed point (Focus) is equal to its distance from a fixed straight line (Directrix). e = 1

Graph				
Type	**Right**	**Left**	**Upwards**	**Downwards**
Equation	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
Vertex	$(0, 0)$			
Focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
Tangent at Vertex	$x = 0$		$y = 0$	
Length of Latus Rectum	$4a$			
Equation of LR	$x = a$	$x = -a$	$y = a$	$y = -a$

Ellipse

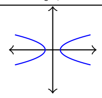
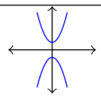
An ellipse is the locus of a point which moves so that its distance from a fixed point (Focus) bears a constant ratio (eccentricity 'e'), which is less than unity, to its distance from a fixed line (Directrix).

Graph		
Type	**Horizontal**	**Vertical**
Equation	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$
Center	$(0, 0)$	
Foci	$(\pm ae, 0)$	$(0, \pm ae)$
Major Vertices	$(\pm a, 0)$	$(0, \pm a)$
Minor Vertices	$(0, \pm b)$	$(\pm b, 0)$
Length of Major Axis	$2a$	
Length of Minor Axis	$2b$	
Equations of Directrices	$x = \pm\frac{a}{e}$	$y = \pm\frac{a}{e}$
Length of Latus Rectum	$\frac{2b^2}{a}$	

$$b^2 = a^2(1 - e^2)$$

Hyperbola

A hyperbola is the locus of a point which moves so that its distance from a fixed point (Focus) bears the a constant ratio (eccentricity 'e'), which is greater than unity, to its distance from a fixed line (Directrix).

Graph		
Type	**Horizontal**	**Vertical**
Equation	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$
Center	$(0, 0)$	
Foci	$(\pm ae, 0)$	$(0, \pm ae)$
Vertices	$(\pm a, 0)$	$(0, \pm a)$
Length of Transverse Axis	$2a$	
Length of Conjugate Axis	$2b$	
Equations of Directrices	$x = \pm\frac{a}{e}$	$y = \pm\frac{a}{e}$
Equation of Transverse Axis	$y = 0$	$x = 0$
Equation of Conjugate Axis	$x = 0$	$y = 0$
Length of Latus Rectum	$\frac{2b^2}{a}$	
Focal Radii	$ex \pm a$	$ey \pm a$

$$b^2 = a^2(e^2 - 1)$$

Tangency

Line	**Curve**	**Condition**	**Tangent**	**Point Of Contact**
$y = mx + c$	$y^2 = 4ax$	$c = \frac{a}{m}$	$y = mx + \frac{a}{m}$	$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$
$y = mx + c$	$y^2 = -4ax$	$c = \frac{-a}{m}$	$y = mx - \frac{a}{m}$	$\left(\frac{-a}{m^2}, \frac{-2a}{m}\right)$
$y = mx + c$	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$c^2 = a^2m^2 + b^2$	$y = mx \pm \sqrt{a^2m^2 + b^2}$	$\left(\frac{\pm a^2m}{\sqrt{a^2m^2 + b^2}}, \frac{\mp b^2}{\sqrt{a^2m^2 + b^2}}\right)$
$y = mx + c$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$c^2 = a^2m^2 - b^2$	$y = mx \pm \sqrt{a^2m^2 - b^2}$	$\left(\frac{\pm a^2m}{\sqrt{a^2m^2 - b^2}}, \frac{\pm b^2}{\sqrt{a^2m^2 - b^2}}\right)$

Inverse Trigonometric Functions

Function	Domain	Principal Values
$y = \sin^{-1} x$	$-1 \le x \le 1$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$y = \cos^{-1} x$	$-1 \le x \le 1$	$[0, \pi]$
$y = \tan^{-1} x$	All real numbers	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$y = \cot^{-1} x$	All real numbers	$(0, \pi)$
$y = \sec^{-1} x$	$x \ge 1$ or $x \le -1$	$(0, \pi]\,,\ y \neq \frac{\pi}{2}$
$y = \csc^{-1} x$	$x \ge 1$ or $x \le -1$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right],\ y \neq 0$

$$\sin^{-1} x = \cos^{-1} \sqrt{1-x^2} = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$
$$= \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \csc^{-1}\frac{1}{x}$$

$$\cos^{-1} x = \sin^{-1} \sqrt{1-x^2} = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$$
$$= \csc^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \sec^{-1}\frac{1}{x}$$

$$\tan^{-1} x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$$
$$= \csc^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right) = \sec^{-1}\sqrt{1+x^2} = \cot^{-1}\frac{1}{x}$$

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right),\ xy < 1$$
$$\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right),\ xy > -1$$

$$2\tan^{-1} x = \tan^{-1}\frac{2x}{1-x^2} = \sin^{-1}\frac{2x}{1+x^2} = \cos^{-1}\frac{1-x^2}{1+x^2}$$

$$2\sin^{-1} x = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$$
$$2\cos^{-1} x = \cos^{-1}\left(2x^2-1\right)$$
$$3\sin^{-1} x = \sin^{-1}\left(3x-4x^3\right)$$
$$3\cos^{-1} x = \cos^{-1}\left(4x^3-3x\right)$$
$$3\tan^{-1} x = \tan^{-1}\frac{3x-x^3}{1-3x^3}$$

$$\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$$
$$\sin^{-1} x - \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)$$
$$\cos^{-1} x + \cos^{-1} y = \cos^{-1}\left(xy - \sqrt{1-x^2}\cdot\sqrt{1-y^2}\right)$$
$$\cos^{-1} x - \cos^{-1} y = \cos^{-1}\left(xy + \sqrt{1-x^2}\cdot\sqrt{1-y^2}\right)$$

Differentiation

Basic:

$$\frac{\mathrm{d}}{\mathrm{d}x}(c) = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(ax+b)^n = na(ax+b)^{n-1}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(x) = 1$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{ax+b}\right) = \frac{a}{x\sqrt{ax+b}}$$

Trigonometric:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) = \sec^2 x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cot x) = -\csc^2 x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sec x) = \sec x \tan x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\csc x) = -\csc x \cot x$$

Logarithmic and Exponential:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log_a x) = \frac{1}{x \log_e a}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log_e x) = \frac{1}{x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(e^x) = e^x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(a^x) = a^x \log_e a$$

Inverse Trigonometric:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\cot^{-1} x\right) = \frac{-1}{1+x^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\csc^{-1} x\right) = \frac{-1}{|x|\sqrt{x^2-1}}$$

Integration

Fundamental Formulae

Basic:

$$\int x^n \, \mathrm{d}x = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$$

$$\int (ax+b)^n \, \mathrm{d}x = \frac{(ax+b)^{n+1}}{a(n+1)} + C\,, n \neq -1$$

$$\int \frac{1}{x} \, \mathrm{d}x = \log|x| + C$$

$$\int \frac{1}{ax+b} \, \mathrm{d}x = \frac{\log|ax+b|}{a} + C$$

$$\int \mathrm{e}^x \, \mathrm{d}x = \mathrm{e}^x + C$$

$$\int a^x \, \mathrm{d}x = \frac{a^x}{\log a} + C$$

Special Forms:

$$\int [f(x)]^n \ f'(x) \, \mathrm{d}x = \frac{[f(x)]^{n+1}}{n+1} + C\,, n \neq -1$$

$$\int \frac{f'(x)}{f(x)} \, \mathrm{d}x = \log|f(x)| + C$$

$$\int \frac{f'(x)}{\sqrt{f(x)}} \, \mathrm{d}x = 2\sqrt{f(x)} + C$$

Trigonometry:

$$\int \sin x \, \mathrm{d}x = -\cos x + C$$

$$\int \cos x \, \mathrm{d}x = \sin x + C$$

$$\int \sec^2 x \, \mathrm{d}x = \tan x + C$$

$$\int \csc^2 x \, \mathrm{d}x = -\cot x + C$$

$$\int \sec x \tan x \, \mathrm{d}x = \sec x + C$$

$$\int \csc x \cot x \, \mathrm{d}x = -\csc x + C$$

$$\int \tan x \, \mathrm{d}x = -\log|\cos x| + C = \log|\sec x| + C$$

$$\int \cot x \, \mathrm{d}x = \log|\sin x| + C$$

$$\int \csc x \, \mathrm{d}x = \log|\csc x - \cot x| + C = \log\left|\tan\frac{x}{2}\right| + C$$

$$\int \sec x \, \mathrm{d}x = \log|\sec x + \tan x| + C = \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C$$

Integration By Parts:

$$\int u \cdot v \, \mathrm{d}x = u \int v \, \mathrm{d}x - \int \frac{\mathrm{d}}{\mathrm{d}x} u \textstyle\int v \, \mathrm{d}x$$

To decide first function use: **L**ogarithmic, **I**nverse Trig, **A**lgebraic, **T**rigo, **E**xponential

For integrals of the form $\int e^{ax} \cos bx$ or $\int e^{ax} \sin bx$:
Integrate by parts taking e^{ax} as the first function, and integrate by parts again, substituting the original integral as I and then simplifying.

$$\int e^x \cdot [f(x) + f'(x)] = e^x f(x) \, \mathrm{d}x$$

Special Integrals:

$$\int \frac{1}{x^2+a^2} \, \mathrm{d}x = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$$

$$\int \frac{1}{x^2-a^2} \, \mathrm{d}x = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + C$$

$$\int \frac{1}{a^2-x^2} \, \mathrm{d}x = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + C$$

For integrals of the form $\int \frac{x^2+1}{x^4+kx^2+1}\,\mathrm{d}x$ or $\int \frac{x^2-1}{x^4+kx^2+1}\,\mathrm{d}x$,

Divide the nummerator and denominator by x^2 and put $\left(x-\frac{1}{x}\right)=t$ and $\left(x+\frac{1}{x}\right)=t$ respectively.

For integrals of the form $\int \frac{1}{ax^2+bx+c}\,\mathrm{d}x$

5. Make coefficient of x^2 unity
6. Complete the square by Adding and Subtracting the square of half the coefficient of x ($^{b^2}/_4$)
7. Use standard form for denominator as $(x+\alpha)^2 \pm \beta^2$

For integrals of the form $\int \frac{px+q}{ax^2+bx+c}\,\mathrm{d}x$,

1. Put $px+q$ as $\frac{\mathrm{d}}{\mathrm{d}x}(Denominator)\cdot k_1 \pm k_2$ where k_1 and k_2 are constants.
2. Split numerator taking individual denominators and evaluate

Irrational Forms:

$$\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \sin^{-1}\frac{x}{a}+C$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,\mathrm{d}x = \log\left|x+\sqrt{x^2+a^2}\right|+C$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,\mathrm{d}x = \log\left|x+\sqrt{x^2-a^2}\right|+C$$

For integrals of the form $\int \frac{1}{\sqrt{ax^2+bx+c}}\,\mathrm{d}x$,

1. Make coefficient of x^2 unity
2. Complete the square by Adding and Subtracting the square of half the coefficient of x ($^{b^2}/_4$)
3. Use standard form for denominator as $\sqrt{(x+\alpha)^2 \pm \beta^2}$

For integrals of the form $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,\mathrm{d}x$,

1. Put $px+q$ as $\frac{\mathrm{d}}{\mathrm{d}x}(Denominator)\cdot k_1 \pm k_2$ where k_1 and k_2 are constants.
2. Split numerator taking individual denominators and evaluate

For integrals of the form $\int \frac{1}{a+b\cos x}\,\mathrm{d}x$, $\int \frac{1}{a+b\sin x}\,\mathrm{d}x$, $\int \frac{1}{a\cos x+b\sin x}\,\mathrm{d}x$

1. Put $\sin x = \frac{2\tan(^x/_2)}{1+\tan^2(^x/_2)}$, $\cos x = \frac{1-\tan^2(^x/_2)}{1+\tan^2(^x/_2)}$
2. Replace $1+\tan^2(^x/_2)$ in numerator with $\sec^2(^x/_2)$
3. Put $\tan(x/2)=t$ so that $\frac{1}{2}\sec^2(^x/_2)\mathrm{d}x=\mathrm{d}t$
4. Evaluate using standard integrals

More Irrational Integrals:

$$\int \sqrt{a^2-x^2}\,\mathrm{d}x = \frac{x}{2}\cdot\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+C$$

$$\int \sqrt{x^2+a^2}\,\mathrm{d}x = \frac{x}{2}\cdot\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+C$$

$$\int \sqrt{x^2-a^2}\,\mathrm{d}x = \frac{x}{2}\cdot\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$$

Correlation and Regression

Karl Pearson's Co-efficient of Correlation:

Covariance:

$$Cov(x, y) = \frac{1}{n} \sum (x - \bar{x})(y - \bar{y}) = \frac{\sum d_x d_y}{n}$$

Correlation Co-efficient:

$$\rho(x, y) \text{ or } r = \frac{Cov(x, y)}{\sigma_x \sigma_y} = \frac{Cov(x, y)}{\sqrt{\sigma_x^2 \sigma_y^2}}$$

$$\rho(x, y) \text{ or } r = \frac{\sum d_x d_y}{n \cdot \sigma_x \sigma_y} = \frac{\sum d_x d_y}{\sqrt{(\sum d_x^2)(\sum d_y^2)}}$$