Conic Assignment

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Problem Statement **- The normal to the curve** $x_2 = 4y$ **passing (1,2) is:**

(a)x+y=3 **(b)x-y=3**
(c)x+y=1 **(d)x-y=1**

Solution

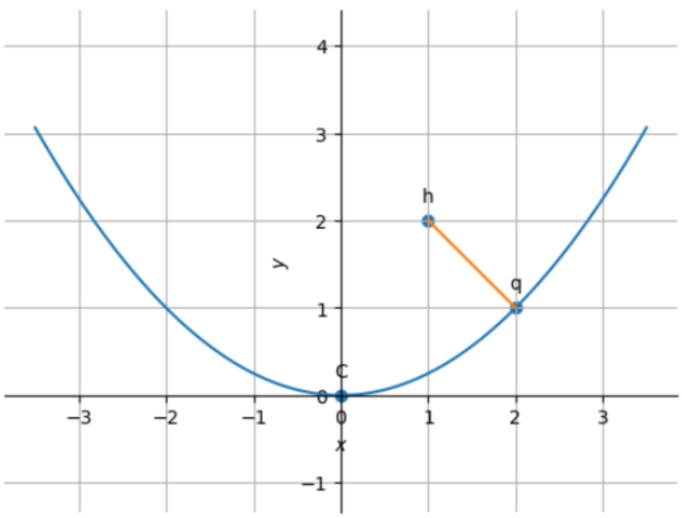

Figure 1: Tangents from A to circle through B, C and D

The given equation of parabola $x^2 = 4y$ can be written in the general quadratic form as

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1}$$

where

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{2}$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \tag{3}$$

$$f = 0 \tag{4}$$

The parabola in (1) can be expressed in standard form (center/vertex at origin, major-axis - x axis) as

$$\mathbf{y}^\top \mathbf{D}\mathbf{y} = -2\eta \mathbf{e}_1^\top \mathbf{y} \qquad |V| = 0 \tag{5}$$

where

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \quad \text{(Affine Transformation)} \tag{6}$$

$$\mathbf{P}^\top \mathbf{V}\mathbf{P} = \mathbf{D}. \quad \text{(Eigenvalue Decomposition)} \tag{7}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{8}$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix}, \quad \mathbf{P}^\top = \mathbf{P}^{-1}, \tag{9}$$

$$\eta = \mathbf{u}^\top \mathbf{p}_1 \tag{10}$$

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{11}$$

To find $\mathbf{c}$ which is the center of the parabola in (1), substitute (6) in (1)

$$(\mathbf{P}\mathbf{y} + \mathbf{c})^T \mathbf{V} (\mathbf{P}\mathbf{y} + \mathbf{c}) + 2\mathbf{u}^T (\mathbf{P}\mathbf{y} + \mathbf{c}) + f = 0, \tag{12}$$

yielding

$$\begin{aligned} \mathbf{y}^T \mathbf{P}^T \mathbf{V}\mathbf{P}\mathbf{y} + 2(\mathbf{V}\mathbf{c} + \mathbf{u})^T \mathbf{P}\mathbf{y} + \mathbf{c}^T \mathbf{V}\mathbf{c} \\ + 2\mathbf{u}^T \mathbf{c} + f = 0 \end{aligned} \tag{13}$$

From (13) and (7),

$$\begin{aligned} \mathbf{y}^T \mathbf{D}\mathbf{y} + 2(\mathbf{V}\mathbf{c} + \mathbf{u})^T \mathbf{P}\mathbf{y} + \mathbf{c}^T (\mathbf{V}\mathbf{c} + \mathbf{u}) \\ + \mathbf{u}^T \mathbf{c} + f = 0 \end{aligned} \tag{14}$$

For a parabola $|\mathbf{V}| = 0, \lambda_1 = 0$ and

$$\mathbf{V}\mathbf{p}_1 = 0, \mathbf{V}\mathbf{p}_2 = \lambda_2 \mathbf{p}_2. \tag{15}$$

where $\mathbf{p}_1, \mathbf{p}_2$ are the eigenvectors of $\mathbf{V}$ such that (7)

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix}, \tag{16}$$

Substituting (16) in (14),

$$\begin{gathered} \mathbf{y}^T \mathbf{D}\mathbf{y} + 2\left(\mathbf{c}^T \mathbf{V} + \mathbf{u}^T\right) \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} \mathbf{y} \\ + \mathbf{c}^T (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^T \mathbf{c} + f = 0 \\ \implies \mathbf{y}^T \mathbf{D}\mathbf{y} \\ + 2\left(\left(\mathbf{c}^T \mathbf{V} + \mathbf{u}^T\right) \mathbf{p}_1 \left(\mathbf{c}^T \mathbf{V} + \mathbf{u}^T\right) \mathbf{p}_2\right) \mathbf{y} \\ + \mathbf{c}^T (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^T \mathbf{c} + f = 0 \\ \implies \mathbf{y}^T \mathbf{D}\mathbf{y} \\ + 2\left(\mathbf{u}^T \mathbf{p}_1 \quad \left(\lambda_2 \mathbf{c}^T + \mathbf{u}^T\right) \mathbf{p}_2\right) \mathbf{y} \\ + \mathbf{c}^T (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^T \mathbf{c} + f = 0 \text{ from (15)} \end{gathered}$$

$$\begin{aligned} \implies \lambda_2 y_2^2 + 2\left(\mathbf{u}^T \mathbf{p}_1\right) y_1 + 2y_2 (\lambda_2 \mathbf{c} + \mathbf{u})^T \mathbf{p}_2 \\ + \mathbf{c}^T (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^T \mathbf{c} + f = 0 \end{aligned}$$

which is the equation of a parabola. Thus, (17) can be expressed as (5) by choosing

$$\eta = \mathbf{u}^T\mathbf{p}_1 \tag{17}$$

and $\mathbf{c}$ in (14) such that

$$\mathbf{P}^T\left(\mathbf{V}\mathbf{c}+\mathbf{u}\right) = \eta\begin{pmatrix}1\\0\end{pmatrix} \tag{18}$$

$$\mathbf{c}^T\left(\mathbf{V}\mathbf{c}+\mathbf{u}\right)+\mathbf{u}^T\mathbf{c}+f = 0 \tag{19}$$

Multiplying (18) by $\mathbf{P}$ yields

$$\left(\mathbf{V}\mathbf{c}+\mathbf{u}\right) = \eta\mathbf{p}_1, \tag{20}$$

which, upon substituting in (19) results in

$$\eta\mathbf{c}^T\mathbf{p}_1+\mathbf{u}^T\mathbf{c}+f = 0 \tag{21}$$

(20) and (21) can be clubbed together to obtain (22).

$$\begin{pmatrix}\mathbf{u}^\top+\eta\mathbf{p}_1^\top\\ \mathbf{V}\end{pmatrix}\mathbf{c} = \begin{pmatrix}-f\\ \eta\mathbf{p}_1-\mathbf{u}\end{pmatrix} \qquad |V| = 0 \tag{22}$$

Substituting appropriate values from (2), (3), (4), (9), and (10) into (22), the below matrix equation is obtained

$$\begin{pmatrix}0 & -4\\1 & 0\\0 & 0\end{pmatrix}\mathbf{c} = \begin{pmatrix}0\\0\\0\end{pmatrix} \tag{23}$$

$$\tag{24}$$

The augmented matrix for (23) can be expressed as

$$\left(\begin{array}{cc|c}0 & -4 & 0\\1 & 0 & 0\\0 & 0 & 0\end{array}\right) \tag{25}$$

$$\xleftrightarrow{R_1\leftrightarrow R_2}\left(\begin{array}{cc|c}1 & 0 & 0\\0 & -4 & 0\\0 & 0 & 0\end{array}\right) \tag{26}$$

$$\xleftrightarrow{-\frac{R_2}{4}\leftarrow R_2}\left(\begin{array}{cc|c}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{array}\right) \tag{27}$$

$$\implies \mathbf{c} = \begin{pmatrix}0\\0\end{pmatrix} \tag{28}$$

Let the point from which normals are drawn be $\mathbf{h}$. Then, the equation of the normal can be written as

$$\mathbf{x} = \mathbf{h}+\lambda\mathbf{m} \tag{29}$$

Say the point of intersection of (29) with the conic is $\mathbf{q}$. A tangent drawn at $\mathbf{q}$ satisfies the equation

$$\mathbf{n}^\top(\mathbf{V}\mathbf{q}+\mathbf{u}) = 0 \tag{30}$$

Where $\mathbf{n}$ is the direction vector of the tangent and is perpendicular to $\mathbf{m}$ in (29).

In general, the parameter values for points of intersection of a line given by (29) with a conic is given by

$$\lambda_i = \frac{1}{\mathbf{m}^T\mathbf{V}\mathbf{m}}\left(-\mathbf{m}^T\left(\mathbf{V}\mathbf{h}+\mathbf{u}\right) \pm\sqrt{\left[\mathbf{m}^T\left(\mathbf{V}\mathbf{h}+\mathbf{u}\right)\right]^2-\left(\mathbf{h}^T\mathbf{V}\mathbf{h}+2\mathbf{u}^T\mathbf{h}+f\right)\left(\mathbf{m}^T\mathbf{V}\mathbf{m}\right)}\right) \tag{31}$$

Using (31) and (29), the intersection point $\mathbf{q}$ can be written as

$$\mathbf{q} = \mathbf{h}+\lambda_i\mathbf{m} \tag{32}$$

Substituting (32) in (30),

$$\mathbf{n}^\top(\mathbf{V}(\mathbf{h}+\lambda_i\mathbf{m})+\mathbf{u}) = 0 \tag{33}$$

$$\implies \lambda_i\mathbf{n}^\top\mathbf{V}\mathbf{m} = -\mathbf{n}^\top(\mathbf{V}\mathbf{h}+\mathbf{u}) \tag{34}$$

Substituting value of λ_i from (31) in (34)

$$\frac{1}{\mathbf{m}^\top\mathbf{V}\mathbf{m}}\left(-\mathbf{m}^\top\left(\mathbf{V}\mathbf{h}+\mathbf{u}\right) \pm\sqrt{\left[\mathbf{m}^T\left(\mathbf{V}\mathbf{h}+\mathbf{u}\right)\right]^2-\left(\mathbf{h}^T\mathbf{V}\mathbf{h}+2\mathbf{u}^T\mathbf{h}+f\right)\left(\mathbf{m}^T\mathbf{V}\mathbf{m}\right)}\right)\mathbf{n}^\top\mathbf{V}\mathbf{m} = -\mathbf{n}^\top\left(\mathbf{V}\mathbf{h}+\mathbf{u}\right) \tag{35}$$

Rearranging the terms,

$$\pm\sqrt{\left[\mathbf{m}^T\left(\mathbf{V}\mathbf{h}+\mathbf{u}\right)\right]^2-\left(\mathbf{h}^T\mathbf{V}\mathbf{h}+2\mathbf{u}^T\mathbf{h}+f\right)\left(\mathbf{m}^T\mathbf{V}\mathbf{m}\right)}\left(\mathbf{n}^\top\mathbf{V}\mathbf{m}\right) = \left(\mathbf{V}\mathbf{h}+\mathbf{u}\right)^\top\left(\left(\mathbf{n}^\top\mathbf{V}\mathbf{m}\right)\mathbf{m}-\left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)\mathbf{n}\right) \tag{36}$$

Squaring on both sides

$$\left[\left[\mathbf{m}^T\left(\mathbf{V}\mathbf{h}+\mathbf{u}\right)\right]^2-\left(\mathbf{h}^T\mathbf{V}\mathbf{h}+2\mathbf{u}^T\mathbf{h}+f\right)\left(\mathbf{m}^T\mathbf{V}\mathbf{m}\right)\right]\left(\mathbf{n}^\top\mathbf{V}\mathbf{m}\right)^2 = \left[\left(\mathbf{V}\mathbf{h}+\mathbf{u}\right)^\top\left(\left(\mathbf{n}^\top\mathbf{V}\mathbf{m}\right)\mathbf{m}-\left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)\mathbf{n}\right)\right]^2 \tag{37}$$

If $\mathbf{n}$ is taken as $\begin{pmatrix}-\mu\\1\end{pmatrix}$, then $\mathbf{m}$ is $\begin{pmatrix}-1\\-\mu\end{pmatrix}$. Substituting these values in (37) and solving for μ, the different possible normals passing through $\mathbf{h}$ are obtained.

Thus after solving we get the following values for μ = -1, 1/2 - sqrt(3)*I/2, 1/2 + sqrt(3)*I/2

Taking μ=1 we get,

$$\mathbf{n} = \begin{pmatrix}1\\1\end{pmatrix}, \mathbf{m} = \begin{pmatrix}-1\\1\end{pmatrix}$$

By calculating λ_i from (34), we get

$$\lambda_i = -1$$

We find out $\mathbf{q}$ from (32),

$$\text{where } \mathbf{h} = \begin{pmatrix}1\\2\end{pmatrix}, \mathbf{m} = \begin{pmatrix}-1\\1\end{pmatrix}, \lambda_i = -1$$

$$\mathbf{q} = \begin{pmatrix}1\\2\end{pmatrix}+(-1)\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$$

Thus $\mathbf{q}$ satisfies Option(a) i.e. $x+y=3$

Construction

Symbol	Value	Description
$\mathbf{h}$	$\begin{pmatrix}1\\2\end{pmatrix}$	Given point through which Normal is passing
$\mathbf{q}$	$\begin{pmatrix}2\\1\end{pmatrix}$	Foot of Normal
$\mathbf{m}$	$\begin{pmatrix}-1\\1\end{pmatrix}$	Direction Vector of Normal
$\mathbf{n}$	$\begin{pmatrix}1\\1\end{pmatrix}$	Direction Vector of Tangent at $\begin{pmatrix}q\end{pmatrix}$
$\mathbf{P}$	$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$	eigenvectors of $\mathbf{V}$