Screening under Fixed-wage Employment*

Nan Zhao†

Boston University

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Abstract

I study a discrete time principal-agent model where the agent's effort and type are both private information. The wage is exogenously fixed and the principal designs a firing policy to incentivize the agent to work. In each period, the agent works on a project with binary outcomes. The high type has a higher probability of getting a good outcome if he exerts high effort and both types get a bad outcome for sure with low effort. The outcome in each period is publicly observed. I show that in the optimal contract, the principal hires the high type for sure and hires the low type with some probability at the beginning. Conditional on being hired, the low type's contract is more preferred by both types.

Keywords: Dynamic mechanism design, moral hazard, adverse selection, fixed wage

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†zhaonan@bu.edu

1 Introduction

A central question in incentive management is how to identify causes of employee performance. To provide proper incentives, it would be ideal to know whether a poor performance is due to the employee's incompetence, lack of effort, or just bad luck. But in practice, it may be hard for the employer to elicit this information. Especially in innovative industries where uncertainty prevails, "maintaining a healthy balance between tolerating productive failures and rooting out incompetence is not easy."[1] Thus, it is important to study how to optimally provide incentives in an environment where the employee's ability and effort are both unobservable.

To address this question, I construct a dynamic principal-agent model in which the agent's type and effort are both private information. In each period, the agent works on a project with binary outcomes. If he exerts high effort, there is some probability that the outcome is good. The probability of a good outcome is higher for a high-type agent. If the agent exerts low effort, then the outcome is bad for sure no matter of his type. It is costly for the agent to exert high effort. The outcomes are publicly observed. The principal has full commitment power and designs a firing policy. Specifically, the principal decides whether to fire the agent at each time based on his overall performance. I focus on the scenario where the wage is fixed and the principal only uses the firing policy to incentivize the agent.

The threat of firing is commonly used as an incentive tool in practice. In academia, assistant professors are evaluated at a certain time and will be fired or tenured afterward. Within a firm, employees of a same position receive similar wages. If someone's performance is bad, he is more likely to be fired instead of being offered a lower wage. There are several reasons why the threat of firing can be more effective in incentivizing the agent. First, it may not be feasible to punish an employee by decreasing his wage. Especially when the performance cannot be verified by a third-party, the employer needs to pay equally to everyone doing a similar job. Secondly, money transfer is not feasible within some relationships. For example, the interaction between headquarters and division managers or between government employees. Finally, for some people the psychological gain from being employed may

[1] Pisano (2019).

be more important than monetary compensation. Therefore, the firing policy can be more important than payment schemes.

I first study a benchmark where the agent's ability is public information. To induce the agent to exert high effort, the firing probability in each period depends on the overall performance of the agent. Specifically, I show that a quota mechanism is optimal. The agent is promised a quota for bad outcomes at the beginning. The quota decreases given a bad outcome and increases given a good outcome. The agent will be fired once the quota drops to zero. When it increases to a certain value, the agent will be tenured in the sense that he will never be fired no matter of future outcomes. In addition, the firing probability does not only depend on the number of good/bad outcomes, but also their orders. I show that early good outcomes are rewarded. Specifically, the agent will be better off if he achieves a good outcome today and a bad outcome tomorrow instead of the reverse.

An interesting feature is that the initial quota is not monotone in the agent's ability. When the agent's ability is very high, he is unlikely to get a bad outcome when exerting high effort. Therefore, the principal is less tolerant of his failures and the initial quota is low. On the other hand, when the agent's ability is very low, the productivity is low even if the agent always exerts high effort. As a result, the principal does not value this relationship by much and the quota for bad outcomes is also low. In conclusion, the agent is better off when his ability is not too high or too low.

When the agent's ability is unknown, a simple quota mechanism does not work. Different from a standard screening problem, the principal cannot use monetary transfer to induce truthful report. In other words, it is impossible to offer one contract with a lower wage and higher tolerance for bad outcomes and the other one with a higher wage and lower tolerance.

Even though monetary transfer is not allowed, I show that the optimal contract is separating. In the high type's contract, the agent is always hired at the beginning. Compared to the contract in the benchmark, the agent is more likely to be tenured given a sequence of good outcomes and also more likely to be fired given a sequence of bad outcomes. In other words, both reward and punishment are larger. In the low type's contract, the agent is hired only with some probability at the beginning. But conditional on being hired, he is more tolerated for bad outcomes.

Intuitively, one possible way to deter the low type from mimicking the high type is to increase reward and punishment. Since the low type has a lower probability of getting a good outcome, a contract with larger reward for good outcomes and larger punishment for bad outcomes is less desirable for him. On the other hand, since the high type values the employment more than the low type, hiring the agent with a smaller probability at the beginning deters the high type from mimicking the low type.

The form of the optimal contract is not uncommon in practice. One example is the dual-track system in many universities. Tenure-track faculties receive a higher wage but also face a higher standard of publication. Non-tenure-track faculties, on the other hand, receive a lower wage but are very unlikely to be fired.

Related Literature. This paper contributes to the growing literature on dynamic mechanism design without money transfer. Guo and Hörner (2020) study a model of dynamic adverse selection where the principal uses future allocation decisions to incentivize the agent to report truthfully in each period. The agent's private information follows a Markov chain. They show that inefficiency is backloaded and the agent is eventually fired or tenured. Li et al. (2017) get similar results by studying a dynamic model of empowerment. The principal decides whether to empower the agent at the beginning of each period and the agent chooses between his preferred project and the principal's preferred project if he is empowered. The principal's preferred project is not always available and only the agent observes this information. The main difference of my paper is that the agent has persistent private information. In addition to inducing the agent to take preferred actions in each period, the principal also needs to screen different types of the agent at the beginning. Deb et al. (2018) study a model with persistent private information. But in their model, the only objective of the principal is to differentiate different types of the agent and the principal only makes one hiring decision at the end.

Along this line of literature, Guo (2016) studies the optimal delegation of experimentation, Balseiro et al. (2019) consider dynamic mechanism design with multiple agents, Bird and Frug (2019) study a model where both the agent's preferred action and the principal's preferred action arrive stochastically over time, Escobar et al. (2019) investigate a delegation problem where the agent learns the state of the world over time, and Chen (2018) studies a continuous-time model in which the privately observed state evolves according to a Brownian motion. Some other papers study a similar setting but without commitment. Lipnowski and Ramos (2020) focus on a repeated game where the principal decides whether to delegate and the agent chooses to initiate a project or not after observing its quality. Fershtman (2017) studies dynamic delegation to multiple agents. Rantakari (2017) investigates a model where the agent can report the quality of the project in each period and the principal is able to verify it if the project is initiated.

Many other papers also study a fixed-wage environment. Chen and Ishida (2018) study a continuous-time model with both adverse selection and moral hazard. The high type's success rate depends on his effort and the low type can never achieve a success. The only contractible decision for the principal is to set a deadline. The major difference of my paper is that there is a different project in each period and both types can achieve a success. Also, the principal's firing decision may depend on the entire history of performance. Aghion and Jackson (2016) consider a model where the principal learns the agent's type over time and replaces the agent when the belief is low. The agent's only objective is to be hired for as long as possible. Kuvalekar and Lipnowski (2020) study a continuous-time game in which the agent's action affects the learning of the agent's type. The agent always wants to be employed but the principal only wants to hire a well-matched agent. Their focus is on the relationship between productivity and job insecurity. Sun and Wei (2019) extend it by introducing information asymmetry and moral hazard.

Finally, my paper is related to the literature on dynamic contracting with both moral hazard and adverse selection. Cvitanić et al. (2013) study a continuous-time model where the agent's effort affects the output process and the agent's type determines the cost of effort. They develop a general method to characterize the optimal contract in a continuous-time setting. Ulbricht (2016) considers a similar problem in a model of delegated search. In my paper, the type of the agent affects his performance and the focus is on how to separate the effect of effort from the effect of type. In addition, the model is of discrete time and there is no money transfer. Halac et al. (2016) study a model of experimentation where the type stands for the ability. Some earlier works in this literature include Gershkov and Perry

(2012), Sannikov (2007), and Sung (2005).

2 The Model

Time $t = 0, 1, 2, \ldots$ is discrete and the horizon is infinite. A principal (she) hires an agent (he) to work on a different project in each period. Conditional on being hired, the agent exerts a private effort $e_t \in \{0, 1\}$ at each time t, where $e_t = 1$ refers to high effort and $e_t = 0$ means low effort. The outcome of each project can be *good* (G) or *bad* (B). If the agent exerts low effort at time t, then the outcome of the project at t must be bad. If the effort is high, on the other hand, the outcome is good with probability θ. θ is private information of the agent and can take two values, θ_h and θ_l, with $0 \leq \theta_l < \theta_h \leq 1$. The common prior is that $\theta = \theta_h$ with probability $p \in (0, 1)$. The outcome of each project is publicly observed at the end of each period.

The principal has full commitment power and designs a firing policy at time 0. By the revelation principle, I focus on direct mechanisms where the agent truthfully report his type at the beginning. A firing policy is a collection $(x_t)_{t=0}^{\infty}$, where $x_t : \{\hat{\theta}_h, \hat{\theta}_l\} \times \{G, B\}^t \times \sigma^{t+1} \to \{0, 1\}$ is a mapping from the public history at time t to a firing decision at t. Specifically, the public history includes the agent's report of type at the beginning, outcomes of previous projects, and realizations of a public randomization device. σ denotes the set of possible realizations of this randomization device in each period and $x_t = 0$ denotes the decision of firing the agent at t. By definition, x_t depends not only on the realization of the randomization device at t, but also on all past realizations up to t. I assume throughout this paper that the firing decision is irreversible. However, it is easy to verify that all of our results remain the same if I allow for rehiring.

Conditional on being hired, the agent receives a fixed payment of $w > 0$ in each period. If the agent exerts high effort, he bears a cost of $c > 0$. The cost of exerting low effort is normalized to 0. The principal gets a net payoff of $h > 0$ if the outcome of a project is good and $l < h$ if the outcome is bad. If the principal fires the agent, both get an outside option of 0. Both parties are risk neutral with a common discount factor $\delta \in (0, 1)$. The agent's utility is therefore given by

$$u = (1-\delta)\sum_{t=0}^{\infty}\delta^t x_t(w - ce_t), \tag{1}$$

and the principal's payoff is given by

$$\pi = (1-\delta)\sum_{t=0}^{\infty}\delta^t x_t(l + (h-l)\theta e_t). \tag{2}$$

To avoid triviality, I maintain the following assumption throughout the paper.

Assumption 1. $\theta_h h + (1-\theta_h)l > 0$.

This assumption means that the relationship is valuable to the principal if the agent is a high type and exerts high effort. When it does not hold, the principal's expected payoff is always negative and it is never optimal for the principal to hire the agent.

3 Benchmark

3.1 Optimal Contract

I first study the benchmark case where the type of the agent is publicly observed. Following Spear and Srivastava (1987), I take the agent's promised utility u_t as a state variable and characterize the value function $\pi(u)$. Denote by q_t the probability of hiring the agent at t. Define u_t^G (u_t^B) as the agent's continuation utility given a good (bad) outcome. Then the principal chooses q_t, u_t^G, and u_t^B to maximize her expected payoff. The continuation utility of the agent cannot be negative, since he can always exert low effort and secure a utility of 0. On the other hand, the largest continuation utility is achieved when the agent is never fired and always exerts low effort, which is given by w. In this paper, I call the situation where the principal promises to never fire the agent as the agent being *tenured*.

The Hamilton-Jacobi-Bellman (HJB) equation is given by

$$\pi_1(u_t) = \sup_{q_t, u_t^G, u_t^B, e_t} q_t \left[(1-\delta)(h\theta e_t + l(1-\theta e_t)) + \delta\theta e_t \pi(u_t^G) + \delta(1-\theta e_t)\pi(u_t^B) \right] \tag{3}$$

$$\begin{aligned} \text{subject to} \quad & u_t^G, u_t^B \in [0, w], \;\; q_t \in [0,1], && \text{(Feasibility)} \\ & e_t \in \arg\max_{e\in[0,1]} \, (1-\delta)(w - ce) + \delta(\theta e u_t^G + (1-\theta e)u_t^B), && \text{(IC)} \\ & u_t = q_t \left[(1-\delta)(w - ce_t) + \delta(\theta e_t u_t^G + (1-\theta e_t)u_t^B) \right], && \text{(PK)} \end{aligned}$$

where $\pi(u)$ is the concavification of $\pi_1(u)$. The concavification is required since I allow for public randomization.

Since there is only one type, Assumption 1 implies that $\theta h + (1-\theta)l > 0$. Then it is in the principal's interest to not fire the agent and induce high effort. However, it is not always possible to do so. Intuitively, the only way to incentivize the agent in this environment is to promise different continuation utilities given different outcomes. Specifically, the difference between u_t^G and u_t^B must be large enough so that the incentive compatibility (IC) constraint is satisfied. Define $u_t^G - u_t^B$ as the *reward* for a good outcome. When $q_t = 1$, by the promise keeping (PK) constraint, u_t^B must decrease when u_t^G increases. In other words, to increase the reward, the principal has to increase u_t^G and decrease u_t^B at the same time. When u_t is too large or too small, the reward is constrained by the feasibility constraint and thus high effort cannot be induced. I show in the following that it is optimal to not fire the agent and induce high effort whenever possible.

Proposition 1. *In any optimal contract, there exists a stochastic time $t^* \geq 0$ such that the agent optimally exerts high effort up to t^*. At t^*, the agent is either fired or tenured. t^* is determined by the history of outcomes and is finite with probability one.*

This result is very similar to Theorem 1 in Guo and Hörner (2020). High effort is frontloaded and it is induced for as long as possible. When δ is very small or c is very large, it is optimal to have $t^* = 0$, Specifically, there exists $\bar{\delta} < 1$ and $\bar{c} > 0$ such that when $\delta < \bar{\delta}$ or $c > \bar{c}$, the principal never hires the agent or tenures the agent immediately. When $\delta > \bar{\delta}$ and $c < \bar{c}$, there exists $u_1 = (1-\delta)w$ and $u_2 = w - \frac{(1-\delta)c}{\theta} > u_1$ such that the value function is linear on $[0, u_1]$ and $[u_2, w]$ and concave in between. The optimal initial utility u^* is within

$[u_1, u_2]$. Whenever $u_t \in [u_1, u_2]$, the agent is induced to exert high effort. The continuation utility increases when the outcome is good and decreases when the outcome is bad. Once u_t drops below u_1, $u_t^B < 0$ and it is impossible to induce high effort with probability one. Therefore, randomization takes place and the agent is fired with a positive probability. If he is not fired, the promised utility increases to u_1 and the agent keeps exerting high effort. Similarly, once $u_t > u_2$, $u_t^G > w$ and it is not possible to induce high effort. Thus, the agent is tenured with a positive probability. If he is not tenured, the promised utility falls back to u_2 and he continues to exert high effort.

Intuitively, when δ is very small, the continuation utility tomorrow is unimportant relative to today's flow payoff. As a result, it is not possible to induce high effort. Then the principal can only choose between never hiring the agent and tenuring the agent immediately. As δ increases, it becomes easier to induce high effort and the relationship is more valuable to the principal. In the extreme case where $\delta \to 1$, $u_1 \to 0$ and $u_2 \to w$. Moreover, both u_t^G and u_t^B converge to u_t. As a result, the agent can be induced to exert high effort for an arbitrary long period of time and the principal's payoff approaches the first best.

Consider the scenario where $\delta > \bar{\delta}$ and $c < \bar{c}$ in the following. As mentioned before, when u_t is too large or too small, it is not possible to induce high effort. As a result, the principal has to randomize in these two regions. However, randomization is not optimal in general. Intuitively, the principal's payoff is maximized when high effort is induced with probability one, thus both firing and tenure are undesirable. Randomization makes the continuation utility closer to 0 or w, which shortens the time period before firing or tenure. By the same logic, the IC constraint should be binding whenever high effort is induced. In other words, the agent is always made indifferent between high effort and low effort.

It is worth noting that the value function is not differentiable when $\delta > \bar{\delta}$. By the above argument, the IC constraint is always binding. Combining it with the promise keeping constraint, I obtain $u_t^G = \frac{1}{\delta}(u_t - (1-\delta)w + \frac{(1-\delta)c}{\theta})$ and $u_t^B = \frac{1}{\delta}(u_t - (1-\delta)w)$. Let $u_t = u_2$. Then $u_t^G = w$ and $u_t^B < u_t$. Since it is optimal to not fire the agent and induce high effort, we have $\pi(u_t) = (1-\delta)(h\theta + l(1-\theta)) + \delta(\theta\pi(u_t^G) + (1-\theta)\pi(u_t^B))$. As a result, $\pi'_-(u_2) = \theta\pi'_-(w) + (1-\theta)\pi'_-(u_t^B)$. Since $\pi(u)$ is linear on $[u_2, w]$, $\pi'_-(w) = \pi'_+(u_2)$. As shown in the proof of Proposition 1, $\pi'_-(u_t^B) > \pi'_-(u_2)$. Therefore, $\pi'_-(u_2) > \pi'_+(u_2)$.

Whether there is a kink at u_1 or not depends on the relationship between c and $\delta\theta w$. Let $u_t = u_1$. Then $u_t^B = 0$ and $u_t^G = \frac{(1-\delta)c}{\delta\theta}$. If $u_t < u_t^G$, i.e., $c > \delta\theta w$, then by the same argument as above, $\pi'_-(u_1) > \pi'_+(u_1)$. If $c \leq \delta\theta w$, on the other hand, it is easy to see that $\pi(u)$ is differentiable at u_1. In addition, by the expression of $\pi(u_t)$, $\pi'(0) \geq 0$ if and only if $\theta h + (1-\theta)l \geq 0$. In other words, when $c \leq \delta\theta w$, the principal hires the agent at the beginning as long as the relationship is valuable for the principal.

In the optimal contract, randomization takes place only when the principal needs to fire the agent or tenure the agent with some probability. Since the firing decision is always public, the only assumption I need to impose regarding the public randomization is that the tenure decision is public, which is plausible in most environment.

3.2 Implementation

In the previous section, I characterize the optimal contract in terms of the continuation utility. Specifically, given today's promised utility u_t, the agent's continuation utility tomorrow evolves to u_t^G with a good outcome and u_t^B with a bad outcome. The agent is fired when $u_t = 0$ and tenured when $u_t = w$. An implementation of this contract specifies a firing and tenuring probability given any history of outcomes. An interesting feature is that not only the number of good outcomes matters, but also the timing of good outcomes. If the agent achieves a good outcome today and a bad outcome tomorrow, his utility will be higher than that when he achieves a bad outcome today and a good outcome tomorrow. Intuitively, the principal values today's payoff more than tomorrow's due to the discounting, thus early successes are rewarded. To implement it, the principal can set a quota for bad outcomes which increases over time aside from the evolvement driven by realized outcomes.

Corollary 1. *Suppose $\delta > \bar{\delta}$. There exists $n \in [0, \frac{1}{1-\delta}]$ such that the optimal contract can be implemented as follows: (i) Initially $n_0 = n$; (ii) $n_t = \frac{1}{\delta}(n_{t-1} - 1 + \frac{c}{\theta w})$ if the outcome is good and $n_t = \frac{1}{\delta}(n_{t-1} - 1)$ if the outcome is bad; (iii) When $n_t < 1$, the agent is fired with probability $1 - n_t$. If he is not fired, $n_t = 1$; (iv) When $n_t > \frac{1}{1-\delta} - \frac{c}{\theta w}$, the agent is tenured with probability $1 - (\frac{1}{1-\delta} - n_t)\frac{\theta w}{c}$. If he is not tenured, $n_t = \frac{1}{1-\delta} - \frac{c}{\theta w}$.*

Intuitively, n_t can be seen as a quota for bad outcomes. n_t is decreased by 1 every time there is a bad outcome and the agent is fired when n_t drops to 0. Each good outcome is rewarded by an increase of $\frac{c}{\theta w}$ in n_t. $\frac{1}{\delta}$ represents the increases of n_t in addition to the evolvement driven by realized outcomes, so that early successes are rewarded. When $n_t = \frac{1}{1-\delta}$, the increase in n_t is large enough to compensate the decrease from a bad outcome. Therefore, the agent will never be fired.

The principal's objective is to choose the initial quota n_0 to maximize her expected payoff. The tradeoff is straightforward. The agent is expected to be fired sooner when n_t is small and be tenured sooner when n_t is large. Both are undesirable since the principal wants to hire the agent and induce high effort. To this end, the principal should choose n_0 to maximize the expected duration before firing or tenure. Another consideration of the principal is the probability of firing and tenure in the end. If $l > 0$, firing is more undesirable than tenure. Therefore, the optimal n_0 should be higher than the one that maximizes the expected duration. If $l < 0$, on the other hand, the optimal n_0 should be decreased.

3.3 Comparative Statics

In this section, I discuss the comparative statics with respect to the agent's type θ. Define $\pi^*(\theta)$ as the principal's expected payoff in the optimal contract and $u^*(\theta)$ as the agent's expected utility in the optimal contract. I first show that the principal is better off when the agent's type is higher.

Corollary 2. $\pi^*(\theta)$ *is increasing in* θ*.*

This result is of no surprise. When the agent's type is higher, there is a larger probability that a good outcome occurs in each period. Moreover, since the agent is more likely to achieve a good outcome, it becomes easier for the principal to induce high effort. In other words, the reward $(u_t^G - u_t^B)$ can be smaller in each period. As a result, the principal can induce the agent to exert high effort for a longer period of time in expectation.

How $u^*(\theta)$ changes with θ is much more complicated. The principal chooses u^* to maximize her expected payoff. Specifically, her payoff depends on the expected duration of high effort and the probability of firing in the end. When θ changes, the optimal choice of u^* is affected by three factors. First, a higher type of agent requires a smaller reward to induce high effort. By the expression of u_t^G and u_t^B, u_t^B is independent of θ and u_t^G decreases in θ. Therefore, when θ is larger, it takes more good outcomes for the agent to be tenured. As a result, the principal has incentives to increase u^* in order to increase the expected duration of high effort. Secondly, a higher type is more likely to achieve a good outcome in each period. Consequently, he is more likely to be tenured if the contract does not change. By the same logic as above, the principal has incentives to decrease u^*. Finally, as θ becomes larger, the principal gets a larger expected payoff in each period when the agent induces high effort. Therefore, the expected duration of high effort becomes more important relative to the firing probability in the end. The effect of this factor on u^* depends on the sign of l. To see it more clearly, suppose $l > 0$ and define u^0 as the initial promised utility that maximizes the expected duration of high effort. Then u^* should be greater than u^0 because tenuring the agent is relatively more desirable than firing him. In other words, the principal is willing to increase the probability of tenure to some extent even though it decreases the expected duration of high effort. As θ increases, since the expected payoff given high effort increases in each period, the principal has incentives to lower u^* to make it closer to u^0. Similarly, if $l < 0$, then $u^* < u^0$ and the principal has incentives to increase u^* when θ increases.

There is no clear relationship between $u^*(\theta)$ and θ in general. For better intuition, we consider a special case where $l < 0$ and $c > \delta w$. Let $\underline{\theta}$ be the smallest θ such that the principal is willing to hire the agent at the beginning. The following result shows that the agent is worse off when his type is very high or very low.

Corollary 3. *Suppose $l < 0$ and $c > \delta w$. Then $u^*(\theta)$ is minimized at $\theta = \underline{\theta}$ and $\theta = 1$ over the interval $[\underline{\theta}, 1]$.*

Since the agent is always made indifferent between high effort and low effort, his expected utility can be characterized by the principal's leniency toward bad outcomes. Intuitively, when θ is small, the relationship is not so valuable to the principal. In other words, the principal's expected payoff is not very large even if the agent exerts high effort. Given $l < 0$, the main objective for the principal is to not tenure the agent. As a result, the principal is less lenient toward bad outcomes and the agent is worse off. When θ is very large, on the other hand, it is very unlikely to have a bad outcome if the agent exerts high effort.

Therefore, one bad outcome is strong evidence that the agent has shirked. Consequently, the principal will be very strict toward bad outcomes. In the extreme case where $\theta = 1$, the principal fires the agent after one bad outcome.

I depict value functions given different θ and the relationship between $u^*(\theta)$ and θ in Figure 1. The pattern is that $u^*(\theta)$ tends to increase in θ when θ is small and decrease in θ when θ is large. A direct implication of this result is that both types may have incentives to mimic the other type.

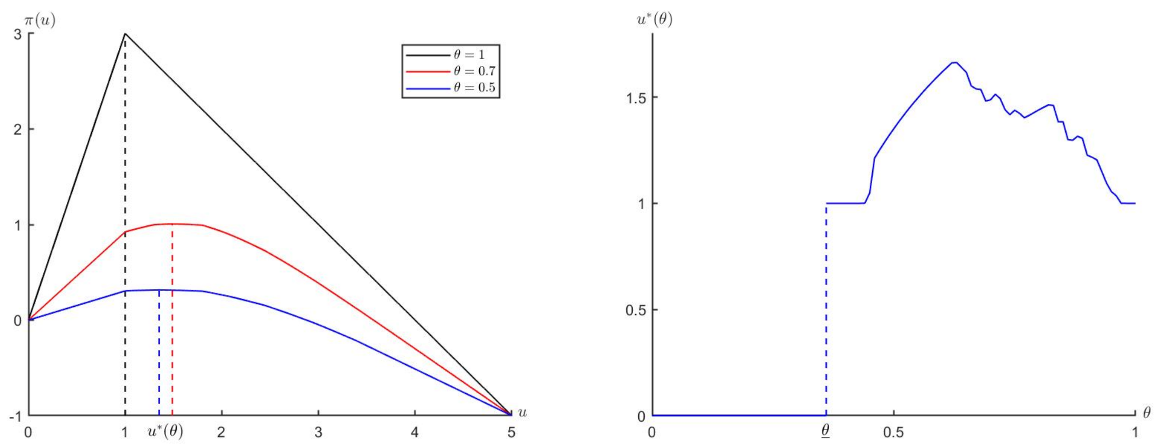

Figure 1: Comparative statics with respect to θ. $(w, c, h, l, \delta) = (5, 4, 3, -1, 0.8)$

4 General Model

In this section, I study the model where θ is private and can take two values, $\theta_l < \theta_h$. The common prior is that $\Pr(\theta = \theta_h) = p$. By the revelation principle, the principal optimally offers a pair of contracts (Γ_h, Γ_l) at the beginning of time 0. It is difficult to characterize the optimal contract under adverse selection for two reasons. First, it is not obvious what is the optimal action profile if an agent takes the other type's contract. The optimal contract in the benchmark is characterized by the agent's continuation utility. But given the same contract, the high type's continuation utility is different from that of the low type. Therefore, we do not know whether the high type will take same actions as the low type if he takes the low type's contract. Secondly, as mentioned in the previous section, it is unclear which type has incentives to mimic the other type. In other words, it is uncertain which IC constraints are binding. In fact, it is possible that both types have incentives to mimic the other type.

To deal with these issues, we need to keep track of both types' continuation utilities given any contract. Specifically, I take (u_h, u_l) as a state variable in this context, where u_h is the high type's continuation utility and u_l is the high type's continuation utility. More specifically, let u_h^l be the agent's utility when he is of high type and the contract is Γ_l and define u_h^h, u_l^h, and u_l^l similarly. Then the screening problem is formulated as

$$\sup_{u_h^h, u_h^l, u_l^h, u_l^l} \quad p_0 \pi_h(u_h^h, u_l^h) + (1-p_0)\pi_l(u_h^l, u_l^l)$$
$$\text{subject to} \quad u_h^h \geq u_h^l \text{ and } u_l^l \geq u_l^h, \tag{4}$$

where $\pi_h(u_h, u_l)$ is the principal's payoff given the contract Γ_h and an agent of high type. $\pi_l(u_h, u_l)$ is defined accordingly. Before proceeding to solve for the optimal contract, I first characterize the feasible set of the state variable (u_h, u_l).

4.1 Feasible Set

Define (u_h, u_l) to be *feasible* if there exists a contract Γ such that the high type optimally gets an expected utility of u_h and the low type gets u_l. Apparently, both u_h and u_l should be within $[0, w]$. Another simple observation is that $u_h \geq u_l$. Intuitively, the high type can always mimic the low type and gets at least the same utility. Whenever the low type exerts high effort with probability p, the high type can exert high effort with probability $p \cdot \frac{\theta_l}{\theta_h}$. Then the probability of a good outcome in each period is identical for both types. Since the cost is weakly smaller for the high type, his expected utility must be weakly larger.

Next I claim that there exists a contract Γ such that $u_h = u_l$, for any $u_l \in [0, w]$. Consider the optimal contract for the high type without adverse selection, as characterized in Proposition 1. As argued before, the high type is always indifferent between low effort and high effort until being fired or tenured. Then it is optimal for him to always exert low effort. On the other hand, if the low type takes this contract, he can also exert low effort all the time and then there is no difference between high type and low type. As a result, we must have $u_h = u_l$.

The remaining task is to characterize the upper bound of u_h given some $u_l \in [0, w]$. Define $u_h^*(u_l)$ to be the largest possible utility of the high type given the low type's utility as u_l. Since the only way to deliver a utility of 0 is to fire the agent and the only way to deliver w is to tenure the agent, we have $u_h^*(0) = 0$ and $u_h^*(w) = w$. By public randomization, $u_h^*(u_l)$ is concave in u_l. Given that $u_h^*(u_l) \in [0, w]$, $u_h^*(u_l)$ must be increasing in u_l.

Similar to the characterization of the value function in equation (3), the HJB equation for $u_h^*(u_l)$ is given by

$$\begin{aligned} u_h(u_l) = & \sup_{q, u_l^G, u_l^B} \max_{e_h \in [0,1]} q \left[(1-\delta)(w - ce_h) + \delta\theta_h e_h u_h^*(u_l^G) + \delta(1-\theta_h e_h) u_h^*(u_l^B) \right] \\ & \text{subject to} \quad u_l^G, u_l^B \in [0, w], \;\; q \in [0, 1], \\ & \qquad\qquad\quad u_l = \max_{e_l \in [0,1]} q \left[(1-\delta)(w - ce_l) + \delta\theta_l e_l u_l^G + \delta(1-\theta_l e_l) u_l^B \right], \end{aligned} \tag{5}$$

where $u_h^*(u_l)$ is the concavification of $u_h(u_l)$.

The constraints of this maximization problem are identical to that in equation (3). Namely, the low type gets a utility of u_l by choosing his best action. The objective function is instead the high type's utility when he chooses his most preferred action. Note that the low type's action does not enter the objective function directly and only his continuation utilities matter.

When the low type is induced to exert low effort, u_l^G does not affect the promise keeping constraint. Therefore, it is optimal to increase u_l^G as much as possible, i.e., to the point that the low type is indifferent between high effort and low effort or $u_l^G = w$. Thus, without loss of generality, it is optimal to induce the low type to exert high effort whenever possible.

Given that the low type exerts high effort, by the promise keeping constraint, the choice variable reduces to $u_l^G - u_l^B$. Intuitively, when the reward is larger, the high type may get a larger utility since he is more likely to achieve a good outcome. But on the other hand, when the rewards is larger, the agent is fired or tenured sooner in expectation. As a result, there is less chance for the high type to make use of his higher ability. This trade-off largely depends on the discount factor δ. When δ is small enough, the reward should be as large as possible to maximize the difference in today's utility. When δ is very large, on the other hand, it is better to make the reward smaller in each period in order to tenure the high type and fire the low type with a larger probability. The following result characterizes how $u_h^*(u_l)$ changes with δ.

Proposition 2. *For any $u_l \in (0, w)$, $u_h^*(u_l)$ increases in δ and $u_h^*(u_l) \to w$ as $\delta \to 1$.*

Intuitively, when δ is larger, the utility in the future becomes more important. As a result, the principal can learn the type of the agent for a longer period of time before making the firing or tenure decision. As $\delta \to 1$, the principal can almost perfectly learn the type of the agent before making the decision. Therefore, she can choose to tenure the high type with a probability close to 1 and fire the low type with any probability. Figure 2 illustrates $u_h^*(u_l)$ given different values of δ.

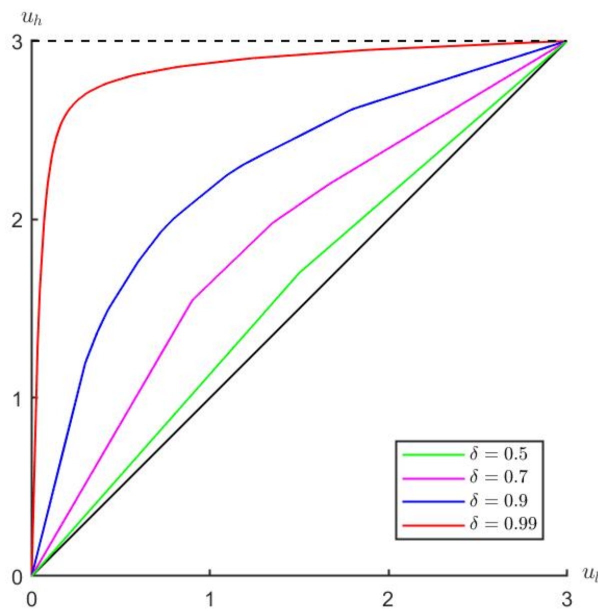

Figure 2: Feasible set given different δ. $(w, c, \theta_h, \theta_l) = (3, 2, 0.8, 0.5)$

Proposition 2 implies that essentially any (u_h, u_l) is feasible as long as $u_h \geq u_l$ and δ is large enough. But it does not mean that any pair can appear in the optimal contract. For example, it is never in the principal's interest to promise a utility of w to the high type agent, since it will give herself a utility of only l. Instead of separating the two types as much as possible, the principal's objective is to induce the agent to exert high effort for as long as possible. In the following, I am going to identify a subset which is more relevant for the optimal contract.

4.2 A Relevant Subset

Denote the optimal contracts without adverse selection as Γ_h^* and Γ_l^*, where Γ_h^* is for the high type and Γ_l^* is for the low type. In Γ_l^*, the low type is induced to be indifferent between high effort and low effort. Thus, if the high type takes Γ_l^*, he must strictly prefer high effort at some point and gets a higher utility than the low type. If the low type takes Γ_h^*, on the other hand, he strictly prefers low effort and gets the same utility as the high type. In other words, given Γ_h^*, the on-path utility pair (u_h, u_l) is represented by the lower bound of the feasible set, $u_h = u_l$. To characterize the high type's utility given Γ_l^*, I first study his optimal strategy when he takes Γ_l^*. Note that the optimal contract for the low type may not be unique. With a slight abuse of notation, I use Γ_l^* in the following to denote any of the optimal contracts which minimizes the high type's utility.

Lemma 1. *If the high type takes Γ_l^*, he always exerts high effort until being fired or tenured.*

Intuitively, the high type does better than the low type only when he exerts high effort. If the low type is induced to always exert high effort, it is natural that the high type also exerts high effort. Define $u_h^0(u_l)$ as the maximum continuation utility of the high type given the contract Γ_l^* and the low type's utility being u_l. Then $u_h^0(u_l) \leq u_h^*(u_l)$. We further show that $u_h^0(u_l)$ is concave in u_l.

Corollary 4. *$u_h^0(u_l)$ is concave in u_l.*

Figure 3 illustrates an example of $u_h^*(u_l)$ and $u_h^0(u_l)$. By the proof of Lemma 1, $(u_h^0)'_-(w) \geq \frac{\theta_l}{\theta_h}$. Therefore, the set bounded by $u_h^0(u_l)$ is a non-trivial subset of the feasible set. Intuitively, since the high type has a higher probability of achieving a good outcome, a larger reward may benefit him. Thus, if we are restricted to contracts where the low type is indifferent between high effort and low effort, then the maximum utility of the high type will be strictly lower.

By definition, if the agent is of low type and his promised utility is u_l, the principal's payoff is maximized at $u_h = u_h^0(u_l)$. When u_h is smaller, the agent can only be induced to exert high effort for a shorter period of time. In the extreme case where $u_h = u_l$, the low type never exerts high effort. If $u_h > u_h^0(u_l)$, on the other hand, the low type must be induced to

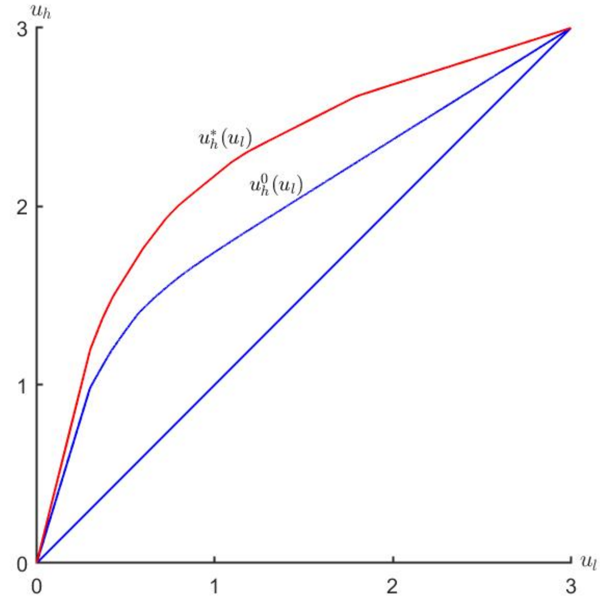

Figure 3: Relevant subset. $(w, c, \theta_h, \theta_l, \delta) = (3, 2, 0.8, 0.5, 0.9)$

strictly prefer high effort at some point, which is also undesirable for the principal. Similarly, if the agent is of high type and has a promised utility of u_h, then the principal's payoff is maximized at $u_l = u_h$. If $u_l < u_h$, then the high type must be induced to strictly prefer high effort at some point, which is undesirable. We formalize this argument as follows.

Lemma 2. *(i) For any $u_h \in (0, w)$, $\pi_h(u_h, u_l)$ increases in u_l; (ii) For any $u_l \in (0, w)$, $\pi_l(u_h, u_l)$ increases in u_h on $[u_l, u_h^0(u_l)]$ and decreases in u_h on $[u_h^0(u_l), u_h^*(u_l)]$.*

By the optimization problem in (4), the principal can always decrease u_h^l and u_l^h without violating any incentive compatibility constraints. Since $\pi_l(u_h, u_l)$ is decreasing in u_h when $u_h > u_h^0(u_l)$, we must have $u_h^l \le u_h^0(u_l^l)$ in the optimal contract. Whether or not $u_h^h \le u_h^0(u_l^h)$ is less obvious. Let u_h^* be the high type's utility given Γ_h^* and u_l^* be the low type's utility given Γ_l^*. We show in the following that when $u_h^* \le u_l^*$, the state variable in both Γ_h and Γ_l are within the subset characterized by $u_l \le u_h \le u_h^0(u_l)$. Furthermore, both types are indifferent between two contracts.

Proposition 3. *In the optimal mechanism, $u_l^l = u_l^h$ and $u_h^l \le u_h^0(u_l^l)$. Furthermore, if $u_h^* \le u_l^*$, then $u_h^h = u_h^l$.*

The condition $u_h^* \leq u_l^*$ is sufficient but not necessary. To guarantee $u_h^h = u_h^l$, by Lemma 2, we just need to ensure that $u_h^h > u_h^0(u_l^l)$ cannot be optimal. Suppose $u_h^h > u_h^l = u_h^0(u_l^l)$. When $u_l^l < u_l^*$, the principal gets better off by increasing both u_l^l and u_l^h and keeping $u_h^l = u_h^0(u_l^l)$. When $u_l^l \geq u_h^*$, $\pi_h(u_h, u_l^l)$ decreases in u_h. Therefore, the principal gets better off by decreasing u_h^h. In fact, as long as u_l^l is not much smaller than u_h^*, $\pi_h(u_h, u_l^l)$ will be decreasing in u_h. As a result, we have $u_h^h = u_h^l$ as long as u_l^* is not much smaller than u_h^*.

The subset bounded by $u_h^0(u_l)$ is relevant in the sense that the state variable in the optimal mechanism is within this subset under certain conditions. Different from the standard screening problem, the low type is made indifferent between two contracts in the optimal mechanism. Intuitively, the best way for the principal to deliver a certain utility to the high type is to make the reward as small as possible, which is exactly the low type's most preferred contract. Therefore, there is no point to make the low type worse off with the high type's contract. When the high type has incentives to mimic the low type, the high type is also made indifferent between two contracts. The logic is very similar. Since it is optimal for the principal to be more lenient toward bad outcomes given a low type, it cannot be optimal to make the high type's contract more lenient than the low type's contract when there is adverse selection.

4.3 Optimal Mechanism

Proposition 3 characterizes the state variable in the optimal mechanism. Under certain conditions, both types are indifferent between two contracts. In other words, the state variable of both contracts is the same. Nevertheless, it does not imply that the contracts are identical. Different contracts may deliver the same utility pair and the principal's preference over these contracts may be affected by the type of the agent. In the following, we solve for the optimal contract for each type separately.

First consider the optimal contract for the low type. The HJB equation is given as below.

$$\pi_0(u_h,u_l) = \sup_{\substack{u_h^G,u_h^B,\\ u_l^G,u_l^B,e_l}} (1-\delta)[h\theta_l e_l + l(1-\theta_l e_l)] + \delta\theta_l e_l \pi_l(u_h^G,u_l^G) + \delta(1-\theta_l e_l)\pi_l(u_h^B,u_l^B) \quad (6)$$

$$\text{s.t.} \quad u_l^B \le u_h^B \le u_h^*(u_l^B),\ u_l^G \le u_h^G \le u_h^*(u_l^G), \qquad \text{(Feasibility)}$$

$$e_l \in \arg\max_{e\in[0,1]} (1-\delta)(w-ce) + \delta(\theta_l e u_l^G + (1-\theta_l e)u_l^B), \qquad (\mathrm{IC}_l)$$

$$u_l = (1-\delta)(w-ce_l) + \delta(\theta_l e_l u_l^G + (1-\theta_l e_l)u_l^B), \qquad (\mathrm{PK}_l)$$

$$u_h = \sup_{e\in[0,1]} (1-\delta)(w-ce) + \delta(\theta_h e u_h^G + (1-\theta_h e)u_h^B), \qquad (\mathrm{PK}_h)$$

where $\pi_l(u_h,u_l)$ is the concavification of $\pi_0(u_h,u_l)$ and (u_h,u_l) is any feasible state variable. This equation is similar to the one in (3), with an additional promise keeping constraint for the high type.

The principal's payoff depends directly on the low type's actions. Specifically, the principal wants to induce the low type to exert high effort for as long as possible. The high type's actions do not matter directly, but his continuation utility matters in the sense that it affects for how long the low type can be induced to exert high effort. For example, when $u_h = u_l$, the low type can never be induced to exert high effort no matter what is the value of u_l. The reason is that the high type can always get a higher utility if the low type is induced to exert high effort. Therefore, we need to keep track of the evolution of u_h in the policy function.

By Proposition 3, we only need to focus on the case where $u_h \le u_h^0(u_l)$. Since it is in the principal's interest to induce high effort, we first investigate the optimal way to induce the low type to exert high effort. When the state variable is on the upper boundary, i.e., $u_h = u_h^0(u_l)$, the continuation utilities should stay on the boundary and the contract will be the same as the optimal one without adverse selection. When $u_h < u_h^0(u_l)$, we can argue that the continuation utilities should also be within this subset. Specifically, if (u_h^B, u_l^B) is outside the subset, then the principal can decrease u_h^B (and thus u_h) and gets a higher payoff, which is in contradiction with Lemma 2. If (u_h^G, u_l^G) is outside the subset and (u_h^B, u_l^B) is within the subset, then the high type must strictly prefer the high effort. As a result, the principal can decrease u_h and u_h^G and gets a higher payoff, which again contradicts to Lemma 2. Therefore, the state variable always evolves within the subset we specified in the previous section.

Another observation is that the high type is also induced to exert high effort whenever the low type is. Intuitively, if the high type is induced to exert low effort, the principal can always increase u_h^G to the point that the high type is indifferent between high effort and low effort and thus increase her payoff. The question is, therefore, whether to induce both types to exert high effort whenever possible, as in the optimal contract without adverse selection. We show in the following that it is not the case.

Lemma 3. *For any $u \in [0, w]$, there exists some $k_u \in (0, 1]$, such that for any $\lambda < k_u$, $\pi_l(\lambda u_h^0(u), \lambda u) = \frac{\lambda}{k_u}\pi_l(k_u u_h^0(u), k_u u)$. For $\lambda \geq k_u$, $\pi_l(\lambda u_h^0(u), \lambda u)$ is achieved by inducing both types to exert high effort.*

This result indicates that the value function is linear on the interval from $(0, 0)$ to $(k_u u_h^0(u), k_u u)$. In other words, it could be optimal for the principal to fire the agent with a positive probability at the very beginning, even if it is possible to induce high effort. This is significantly different from the optimal contract without adverse selection, where high effort is frontloaded. To see why randomization plays an important role here, we first look at the scenario where u_h is close to u_l. As mentioned before, the high type gets a strictly higher utility than the low type if the low type is induced to exert high effort. Therefore, if u_h is close enough to u_l, it is impossible to induce the low type to exert high effort. Then the only options are inducing low effort or taking randomization. If the low type is induced to exert low effort, then the principal should maximize the continuation payoff given a bad outcome. By Lemma 2, u_h^B should be as large as possible. As a result, the high type should also be induced to exert low effort. Then by the expression of u_l^B and u_h^B, the principal's payoff can be equivalently achieved by randomization between (w, w) and (u_h^B, u_l^B). Therefore, it is optimal to take randomization when u_h is close enough to u_l.

To see why randomization can be optimal in general, we consider the principal's payoff from this contract. Specifically, the principal's payoff consists of two parts, the flow payoff when the agent exerts high effort and the termination payoff when the agent is fired or tenured. The next result shows that the second part does not affect the form of the optimal mechanism if the state variable is given.

Lemma 4. *Suppose $lc+\theta_l(h-l)w > 0$. Given any feasible state variable (u_h, u_l), the optimal policy does not depend on the value of l.*

It is obvious that the value of l does affect the form of the optimal mechanism. In general, the principal tends to be more lenient toward bad outcomes when l is larger. Nevertheless, Lemma 4 indicates that the effect is only through the choice of the initial state variable. In other words, once the state variable is fixed, l does not affect the optimal mechanism anymore. Intuitively, when the low type is induced to exert high effort for one more period, the principal gets an additional flow payoff of $\theta_l(h-l)$. At the same time, to compensate for the agent's cost, the principal needs to tenure the agent with a higher probability, which is given by $\frac{c}{w}$. Therefore, the principal also gets an additional termination payoff of $\frac{lc}{w}$. When $lc + \theta_l(h-l)w \leq 0$, the principal never induces high effort in the optimal contract. When $lc + \theta_l(h-l)w > 0$, on the other hand, the principal optimally maximizes the duration of high effort, which is independent of l.

As mentioned before, the low type cannot be induced to exert high effort when the state variable hits the lower bound, i.e., $u_h = u_l$. Therefore, to maximize the duration of high effort, the principal should keep the expectation of $u_h - u_l$ as large as possible. Given that both types exert high effort, by the promise keeping constraints, we can obtain $u_h - u_l = \delta\left[\theta_l(u_h^G - u_l^G) + (1-\theta_l)(u_h^B - u_l^B)\right] + \delta(\theta_h - \theta_l)(u_h^G - u_l^G)$, where the second term is due to the higher probability of success for the high type. Therefore, to maximize the expectation of $u_h - u_l$ in the next period, the reward for the high type $(u_h^G - u_l^G)$ should be as small as possible. Intuitively, the high type gets a premium over the low type in each period where both exert high effort, which is represented by $(\theta_h - \theta_l)(u_h^G - u_l^G)$. Once $u_h - u_l$ is exhausted, the low type cannot be induced to exert high effort anymore. Therefore, the objective of the principal is to minimize the high type's premium in each period. Ideally, it is optimal to make the high type indifferent between high effort and low effort. However, this strategy is not feasible when $u_h - u_l$ is small. Consider the case where $u_h - u_l = (1-\delta)c \cdot \frac{\theta_h - \theta_l}{\theta_l}$ for an example. Since $u_h^G \geq u_l^G$ and $u_h^B \geq u_l^B$, it is easy to verify that the only way to induce the low type to exert high effort is to set $u_h^G = u_l^G$ and $u_h^B = u_l^B$. As a result, the high type strictly prefers high effort. If u_h increases, the principal can increase u_h^B and keep u_h^G unchanged. Then the reward for the high type decreases in the first period. Therefore, it is better to randomize at the beginning between $u_h - u_l = 0$ and $u_h - u_l > (1-\delta)c \cdot \frac{\theta_h - \theta_l}{\theta_l}$.

For better intuition, we compare this problem to the one in the benchmark. When there is no adverse selection, the agent can be induced to exert high effort when u is away from 0 and w. Since any randomization makes u closer to both, it is never optimal to randomize when it is not necessary. By contrast, with adverse selection, the low type can be induced to exert high effort when $u_h - u_l$ is away from 0. In this case, randomization does not change the expectation of $u_h - u_l$. Instead, a larger $u_h - u_l$ provides more flexibility in choosing the high type's reward, which is beneficial for the principal. Therefore, it is optimal to randomize between $u_h - u_l = 0$ and a larger $u_h - u_l$.

Finally, we explain why the randomization is always taken with $(0, 0)$. When the state variable is on the upper boundary, i.e., $u_h = u_h^0(u_l)$, the reward for the high type is given by $u_h^0(u_l^G) - u_h^0(u_l^B)$. By Corollary 4, $u_h^0(u_l)$ is concave in u_l. Therefore, the reward decreases when u_l increases. In other words, given $u_h - u_l$, it is better for the principal to have a larger u_l. As a result, randomization with $(0, 0)$ is optimal.

In the following, we analyze the optimal contract for the high type. The HJB equation is very similar to the one in (6). The only difference is that the principal's payoff now depends on the high type's effort.

$$
\begin{aligned}
\pi_1(u_h, u_l) = \sup_{\substack{u_h^G, u_h^B, \\ u_l^G, u_l^B, e_h}} \; & (1-\delta)[h\theta_h e_h + l(1-\theta_h e_h)] + \delta\theta_h e_h \pi_h(u_h^G, u_l^G) + \delta(1-\theta_h e_h)\pi_h(u_h^B, u_l^B) \\
\text{s.t.} \quad & u_l^B \le u_h^B \le u_h^*(u_l^B), \; u_l^G \le u_h^G \le u_h^*(u_l^G), && \text{(Feasibility)} \\
& e_h \in \arg\max_{e \in [0,1]} \; (1-\delta)(w - ce) + \delta(\theta_h e u_h^G + (1-\theta_h e)u_h^B), && (\text{IC}_h) \\
& u_h = (1-\delta)(w - ce_h) + \delta(\theta_h e_h u_h^G + (1-\theta_h e_h)u_h^B), && (\text{PK}_h) \\
& u_l = \sup_{e \in [0,1]} \; (1-\delta)(w - ce) + \delta(\theta_l e u_l^G + (1-\theta_l e)u_l^B), && (\text{PK}_l)
\end{aligned}
$$

where $\pi_h(u_h, u_l)$ is the concavification of $\pi_1(u_h, u_l)$ and (u_h, u_l) is any feasible state variable. Different from the case in the low type's contract, the high type is induced to exert high effort whenever possible in the optimal contract.

Lemma 5. *Suppose $lc + \theta_h(h-l)w > 0$. In the optimal contract for the high type, the high* *type is induced to exert high effort whenever $u_l \geq (1-\delta)w$ and $u_h \leq w - \frac{(1-\delta)c}{\theta_h}$.*

When $lc + \theta_h(h-l)w \leq 0$, it is not optimal to induce high effort even if there is no adverse selection. Thus, there is no reason to induce high effort in the presence of adverse selection. However, as long as it is beneficial to induce high effort without adverse selection, the principal induces the high type to exert high effort whenever possible in the optimal contract with adverse selection.

The intuition for this result is similar to the one in the benchmark. Conditional on $lc + \theta_h(h-l)w > 0$, the principal's objective is to maximize the duration in which the high type exerts high effort. By the promise keeping constraint, high effort can only be induced to exert high effort when $u_l \geq (1-\delta)w$ and $u_h \leq w - \frac{(1-\delta)c}{\theta_h}$. Any randomization makes the state variable closer to these two regions and therefore cannot be optimal.

The difference in the form of Γ_h and Γ_l results from the behavior around the lower boundary. When u_h is very close to u_l, the high type can be induced to exert high effort but the low type can only be induced to exert low effort. In Γ_h, the principal's objective is to make the high type exert high effort. Therefore, inducing pure actions is optimal. On the contrary, in Γ_l, the principal's objective is to induce the low type to exert high effort. As a result, inducing pure actions is not desirable and randomization can be optimal.

Next we state the main theorem of this paper.

Theorem 1. *There exists $\delta^* < 1$, $c^* > 0$, and $p^* < 1$ such that when $\delta > \delta^*$, $c < c^*$, and $p > p^*$, the principal hires the high type for sure and hires the low type with some probability. Conditional on being hired, the low type's contract is preferred by both types.*

The requirement of a large δ and a small c is similar to that in Proposition 1. Specifically, when δ is small, the continuation utility tomorrow is relatively less important than today's flow payoff. Therefore, it is impossible to induce the agent to exert high effort by promising a larger continuation utility. When c is large, the cost is too large relative to the benefit and thus it is never optimal for the principal to induce high effort. An additional requirement is that p, the prior for the high type, is large enough. Intuitively, when p is very small, it may be optimal for the principal to maximize her payoff with a low type. As a result, the low type would get the same contract as in Proposition 1, in which he is hired with probability one.

Even though both types are made indifferent between the high type's contract and the low type's contract, we show in Theorem 1 that the optimal contract is separating. Specifically, the high type is hired with a higher probability than the low type at the beginning. But once hired, he is more likely to be fired and less likely to be tenured given a same history of outcomes. Define $u_l^G - u_l^B$ as the reward for a good outcome. Then the reward in the low type's contract is the same as the one in the model without adverse selection. The reward in the high type's contract is larger at first and reduces to the level of the one in the model without adverse selection at some point.

By Proposition 3, both IC constraints are binding. In other words, the principal needs to deter the low type from mimicking the high type and also deter the high type from mimicking the low type. Since the low type has a lower probability of getting a good outcome, increasing reward for each good outcome makes the low type worse off. Therefore, the high type's contract features a larger reward in early periods. On the other hand, since the high type values this employment more than the low type, one way to make the high type worse off with the low type's contract is to hire the agent for less periods in expectation. Theorem 1 indicates that it is most efficient to fire the agent with some probability at the beginning.

The optimal contract is reminiscent of the dual-track system in many universities. Tenure-track faculties receive a higher wage, but also face a higher standard. They will be fired if their performance is not good enough within some time period. On the other hand, non-tenure-track faculties receive a lower wage but are very unlikely to be fired.

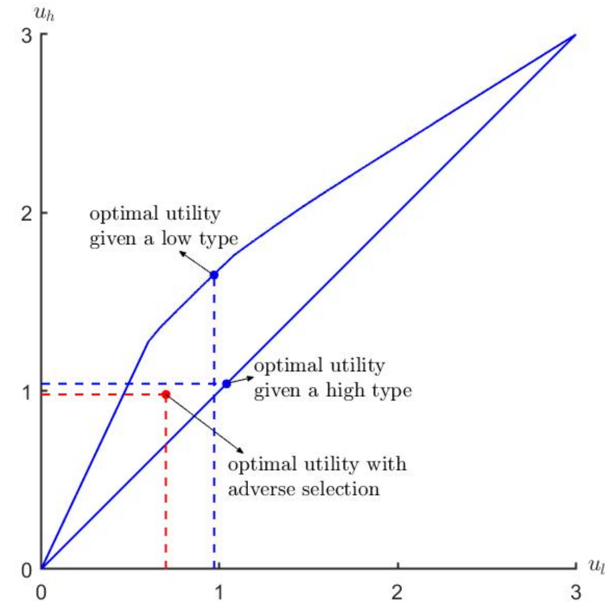

Figure 4: Agent can be worse off with adverse selection.
$(w, c, h, l, \theta_h, \theta_l, \delta, p) = (3, 2, 5, -2, 0.8, 0.5, 0.8, 0.6)$

Compared to the contract in Proposition 1, it is unclear whether the agent becomes better off or worse off when his type is private. In a standard screening problem, the high type earns an information rent because he can mimic the low type and get a larger utility than the low type. But in this environment, we show by a numerical example that both types could get worse off (See Figure 4). In other words, the agent is hurt by his private information. Intuitively, when $l < 0$, the principal has two objectives, increasing the expected duration of high effort and decreasing the probability of tenuring the agent eventually. When there is adverse selection, the cost of inducing high effort becomes larger. As a result, the consideration of increasing the expected duration of high effort becomes less important than decreasing the probability of tenuring the agent. Therefore, the initial promised utility to the agent would be smaller.

5 Concluding Remarks

I study optimal contracting without money transfer when there is dynamic moral hazard and persistent private information. In the optimal contract, both types exert high effort until being fired or tenured. The principal hires the high type for sure at the beginning and hires the low type only with some probability. Conditional on being hired, both types prefer the low type's contract.

A natural extension of this model is to allow for different wages offered to different types. Intuitively, the hiring probability at the beginning plays a similar role as the wage level. Instead of hiring the agent with a smaller probability, the principal could offer a lower wage and provide similar incentives. Thus, a reasonable conjecture is that in the optimal contract, the high type receives a higher wage than the low type but is more likely to be fired given a same history of performance. However, one technical difficulty of this model setup is that the wage also affects the expected duration of high effort. Specifically, w is the maximum promised utility to the agent. When w is smaller, the scope of reward is more restricted and high effort can be induced for less periods in expectation. Therefore, it is unclear what the optimal wage should be even in the model without adverse selection.

The results would change more dramatically if we allow for completely flexible money transfer. In the model without adverse selection, the principal can achieve the first best by rewarding good outcomes and punishing bad outcomes. When the agent's type is private, the high type has incentives to mimic the low type, but not vice versa. As in a standard screening problem, the high type is always hired and receives an information rent in the optimal contract. The low type is hired for less periods such that the high type is indifferent between his contract and the low type's contract.

References

Aghion, P. and M. O. Jackson (2016): "Inducing leaders to take risky decisions: dismissal, tenure, and term limits," *American Economic Journal: Microeconomics*, 8, 1–38.

Balseiro, S. R., H. Gurkan, and P. Sun (2019): "Multiagent mechanism design without money," *Operations Research*, 67, 1417–1436.

Bird, D. and A. Frug (2019): "Dynamic Non-monetary Incentives," *American Economic Journal: Microeconomics*, 11, 111–50.

Chen, C.-H. and J. Ishida (2018): "Dynamic performance evaluation with deadlines: The role of commitment," *The Journal of Industrial Economics*, 66, 377–422.

Chen, Y. (2018): "Dynamic Communication with Commitment," .

Cvitanić, J., X. Wan, and H. Yang (2013): "Dynamics of contract design with screening," *Management science*, 59, 1229–1244.

Deb, R., M. M. Pai, and M. Said (2018): "Evaluating strategic forecasters," *American Economic Review*, 108, 3057–3103.

Escobar, J. F., Q. Zhang, et al. (2019): "Delegating Learning," Tech. rep.

Fershtman, D. (2017): "Dynamic Delegation: Specialization and Favoritism," .

Gershkov, A. and M. Perry (2012): "Dynamic contracts with moral hazard and adverse selection," *The Review of Economic Studies*, 79, 268–306.

Guo, Y. (2016): "Dynamic delegation of experimentation," *American Economic Review*, 106, 1969–2008.

Guo, Y. and J. Hörner (2020): "Dynamic allocation without money," .

Halac, M., N. Kartik, and Q. Liu (2016): "Optimal contracts for experimentation," *The Review of Economic Studies*, 83, 1040–1091.

Kuvalekar, A. and E. Lipnowski (2020): "Job Insecurity," *American Economic Journal: Microeconomics*, 12, 188–229.

Li, J., N. Matouschek, and M. Powell (2017): "Power dynamics in organizations," *American Economic Journal: Microeconomics*, 9, 217–41.

Lipnowski, E. and J. Ramos (2020): "Repeated delegation," *Journal of Economic Theory*, 105040.

Pisano, G. P. (2019): "The Hard Truth About Innovative," *Harvard Business Review*.

Rantakari, H. (2017): "Relational Influence," .

Sannikov, Y. (2007): "Agency problems, screening and increasing credit lines," *Unpublished Paper*.

Spear, S. E. and S. Srivastava (1987): "On repeated moral hazard with discounting," *The Review of Economic Studies*, 54, 599–617.

Sun, J. and D. Wei (2019): "Reputation Concerns Under At-Will Employment," *Available at SSRN 3424122*.

Sung, J. (2005): "Optimal contracts under adverse selection and moral hazard: a continuous-time approach," *The Review of Financial Studies*, 18, 1021–1073.

Ulbricht, R. (2016): "Optimal delegated search with adverse selection and moral hazard," *Theoretical Economics*, 11, 253–278.

Appendix

Proof of Proposition 1. Since we allow for public randomization, $\pi(u)$ is concave. When $q_t < 1$, by the HJB equation, $\pi(u_t) = q_t \pi(\frac{u_t}{q_t})$. Therefore, $q_t < 1$ is optimal if and only if $\pi(u)$ is linear on $[0, \frac{u_t}{q_t}]$. We consider the case where $q_t = 1$ in the following.

Define $\pi_L(u_t)$ as the maximum payoff of the principal when the agent is induced to exert low effort and define $\pi_H(u_t)$ accordingly. First consider the case when the agent is induced to exert low effort.

When $e_t = 0$, u_t^G does not affect the principal's payoff or the promise keeping constraint. Without loss of generality, let $u_t^G = 0$ and then the IC constraint is satisfied. By the promise keeping constraint, $u_t^B = \frac{1}{\delta}(u_t - (1-\delta)w)$. Plugging it into equation (3), we obtain $\pi_H(u_t) = (1-\delta)l + \delta\pi(\frac{1}{\delta}(u_t - (1-\delta)w))$. By the feasibility constraint, the domain of π_L is $[(1-\delta)w, w]$ and $u_t \geq \frac{1}{\delta}(u_t - (1-\delta)w)$. When $u_t = w$, the only way to deliver the promised utility is to never fire the agent and induce low effort. Thus, $\pi(w) = l$. If $\pi_L(u_t) = \pi(u_t)$, then $\pi(u_t) = (1-\delta)\pi(w) + \delta\pi(u_t^B)$. Since $u_t = (1-\delta)w + \delta u_t^B$, by concavity of $\pi(u)$, $\pi(u)$ is linear on $[\frac{1}{\delta}(u_t - (1-\delta)w), w]$. Consequently, it is equivalent for the principal to randomize between w and u_t^B. Therefore, without loss of generality, it is optimal for the principal to never induce low effort with probability one unless $u_t = w$.

When $e_t = 1$, the IC constraint is equivalent to $u_t^G - u_t^B \geq \frac{(1-\delta)c}{\delta\theta}$. By the promise keeping constraint, when u_t^G increases by $1-\theta$, u_t^B needs to decrease by θ. By concavity of $\pi(u)$, $\pi_H(u_t)$ decreases when u_t^G increases. Therefore, the IC constraint should be binding. Combining it with the PK constraint, we have $u_t^B = \frac{1}{\delta}(u_t - (1-\delta)w)$ and $u_t^G = \frac{1}{\delta}(u_t - (1-\delta)w + \frac{(1-\delta)c}{\theta})$. In the following, we use u^G and u^B to denote the continuation utility tomorrow given that today's promised utility is u and the IC constraint is binding. By the HJB equation in (3), $\pi_H(u_t) = (1-\delta)(h\theta + l(1-\theta)) + \delta\theta\pi(\frac{1}{\delta}(u_t - (1-\delta)w + \frac{(1-\delta)c}{\theta})) + \delta(1-\theta)\pi(\frac{1}{\delta}(u_t - (1-\delta)w))$. By the feasibility constraint, $\pi_H(u)$ is well-defined on $[(1-\delta)w, w - \frac{(1-\delta)c}{\theta}]$. When $(1-\delta)w > w - \frac{(1-\delta)c}{\theta}$, it is impossible to induce high effort. As a result, the value function $\pi(u)$ is linear on $[0, w]$. When $l > 0$, it is optimal for the principal to never fire the agent. When $l < 0$, on the other hand, the principal never hires the agent in the optimal contract.

When $(1-\delta)w \leq w - \frac{(1-\delta)c}{\theta}$, the value function is linear on $[0, (1-\delta)w]$ and $[w - \frac{(1-\delta)c}{\theta}, w]$.

Let $u_2 = w - \frac{(1-\delta)c}{\theta}$. Suppose $\pi(u)$ is linear on $[0, w]$. Then by the expression of $\pi_H(u)$, $\pi_H(u) - \pi(u) = (1-\delta)\left[\frac{lc}{w} + \theta(h-l)\right]$, for any $u \in [u_1, u_2]$. If $\frac{lc}{w} + \theta(h-l) > 0$, then $\pi_H(u) - \pi(u) > 0$, which contradicts to the definition of $\pi(u)$. Consequently, $\pi(u)$ is linear on $[0, w]$. If $\frac{lc}{w} + \theta(h-l) \leq 0$, on the other hand, $\pi(u)$ is linear on $[0, w]$ and the principal optimally fires the agent or tenures the agent at the beginning. In the following, we investigate the case where $\frac{lc}{w} + \theta(h-l) > 0$ and $\pi(u)$ is not linear.

Define $\hat{u} > 0$ as the smallest v such that $\pi(u)$ is linear on $[v, w]$. Then it must be optimal to induce high effort given $u_t = \hat{u}$, i.e., $\pi_H(\hat{u}) = \pi(\hat{u})$. Suppose $\hat{u} < u_2$. By the expression of $\pi_H(u)$, $\pi'_{h+}(\hat{u}) = \theta\pi'_+(\hat{u}^G) + (1-\theta)\pi'_+(\hat{u}^B)$. Since $\hat{u}^B < \hat{u}$, by definition of $\hat{u}$, $\pi'_+(\hat{u}^B) > \pi'_+(\hat{u})$. If $\theta < 1$, then $\pi'_{h+}(\hat{u}) > \pi'_+(\hat{u})$. In other words, there exists $\epsilon > 0$ such that $\pi_H(\hat{u} + \epsilon) > \pi(\hat{u} + \epsilon)$, which contradicts to the definition of $\pi(u)$. Therefore, $\hat{u} = u_2$ when $\theta < 1$. If $\theta = 1$, then $\pi'_{h+}(u) = \pi'_+(u^G)$. As a result, whenever $u < u^G$, inducing high effort is optimal and $\pi(u)$ is linear on $[u, w]$. Thus, $\hat{u} = (1-\delta)w$ or $\hat{u} = \hat{u}^G > (1-\delta)w$. According to definition of $\tilde{u}$ as given below, $\hat{u} = \tilde{u}$. In conclusion, $\pi(u)$ is linear on both $[0, \hat{u}]$ and $[\hat{u}, w]$ when $\theta = 1$.

Let $u_1 = (1-\delta)w$. Define $\tilde{u}$ as the largest v such that $\pi(u)$ is linear on $[0, v]$. Then $\pi_H(\tilde{u}) = \pi(\tilde{u})$. Consider the relationship between $\tilde{u}^G$ and $\tilde{u}$. If $\tilde{u}^G > \tilde{u}$, by the same argument as above, $\tilde{u} = u_1$. If $\tilde{u}^G < \tilde{u}$, then by the expression of $\pi_H(u)$, $\pi'_+(\tilde{u}) = \pi'_+(0)$. Therefore, there exists $\epsilon > 0$ such that $\pi(u)$ is linear on $[0, \tilde{u} + \epsilon]$, which contradicts to the definition of $\tilde{u}$. If $\tilde{u}^G = \tilde{u}$, then $\pi'_+(\tilde{u}) = \pi'_+(0)$ and $\pi(u)$ is differentiable at $\tilde{u}$. Given $u_t = u_1$, we have $u_t^G = \frac{(1-\delta)c}{\delta\theta}$. Then $u_t^G > u_1$ if and only if $c > \delta\theta w$. As a result, if $c > \delta\theta w$, then $\tilde{u} = u_1$ and $\pi(u)$ is not differentiable at u_1. If $c \leq \delta\theta w$, then $\tilde{u}$ should satisfy $\tilde{u} = \tilde{u}^G$ and $\pi(u)$ is differentiable at $\tilde{u}$. By the expression of $\tilde{u}^G$, we have $\tilde{u} = w - \frac{c}{\theta} < u_2$.

Next we show that it is optimal to induce high effort when $u_t \in (\tilde{u}, u_2)$. Suppose there exists $u_c \in (\tilde{u}, u_2)$ such that $\pi_H(u_c) < \pi(u_c)$. Then it is optimal to randomize given $u_t = u_c$. In other words, $\pi(u)$ is linear around u_c. Define u_a and u_b as the left and right endpoint of this linear part. Then $u_a, u_b \in [\tilde{u}, u_2]$ and there exists $\lambda \in (0, 1)$ such that $u_c = \lambda u_a + (1-\lambda)u_b$ and $\pi(u_c) = \lambda\pi(u_a) + (1-\lambda)\pi(u_b)$. Since $\pi(u_a)$ and $\pi(u_b)$ cannot be achieved by randomization, we must have $\pi_H(u_a) = \pi(u_a)$ and $\pi_H(u_b) = \pi(u_b)$. By expression of u_t^G and u_t^B, we get $u_c^G = \lambda u_a^G + (1-\lambda)u_b^G$ and $u_c^B = \lambda u_a^B + (1-\lambda)u_b^B$. Then by concavity of $\pi(u)$ and expression of $\pi_H(u)$, $\pi_H(u_c) \geq \lambda\pi_H(u_a) + (1-\lambda)\pi_H(u_b)$. It contradicts to the assumption that $\pi_H(u_c) < \pi(u_c)$.

Whether there is a relationship or not depends on $\pi'(0)$. When $\pi'(0) < 0$, the principal never hires the agent. Similarly, when $\pi'(w) > 0$, the principal tenures the agent immediately. When $\pi'(0) \geq 0 \geq \pi'(w)$, the optimal initial promised utility is within $[\tilde{u}, u_2]$. The agent is induced to exert high effort at first. Once $u_t < \tilde{u}$, the agent is fired with some probability. If he is not fired, he continues to exert high effort. Similarly, when $u_t > u_2$, the agent is tenured with some probability. If he is not tenured, then again he is induced to exert high effort.

Finally, we show that t^* is finite with probability 1. Denote the initial promised utility by u^*. If $u^* = u_2$, since $u_t - u_t^B = \frac{1-\delta}{\delta}(w - u_t)$ is decreasing in u_t, there exists $N > 0$ such that the agent is fired with a positive probability after N consecutive bad outcomes. When $u^* < u_2$, the agent is fired with a larger probability given the same history. When $u^* > u_2$, $t^* = 0$ with some probability and $u^* = u_2$ otherwise. Therefore, there exists $N > 0$ and $\epsilon > 0$ such that $\mathbb{P}(t^* \leq N \mid u^* = u) > \epsilon$ for any $u \in [0, w]$. By induction, it is easy to show that $\mathbb{P}(t^* > kN \mid u^* = u) < (1-\epsilon)^k$ for any $k \in \mathbb{N}$ and $u \in [0, w]$. Therefore, $\mathbb{E}(t^* \mid u^* = u) < \sum_{k=1}^{\infty} kN\epsilon(1-\epsilon)^{k-1}$ is finite. As a result, t^* is finite with probability 1. □

Proof of Corollary 1. Since $\delta > \bar{\delta}$, it is optimal for the principal to hire the agent and induce high effort at the beginning. Define $u_1 = (1-\delta)w$ and $u_2 = w - \frac{(1-\delta)c}{\theta}$, as in the proof of Proposition 1. Then the optimal initial utility is within $[u_1, u_2]$. Denote it by u^*.

By the proof of Proposition 1, the continuation utility in the optimal contract is given by $u_t^G = \frac{1}{\delta}(u_t - (1-\delta)w + \frac{(1-\delta)c}{\theta})$ and $u_t^B = \frac{1}{\delta}(u_t - (1-\delta)w)$. Since the value function is linear on $[0, u_1]$ and $[u_2, w]$, the agent is fired with probability $\frac{u_1 - u_t}{u_1}$ when $u_t < u_1$ and tenured with probability $\frac{u_t - u_2}{w - u_2}$ when $u_t > u_2$. Divide everything by $(1-\delta)w$ and let $n_0 = \frac{u^*}{(1-\delta)w}$ and $n_t = \frac{u_t}{(1-\delta)w}$. Then we have the implementation as in Corollary 1. □

Proof of Corollary 2. Let $\theta_1 < \theta_2$. Denote the optimal contract for a type θ_1 agent by Γ_1. Furthermore, define $\Gamma_1(u)$ as the optimal contract conditional on the agent's promised utility being u. Without loss of generality, assume that the agent is not fired or tenured at the beginning in Γ_1. The proof proceeds in two steps. First, by Lemma 1, given Γ_1, the type θ_2 agent optimally exerts high effort whenever he is hired and not tenured. Secondly, we show in the following that the principal is better off with a type θ_2 agent given that the contract is Γ_1 and the agent always exerts high effort before being fired or tenured.

Define $\pi_l(u)$ as the principal's payoff when the contract is $\Gamma_1(u)$ and the agent is of type θ_1. Correspondingly, define $\pi_h(u)$ as the principal's payoff given that the contract is $\Gamma_1(u)$ and the agent is of type θ_2. By definition, $\pi_l(u)$ is the value function when the agent's type is θ_1. Our objective is to show that $\pi_h(u^*) > \pi_l(u^*)$, where u^* is the type θ_1 agent's utility in the optimal contract. More generally, we will show that $\pi_h(u) > \pi_l(u)$ for all $u \in (0, w)$.

As argued above, given $\Gamma_1(u)$, both types exert high effort until being fired or tenured. Since the principal randomizes when $u < u_1$ or $u > u_2$, $\pi_h(u)$ is linear on $[0, u_1]$ and $[u_2, w]$. Therefore, $\pi_h(u)$ is the solution to the following equation:

$$\pi_h(u) = \begin{cases} \frac{u}{u_1}\pi_h(u_1) & \text{if } u < u_1, \\ (1-\delta)\left[h\theta_2 + l(1-\theta_2)\right] + \delta\left[\theta_2\pi_h(u^G) + (1-\theta_2)\pi_h(u^B)\right] & \text{if } u \in [u_1, u_2], \\ \frac{u-u_2}{w-u_2}l + \frac{w-u}{w-u_2}\pi_h(u_2) & \text{if } u > u_2, \end{cases} \tag{7}$$

where $u^G = \frac{1}{\delta}(u-(1-\delta)w+\frac{(1-\delta)c}{\theta_1})$, $u^B = \frac{1}{\delta}(u-(1-\delta)w)$, $u_1 = (1-\delta)w$, and $u_2 = w - \frac{(1-\delta)c}{\theta_1}$, as specified in Γ_1. Define an operator T such that

$$(Tf)(u) = \begin{cases} \frac{u}{u_1}\left[(1-\delta)\left(h\theta_2 + l(1-\theta_2)\right) + \delta\theta_2 f(u_1^G)\right] & \text{if } u < u_1, \\ (1-\delta)\left[h\theta_2 + l(1-\theta_2)\right] + \delta\left[\theta_2 f(u^G) + (1-\theta_2)f(u^B)\right] & \text{if } u \in [u_1, u_2], \\ \frac{u-u_2}{w-u_2}l + \frac{w-u}{w-u_2}\left[(1-\delta)\left(h\theta_2 + l(1-\theta_2)\right) + \delta\left(\theta_2 l + (1-\theta_2)f(u_2^B)\right)\right] & \text{if } u > u_2, \end{cases} \tag{8}$$

where $u_1^G = \frac{1}{\delta}(u_1 - (1-\delta)w + \frac{(1-\delta)c}{\theta_1})$ and $u_2^B = \frac{1}{\delta}(u_2 - (1-\delta)w)$, and f is any bounded function on $[0, w]$. Then $\pi_h(u)$ is a fixed point of T. By the expression in (8), it is obvious that T satisfies Blackwell's sufficient conditions for a contraction. By Contraction Mapping Theorem, T has a unique fixed point and $T^n f \to \pi_h$ for any bounded function f.

By the proof of Proposition 1, it is always strictly better for the principal to induce high effort rather than low effort. In other words, given the contract as Γ_1 and the agent's type as θ_1, a good outcome today is always better than a bad outcome given any history. Therefore,

$\delta h + (1-\delta)\pi_l(u^G) > \delta l + (1-\delta)\pi_l(u^B)$ for any $u \in [u_1, u_2]$. Let f be any bounded function such that $f(u) \geq \pi_l(u)$ for all u. Then by equation (8), $(Tf)(u) > \pi_l(u)$ for all $u \in (0, w)$. According to the implication of Contraction Mapping Theorem, we must have $\pi_h(u) > \pi_l(u)$ for all $u \in (0, w)$.

In conclusion, given Γ_1, the principal gets a higher payoff when the agent is of a higher type. Since Γ_1 is the optimal contract for the low type, we must have $\pi^*(\theta_2) \geq \pi^*(\theta_1)$. $\square$

Proof of Corollary 3. Since $l < 0$, the principal does not hire the agent when θ is very small. Given $c > \delta w$, by the proof of Proposition 1, $\pi(u)$ is linear on $[0, u_1]$ and $\pi'_+(u_1) < \pi'_-(u_1)$. By definition of $\underline{\theta}$, $\pi(u_1) = 0$ and $\pi(u) < 0$ for all $u > u_1$. Therefore, $u^*(\underline{\theta}) = u_1$.

When $\theta = 1$, by the proof of Proposition 1, $\pi(u)$ is linear on $[0, u_1]$ and $[u_1, w]$. Then $u^*(1) = u_1$. For $\theta \in (\underline{\theta}, 1)$, since $\pi(u)$ is linear on $[0, u_1]$, $u^*(\theta) \geq u_1$. $\square$

Proof of Proposition 2. Similar to the proof of Corollary 2, it is easy to see that $u_h^*(u_l)$ is a fixed point of a contraction mapping. Denote this mapping by T_δ. Let $\delta_1 < \delta_2$. Denote the fixed point of T_{δ_1} by $u_h^1(\cdot)$. Given any bounded function $f \geq u_h^1$, we show in the following that $T_{\delta_2} f \geq u_h^1$. Then we can conclude by Contraction Mapping Theorem that the fixed point of T_{δ_2} is greater than $u_h^1(\cdot)$.

Take any $u_l \in (0, w)$. If $u_h^1(u_l)$ is delivered by randomization, we only need to consider each of the realized utility. Thus, without loss of generality, we assume that $u_h^1(u_l)$ is not delivered by randomization. Denote the choice variables of $u_h^1(u_l)$ by u_l^G and u_l^B. Then according to the HJB equation, $u_l = (1-\delta_1)(w - ce_l) + \delta_1 \theta_l e_l u_l^G + \delta_1(1-\theta_l e_l)u_l^B$ and $u_h^1(u_l) = (1-\delta_1)(w - ce_h) + \delta_1 \theta_h e_h u_h^1(u_l^G) + \delta_1(1-\theta_h e_h)u_h^1(u_l^B)$. If $u_h = 0$, then it is optimal to increase u_l^B as much as possible. Thus, $e_l = 0$ is optimal for the low type. By concavity of $u_h^1(u_l)$, $u_h^1(u_l)$ must be linear on $[u_l^B, w]$. Therefore, u_l can also be delivered by randomization. Then without loss of generality, we can assume $e_h = 1$. Now consider the scenario where $\delta = \delta_2$. Suppose the optimal action for the low type changes when δ increases from δ_1 to δ_2. Then we can divide the increase of δ into two steps. In the first step, δ is increased from δ_1 to the point where the low type is indifferent between high effort and low effort, and in the second step, it further increases to δ_2. Then the low type's optimal action does not change in both steps. As a result, we can assume without loss of generality that

e_l does not change. Let $u_l' = (1-\delta_2)(w-ce_l) + \delta_2\theta_l e_l u_l^G + \delta_2(1-\theta_l e_l)u_l^B$. If $u_l' = u_l$, then keeping both u_l^G and u_l^B unchanged is a feasible policy. Since the high type may still choose $e_h = 1$, we have

$$\begin{aligned}(T_{\delta_2}f)(u_l) &\geq (1-\delta_2)(w-c) + \delta_2\theta_h f(u_l^G) + \delta_2(1-\theta_h)f(u_l^B) \\ &\geq (1-\delta_2)(w-c) + \delta_2\theta_h u_h^1(u_l^G) + \delta_2(1-\theta_h)u_h^1(u_l^B). \end{aligned} \tag{9}$$

By the expression of u_l,

$$\begin{aligned}(T_{\delta_2}f)(u_l) - u_l &\geq (1-\delta_2)c(e_l-1) + \delta_2\left[\theta_h u_h^1(u_l^G) + (1-\theta_h)u_h^1(u_l^B) - \theta_l e_l u_l^G - (1-\theta_l e_l)u_l^B\right] \\ &\geq (1-\delta_1)c(e_l-1) + \delta_1\left[\theta_h u_h^1(u_l^G) + (1-\theta_h)u_h^1(u_l^B) - \theta_l e_l u_l^G - (1-\theta_l e_l)u_l^B\right] \\ &= u_h^1(u_l) - u_l. \end{aligned} \tag{10}$$

When $u_l' > u_l$, choose $q < 1$ such that $qu_l' = u_l$ and keep both u_l^G and u_l^B unchanged. Then $q(1-\delta_2)(w-ce_l) + q\delta_2\theta_l e_l u_l^G + q\delta_2(1-\theta_l e_l)u_l^B = (1-\delta_1)(w-ce_l) + \delta_1\theta_l e_l u_l^G + \delta_1(1-\theta_l e_l)u_l^B$. Since $q(1-\delta_2) < 1-\delta_1$, we have $q\delta_2 > \delta_1$. Similar to equation (9) and (10),

$$\begin{aligned}(T_{\delta_2}f)(u_l) - u_l &\geq q(1-\delta_2)c(e_l-1) + q\delta_2\left[\theta_h u_h^1(u_l^G) + (1-\theta_h)u_h^1(u_l^B) - \theta_l e_l u_l^G - (1-\theta_l e_l)u_l^B\right] \\ &\geq (1-\delta_1)c(e_l-1) + \delta_1\left[\theta_h u_h^1(u_l^G) + (1-\theta_h)u_h^1(u_l^B) - \theta_l e_l u_l^G - (1-\theta_l e_l)u_l^B\right] \\ &= u_h^1(u_l) - u_l. \end{aligned}$$

When $u_l' < u_l$, choose $\lambda \in (0,1)$ such that $\lambda u_l' + (1-\lambda)w = u_l$, i.e.,

$$\begin{aligned} &\lambda\left[(1-\delta_2)(w-ce_l) + \delta_2\theta_l e_l u_l^G + \delta_2(1-\theta_l e_l)u_l^B\right] + (1-\lambda)w \\ =& (1-\delta_1)(w-ce_l) + \delta_1\theta_l e_l u_l^G + \delta_1(1-\theta_l e_l)u_l^B. \end{aligned}$$

By rearrangement,

$$(1-\lambda\delta_2)(w-ce_l) + \lambda\delta_2(\theta_l e_l u_l^G + (1-\theta_l e_l)u_l^B) \leq (1-\delta_1)(w-ce_l) + \delta_1(\theta_l e_l u_l^G + (1-\theta_l e_l)u_l^B).$$

Since $u_l' < u_l$, we have $(1-\delta)(w-ce_l) + \delta(\theta_l e_l u_l^G + (1-\theta_l e_l)u_l^B)$ decrease in δ. Therefore,

$\lambda\delta_2 \geq \delta_1$. Given the choice variable (u_l^G, u_l^B, λ), we obtain

$$\begin{aligned}(T_{\delta_2} f)(u_l) - u_l &\geq \lambda(1-\delta_2)c(e_l - 1) + \lambda\delta_2\left[\theta_h u_h^1(u_l^G) + (1-\theta_h)u_h^1(u_l^B) - \theta_l e_l u_l^G - (1-\theta_l e_l)u_l^B\right] \\ &\geq (1-\delta_1)c(e_l-1) + \delta_1\left[\theta_h u_h^1(u_l^G) + (1-\theta_h)u_h^1(u_l^B) - \theta_l e_l u_l^G - (1-\theta_l e_l)u_l^B\right] \\ &= u_h^1(u_l) - u_l.\end{aligned}$$

In conclusion, $(T_{\delta_2} f)(u_l) \geq u_h^1(u_l)$ for all $u_l \in [0, w]$. Therefore, the fixed point of T_{δ_2} is greater than the fixed point of T_{δ_1}.

Next we show that $u_h^*(u_l) \to w$ as $\delta \to 1$. Consider the following strategy of the principal. For some $N \in \mathbb{N}$, the principal does not fire or tenure the agent prior to N. At time $t = N$, the principal either fires or tenures the agent and the firing probability depends on the number of good outcomes achieved. Specifically, let k be the number of good outcomes. The agent is fired with probability 1 if $k < N(\theta_l - d)$, with some probability p if $k \in [N(\theta_l - d), N(\theta_l + d)]$, and with probability 0 if $k > N(\theta_l + d)$, where $d = \frac{1}{2}(\theta_h - \theta_l)$.

If the agent always exerts high effort before N, then k follows a binomial distribution which converges to a normal distribution as $N \to \infty$ and the variance is linear in N. Therefore, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\mathbb{P}(k \in [N(\theta_l - d), N(\theta_l + d)]) \geq 1 - \epsilon$ if the low type always exerts high effort and $\mathbb{P}(k \in [N(\theta_h - d), N(\theta_h + d)]) \geq 1 - \epsilon$ if the high type always exerts high effort. Pick any $\epsilon < \min(\frac{u_l}{2w}, 1 - \frac{u_l}{w})$ and let δ be large enough such that $(1-\delta)Nw < \frac{u_l}{2}$ and $\delta^N(1-\epsilon)w > u_l$. Then there exists $p \in (0,1)$ such that the utility of the low type given the above contract is u_l. To be more specific, if $p = 0$, the agent's utility must be smaller than $(1-\delta)Nw + \delta^N \epsilon w < u_l$. If $p = 1$, the agent can always exert high effort and get a utility larger than $\delta^N(1-\epsilon)w > u_l$.

Now consider the high type. If he always exerts high effort before N, then by definition, his expected utility given the above contract is greater than $\delta^N(1-\epsilon)w$. For any $\eta > 0$, pick any $\epsilon < \min(\frac{\eta}{2w}, \frac{u_l}{2w}, 1 - \frac{u_l}{w})$. Then there exists $\bar{\delta} < 1$ such that whenever $\delta > \bar{\delta}$, we have $\delta^N(1-\epsilon)w > w - \eta$, $(1-\delta)Nw < \frac{u_l}{2}$, and $\delta^N(1-\epsilon)w > u_l$. As shown above, for any $\delta > \bar{\delta}$, there exists $p \in (0,1)$ such that the low type gets exactly u_l.

In conclusion, for any $\eta > 0$, there exists $\bar{\delta} < 1$ such that $(u_l, w - \eta)$ is feasible whenever $\delta > \bar{\delta}$. Therefore, $u_h^*(u_l) \to w$ as $\delta \to 1$. $\square$

Proof of Lemma 1 and Corollary 4. Define $u_h^0(u_l)$ as the maximum continuation utility of the high type given the contract Γ_l^* and the low type's utility being u_l. When it is never possible or optimal to induce the low type to exert high effort, i.e., $(1-\delta)w > w - \frac{(1-\delta)c}{\theta_l}$ or $lc + \theta_l(h-l)w \leq 0$, the principal optimally induces low effort or randomizes. One specific optimal contract is always randomizing between 0 and w. Given this contract, $u_h^0(u_l) = u_l$ for all $u_l \in [0, w]$. Apparently, it minimizes the high type's utility and $u_h^0(u_l)$ is concave.

Next consider the case where $(1-\delta)w \leq w - \frac{(1-\delta)c}{\theta_l}$ and $lc + \theta_l(h-l)w > 0$. Let Γ_l^* be an optimal contract where the principal does not randomize when $u_l \in [u_1, u_2]$ and randomizes otherwise, where $u_1 = (1-\delta)w$ and $u_2 = w - \frac{(1-\delta)c}{\theta_l}$. We will verify later that Γ_l^* minimizes the high type's utility out of all optimal contracts. By definition, $u_h^0(u_l)$ is the solution to the following equation:

$$u_h^0(u_l) = \begin{cases} \frac{u_l}{u_1} u_h^0(u_1) & \text{if } u_l < u_1, \\ \sup_{e_h} \{(1-\delta)(w - ce_h) + \delta \left[\theta_h e_h u_h^0(u_l^G) + (1-\theta_h e_h) u_h^0(u_l^B)\right]\} & \text{if } u_l \in [u_1, u_2], \\ \frac{u_l - u_2}{w - u_2} w + \frac{w - u_l}{w - u_2} u_h^0(u_2) & \text{if } u_l > u_2, \end{cases}$$

where $u_1 = (1-\delta)w$ and $u_2 = w - \frac{(1-\delta)c}{\theta_l}$. Similar to the proof of Corollary 2, it is easy to see that $u_h^0(u_l)$ is a fixed point of a contraction mapping. Denote this mapping by T. Let f be any bounded function on $[0, w]$. Then

$$(Tf)(u_l) = \begin{cases} \frac{u_l}{u_1} (Tf)(u_1) & \text{if } u_l < u_1, \\ \sup_{e_h} \{(1-\delta)(w - ce_h) + \delta \left[\theta_h e_h f(u_l^G) + (1-\theta_h e_h) f(u_l^B)\right]\} & \text{if } u_l \in [u_1, u_2], \\ \frac{u_l - u_2}{w - u_2} w + \frac{w - u_l}{w - u_2} (Tf)(u_2) & \text{if } u_l > u_2. \end{cases} \tag{11}$$

Let $f^{(0)}(u_l) = u_l$ and $f^{(n)}(u_l) = (Tf^{(n-1)})(u_l)$ for all $n \in \mathbb{N}^*$. Since $f^{(1)} \geq f^{(0)}$ and T satisfies the monotonicity condition, $f^{(n)}$ is an increasing sequence. By Contraction Mapping Theorem, $f^{(n)} \to u_h^0$. Apparently, $f^{(0)}$ is concave and $(f^{(0)})'_-(w) = 1 \geq \frac{\theta_l}{\theta_h}$. In the following, we show by induction that $f^{(n)}$ is concave and $(f^{(n)})'_-(w) \geq \frac{\theta_l}{\theta_h}$ for any $n \in \mathbb{N}$.

Suppose $f^{(n)}$ is concave and $(f^{(n)})'_-(w) \geq \frac{\theta_l}{\theta_h}$. Then $\theta_h(f^{(n)}(u_l^G) - f^{(n)}(u_l^B)) \geq (1-\delta)c$. Therefore, it is optimal for the high type to always exert high effort when he is not fired or tenured, i.e., $(Tf^{(n)})(u_l) = (1-\delta)(w-c) + \delta(\theta_h f^{(n)}(u_l^G) + (1-\theta_h) f^{(n)}(u_l^B))$ for $u_l \in [u_1, u_2]$. Since u_l^G and u_l^B are both linear in u_l and f is concave, it is obvious that $Tf^{(n)}$ is concave on $[u_1, u_2]$. By the expression of T, $(Tf^{(n)})'_+(u_1) = \theta_h (f^{(n)})'_+(u_1^G) + (1-\theta_h)(f^{(n)})'_+(0)$. By $f^{(n)}$ concave, $(f^{(n)})'_+(u_1^G) \leq (f^{(n)})'_+(0)$. Therefore, $(Tf^{(n)})'_+(u_1) \leq (f^{(n)})'_+(0)$. Since $(Tf^{(n)})(u_1) \geq f^{(n)}(u_1)$, $(Tf^{(n)})'_+(0) \geq (f^{(n)})'_+(0)$. As a result, $(Tf^{(n)})'_+(u_1) \leq (Tf^{(n)})'_+(0)$. Similarly, we have $(Tf^{(n)})'_-(u_2) = \theta_h (f^{(n)})'_-(w) + (1-\theta_h)(f^{(n)})'_-(u_2^B) \geq (f^{(n)})'_-(w)$. Since $(Tf^{(n)})(u_2) \geq f^{(n)}(u_2)$, $(Tf^{(n)})'_-(w) \leq (f^{(n)})'_-(w)$. Therefore, $(Tf^{(n)})'_-(u_2) \geq (Tf^{(n)})'_-(w)$. In conclusion, $Tf^{(n)}$ is concave. By $(f^{(n)})'_-(w) \geq \frac{\theta_l}{\theta_h}$, $f^{(n)}(u_2^B) \leq w - \frac{(1-\delta)c}{\delta\theta_h}$. As a result, $(Tf^{(n)})(u_2) = (1-\delta)(w-c) + \delta\theta_h w + \delta(1-\theta_h) f^{(n)}(u_2^B) \leq w - \frac{(1-\delta)c}{\theta_h}$. Then we can conclude that $(Tf^{(n)})'_-(w) \geq \frac{\theta_l}{\theta_h}$.

By the proof of Proposition 1, possible variants of Γ_l^* are to induce the low type to exert low effort when $u > u_2$ or to randomize when $u \in [u_1, u_2]$. To minimize the high type's utility, it is never optimal to induce the low type to exert low effort since it gives the high type more options. Instead, taking randomization is always better. We only need to show that randomization within $[u_1, u_2]$ does not affect the high type's utility. Define $\pi_l^*(u_l)$ as the value function when the agent is known to be a low type. Randomization within $[u_1, u_2]$ can arise in the low type's optimal contract only when $\pi_l^*(u_l)$ is linear on some interval within $[u_1, u_2]$. If it is the case, then the principal can either randomize on these linear parts or induce high effort directly. To show that randomization does not make a difference, we only need to show that $u_h^0(u_l)$ is linear whenever $\pi_l^*(u_l)$ is linear. Suppose $\pi_l^*(u_l)$ is linear on $[u_a, u_b]$, where $u_1 \leq u_a < u_b \leq u_2$. Then by concavity of $\pi_l^*(u_l)$, $\pi_l^*(u_l)$ must be linear on both $[u_a^B, u_b^B]$ and $[u_a^G, u_b^G]$. Let f be any bounded function on $[0, w]$ which has the same linear intervals as $\pi_l^*(u_l)$, i.e., f is linear on $[u_a, u_b]$ whenever $\pi_l^*(u_l)$ is linear on $[u_a, u_b]$. Since $\pi_l^*(u_l)$ is linear on $[u_a^B, u_b^B]$ and $[u_a^G, u_b^G]$ by our earlier argument, f is also linear on both intervals. Then by definition of T in (11), $(Tf)(u_l)$ is linear on $[u_a, u_b]$. By Contraction Mapping Theorem, $u_h^0(u_l)$ must be linear on $[u_a, u_b]$. Therefore, all forms of the optimal contract give the same $u_h^0(u_l)$.

Finally We show that $u_h^0(u_l)$ is concave. Similar to the method above, we only need to show that Tf is concave given f is concave, where T is defined as in (11) except that it now includes the option to not randomize when $u > u_2$. Since any randomization within

$[u_2, w]$ is feasible, $u_h^0(u_l)$ is concave on $[u_2, w]$. By the same argument as above, $u_h^0(u_l)$ is also concave on $[0, u_2]$. The only thing left to show is that $u_h^0(u_l)$ is concave at u_2. Consider any $[u_a, u_b]$ such that $u_a < u_2 < u_b$ and the high type optimally exerts high effort at u_b. Then $(Tf)(u_b) = (1-\delta)(w-c) + \delta\theta_h w + \delta(1-\theta_h)f(u_b^B)$. $(Tf)(u_a)$ and $(Tf)(u_2)$ can be expressed in a similar way, with $u_2^G = w$ and $u_a^G < w$. Let $u_2 = \lambda u_a + (1-\lambda)u_b$, where $\lambda \in (0,1)$. By the expression of u^B, $u_2^B = \lambda u_a^B + (1-\lambda)u_b^B$. By concavity of f and the fact that $f(u_a^G) < w$, we have $(Tf)(u_b) > \lambda(Tf)(u_a) + (1-\lambda)(Tf)(u_b)$. Therefore, $u_h^0(u_l)$ is concave on $[0, w]$. □

Proof of Lemma 2. Let $\Gamma_h^*(u_h)$ be the optimal contract given that the agent is known to be a high type and the promised utility is u_h. If the low type takes this contract, he can always exert low effort until being fired. By the proof of Proposition 1, the high type is always indifferent between high effort and low effort given the contract Γ_h^*. Therefore, he can always exert low effort and still get u_h. Since there is no difference between the high type and the low type when they exert low effort, the low type can also get u_h. As a result, $\pi_h(u_h, u_l)$ is maximized at $u_l = u_h$ given u_h fixed, i.e., $\pi_h(u_h, u_h) \geq \pi_h(u_h, u_l)$ for any u_l. Let $u_l^1 < u_l^2 < u_h$. By concavity of π_h, $\pi_h(u_h, u_l^2) \geq \min(\pi_h(u_h, u_l^1), \pi_h(u_h, u_h))$. Therefore, $\pi_h(u_h, u_l^2) \geq \pi_h(u_h, u_l^1)$.

Similarly, let $\Gamma_l^*(u_l)$ be the optimal contract given that the agent is known to be a low type and the promised utility is u_l. By definition of $u_h^0(u_l)$, $\pi_l(u_h, u_l)$ is maximized at $u_h = u_h^0(u_l)$ given u_l fixed, i.e., $\pi_l(u_h^0(u_l), u_l) \geq \pi_l(u_h, u_l)$ for any u_h. Let $u_h^1 < u_h^2 < u_h^0(u_l)$. By concavity of π_l, $\pi_l(u_h^2, u_l) \geq \min(\pi_l(u_h^1, u_l), \pi_l(u_h^0(u_l), u_l))$. Therefore, $\pi_l(u_h^2, u_l) \geq \pi_l(u_h^1, u_l)$. Similarly, let $u_h^1 > u_h^2 > u_h^0(u_l)$. By the same argument, $\pi_l(u_h^2, u_l) \geq \min(\pi_l(u_h^1, u_l), \pi_l(u_h^0(u_l), u_l))$ and thus $\pi_l(u_h^2, u_l) \geq \pi_l(u_h^1, u_l)$. □

Proof of Proposition 3. The IC constraints require that $u_h^h \geq u_h^l$ and $u_l^l \geq u_l^h$. Together with the feasibility constraint, we have $u_h^h \geq u_h^l \geq u_l^l \geq u_l^h$. By Lemma 2, $\pi_h(u_h, u_l)$ increases in u_l. Therefore, it is optimal to increase u_l^h until $u_l^h = u_l^l$.

Similarly, since $\pi_l(u_h, u_l)$ increases in u_h on $[u_l, u_h^0(u_l)]$, it is optimal to increase u_h^l to the point that $u_h^l = u_h^0(u_l^l)$ or $u_h^l = u_h^h$. Thus, the only scenario where $u_h^h > u_h^l$ can be in the optimal mechanism is $u_h^h > u_h^0(u_l^l) = u_h^l$. Now suppose $u_h^* \leq u_l^*$ and $u_h^h > u_h^0(u_l^l) = u_h^l$. If $u_l^l < u_l^*$, then by definition, $\pi_l(u_h^0(u_l), u_l)$ increases in u_l when $u_l < u_l^*$. Since $u_h^0(u_l^l) < u_h^h$, the principal can increase u_l^l such that her payoff is increased and no constraints are violated. If $u_l^l \geq u_h^*$, then $\pi_h(u_h, u_h)$ is decreasing in u_h when $u_h > u_h^*$. Together with Lemma 2, we have $\pi_h(u_h^h, u_l^h) \leq \pi_h(u_h^h, u_h^h) \leq \pi_h(u_l^h, u_l^h)$. By concavity of π_h, $\pi_h(u_h, u_l^h)$ decreases in u_h. Therefore, the principal can decrease u_h^h to u_h^l and then her payoff will be increased. $\square$

Proof of Lemma 3. Consider any (u_h, u_l) such that $u_h \leq u_h^0(u_l)$. When $lc + \theta_l(h-l)w \leq 0$ or $(1-\delta)w > w - \frac{(1-\delta)c}{\theta_l}$, it is never optimal to induce the low type to exert high effort. Therefore, $\pi_l(u_l, u_l) = \pi_l(u_h^0(u_l), u_l) = \frac{u_l}{w} \cdot l$. As a result, any kind of randomization is optimal. Specifically, $\pi_l(\lambda u_h^0(u), \lambda u) = \lambda \pi_l(u_h^0(u), u)$ for any $\lambda \leq 1$. We focus on the case where $lc + \theta_l(h-l)w > 0$ and $(1-\delta)w \leq w - \frac{(1-\delta)c}{\theta_l}$ in the following.

We first investigate the case where randomization does not take place. There are four possible levels of effort, $(e_h, e_l) = (1,1), (1,0), (0,1)$, or $(0,0)$. When the low type is induced to exert low effort, $\pi_l(u_h, u_l) = (1-\delta)l + \delta\pi_l(u_h^B, u_l^B)$, where $u_l^B = \frac{1}{\delta}(u_l - (1-\delta)w)$ by the promise keeping constraint. By Lemma 2, u_h^B should be as large as possible unless it reaches $u_h^0(u_l^B)$. By the promise keeping constraint for the high type, u_h^B is largest when the high type is also induced to exert low effort, with a value of $\bar{u}_h^B = \frac{1}{\delta}(u_h - (1-\delta)w)$. If $\bar{u}_h^B \leq u_h^0(u_l^B)$, then $\pi_l(u_h, u_l)$ can equivalently be achieved by randomization between (w, w) and $(\bar{u}_h^B, u_l^B)$. If $\bar{u}_h^B > u_h^0(u_l^B)$, then $\pi_l(u_h, u_l) = (1-\delta)l + \delta\pi_l(u_h^0(u_l^B), u_l^B) = (1-\delta)l + \delta\pi_l^*(u_l^B)$, where $\pi_l^*(u_l)$ is the value function when there is no adverse selection. At the same time, there exists $\tilde{u}_l > u_l^B$ and $\mu > \delta$ such that $(u_h, u_l) = \mu \cdot (u_h^0(\tilde{u}_l), \tilde{u}_l) + (1-\mu) \cdot (w, w)$. By randomization, $\pi_l(u_h, u_l) \geq (1-\mu)l + \mu\pi_l(u_h^0(\tilde{u}_l), \tilde{u}_l) = (1-\mu)l + \mu\pi_l^*(\tilde{u}_l)$. By concavity of π_l^*, $\mu\pi_l^*(\tilde{u}_l) \geq (\mu-\delta)l + \delta\pi_l^*(u_l^B)$. Therefore, inducing low effort cannot be better than taking randomization. Without loss of generality, we only need to consider $e_l = 1$. When $e_l = 1$, $\pi_l(u_h, u_l) = (1-\delta)(h\theta_l + l(1-\theta_l)) + \delta\theta_l\pi_l(u_h^G, u_l^G) + \delta(1-\theta_l)\pi_l(u_h^B, u_l^B)$. Since $u_h \leq u_h^0(u_l)$, it is impossible that $u_h^B > u_h^0(u_l^B)$ and $u_h^G > u_h^0(u_l^G)$. Furthermore, if $u_h^B > u_h^0(u_l^B)$, the principal can always decrease u_h^B and increase u_h^G, such that $u_h^B \leq u_h^0(u_l^B)$, $u_h^G \leq u_h^0(u_l^G)$, and the promise keeping constraint is satisfied. The same logic applies when $u_h^G > u_h^0(u_l^G)$. By Lemma 2, we must have $u_h^B \leq u_h^0(u_l^B)$ and $u_h^G \leq u_h^0(u_l^G)$. If $e_h = 0$, then it is optimal to increase u_h^G to the point that $u_h^G = u_h^0(u_l^G)$ or that the high type is indifferent between high effort and low effort. By Lemma 1, the high type optimally exerts high effort when $u_h^B = u_h^0(u_l^B)$ and $u_h^G = u_h^0(u_l^G)$. Therefore, the high type should also be induced to exert high effort. In summary, in the optimal contract for the low type, either both types are induced to exert high effort or randomization takes place. Moreover, the state variable always evolves within the relevant subset.

When both types are induced to exert high effort, by the IC constraint, $u_l^G - u_l^B \geq \frac{(1-\delta)c}{\delta\theta_l}$. Since $u_h^G \geq u_l^G$ and $u_h^B \geq u_l^B$, the high type's utility

$$u_h \geq (1-\delta)(w-c) + \delta\theta_h u_l^G + \delta(1-\theta_h)u_l^B \geq u_l + \delta(\theta_h - \theta_l)(u_l^G - u_l^B) \geq u_l + (1-\delta)c \cdot \frac{\theta_h - \theta_l}{\theta_l}.$$

Therefore, when $u_h - u_l < (1-\delta)c \cdot \frac{\theta_h - \theta_l}{\theta_l}$, the principal must use randomization.

Suppose randomization is optimal given some (u_h, u_l). Let $(u_h, u_l) = \sum_{i=1}^n \lambda_i \cdot (u_h^i, u_l^i)$ be the optimal randomization, where $\sum_{i=1}^n \lambda_i = 1$ and $\lambda_i > 0$ for all i. Apparently, (u_h^i, u_l^i) must be within the relevant subset for all i. Without loss of generality, we assume that randomization does not take place at any (u_h^i, u_l^i). Suppose that there exists $j \neq k$, such that both types are induced to exert high effort at (u_h^j, u_l^j) and (u_h^k, u_l^k). Consider a new state variable $(\tilde{u}_h, \tilde{u}_l) := \frac{\lambda_j (u_h^j, u_l^j) + \lambda_k (u_h^k, u_l^k)}{\lambda_j + \lambda_k}$. Then $(\tilde{u}_h, \tilde{u}_l)$ is a linear combination of (u_h^j, u_l^j) and (u_h^k, u_l^k). By concavity of $u_h^0(u_l)$, $(\tilde{u}_h, \tilde{u}_l)$ must be within the relevant subset as well. Let $(\tilde{u}_h, \tilde{u}_l)^G = \frac{\lambda_j (u_h^j, u_l^j)^G + \lambda_k (u_h^k, u_l^k)^G}{\lambda_j + \lambda_k}$ and $(\tilde{u}_h, \tilde{u}_l)^B = \frac{\lambda_j (u_h^j, u_l^j)^B + \lambda_k (u_h^k, u_l^k)^B}{\lambda_j + \lambda_k}$, where $(u_h, u_l)^G$ denotes the vector (u_h^G, u_l^G) and $(u_h, u_l)^B$ is defined accordingly. Then both types exert high effort at $(\tilde{u}_h, \tilde{u}_l)$ and the promise keeping constraints are satisfied. By concavity of $\pi_l(u_h, u_l)$, inducing high effort at $(\tilde{u}_h, \tilde{u}_l)$ is weakly better than randomizing between (u_h^j, u_l^j) and (u_h^k, u_l^k). Therefore, $(u_h, u_l) = \sum_{i \neq j,k} \lambda_i \cdot (u_h^i, u_l^i) + (\lambda_j + \lambda_k)(\tilde{u}_h, \tilde{u}_l)$ is also an optimal randomization. Without loss of generality, there exists a single j such that high effort is induced at (u_h^j, u_l^j). Since randomization does not take place at any (u_h^i, u_l^i), the only possibility is that $(u_h^i, u_l^i) = (0,0)$ or $(u_h^i, u_l^i) = (w, w)$. In summary, the optimal randomization must take the form of $(u_h, u_l) = \lambda_1 \cdot (0,0) + \lambda_2 \cdot (w,w) + \lambda_3 \cdot (u_h^1, u_l^1)$, where high effort is induced at (u_h^1, u_l^1), $\lambda_1 + \lambda_2 + \lambda_3 = 1$, and $\lambda_i \geq 0$ for $i = 1,2,3$. In the non-trivial case where $u_h > u_l$, $\lambda_3 > 0$.

We first argue that $\lambda_1 > 0$ when $u_l < (1-\delta)w$ and $\lambda_2 > 0$ when $u_l > w - \frac{(1-\delta)c}{\theta_l}$. Let $u_l < (1-\delta)w$. Since high effort is induced at (u_h^1, u_l^1), we must have $u_l^1 \geq (1-\delta)w$. By $(u_h, u_l) = \lambda_1 \cdot (0,0) + \lambda_2 \cdot (w,w) + \lambda_3 \cdot (u_h^1, u_l^1)$, $\lambda_1 > 0$. The proof for the other part is the same. A direct implication is that π_l is linear from $(0,0)$ to (u_h, u_1) for any u_h and from

(u_h, u_2) to (w, w) for any u_h.

We show in the following that $\lambda_2 = 0$ when $u_l^1 < w - \frac{(1-\delta)c}{\theta_l}$. By Lemma 4, we can assume $l = 0$ without loss of generality. Then it is easy to see $\pi_l(u_l, u_l) = 0$ for any $u_l \in [0, w]$. Consider the scenario where the principal either induces high effort or fires the agent with a positive probability when $u_l \leq w - \frac{(1-\delta)c}{\theta_l}$. Denote her maximum payoff in this situation by $\tilde{\pi}_l(u_h, u_l)$. Then by definition,

$$
\begin{aligned}
\tilde{\pi}_l(u_h, u_l) = \sup_{\lambda_1, \lambda_2, u_h^G, u_h^B, u_l^G, u_l^B} \quad & (1 - \lambda_1 - \lambda_2) \left[(1-\delta)h\theta_l + \delta\theta_l\tilde{\pi}_l(u_h^G, u_l^G) + \delta(1-\theta_l)\tilde{\pi}_l(u_h^B, u_l^B)\right] \\
\text{s.t.} \quad & u_l^B \leq u_h^B \leq u_h^*(u_l^B), \ u_l^G \leq u_h^G \leq u_h^*(u_l^G), \\
& \frac{u_l - \lambda_2 w}{1 - \lambda_1 - \lambda_2} = (1-\delta)(w-c) + \delta(\theta_l u_l^G + (1-\theta_l)u_l^B) \geq (1-\delta)w + \delta u_l^B, \\
& \frac{u_h - \lambda_2 w}{1 - \lambda_1 - \lambda_2} = (1-\delta)(w-c) + \delta(\theta_h u_h^G + (1-\theta_h)u_h^B) \geq (1-\delta)w + \delta u_h^B, \\
& \lambda_2 \cdot \mathbb{1}_{\{\frac{u_l - \lambda_2 w}{1-\lambda_1-\lambda_2} < w - \frac{(1-\delta)c}{\theta_l}\}} = 0,
\end{aligned}
$$

where λ_1 is the probability of firing the agent and λ_2 is the probability of tenuring the agent. The last constraint means that we do not consider randomization with (w, w) unless $u_l > w - \frac{(1-\delta)c}{\theta_l}$. Specifically, consider any (u_h, u_l) where $u_l < w - \frac{(1-\delta)c}{\theta_l}$. Then the principal either induces high effort or fires the agent with a positive probability. Suppose the latter is optimal, and conditional on not being fired, the state variable is $(\tilde{u}_h, \tilde{u}_l)$. If $\tilde{u}_l \leq w - \frac{(1-\delta)c}{\theta_l}$, then both types are induced to exert high effort at $(\tilde{u}_h, \tilde{u}_l)$ and $\lambda_2 = 0$. If $\tilde{u}_l > w - \frac{(1-\delta)c}{\theta_l}$, then the agent is tenured with a positive probability. Conditional on not being tenured, the new state variable (u_h^1, u_l^1) should satisfy $u_l^1 = w - \frac{(1-\delta)c}{\theta_l}$ and both types are induced to exert high effort at (u_h^1, u_l^1).

We want to show that $\tilde{\pi}_l(u_h, u_l) = \pi_l(u_h, u_l)$. Specifically, since we have shown that inducing low effort from any type is not optimal, we only need to show that $\tilde{\pi}_l(u_h, u_l)$ is concave. By $\delta < 1$, it is easy to verify that $\tilde{\pi}_l(u_h, u_l)$ is a fixed point of a contraction mapping. Denote this mapping by T. Let f be any bounded function on the relevant subset such that f is concave, continuous, and $f(u_l, u_l) = 0$. Then apparently we have $(Tf)(u_l, u_l) = 0$. By Maximum Theorem, Tf is also continuous. By Contraction Mapping Theorem, we only need to show that (Tf) is concave.

First consider any (u_h, u_2) where $u_2 = w - \frac{(1-\delta)c}{\theta_l}$. If it is delivered by randomization, i.e., $\lambda_1 > 0$, then $(Tf)(u_h, u_2) = (1-\lambda_1) \cdot (Tf)(\frac{u_h}{1-\lambda_1}, \frac{u_2}{1-\lambda_1})$. Since $\frac{u_2}{1-\lambda_1} > u_2$, the optimal way to deliver $(\frac{u_h}{1-\lambda_1}, \frac{u_2}{1-\lambda_1})$ is to randomize between (w, w) and (u_h', u_2) for some $u_h' > u_h$. Therefore, there exists $\lambda_2 \in (0, 1-\lambda_1)$ such that $(u_h, u_2) = \lambda_1 \cdot (0,0) + \lambda_2 \cdot (w, w) + (1-\lambda_1-\lambda_2) \cdot (u_h', u_2)$ and $(Tf)(u_h, u_2) = (1-\lambda_1-\lambda_2) \cdot (Tf)(u_h', u_2)$. By rearrangement, $(u_h, u_2) = (\lambda_1+\lambda_2) \cdot (u_2, u_2) + (1-\lambda_1-\lambda_2) \cdot (u_h', u_2)$ and $(Tf)(u_h, u_2) = (\lambda_1+\lambda_2) \cdot (Tf)(u_2, u_2) + (1-\lambda_1-\lambda_2) \cdot (Tf)(u_h', u_l)$. If it is optimal to induce high effort at (u_h, u_2), then $(Tf)(u_h, u_2) \geq (1-\lambda_1)(Tf)(\frac{u_h}{1-\lambda_1}, \frac{u_2}{1-\lambda_1})$ for any $\lambda_1 > 0$. By the same logic as above, for any $u_h' > u_h$ and $\lambda > 0$ such that $\lambda u_h' + (1-\lambda)u_2 = u_h$, $(Tf)(u_h, u_2) \geq \lambda \cdot (Tf)(u_h', u_2) + (1-\lambda) \cdot (Tf)(u_2, u_2)$. In conclusion, Tf is concave on the line $u_2 = w - \frac{(1-\delta)c}{\theta_l}$. Moreover, Tf is linear from (u_2, u_2) to some $(\bar{u}_h, u_2)$ if and only if it is optimal to fire the agent with a positive probability given any (u_h, u_2) such that $u_h < \bar{u}_h$.

Define $\bar{u}_h$ as the largest u_h such that Tf is linear from (u_2, u_2) to (u_h, u_2). Then for any $\lambda_i > 0$ $(i = 1, 2, 3)$ such that $\lambda_1 + \lambda_2 + \lambda_3 = 1$, $(Tf)(\lambda_1 \cdot (0,0) + \lambda_2 \cdot (w, w) + \lambda_3 \cdot (\bar{u}_h, u_2)) = \lambda_1(Tf)(0,0) + \lambda_2(Tf)(w, w) + \lambda_3(Tf)(\bar{u}_h, u_2)$. As a result, (Tf) is concave within the triangle formed by $(0,0)$, (w, w), and $(\bar{u}_h, u_2)$. We next argue that $\bar{u}_h = u_h^0(u_2)$ when δ is large enough.

Finally we show that it is concave when $u_l < w - \frac{(1-\delta)c}{\theta_l}$. Suppose (Tf) is not concave around some (u_h, u_l) where $u_l < w - \frac{(1-\delta)c}{\theta_l}$. Then there exists $(\hat{u}_h, \hat{u}_l)$, $(\tilde{u}_h, \tilde{u}_l)$, and $\mu \in (0,1)$ such that $(u_h, u_l) = \mu(\hat{u}_h, \hat{u}_l) + (1-\mu)(\tilde{u}_h, \tilde{u}_l)$ and $(Tf)(u_h, u_l) < \mu(\hat{u}_h, \hat{u}_l) + (1-\mu)(\tilde{u}_h, \tilde{u}_l)$. [To be finished] $\square$

Proof of Lemma 4. Suppose $l \neq 0$. We show in the following that the optimal policy is the same as in the case where $l = 0$. Consider the HJB equation in (6) and ignore concavification for a moment. By the analysis in the first paragraph of the proof of Lemma 3, we only need to consider the case where $e_l = 1$. Then by rearrangement, if $l > 0$, we have

$$\pi_l(u_h, u_l)\frac{w}{l} = \sup_{u_h^G, u_h^B, u_l^G, u_l^B} (1-\delta)[h\theta_l + l(1-\theta_l)]\frac{w}{l} + \delta\theta_l\pi_l(u_h^G, u_l^G)\frac{w}{l} + \delta(1-\theta_l)\pi_l(u_h^B, u_l^B)\frac{w}{l}$$

subject to the original feasibility constraints, IC_l, PK_l, and PK_h. If $l < 0$, the supremum should be changed to the infimum. Let $f(u_h, u_l) = \pi_l(u_h, u_l)\frac{w}{l}$. Then f is the solution to a new HJB equation, with $(1-\delta)[h\theta_l + l(1-\theta_l)]$ replaced by $(1-\delta)[h\theta_l + l(1-\theta_l)]\frac{w}{l}$ in the objective function and the boundary conditions being $f(0,0) = 0$ and $f(w,w) = w$. Apparently, the optimal choice variables are the same in two HJB equations. Let $g(u_h, u_l) = f(u_h, u_l) - u_l$. By the promise keeping constraint for the low type, $\delta\theta_l u_l^G + \delta(1-\theta_l)u_l^B = u_l - (1-\delta)(w-c)$. Therefore, when $l > 0$, we obtain

$$g(u_h, u_l) = \sup_{u_h^G, u_h^B, u_l^G, u_l^B} (1-\delta)[h\theta_l + l(1-\theta_l)]\frac{w}{l} - (1-\delta)(w-c) + \delta\theta_l g(u_h^G, u_l^G) + \delta(1-\theta_l)g(u_h^B, u_l^B)$$

subject to the same constraints as in (6) and the boundary conditions $g(0,0) = g(w,w) = 0$. The supremum is again replaced by the infimum when $l < 0$.

Let $h(u_h, u_l) = g(u_h, u_l) \cdot \frac{hl\theta_l}{lc+\theta_l(h-l)w}$. Since $lc + \theta_l(h-l)w > 0$, we have $l > 0$ if and only if $\frac{hl\theta_l}{lc+\theta_l(h-l)w} > 0$. Therefore,

$$h(u_h, u_l) = \sup_{u_h^G, u_h^B, u_l^G, u_l^B} (1-\delta)h\theta_l + \delta\theta_l h(u_h^G, u_l^G) + \delta(1-\theta_l)h(u_h^B, u_l^B)$$

subject to the same constraints as before and the boundary conditions $h(0,0) = h(w,w) = 0$, which is exactly the HJB equation when $l = 0$. In other words, the value of l does not affect the relative magnitude of the value function at two different points. Therefore, the optimal choice variables are the same for different values of l.

By the analysis in the first paragraph of the proof of Lemma 5, when randomization does not take place, it is optimal to induce $e_h = 1$ in the high type's contract. As a result,

$$\pi_h(u_h, u_l) = \sup_{u_h^G, u_h^B, u_l^G, u_l^B} (1-\delta)[h\theta_h + l(1-\theta_h)] + \delta\theta_h\pi_h(u_h^G, u_l^G) + \delta(1-\theta_h)\pi_h(u_h^B, u_l^B)$$

subject to the feasibility constraint, IC_h, PK_h, and PK_l as in the original HJB equation. By the same transformation as above, we can conclude that the high type's contract does not depend on the value of l given any fixed (u_h, u_l). $\square$

Proof of Lemma 5. Similar to the proof of Lemma 3, we first look at the optimal strategy when randomization does not take place. If the high type is induced to exert low effort, then $\pi_h(u_h, u_l) = (1-\delta)l + \delta\pi_l(u_h^B, u_l^B)$, where $u_h^B = \frac{1}{\delta}(u_h - (1-\delta)w)$ by the promise keeping constraint. By Lemma 2, u_l^B should be as large as possible whenever it is not larger than

u_h^B. By the promise keeping constraint for the low type, u_l^B is largest when the low type is also induced to exert low effort, where $u_l^B = \frac{1}{\delta}(u_l - (1-\delta)w)$. Since $u_l \le u_h$, we must have $u_l^B \le u_h^B$. Therefore, it is optimal to induce the low type to induce low effort as well. By the expression of $\pi_h(u_h, u_l)$, the principal can randomize between (w, w) and (u_h^B, u_l^B) and get the same utility. Without loss of generality, we can assume that the principal never induces the high type to exert low effort. When the high type is induced to exert high effort, $\pi_h(u_h, u_l) = (1-\delta)(h\theta_h + l(1-\theta_h)) + \delta\theta_h\pi_h(u_h^G, u_l^G) + \delta(1-\theta_h)\pi_h(u_l^G, u_l^B)$. By Lemma 2, a larger u_l^G benefits the principal. Thus, if the low type is induced to exert low effort, the principal can always increase u_l^G to the point that $u_l^G = u_h^G$ or the low type is indifferent between both effort levels. Without loss of generality, the low type is also induced to exert high effort unless $u_l^G = u_h^G$.

By Lemma 4, we assume $l = 0$ without loss of generality. Let $u_1 = (1-\delta)w$ and $u_2 = w - \frac{(1-\delta)c}{\theta_h}$. When $u_1 > u_2$, by the proof of Proposition 1, the high type can never be induced to exert high effort. Thus, we focus on the case where $u_1 \le u_2$. When $u_l \ge u_1$ and $u_h \le u_2$, there exists u_h^G and u_h^B such that the high type is induced to exert high effort and his promise keeping constraint is satisfied. Since $u_l \le u_h$, there must exist $u_l^G \le u_h^G$ and $u_l^B \le u_h^B$ such that the low type's promise keeping constraint is also satisfied. Therefore, it is possible to induce the high type to exert high effort whenever $u_l \ge u_1$ and $u_h \le u_2$. We show in the following that it is optimal to do so. Specifically, define $\hat{\pi}_h(u_h, u_l)$ as the principal's maximum payoff when she induces the high type to exert high effort whenever possible. Then for any feasible (u_h, u_l) such that $u_l \ge u_1$ and $u_h \le u_2$,

$$\begin{aligned}
\hat{\pi}_h(u_h, u_l) = \sup_{u_h^G, u_h^B, u_l^G, u_l^B} \; & (1-\delta)h\theta_h + \delta\theta_h\hat{\pi}_h(u_h^G, u_l^G) + \delta(1-\theta_h)\hat{\pi}_h(u_h^B, u_l^B) \qquad (12)\\
\text{s.t.} \quad & u_l^B \le u_h^B \le u_h^*(u_l^B),\ u_l^G \le u_h^G \le u_h^*(u_l^G),\\
& u_h = (1-\delta)(w-c) + \delta(\theta_h u_h^G + (1-\theta_h)u_h^B) \ge (1-\delta)w + \delta u_h^B,\\
& u_l = \sup_{e\in[0,1]} (1-\delta)(w - ce) + \delta(\theta_l e u_l^G + (1-\theta_l e)u_l^B).
\end{aligned}$$

Since it is optimal to randomize when the high type cannot be induced to exert high effort, we define $\hat{\pi}_h(u_h, u_l) = \frac{u_l}{u_1}\hat{\pi}_h(u_h \cdot \frac{u_1}{u_l}, u_1)$ for $u_l < u_1$. When $u_h > u_2$, we similarly define

$\hat{\pi}_h(u_h, u_l) = \frac{w-u_h}{w-u_2}\hat{\pi}_h(w-u_2, w-(w-u_l)\cdot\frac{w-u_2}{w-u_h})$. By definition, we only need to show that $\hat{\pi}_h(u_h, u_l) = \pi_h(u_h, u_l)$. Since we have shown that inducing low effort is not optimal, the only thing left to show is that $\hat{\pi}_h(u_h, u_l)$ is concave.

By definition, it is easy to verify that $\hat{\pi}_h(u_h, u_l)$ is a fixed point of a contraction mapping. Denote this mapping by T. Let f be any concave function defined on the feasible set such that $f(u, u) = \pi_h^*(u)$, where $\pi_h^*(u)$ is the principal's value function when there is no adverse selection. In addition, let $f(u_h, u_l)$ increase in u_l for any given u_h. Then it is obvious that $(Tf)(u, u) = \pi_h^*(u)$ and $(Tf)(u_h, u_l)$ also increases in u_l. By Contraction Mapping Theorem, we only need to show that Tf is also concave.

We first look at the case where $u_l > u_1$ and $u_h < u_2$. Consider any feasible (u_h^1, u_l^1) and (u_h^2, u_l^2) such that $(u_h, u_l) = \lambda\cdot(u_h^1, u_l^1) + (1-\lambda)\cdot(u_h^2, u_l^2)$, where $\lambda \in (0,1)$, $u_l^i > u_1$, and $u_h^i < u_2$ $(i = 1, 2)$. By definition of T, the high type is induced to exert high effort at both (u_h^1, u_l^1) and (u_h^2, u_l^2). Then by the expression in (12), we have $(Tf)(u_h^i, u_l^i) = (1-\delta)h\theta_h + \delta\theta_h f(u_h^{iG}, u_l^{iG}) + \delta(1-\theta_h)f(u_h^{iB}, u_l^{iB})$ $(i = 1, 2)$, where u_h^{1G} is the high type's continuation utility at (u_h^1, u_l^1) given a good outcome and other notations are defined similarly. Let $(u_h^G, u_l^G, u_h^B, u_l^B) = \lambda(u_h^{1G}, u_l^{1G}, u_h^{1B}, u_l^{1B}) + (1-\lambda)(u_h^{2G}, u_l^{2G}, u_h^{2B}, u_l^{2B})$. Since the feasible set is concave, $(u_h^G, u_l^G, u_h^B, u_l^B)$ satisfies the feasibility constraint. By the IC constraint, the high type optimally exerts high effort at (u_h, u_l). Then it is easy to verify that $(u_h^G, u_l^G, u_h^B, u_l^B)$ satisfies the high type's promise keeping constraint. Regarding the low type's promise keeping constraint, we have

$$
\begin{aligned}
&\sup_{e\in[0,1]} (1-\delta)(w-ce) + \delta(\theta_l e u_l^G + (1-\theta_l e)u_l^B) \\
\le\ &\lambda \sup_{e\in[0,1]} (1-\delta)(w-ce) + \delta(\theta_l e u_l^{1G} + (1-\theta_l e)u_l^{1B}) \\
&+ (1-\lambda) \sup_{e\in[0,1]} (1-\delta)(w-ce) + \delta(\theta_l e u_l^{2G} + (1-\theta_l e)u_l^{2B}) \\
=\ &\lambda u_l^1 + (1-\lambda)u_l^2 = u_l.
\end{aligned}
$$

When the inequality holds, we can increase u_l^G and u_l^B such that the PK$_l$ constraint is satisfied without violating the feasibility constraint. Since $(Tf)(u_h, u_l)$ increases in u_l, we have $(Tf)(u_h, u_l) \ge (1-\delta)h\theta_h + \delta\theta_h f(u_h^G, u_l^G) + \delta(1-\theta_h)f(u_h^B, u_l^B)$. By concavity of f, it is easy to see that $(Tf)(u_h, u_l) \geq \lambda(Tf)(u_h^1, u_l^1) + (1-\lambda)(Tf)(u_h^2, u_l^2)$. As a result, Tf is concave in the region given by $u_l > u_1$ and $u_h < u_2$.

By the same logic, $(Tf)(u_h, u_1)$ is concave in u_h and $(Tf)(u_2, u_l)$ is concave in u_l. Consider the case where $u_l < u_1$. Let (u_h^1, u_l^1) and (u_h^2, u_l^2) be two feasible state variables such that $u_l^i < u_1$ $(i = 1, 2)$ and $(u_h, u_l) = \lambda \cdot (u_h^1, u_l^1) + (1-\lambda) \cdot (u_h^2, u_l^2)$ for some $\lambda \in (0, 1)$. It is obvious that $(Tf)(u_h, u_l) = \lambda(Tf)(u_h^1, u_l^1) + (1-\lambda)(Tf)(u_h^2, u_l^2)$ when $u_l^1 = 0$ or $u_l^2 = 0$. We assume $u_h^i > 0$ $(i = 1, 2)$ in the following. By definition, $(Tf)(u_h^i, u_l^i) = \frac{u_l^i}{u_1}(Tf)(u_h^i \cdot \frac{u_1}{u_l^i}, u_1)$ $(i = 1, 2)$ and $(Tf)(u_h, u_l) = \frac{u_l}{u_1}(Tf)(u_h \cdot \frac{u_1}{u_l}, u_1)$. Let $\mu = \lambda \cdot \frac{u_l^1}{u_l}$. By $u_l = \lambda u_l^1 + (1-\lambda)u_l^2$, we have $\mu \in (0, 1)$ and $1 - \mu = (1-\lambda)\frac{u_l^2}{u_l}$. Furthermore, by $u_h = \lambda u_h^1 + (1-\lambda)u_h^2$, we can obtain $u_h \cdot \frac{u_1}{u_l} = \mu u_h^1 \cdot \frac{u_1}{u_l^1} + (1-\mu)u_h^1 \cdot \frac{u_1}{u_l^1}$. By concavity of $(Tf)(u_h, u_1)$,

$$
\begin{aligned}
(Tf)(u_h, u_l) &\geq \frac{u_l}{u_1}\left[\mu(Tf)(u_h^1 \cdot \frac{u_1}{u_l^1}, u_1) + (1-\mu)(Tf)(u_h^2 \cdot \frac{u_1}{u_l^2}, u_1)\right] \\
&= \lambda \cdot \frac{u_l^1}{u_1}(Tf)(u_h^1 \cdot \frac{u_1}{u_l^1}, u_1) + (1-\lambda) \cdot \frac{u_l^2}{u_1}(Tf)(u_h^2 \cdot \frac{u_1}{u_l^2}, u_1) \\
&= \lambda(Tf)(u_h^1, u_l^1) + (1-\lambda)(Tf)(u_h^2, u_l^2).
\end{aligned}
$$

Therefore, Tf is concave when $u_l < u_1$. By the same argument, Tf is concave when $u_h > u_2$. As a result, the only thing left to show is that Tf is concave at any (u_h, u_1) and (u_2, u_l).

$\square$