Dynamic Incentive Provision When Evaluation Takes Time*

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Abstract

I study a dynamic principal-agent model in which the agent continuously works on a project which may yield a success. The principal cannot observe the success, but she observes imperfect signals over time after the agent stops working. Making payments at a later time increases the principal's informativeness, but is also more costly due to the agent's relative impatience. I derive optimal contracts in two different settings. First, if success is observed by the agent, he is induced to exert full effort until success and report it truthfully. The principal makes deferred payments after the agent's report of success. Before the report, the principal makes an increasing flow payment starting from a certain positive time. Secondly, if success is unobserved by the agent, the principal sets a deadline and makes a deferred payment after the deadline. To reduce the agent's procrastination rent, the principal either terminates the project randomly or makes payments before the deadline, depending on the information structure.

Keywords: Dynamic contracting, deferred evaluation, unobservable success, moral hazard, deadline effect

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1 Introduction

In many principal-agent relationships, evaluation of the agent's performance takes time. For example, when a researcher develops a new theory, people may learn its value over time as they find ways to apply it. Similarly, the value of a new product may not be immediately recognized when it is first introduced. Especially when its performance is stochastic, it takes time to better evaluate its overall performance. In the pharmaceutical industry, effectiveness and side effects of a new drug or vaccine may be learned only after a long time of use. Similarly, in the financial sector, the performance of a new trading strategy needs to be tested under different market conditions.

In many of these examples, it takes time for the agent to complete the project and the principal can only evaluate his performance after he is finished. With this lack of continuous monitoring, the principal's tools for incentivizing the agent to work over time could be limited. One natural way to provide incentives is to defer payments until the principal has a precise evaluation of the agent's performance. However, deferring payments can be costly for the principal. For example, if the principal is a firm and the agent is an employee, it seems natural to expect the principal to be more patient than the agent. Then the principal faces a trade-off between paying later with a more precise evaluation and paying earlier with a less precise evaluation.

In this paper, I study optimal incentive provision in an environment where the evaluation of the agent's performance takes time and the principal is more patient than the agent. The optimal contract features a deadline, deferred payments after the deadline, and payments before the deadline. Payments after the deadline provide incentives for the agent to work continuously and payments before the deadline induces the agent to be truth-telling.

To be more specific, I construct a dynamic model where an agent continuously works on a project. The project succeeds at a Poisson rate which depends on the agent's instant effort. The principal cannot observe when success arrives. After the agent stops working, the principal observes imperfect signals over time on whether success has been achieved. The principal is more patient than the agent and has full commitment power. Both are assumed to be risk-neutral and the agent is protected by limited liability.

I first study a model where the agent observes success and can report it to the principal. In this model, the principal needs to induce both effort and truthful reporting. In the optimal contract, the agent exerts full effort until achieving a success and reports it immediately. The principal makes deferred payments after the agent's report. The expected amount of payments increases with the time of the report and the expected delay decreases. Before the report, the principal makes a constant flow payment starting from a certain positive time.

The optimal contract has three main features. First, payments after the report of success are made with delays. Intuitively, it takes time to differentiate the agent who has achieved success from the one who has not. If payments were made immediately after the report, the agent would have no incentives to work because he could report a fake success and get all the payments. In practice, stock options are one commonly observed example of deferred rewards. The value of a stock option depends on whether there was really a success and the employee's reward is determined by the firm's performance at a future time.

Secondly, the principal makes a constant flow payment before the report of success. Payments before the report are not useful in incentivizing the agent to work because the agent can always get these payments even if he shirks. The reason that the principal makes payments before the report is to induce the agent to not report a fake success. Intuitively, the agent has incentives to report a fake success because the principal cannot perfectly verify whether a success has been achieved. Payments before the report rewards the agent for being honest because only the truthful agent may get these payments.

Thirdly, the expected amount of payments after the report increases with the time of the report and the expected delay decreases. To induce effort, the expected payment conditional on success needs to be larger than the expected payment conditional on no success by a certain amount. When the delay is smaller, the principal has less information and it is harder to differentiate a successful agent from a non-successful agent. As a result, the expected amount of payments needs to be larger. When the delay is larger, the expected payment to the agent can be smaller but the principal's cost increases due to the agent's relative impatience. This trade-off leans toward reducing the delay and increasing the expected payment when the report of success is at a later time. The reason is that payments after the report at $t > 0$ have a positive effect on incentive provision before t. Specifically, when the expected payment after the report at t is larger, the agent has a stronger incentive to keep working at any time before t because there is some probability that he will succeed at time t. As a result, the agent has less incentives to report a fake success before t. This positive effect increases with the time of the report. Therefore, the expected payment after the report increases over time and the expected delay of payments decreases.

I extend the model to incorporate the possibility that success is made impossible due to some exogenous events. For example, it is possible that the agent's methodology is proven to be wrong, there is an unexpected cut on the funding, or the product is successfully developed by a competitor. Specifically, I assume that a breakdown arrives at an exogenous Poisson rate before success and the project has to be terminated after the breakdown. Most of the previous results still hold in this model with breakdown. One major difference is, instead of a flow payment before the agent's report of success, the principal makes a lump-sum payment at the time of the breakdown. Intuitively, these two types of payments make no difference for the agent who has not achieved a success. But for the agent who has achieved a success but did not report it, he can only claim the flow payment. To reduce the agent's incentive to hide a success, the principal prefers to make payments at the time of the breakdown.

The previous results depend crucially on the assumption that the agent observes success. However, there are also situations where the agent does not observe success. For example, if a researcher develops a new theory, even he may be unable to recognize the theory's value. When the agent does not observe success, there is no reporting problem and the principal only needs to induce effort. Since the principal can only evaluate the agent's performance after he stops working, the agent has incentives to procrastinate. Therefore, the principal's main objective in this scenario is to reduce the agent's procrastination rent.

In the optimal contract, the principal sets a stochastic deadline to terminate the project and makes a deferred payment after the deadline. No payments are made before the deadline. Similar to the model where the agent observes success, a deferred payment after the deadline incentivizes the agent to work. However, since there is no reporting problem, the principal does not make any payments before the deadline. The reason is that the principal observes nothing before the deadline and the agent can collect all payments before the deadline by shirking. To reduce the agent's incentive to procrastinate, the principal randomly terminates the project from the beginning. As a result, the agent always exerts full effort for fear of being terminated before success.

The results would be different if the principal can observe signals on the agent's performance before the deadline. Intuitively, payments before the deadline can now be conditional on signal realizations and may incentivize the agent to work. To illustrate this effect, I add the possibility of breakdown as in the first model. Then the principal can learn from (the absence of) the breakdown before the deadline. I show that in the optimal contract with a deterministic deadline, the agent exerts full effort until the deadline and the principal makes a deferred payment after the deadline. Different from the model without breakdown, the principal may make additional payments before the deadline, depending on parameter values. Specifically, when the deadline is relatively small, one payment before the deadline is optimal. When the deadline is relatively large, it is optimal to make two lump-sum payments and a flow payment in between. Payments before the deadline play a similar role as random termination in the model without breakdown. To secure these payments, the agent has incentives to work from the beginning in order to achieve an early success.

The optimal payment scheme when the agent does not observe success is very different from the one when he observes success. In the model where the agent observes success, he gets an information rent because of the option to report a fake success. To induce truthful reporting, the principal rewards the agent for being honest by making payments before the report of success. On the other hand, the agent gets a procrastination rent in the model where he does not observe success. To incentivize the agent to work from the beginning, the principal terminates the project randomly or makes payments before the deadline, depending on the information structure before the deadline.

Related Literature. This paper contributes to the literature on persistent moral hazard. Hoffmann et al. (2020) study a similar model to mine where the principal observes signals over time on the agent's action. They allow for a general information structure and characterize the optimal timing of pay. The major difference is that the agent only takes one action in their model. By contrast, I focus on the dynamics of the principal-agent relationship before the agent stops working. Georgiadis and Szentes (2020) consider a similar setting as in Hoffmann et al. (2020) but focus on a risk-averse agent with a continuous effort space. Dai et al. (2020), Li and Yang (2020), and Varas et al. (2020) study scenarios where the monitoring technology is endogenously chosen by the principal. Several other papers study a dynamic environment with some similarities to my model, but without the need for delayed evaluation. Biais et al. (2010) consider a model where breakdowns arrive according to a Poisson process and an agent exerts private effort to prevent breakdowns. Chen et al. (2020) extend Biais et al. (2010) by allowing for costly monitoring. Myerson (2015) considers a similar setting but allows the principal to replace the agent in each period. More generally, Sannikov (2014) studies optimal contracting with a risk-averse agent where the agent's effort affects output at all future times. Hopenhayn and Jarque (2010), Jarque (2010), Varas (2018), and Zhu (2018) consider different settings where the agent's actions have a long-run effect. In my model, unlike these papers, the principal can only evaluate the agent's performance after the agent stops working. Also, the agent has dynamically arriving private information on the state of the world and the principal needs to induce both effort and truthful reporting.

This paper is also related to the literature on dynamic private information. Green and Taylor (2016) consider a multistage environment where the agent's intermediate progress is not observed by the principal. The principal eventually learns it when the agent achieves a second breakthrough. Klein (2016) studies an experimentation model where the principal cannot directly distinguish between genuine and fake success. However, the principal may be able to distinguish almost perfectly by observing certain signals. Similarly, Boleslavsky and Taylor (2020) study a setting where the principal does not observe whether a project is real or fake. In their model, it takes time to develop a real project but it is costly to develop a fake project. My paper also has the feature that the principal cannot immediately verify if a success is true, but I focus on the principal's trade-off between the cost of learning and the agent's information rent. More generally, Williams (2011) studies a contracting problem where the agent privately observes the realization of a Markov diffusion process. Madsen (2018) considers a model where the agent has incentives to hide the arrival of failure in order to prolong his employment. Escobar and Zhang (2019) study dynamic incentive provision with evolving private information and no monetary transfer.

One result of this paper is that the principal optimally pays the agent at the time of breakdown. Rewarding failure is not a new feature in the literature. Manso (2011) shows that the principal may reward early failures to incentivize the agent to experiment. In Kuvalekar and Ravi (2019), Hidir (2017), and Chade and Kovrijnykh (2016), the principal rewards failure because it indicates the agent's effort and reveals the quality of the project. Unlike these models, in this paper, the reason for rewarding failure is that it reduces the agent's incentive to report a fake success by rewarding him for being honest.

2 The Model

2.1 Setup

Time $t \geq 0$ is continuous and the horizon is infinite. A principal (she) contracts with an agent (he) to work on a project. At each time t, the agent exerts private effort $a_t \in [0, 1]$, with a flow cost ca_t ($c > 0$). The project succeeds at Poisson rate $\lambda_s a_t$ where $\lambda_s > 0$. Once it succeeds, the principal receives a positive benefit $\pi > 0$. Specifically, π may denote the expected value of all future payoffs and may not be immediately observed by the principal after success. I assume that success is not observable by the principal throughout the paper. I study the scenario where the agent observes success in Section 3 and the scenario where the agent does not observe success in Section 4. When the agent observes success, he can make an unverifiable report of success to the principal.

Before achieving a success, an exogenous breakdown arrives at Poisson rate $\lambda_f \geq 0$, where λ_f is independent of the agent's effort. The project will be terminated once a breakdown arrives. In reality, the breakdown could be any shock which makes the project infeasible. For example, it could be that the agent's methodology has been proven to be wrong or there is a cut on funding such that the principal can no longer support this project. I will study the case where there are no breakdowns in Sections 3.3 and 4.1.

The principal may irreversibly end the project before a breakdown occurs. If the project is ended by the principal, the agent has to stop working and the principal starts evaluating the agent's performance. Specifically, public signals on whether a success has been achieved arrive over time after the project is ended. I will describe evaluation technologies in more detail in section 2.2. The principal has full commitment power and offers a contract to the agent at time $t = 0$. A contract is denoted by a tuple, $\Gamma := (a_t, W_t, T)_{t\geq 0}$, where a_t is the recommended effort for the agent at time t, W_t is the cumulative payment up to t, and T is the time when the principal ends the project. Payments can still be made after T. I assume that the principal has access to randomization device. The agent is protected by limited liability in the sense that every realization of $\{W_t\}_{t\geq 0}$ is nonnegative and nondecreasing. The agent's outside option is normalized to zero. Consequently, the agent always accepts the contract.

Both parties are risk-neutral. The agent is assumed to be more impatient than the principal, with discount factors $\rho > r > 0$. This assumption rules out the possibility of indefinitely postponing payments. Otherwise the principal can approximately achieve the first best. To make the problem non-trivial, I maintain the following assumption throughout the paper.

Assumption 1. *$c < \lambda_s \pi$.*

$\lambda_s \pi$ is the marginal benefit of effort. When the cost of effort is greater than the expected benefit, the project is not profitable and the principal will not contract with the agent. By Assumption 1, I have the following result.

Proposition 1. *In the first best, the agent exerts full effort ($a_t = 1$) until a success or a breakdown .*

Since there is no uncertainty in the quality of the project and the arrival rate of success or breakdown does not change over time, the environment does not change until either a success or a breakdown occurs. Therefore, if it is optimal to make full effort at the beginning, then it is optimal to make full effort until a success or a breakdown.

Let $\phi_t := W_t - \lim_{\tau\to t^-} W_\tau$ be the lump-sum payment at time t. For technical convenience, I assume that W_t is piecewise continuous and $W_t - \phi_t$ is absolutely continuous. This assumption allows for all payment schemes with finitely many lump-sum payments and finitely many jumps in the flow rate. By absolute continuity, $W_t - \phi_t$ is differentiable almost everywhere. Let $w_t = (W_t - \phi_t)'_+$. Then $W_t = \int_0^t w_\tau \, d\tau + \Sigma_{\{\tau \leq t | \phi_\tau > 0\}} \phi_\tau$. Define τ_f as the time of breakdown. Then the expected utility of the principal is given by

$$\pi_0 = r \cdot \mathbb{E}\left[\int_0^{T \wedge \tau_f} \lambda_s a_t e^{-rt - \int_0^t \lambda_s a_\tau d\tau} \pi \, dt - \int_0^\infty e^{-rt} \, dW_t\right], \tag{1}$$

and the expected utility of the agent is given by

$$u_0 = \rho \cdot \mathbb{E}\left[\int_0^\infty e^{-\rho t} \, dW_t - \int_0^{T \wedge \tau_f} e^{-\rho t} c a_t \, dt\right]. \tag{2}$$

2.2 Evaluation Technologies

The evaluation of the agent's performance starts only after the project is ended. Specifically, signals on whether success has been achieved arrive over time after T. Denote by $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}^S)$ the filtered probability space of signals conditional on a success. Let $\mathbb{P}^{NS}$ be the probability measure conditional on no success. The probability space is assumed to be independent of T, which means that the evaluation technology does not depend on when the project is ended. Define $\mathbb{P}_t^S$ as the restriction of $\mathbb{P}^S$ to $\mathcal{F}_t$ and define $\mathbb{P}_t^{NS}$ accordingly. I impose the following assumption on the evaluation technology.

Assumption 2. *For any finite t, $\inf_{\{A_t \in \mathcal{F}_t | \mathbb{P}_t^S(A_t) > 0\}} \frac{\mathbb{P}_t^{NS}(A_t)}{\mathbb{P}_t^S(A_t)} > 0$.*

Intuitively, when $\mathbb{P}_t^{NS}(A_t) = 0$ and $\mathbb{P}_t^S(A_t) > 0$, the principal learns perfectly that there was a success by observing any signal realizations in A_t. Therefore, Assumption 2 indicates that it is impossible for the principal to learn almost perfectly within finite time that there was a success. This assumption rules out the trivial case where the principal never makes payments before learning almost perfectly.

A direct implication of Assumption 2 is that $\mathbb{P}_t^S$ is absolutely continuous with respect to $\mathbb{P}_t^{NS}$. By Radon-Nikodym theorem, there exists a $\mathcal{F}_t$-measurable function L_t such that $\mathbb{P}_t^S(A_t) = \int_{A_t} L_t \, d\mathbb{P}_t^{NS}$ for any $A_t \in \mathcal{F}_t$. By definition, when L_t is larger, the principal is more confident that there was a success. Therefore, if the principal decides to make payments at time $T + t$, it is optimal to make them conditional on signal realizations maximizing L_t. To ensure existence of an optimal contract, as in Hoffmann et al. (2020), I need to impose the following condition.

Assumption 3. $\mathbb{P}_t^S(\arg\max_{\omega\in\Omega} L_t(\omega)) > 0$ *for any* t.

This assumption requires that the set of signal realizations which maximize the principal's informativeness occurs with a positive probability. Otherwise the maximum informativeness cannot be achieved and the optimal contract does not exist.

3 Agent Observes Success

In this section, I study the scenario where the agent observes success. By the dynamic revelation principle, it is without loss of generality to focus on direct mechanisms where the agent reports success truthfully (See Myerson (1986)). I first formulate the agent's incentives in the following.

3.1 Incentive Compatibility

In this model, the principal needs to incentivize the agent to exert effort and report success truthfully. Accordingly, there are three IC constraints at each time t. The first one is the standard moral hazard constraint. The second one makes the agent not report a fake success and the third one keeps him from not reporting a success. To state these IC constraints formally, I define u_t as the on-path continuation utility of the agent before success. Also, define b_t as the on-path expected discounted payments when a success arrives at time t and l_t as the on-path expected payments when a breakdown arrives at time t. b_t and l_t can be thought of as reward for success and compensation for breakdown. Then the Hamilton-Jacobi-Bellman (HJB) equation is given by

$$u_t = \sup_{a_t}\{\rho(a_t\lambda_s b_t dt + \lambda_f l_t dt - ca_t dt + \phi_t + w_t dt) + (1-\rho dt)(1-\lambda_s a_t dt)(1-\lambda_f dt)u_{t+dt}\} + o(dt), \tag{3}$$

where dW_t is the payment at time t conditional on no success and no breakdown. Note that u_t may not be continuous in this setup. Since $\lim_{dt\to 0^+} u_{t+dt} = u_t - \rho\phi_t$, if there is a lump-sum payment $\phi_t > 0$, then the continuation utility jumps down at t. In the limit,

$$\lim_{dt\to 0^+} \frac{u_t - u_{t+dt} - \rho\phi_t}{dt} = \sup_{a_t}\{a_t(\rho\lambda_s b_t - \lambda_s u_t + \lambda_s \rho\phi_t - \rho c)\} + \rho(w_t + \lambda_f l_t) - (\rho + \lambda_f)(u_t - \rho\phi_t). \tag{4}$$

The agent chooses a_t to maximize his expected utility. By equation (4), the first IC constraint is formulated as

$$(\text{IC}_1) \qquad a_t = \begin{cases} 1 & \text{if } b_t \geq \frac{\lambda_s(u_t - \rho\phi_t) + \rho c}{\rho\lambda_s} \\ 0 & \text{if } b_t < \frac{\lambda_s(u_t - \rho\phi_t) + \rho c}{\rho\lambda_s} \end{cases} \tag{5}$$

Since both utilities are linear in a_t, optimally $a_t = 0$ or $a_t = 1$.

Define $\tilde{u}_t$ as the continuation utility when the agent has achieved a success by t but does not report it, and define $\tilde{b}_t$ as expected discounted payments when the agent reports a success at t but has not achieved it. Then the other two IC constraints can be formulated as

$$(\text{IC}_2) \qquad u_t - \rho\phi_t \geq \rho\tilde{b}_t \tag{6}$$

and

$$(\text{IC}_3) \qquad \rho b_t \geq \tilde{u}_t - \rho\phi_t. \tag{7}$$

The principal's optimization problem is not trivial since she cannot decrease $\tilde{b}_t$ without incurring additional cost. Specifically, to satisfy IC_2, the principal needs to increase u_t or decrease $\tilde{b}_t$. Increasing u_t requires a larger expected payment, which is costly for the principal. On the other hand, decreasing $\tilde{b}_t$ requires a larger delay of payments after report of success, which is also costly since the principal is more patient. This trade-off plays an important role in the derivation of the optimal contract.

3.2 After Report of Success

It is optimal for the principal to end the project immediately after the agent's report of success. Intuitively, the on-path agent has already achieved success by the time of report.

Therefore, ending the project does not decrease his expected utility. On the other hand, the off-path agent who has not achieved success may have incentives to keep working. Then ending the project immediately minimizes the off-path agent's expected utility, which is desirable for incentive compatibility. In addition, since evaluation starts after the project is ended, it is optimal for the principal to end the project and acquire information as soon as possible.

The more difficult question is how to arrange payments after report of success. Since the principal is more patient than the agent, paying at a later time is more costly. On the other hand, paying at a later time increases the principal's informativeness and thus reduces the agent's information rent. The following lemma shows that it is optimal to make a single payment if the evaluation takes the form of learning from bad news.

Lemma 1. *Suppose a bad news arrives at Poisson rate $\lambda > 0$ conditional on no success and no signals would arrive conditional on a success. Then the principal optimally makes one payment after report of success.*

Intuitively, the principal's cost of delaying payments can be characterized by the function $e^{(\rho-r)t}$, which is convex in the delay of payments. On the other hand, the information arrives gradually in the sense that the informativeness is concave in t. As a result, combining payments at different times is always desirable for the principal.

In general, a direct implication of Theorem 1 in Hoffmann et al. (2020) shows that for any evaluation technology, the principal optimally makes one or two payments after report of success.

3.3 Optimal Contracts with No Breakdowns

I first investigate a simple case where no breakdowns occur, i.e., $\lambda_f = 0$. To begin with, I ignore IC_3 and study the optimization problem with the other two IC constraints. I show in the end that IC_3 does not bind in this context.

From the analysis in the previous section, it is easy to show that IC_2 is binding. Intuitively, the only benefit of delaying payments after report is to reduce $\tilde{b}_t$. Since $\tilde{b}_t$ only appears in IC_2, it is optimal to delay payments to the point that IC_2 is binding. Let $\varphi_t(b_t, \tilde{b}_t)$

be the minimum cost of delivering b_t to the agent who has achieved a success and delivering $\tilde{b}_t$ to the one who has not achieved a success, after report of success at t. Then $\varphi_t(b_t, \tilde{b}_t)$ increases in b_t and decreases in $\tilde{b}_t$. Specifically, the amount of payments is determined by b_t and the expected delay of payments decreases in $\frac{b_t}{\tilde{b}_t}$. By IC_2 binding, φ_t can be written as a function of b_t and $u_t - \rho\phi_t$, i.e., $\varphi_t(b_t, \frac{u_t - \rho\phi_t}{\rho})$.

Suppose IC_3 is not binding. Then the only remaining IC constraint is IC_1. Following Spear and Srivastava (1987), I formulate the principal's problem recursively and take the agent's promised utility as the state variable. This method is valid in our environment since IC_1 can be fully characterized by variables at each time t. In other words, once u_t is given, the history does not matter. Define $\pi(u)$ as the maximum expected utility of the principal before success or report and given that the agent's expected utility is u. Then the HJB equation for $\pi(u)$ is given by

$$\begin{aligned}\pi(u_t) = \sup_{a_t, b_t, \phi_t, w_t} & \{r(\lambda_s a_t \pi dt - \phi_t - w_t dt - \lambda_s a_t \varphi_t(b_t, \frac{u_t - \rho\phi_t}{\rho})dt) \\ & + (1 - rdt)(1 - \lambda_s a_t dt)\pi(u_{t+dt})\} + o(dt) \qquad (8) \\ & \text{subject to equation (3) and IC}_1.\end{aligned}$$

Payments ϕ_t and w_t are made before report of success, therefore, they are not useful in incentivizing the agent to work. However, by equation (3), paying ϕ_t or w_t decreases u_{t+dt} and thus may increase the principal's expected utility in the next period. When u_t is very large, $\pi'(u_t)$ can be very negative. Then the benefit of decreasing u_t is large enough to compensate for the cost, making $\phi_t > 0$ optimal. On the other hand, $\phi_t = w_t = 0$ is optimal when u_t is small.

Next I claim that IC_1 is binding whenever $a_t = 1$, i.e., $b_t = \frac{\lambda_s(u_t - \rho\phi_t) + \rho c}{\rho\lambda_s}$. Intuitively, the only benefit of making $b_t > \frac{\lambda_s(u_t - \rho\phi_t) + \rho c}{\rho\lambda_s}$ is to decrease u_{t+dt}. However, since b_t is paid with a delay, increasing b_t is more costly than increasing w_t. In other words, if $b_t > \frac{\lambda_s(u_t - \rho\phi_t) + \rho c}{\rho\lambda_s}$, the principal can get better off by decreasing b_t and increasing w_t such that u_{t+dt} is unchanged and IC_2 is still satisfied.

Suppose $\phi_t = 0$, then by IC_1 binding, φ_t can be written as a function of u_t. Denote this function by $\varphi(u_t)$. Since $\frac{b_t}{\tilde{b}_t} = \frac{\lambda_s u_t + \rho c}{\lambda_s u_t}$ decreases in u_t, the expected delay of payments after report decreases in u_t. Specifically, the delay goes to infinity as $u_t \to 0$ and goes to 0 as $u_t \to \infty$. Therefore, as $u_t \to 0$, $\varphi(u_t) \to \infty$ so inducing the agent to work is very costly. As a result, $a_t = 0$ when u_t is small enough. As u_t increases, the expected delay of b_t decreases and $a_t = 1$ becomes optimal. The results are summarized in the next proposition.

Proposition 2. *There exists $c^* > 0$, such that when $c < c^*$, the value function $\pi(u)$ is characterized as follows: There exists $0 < u_1 < u_2$, $C_1 > 0$, and $C_2 < 0$, such that*

$$\pi(u) = \begin{cases} C_1 u^{\frac{r}{\rho}} & \text{if } u < u_1 \\ \pi_1(u) = \frac{r\lambda_s \pi}{r+\lambda_s} - \frac{r\lambda_s}{\rho} u^{\frac{r+\lambda_s}{\rho}} \int_u^{u_2} v^{-\frac{r+\lambda_s}{\rho}-1} \varphi(v) dv + C_2 u^{\frac{r+\lambda_s}{\rho}} & \text{if } u_1 \le u \le u_2 \\ \pi(u_2) - \frac{r}{\rho}(u - u_2) & \text{if } u > u_2 \end{cases} \tag{9}$$

The corresponding choice variables are given by:

- $a = 0$ *if* $u < u_1$ *and* $a = 1$ *if* $u \ge u_1$.
- $b = 0$ *if* $u < u_1$, $b = \frac{\lambda_s u + \rho c}{\rho \lambda_s}$ *if* $u_1 \le u \le u_2$, *and* $b = \frac{\lambda_s u_2 + \rho c}{\rho \lambda_s}$ *if* $u > u_2$.
- $\phi = 0$ *if* $u \le u_2$ *and* $\phi = \frac{u - u_2}{\rho}$ *if* $u > u_2$.
- $w = 0$ *if* $u < u_2$ *and* $w = u_2$ *if* $u \ge u_2$.

The determination of u_1, u_2, C_1, and C_2 are given in Appendix.

When $u < u_1$, the principal makes no payments and the agent does not work. Intuitively, when u is very small, the agent has a strong incentive to report a fake success. To make him truth-telling, payments after report of success have to be made with a large delay. Since the principal is more patient, it is very costly to induce the agent to work. As u increases, the delay of payments after report of success can be reduced. Therefore, the cost of inducing the agent to work becomes smaller and $a = 1$ is optimal. Specifically, payments after report of success induce the agent to work and the delay of payments induces the agent to not report a fake success.

When $u > u_2$, the principal makes a lump-sum payment directly such that the agent's continuation utility drops to u_2. Intuitively, to deliver $u > 0$ to the agent, the expected payments must be more than compensating for the cost. Therefore, the question is when to make these additional payments. A first observation is that it is not optimal to make additional payments after report of success. Since the principal is more patient than the agent, it is always better to make payments earlier than later. In addition, paying after report increases the agent's incentive to report a fake success, which is undesirable. Therefore, payments after report should be just enough to compensate for the cost and all additional payments should be made before report. Then the principal only needs to decide the timing of payments before report. The benefit of paying today is that it reduces cost for the principal since she is more patient than the agent. On the other hand, paying tomorrow reduces the agent's incentive to report a fake success today after claiming the payments, thus the principal can decrease the delay of payments after today's report. When u is relatively small, the delay of payments after report is relatively large thus the benefit of decreasing the delay is large. As a result, the principal makes no payments before report. By contrast, when u is large and the delay is small, the benefit of further reducing the delay is relatively small. Then the principal prefers to pay immediately before report. At $u = u_2$, the benefit equals to the cost. Therefore, the principal makes a flow payment before report such that the agent's continuation utility stays at u_2.

The principal's payoff increases in u when u is small and decreases in u when u is large. Intuitively, the delay of payments after report decreases as u increases. But at the same time, the principal needs to pay more when u increases. When u is small and the delay of payments is large, reducing the delay is more important for the principal. Therefore, the principal gets better off as u increases. The effect is reversed when u is large.

In the optimal contract, the principal chooses initial utility u_0 such that $\pi(u)$ is maximized. I show that u_0 is in between u_1 and u_2. As a result, the agent always exerts full effort before success. Since the agent is always made indifferent between working and shirking, by the promise keeping constraint, u_t increases over time. Once u_t reaches u_2, the principal makes a flow payment such that u_t keeps constant thereafter.

Theorem 1. *There exists $c^* > 0$, such that when $c < c^*$, the agent exerts full effort until a success and reports it immediately in the optimal contract. The principal pays with a delay after the agent's report of success. In addition, there exists $t^* > 0$, such that the principal makes a constant flow payment before report of success starting from t^*. When $c \geq c^*$, it is* *optimal for the principal to not contract with the agent.*

Theorem 1 follows directly from Proposition 2. Intuitively, payments after report incentivizes the agent to work. Payments before report and the delay of payments after report make the agent report truthfully.

In the optimal contract, the effort process is efficient. Intuitively, early termination is not effective in reducing the information rent. The agent gets an information rent because evaluation of the agent's performance takes time and it is costly for the principal to delay the payment. Early termination makes the agent worse off on-path, which increases his incentive to report a fake success. As a result, the principal has to increase the delay of payments after report, which is undesirable.

Figure 1 illustrates the evolution of payments and delay in the optimal contract. As mentioned before, the agent's continuation utility increases over time by the promise keeping constraint. Accordingly, payments after report should increase over time and the delay of payments decreases over time.

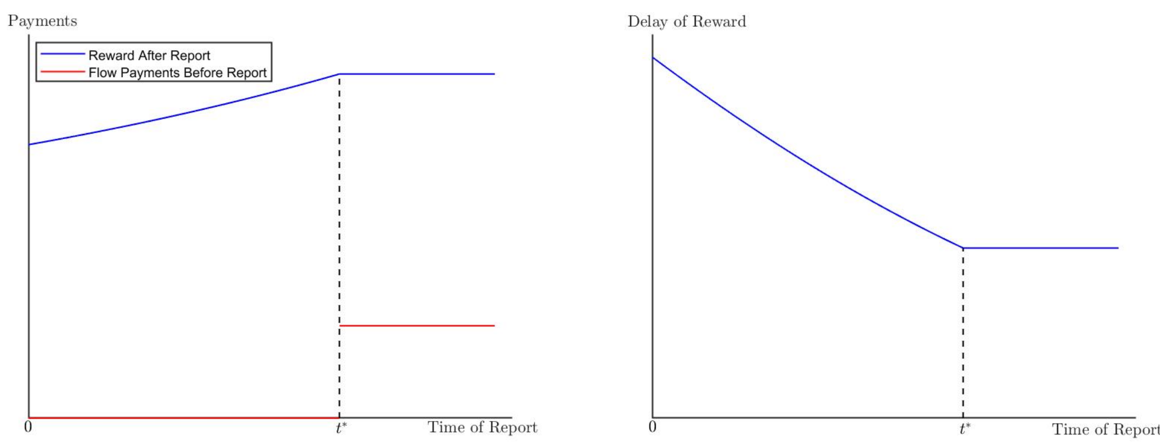

Figure 1: Evolution of payments and delay in Theorem 1

Corollary 1. *Suppose $c < c^*$. In the optimal contract, u_t and b_t increase over time. The expected delay of payments after report decreases over time.*

Intuitively, if the principal only makes payments after report that are just enough to compensate for the agent's cost of effort, then the agent has incentives to report a fake success. There are two different ways to make the agent report truthfully. One is to make him better off when he does not report a fake success and the other is to make him worse off when he reports a fake success. To make him better off on path, the principal either increases payments after report or pays before report. As argued previously, it is better to pay before report. To make the agent worse off when he reports a fake success, the principal increases the delay of payments after report. Thus, the trade-off for the principal is whether to increase the delay or to increase payments before report. I claim that the benefit of making payments before report increases over time. Intuitively, payment at time t makes the agent better off at any time before t. Therefore, it does not only reduces the delay after report at t, but also reduces the delay after any report before t. As a result, the delay decreases over time and the agent's continuation utility increases over time. To induce the agent to work, payments after report increase over time accordingly.

3.4 Optimal Contracts with Breakdown

In this section, I study the general model where a breakdown may occur before success, i.e. $\lambda_f > 0$. I again first ignore IC_3 and check it later. By the same analysis as in the previous section, IC_1 and IC_2 are both binding. I still use $\pi(u)$ to denote the principal's value function. Then the HJB equation is given by

$$\begin{aligned}\pi(u_t) = &\sup_{a_t,\phi_t,w_t,l_t} \{r(\lambda_s a_t \pi dt - \phi_t - w_t dt - \lambda_f l_t dt - \lambda_s a_t \varphi_t(b_t, \tilde{b}_t)dt) \\ &+ (1 - rdt)(1 - \lambda_s a_t dt)(1 - \lambda_f dt)\pi(u_{t+dt})\} + o(dt) \qquad (10)\\ &\text{subject to equation (3),}\end{aligned}$$

where $b_t = \frac{\lambda_s(u_t - \rho\phi_t) + \rho c}{\rho\lambda_s}$ and $\tilde{b}_t = \frac{u_t - \rho\phi_t}{\rho}$.

A first observation is that w_t and l_t play a same role in the value function. They are both payments before report which incentivize the agent to not report a fake success. The only difference between w_t and l_t is the effect on IC_3. When the agent achieves a success and does not report it, he could claim payments w_t in the future but can never get l_t, which is paid at the time of breakdown. Therefore, paying l_t instead of w_t reduces the agent's incentive to hide a success. When IC_3 is not binding, w_t and l_t are equivalent.

The introduction of potential breakdown essentially makes the principal and the agent discount future payoffs by a larger rate before success. In other words, since a breakdown may occur at any time before success, there is a smaller probability that the principal and the agent can get the continuation utility tomorrow. Thus, the continuation utility becomes less important. The next result shows that the only effect of potential breakdown is increasing discount factors, as long as IC_3 never binds.

Proposition 3. *Suppose IC_3 does not bind at any time. If the evaluation technology does not change, then the optimal contract is the same as if there was no breakdown and the discount factors were $\rho + \lambda_f$ for the agent and $r + \lambda_f$ for the principal.*

However, the potential breakdown does have an effect on IC_3. As shown in the previous section, when there are no breakdowns, the agent's continuation utility increases at rate ρ before achieving u_2, i.e., $u_t = u_0 e^{\rho t}$. At $u_t = u_2$, the principal starts to make a flow payment and everything keeps constant thereafter. By IC_1 binding, the payment after report $b_t = \frac{\lambda_s u_t + \rho c}{\rho \lambda_s}$ increases at a smaller rate than u_t. Thus, it increases slower than ρ so the agent has no incentives to report success at a later time. But in the general model with $\lambda_f > 0$, u_t increases at rate $\rho + \lambda_f$ before achieving u_2. Even though b_t still increases at a smaller rate than u_t, it could increase faster than ρ. As a result, IC_3 can be binding for some t. Specifically, when u_t is large, the increasing rate of b_t is close to that of u_t and may thus be greater than ρ. By the determination of u_2 in the proof of Proposition 2, u_2 increases in λ_s. Therefore, IC_3 could be binding when λ_s is large or λ_f is larger.

Next I investigate the scenario where IC_3 binds for some t. As mentioned above, paying l_t instead of w_t reduces the agent's incentive to hide a success. Without loss of generality, I assume $w_t = 0$ for all t.

Once u_t reaches u_2, b_t keeps constant over time so IC_3 does not bind. Therefore, I only focus on $u_t < u_2$. There are two ways to satisfy IC_3 when it is binding. One is to make payments before report such that u_t increases at a smaller rate and $b_t = \frac{\lambda_s u_t + \rho c}{\rho \lambda_s}$ increases at rate ρ. The other one is to increase b_t directly to reduce the agent's incentive to hide a success at t. I show that when u_t is close to u_2, it is better to make payments before report. Intuitively, when u_t goes to u_2, $\pi'(u_t)$ goes to $-\frac{r}{\rho}$ so the cost of paying before report becomes very small. Therefore, the principal starts to pay at the time of breakdown before $u_t = u_2$. The following result shows that the amount of this payment increases in the time of breakdown.

Theorem 2. *There exists $c^{**} > 0$, such that when $c < c^{**}$, the agent exerts full effort until success or breakdown and reports the success immediately in the optimal contract. The principal pays with a delay after the agent's report. In addition, there exists $t^* > 0$, such that the principal makes a lump-sum payment at the time of breakdown if it occurs after t^*. When $c \geq c^{**}$, it is optimal for the principal to not contract with the agent.*

Figure 2 illustrates the evolution of payments and delay in the optimal contract. At $u_t = u_2$, the principal makes a lump-sum payment at time of breakdown such that u_t keeps constant over time. As a result, everything is constant before success or breakdown. Before achieving u_2, u_t increases over time. As a result, payments increase over time and delay of payments decreases over time.

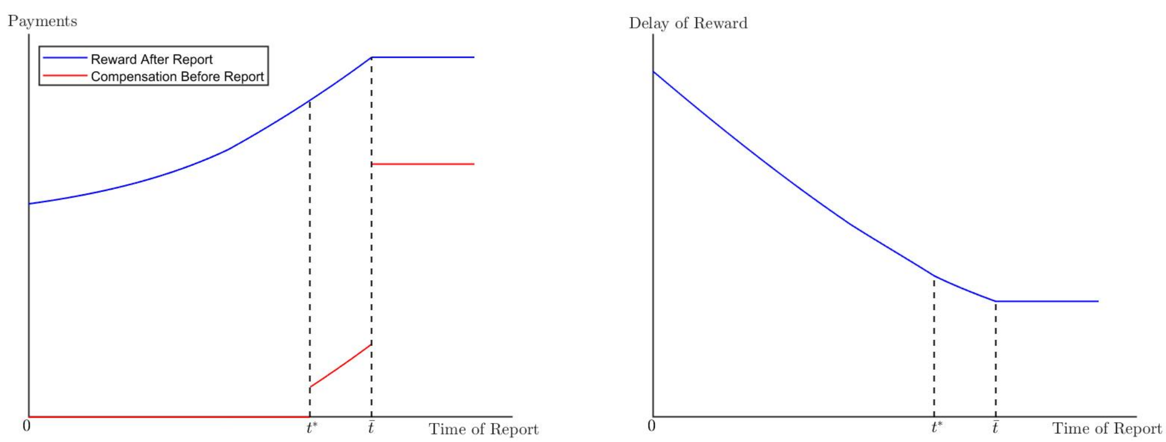

Figure 2: Evolution of payments and delay in Theorem 2

Corollary 2. *Suppose $c < c^{**}$. In the optimal contract, u_t, b_t, and l_t increase over time. The expected delay of payments after report decreases over time.*

One different feature of Theorem 2 is that payment at time of breakdown strictly increases within some time region. This is because of IC_3. Within this region, both IC_1 and IC_3 are binding. Then payments after report can be expressed as $b_t = \frac{\lambda_s u_t + \rho c}{\rho \lambda_s}$. Without payments before report, u_t increases at rate $\rho + \lambda_f$ and b_t increases at a rate greater than ρ. As u_t gets larger, the increasing rate of b_t gets closer to that of u_t, which means that b_t increases faster over time. As a result, to satisfy IC_3, the principal has to make a larger payment before report such that the increasing rate of u_t gets smaller.

3.5 Rewarding Success and Compensating Failure

In the optimal contract, payment after report can be seen as a reward for success. It incentivizes the agent to exert effort over time. Since the principal cannot observe success directly, a delay of reward is necessary to make the agent report truthfully. In reality, it is common to deliver reward in the form of stock options, which can only be realized in a future time. The delay of reward decreases over time since the agent has a larger incentive to report a fake success early on.

Another feature of the optimal contract is that the principal may also pay the agent at time of breakdown. Apparently, the compensation for breakdown is not effective in incentivizing the agent to work. The reason for this payment is also to induce the agent to report truthfully. Intuitively, compensation for breakdown makes the agent better off when he does not report a fake success. In other words, knowing that he will be compensated when a breakdown occurs, the agent is more willing to keep working instead of reporting a fake success in the hope of not being caught by the principal.

In this principal-agent relationship, the agent gets better off over time even if he has not achieved a success. Intuitively, the agent has no incentives to procrastinate in this setting. Therefore, the principal does not need to punish the agent for not having a success. The reason for making the agent better off over time is similar to the reason for compensating breakdown. Having a more promising future, the agent is less likely to report a fake success at an early stage.

4 Agent Does Not Observe Success

In this section, I extend the model to the case where the agent cannot observe success. This assumption can be plausible in examples where a researcher comes up with an innovative theory. The value of this theory may not be recognized by anyone until people find a way to apply it in practice.

When the agent does not observe success, he has no private information except for the history of effort. Therefore, the agent does not need to make any report to the principal. Therefore, the only objective of the principal is to induce the agent to work. The principal sets a deadline and designs a payment scheme conditional on public history, which may include the time of breakdown and signals after the deadline.

4.1 Optimal Contracts with No Breakdowns

I first study the scenario where no breakdowns would occur, i.e. $\lambda_f = 0$. In this setup, the principal observes no signals before the deadline. As a result, both the deadline and payments before the deadline are unconditional on the agent's actions.

A first observation is that the principal does not make any payments before the deadline. Since the agent can always claim all the payments before the deadline, these payments are useless in incentivizing the agent to work. The payment scheme after the deadline is very similar to the one in the previous section. Specifically, payments must be with a delay such that the agent who has achieved a success gets more in expectation than the one who has not succeeded. Hoffmann et al. (2020) show that the principal optimally makes one payment after the deadline.

If the deadline is deterministic, then nothing happens before the deadline and it does not matter whether success arrives earlier or later as long as it is before the deadline. Therefore, the agent has incentives to backload his effort. In other words, he has a stronger incentive to work when he is closer to the deadline. Apparently, it is not optimal for the principal to induce the agent to shirk at the beginning. Therefore, in the optimal contract, the agent is induced to exert full effort until the deadline. The payment after the deadline makes the agent indifferent between working and shirking at time 0. Define T^* as the optimal deterministic deadline and T^{FB} as the first-best deadline. Then I can show that $T^* < T^{FB}$.

Lemma 2. *The optimal deterministic deadline is smaller than the first-best deadline.*

Intuitively, the marginal benefit of effort at the deadline equals to the marginal cost at the deadline in the first best. However, in the optimal contract, the marginal benefit of effort at the deadline equals to the marginal cost at the beginning.

Next I argue that the deterministic deadline is not optimal. Consider an alternative contract where the principal terminates the project at $T^* - \tau$ with probability ϵ and terminates at T^* with probability $1 - \epsilon$. At the same time, the principal decreases the payment after termination at T^* such that the agent is indifferent between working and shirking at $T^* - \tau$. Since the agent has a stronger incentive to work when he is closer to the deadline, he strictly prefers working when $t \in (T^* - \tau, T^*)$. To incentive the agent to work before $T^* - \tau$, the principal makes a payment after termination at $T^* - \tau$ such that the agent is indifferent between working and shirking at time 0. Then the agent always exerts full effort until termination. In this new contract, the principal's cost is reduced. Intuitively, the agent is made indifferent at time 0 in both contracts. But in the new contract with stochastic deadline, part of the payments are made at an earlier time, which reduces the cost. On the other hand, the expected probability of success is smaller in the new contract since there is a probability that the project is terminated earlier. When ϵ goes to 0, the decrease in probability of success becomes negligible but the cost reduction is unaffected. Therefore, the contract with stochastic deadline is better for the principal.

Naturally, the principal can repeat this process and terminate the project with a small but positive probability at any time. To maximize the probability of success, the principal would like to make each $\epsilon > 0$ as small as possible. Then similar to Mirrlees (1999), the optimal contract does not exist in this setup. Instead, the supremum of the principal's payoff can be approximated by letting $\epsilon \to 0$. To ensure the existence of the optimal contract, I impose an upper bound on the aggregate payment made to the agent. I state the above results formally as below.

Proposition 4. *Suppose the aggregate payment is bounded above by B. There exists $B^* > 0$ such that when $B \geq B^*$, the principal optimally sets a stochastic deadline. Specifically, there exists $\bar{T} \in (T^*, T^{FB})$ such that the distribution function of the deadline, $F(t)$, strictly increases on $[0, \bar{T}]$, is continuous on $[0, \bar{T})$, and jumps to 1 at $\bar{T}$.*

When B is relatively small, the principal is constrained from terminating with a small probability and paying a large amount. Therefore, the benefit of random termination is smaller. As a result, the principal does not start terminating from the beginning. When B is very small, a deterministic deadline can be optimal.

In the optimal contract, the agent is made indifferent between working and shirking at any point. Intuitively, since the project could be terminated at any time, an earlier success is better for the agent thus he starts working from the beginning.

Ideally, to induce the agent to always exert effort, the most efficient way would be making payments whenever he achieves a success. However, in this model, evaluation of the agent's performance starts only after the termination of the project. Therefore, it is impossible for the principal to provide instantaneous incentives just by payments. With a stochastic deadline, the agent is induced to work for fear of being terminated before success.

It is not surprising that $\bar{T}$ is greater than T^*, since the cost of the principal is reduced with a stochastic deadline. I also show that $\bar{T}$ is always smaller than the first-best deadline, even as $B \to \infty$. The reason is that the agent still gets a procrastination rent in this optimal contract, even though he is always indifferent between working and shirking. Intuitively, if the agent did not exert effort for some time period, then his belief that a success has been achieved would be smaller. Therefore, he would strictly prefer working in the next period.

4.2 Optimal Contracts with Breakdown

Next I investigate the scenario where a breakdown may occur before success, i.e., $\lambda_f > 0$. A major difference of this model is that the agent's performance can be evaluated before the deadline. Specifically, the principal learns perfectly that no success has been achieved by observing breakdown and the belief of having achieved a success increases over time given no breakdown. A direct implication is that the principal may use payments before the deadline to incentivize the agent to work. Intuitively, an early success matters for the agent since it prevents breakdowns and secures all payments in the future. In this part, I focus on the design of the optimal payment scheme by assuming a deterministic deadline.

In terms of the methodology, the dynamic programming techniques cannot be applied in this model. It is because that beliefs are also state variables and the principal's belief may differ from the agent's belief off path. Instead, I use Pontryagin's maximum principle to find a necessary condition for $(a_t)_{t\geq 0}$ to be optimal, then maximize the principal's expected utility subject to this necessary condition. Finally, I show that the condition is also sufficient.

4.2.1 The Agent's Problem

Define p_t as the on-path belief that a success has been achieved. By Bayes' rule,

$$p_t = \frac{\int_0^t e^{-\int_0^s(\lambda_f+\lambda_s a_\tau)d\tau}\lambda_s a_s ds}{1-\int_0^t e^{-\int_0^s(\lambda_f+\lambda_s a_\tau)d\tau}\lambda_f ds}, \tag{11}$$

Denote the denominator by q_t, which is the probability that no breakdown arrives before t. The differential form of p_t is given by

$$\dot{p}_t = (1-p_t)\lambda_s a_t + p_t(1-p_t)\lambda_f, \tag{12}$$

where the first term denotes the increase in p_t generated by additional effort, and the second term characterizes learning from breakdown. In other words, even if the agent exerts no effort, the belief still increases when no breakdown occurs. Define W_t^A as the cumulative payments up to t discounted by the agent's discount factor, given no breakdown before t. Since a breakdown is conclusive evidence that no success has been achieved and there is no reporting problem, no payments will be made at the time of breakdown. Define W^S as the expected time-0 ρ-discounted value of payments after the deadline for the agent who has achieved a success and define W^{NS} as the one for the agent without success. Denote $\Delta W := W^S - W^{NS}$. Let $\bar{W} := W_T^A + W^S$ be the aggregate expected payment for the agent who achieved success before the deadline. For convenience, let $y_t := e^{-\int_0^t(\lambda_f+\lambda_s a_s)ds}$ be the probability that no success or breakdown occurs by t. Then the agent's utility is given by

$$u_0 = \int_0^T e^{-\rho t}(-ca_t)q_t dt + \bar{W} - \int_0^T \lambda_f y_t(\bar{W} - W_t^A)dt - y_T\Delta W. \tag{13}$$

Specifically, the first term is the expected cost of effort. By definition, q_t is the probability that no breakdown arrives before t. Once a breakdown arrives, the project is terminated and there will be no more effort. The second term, $\bar{W}$, is the expected payment for the agent conditional on success before T. However, the agent may not get all of it. First, if a breakdown occurs at t, then the agent gets no more payments after t, which results in a loss of $\bar{W} - W_t^A$. Note that y_t is the probability that no success or breakdown occurs before t and $\lambda_f dt$ is the instant probability that a breakdown occurs. Secondly, if no success or breakdown occurs before T, then the agent only gets W^{NS} after T, which results in a loss of ΔW.

Given any $(W_t^A)_{0\leq t\leq T}$, W_S, and W^{NS}, the agent's objective is to maximize u_0 by choosing the effort process $(a_t)_{t\geq 0}$. The first-best effort process is as given below.

Lemma 3. *The agent optimally exerts full effort until* $p_t = \frac{\lambda_s \pi - c}{(\lambda_s+\rho)\pi}$.

The agent's belief of success increases over time. Therefore, his work incentive decreases over time and the effort is frontloaded. In the following analysis, I also focus on a frontloaded effort process. I will show the optimality of such effort process in the end.

To induce frontloaded effort, it is optimal to terminate the project at the time when the agent stops working. In other words, the agent should be induced to always exert full effort before the deadline. Let $\Delta W \coloneqq W^S - W^{NS}$. By Pontryagin's maximum principle, a necessary condition for the agent to exert full effort up until T is as given below.

Lemma 4. *If the agent optimally exerts full effort up until T, then* $\lambda_s(1-p_t)\xi_t - ce^{-\rho t} \geq 0$ *for all* $t \leq T$, *where*

$$\xi_t = \int_t^T \big(\bar{W} - W_s^A - \int_s^T ce^{-\rho\tau} d\tau\big)\lambda_f e^{-(\lambda_f+\lambda_s)(s-t)} ds + e^{-(\lambda_f+\lambda_s)(T-t)} \cdot \Delta W \tag{14}$$

and p_t is given by (11) substituting $a_\tau = 1\ \forall \tau \leq t$.

Intuitively, ξ_t is the expected loss of utility if no success was achieved by t. It consists of two parts. The first part is the loss when a breakdown occurs at some point in the future. Once a breakdown occurs, the agent loses all future payments $\bar{W} - W_s$ and saves the future cost of effort since he will not work anymore. The second part is the expected loss if no success will be achieved before the deadline. Therefore, $\lambda_s(1-p_t)\xi_t$ characterizes the marginal benefit of effort at t and $\lambda_s(1-p_t)\xi_t - ce^{-\rho t} \geq 0$ denotes the local IC constraint.

4.3 The Principal's Problem

The principal's objective is to minimize expected cost subject to the necessary condition characterized in Lemma 4. Define the *incentive* of the agent as

$$\mu_t = \lambda_s \xi_t - \frac{ce^{-\rho t}}{1 - p_t}. \tag{15}$$

Then by Lemma 4, the agent exerts full effort at time t if and only if $\mu_t \geq 0$. I next investigate how μ_t changes over time. Suppose there is a single payment at time τ. Then $\xi_t = 0$ for $t \geq \tau$. In other words, payment at τ can only incentivize the agent to work before τ. Three factors may affect how incentives change over time. First, the belief that a success has arrived increases over time. Consequently, additional effort is more likely to be redundant and there is less incentive to work. Secondly, when the agent is closer to the payment date, the discounted value of the payment increases. As a result, the agent is more willing to work. Thirdly, as the payment date approaches, the probability that a breakdown occurs before the payment decreases. Therefore, a success is less beneficial and the agent has less incentives to work.

When t is very small and far away from τ, the second effect dominates and the incentive increases over time. On the other hand, when t is very close to τ, the third effect dominates and the incentive decreases over time. Formally, I am able to show that μ_t is concave in t.

Lemma 5. *Suppose there is a single payment at τ. Then μ_t is concave on $[0, \tau]$ if $\xi_0 \geq 0$.*

Note that μ_t is not concave around τ. At $t = \tau$, the effect of the payment disappears and μ_t' jumps up. If there are multiple payments, Lemma 5 directly implies that μ_t is concave around t if there are no payments around t.

An important feature of this model is that the local IC constraints may not be binding everywhere. Intuitively, the payment at time τ provides no incentives for the agent to work at τ, but instead provides incentives for the agent to work at any time before τ. As a result, even if $\mu_t > 0$ for some t, it may not be possible for the principal to lower the payment without violating other IC constraints.

There is a trade-off for the principal regarding the timing of payments. Paying earlier is less costly since the principal is more patient than the agent. However, it also reduces the agent's incentive to work because there will be a larger probability that he can get the payment without achieving a success. The next lemma shows that lump-sum payments are more efficient in incentive provision.

Lemma 6. *Suppose there are two separate payments at t_1 and t_2 where $t_1 < t_2 \leq T$ or $T \leq t_1 < t_2$. Then there exists a single payment at $t' \in (t_1, t_2)$, such that the cost for the principal is smaller and μ_t is larger for all $t \in [0, t_1 \wedge T]$.*

Lemma 6 implies that if there are multiple payments, then some local IC constraints in between must be binding. The proof is similar to that of Lemma 1. Specifically, the incentive provided by a payment is concave in the time of payment and the cost is convex in the time of payment.

Next I present the optimal payment scheme subject to the necessary condition.

Proposition 5. *Suppose the principal minimizes the expected cost subject to $\mu_t \geq 0$ for $t \in [0, T]$. Then the optimal payment scheme takes one of the following forms:*

(i) A single lump-sum payment after the deadline;

(ii) A lump-sum payment before the deadline and another lump-sum payment after the deadline;

(iii) Two lump-sum payments before the deadline, a flow payment in between, and another lump-sum payment after the deadline.

Specifically, when $\rho - r$ is very small or very large, a single payment after the deadline is optimal. When $\rho - r$ is moderate, (ii) or (iii) can be optimal depending on parameter values.

First consider the incentive at the deadline. To make $\mu_T \geq 0$, there must be a payment after the deadline with a delay. Similar to the analysis in previous sections, the optimal delay of payment depends on the difference in discount factors, $\rho - r$. When $\rho - r$ is very large, it is very costly for the principal to defer payments so the delay should be small. To sufficiently differentiate the agent who has achieved a success from the one who has not, the amount of payment has to be large. As a result, the agent's loss from breakdown can be large enough such that he has a stronger incentive to work at an earlier time. In other words, in order to prevent breakdown, the agent optimally frontloads his effort to achieve a success as soon as possible, even though it incurs more cost. Therefore, one payment after the deadline is enough to induce the agent to exert full effort until the deadline.

As $\rho - r$ decreases, the delay of payment after deadline increases. Accordingly, the amount of payment decreases. As a result, the agent's loss from breakdown is smaller and his incentives to work may decrease. Especially, at the beginning when the agent is far away from the deadline, the discounted value of the payment is so small that he may be not willing to work. In this scenario, the principal needs to make payments before the deadline to induce the agent to work from the beginning. When T is relatively small, one payment before the deadline is enough to keep the agent working all the time. However, when T is large, multiple payments before the deadline may be required. Intuitively, the first payment can induce the agent to work for a period of time starting from the beginning. But if the agent is still far away from the deadline after claiming this payment, his incentive to work may decrease again such that he is not willing to keep working.

If there are multiple lump-sum payments before the deadline, then by Lemma 5, the incentive is strictly positive between each of the two consecutive payments. Thus, it is possible to combine some of the payments so that the incentive in between is still positive. By Lemma 6, the principal can reduce the cost in this way until all the incentives in between become zero. Therefore, optimally there are only two lump-sum payments with a flow payment in between, such that the agent is always indifferent between working and shirking when the flow payment is made.

Finally, when $\rho - r$ is very small, the optimal delay of payment is very large. Now consider the incentive at time 0. To make $\mu_0 \geq 0$, the optimal payment date may be after the deadline, as long as $\rho - r$ is small enough. Then the principal never makes any payments before the deadline. By Lemma 6, optimally there is only one payment after the deadline.

4.4 Optimal Contract

Given the optimal payment scheme to induce the agent to work, I only need to further determine the optimal deadline. Apparently, the optimal deadline strictly increases in the benefit of success, π. When π is very small, the agent never works and no payments are made in the optimal contract. Denote the cutoff by $\underline{\pi}$ and suppose $\pi > \underline{\pi}$ in the following.

With Proposition 5, I can directly state the optimal contract as follows.

Theorem 3. *In the optimal contract, the agent's effort is frontloaded. The payment scheme takes one of the following forms:*

(i) A single lump-sum payment after the deadline;

(ii) A lump-sum payment before the deadline and another lump-sum payment after the deadline;

(iii) Two lump-sum payments before the deadline, a flow payment in between, and another lump-sum payment after the deadline.

To prove the theorem, I only need to show two more things. First, the proposed payment scheme is sufficient in inducing full effort up to T, and secondly, it is optimal to induce frontloaded effort. For sufficiency, I show that the necessary condition pins down an essentially unique (except for deviations on a set of measure zero) effort process. Then by the existence of a best response, this effort process must be optimal. Intuitively, if the agent does not work in some periods, then his belief of having achieved a success will be smaller. As a result, he has stronger incentives to work and $a_t = 0$ cannot satisfy the local IC constraint.

The optimality of frontloaded effort results from the monotonicity of the belief. Suppose it is not optimal to induce the agent to work at some time t. Then at any $t' > t$, the belief that a success has arrived is higher than the belief at t. Thus, it is not optimal to induce the agent to work at t' as well.

Similar to the model without breakdown, the main concern of the principal is how to incentivize the agent at the beginning. Especially when the benefit of success is large and the principal intends to induce the agent to work for a long period of time, the agent may have a strong incentive to procrastinate and only exert effort when the deadline approaches. The general solution is to make early success matter for the agent. When there is no breakdown, evaluation of the agent's performance takes place only after the deadline. As a result, the principal can only use a stochastic deadline to incentivize the agent. Specifically, if the agent shirks, then there is some probability that he will not have a second chance. In the model with breakdown, the principal can learn the agent's performance before the deadline. Therefore, the principal can use periodic reward to provide instant feedback and thus make the agent keep working.

I restricted my attention to deterministic deadlines in the model with breakdown to better study the optimal payment scheme. In general, a stochastic deadline could still be optimal. For example, if the evaluation technology after the deadline is much more informative than the learning before the deadline, then as in the model without breakdown, a stochastic deadline can be very efficient in incentivizing the agent. But if the principal can only learn from breakdown even after the deadline, then there is no point to terminate at an earlier time.

5 Concluding Remarks

I study optimal incentive provision when evaluation of the agent's performance takes time. Specifically, the principal observes imperfect signals over time after the agent stops working. The optimal contract takes different forms depending on whether the agent observes success.

If the agent observes success, then he exerts full effort until a success and reports it immediately. The principal makes deferred payments after the report of success. In addition, a lump-sum payment is made at the time of breakdown if it occurs after a threshold date and before the report. Payments after the report incentivize the agent to exert effort and payments at the time of breakdown induce truthful reporting. The expected amount of payments increases with the date at which the agent reports success and the expected delay of payments after the report decreases with the date of the report. The agent gets better off over time even before success.

If the agent does not observe success, then the principal sets a stochastic deadline when there is no breakdown. The project is terminated with a small probability at each instant before the final deadline. In the model with breakdown, the principal may make small payments along the way before the deadline. Both the stochastic deadline and payments before the deadline induce the agent to work continuously over time and reduce the agent's procrastination rent.

I maintained throughout the paper the assumption that evaluation only takes place after the agent stops working. This assumption is not crucial in the model where the agent observes success. Before the agent's report, observing that a success has not been achieved only proves that the agent has been truth-telling. The only possible benefit of evaluating before the report is that it reduces the agent's incentive to hide a success. But it does not have a substantial effect on the optimal contract. However, the assumption is important in the model where the agent does not observe success. If the principal had more information before the agent stops working, then she could better incentivize the agent by making additional payments before the deadline.

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Appendix

Proof of Proposition 1. Suppose the principal works on the project herself and bears the cost. Define u_t as the continuation utility before the success or the breakdown. Then the HJB equation is given by

$$u_t = \sup_{a_t}\{r\lambda_s a_t \pi dt - r a_t c dt + (1 - r dt)(1 - \lambda_s a_t dt)(1 - \lambda_f dt)u_{t+dt}\} + o(dt). \tag{16}$$

In the absence of success and breakdown, the environment does not change and thus the continuation utility keeps the same, i.e., $u_{t+dt} = u_t$. By rearrangement,

$$0 = \sup_{a_t}\{r\lambda_s a_t \pi - r a_t c - (r + \lambda_f + \lambda_s a_t)u_t\}. \tag{17}$$

The maximization requires $a_t = 1$ if $u_t \leq \frac{r(\lambda_s \pi - c)}{\lambda_s}$ and $a_t = 0$ otherwise. In addition, $u_t = \frac{r(\lambda_s \pi - c)}{\lambda_f + \lambda_s + r}$ when $a_t = 1$ and $u_t = 0$ when $a_t = 0$.

If $c \leq \lambda_s \pi$, then $\frac{r(\lambda_s \pi - c)}{\lambda_f + \lambda_s + r} \leq \frac{r(\lambda_s \pi - c)}{\lambda_s}$. Thus, $a_t = 1$ is optimal.

If $c > \lambda_s \pi$, then $u_t = 0 > \frac{r(\lambda_s \pi - c)}{\lambda_s}$ and $a_t = 0$ is optimal. □

Proof of Lemma 1. Suppose there are at least two lump-sum payments, ϕ_{t_1} with a delay of t_1 and ϕ_{t_2} with a delay of t_2. Consider another payment ϕ_Δ with a delay of Δ, such that

$$\begin{cases} e^{-\rho t_1}\phi_{t_1} + e^{-\rho t_2}\phi_{t_2} = e^{-\rho\Delta}\phi_\Delta \\ t_1 e^{-\rho t_1}\phi_{t_1} + t_2 e^{-\rho t_2}\phi_{t_2} = \Delta e^{-\rho\Delta}\phi_\Delta \end{cases} \tag{18}$$

The existence of Δ and ϕ_Δ can be easily verified. By the first equation of (18), b_t is unchanged if ϕ_{t_1} and ϕ_{t_2} are replaced by ϕ_Δ.

Since $e^{(\rho - r)x}$ is strictly concave in x, by Jensen's Inequality,

$$\begin{aligned} e^{-rt_1}\phi_{t_1} + e^{-rt_2}\phi_{t_2} &= e^{-\rho\Delta}\phi_\Delta \left(e^{(\rho-r)t_1}\frac{e^{-\rho t_1}\phi_{t_1}}{e^{-\rho\Delta}\phi_\Delta} + e^{(\rho-r)t_2}\frac{e^{-\rho t_2}\phi_{t_2}}{e^{-\rho\Delta}\phi_\Delta} \right) \\ &> e^{-\rho\Delta}\phi_\Delta e^{(\rho-r)(t_1\frac{e^{-\rho t_1}\phi_{t_1}}{e^{-\rho\Delta}\phi_\Delta} + t_2\frac{e^{-\rho t_2}\phi_{t_2}}{e^{-\rho\Delta}\phi_\Delta})} \\ &= e^{-r\Delta}\phi_\Delta. \end{aligned} \tag{19}$$

Similarly, by the strict concavity of $e^{-\lambda_f x}$ in x,

$$\begin{aligned}
e^{-(\rho+\lambda_f)t_1}\phi_{t_1} + e^{-(\rho+\lambda_f)t_2}\phi_{t_2} &= e^{-\rho\Delta}\phi_\Delta \left(e^{-\lambda_f t_1}\frac{e^{-\rho t_1}\phi_{t_1}}{e^{-\rho\Delta}\phi_\Delta} + e^{-\lambda_f t_2}\frac{e^{-\rho t_2}\phi_{t_2}}{e^{-\rho\Delta}\phi_\Delta} \right) \\
&> e^{-\rho\Delta}\phi_\Delta e^{-\lambda_f (t_1 \frac{e^{-\rho t_1}\phi_{t_1}}{e^{-\rho\Delta}\phi_\Delta} + t_2 \frac{e^{-\rho t_2}\phi_{t_2}}{e^{-\rho\Delta}\phi_\Delta})} \\
&= e^{-(\rho+\lambda_f)\Delta}\phi_\Delta. \qquad (20)
\end{aligned}$$

In other words, both φ_t and $\tilde{b}_t$ decrease with the single payment at Δ, which proves that multiple payments cannot be optimal.

The proof with the flow payment follows the same argument. □

Proof of Proposition 2. We focus on the case where $a_t = 1$ is optimal for some t. Otherwise the agent never works and the principal makes no payments. We give the condition under which $a_t = 1$ can be optimal later in our proof. We first ignore IC_3 and check it later.

Consider the payment scheme after report at t. Let $I_\tau := \sup_{\{A_\tau \in \mathcal{F}_\tau | \mathbb{P}_\tau^{NS}(A_\tau)>0\}} \frac{\mathbb{P}_\tau^S(A_\tau)}{\mathbb{P}_\tau^{NS}(A_\tau)}$. Then I_τ increases over time. By Assumption 2, I_τ is finite for any τ. By Assumption 3, the maximum can be achieved. Denote by B_τ the expected discounted payment at time $t+\tau$ for the agent who has not achieved a success. Then for the agent who has achieved a success, the expected discounted payment would be $B_\tau I_\tau$. By definition, $b_t = \sum B_\tau I_\tau$ and $\tilde{b}_t = \sum B_\tau$. Define a distribution function f such that $f(\tau) = \frac{B_\tau I_\tau}{b_t}$. By definition, f is determined by $\frac{b_t}{\tilde{b}_t}$. The cost of delay can therefore be characterized as $\sum e^{(\rho-r)\tau} f(\tau)$. Define Δ_t such that $e^{(\rho-r)\Delta_t} = \sum e^{(\rho-r)\tau} f(\tau)$. Given that I_τ increases over time and $\rho > r$, Δ_t increases in $\frac{b_t}{\tilde{b}_t}$. Since $\frac{b_t}{\tilde{b}_t} = \frac{\lambda_s u_t + \rho c}{\lambda_s u_t}$ decreases in u_t, we have Δ_t decreasing in u_t. We call Δ_t the expected delay of payments at time t for convenience.

First we show that IC_1 is binding whenever $a_t = 1$. Suppose $b_t > \frac{\lambda_s(u_t - \rho\phi_t) + \rho c}{\rho\lambda_s}$. Then there exists $\epsilon > 0$, such that $b_t - \epsilon > \frac{\lambda_s(u_t - \rho\phi_t) + \rho c}{\rho\lambda_s}$. Let $\hat{b}_t = b_t - \epsilon$ and $\hat{w}_t = w_t + \lambda_s\epsilon$, and keep other variables unchanged. By equation (3), u_{t+dt} keeps unchanged. Then by equation (8), we have $\hat{\pi}(u_t) - \pi(u_t) = -r(\lambda_s(\varphi_t(\hat{b}_t, \frac{u_t - \rho\phi_t}{\rho}) - \varphi_t(b_t, \frac{u_t - \rho\phi_t}{\rho}))dt + (\hat{w}_t - w_t)dt)$. Since $\hat{b}_t < b_t$ and $\tilde{b}_t$ does not change, the expected delay of payments after report decreases. Therefore, $\varphi_t(\hat{b}_t, \frac{u_t - \rho\phi_t}{\rho}) \leq \frac{\hat{b}_t}{b_t}\varphi_t(b_t, \frac{u_t - \rho\phi_t}{\rho}) = \varphi_t(b_t, \frac{u_t - \rho\phi_t}{\rho}) - \frac{\epsilon}{b_t}\varphi_t(b_t, \frac{u_t - \rho\phi_t}{\rho})$. Since payments after report are made with a delay and the principal is more patient than the agent, $\varphi_t(b_t, \frac{u_t - \rho\phi_t}{\rho}) > b_t$.

Then $\varphi_t(\hat{b}_t, \frac{u_t-\rho\phi_t}{\rho}) - \varphi_t(b_t, \frac{u_t-\rho\phi_t}{\rho}) < \epsilon$. As a result, $\hat{\pi}(u_t) - \pi(u_t) > 0$, which contradicts to the optimality condition.

We characterize the value function $\pi(u)$ in several steps.

Step 1. We first show that $\lim_{u\to\infty} \pi'_+(u) = -\frac{r}{\rho}$, and $\exists u \in \mathbb{R}$, s.t. $\pi'_+(u) = -\frac{r}{\rho}$.

When $\phi_t > 0$, $\pi(u_t) - r\phi_t = \pi(u_t - \rho\phi_t)$. Therefore, $\pi'_+(u) \geq -\frac{r}{\rho}$, $\forall u$.

Suppose $\pi'_+(u) > -\frac{r}{\rho}$, $\forall u$. Then $\phi_t = 0$, $\forall t$. Moreover,

$$0 = \sup_{w_t}\{-rw_t dt + \pi(u_{t+dt}) - \pi(u_t) + A + o(dt)\}, \tag{21}$$

where A is independent of w_t, and

$$u_{t+dt} - u_t = -\rho w_t dt + B + o(dt), \tag{22}$$

where B is independent of w_t. Dividing (21) by dt and taking the limit as $dt \to 0$, we have $w_t > 0$ only if $\pi'_+(u_t) \leq -\frac{r}{\rho}$. Therefore, $w_t = 0$, $\forall t$.

Since IC_1 is binding, the agent is indifferent between $a_t = 0$ and $a_t = 1$. In other words, the agent can also get u_t by choosing $a_t = 0$. Therefore,

$$u_{t+dt} = u_t + \rho u_t dt. \tag{23}$$

If $a_t = 0$, by rearrangement of equation (8),

$$\pi'(u_t)\rho u_t = r\pi(u_t). \tag{24}$$

Apparently, $\pi(u) \to -\infty$ as $u \to \infty$. Therefore, $\lim_{u\to\infty} \pi'(u) < 0$. Divide both sides of (24) by u_t and take the limit as $u_t \to \infty$, we get $\rho \lim_{u\to\infty} \pi'(u) = r \lim_{u\to\infty} \pi'(u)$ by L'Hôpital's Rule, which is a contradiction. Therefore, we must have $a_t = 1$ when u_t is large enough.

If $a_t = 1$, by the same argument, we get

$$\pi'(u_t)\rho u_t = -r\lambda_s\pi + r\lambda_s\varphi(u_t) + (r + \lambda_s)\pi(u_t). \tag{25}$$

Given that IC_1 is binding and $\phi_t = 0$, $\frac{b_t}{\bar{b}_t} = \frac{\lambda_s u_t + \rho c}{\lambda_s u_t}$. As $u_t \to \infty$, $\frac{b_t}{\bar{b}_t} \to 1$ and the expected delay of payments goes to 0. Therefore, $\lim_{u_t\to\infty}\frac{\varphi(u_t)}{u_t}=\lim_{u_t\to\infty}\frac{b_t}{u_t}=\frac{1}{\rho}$. Dividing (25) by u_t and taking the limit as $u_t\to\infty$, we obtain $\lim_{u\to\infty}\pi'(u)=-\frac{r}{\rho}\cdot\frac{\lambda_s}{\lambda_s+r-\rho}$. When $\lambda_s+r-\rho\le 0$, it contradicts to $\lim_{u\to\infty}\pi'(u)<0$. If $\lambda_s+r-\rho>0$, then $\lim_{u\to\infty}\pi'(u)<-\frac{r}{\rho}$, which is also a contradiction.

Step 2. Define u_2 as the smallest u such that $\pi'_+(u)=-\frac{r}{\rho}$. A direct implication is that it is optimal to set $\phi_t=u_t-u_2$ whenever $u_t>u_2$ and $\phi_t=0$ otherwise. If $u_2=0$, then the agent never works. We focus on the case where $u_2>0$.

To look for the optimal w_t, we show that if $u_t=u_2$, then $u_{\hat{t}}=u_2$ for any $\hat{t}>t$. Suppose there exists $t_1>t$ such that $u_{t_1}>u_2$. Since u_t cannot jump up, there must be some $t_2\in(t,t_1)$ such that $u_{t_2}\in(u_2,u_{t_1})$. Then it implies that $\phi_{t_2}<u_{t_2}-u_2$, which is a contradiction. Similarly, suppose there exists $t_1>t$ such that $u_{t_1}<u_2$. Since $\phi=0$ whenever $u\le u_2$, there must be some $t_2\in(t,t_1)$ such that $u_{t_2}\in(u_{t_1},u_2)$, which implies $w_{t_2}>0$. However, we showed in step 1 that $w_t=0$ whenever $\pi'_+(u_t)>-\frac{r}{\rho}$, which contradicts to $w_{t_2}>0$. To keep u_t constant at u_2, by equation (3), we obtain $w_t=u_2$.

To pin down u_2, we consider two alternative strategies at $u_t=u$. One is to set $w_t=u$ and keep u_t constant, the other is to set $w_t=0$ and $w_{t+dt}=u_{t+dt}$. The utility of the first strategy is given by

$$\hat{\pi}(u)=\frac{r\lambda_s\pi-ru-r\lambda_s\varphi(u)}{r+\lambda_s},\tag{26}$$

and the utility of the second strategy is given by

$$\begin{aligned}\tilde{\pi}(u)&=r\lambda_s\pi dt-r\lambda_s\varphi(u)dt+(1-rdt)(1-\lambda_s dt)\hat{\pi}(u_{t+dt})\\&=r\lambda_s\pi dt-r\lambda_s\varphi(u)dt-(r+\lambda_s)\hat{\pi}(u)dt+\frac{r\lambda_s\pi-ru_{t+dt}-r\lambda_s\varphi(u_{t+dt})}{r+\lambda_s}\\&=rudt+\frac{r\lambda_s\pi-r(u+\rho udt)-r\lambda_s\varphi(u_{t+dt})}{r+\lambda_s}\\&=\hat{\pi}(u)+\frac{r\lambda_s\left(\varphi(u)-\varphi(u_{t+dt})\right)}{r+\lambda_s}-\frac{r\rho udt}{r+\lambda_s}+rudt\\&=\hat{\pi}(u)-\frac{r\lambda_s\varphi'_+(u)\rho udt}{r+\lambda_s}-\frac{r\rho udt}{r+\lambda_s}+rudt.\end{aligned}\tag{27}$$

Therefore, $\hat{\pi}(u)\ge\tilde{\pi}(u)$ if and only if $\rho\lambda_s\varphi'_+(u)\ge\lambda_s+r-\rho$. The existence of right derivatives can be justified by convexity of $\varphi(u)$. We show that $\varphi(u)$ is convex in the following. Consider any $u^a < u^b$ and $\lambda \in (0,1)$. Suppose the optimal payment scheme is $(B^a_\tau)_{\tau\geq 0}$ given $u_t = u^a$ and $(B^b_\tau)_{\tau\geq 0}$ given $u_t = u^b$. Consider a new payment scheme $(\lambda B^a_\tau + (1-\lambda)B^b_\tau)_{\tau\geq 0}$. Since b_t and $\tilde{b}_t$ are both linear in u_t, this new payment scheme is feasible given $u_t = \lambda u^a + (1-\lambda)u^b$. At the same time, the cost for the principal is $\lambda\varphi(u^a) + (1-\lambda)\varphi(u^b)$ given this new payment scheme. Therefore, $\varphi(\lambda u^a + (1-\lambda)u^b) \leq \lambda\varphi(u^a) + (1-\lambda)\varphi(u^b)$.

As $u_t \to 0$, $\frac{b_t}{\tilde{b}_t} = \frac{\lambda_s u_t + \rho c}{\lambda_s u_t} \to \infty$. By Assumption 2, the expected delay of payments goes to infinity. Since the amount of payment $b_t \geq \rho c$ is strictly positive, we have $\lim_{u\to 0}\varphi(u) \to \infty$. Then $\lim_{u\to 0}\varphi'_+(u) \to -\infty$. On the other hand, as $u \to \infty$, $\frac{b_t}{\tilde{b}_t} \to 1$. Therefore, the expected delay of payments goes to 0. By $b_t = \frac{\lambda_s u_t + \rho c}{\rho \lambda_s}$, $\lim_{u\to\infty}\varphi'_+(u) \to \frac{1}{\rho}$. By convexity of $\varphi(u)$, $\varphi'_+(u)$ is non-decreasing in u. We show in the following that $\varphi'_+(u)$ is right-continuous. Fix $a > 0$. By monotonicity of $\varphi'_+(u)$, $\lim_{u\to a^+}\varphi'_+(u)$ exists. Denote it by A. Then $\varphi'_+(u) \geq A$ for any $u > a$. As a result, $\varphi(y) - \varphi(x) \geq A(y-x)$ for any $y > x > a$. It is well-known that a convex function must be continuous. Therefore, taking the limit as $x \to a$, we have $\frac{\varphi(y)-\varphi(a)}{y-a} \geq A$. Let $y \to a$ and we can conclude that $\varphi'_+(a) \geq A$, which proves that $\varphi'_+(u)$ is right-continuous. By right-continuity, there exists a smallest $u > 0$ such that $\rho\lambda_s\varphi'_+(u) \geq \lambda_s + r - \rho$, which by definition is u_2.

At $u_t = u_2$, we must have $a_t = 1$, otherwise the agent never works after t and it is optimal to pay the agent u_t directly. Since the agent's utility keeps constant at u_2, it is easy to calculate $\pi(u_2)$ based on the optimal φ_t and w_t we derived before. The expression is given by

$$\pi(u_2) = \frac{r}{r+\lambda_s}(\lambda_s\pi - u_2 - \lambda_s\varphi(u_2)). \tag{28}$$

Since $\phi_t = w_t = 0$ whenever $u < u_2$, the objective function is well-behaved. Given $(u_2, \pi(u_2))$ as the termination condition, the value function must exist on $[0, u_2]$. As shown before, $a_t = 1$ is optimal at $u_t = u_2$. Thus, equation (25) is satisfied around and to the left of u_2. Plugging in $u_t = u_2$, we directly obtain $\pi'_-(u_2) = -\frac{r}{\rho}$, which confirms the differentiability of $\pi(u)$. Denote the solution to (25) by $\pi_1(u)$, then

$$\pi_1(u) = \frac{r\lambda_s\pi}{r+\lambda_s} - \frac{r\lambda_s}{\rho}u^{\frac{r+\lambda_s}{\rho}}\int_u^{u_2} v^{-\frac{r+\lambda_s}{\rho}-1}\varphi(v)dv + C_2 u^{\frac{r+\lambda_s}{\rho}}, \tag{29}$$

where C_2 is a constant. The first term is the expected benefit from success when the agent always exerts full effort, the second one represents expected payments upon success before u_2, and the last term is the expected payment after achieving u_2. By $\pi_1(u_2) = \pi(u_2)$, we can get the expression $C_2 = -\frac{ru_2 + r\lambda_s\varphi(u_2)}{r+\lambda_s} u_2^{-\frac{r+\lambda_s}{\rho}} < 0$.

Step 3. Next we check the concavity of $\pi_1(u)$. Specifically, given $0 < \hat{u}_1 < \hat{u}_2 \leq u_2$ and $\hat{u} = \lambda\hat{u}_1 + (1-\lambda)\hat{u}_2$, where $\lambda \in (0,1)$, we show that $\pi_1(\hat{u}) > \lambda\pi_1(\hat{u}_1) + (1-\lambda)\pi_1(\hat{u}_2)$.

Define $\hat{t}_1$, $\hat{t}_2$, and $\hat{t}$ as the time when the continuation utility achieves u_2, given the initial utility $\hat{u}_1$, $\hat{u}_2$, and $\hat{u}$. By definition, $\pi_1(\hat{u})$ is the principal's expected utility when the agent always exerts full effort and the principal starts to pay w_t at $\hat{t}$. Consider an alternative way to deliver $\hat{u}$. The principal still pays $\varphi(u_t)$ to induce the agent to exert full effort, but now she starts to pay $\hat{w}_t = (1-\lambda)u_2 dt$ from $\hat{t}_2 < \hat{t}$ and the flow rate jumps to $\hat{w}_t = u_2$ at $\hat{t}_1$. Denote the expected utility given by this strategy as $\pi_1^0(\hat{u})$. By the analysis in step 2, paying w_t before achieving u_2 is suboptimal. Therefore, $\pi_1^0(\hat{u}) \leq \pi_1(\hat{u})$. We focus on this new strategy from now on.

When $w_t = 0$, $u_{t+dt} = (1+\rho dt)u_t$, which is linear in u_t. As a result, $\hat{u}_t = \lambda\hat{u}_{1,t} + (1-\lambda)\hat{u}_{2,t}$ for all $t \leq \hat{t}_2$. Starting from $\hat{t}_2$, $\hat{u}_{2,t} = u_2$ is constant, $\hat{u}_{1,t+dt} = (1+\rho dt)\hat{u}_{1,t}$, and $\hat{u}_{t+dt} = \hat{u}_t + \lambda\rho\hat{u}_{1,t}dt$ since $\hat{w}_t = (1-\lambda)u_2 dt$. Consequently, we still have $\hat{u}_t = \lambda\hat{u}_{1,t} + (1-\lambda)\hat{u}_{2,t}$ when $t \in (\hat{t}_2, \hat{t}_1)$. Starting from $\hat{t}_1$, both $\hat{u}_t$ and $\hat{u}_{1,t}$ equal to u_2.

There are three terms in π_1, the expected benefit from success, expected payments conditional on success, and expected payments before report of success. Since the effort process is the same, the expected benefit is the same in each π_1. By $\hat{w}_t = (1-\lambda)u_2 dt$ and $\hat{l}_{1,t} = 0$ when $t \in (\hat{t}_2, \hat{t}_1)$, the expected payment before report of success in $\pi_1^0(\hat{u})$ is a linear combination of that in $\pi_1(\hat{u}_1)$ and $\pi_1(\hat{u}_2)$. To show that $\pi_1^0(\hat{u}) \geq \lambda\pi_1(\hat{u}_1) + (1-\lambda)\pi_1(\hat{u}_2)$, we only need to show that the expected payment conditional on success in $\pi_1^0(\hat{u})$ is smaller than the linear combination of that in $\pi_1(\hat{u}_1)$ and $\pi_1(\hat{u}_2)$. Since $\hat{u}_t = \lambda\hat{u}_{1,t} + (1-\lambda)\hat{u}_{2,t}$ for all t, by convexity of $\varphi(u)$, $\varphi(\hat{u}_t) \geq \lambda\varphi(\hat{u}_{1,t}) + (1-\lambda)\varphi(\hat{u}_{2,t})$ for all t.

To conclude, $\pi_1(\hat{u}) \geq \pi_1^0(\hat{u}) \geq \lambda\pi_1(\hat{u}_1) + (1-\lambda)\pi_1(\hat{u}_2)$.

Step 4. It is not optimal to induce the agent to work for all u. When $u \to 0$, $\varphi(u) \to \infty$ and inducing $a_t = 1$ becomes arbitrarily costly. Therefore, $a_t = 0$ is optimal when u is small enough. Solving the differential equation (24), we obtain

$$\pi_0(u) = C_1 u^{\frac{r}{\rho}}, \tag{30}$$

where C_1 is a constant. When $C_1 \leq 0$, the value function decreases in u and it is optimal to induce $a_t = 0$ for all t. When $C_1 > 0$, inducing $a_t = 1$ is optimal for some u_t.

Define u_1 as the point where the agent is indifferent between $a_t = 0$ and $a_t = 1$. By value matching and smooth pasting, $\pi_0(u_1) = \pi_1(u_1)$ and $\pi_0'(u_1) = \pi_1'(u_1)$. Combining equation (24) and (25), we get

$$\pi_0(u_1) = \pi_1(u_1) = r\pi - r\varphi(u_1). \tag{31}$$

We can solve for u_1 by the second equality and then pin down C_1 with the first equality.

We next show that the solution exists and $C_1 > 0$. By definition, $\pi_1(u_2) > -u_2$. By (28), $r(\pi - \varphi(u_2)) > -u_2$. Then $\pi_1(u_2) - r(\pi - \varphi(u_2)) = \frac{r}{r+\lambda_s}(-r(\pi - \varphi(u_2)) - u_2) < 0$. On the other hand, $\varphi(u) \to \infty$ as $u \to 0$. By (29), it can be shown that $\pi_1(u) + r\varphi(u) \to \infty$ as $u \to 0$. Therefore, there exists a solution to the equation $\pi_1(u) = r\pi - r\varphi(u)$ within $(0, u_2)$. To ensure that $C_1 > 0$, we only need to have $\pi > \varphi(u)$ for some u. Since $\varphi(u)$ decreases in c, there exists $c^* > 0$ such that $C_1 > 0$ when $c < c^*$. By $C_1 > 0$ and $\rho > r$, $\pi_0(u)$ is concave.

Step 5. Finally we check whether IC_3 is satisfied. If the agent achieves a success at t where $u_t = u_2$, then he gets $b_t = \frac{\lambda_s u_2 + \rho c}{\rho \lambda_s}$ when he reports it. If he reports it at $t + dt$, then he gets $u_2 dt + (1 - \rho dt)\frac{\lambda_s u_2 + \rho c}{\rho \lambda_s}$. Since $\frac{\lambda_s u_2 + \rho c}{\rho \lambda_s} > u_2$, the agent prefers to report earlier. If the agent achieves a success before $u_t = u_2$, then he gets $\frac{\lambda_s u_t + \rho c}{\rho \lambda_s}$ by reporting immediately and $(1 - \rho dt)\frac{\lambda_s u_{t+dt} + \rho c}{\rho \lambda_s}$ by reporting at $t + dt$. Since $u_{t+dt} = u_t + \rho u_t dt$, we can obtain $\frac{\lambda_s u_t + \rho c}{\rho \lambda_s} > (1 - \rho dt)\frac{\lambda_s u_{t+dt} + \rho c}{\rho \lambda_s}$. In conclusion, IC_3 is always satisfied. $\square$

Proof of Theorem 1. By the proof of Proposition 2, $\pi'(u_1) = C_1 \frac{r}{\rho} u^{\frac{r}{\rho} - 1} > 0$ and $\pi'(u_2) = -\frac{r}{\rho} < 0$. Since $\pi(u)$ is concave on $(0, u_2)$, the optimal initial utility $u_0 \in (u_1, u_2)$. Given that IC_1 always binds and $w_t = \phi_t = 0$ whenever $u_t < u_2$, by (23), u_t increases over time until it reaches u_2. The rest follows from the optimal choice specified in Proposition 2. $\square$

Proof of Corollary 1. Since all variables are constant once u_t reaches u_2, we only need to prove the case where $u_t < u_2$. By the proof of Theorem 1, u_t increases over time. By IC_1

binding, $b_t = \frac{\lambda_s u_t + \rho c}{\rho \lambda_s}$. Therefore, b_t is increasing over time. By the proof of Proposition 2, the expected delay decreases in u_t. By u_t increasing, the expected delay decreases over time. □

Proof of Proposition 3. Normalize the utilities by $\bar{u}_0 = \frac{u_0}{\rho}$ and $\bar{\pi}_0 = \frac{\pi_0}{r}$. Then the value function can be rewritten as

$$\begin{aligned}\bar{\pi}(\bar{u}_t) = \sup_{a_t,\phi_t,w_t,l_t} &\{\lambda_s a_t \pi dt - \phi_t - w_t dt - \lambda_f l_t dt - \lambda_s a_t \varphi_t(b_t, \tilde{b}_t)dt \\ &+ (1 - rdt)(1 - \lambda_s a_t dt)(1 - \lambda_f dt)\bar{\pi}(\bar{u}_{t+dt})\} + o(dt) \\ &\text{subject to equation (3).}\end{aligned}$$

Since IC_3 never binds, there is no difference between paying w_t and l_t. Without loss of generality, we can assume $l_t = 0$.

As shown in the proof of Proposition 2, the cost function φ_t is determined by the evaluation technology and the difference in discount factors, $\rho - r$. Therefore, φ_t is not affected by λ_f. By $(1 - rdt)(1 - \lambda_f dt) = 1 - (r + \lambda_f)dt$, the objective function is the same as the one in which there are no breakdowns and the discount factors are $\rho + \lambda_f$ and $r + \lambda_f$.

Finally, by (3), the evolution of $\bar{u}_t$ is given by

$$\bar{u}_t = \sup_{a_t}\{a_t \lambda_s b_t dt - c a_t dt + \phi_t + w_t dt + (1 - \rho dt)(1 - \lambda_s a_t dt)(1 - \lambda_f dt)\bar{u}_{t+dt}\} + o(dt).$$

By $(1 - \rho dt)(1 - \lambda_f dt) = 1 - (\rho + \lambda_f)dt$, we can conclude that the value function is the same as the one in which there are no breakdowns and the discount factors are $\rho + \lambda_f$ and $r + \lambda_f$. □

Proof of Theorem 2 and Corollary 2. When IC_3 is not binding, by Proposition 3, the form of the optimal contract is the same as the one in Theorem 1.

When IC_3 is binding, define $\tilde{\pi}(u, b)$ as the principal's maximum utility given $u_0 = u$ and $b_0 = b$. Define $\bar{\pi}(u) := \sup_{b \geq 0} \tilde{\pi}(u, b)$. Note that $\bar{\pi}(u) = \pi(u)$ when IC_3 does not bind. In general, $\bar{\pi}(u) \leq \pi(u)$. At $u_t = u_2$, the principal optimally sets $l_t > 0$ such that b_t does not change over time. Since the agent who has achieved a success can never get l_t, he has no incentives to report success at a later time. In other words, IC_3 does not bind when $u \geq u_2$. Therefore, $\bar{\pi}(u) = \pi(u)$ for all $u \geq u_2$. As a result, $\bar{\pi}'_+(u_2) = -\frac{r}{\rho}$. By concavity, $\bar{\pi}'_+(u) > -\frac{r}{\rho}$ for $u < u_2$. Therefore, $\phi_t = 0$ for $t < t_1$. By the same argument as in the proof of Proposition 2, we have $l_t = 0$ for $t < t_1$.

We focus on the scenario where c is small enough such that it is optimal to induce $a = 1$ at time $t = 0$. We first show that u_t cannot decrease. Suppose $u_{t+dt} < u_t$ for some t. Then the principal must be able to decrease b_t, l_t, or ϕ_t such that $u_{t+dt} = u_t$ and no IC constraints are violated. By $\bar{\pi}'_+(u) > -\frac{r}{\rho}$, the principal's payoff increases.

Denote by t_1 the first moment that IC_3 is binding. We first show that $a_t = 1$ and IC_1 is binding for all $t \leq t_1$. Define $\pi_t(u_t)$ as the principal's continuation utility at time t. Since IC_3 does not bind, $\pi_t(u_t) = \bar{\pi}(u_t)$. Given that u_t increases and $\bar{\pi}(u)$ is concave, optimally $a_t = 1$. Suppose IC_1 is not binding around some $t_0 < t_1$. Then there exists $\epsilon > 0$ such that $b_{t_0} - \epsilon > \frac{\lambda_s u_{t_0} + \rho c}{\rho \lambda_s}$ and $b_{t_0} - \epsilon > (1 - \rho dt) b_{t_0+dt}$ as $dt \to 0$. Let $\hat{b}_t = b_t - \epsilon$ and $\hat{l}_t = l_t + \frac{\lambda_s}{\lambda_f}\epsilon$. Then by the same argument as in the proof of Proposition 2, the principal's expected utility increases. Given that IC_1 binds, b_t increases over t when $t < t_1$.

We next show that $a_t = 1$ for $t > t_1$. Suppose the agent exerts full effort up to time t. Then $b_t \geq \frac{\lambda_s u_t + \rho c}{\rho \lambda_s}$. By IC_3 binding, $b_{t+dt} = b_t + \rho b_t dt$. By the promise keeping constraint, $u_{t+dt} \leq u_t + (\rho + \lambda_f) u_t dt$. Therefore, when $u_t < \frac{\rho^2 c}{\lambda_s \lambda_f}$, $b_{t+dt} > \frac{\lambda_s u_{t+dt} + \rho c}{\rho \lambda_s}$. When $u_t > \frac{\rho^2 c}{\lambda_s \lambda_f}$, it is impossible to induce the agent to exert effort without payments before report. The following lemma shows that it is always optimal to pay before report such that $a_t = 1$.

Lemma 7. *Suppose* $u > \frac{\rho^2 c}{\lambda_s \lambda_f}$ *and* $\frac{(\rho+\lambda_f)c}{\lambda_s \lambda_f} \leq b < \frac{\lambda_s u + \rho c}{\rho \lambda_s}$. *Then* $\tilde{\pi}(u, b) = -\frac{r}{\rho}(u - \frac{\rho \lambda_s b - \rho c}{\lambda_s}) + \tilde{\pi}(\frac{\rho \lambda_s b - \rho c}{\lambda_s}, b)$.

Proof. By $b < \frac{\lambda_s u + \rho c}{\rho \lambda_s}$, $a_0 = 0$ if $\phi_0 = 0$. Suppose $\phi_s = 0$ and $a_s = 0$ for all $s \in [0, t)$. By $u > \frac{\rho^2 c}{\lambda_s \lambda_f}$, we have $b_t < \frac{\lambda_s u_t + \rho c}{\rho \lambda_s}$. Thus, we must have $\phi_t > 0$ for some t, otherwise $a_t = 0$ for all t and it is optimal to pay directly at the beginning. Define t_0 as the smallest t such that $\phi_t > 0$. Suppose $t_0 > 0$. Then $a_s = 0$ for $s \in [0, t_0)$. Consider an alternative strategy in which $\hat{\phi}_0 = du > 0$ and $\hat{\phi}_{t_0} = \phi_{t_0} - e^{(\rho+\lambda_b)t_0} du$, where du is small enough. Then we still have $a_s = 0$ for $s \in [0, t_0)$. By equation (3), u_{t_0} is unchanged. Therefore, with this new strategy, the principal's payoff increases by $-du + e^{-(r+\lambda_f)t_0} e^{(\rho+\lambda_f)t_0} du > 0$. As a result, optimally $\phi_0 > 0$.

By $b > \frac{(\rho+\lambda_f)c}{\lambda_s\lambda_f}$, we obtain $\frac{\rho\lambda_s b-\rho c}{\lambda_s} > \frac{\rho^2 c}{\lambda_s\lambda_f}$. Thus, $\tilde{\pi}(u,b) = -\frac{r}{\rho}(u - \frac{\rho\lambda_s b-\rho c}{\lambda_s}) + \tilde{\pi}(\frac{\rho\lambda_s b-\rho c}{\lambda_s}, b)$. $\square$

Intuitively, when the bonus payment is not enough to induce $a = 1$, the principal optimally makes a lump-sum payment in an amount that the continuation utility is decreased to the point where $a = 1$ can be induced. Therefore, by Lemma 7, we can conclude that $a_t = 1$ for all t.

Define t_3 as the last moment that IC_3 is binding. We show in the following that $u_{t_3} = u_2$. Suppose $u_{t_3} < u_2$. If IC_1 is not binding at t_3, then the principal can increase her payoff by decreasing b_{t_3}. If IC_1 is binding, then it must be the case that $u_{t_3} > \frac{\rho^2 c}{\lambda_s\lambda_f}$. To ensure $a = 1$ around t_3, l_{t_3} needs to be positive. Since IC_3 is not binding at $t_3 + dt$, there exists $\epsilon > 0$ such that $\epsilon < l_{t_3}$ and $\arg\max_{b\geq 0} \tilde{\pi}(u_{t3+dt} + \rho\lambda_f\epsilon, b) < (1+\rho dt)b_{t_3} + o(dt)$. By $\bar{\pi}'_+(u) > -\frac{r}{\rho}$, the principal can increase her payoff by decreasing l_{t_3}. Given that IC_3 is binding until $u_t = u_2$, b_t increases over time.

Define $t_2 \geq t_1$ as the first moment after t_1 such that IC_1 is binding on $[t_2, t_2 + dt]$. Then $u_{t_2} > \frac{\rho^2 c}{\lambda_s\lambda_f}$. To pin down the optimal process of u_t, l_t, and ϕ_t, we present the following lemma.

Lemma 8. *Suppose* $0 < u < u_2$ *and* $b > \frac{\lambda_s u+\rho c}{\rho\lambda_s}$. *Then* $\tilde{\pi}'_{u^+}(u,b) > -\frac{r}{\rho}$.

Proof. If $a = 0$ is optimal, then $\tilde{\pi}(u,b)$ is increasing in u. The statement is obviously true. We focus on the case where $a = 1$ is optimal.

Given $a = 1$, the cost of payments after report within time period dt is characterized by $c_1 = r\lambda_s\varphi(b,u)dt + o(dt)$. Consider $\tilde{\pi}(u+du,b)$. A feasible strategy is to induce $a = 1$ within time dt and then make a lump-sum payment at time dt, such that the continuation utility is the same as in $\tilde{\pi}(u,b)$. Then the amount of lump-sum payment is $du + (\rho+\lambda_f)dudt$. As a result, the cost of payment within time dt is given by $c_2 = \frac{r}{\rho}(du + (\rho - r - \lambda_s)dtdu) + r\lambda_s\varphi(b, u+du)dt + o(dt)$. To show that $\tilde{\pi}'_{u^+}(u,b) > -\frac{r}{\rho}$, we only need to show $c_2 < c_1 + \frac{r}{\rho}du$. As $du \to 0$, it is equivalent to $-\rho\lambda_s\varphi'_{2+}(b,u) > \rho - r - \lambda_s$. By definition, $\varphi'_{2+}(b,u) \leq 0$. Since $\varphi(b,u)$ is homogeneous of degree one and is convex in u, $\varphi'_{2+}(b,u)$ decreases in b, i.e., $\varphi'_{2+}(b,u) \leq \varphi'_{2+}(\frac{\lambda_s u+\rho c}{\rho\lambda_s}, u)$. By $\varphi(b,u)$ increasing in b, $\varphi'_{2+}(\frac{\lambda_s u+\rho c}{\rho\lambda_s}, u) < \varphi'_+(u)$, where $\varphi(u)$ is defined as the cost when IC_1 is always binding. Therefore, $\varphi'_{2+}(b,u) < \varphi'_+(u)$. By definition of u_2 as in Proposition 2, $\rho\lambda_s\varphi'_+(u) < \lambda_s + r - \rho$ for any $u < u_2$. Consequently, $-\rho\lambda_s\varphi'_{2+}(b,u) > -\rho\lambda_s\varphi'_+(u) > \rho - r - \lambda_s$. □

The intuition is very similar to that of u_2. The benefit of paying the agent later is to increase the agent's continuation utility today and thus decrease the delay of bonus. The cost, on the other hand, is due to the difference in discount factors. When $u < u_2$, the benefit outweighs the cost and it is optimal to set $\phi = 0$.

By Lemma 8, $\phi_t = l_t = 0$ for $t < t_2$. When $t \geq t_2$, we still have $\phi_t = 0$. However, l_t must be positive to make both IC_1 and IC_3 binding. Specifically, by IC_3 binding at t, $b_{t+dt} = b_t + \rho b_t dt + o(dt)$. By IC_1 binding at $t + dt$, $u_{t+dt} = \frac{(b_t+\rho b_t dt)\rho\lambda_s - \rho c}{\lambda_s} + o(dt)$. By equation (3), $l_t = \frac{1}{\rho\lambda_f}(\lambda_f u_t - \frac{\rho^2 c}{\lambda_s})$, which is increasing in u_t. As u_t gets to u_2, l_t jumps up to $\frac{\rho+\lambda_f}{\rho\lambda_f}u_2$, which keeps u_t constant thereafter.

Next we show that $l_0 = 0$. In other words, it is not optimal to make both IC_1 and IC_3 binding at the beginning. Suppose both are binding and $u_0 \geq \frac{\rho^2 c}{\lambda_s\lambda_f}$. Then $l_t > 0$ for all t. Consider an alternative strategy where b_0 is increased by ϵ and all future b_t are increased such that IC_3 binds until $u_t = u_2$. Let ϵ be of the same magnitude with dt. Denote variables associated with the new strategy by $\hat{b}_{dt}$ and $\hat{u}_{dt}$. Then $\hat{b}_{dt} = b_{dt} + \epsilon + \rho\epsilon dt + o(dt) = b_{dt} + \epsilon + o(dt)$. Since IC_3 binds, $\hat{u}_{dt} = u_{dt} + \rho\epsilon + o(dt)$. By equation (3), $\lambda_f l_t dt$ decreases by $\epsilon\lambda_s dt + \epsilon + o(dt) = \epsilon + o(dt)$. Therefore, the difference in the principal's payoff is

$$\begin{aligned}\Delta\pi = & -r\left[\lambda_s\varphi(b_0+\epsilon, u_0)dt - \lambda_s\varphi(b_0, u_0)dt - \epsilon\right] \\ & + (1 - rdt)\left[\tilde{\pi}(\hat{u}_{dt}, \hat{b}_{dt}) - \tilde{\pi}(u_{dt}, b_{dt})\right] + o(dt) \\ = & \; r\epsilon + \tilde{\pi}(\hat{u}_{dt}, \hat{b}_{dt}) - \tilde{\pi}(u_{dt}, b_{dt}) + o(dt). \end{aligned} \tag{32}$$

By Lemma 8,

$$\tilde{\pi}(\hat{u}_{dt}, \hat{b}_{dt}) \geq \tilde{\pi}(\hat{u}_{dt}, b_{dt}) > \tilde{\pi}(u_{dt}, b_{dt}) - \frac{r}{\rho}(\hat{u}_{dt} - u_{dt}) + o(dt) = \tilde{\pi}(u_{dt}, b_{dt}) - r\epsilon + o(dt).$$

Thus, $\Delta\pi > 0$, which proves that $l_0 > 0$ is not optimal. By the same logic, we are able to show that $t_1 < t_2$.

What is remaining to show is that l_t increases gradually before jumping to $\frac{\rho+\lambda_f}{\rho\lambda_f}u_2$, or

$t_2 < t_3$. Let $\tilde{t} = t_3 - dt$. We first show that $\tilde{\pi}(u_2, b_2) = \tilde{\pi}(u_{\tilde{t}}, b_{\tilde{t}}) - \frac{r}{\rho}(u_2 - u_{\tilde{t}}) + o(dt)$, where $b_2 = \frac{\lambda_s u_2 + \rho c}{\rho \lambda_s}$, $b_{\tilde{t}} = (1 - \rho dt) b_2 + o(dt)$, and $u_{\tilde{t}} = \frac{\rho \lambda_s b_{\tilde{t}} - \rho c}{\lambda_s}$. Consider $\tilde{\pi}(u_2, b_2)$ and $\tilde{\pi}(u_{\tilde{t}}, b_{\tilde{t}})$. The continuation payoff of the principal is the same given these two pairs of state variables. Thus, the difference between $\tilde{\pi}(u_2, b_2)$ and $\tilde{\pi}(u_{\tilde{t}}, b_{\tilde{t}})$ is in the flow payoff within dt. Since $b_2 - b_{\tilde{t}} = O(dt)$, the difference in payments after report is $o(dt)$. Therefore, $\tilde{\pi}(u_2, b_2) - \tilde{\pi}(u_{\tilde{t}}, b_{\tilde{t}}) = -r\lambda_f(l_2 - l_{\tilde{t}})dt + o(dt)$, where $l_2 = \frac{\rho + \lambda_f}{\rho \lambda_f} u_2$ and $l_{\tilde{t}} = \frac{1}{\rho \lambda_f}(\lambda_f u_{\tilde{t}} - \frac{\rho^2 c}{\lambda_s})$, as shown before. Plugging the expression of $b_{\tilde{t}}$ and b_2 into the definition of $u_{\tilde{t}}$, we obtain $u_{\tilde{t}} = u_2 - \frac{\rho(\lambda_s u_2 + \rho c)}{\lambda_s} dt$. By rearrangement, $\tilde{\pi}(u_2, b_2) = \tilde{\pi}(u_{\tilde{t}}, b_{\tilde{t}}) - \frac{r}{\rho}(u_2 - u_{\tilde{t}}) + o(dt)$.

Suppose $t_2 = t_3$. Consider an alternative strategy where $b_{t_3 - dt}$ is decreased by ϵ, such that IC_1 binds at $t_3 - dt$. Then ϵ must be of the same magnitude as dt. Further decrease all future b_t such that IC_3 binds until $u_t = u_2$. Denote the continuation utility by $\hat{u}$ and the bonus at t_3 by $\hat{b}$. Then by the result above, $\tilde{\pi}(u_2, b_2) - \tilde{\pi}(\hat{u}, \hat{b}) = -\frac{r}{\rho}(u_2 - \hat{u})dt + o(dt)$. Since $b_{t_3 - dt}$ is decreased by ϵ, b_{t_3} should decrease by $\epsilon + o(dt)$ and u_{t_3} decreases by $\rho\epsilon + o(dt)$. Therefore, $\tilde{\pi}(u_2, b_2) - \tilde{\pi}(\hat{u}, \hat{b}) = -r\epsilon + o(dt)$. By equation (32), $\Delta\pi = o(dt)$. Now decrease b_t in between t_1 and $t_3 - dt$ such that IC_3 binds. At the same time, adjust l_t so that the process $\{u_t\}_{t \geq 0}$ is unchanged up to $t_3 - dt$. The decrease of each b_t should be of the same magnitude as dt. Suppose b_t is decreased by η, then l_t increases by $\frac{\lambda_s}{\lambda_f}\eta$. The flow payoff of the principal is therefore increased by

$$D\pi_f = -r\left[\lambda_s \varphi(b_t - \eta, u_t) - \lambda_s \varphi(b_t, u_t) + \lambda_s \eta\right] dt > r\lambda_s \left[\varphi(\eta, u_t) - \eta\right] dt > 0.$$

Since $t_1 < t_2$ and the flow payoff is increased in the whole range, the increase in the principal's expected payoff is of the same magnitude as dt. Therefore, the principal's expected payoff increases with this new strategy, which shows that $t_2 = t_3$ is not optimal.

Finally, we argue that randomization is not optimal. Given $(\hat{u}, \hat{b}) = \lambda(\hat{u}_1, \hat{b}_1) + (1 - \lambda)(\hat{u}_2, \hat{b}_2)$, where $\lambda \in (0, 1)$, we want to show that $\tilde{\pi}(\hat{u}, \hat{b}) \geq \lambda\tilde{\pi}(\hat{u}_1, \hat{b}_1) + (1 - \lambda)\tilde{\pi}(\hat{u}_2, \hat{b}_2)$. Since the agent is always induced to exert high effort, we only need to show that the cost is convex in (u, b). Choose $\hat{l}_t$ such that $\hat{l}_t = \lambda \hat{l}_{1t} + (1 - \lambda)\hat{l}_{2t}$. Since everything is linear in l_t, it is sufficient to show that $\varphi(b, u)$ is convex. The convexity of $\varphi(b, u)$ can be proved in the same way as proving the convexity of $\varphi(u)$. $\square$

Proof of Lemma 2. First consider the first-best scenario where the agent gets the benefit from success. Denote by p_t the agent's belief that success has been achieved. Since p_t increases over time, the marginal benefit of effort decreases over time. Therefore, the agent's effort should be frontloaded. See Lemma 3 for the formal proof of this statement. Let T be the time at which the agent stops working. Then his expected utility is

$$u_0 = (1 - e^{-\lambda_s T})e^{-rT}\pi - c\int_0^T e^{-rt}dt.$$

Taking the first-order derivative, T^{FB} is the solution to

$$[-re^{-rT} + (\lambda_s + r)e^{-(\lambda_s+r)T}]\pi = ce^{-rT}.$$

By rearrangement, $(\lambda_s + r)e^{-\lambda_s T^{FB}} - r = \frac{c}{\pi}$.

Next consider the principal-agent relationship with a deterministic deadline. Denote the deadline by T. Define b_T as the agent's expected payoff at T when there was success and $\hat{b}_T$ as that when there was no success. Then the marginal benefit of effort at $t \leq T$ is $\lambda_s e^{-\lambda_s T}e^{-\rho(T-t)}(b_T - \hat{b}_T)$. Since the marginal cost is constant at c, the agent's incentive to work is smallest at the beginning. In the optimal contract, the agent is always induced to work from the beginning, otherwise the principal can move forward the deadline and be better off. Therefore, the binding IC constraint is $\lambda_s e^{-\lambda_s T}e^{-\rho T}(b_T - \hat{b}_T) = c$. For convenience, denote $\Delta b_T = b_T - \hat{b}_T$. Let ψ_T be the principal's minimum expected cost of payments at T. Then $\psi_T = (1 - e^{-\lambda_s T})e^{(\rho-r)d_T}b_T + e^{-\lambda_s T}e^{(\rho-r)d_T}\hat{b}_T = e^{(\rho-r)d_T}(b_T - e^{-\lambda_s T}\Delta b_T)$, where d_T is the delay of the payment after the deadline. Fixing the value of Δb_T, we have ψ_T increase in T because $e^{-\lambda_s T}$ decreases in T. Rewrite ψ_T as $\psi_T = k_T\Delta b_T$. Then k_T increases in T. The principal's expected payoff is given by

$$\pi_0 = (1 - e^{-\lambda_s T})e^{-rT}\pi - e^{-rT}k_T\Delta b_T.$$

Plugging in the IC constraint, we obtain

$$\pi_0 = (1 - e^{-\lambda_s T})e^{-rT}\pi - \frac{k_T c}{\lambda_s}e^{(\rho-r)T}(e^{\lambda_s T} - 1).$$

Taking the first-order derivative and rearranging, the optimal deterministic deadline is given by

$$(\lambda_s+r)e^{-\lambda_s T^*}-r=\frac{k_{T^*}\cdot c}{\pi}[(\rho-r)e^{\rho T^*}\frac{e^{\lambda_s T^*}-1}{\lambda_s}+e^{(\rho+\lambda_s)T^*}]+\frac{k'_{T^*}\cdot c}{\pi}e^{(\rho-r)T^*}\frac{e^{\lambda_s T^*}-1}{\lambda_s}. \quad (33)$$

It is obvious that the right hand side of (33) is larger than $\frac{c}{\pi}$. Therefore, $T^* < T^{FB}$. □

Proof of Proposition 4. By the same argument as in the proof of Lemma 2, the agent always exerts full effort until the deadline in the optimal contract. Denote T as the largest possible deadline. In other words, the project is terminated for sure by time T, but there is a positive probability that it is not terminated by time $T-\tau$ for any $\tau > 0$. Define $F(t)$ as the distribution function of the deadline, where $F(T) = 1$. Similar to the proof of Lemma 2, let b_t be the agent's expected utility when the deadline is at t and there was success before t, and let $\hat{b}_t$ be the counterpart when there was no success. Define $\Delta b_t = b_t - \hat{b}_t$. Then the principal's minimum expected cost of payments after the deadline at t can be expressed as $\psi_t = k_t \cdot \Delta b_t$, where k_t increases in t by the proof of Lemma 2. I first show that $F(t)$ is strictly increasing on $[0,T]$ when B is large enough.

Suppose $F(t_1) = F(t_2^-) < F(t_2)$ for some $t_1 < t_2$, where $F(t_2^-)$ is the left limit at t_2.[1] By $F(t_1) = F(t_2^-)$, the marginal benefit of effort at $t \in (t_1,t_2)$ is given by $e^{-\lambda_s t_2}e^{-\rho(t_2-t)}(u_{t_2}-\hat{u}_{t_2})$, where u_{t_2} is the agent's expected utility right before t_2 when there was success, $\hat{u}_{t_2}$ is the counterpart when there was no success, and $e^{-\lambda_s t_2}$ denotes the probability that no success is achieved by t_2. Therefore, the marginal benefit of effort increases over time. Since the marginal cost is constant and the agent is always induced to exert effort, he must strictly prefer working at any $t \in (t_1,t_2)$.

Consider an alternative policy $\hat{F}(t)$ such that $\hat{F}(t) = F(t_1)+\epsilon$ for $t \in [t_2-\tau, t_2)$, $\hat{F}(t) = F(t)$ for $t < t_2-\tau$, and $\frac{\hat{F}(t)-\hat{F}(t_2^-)}{1-\hat{F}(t_2^-)} = \frac{F(t)-F(t_2^-)}{1-F(t_2^-)}$ for $t \geq t_2$, where $\epsilon \in (0, 1-F(t_1))$ and $\tau \in (0, t_2-t_1)$. In words, the project is terminated at $t_2-\tau$ with probability ϵ, but conditional on not being terminated, the distribution of the deadline is not affected. Given that the agent strictly prefers working at $t_2-\tau$, we can decrease Δb_{t_2} such that the agent still prefers working on $(t_2-\tau, t_2)$. Denote the decrease in Δb_{t_2} by d_1. Let

[1] The argument would be the same if $F(t_1) = F(t_2) < F(t_2+\eta)$ for any $\eta > 0$.

$\Delta b_{t_2-\tau} = d_2$ such that the marginal benefit of effort at t_1 is unchanged. Then by definition, $e^{-\lambda_s(t_2-\tau)}\epsilon e^{-\rho(t_2-\tau-t_1)}d_2 = e^{-\lambda_s t_2}p_\epsilon e^{-\rho(t_2-t_1)}d_1$, where $p_\epsilon = \hat{F}(t) - \hat{F}(t_2^-) > 0$. By rearrangement, $e^{(\rho+\lambda_s)\tau}\epsilon d_2 = p_\epsilon d_1$. Next consider the principal's expected cost of payments. With the new policy, the change in the expected cost is given by $e^{-r(t_2-\tau)}\epsilon k_{t_2-\tau}d_2 - e^{-rt_2}p_\epsilon k_{t_2}d_1 = e^{-r(t_2-\tau)}\epsilon k_{t_2-\tau}d_2 - e^{-rt_2}e^{(\rho+\lambda_s)\tau}\epsilon k_{t_2}d_2$. Since $\rho + \lambda_s > r$ and $k_{t_2} > k_{t_2-\tau}$, the expected cost decreases in d_2. Therefore, the principal optimally decreases Δb_{t_2} and increases $\Delta b_{t_2-\tau}$ to the point that the maximum payment after the deadline at $t_2 - \tau$ reaches B or the agent is indifferent between working and shirking at $t_2 - \tau$. Suppose the maximum payment is smaller than B. Then the principal can decrease ϵ and increase $\Delta b_{t_2-\tau}$ such that the cost is unchanged. Since the expected probability of success decreases in ϵ, the principal's expected payoff increases as ϵ decreases. Therefore, the maximum payment must be B in the optimal contract.

Next I identify the condition under which the new policy is better than the original one for the principal. Given any $\epsilon > 0$, the principal's expected benefit from success is decreased by $L = \epsilon[e^{-\lambda_s(t_2-\tau)}(1-e^{-\lambda_s\tau})e^{-rt_2}\pi - (1-e^{-\lambda_s(t_2-\tau)})(e^{-r(t_2-\tau)} - e^{-rt_2})\pi]$. As long as $t_2 \leq T^{FB}$, $L > 0$ for any $\tau > 0$. When there is no bound on the aggregate payment, i.e. $B = \infty$, the principal can decrease Δb_{t_2} and increase $\Delta b_{t_2-\tau}$ to the point that the agent is indifferent between working and shirking at $t_2 - \tau$, for any $\epsilon > 0$. Therefore, it is optimal to set ϵ positive but as small as possible. In general, the expected cost of payments in the new policy decreases in B. There exists $B_{t_2} > 0$ such that the new policy is better for the principal whenever $B > B_{t_2}$. In other words, if $B > B_{t_2}$, it is impossible that $F(t_1) = F(t_2^-)$ in the optimal contract. Let $B^* = \sup_{t\in[0,T]} B_t$. Then as long as $B > B^*$, $F(t)$ must be strictly increasing on $[0, T]$.

I next argue that $F(t)$ has no jumps before T. Suppose $F(t^-) < F(t)$ for some $t < T$. As mentioned before, the agent is always induced to exert full effort. Specifically, he prefers working right after t. Since there is a positive probability that the deadline is at t, then he must strictly prefer working on an interval before t, say $(t - \tau, t)$. Then by the same argument as above, the principal can set the deadline at $t - \tau$ with a positive probability and be better off. A direct implication of this argument is that the agent is induced to be indifferent between working and shirking almost everywhere.

By the analysis above, the aggregate payment after the deadline at almost every $t < T$ equals B. Otherwise, suppose it is smaller than B on some interval $[t_1, t_2]$. Then the principal can decrease $F(t)$ and increase Δb_t for $t \in [t_1, t_2]$ such that the expected cost of payments is unchanged but the expected benefit from success is increased. However, it cannot be optimal to always pay the largest amount. Therefore, the payment after the deadline at T must be smaller than B and there is a positive probability that the deadline is at T.

Finally I show that optimally $T \in (T^*, T^{FB})$. Denote the optimal choice of T by $\bar{T}$. Suppose $\bar{T} \leq T^*$. Consider an alternative distribution $\hat{F}(t)$ such that $\hat{F}(t) = F(t)$ for $t < \bar{T}$, $\hat{F}(t) = F(\bar{T}^-)$ for $t \in [\bar{T}, \hat{T})$, and $\hat{F}(t) = 1$ for $t \geq \hat{T}$, where $\hat{T} = T^* + \tau$ for some $\tau > 0$. In words, the deadline at $\bar{T}$ is moved to $\hat{T}$. Set $\Delta b_{\hat{T}}$ such that the agent is indifferent between working and shirking at $\bar{T}$. Conditional on not terminated by $\bar{T}$, the increase in the expected benefit from success of this new policy is the same as moving a deterministic deadline from $\bar{T}$ to $\hat{T}$. On the other hand, to make the agent indifferent at $\bar{T}$, $\lambda_s e^{-\lambda_s \hat{T}} e^{-\rho(\hat{T}-\bar{T})} \Delta b_{\hat{T}} = \lambda_s e^{-\lambda_s \bar{T}} \Delta b_{\bar{T}} = c$. Therefore, the increase in the expected cost is smaller. As a result, the new policy is better off for the principal for some $\tau > 0$. Therefore, $\bar{T} > T^*$. On the other hand, the cost of inducing additional effort is larger than the marginal cost in the first best. As a result, $\bar{T} < T^{FB}$. □

Proof of Lemma 3. Suppose the agent works and gets the benefit from success. The continuation utility before the breakdown only depends on the belief that success has been achieved. Denote it by $u(p)$. At each time, the agent chooses to stop working or not. If he stops working, he gets an expected benefit of $p\pi$. Otherwise, he chooses an effort level $a \in [0, 1]$. By the evolvement of p_t in (12), we have

$$\begin{aligned}
u(p_t) &= \max\{\rho p_t \pi, \sup_{a_t \in [0,1]} \{-\rho c a_t dt + (1 - \rho dt)\big(1 - (1 - p_t)\lambda_b dt\big) u(p_{t+dt})\} + o(dt)\} \\
&= \max\{\rho p_t \pi, \sup_{a_t \in [0,1]} \{-\rho c a_t dt + (1 - \rho dt)\big(1 - (1 - p_t)\lambda_b dt\big) \\
&\qquad \cdot \big(u(p_t) + u'(p_t)\big((1 - p_t)\lambda_s a_t + p_t(1 - p_t)\lambda_b\big) dt\big)\} + o(dt)\}.
\end{aligned}$$

We use the "guess and verify" approach to solve for this differential equation. Suppose there exists $p^* \in [0, 1]$ such that the agent exerts full effort when $p < p^*$ and stops working when

$p \geq p^*$. Then $u(p) = \rho p \pi$ for any $p \geq p^*$. At $p_t \geq p^*$, if the agent exerts full effort for time period dt and then stops working, his expected utility is given by

$$
\begin{aligned}
\hat{u} = & -\rho c dt + (1-\rho dt)\big(1-(1-p_t)\lambda_b dt\big)\rho p_{t+dt}\pi + o(dt) \\
= & -\rho c dt + (1-\rho dt)\big(1-(1-p_t)\lambda_b dt\big)\rho \pi \big(p_t + \big((1-p_t)\lambda_s + p_t(1-p_t)\lambda_b\big)dt\big) + o(dt) \\
= & \rho p_t \pi + (1-p_t)\lambda_s \rho \pi dt - \rho^2 p_t \pi dt - \rho c dt + o(dt).
\end{aligned}
$$

As a result, $\hat{u} \leq \rho p_t \pi$ if and only if $p_t \geq \frac{\lambda_s \pi - c}{(\lambda_s+\rho)\pi}$. Therefore, we must have $p^* = \frac{\lambda_s \pi - c}{(\lambda_s+\rho)\pi}$. When $p \leq p^*$, given that $a_t = 1$, the differential equation is given by

$$
u'(p)\big((1-p)\lambda_s + p(1-p)\lambda_b\big) = \big(\rho + (1-p)\lambda_b\big)u(p) + \rho c, \tag{34}
$$

with termination condition $u(p^*) = \rho p^* \pi$.

We finally verify that $a_t = 0$ can never be optimal. At $p_t \geq p^*$, if the agent shirks for a period of dt and then stops working, his expected utility is given by

$$
\begin{aligned}
\tilde{u} &= (1-\rho dt)\big(1-(1-p_t)\lambda_b dt\big)\rho p_{t+dt}\pi + o(dt) \\
&= (1-\rho dt)\big(1-(1-p_t)\lambda_b dt\big)\rho \pi \big(p_t + \big(p_t(1-p_t)\lambda_b\big)dt\big) + o(dt) \\
&= \rho p_t \pi - \rho^2 p_t \pi dt + o(dt) < \rho p_t \pi.
\end{aligned}
$$

Therefore, $a_t = 0$ is not optimal when $p_t \geq p^*$. We next show that the agent's effort is frontloaded. Suppose $a_t = 0$ on some interval $[t_1, t_2]$. Define $u^S_{t_1}$ as the continuation utility conditional on success and $u^N_{t_1}$ as the one conditional on no success. Then $u^S_{t_1} > u^N_{t_1} \geq 0$ and $u_{t_1} = p_t u^S_{t_1} + (1-p_t)u^N_{t_1}$. Consider an alternative effort process where all effort after t_1 is moved forward by $t_2 - t_1$. Denote the new continuation utilities by $\hat{u}^S_{t_1}$ and $\hat{u}^N_{t_1}$. Then by definition, $\hat{u}^S_{t_1} = e^{\rho(t_2-t_1)}u^S_{t_1} > u^S_{t_1}$ and $\hat{u}^N_{t_1} = e^{\rho(t_2-t_1)}u^N_{t_1} \geq u^N_{t_1}$. Therefore, the new effort process is better. □

Proof of Lemma 4. Consider the maximization of the utility function as defined in (13). The state variables are y_t and q_t, and the control variable is a_t. By the expression of y_t and q_t, $\dot{y}_t = -(\lambda_b + \lambda_s a_t)y_t$ and $\dot{q}_t = -\lambda_b y_t$. Denote the corresponding costate variables by $-\xi_t$ and

$-\eta_t$. Then the Hamiltonian is given by

$$\mathcal{H}(a_t, y_t, q_t, \xi_t, \eta_t, t) = -\lambda_b y_t(\bar{W} - W_t^A) - e^{-\rho t} c a_t q_t + \xi_t(\lambda_b + \lambda_s a_t) y_t + \eta_t \lambda_b y_t. \tag{35}$$

The evolution of costate variables are

$$\dot{\xi}_t = \frac{\partial \mathcal{H}}{\partial y} = \xi_t(\lambda_b + \lambda_s a_t) + \eta_t \lambda_b - \lambda_b(\bar{W} - W_t^A) \tag{36}$$

and

$$\dot{\eta}_t = \frac{\partial \mathcal{H}}{\partial q} = -e^{-\rho t} c a_t. \tag{37}$$

The maximization condition requires that $a_t = 1$ if and only if $\lambda_s \xi_t y_t \geq e^{-\rho t} c q_t$. By definition, we can easily verify that $y_t = q_t(1 - p_t)$. Therefore, $\lambda_s(1 - p_t)\xi_t \geq c e^{-\rho t}$ for all $t \leq T$.

By the transversality condition, $\xi_T = \Delta W$ and $\eta_T = 0$, where ΔW comes from the terminal term in the utility function (13). Given $a_t = 1$ for all t, we can solve for the differential equation (36) as

$$\xi_t = \int_t^T \big(\bar{W} - W_s^A - \eta_s\big) \lambda_b e^{-(\lambda_b + \lambda_s)(s-t)} ds + e^{-(\lambda_b + \lambda_s)(T-t)} \Delta W.$$

Similarly, solving the differential equation (37) and plugging in the transversality condition, we have $\eta_t = \int_t^T c e^{-\rho\tau} d\tau$ for $t \leq T$. Therefore,

$$\xi_t = \int_t^T \big(\bar{W} - W_s^A - \int_s^T c e^{-\rho\tau} d\tau\big) \lambda_b e^{-(\lambda_b + \lambda_s)(s-t)} ds + e^{-(\lambda_b + \lambda_s)(T-t)} \Delta W.$$

$\square$

Proof of Lemma 5. Denote the payment by ϕ_T. If $T \leq \bar{t}$, then

$$\begin{aligned} \xi_t &= \int_t^T (e^{-\rho T} \phi_T - \int_s^{\bar{t}} c e^{-\rho\tau} d\tau) \lambda_b e^{-(\lambda_b + \lambda_g)(s-t)} ds \\ &= \frac{\lambda_b}{\lambda_b + \lambda_g} (1 - e^{-(\lambda_b + \lambda_g)(T-t)})(e^{-\rho T} \phi_T + \frac{c}{\rho} e^{-\rho \bar{t}}) - \frac{c}{\rho} \cdot \frac{\lambda_b}{\rho + \lambda_b + \lambda_g} e^{-\rho t} (1 - e^{-(\rho + \lambda_b + \lambda_g)(T-t)}). \end{aligned}$$

Taking the second order derivative,

$$\xi_t'' = -\lambda_b(\lambda_b+\lambda_g)e^{-(\lambda_b+\lambda_g)(T-t)}(e^{-\rho T}\phi_T + \frac{c}{\rho}e^{-\rho\bar{t}})$$
$$-\frac{\rho c\lambda_b}{\rho+\lambda_b+\lambda_g}e^{-\rho t} + \frac{c}{\rho}\cdot\frac{\lambda_b(\lambda_b+\lambda_g)^2}{\rho+\lambda_b+\lambda_g}e^{-\rho T}e^{-(\lambda_b+\lambda_g)(T-t)}.$$

By $\xi_0 \geq 0$ and $e^{-(\lambda_b+\lambda_g)T} > e^{-(\rho+\lambda_b+\lambda_g)T}$, we obtain $\frac{\lambda_b}{\lambda_b+\lambda_g}(e^{-\rho T}\phi_T + \frac{c}{\rho}e^{-\rho\bar{t}}) > \frac{c}{\rho}\cdot\frac{\lambda_b}{\rho+\lambda_b+\lambda_g}$. As a result, $\xi_t'' < -\lambda_b(\lambda_b+\lambda_g)e^{-(\lambda_b+\lambda_g)(T-t)}(e^{-\rho T}\phi_T + \frac{c}{\rho}e^{-\rho\bar{t}}) + \frac{c}{\rho}\cdot\frac{\lambda_b(\lambda_b+\lambda_g)^2}{\rho+\lambda_b+\lambda_g}e^{-(\lambda_b+\lambda_g)(T-t)} < 0$. The proof with $T > \bar{t}$ is very similar and thus we omit the detail.

On the other hand, since the agent exerts full effort by t,

$$p_t = \frac{\lambda_g(1-e^{-(\lambda_b+\lambda_g)t})}{\lambda_g+\lambda_b e^{-(\lambda_b+\lambda_g)t}}. \tag{38}$$

Thus, $\frac{e^{-\rho t}}{1-p_t} = e^{(\lambda_b+\lambda_g-\rho)t}(\frac{\lambda_g}{\lambda_b+\lambda_g} + \frac{\lambda_b}{\lambda_b+\lambda_g}e^{-(\lambda_b+\lambda_g)t})$, which is apparently convex.

In conclusion, $\mu_t = \lambda_g\xi_t - \frac{ce^{-\rho t}}{1-p_t}$ is concave. □

Proof of Lemma 6. We first prove (ii). Suppose there are two lump-sum payments, ϕ_{t_1} at t_1 and ϕ_{t_2} at t_2, with $t_1 < t_2$. Define $W_{t_1} = e^{-\rho t_1}\phi_{t_1}$ and $W_{t_2} = e^{-\rho t_2}\phi_{t_2}$. Further, define p_1 as the probability that the agent cannot get ϕ_{t_1} given the effort process. Then

$$p_1 = \int_0^{t_1} \lambda_b e^{-\int_0^s(\lambda_b+\lambda_g a_\tau)d\tau}ds. \tag{39}$$

Define p_2 correspondingly. Choose $t' \in (t_1, t_2)$ and $\phi_{t'}$ such that

$$\begin{cases} W_{t_1} + W_{t_2} = W_{t'} \\ t_1 W_{t_1} + t_2 W_{t_2} = t' W_{t'} \end{cases} \tag{40}$$

where $W_{t'} := e^{-\rho t'}\phi_{t'}$. Define p' in the same way as p_1. First we consider the case where $t_1 < t_2 \leq \bar{t}$. By $a_\tau = 1$, $p_1 = \frac{\lambda_b}{\lambda_b+\lambda_g}(1-e^{-(\lambda_b+\lambda_g)t_1})$. p_2 and p' are given correspondingly. By

Jensen's inequality,

$$
\begin{aligned}
e^{-(\lambda_b+\lambda_g)t_1}W_{t_1}+e^{-(\lambda_b+\lambda_g)t_2}W_{t_2} &= W_{t'}(e^{-(\lambda_b+\lambda_g)t_1}\frac{W_{t_1}}{W_{t'}}+e^{-(\lambda_b+\lambda_g)t_2}\frac{W_{t_2}}{W_{t'}}) \\
&> W_{t'}e^{-(\lambda_b+\lambda_g)(t_1\frac{W_{t_1}}{W_{t'}}+t_2\frac{W_{t_2}}{W_{t'}})} \\
&= W_{t'}e^{-(\lambda_b+\lambda_g)t'}. \qquad (41)
\end{aligned}
$$

As a result, $p_1W_{t_1}+p_2W_{t_2}<p'W_{t'}$. By the expression of ξ_t in (14), ξ_0 is increased with the single payment at t'.

Next consider the incentive at $t\le t_1$. Define $p_{1t}:=\int_t^{t_1}\lambda_b e^{-\int_t^s(\lambda_b+\lambda_g a_\tau)d\tau}ds$. Then $p_{1t}=e^{(\lambda_b+\lambda_g)t}(p_1-\frac{\lambda_b}{\lambda_b+\lambda_g}(1-e^{-(\lambda_b+\lambda_g)t}))$. Define p_{2t} and p'_t in the same way. By $W_{t_1}+W_{t_2}=W_{t'}$ and $p_1W_{t_1}+p_2W_{t_2}<p'W_{t'}$, we have $p_{1t}W_{t_1}+p_{2t}W_{t_2}<p'_tW_{t'}$. According to the expression of ξ_t in (14), ξ_t is also increased with the single payment at t'.

Finally consider the cost for the principal, which can be expressed by $(1-p_1)e^{(\rho-r)t_1}W_{t_1}+(1-p_2)e^{(\rho-r)t_2}W_{t_2}$ and $(1-p')e^{(\rho-r)t'}W_{t'}$, respectively. Again, by Jensen's inequality,

$$
(1-p_1)e^{(\rho-r)t_1}W_{t_1}+(1-p_2)e^{(\rho-r)t_2}W_{t_2}>(1-p')e^{(\rho-r)t'}W_{t'}. \qquad (42)
$$

In other words, the cost is reduced with the single payment at t'.

The proof for the case where $\bar{t}\le t_1<t_2$ is essentially the same and we omit the detail.

Now we prove (i). Without loss of generality, we focus on the single payment. Denote the time of the payment by t_0 and the time-0 discounted amount by W_{t_0}. To make ensure that $\mu_t\ge 0$, we need $p_{t_0}W_{t_0}\ge C$, where C is a constant and $p_{t_0}:=\int_t^{t_0}\lambda_b e^{-\int_t^s(\lambda_b+\lambda_g a_\tau)d\tau}ds$. Define $p_0:=\int_0^{t_0}\lambda_b e^{-\int_0^s(\lambda_b+\lambda_g a_\tau)d\tau}ds$. Then the principal's objective is to minimize $(1-p_0)e^{(\rho-r)t_0}W_{t_0}$. Since the constraint must be binding, the principal is essentially minimizing $\frac{(1-p_0)}{p_{t_0}}e^{(\rho-r)t_0}$.

If $t_0\le\bar{t}$, then $p_{t_0}=\frac{\lambda_b}{\lambda_b+\lambda_g}(1-e^{-(\lambda_b+\lambda_g)(t_0-t)})$ and $p_0=\frac{\lambda_b}{\lambda_b+\lambda_g}(1-e^{-(\lambda_b+\lambda_g)t_0})$. For convenience, define $A_t:=e^{-(\lambda_b+\lambda_g)t}$. By first-order conditions, $e^{-(\lambda_b+\lambda_g)(t_1^*-t)}$ is the solution to the equation $(\lambda_bA_t+\lambda_g)\frac{x}{1-x}=\frac{\rho-r}{\lambda_b+\lambda_g}(\lambda_bA_tx+\lambda_g)$, where t_1^* is the optimal payment date given $t_0\le\bar{t}$. It is easy to show that this equation has a unique solution in between 0 and 1. Since the cost goes to infinity as $x\to 0$ or $x\to 1$, t_1^* must minimize the cost. If $t_1^*>\bar{t}$, then it implies that the optimal payment date should be after $\bar{t}$.

If $t_0 > \bar{t}$, then $p_{t_0} = \frac{\lambda_b}{\lambda_b+\lambda_g}(1 - e^{-(\lambda_b+\lambda_g)(\bar{t}-t)}) + e^{-(\lambda_b+\lambda_g)(\bar{t}-t)}(1 - e^{-\lambda_b(t_0-\bar{t})})$ and $p_0 = \frac{\lambda_b}{\lambda_b+\lambda_g}(1 - e^{-(\lambda_b+\lambda_g)\bar{t}}) + e^{-(\lambda_b+\lambda_g)\bar{t}}(1 - e^{-\lambda_b(t_0-\bar{t})})$. By first-order conditions, $e^{-\lambda_b(t_2^*-t)}$ is the solution to the equation

$$\frac{(\lambda_b+\lambda_g)(\lambda_b e^{-\lambda_g\bar{t}-\lambda_b t} + \lambda_g e^{-\lambda_g(\bar{t}-t)})}{\lambda_b + \lambda_g e^{-(\lambda_b+\lambda_g)(\bar{t}-t)} - (\lambda_b+\lambda_g)e^{-\lambda_g(\bar{t}-t)}x} \cdot x = \frac{\rho - r}{\lambda_b}(\lambda_g - \lambda_g e^{-(\lambda_b+\lambda_g)\bar{t}} + (\lambda_b+\lambda_g)e^{-\lambda_g\bar{t}-\lambda_b t}x). \tag{43}$$

It is easy to verify that there exists one positive solution. When $x = 0$, LHS<RHS, which means that postponing the payment indefinitely is not optimal. When $x = 1$, if LHS<RHS, then the cost is increasing in t^* in the whole region, which means that optimally the payment should be before $\bar{t}$. By the analysis above, t_1^* is optimal. If LHS>RHS when $x = 1$, on the other hand, there exists a unique solution between 0 and 1. Then t_2^* is optimal . Specifically, when $t = \bar{t}$, LHS>RHS when $x = 1$, which is consistent with our analysis.

In conclusion, if $\frac{(\lambda_b+\lambda_g)(\lambda_b e^{-\lambda_g\bar{t}-\lambda_b t}+\lambda_g e^{-\lambda_g(\bar{t}-t)})}{\lambda_b+\lambda_g e^{-(\lambda_b+\lambda_g)(\bar{t}-t)}-(\lambda_b+\lambda_g)e^{-\lambda_g(\bar{t}-t)}} < \frac{\rho-r}{\lambda_b}(\lambda_g - \lambda_g e^{-(\lambda_b+\lambda_g)\bar{t}} + (\lambda_b+\lambda_g)e^{-\lambda_g\bar{t}-\lambda_b t})$, then the optimal payment date is before the termination and given by t_1^*. If $t_1^* > \bar{t}$, then the optimal payment date is after the termination and given by t_2^*. Otherwise, we need to compute the expected cost induced by t_1^* and t_2^* respectively, and pick the smaller one. □

Proof Sketch of Proposition 4. By Lemma 5, we first look at μ_0 and $\mu_{\bar{t}}$.

Case 1. $\mu_{\bar{t}} > 0$.

Then there exists some $t < \bar{t}$ such that $\mu_t = 0$. Denote the largest one by t. Suppose $t > 0$. By Lemma 5, there must be a lump-sum payment at t, otherwise $\mu_{t-\epsilon} < 0$ for ϵ small enough. Also, another lump-sum payment after $\bar{t}$ is required to ensure $\mu_{\bar{t}} > 0$. Then by Lemma 6, the principal can delay the payment at t and advance the payment after $\bar{t}$, such that all the incentives are satisfied and the cost is smaller. Thus, $t = 0$. By concavity, the only binding constraint is $\mu_0 = 0$. By Lemma 6, the single payment is optimal.

As $\bar{t} \to 0$, payments before the termination have no effects on μ_t and thus all payments are after $\bar{t}$. Therefore, there exists $T_1 > 0$, such that a single payment is optimal if $\bar{t} < T_1$. If $\lim_{\bar{t}\to 0} \mu_0' > 0$, then $\mu_{\bar{t}} > 0$ when $\bar{t}$ is small enough, which implies that $\mu_{\bar{t}} > 0$ is optimal. On the other hand, if $\lim_{\bar{t}\to 0} \mu_0' \leq 0$, then we must have $\mu_{\bar{t}} = 0$ when $\bar{t}$ is small enough.

Case 2. $\mu_{\bar{t}} = 0$ and $\mu_0 > 0$.

Let t be the smallest number such that $\mu_t = 0$. Suppose $t < \bar{t}$. By Lemma 5, there must be a lump-sum payment at t. However, since $\mu_0 > 0$, $\mu_{\hat{t}} > 0$ for all $\hat{t} \in (0, t)$. Thus, decreasing the amount of the payment at t by a little bit can ensure $\mu_{\hat{t}} > 0$ for all $\hat{t}$. As a result, the only binding constraint is $\mu_{\bar{t}} = 0$. By Lemma 6, a single payment after $\bar{t}$ is optimal. We then need to check the incentive at 0. When $\lambda_b + \lambda_g \geq \rho$, $\mu_0 > 0$ as $\bar{t} \to \infty$. Therefore, there exists $T_2 > 0$, such that a single payment is optimal whenever $\bar{t} > T_2$.

Case 3. $\mu_{\bar{t}} = \mu_0 = 0$.

Given the optimal payment subject to $\mu_{\bar{t}} = 0$, μ_0 may be negative. For example, when $\lambda_b + \lambda_g < \rho$ and $\bar{t} \to \infty$. Under this circumstance, both constraints bind, i.e., $\mu_{\bar{t}} = \mu_0 = 0$. There are two ways to increase the incentive at $t = 0$. The first one is to increase and bring forward the payment after the termination, and the second one is to make another payment before the termination. As $\bar{t} \to \infty$, the benefit of increasing and advancing the payment after the termination goes to zero. Therefore, when $\bar{t}$ is large enough, a lump-sum payment before the termination is optimal. Since this payment has no effects on $\mu_{\bar{t}}$, the payment date should be the one given in Lemma 6. Denote it by t_1.

Given these two payments, if $\mu_{t_1} \geq 0$, then by concavity $\mu_t \geq 0$ for all t. If $\mu_{t_1} < 0$, then the constraint at t_1 is also binding. Note that the date of the first payment should still be t_1. Again, there are two ways to increase the incentive at t_1. When $\bar{t}$ is large enough, it is optimal to make payments after t_1 and before the termination.

By the expression of ξ_t, the lump-sum payment changes the first-order derivative μ'_t, but the flow payment only changes the second-order derivative μ''_t. If there are no lump-sum payments in between t_1 and $\bar{t}$, then either the constrains are not binding anywhere or they are binding everywhere in between. If the first case is true, then by Lemma 6, it is better to combine the payments. Therefore, the constraints must be binding everywhere in between t_1 and $\bar{t}$. As a result, the flow payment goes from t_1 up until $\bar{t}$, and the payment after the termination makes sure that $\mu'_{\bar{t}} = 0$.

Now suppose there are lump-sum payments in between t_1 and $\bar{t}$. Denote the date of the first lump-sum payment after t_1 by t_2. Then the constraints between t_1 and t_2 must be binding. Otherwise it is better to combine the payments at t_1 and t_2. As a result, there is a flow payment in between t_1 and t_2, such that $\mu_t = 0$ for $t \in [t_1, t_2]$. Since there is a lump-sum payment at t_2, the right derivative at t_2 is positive. Suppose there is another lump-sum payment at t_3 in between t_2 and $\bar{t}$. Then $\mu_t > 0$ for $t \in (t_2, t_3)$. By Lemma 6, it is better to combine payments at t_2 and t_3. Therefore, no payments are in between t_2 and $\bar{t}$.

In conclusion, when $\lambda_b + \lambda_g < \rho$, there exists $T_4 > T_3 > 0$, such that the optimal payment scheme consists of two lump-sum payments when $\bar{t} \in (T_3, T_4)$, and a combination of lump-sum payments and flow payments when $\bar{t} > T_4$.

Finally we show that the flow payment is increasing when $\lambda_b + \lambda_g < \rho$. Taking the derivative of ξ_t, we have $\xi_t' = -\lambda_b(W_\infty - W_t - \int_t^{\bar{t}} ce^{-\rho\tau} d\tau) + (\lambda_b + \lambda_g)\xi_t$. Since $\mu_t = 0$ and $\mu_t' = 0$, $\lambda_g \xi_t = \frac{ce^{-\rho t}}{1-p_t}$ and $\lambda_g \xi_t' = (\frac{ce^{-\rho t}}{1-p_t})_t'$. Therefore,

$$\lambda_b(W_\infty - W_t - \int_t^{\bar{t}} ce^{-\rho\tau} d\tau) = \frac{\lambda_b + \lambda_g}{\lambda_g} \cdot \frac{ce^{-\rho t}}{1 - p_t} - \frac{1}{\lambda_g}(\frac{ce^{-\rho t}}{1 - p_t})_t'. \tag{44}$$

Taking the derivative on both sides, we get $\lambda_b e^{-\rho t}(w_t + c) = (\frac{\lambda_b+\lambda_g}{\lambda_g} \cdot \frac{ce^{-\rho t}}{1-p_t})_t' - \frac{1}{\lambda_g}(\frac{ce^{-\rho t}}{1-p_t})_t''$, where w_t is the flow payment rate. Substituting the expression of p_t in (38), w_t can be expressed as

$$w_t = \frac{\rho c(\rho + \lambda_b + \lambda_g)}{\lambda_g(\lambda_b + \lambda_g)} + \frac{\rho c(\rho - \lambda_b - \lambda_g)}{\lambda_b(\lambda_b + \lambda_g)} \cdot e^{(\lambda_b + \lambda_g)t}, \tag{45}$$

which is increasing over t. □

Proof of Theorem 3. We first prove that the necessary condition is sufficient. Given any payment scheme, suppose $a_t = 1$ for all $t \leq \bar{t}$ satisfies the necessary condition in Lemma 4. We want to show that it is essentially unique.

Suppose $a < 1$ on a set of positive measure. Denote the corresponding variables as $\hat{a}_t$, $\hat{p}_t$, and $\hat{\xi}_t$, and denote the original variables by a_t, p_t, and ξ_t. By definition, $p_0 = \hat{p}_0 = 0$. By the expression of p_t in (12), when $p_t = \hat{p}_t$, we must have $p_{t+dt} \geq \hat{p}_{t+dt}$ since $a_t \geq \hat{a}_t$. By continuity of p_t, $p_t \geq \hat{p}_t$ for all t.

Define $\hat{t}$ as the largest t such that $t \leq T$ and $\hat{a}_t < 1$. By (14), $\xi_{\hat{t}} = \hat{\xi}_{\hat{t}}$. Since $\hat{a}_t < a_t$ on a set of positive measure, $\hat{p}_{\hat{t}} < p_{\hat{t}}$. Then $\lambda_s(1 - \hat{p}_{\hat{t}})\hat{\xi}_{\hat{t}} > \lambda_s(1 - p_{\hat{t}})\xi_{\hat{t}} \geq ce^{-\rho t}$. The necessary condition therefore requires $\hat{a}_{\hat{t}} = 1$, which is a contradiction.

In conclusion, if $a_t = 1$ for all $t \leq T$ satisfies the necessary condition, then it is the essentially unique effort process satisfying the necessary condition. Since W_t is piecewise continuous, there exists a best reply to the agent's optimal control problem. Therefore,

$a_t = 1$ for all $t \leq T$ must be the best reply.

Next we show that the optimal effort process must be frontloaded. Since both utilities are linear in a_t, without loss of generality we assume that $a_t = 1$ or $a_t = 0$ for all t. Suppose the effort is not frontloaded, then there exists a non-degenerate interval (t_1, t_2), such that $a_t = 0$ for a.e. $t \in (t_1, t_2)$ and $a_t = 1$ for a.e. $t \in (t_2, T)$.

Consider an alternative payment scheme $\tilde{W}$, with $\tilde{W}_t = W_t$ for $t \in [0, t_1]$, $\tilde{W}_t = W_t + e^{\rho(t_2 - t_1)}(W_{t_2+t-t_1} - W_{t_2})$ for $t \in (t_1, t_2)$, and $\tilde{W}_t = \tilde{W}_{t_2} + e^{\rho(t_2-t_1)}(W_{t_2+t-t_1} - W_{2t_2-t_1})$ for $t > t_2$. In words, the new payment scheme brings forward the payments after t_2 by $t_2 - t_1$. At the same time, set $\tilde{T} = T - (t_2 - t_1)$. Define $\tilde{a}_t$ as the new effort process and define $\tilde{\xi}_t$ and $\tilde{p}_t$ correspondingly. We first assume $\tilde{a}_t = a_t$ for all $t \in [0, t_1]$ and check it later. Then $\tilde{p}_{t_1} < p_{t_2}$. Since $a_t = 1$ on (t_2, T) and $e^{\rho t}\tilde{\xi}_t \geq e^{\rho(t+t_2-t_1)}\xi_{t+t_2-t_1}$ for $t \geq t_1$, we must have $\tilde{a}_t = 1$ on $(t_1, \tilde{T})$. By $\mu_{t_2} \geq \mu_{t_1}$ and $p_{t_2} \leq p_{t_1}$, we obtain $e^{\rho t_2}\xi_{t_2} \geq e^{\rho t_1}\xi_{t_1}$. Together with $e^{\rho t_1}\tilde{\xi}_{t_1} \geq e^{\rho t_2}\xi_{t_2}$, we conclude $\tilde{\xi}_{t_1} \geq \xi_{t_1}$.

By (14), for any $t < t_1$, ξ_t can be expressed as

$$\xi_t = \int_t^{t_1} (\bar{W} - W_s^A - \int_s^T c a_\tau e^{-\rho\tau} d\tau) \lambda_b e^{-\int_t^s (\lambda_b + \lambda_s a_\tau) d\tau} ds + e^{-\int_t^{t_1} (\lambda_b + \lambda_s a_\tau) d\tau} \xi_{t_1}. \tag{46}$$

Note that $\bar{W} - W_s^A - \int_s^T c a_\tau e^{-\rho\tau} d\tau > 0$ whenever $a_s > 0$. Otherwise the agent gets a negative payoff when he achieves success at s so that $a_s > 0$ cannot be optimal. Specifically, $\bar{W} - W_{t_2}^A - \int_{t_2}^T c a_\tau e^{-\rho\tau} d\tau > 0$. Since the new process brings forward payments as well as effort after t_2 by $t_2 - t_1$, we have $\tilde{\bar{W}} - \tilde{W}_s^A - \int_s^T c\tilde{a}_\tau e^{-\rho\tau} d\tau > \bar{W} - W_s^A - \int_s^T c a_\tau e^{-\rho\tau} d\tau$ for any $s \leq t_1$. Also, since we assumed $\tilde{a}_t = a_t$ for all $t \in [0, t_1]$, then by $\tilde{\xi}_{t_1} \geq \xi_{t_1}$, we conclude that $\tilde{\xi}_t \geq \xi_t$ for all $t \geq t_1$.

If $a_t = 1$ for all $t \leq t_1$, then $\tilde{a}_t = 1$ must satisfy the necessary condition. Thus, $\tilde{a}_t = a_t$ for all $t \in [0, t_1]$. If $\{t \leq t_1 : a_t = 0\}$ has a positive measure, then we can repeat the above procedure in which the payments and the deadline are moved forward. By the same argument, ξ will be larger with the new process and $a = 1$ will still be optimal. Repeat this procedure until we obtain $\tilde{a}_t = 1$ for all t. Then this effort process satisfies the necessary condition.

Finally we show that the principal is better off with the new contract. The principal's expected benefit from success must be larger than the expected cost of aggregate payments, otherwise she would not contract with the agent. Denote the principal's continuation utility at t_2 conditional on success by $\pi_{t_2}^S$ and the continuation utility conditional on no success by $\pi_{t_2}^N$. Then $pi_{t_2} = p_{t_2}\pi_{t_2}^S + (1 - p_{t_2})\pi_{t_2}^N$ and $\pi_{t_2}^S > \pi_{t_2}^N > 0$. With the new policy, the discount is smaller and thus both $\pi_{t_2}^S$ and $\pi_{t_2}^N$ are larger. Therefore, the principal is better off. $\square$